Milan J. Math. Vol. 79 (2011) 657–687
DOI 10.1007/s00032-011-0167-2
Published online November 3, 2011

Milan Journal of Mathematics

On Caffarelli–Kohn–Nirenberg-type Inequalities for the Weighted Biharmonic Operator in Cones

Paolo Caldiroli and Roberta Musina

Abstract. We investigate Caffarelli–Kohn–Nirenberg-type inequalities for the weighted biharmonic operator in cones, both under Navier and Dirichlet boundary conditions. Moreover, we study existence and qualitative properties of extremal functions. In particular, we show that in some cases extremal functions do change sign; when the domain is the whole space, we prove some breaking symmetry phenomena.

Mathematics Subject Classification (2010). Primary 26D10; Secondary 47F05.

Keywords. Caffarelli–Kohn–Nirenberg–type inequalities, weighted biharmonic operator, dilation invariance, breaking positivity, breaking symmetry.

1. Introduction

In this paper we study second order interpolation inequalities with weights being powers of the distance from the origin, and involving functions defined on dilation invariant domains. More precisely, for any regular domain Σ in the unit sphere $\mathbb{S}^{n-1}$ we denote by $\mathcal{C}_\Sigma$ the cone

$$\mathcal{C}_\Sigma := \{ \, r\sigma \mid r > 0 \, , \, \sigma \in \Sigma \, \} \, . \tag{1.1}$$

We are mainly interested in a class of inequalities of the form

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx \geq C \left(\int_{\mathcal{C}_\Sigma} |x|^{-\beta} |u|^q dx \right)^{2/q} \text{ for any } u \in C^2_c(\overline{\mathcal{C}_\Sigma} \setminus \{0\}), \tag{1.2}$$

where $q > 2$ and $\alpha \in \mathbb{R}$ are given parameters, and where $C^2_c(\overline{\mathcal{C}_\Sigma} \setminus \{0\})$ is the space of functions in $C^2(\overline{\mathcal{C}_\Sigma})$ vanishing on $\partial\mathcal{C}_\Sigma$ and in a neighborhood of 0 and of ∞. The best constant in (1.2) is given by

$$S_q(\mathcal{C}_\Sigma;\alpha) := \inf_{\substack{u\in C^2_c(\overline{\mathcal{C}_\Sigma}\setminus\{0\})\\ u\neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u|^q dx\right)^{2/q}}\,. \tag{1.3}$$

A simple rescaling argument shows that $S_q(\mathcal{C}_\Sigma;\alpha)$ vanishes unless β is such that

$$\frac{n-\alpha}{2} + \frac{n-\beta}{q} = n-2. \tag{1.4}$$

Therefore, from now on we will assume that (1.4) holds. We point out that condition (1.4) defines the "weighted critical hyperbola" (case $p=2$) introduced in [9] and [17] in the context of solvability of Hardy–Henon-type elliptic systems in bounded domains. See also [5] for related results on symmetry breaking.

If $n \geq 5$, then the Sobolev embedding theorem implies that a necessary condition to have $S_q(\mathcal{C}_\Sigma;\alpha) > 0$ is that $q \leq 2^{**}$, where 2^{**} is the critical Sobolev exponent:

$$2^{**} = \frac{2n}{n-4}\,.$$

Our goal is to estimate the best constant $S_q(\mathcal{C}_\Sigma;\alpha)$ under the above assumptions on β and q. Moreover, we will study existence and qualitative properties of functions achieving $S_q(\mathcal{C}_\Sigma;\alpha)$ on a suitable function space.

Let us notice that (1.2) can not be obtained by iterating the first order Caffarelli-Kohn-Nirenberg inequalities in [4], see Remark 2.3. We quote [15], [23] and references there-in, for a related interpolation inequality due to C.S. Lin.

If $q=2$, then (1.4) gives $\beta = \alpha - 4$ and (1.2) becomes

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx \geq C \int_{\mathcal{C}_\Sigma} |x|^{\alpha-4}|u|^2 dx \quad \text{for any } u \in C^2_c(\overline{\mathcal{C}_\Sigma}\setminus\{0\}). \tag{1.5}$$

A first version of this inequality has been introduced by F. Rellich in 1953 (see [19] and [20]) in case $\mathcal{C}_\Sigma = \mathbb{R}^n\setminus\{0\}$ and $\alpha = 0$. For general cones $\mathcal{C}_\Sigma$ and parameters $\alpha\in\mathbb{R}$ we refer to [7], where it is proved that the best constant in (1.5) is exactly the square of the distance of $-\gamma_\alpha$ from the Dirichlet spectrum $\Lambda(\Sigma)$ of the Laplace-Beltrami operator on Σ, where

$$\gamma_\alpha = \left(\frac{n-2}{2}\right)^2 - \left(\frac{\alpha-2}{2}\right)^2. \tag{1.6}$$

For instance, taking $\Sigma = \mathbb{S}^{n-1}$ or $\Sigma =$ half-sphere we have

$$\begin{aligned} S_2(\mathbb{R}^n\setminus\{0\};\alpha) &= \min_{k\in\mathbb{N}\cup\{0\}} |\gamma_\alpha + k(n-2+k)|^2 \\ S_2(\mathbb{R}^n_+;\alpha) &= \min_{k\in\mathbb{N}} |\gamma_\alpha + k(n-2+k)|^2\,, \end{aligned}$$

where $\mathbb{R}^n_+$ denotes any homogeneous half-space.

In our first theorem we show that $S_q(\mathcal{C}_\Sigma,\alpha) > 0$ whenever the best constant in the weighted Rellich inequality is positive.

Theorem 1.1. *Let $\alpha \in \mathbb{R}$ and let $\Sigma \subseteq \mathbb{S}^{n-1}$ be a domain of class C^2. Let $q > 2$ be a given exponent, and assume that $q \le 2^{**}$ if $n \ge 5$. Then $S_q(\mathcal{C}_\Sigma;\alpha) > 0$ if and only if $-\gamma_\alpha \notin \Lambda(\Sigma)$.*

If $-\gamma_\alpha$ is not a Dirichlet eigenvalue on Σ, then we can define the Hilbert space $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ as the completion of $C^2_c(\overline{\mathcal{C}_\Sigma} \setminus \{0\})$ with respect to the norm

$$\|u\|_{2,\alpha} = \left(\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx \right)^{1/2} . \tag{1.7}$$

If $n \ge 5$ and $\alpha = 0$, then $\mathcal{N}^2(\mathbb{R}^n \setminus \{0\}; 0) = \mathcal{D}^2(\mathbb{R}^n)$, see Remark 2.2. In general, it holds that

$$S_q(\mathcal{C}_\Sigma;\alpha) = \inf_{\substack{u \in \mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)\\ u \ne 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-\beta} |u|^q dx \right)^{2/q}} . \tag{1.8}$$

In the rest of the paper we study the existence of extremals for $S_q(\mathcal{C}_\Sigma;\alpha)$ and their qualitative properties.

When $q = 2$ it was shown in [7] that the best constant $S_2(\mathcal{C}_\Sigma;\alpha)$ is never attained in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$. Another remarkable case is $\Sigma = \mathbb{S}^{n-1}$, $n \ge 5$ and $q = 2^{**}$. Then $\mathcal{C}_\Sigma = \mathbb{R}^n \setminus \{0\}$, $\beta = 0$ and $S_{2^{**}}(\mathbb{R}^n \setminus \{0\}; 0)$ equals the Sobolev constant

$$S^{**} = \inf_{\substack{u \in \mathcal{D}^2(\mathbb{R}^n)\\ u \ne 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |u|^{2^{**}} dx \right)^{2/2^{**}}} . \tag{1.9}$$

It is well known that the best constant S^{**} is achieved by an explicitly known radially symmetric and positive function, see for instance [22].

In the next results we study the attainability of $S_q(\mathcal{C}_\Sigma;\alpha)$. By standard arguments, extremals for $S_q(\mathcal{C}_\Sigma;\alpha)$ are, up to a Lagrange multiplier, ground state solutions of the equation

$$\Delta(|x|^\alpha \Delta u) = |x|^{-\beta} |u|^{q-2} u \quad \text{in } \mathcal{C}_\Sigma \tag{1.10}$$

under Navier boundary conditions $u = \Delta u = 0$ on $\partial\mathcal{C}_\Sigma$, in case Σ is properly contained in $\mathbb{S}^{n-1}$.

Notice that the minimization problem (1.9) is noncompact, due to the action of the group of dilations in $\mathbb{R}^n$. However, when q is subcritical the infimum $S_q(\mathcal{C}_\Sigma;\alpha)$ is always achieved:

Theorem 1.2. *Let $q > 2$ be a given exponent such that $q < 2^{**}$ if $n \ge 5$. Let Σ be a domain in $\mathbb{S}^{n-1}$ of class C^2, with $-\gamma_\alpha \notin \Lambda(\Sigma)$. Then $S_q(\mathcal{C}_\Sigma;\alpha)$ is achieved in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$.*

When $n \ge 5$ and $q = 2^{**}$ it holds that $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) \le S^{**}$ for any cone $\mathcal{C}_\Sigma$ and for any admissible exponent α, see Proposition 2.4. In this case the group of translations in $\mathbb{R}^n$ may produce lack of compactness and nonexistence phenomena. As usual, the strict inequality guarantees the compactness of all minimizing sequences.

Theorem 1.3. *Let $n \geq 5$ and let Σ be a domain in $\mathbb{S}^{n-1}$ of class C^2. Assume that $-\gamma_\alpha \notin \Lambda(\Sigma)$. If $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$, then $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha)$ is achieved in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$.*

The above stated theorems constitute the second order version of well known results related to the classical Caffarelli-Kohn-Nirenberg inequalities [4] for first order operators, see for instance [6], [8] and the references therein.

However, when we push further the study of minimization problems (1.8), some relevant differences appear. Firstly we can show that in the case of critical exponent the strict inequality $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ holds in the following cases.

Theorem 1.4. *If $n \geq 6$ and $|\alpha - 2| > 2$, then $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ for every $\Sigma \subseteq \mathbb{S}^{n-1}$. If $n = 5$ and $2 < |\alpha - 2| < \sqrt{13}$, then $S_{2^{**}}(\mathbb{R}^5 \setminus \{0\};\alpha) < S^{**}$.*

The previous result is discussed separately for dimensions $n \geq 6$ in Theorem 2.5, whereas for $n = 5$ is a special case of an estimate proved in Theorem 2.9.

The difference between the case $n = 5$ and $n \geq 6$ seems to be not purely techical. There is indeed a deep connection between the validity of the strict inequality $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ and the existence of ground state solutions for the following Dirichlet problem:

$$\begin{cases} \Delta^2 u + \lambda \Delta u = |u|^{2^{**}-2}u & \text{in } B \\ u = |\nabla u| = 0 & \text{on } \partial B. \end{cases} \tag{1.11}$$

Here $B \subset \mathbb{R}^n$ is the unit ball and λ is a given real parameter. As a by-product of our computations we can prove a Brezis–Nirenberg-type result for problem (1.11) in the spirit of the celebrated paper [3], see Appendix A. By adapting a terminology which has been introduced by Pucci and Serrin in [18], we can assert that $n = 5$ is the unique **weakly critical dimension** for problem (1.11).

When $\mathcal{C}_\Sigma = \mathbb{R}^n \setminus \{0\}$ **breaking symmetry** can be observed as well. In particular, from the results in Section 5 it follows that minimizers for $S_q(\mathcal{C}_\Sigma;\alpha)$ may be not radially symmetric. In Theorems 5.1 and 5.2 we show that breaking symmetry occurs, for instance, when $-\gamma_\alpha$ is close to a Dirichlet eigenvalue on the sphere or when $|\alpha|$ is large enough.

Even in correspondence of the critical exponent, breaking symmetry occurs: for $|\alpha|$ large enough there exist minimizers both for $S_{2^{**}}(\mathbb{R}^n \setminus \{0\};\alpha)$ and for the corresponding radial best constant $S^{\mathrm{rad}}_{2^{**}}(\mathbb{R}^n;\alpha)$, defined in (2.10), and the minimizers are different as $S_{2^{**}}(\mathbb{R}^n \setminus \{0\};\alpha) < S^{\mathrm{rad}}_{2^{**}}(\mathbb{R}^n;\alpha)$. A similar breaking symmetry phenomenon does not occur, for instance, in dealing with critical exponents in first-order Caffarelli-Kohn-Nirenberg inequalities: in that case, the best constant is not achieved, or all the minimizers are radially symmetric. We refer to [8], [11], [10] for breaking symmetry in first order Caffarelli-Kohn-Nirenberg inequalities.

Another striking difference with respect to similar first order problems, is a **breaking positivity** phenomenon. Indeed, in Section 4 we show that, in general, no extremal for (1.8) has constant sign, see Theorem 4.1.

In Section 6 we take Σ to be a proper domain in the sphere and we deal with the infimum

$$S_q^D(\mathcal{C}_\Sigma;\alpha) := \inf_{\substack{u\in C^2_c(\mathcal{C}_\Sigma)\\ u\neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha|\Delta u|^2dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u|^q dx\right)^{2/q}}\,. \tag{1.12}$$

Differently from the Navier case, it turns out that $S_q^D(\mathcal{C}_\Sigma;\alpha)$ is always positive, whenever Σ has compact closure in $\mathbb{S}^{n-1}$, with no restriction on α. We also show existence of extremals (see Theorem 6.1) which give rise to solutions of (1.10) satisfying Dirichlet boundary conditions $u = |\nabla u| = 0$ on $\partial\mathcal{C}_\Sigma$.

2. Inequalities

In this Section we prove Theorem 1.1 and some related results. We start by noticing that

$$S_q(\mathcal{C}_\Sigma;\alpha) = S_q(\mathcal{C}_\Sigma;4-\alpha) \quad \text{and} \quad S_q^D(\mathcal{C}_\Sigma;\alpha) = S_q^D(\mathcal{C}_\Sigma;4-\alpha)\,. \tag{2.1}$$

To check (2.1) use (as in [7], where $q = 2$ is assumed) the transform $u \mapsto \hat{u}$ given by

$$\hat{u}(x) = |x|^{2-n}u(|x|^{-2}x).$$

The proof of Theorem 1.1 is based on the Emden-Fowler transform $u \mapsto w = Tu$, defined by

$$u(x) = |x|^{\frac{4-n-\alpha}{2}}\, w\left(-\log|x|, \frac{x}{|x|}\right). \tag{2.2}$$

Notice that T maps functions $u\colon \overline{\mathcal{C}_\Sigma}\setminus\{0\} \to \mathbb{R}$ into functions w on the cylinder

$$\mathcal{Z}_\Sigma := \{(s,\sigma) \in \mathbb{R}\times\mathbb{S}^{n-1} \mid s\in\mathbb{R},\ \sigma\in\Sigma\}.$$

In [7] it is noticed that for every $u \in C^2_c(\overline{\mathcal{C}_\Sigma}\setminus\{0\})$ one has $w\in C^2_c(\overline{\mathcal{Z}_\Sigma})$ and

$$\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u|^q dx = \int_{\mathcal{Z}_\Sigma} |w|^q ds d\sigma \tag{2.3}$$

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha|\Delta u|^2dx = \int_{\mathcal{Z}_\Sigma} \left(|L_\alpha w|^2 + |w_{ss}|^2 + 2|\nabla_\sigma w_s|^2 + 2\overline{\gamma}_\alpha|w_s|^2\right) ds d\sigma\,, \tag{2.4}$$

where

$$L_\alpha w = -\Delta_\sigma w + \gamma_\alpha w\,, \quad \overline{\gamma}_\alpha = \left(\frac{n-2}{2}\right)^2 + \left(\frac{\alpha-2}{2}\right)^2\,, \tag{2.5}$$

and γ_α is defined in (1.6). For every $\gamma\in\mathbb{R}$ we introduce also the value

$$m_N(\Sigma;\gamma) = \inf_{\substack{\varphi\in H^2\cap H^1_0(\Sigma)\\ \varphi\neq 0}} \frac{\displaystyle\int_\Sigma |-\Delta_\sigma\varphi + \gamma\varphi|^2 d\sigma}{\displaystyle\int_\Sigma \varphi^2 d\sigma}\,. \tag{2.6}$$

The following facts hold (see Proposition 1.1 and Theorem 2.1 in [7]).

Lemma 2.1. (i) *For every $\gamma \in \mathbb{R}$ one has that $m_N(\Sigma;\gamma) = \mathrm{dist}(-\gamma, \Lambda(\Sigma))^2$. Moreover $\varphi \in H^2 \cap H^1_0(\Sigma)$ is a minimizer for $m_N(\Sigma;\gamma)$ if and only if φ is an eigenfunction of $-\Delta_\sigma$ relative to the eigenvalue achieving the minimal distance of $-\gamma$ from $\Lambda(\Sigma)$.*

(ii) *For every $\alpha \in \mathbb{R}$ one has that $S_2(\mathcal{C}_\Sigma;\alpha) = m_N(\Sigma;\gamma_\alpha)$ with γ_α given by* (1.6).

2.1. Proof of Theorem 1.1

Assume that $-\gamma_\alpha \in \Lambda(\Sigma)$, and take a nontrivial $\varphi \in H^1_0(\Sigma)$ in the kernel of the operator L_α. Test the quotient in (1.3) with

$$u(x) = |x|^{\frac{4-n-\alpha}{2}}\,\eta(-\log|x|)\,\varphi\left(\frac{x}{|x|}\right),$$

where $\eta \in C^\infty_c(\mathbb{R})$, $\eta \neq 0$ is an arbitrary function. Using (2.3)–(2.4) we readily get

$$S_q(\mathcal{C}_\Sigma;\alpha) \le C_\varphi \frac{\displaystyle\int_{-\infty}^{\infty} |\eta''|^2 ds + \int_{-\infty}^{\infty} |\eta'|^2 ds}{\left(\displaystyle\int_{-\infty}^{\infty} |\eta|^q ds\right)^{2/q}},$$

where the constant $C_\varphi > 0$ does not depend on η. Thus $S_q(\mathcal{C}_\Sigma;\alpha) = 0$, by a simple rescaling argument.

Next, assume that $-\gamma_\alpha \notin \Lambda(\Sigma)$. By the results in [7], it turns out that the space $H^2 \cap H^1_0(\mathcal{Z}_\Sigma)$ has an equivalent norm given by

$$\|w\|^2_{H^2\cap H^1_0(\mathcal{Z}_\Sigma;\alpha)} = \int_{\mathcal{Z}_\Sigma} \left(|L_\alpha w|^2 + |w_{ss}|^2 + 2|\nabla_\sigma w_s|^2 + 2\overline{\gamma}_\alpha |w_s|^2\right) ds d\sigma.$$

Moreover, the operator T is an isomorphism between the spaces $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ and $H^2 \cap H^1_0(\mathcal{Z}_\Sigma)$ and (2.3)–(2.4) hold for every $u \in \mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$. In addition, thanks to the Sobolev embedding theorem for $H^2(\mathcal{Z}_\Sigma)$ (see [1]) and by (2.3)–(2.4), we infer that $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ is continuously embedded into $L^q(\mathcal{C}_\Sigma;|x|^{-\beta}dx)$, namely, $S_q(\mathcal{C}_\Sigma;\alpha) > 0$. $\square$

Remark 2.2. Let $\alpha \in \mathbb{R}$, $q > 2$ with $q \le 2^{**}$ if $n \ge 5$, and $\beta = n - q\,\frac{n-4+\alpha}{2}$. If $n > 4-\alpha$, then $C^2_c(\mathbb{R}^n) \subset L^q(\mathbb{R}^n;|x|^{-\beta}dx)$ and

$$S_q(\mathbb{R}^n\setminus\{0\};\alpha) = \inf_{\substack{u\in C^2_c(\mathbb{R}^n)\\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |x|^{-\beta}|u|^q dx\right)^{2/q}}. \tag{2.7}$$

Moreover if $-\gamma_\alpha \notin \Lambda(\mathbb{S}^{n-1})$ and $n > 4-\alpha$, then $C^2_c(\mathbb{R}^n)$ is dense in $\mathcal{N}^2(\mathbb{R}^n\setminus\{0\};\alpha)$. These facts can be proved in a standard way.

Remark 2.3. C.S. Lin proved in [15] several interpolation inequalities involving weighted L^p norms of the derivatives of functions $u \in C^\infty_c(\mathbb{R}^n)$. In particular, in case $n \ge 5$ he proved that for any $\alpha \in \mathbb{R}$ and for any $q \in (2, 2^{**}]$, there exists $C_L > 0$ such that

$$\max_{i,j=1,\dots,n} \int_{\mathbb{R}^n} |x|^\alpha |\partial_{ij} u|^2 dx \ge C_L \left(\int_{\mathbb{R}^n} |x|^{-\beta}|u|^q dx\right)^{2/q}$$

for any $u \in C^2_c(\mathbb{R}^n \setminus \{0\})$, where β is given by (1.4). Clearly, the best constant C_L controls $S_q(\mathbb{R}^n \setminus \{0\};\alpha)$ from above, but it always happens that

$$0 = S_q(\mathbb{R}^n \setminus \{0\};\alpha) < C_L(\alpha)$$

when $-\gamma_\alpha = k(n-2+k)$ for some positive integer k. More precisely, the functions

$$u \mapsto \max_{i,j=1,\ldots,n} \int_{\mathbb{R}^n} |x|^\alpha |\partial_{ij}u|^2 dx\ , \quad u \mapsto \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2\, dx$$

define two equivalent norms in $C^\infty_c(\mathbb{R}^n)$ if and only if $-\gamma_\alpha \notin \Lambda(\mathbb{S}^{n-1})$. Notice that the present remark improves Lemma 3.1 in [23] (in case $p = k = 2$), where $4-n < \alpha \le 0$ is assumed.

2.2. Large and strict inequalities in the limiting case

In this subsection we take $n \ge 5$ and $q = 2^{**}$. Let $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha)$, $S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha)$ be the infima defined in (1.3), (1.12) respectively. In particular,

$$S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) = \inf_{\substack{u\in C^2_c(\mathcal{C}_\Sigma)\\ u\neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |u|^{2^{**}} dx\right)^{2/2^{**}}}\ .$$

Proposition 2.4. *Let Σ be a domain in $\mathbb{S}^{n-1}$ of class C^2, $n \ge 5$, and let $\alpha \in \mathbb{R}$. Then*

$$S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) \le S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) \le S^{**}\ ,$$

where S^{**} is the Sobolev constant, given by (1.9).

Proof. The first inequality is trivial. To prove that $S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) \le S^{**}$ we fix a point $x_0 \in \mathcal{C}_\Sigma$. For an arbitrary $u \in C^2_c(\mathbb{R}^n)$, $u \neq 0$ and for any integer $h > 0$ we put

$$u_h(x) = u(h(x-x_0)).$$

If h is large enough, then the support of u_h is compactly contained in $\mathcal{C}_\Sigma$, and hence

$$\begin{aligned} S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) &\le \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u_h|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}} dx\right)^{2/2^{**}}} \\ &= \frac{\displaystyle\int_{\mathbb{R}^n} \left|\frac{y}{h}+x_0\right|^\alpha |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} \left|\frac{y}{h}+x_0\right|^{\frac{n\alpha}{n-4}} |u|^{2^{**}} dx\right)^{2/2^{**}}} \\ &= \frac{\displaystyle\int_{\mathbb{R}^n} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |u|^{2^{**}} dx\right)^{2/2^{**}}} + o(1) \end{aligned}$$

as $h \to \infty$. Since u was arbitrarily chosen, the conclusion follows. □

As concerns the validity of the strict inequality $S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ we have the following result.

Theorem 2.5. *If $n \ge 6$ and $|\alpha - 2| > 2$, then $S^D_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ for every $\Sigma \subset \mathbb{S}^{n-1}$.*

Proof. Let $a = -\alpha/2$. We notice that $|\alpha-2| > 2$ is equivalent to say that $C_a := a(a+2)(n-2)/n > 0$. By Lemma B.1 in Appendix B, there exists $T_a \in (0,1)$ such that for $0 < t \le T_a$ and for every radial mapping $u \in C^2_c(B)$, where B is the unit open ball in $\mathbb{R}^n$, one has

$$\int |tx+e|^{-2a} \left|\Delta\left(|tx+e|^a u\right)\right|^2 \le \int |\Delta u|^2 - C_a t^2 \int |\nabla u|^2. \tag{2.8}$$

Fix a point $e \in \Sigma$. Let $t_0 > 0$ be such that $e + t_0 B \subset \mathcal{C}_\Sigma$ and put $t = \frac{1}{2}\min\{t_0, T_a\}$. By Lemma A.2 in Appendix A, there exists a radially symmetric function $u \in C^2_c(B)$ such that

$$\int |\Delta u|^2 - C_a t^2 \int |\nabla u|^2 < S^{**}\left(\int |u|^{2^{**}}\right)^{2/2^{**}}. \tag{2.9}$$

Define

$$v(x) = |x|^{-\frac{\alpha}{2}} u\left(\frac{x-e}{t}\right)$$

and notice that $v \in C^2_c(\mathcal{C}_\Sigma)$ verifies

$$\int |x|^{\frac{n\alpha}{n-4}} |v|^{2^{**}} = t^n \int |u|^{2^{**}},$$

$$\int |x|^\alpha |\Delta v|^2 = t^{n-4}\int |tx+e|^{-2a}\left|\Delta\left(|tx+e|^a u\right)\right|^2.$$

Then, by (2.8) and (2.9)

$$\begin{aligned} S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) &\le \frac{\displaystyle\int |x|^\alpha |\Delta v|^2}{\left(\displaystyle\int |x|^{\frac{n\alpha}{n-4}}|v|^{2^{**}}\right)^{2/2^{**}}} = \frac{\displaystyle\int |tx+e|^{-2a}\left|\Delta\left(|tx+e|^a u\right)\right|^2}{\left(\displaystyle\int |u|^{2^{**}}\right)^{2/2^{**}}} \\ &\le \frac{\displaystyle\int |\Delta u|^2 - C_a t^2 \int |\nabla u|^2}{\left(\displaystyle\int |u|^{2^{**}}\right)^{2/2^{**}}} < S^{**}. \end{aligned}$$

$\square$

In dimension $n = 5$ we have a partial result on the whole space.

Theorem 2.6. *If $2 < |\alpha - 2| < \sqrt{13}$, then $S_{2^{**}}(\mathbb{R}^5 \setminus \{0\};\alpha) < S^{**}$.*

Theorem 2.6 is a special case of an estimate which will be proved in the next subsection (see Theorem 2.9).

2.3. Radially symmetric functions

For any $\alpha \in \mathbb{R}$ and $q \geq 2$ we define

$$S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) := \inf_{\substack{u\in C_c^2(\mathbb{R}^n\setminus\{0\})\\ u\neq 0,\ u=u(|x|)}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^{\alpha}|\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |x|^{-\beta}|u|^q dx\right)^{2/q}} \tag{2.10}$$

where $\beta = n - q\frac{n-4+\alpha}{2}$. Notice that there is no upper bound on q even in large dimensions. Arguing as for (2.1), one can easily check that

$$S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) = S_q^{\mathrm{rad}}(\mathbb{R}^n;4-\alpha)\,.$$

In case $q = 2$, it was proved in [7] that

$$S_2^{\mathrm{rad}}(\mathbb{R}^n;\alpha) = \gamma_\alpha^2 = \frac{(n-4+\alpha)^2(n-\alpha)^2}{16}\,. \tag{2.11}$$

In particular, if $\alpha \neq 4-n$ and $\alpha \neq n$, then $S_2^{\mathrm{rad}}(\mathbb{R}^n;\alpha) > 0$, and we can suitably define a Hilbert space of radially symmetric functions $\mathcal{N}^2_{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ endowed with the norm (1.7).

The next theorem provides a second order Caffarelli–Kohn–Nirenberg-type inequality for radially symmetric maps. We only need to assume $\gamma_\alpha \neq 0$. In particular, q can be supercritical and $-\gamma_\alpha$ might be a Dirichlet eigenvalue on the sphere.

Theorem 2.7. *Let $q > 2$ be a given exponent. Then $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) > 0$ if and only if $\alpha \notin \{4-n, n\}$.*

Proof. To any radial function $u \in C_c^2(\mathbb{R}^n \setminus \{0\})$ we associate a function $w \in C_c^2(\mathbb{R})$ via the Emden-Fowler transform defined in (2.2). Thus

$$u(x) = |x|^{\frac{4-n-\alpha}{2}} w(-\log|x|) \tag{2.12}$$

$$\int_{\mathbb{R}^n} |x|^{-\beta}|u|^q dx = \omega_n \int_{-\infty}^{\infty} |w|^q ds$$

$$\int_{\mathbb{R}^n} |x|^{\alpha}|\Delta u|^2 dx = \omega_n \int_{-\infty}^{\infty} \left(|w''|^2 + 2\overline{\gamma}_\alpha|w'|^2 + \gamma_\alpha^2|w|^2\right) ds$$

where ω_n is the measure of $\mathbb{S}^{n-1}$, γ_α is defined in (1.6) and $\overline{\gamma}_\alpha$ in (2.5) (compare with (2.3) and (2.4)). In particular, $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) = \omega_n^{(q-2)/q}\mu_q(\alpha)$, where

$$\mu_q(\alpha) = \inf_{\substack{w\in C_c^2(\mathbb{R})\\ w\neq 0}} \frac{\displaystyle\int_{-\infty}^{\infty} \left(|w''|^2 + 2\overline{\gamma}_\alpha|w'|^2 + \gamma_\alpha^2|w|^2\right) ds}{\left(\displaystyle\int_{-\infty}^{\infty} |w|^q ds\right)^{2/q}}.$$

If $\gamma_\alpha = 0$, then clearly $\mu_q(\alpha) = 0$, via rescaling. Conversely, notice that $\alpha \notin \{4-n, n\}$ if and only if $\gamma_\alpha \neq 0$ and in this case the space $H^2(\mathbb{R})$ admits as an equivalent norm

$$\|w\|_\alpha^2 = \int_{-\infty}^{\infty} \left(|w''|^2 + 2\overline{\gamma}_\alpha|w'|^2 + \gamma_\alpha^2|w|^2\right) ds. \tag{2.13}$$

Thus $\mu_q(\alpha) > 0$ since $H^2(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ by Sobolev embedding theorem, and hence $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) > 0$. □

We conclude this section with an existence result.

Theorem 2.8. *Let $q > 2$ be a given exponent, and assume that $\alpha \notin \{4-n, n\}$. Then $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ is achieved in $\mathcal{N}^2_{\mathrm{rad}}(\mathbb{R}^n;\alpha)$.*

Proof. Since $\gamma_\alpha \neq 0$, then the Emden-Fowler transform induces an isometry between $\mathcal{N}^2_{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ and the Sobolev space $H^2(\mathbb{R})$, endowed with the equivalent norm in (2.13). It is standard to show the existence of some $\underline{w} \in H^2(\mathbb{R})$ such that $\|\underline{w}\|_{L^q} = 1$ and $\|\underline{w}\|^2_\alpha = \mu_q(\alpha)$ (see [21]). Then the corresponding function $\underline{u}$ defined by (2.12) belongs to $\mathcal{N}^2_{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ and achieves $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$. □

2.4. Estimates on $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$

In this subsection we provide some estimates on the infima $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$. We start with the limiting case $n \geq 5$ and $q = 2^{**}$.

Theorem 2.9. *If $n \geq 5$ and*

$$2 < |\alpha - 2| < \sqrt{4 + 2\,\frac{(n-2)^2(n-4)}{n-3}}, \tag{2.14}$$

then $S_{2^{**}}^{\mathrm{rad}}(\mathbb{R}^n;\alpha) < S^{**}$.

Proof. Set $a = -\alpha/2$. Let $U \in \mathcal{D}^2(\mathbb{R}^n)$ be the radial mapping defined by

$$U(x) = \left(1 + |x|^2\right)^{\frac{4-n}{2}}\ . \tag{2.15}$$

Our aim is to test $S_q^{\mathrm{rad}}(\mathbb{R}^n;-2a)$ with $|x|^aU$. In order to simplify notations we put

$$J = \int |x|^{-2}|\nabla U|^2\ ,\ I = \int |x|^{-4}|U|^2\,.$$

We compute

$$\begin{aligned}\int |x|^{-2a}|\Delta(|x|^aU)|^2 &= \int |\Delta U|^2 + 4a^2\,J + a^2(n-2+a)^2\,I \\ &\quad + 4a^2(n-2+a)\int |x|^{-4}U(x\cdot\nabla U) \\ &\quad + 4a\int |x|^{-2}(x\cdot\nabla U)\Delta U + 2a(n-2+a)\int |x|^{-2}U\Delta U.\end{aligned}$$

Since U is radial, then

$$\int |x|^{-4}U(x\cdot\nabla U) = -\frac{n-4}{2}\,I\ ,\quad \int |x|^{-2}(x\cdot\nabla U)\Delta U = \frac{n}{2}\,J$$
$$\int |x|^{-2}U\Delta U = -\,J - (n-4)\,I.$$

We infer that

$$\int |x|^{-2a}|\Delta(|x|^aU)|^2 = \int |\Delta U|^2 + 2a(a+2)\,J - a(a+2)(n-2+a)(n-4-a)\,I.$$

Integrating by parts twice we get

$$J = \frac{(n-2)(n-4)^2}{4(n-3)}\, I,$$

that leads to

$$\int |x|^{-2a} |\Delta(|x|^a U)|^2 = \int |\Delta U|^2 + a(a+2)\left[a^2 + 2a - \frac{(n-2)^2(n-4)}{2(n-3)}\right] I.$$

Since U achieves the best Sobolev constant S^{**} (see [22]), we have

$$\int |\Delta U|^2 = S^{**}\left(\int |U|^{2^{**}}\right)^{2/2^{**}}, \tag{2.16}$$

and then

$$S^{\mathrm{rad}}_{2^{**}}(\mathbb{R}^n;\alpha) \le S^{**} + Ca(a+2)\left[a^2 + 2a - \frac{(n-2)^2(n-4)}{2(n-3)}\right] I,$$

where $C > 0$ is a power of the $L^{2^{**}}$ norm of U. Since

$$a(a+2)\left[a^2 + 2a - \frac{(n-2)^2(n-4)}{2(n-3)}\right] < 0$$

if and only if (2.14) holds, the conclusion follows. □

Our next goal is to provide the asymptotic behavior of $S^{\mathrm{rad}}_q(\mathbb{R}^n;\alpha)$ as $|\alpha| \to \infty$. We first point out a useful lemma.

Lemma 2.10. *Let $q \ge 2$ and $\alpha, \widetilde{\alpha} \in \mathbb{R} \setminus \{4-n\}$ be given. Then*

$$S^{\mathrm{rad}}_q(\mathbb{R}^n;\alpha) = |\tau(\alpha,\tilde{\alpha})|^{3+\frac{2}{q}} \inf_{\substack{u \in C^2_c(\mathbb{R}^n\setminus\{0\})\\ u=u(|x|),\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^{\widetilde{\alpha}}|\Delta u|^2 - g(\alpha,\widetilde{\alpha})\int_{\mathbb{R}^n} |x|^{\widetilde{\alpha}-2}|\nabla u|^2}{\left(\displaystyle\int_{\mathbb{R}^n} |x|^{-\beta_{\tilde{\alpha}}}|u|^q\right)^{2/q}},$$

where

$$\begin{aligned} \tau(\alpha,\tilde{\alpha}) &:= \frac{n-4+\alpha}{n-4+\widetilde{\alpha}}\,, \\ g(\alpha,\widetilde{\alpha}) &:= (n-2)\,\frac{(\widetilde{\alpha}-\alpha)[\widetilde{\alpha}\alpha - 2(\widetilde{\alpha}+\alpha) - n(n-4)]}{(n-4+\alpha)^2}. \end{aligned} \tag{2.17}$$

Proof. Fix $u \in C^\infty_c(\mathbb{R}^n \setminus \{0\})$ radially symmetric, put $\tau = \tau(\alpha,\tilde{\alpha})$ and define

$$\widetilde{u}(r) = u(r^{1/\tau})\,. \tag{2.18}$$

Direct computation leads to

$$\int_{\mathbb{R}^n} |x|^{-\beta_\alpha}|u|^q = |\tau|^{-1}\int_{\mathbb{R}^n} |x|^{-\beta_{\tilde{\alpha}}}|\widetilde{u}|^q \tag{2.19}$$

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{\alpha}|\Delta u|^2 &= |\tau|^3 \int_{\mathbb{R}^n} |x|^{\widetilde{\alpha}}\left|\Delta\widetilde{u} - (1-\tau^{-1})(n-2)|x|^{-2}x\cdot\nabla\widetilde{u}\right|^2 dx \\ &= |\tau|^3\left[\int_{\mathbb{R}^n} |x|^{\widetilde{\alpha}}|\Delta\widetilde{u}|^2 - g(\alpha,\widetilde{\alpha})\int_{\mathbb{R}^n} |x|^{\widetilde{\alpha}-2}|\nabla\widetilde{u}|^2\right]. \end{aligned} \tag{2.20}$$

The conclusion readily follows. □

Notice that $g \equiv 0$ in the two-dimensional case. Therefore the following immediate corollary holds.

Corollary 2.11. *Assume $n = 2$ and fix $q \geq 2$. Then for any $\alpha \neq 2$ the ratio*

$$\frac{S_q^{\mathrm{rad}}(\mathbb{R}^2;\alpha)}{|\alpha-2|^{3+\frac{2}{q}}}$$

is a constant, independent on α.

Next assume $n \geq 3$. We will say that $\alpha, \widetilde{\alpha}$ are *conjugate* if

$$(\alpha-2)(\widetilde{\alpha}-2) = (n-2)^2. \tag{2.21}$$

Notice that $\alpha = n$ and $\widetilde{\alpha} = 4-n$ are self-conjugate. If $\alpha, \widetilde{\alpha} \neq 4-n$ are conjugate, then $g(\alpha,\widetilde{\alpha}) = 0$ and

$$|\tau(\alpha,\widetilde{\alpha})| = \left|\frac{n-2}{\widetilde{\alpha}-2}\right| = \left|\frac{\alpha-2}{n-2}\right|. \tag{2.22}$$

In this case

$$S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) = |\tau(\alpha,\widetilde{\alpha})|^{3+\frac{2}{q}} S_q^{\mathrm{rad}}(\mathbb{R}^n;\widetilde{\alpha}). \tag{2.23}$$

Corollary 2.12. *Assume $n \geq 3$ and fix $q \geq 2$. Then*

$$\lim_{|\alpha|\to\infty} \frac{S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)}{|\alpha-2|^{3+\frac{2}{q}}} = \frac{S_q^{\mathrm{rad}}(\mathbb{R}^n;2)}{(n-2)^{3+\frac{2}{q}}}.$$

Proof. For any $\alpha \neq 4-n$ we let $\widetilde{\alpha}$ to be its conjugate exponent. When $|\alpha| \to \infty$, then $\widetilde{\alpha} \to 2$ by (2.21), and hence $S_q^{\mathrm{rad}}(\mathbb{R}^n;\widetilde{\alpha}) \to S_q^{\mathrm{rad}}(\mathbb{R}^n;2)$ (use the continuity Lemma B.3 in Appendix B.1). Then the conclusion follows from (2.22)–(2.23). □

3. Existence

In this Section we prove Theorems 1.2 and 1.3. We will always assume:

$$\begin{cases} \Sigma \subseteq \mathbb{S}^{n-1} \text{ is of class } C^2 \text{ and } -\gamma_\alpha \notin \Lambda(\Sigma) \\ q > 2 \text{ and } q \leq 2^{**} \text{ if } n \geq 5 \\ \beta = n - q\,\dfrac{n-4+\alpha}{2}. \end{cases}$$

In particular, $S_q(\mathcal{C}_\Sigma;\alpha) > 0$ by Theorem 1.1. We need the following result.

Lemma 3.1. *Let $(u_h) \subset \mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ be a minimizing sequence for $S_q(\mathcal{C}_\Sigma;\alpha)$. If $u_h \rightharpoonup u$ weakly in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ and $u \neq 0$, then u is a minimizer for $S_q(\mathcal{C}_\Sigma;\alpha)$ and $u_h \to u$ strongly in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$.*

The proof is standard. One can adapt to our situation a well known argument (see, e.g., [21], Chapt. 1, Sect. 4).

3.1. ε-compactness

To prove the existence results stated in the introduction we need an ε-compactness criterion for sequences of approximating solutions to (1.10). We start by pointing out an immediate consequence of Rellich Theorem.

Lemma 3.2. *Let A be a domain with compact closure in $\mathbb{R}^n \setminus \{0\}$. Then $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ is compactly embedded into $H^1(\mathcal{C}_\Sigma \cap A)$.*

In the next result we let $\mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$ to be the topological dual space of $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ and we use Theorem 1.1 to fix a small number $\varepsilon_0 > 0$ such that

$$\varepsilon_0^{\frac{q-2}{q}} < S_q(\mathcal{C}_\Sigma;\alpha)\,. \tag{3.1}$$

Proposition 3.3. *Let $u_h \in \mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$, $f_h \in \mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$ be given sequences, such that $f_h \to 0$ in $\mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$, $u_h \rightharpoonup 0$ weakly in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$ and moreover*

$$\Delta\left(|x|^\alpha \Delta u_h\right) = |x|^{-\beta}|u_h|^{q-2}u_h + f_h \tag{3.2}$$

$$\int_{\mathcal{C}_\Sigma \cap B_R} |x|^{-\beta}|u_h|^q\, dx \le \varepsilon_0 \tag{3.3}$$

for some $R > 0$, where $\varepsilon_0 > 0$ satisfies (3.1). *Then*

$$\int_{\mathcal{C}_\Sigma \cap B_{R'}} |x|^{-\beta}|u_h|^q\, dx \to 0 \quad \textit{for any } R' \in (0,R).$$

Proof. Fix $R' \in (0,R)$ and take a cut-off function $\varphi \in C^\infty_c(B_R)$ such that $\varphi \equiv 1$ on $B_{R'}$. Notice that

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha \Delta u_h \Delta(\varphi^2 u_h)\, dx = \int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta(\varphi u_h)|^2\, dx + o(1)$$

by Lemma 3.2, as φ and its derivatives have compact supports in $\mathbb{R}^n \setminus \{0\}$. Therefore, using $\phi^2 u_h$ as a test function in (3.2) and using Hölder inequality we get

$$\begin{aligned}\int_{\mathcal{C}_\Sigma} |x|^\alpha\, |\Delta(\varphi u_h)|^2\, dx &= \int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u_h|^{q-2}|(\varphi u_h)|^2\, dx + o(1)\\ &\le \left(\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u_h|^q\, dx\right)^{\frac{q-2}{q}} \left(\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|\varphi u_h|^q\, dx\right)^{\frac{2}{q}} + o(1).\end{aligned}$$

The left hand side in the above inequality can be bounded from below by using the definition of $S_q(\mathcal{C}_\Sigma;\alpha)$. Thus, from (3.3) we infer

$$\left(\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|\varphi u_h|^q\, dx\right)^{\frac{2}{q}} S_q(\mathcal{C}_\Sigma;\alpha) \le \varepsilon_0^{\frac{q-2}{q}} \left(\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|\varphi u_h|^q\, dx\right)^{\frac{2}{q}}\,.$$

The conclusion readily follows from (3.1), since $\varphi \equiv 1$ on $B_{R'}$. $\square$

3.2. Proof of Theorem 1.2

Using Ekeland's variational principle (see [21] Chapt. 1, Sect. 5) we can find a minimizing sequence $u_h \in \mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$, such that (3.2) holds for a sequence $f_h \to 0$ in $\mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$ and such that

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u_h|^2 \, dx = \int_{\mathcal{C}_\Sigma} |x|^{-\beta} |u_h|^q \, dx + o(1) = S_q(\mathcal{C}_\Sigma;\alpha)^{\frac{q}{q-2}} + o(1).$$

Since u_h is bounded in $\mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$, we can assume that $u_h \rightharpoonup u$ weakly in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$. Up to a rescaling, we can also assume that

$$\int_{\mathcal{C}_\Sigma \cap B_2} |x|^{-\beta} |u_h|^q \, dx = \frac{1}{2} S_q(\mathcal{C}_\Sigma;\alpha)^{\frac{q}{q-2}}. \tag{3.4}$$

We claim that $u \neq 0$. Indeed, if $u_h \rightharpoonup 0$, then

$$\int_{\mathcal{C}_\Sigma \cap B_1} |x|^{-\beta} |u_h|^q \, dx = o(1)$$

by Proposition 3.3. On the other hand,

$$\int_{\mathcal{C}_\Sigma \cap \{1<|x|<2\}} |x|^{-\beta} |u_h|^q \, dx = o(1)$$

by Lemma 3.2 and by Rellich Theorem, contradicting (3.4). Thus the minimizing sequence u_h converges weakly to a non trivial limit. Then we can apply Lemma 3.1 to conclude. □

3.3. Proof of Theorem 1.3

We put here $S(\alpha) = S_{2^{**}}(\mathcal{C}_\Sigma;\alpha)$ to simplify notations. We select a minimizing sequence u_h as in the proof of Theorem 1.2. In particular there exists a sequence $f_h \to 0$ in $\mathcal{N}^{-2}(\mathcal{C}_\Sigma;\alpha)$ such that u_h satisfies

$$\Delta\left(|x|^\alpha \Delta u_h\right) = |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}-2} u_h + f_h \tag{3.5}$$

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u_h|^2 \, dx = \int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}} \, dx + o(1) = S(\alpha)^{\frac{n}{4}} + o(1) \tag{3.6}$$

$$\int_{\mathcal{C}_\Sigma \cap B_2} |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}} \, dx = \frac{1}{2} S(\alpha)^{\frac{n}{4}}. \tag{3.7}$$

As before, we have to prove that u_h cannot converge weakly to 0. By contradiction, assume that $u_h \rightharpoonup 0$ weakly in $\mathcal{N}^2(\mathcal{C}_\Sigma;\alpha)$. Then we can argue as in the proof of Theorem 1.2 to get

$$\int_{\mathcal{C}_\Sigma \cap B_1} |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}} \, dx = o(1)$$

and hence

$$\int_{\mathcal{C}_\Sigma \cap \{1<|x|<2\}} |x|^{\frac{n\alpha}{n-4}} |u_h|^{2^{**}} \, dx = \frac{1}{2} S(\alpha)^{\frac{n}{4}} + o(1) \tag{3.8}$$

by (3.7). Now we take a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\varphi \equiv 1$ on $B_2 \setminus B_1$ and we use $\varphi^2 u_h$ as test function in (3.5). Using Lemma 3.2, Hölder inequality and (3.6) we have

$$\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta(\varphi u_h)|^2 \, dx \leq S(\alpha) \left(\int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |\varphi u_h|^{2^{**}} \, dx \right)^{\frac{n-4}{n}} + o(1).$$

Let us define

$$F_h = \Delta(|x|^{\alpha/2} \varphi u_h) - |x|^{\alpha/2} \Delta(\varphi u_h).$$

Using Lemma 3.2 and Rellich Theorem one plainly gets that $F_h \to 0$ strongly in $L^2(\mathbb{R}^n)$. Thus, by Sobolev inequality,

$$\begin{aligned} \int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta(\varphi u_h)|^2 \, dx &= \int_{\mathcal{C}_\Sigma} |\Delta(|x|^{\frac{\alpha}{2}} \varphi u_h)|^2 \, dx + o(1) \\ &\geq S^{**} \left(\int_{\mathcal{C}_\Sigma} \left| |x|^{\frac{\alpha}{2}} \varphi u_h \right|^{2^{**}} \, dx \right)^{\frac{n-4}{n}} + o(1). \end{aligned}$$

Putting together these informations we conclude that

$$S^{**} \left(\int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |\varphi u_h|^{2^{**}} \, dx \right)^{\frac{n-4}{n}} \leq S(\alpha) \left(\int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |\varphi u_h|^{2^{**}} \, dx \right)^{\frac{n-4}{n}} + o(1).$$

Thus

$$o(1) = \int_{\mathcal{C}_\Sigma} |x|^{\frac{n\alpha}{n-4}} |\varphi u_h|^{2^{**}} \, dx \geq \int_{\mathcal{C}_\Sigma \cap \{1<|x|<2\}} |x|^{\frac{n\alpha}{n-4}} |\varphi u_h|^{2^{**}} \, dx,$$

as $0 < S(\alpha) < S^{**}$ by assumption and $\varphi \equiv 1$ on the annulus $B_2 \setminus B_1$. Since this conclusion contradicts (3.8), we infer that the weak limit of the minimizing sequence u_h cannot vanish. Then we can apply Lemma 3.1 to conclude. □

From Theorems 1.3, 2.5 and 2.9 we infer the next existence result.

Theorem 3.4. *Let $n \geq 5$ and let Σ be a domain in $\mathbb{S}^{n-1}$ of class C^2. Assume that $-\gamma_\alpha \notin \Lambda(\Sigma)$. The best constant $S_{2^{**}}(\mathcal{C}_\Sigma; \alpha)$ is achieved if one of the following conditions holds:*

(i) $n \geq 6$ *and* $|\alpha - 2| > 2$
(ii) $n = 5$, $\Sigma = \mathbb{S}^4$ *and* $2 < |\alpha - 2| < \sqrt{13}$.

Remark 3.5. Assume $\Sigma = \mathbb{S}^{n-1}$ and $n \geq 6$. By Proposition 2.4 it results that $S_{2^{**}}(\mathbb{R}^n \setminus \{0\}; \alpha) \leq S^{**}$, while $S_{2^{**}}^{\mathrm{rad}}(\mathbb{R}^n; \alpha)$ diverges as $|\alpha| \to \infty$. Thus, for $|\alpha|$ large enough, extremals for $S_{2^{**}}(\mathbb{R}^n \setminus \{0\}; \alpha)$ do exist, but none of them is radially symmetric. This breaking symmetry phenomenon is definitively new with respect to the Caffarelli-Kohn-Nirenberg first order inequalities. Breaking symmetry will be studied in more detail in Section 5.

4. Breaking positivity

In this section we illustrate a surprising phenomenon that is completely new with respect to similar first order problems. Namely, we show that all functions achieving the best constant $S_q(\mathcal{C}_\Sigma;\alpha)$ might be forced to change sign. In particular, extremal functions for $S_q(\mathcal{C}_\Sigma;\alpha)$ cannot be positive if q is close to 2 and

$$-\gamma_\alpha > \frac{\lambda_1+\lambda_2}{2}\,, \tag{4.1}$$

where λ_1 and λ_2 are the two first eigenvalues of $-\Delta_\sigma$ in $H^1_0(\Sigma)$. To this goal we introduce the infima

$$S_q^+(\mathcal{C}_\Sigma;\alpha) = \inf_{\substack{u\in C^2(\overline{\mathcal{C}_\Sigma}\setminus\{0\})\\ u\ge 0,\ u\not\equiv 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma}|x|^\alpha|\Delta u|^2dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma}|x|^{-\beta}u^q dx\right)^{2/q}}.$$

Let us state the main result of this section.

Theorem 4.1. *Assume* (4.1). *Then there exists $q_\alpha > 2$ such that*

$$S_q(\mathcal{C}_\Sigma;\alpha) < S_q^+(\mathcal{C}_\Sigma;\alpha)$$

for all $q\in[2,q_\alpha)$. In particular, if $q\in(2,q_\alpha)$, extremal functions for $S_q(\mathcal{C}_\Sigma;\alpha)$ cannot be positive.

Proof. In order to prove Theorem 4.1 we will use once more the Emden-Fowler transform T already introduced in (2.2).
Besides the infimum $m_N(\Sigma;\gamma)$ in (2.6), we define also

$$m_N^+(\Sigma;\gamma) = \inf_{\substack{\varphi\in H^2\cap H^1_0(\Sigma)\\ \varphi\ge 0,\ \varphi\not\equiv 0}} \frac{\displaystyle\int_\Sigma|-\Delta_\sigma\varphi+\gamma\varphi|^2d\sigma}{\displaystyle\int_\Sigma\varphi^2d\sigma}\,.$$

The following facts hold:

(i) $S_2^+(\mathcal{C}_\Sigma;\alpha) = m_N^+(\Sigma;\gamma_\alpha)$.
(ii) If $-\gamma\le\frac{\lambda_1+\lambda_2}{2}$, then $m_N^+(\Sigma;\gamma)=m_N(\Sigma;\gamma)$.
(iii) If $-\gamma>\frac{\lambda_1+\lambda_2}{2}$, then $m_N^+(\Sigma;\gamma)>m_N(\Sigma;\gamma)$. In particular $m_N^+(\Sigma;\gamma)>0$ for all $\gamma\in\mathbb{R}$, $\gamma\ne\lambda_1$.

Theorem 4.1 is an immediate consequence of (i)–(iii) and of the continuity Lemma B.2 in Appendix B.1. Claim (i) easily follows from the computations in Section 2 on the Emden-Fowler transform. To prove (ii), notice that if $-\gamma\le\frac{\lambda_1+\lambda_2}{2}$, then $\mathrm{dist}(-\gamma,\Lambda(\Sigma)) = |\gamma+\lambda_1|$. Thus, by (i) in Lemma 2.1, $m_N(\Sigma;\gamma)$ is achieved by an eigenfunction φ_1 corresponding to λ_1. Since one can take $\varphi_1\ge 0$, it follows that $m_N^+(\Sigma;\gamma)\le m_N(\Sigma;\gamma)$. The opposite inequality is trivial.

Now we check (iii). Assume that $m_N^+(\Sigma;\gamma)\le m_N(\Sigma;\gamma)$. Then equality holds. By a standard argument one can plainly check that $m_N^+(\Sigma;\gamma)$ is attained, namely there exists $\varphi \in H^2 \cap H^1_0(\Sigma)$ such that

$$\int_\Sigma |-\Delta_\sigma \varphi + \gamma\varphi|^2 d\sigma = m_N^+(\Sigma;\gamma)\,, \quad \int_\Sigma \varphi^2 d\sigma = 1\,, \quad \varphi \ge 0\,.$$

Then φ is an extremal for $m_N(\Sigma;\gamma)$, too. By (i), φ is an eigenfunction of $-\Delta_\sigma$. Since $\varphi \ge 0$, it must be $-\Delta_\sigma\varphi = \lambda_1\varphi$ and, again by (i), $\mathrm{dist}(-\gamma, \Lambda(\Sigma)) = |\gamma + \lambda_1|$, that is, $-\gamma \le \frac{\lambda_1+\lambda_2}{2}$. Theorem 4.1 is completely proved. □

Specializing Theorem 4.1 to the case $\Sigma = \mathbb{S}^{n-1}$, when $\lambda_1 = 0$ and $\lambda_2 = n-1$, we immediately obtain the next result.

Corollary 4.2. *Assume that*

$$|\alpha - 2| > \sqrt{(n-1)^2+1}.$$

Then there exists $q_\alpha > 2$ such that $S_q(\mathbb{R}^n\backslash\{0\};\alpha) < S_q^+(\mathbb{R}^n\backslash\{0\};\alpha)$ for all $q \in [2, q_\alpha)$. In particular, if $q \in (2, q_\alpha)$, extremal functions for $S_q(\mathbb{R}^n \setminus \{0\};\alpha)$ cannot be positive.

5. Breaking symmetry

In this section we discuss some conditions for breaking symmetry. We use the the constants $S_q(\mathbb{R}^n \setminus \{0\};\alpha)$ and $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ already defined in (2.7) and (2.10), respectively.

As a first condition, we have that if $-\gamma_\alpha$ is close enough to the spectrum $\Lambda(\mathbb{S}^{n-1})$, then breaking symmetry occurs.

Theorem 5.1. *For every $q > 2$ and for every $k \in \mathbb{N}$ there exists $\delta > 0$ such that if*

$$0 < |\gamma_\alpha + k(n-2+k)| < \delta,$$

then $S_q(\mathbb{R}^n \setminus \{0\};\alpha) < S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$.

Proof. Fix $k \in \mathbb{N}$, let $\lambda = k(n-2+k)$ and α_0 be such that $-\gamma_{\alpha_0} = \lambda$. Since $\lambda \in \Lambda(\mathbb{S}^{n-1})$, by Theorem 1.1 it turns out that $S_q(\mathbb{R}^n \setminus \{0\};\alpha_0) = 0$. In general, if $\alpha \to \alpha_0$, then

$$S_q(\mathbb{R}^n \setminus \{0\};\alpha_0) \ge \limsup_{\alpha\to\alpha_0} S_q(\mathbb{R}^n \setminus \{0\};\alpha)$$

(see Remark B.4). Hence we have that $S_q(\mathbb{R}^n \setminus \{0\};\alpha) \to 0 \quad$ as $\alpha \to \alpha_0$. We also have $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) \to S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha_0)$ as $\alpha \to \alpha_0$ by Lemma B.3, and $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha_0) > 0$ by Theorem 2.7. Hence the conclusion follows. □

As a second condition, we show that if $|\alpha|$ is large, then again breaking symmetry occurs. More precisely we have the following result.

Theorem 5.2. *Let $q > 2$, $q \le 2^{**}$ when $n \ge 5$, and let $\alpha \in \mathbb{R}$. If*

$$|\gamma_\alpha| > \frac{n-1}{q-2}\left(1 + \sqrt{q-1}\right), \tag{5.1}$$

then $S_q(\mathbb{R}^n \setminus \{0\};\alpha) < S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$.

Proof. Assume that $S_q(\mathbb{R}^n \setminus \{0\};\alpha) = S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$ for some $\alpha \in \mathbb{R}$ such that $\gamma_\alpha \neq 0$. We claim that in this case

$$|\gamma_\alpha| \leq \frac{n-1}{q-2}\left(1+\sqrt{q-1}\right). \tag{5.2}$$

We start by noticing that $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) > 0$ by Theorem 2.7. Thus also $S_q(\mathbb{R}^n \setminus \{0\};\alpha) > 0$, and hence $-\gamma_\alpha$ does not belong to the spectrum of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$, by Theorem 1.1. In particular, the Hilbert space $\mathcal{N}^2(\mathbb{R}^n \setminus \{0\};\alpha)$ is well defined. We introduce the following functionals on $\mathcal{N}^2(\mathbb{R}^n \setminus \{0\};\alpha) \setminus \{0\}$:

$$A(u) := \int |x|^\alpha |\Delta u|^2, \quad B(u) := \left(\int |x|^{-\beta}|u|^q\right)^{2/q}, \quad R(u) := \frac{A(u)}{B(u)}.$$

Let $\underline{u}$ be the radially symmetric solution to the minimization problem (2.10) given by Theorem 2.7. Thus $\underline{u}$ achieves also $S_q(\mathbb{R}^n \setminus \{0\};\alpha)$, that is, $\underline{u}$ minimizes the functional $R(u)$ on $\mathcal{N}^2(\mathbb{R}^n \setminus \{0\};\alpha) \setminus \{0\}$. In particular,

$$R'(\underline{u}) \cdot v = 0, \quad R''(\underline{u})[v,v] \geq 0 \quad \text{for any } v \in \mathcal{N}^2(\mathbb{R}^n \setminus \{0\};\alpha). \tag{5.3}$$

In order to simplify notation we can assume that $B(\underline{u}) = 1$. Then by direct computations based on (5.3) one gets

$$B''(\underline{u})[v,v]\left(\int |x|^\alpha |\Delta \underline{u}|^2\right) \leq A''(\underline{u})[v,v] = 2\int |x|^\alpha |\Delta v|^2 \tag{5.4}$$

$$B''(\underline{u})[v,v] = 2(2-q)\left(\int |x|^{-\beta}|\underline{u}|^{q-2}\underline{u}v\right)^2 + 2(q-1)\int |x|^{-\beta}|\underline{u}|^{q-2}|v|^2.$$

Now we choose the test function v, that is, $v(r\sigma) = \underline{u}(r)\varphi(\sigma)$, where $\varphi \in H^1(\mathbb{S}^{n-1})$ is an eigenfunction of the Laplace-Beltrami operator on the sphere, relatively to the first positive eigenvalue and normalized with respect to the L^2 norm. Hence

$$-\Delta_\sigma \varphi = (n-1)\varphi, \quad \int_{\mathbb{S}^{n-1}} \varphi \, d\sigma = 0, \quad \int_{\mathbb{S}^{n-1}} |\varphi|^2 \, d\sigma = \omega_n \tag{5.5}$$

where ω_n denotes the measure of $\mathbb{S}^{n-1}$. Since φ has zero mean value, then

$$B''(\underline{u})[\underline{u}\varphi, \underline{u}\varphi] = 2(q-1)\int |x|^{-\beta}|\underline{u}|^q = 2(q-1).$$

Then, taking into account that $\Delta v = (\Delta \underline{u})\varphi + |x|^{-2}\underline{u}\Delta_\sigma \varphi$ and using (5.5) we compute

$$\begin{aligned}
\int |x|^\alpha |\Delta(\underline{u}\varphi)|^2 &= \int |x|^\alpha |(\Delta \underline{u} - (n-1)|x|^{-2}\underline{u})|^2 \\
&= \int |x|^\alpha |\Delta \underline{u}|^2 + (n-1)^2 \int |x|^{\alpha-4}|\underline{u}|^2 \\
&\quad -2(n-1)\int |x|^{\alpha-2}\underline{u}\Delta \underline{u} \\
&\leq \int |x|^\alpha |\Delta \underline{u}|^2 + (n-1)^2 \int |x|^{\alpha-4}|\underline{u}|^2 \\
&\quad +2(n-1)\left(\int |x|^{\alpha-4}|\underline{u}|^2\right)^{1/2}\left(\int |x|^\alpha |\Delta \underline{u}|^2\right)^{1/2}
\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus from (5.4) we get

$$
\begin{aligned}
(q-2)\int |x|^{\alpha}|\Delta \underline{u}|^2 \quad &\le \quad (n-1)^2\int |x|^{\alpha-4}|\underline{u}|^2 \\
&\quad +2(n-1)\left(\int |x|^{\alpha-4}|\underline{u}|^2\right)^{1/2}\left(\int |x|^{\alpha}|\Delta \underline{u}|^2\right)^{1/2}.
\end{aligned}
$$

Thus $(q-2)\xi^2 \le (n-1)^2 + 2(n-1)\xi$, where

$$
\xi := \left(\frac{\displaystyle\int |x|^{\alpha}|\Delta \underline{u}|^2}{\displaystyle\int |x|^{\alpha-4}|\underline{u}|^2}\right)^{1/2} \quad \ge \quad |\gamma_\alpha|
$$

by (2.11). Inequality (5.2) readily follows via elementary calculus. □

Remark 5.3. Assume $n \le 4$ and fix any $\alpha \notin \{4-n, n\}$. If $q > 2$ is large enough, then breaking symmetry occurs, that is, $S_q(\mathbb{R}^n \setminus \{0\};\alpha) < S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha)$.

6. Dirichlet boundary conditions

In this section we assume that Σ is a domain of class C^2 with compact closure in $\mathbb{S}^{n-1}$. In particular, $\partial\mathcal{C}_\Sigma \setminus \{0\}$ is not empty. Our aim is to study study minimization problems of the form (1.12). First of all we recall that

$$
\inf_{\substack{u\in C^2_c(\mathcal{C}_\Sigma)\\ u\ne 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{\alpha}|\Delta u|^2 dx}{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{\alpha-4}|u|^2 dx} > 0\,, \tag{6.1}
$$

see [7], whatever $\alpha \in \mathbb{R}$ is. Thus we can define the Hilbert space $\mathcal{D}^2(\mathcal{C}_\Sigma;\alpha)$ as the completion of $C^2_c(\mathcal{C}_\Sigma)$ with respect to the norm defined in (1.7). In particular, if $q > 2$, $q \le 2^{**}$ if $n \ge 5$, by density we have that

$$
S_q^D(\mathcal{C}_\Sigma;\alpha) = \inf_{\substack{u\in \mathcal{D}^2(\mathcal{C}_\Sigma;\alpha)\\ u\ne 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{\alpha}|\Delta u|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u|^q dx\right)^{2/q}}\,. \tag{6.2}
$$

Our main results about the existence of minimizers for problems (6.2) are summarized in the next theorem.

Theorem 6.1. *Let $\alpha \in \mathbb{R}$ and let Σ be a domain of class C^2 properly contained in $\mathbb{S}^{n-1}$. Let $q > 2$, and $q \le 2^{**}$ if $n \ge 5$. Then $S_q^D(\mathcal{C}_\Sigma;\alpha) > 0$ and moreover:*

(i) *If $n \le 4$ or $q < 2^{**}$, then $S_q^D(\mathcal{C}_\Sigma;\alpha)$ is achieved in $\mathcal{D}^2(\mathcal{C}_\Sigma;\alpha)$.*
(ii) *As $n \ge 5$, if $S_{2^{**}}^D(\mathcal{C}_\Sigma;\alpha) < S^{**}$, then $S_{2^{**}}^D(\mathcal{C}_\Sigma;\alpha)$ is achieved in $\mathcal{D}^2(\mathcal{C}_\Sigma;\alpha)$.*
(iii) *If $n \ge 6$, then $S_{2^{**}}^D(\mathcal{C}_\Sigma;\alpha) < S^{**}$.*
(iv) *If $S_q^D(\mathcal{C}_\Sigma;\alpha)$ is attained, then $S_q^N(\mathcal{C}_\Sigma;\alpha) < S_q^D(\mathcal{C}_\Sigma;\alpha)$.*

Proof. Parts (i)–(iii) can be proved by repeating the same argument developed in Sections 2 and 3. As far as concerns (iv), we point out that the large inequality $S_q^N(\mathcal{C}_\Sigma;\alpha) \le S_q^D(\mathcal{C}_\Sigma;\alpha)$ always holds true. Moreover, if $u \in \mathcal{D}^2(\mathcal{C}_\Sigma;\alpha)$ is a minimizer for $S_q^D(\mathcal{C}_\Sigma;\alpha)$ and equality holds, u would be a solution of

$$\Delta(|x|^\alpha \Delta u) = \lambda |x|^{-\beta}|u|^{q-2}u \text{ in } \mathcal{C}_\Sigma$$

for some $\lambda > 0$ and it would satisfy both Neumann and Dirichlet boundary conditions. Hence it would be $u = 0$, which is impossible. □

Appendix A. Remarks on a Brezis–Nirenberg-type problem

In this section we deal with the Dirichlet problem

$$\begin{cases} \Delta^2 u + \lambda \Delta u = |u|^{2^{**}-2}u & \text{in } B \\ u = u(|x|)\ ,\ u \neq 0 & \\ u = |\nabla u| = 0 & \text{on } \partial B \end{cases} \tag{A.1}$$

where $\lambda \in \mathbb{R}$ is a given parameter and B is the unit ball in $\mathbb{R}^n$, $n \ge 5$. Since a detailed analysis of problem (A.1) would lead us far from our purposes, we limit ourself to investigate those features of problem (A.1) that have some relevance with the questions under investigation in the present paper.

We point out that the fourth order differential equation in (A.1) contains a leading term with critical growth and a linear term involving the Laplacian. In the spirit of the result by Brezis-Nirenberg [3], this last term provides a perturbation of a dilation invariant problem which allows us to recover compactness, when the parameter λ stays in a suitably restricted range.

We start our analysis by pointing out a non-existence result.

Theorem A.1. *If $\lambda \le 0$, then problem* (A.1) *has no solution. If $n = 5$ and $\lambda \le 21/8$, then problem* (A.1) *has no solution.*

Proof. For $\lambda = 0$ the result is already known, see for instance [12] or [13]. If $\lambda \neq 0$, the proof is based on a Pohozaev identity that has to be coupled with a Hardy-type inequality in the lowest dimensional case.

Let u be a solution of (A.1). We put $r = |x|$ and we denote by u_r the radial derivatives of u, namely $u_r = r^{-1}x \cdot \nabla u$. Testing (A.1) with $2ru_r - u$ one infers the following Pohozaev identity (use for instance the computations in [13], pagg. 250–252):

$$2\lambda \int_B |\nabla u|^2 = \omega_n \int_{\partial B} |\Delta u|^2 (x \cdot \nu) = \omega_n |u_{rr}(1)|^2, \tag{A.2}$$

where ω_n is the measure of $\mathbb{S}^{n-1}$. Thus $\lambda > 0$.

Now we assume $n = 5$. We will prove below that

$$5\int_B r^2|\Delta u|^2 - 6\int_B |\nabla u|^2 - 2\lambda \int_B |\nabla u|^2 = -\lambda \int_B r^2 |\nabla u|^2 - \frac{7}{5}\int_B r^2|u|^{10}. \tag{A.3}$$

From (A.3) and using Lemma B.6 with $\alpha = 2$ we get

$$0 > 5\int_B r^2|\Delta u|^2 - 6\int_B |\nabla u|^2 - 2\lambda\int_B |\nabla u|^2 \geq \left(\frac{21}{4} - 2\lambda\right)\int_B |\nabla u|^2,$$

that implies $\lambda > 21/8$ and concludes the proof.
It remains to check (A.3). If η and φ are radial and smooth enough, then

$$\begin{aligned} &\int_B \varphi\eta_r = \omega_4\varphi(1)\eta(1) - \int_B (\varphi_r + 4r^{-1}\varphi)\eta \ , \\ &2\int_B \eta u_r \Delta u = \int_B (4r^{-1}\eta - \eta_r)|\nabla u|^2. \end{aligned} \tag{A.4}$$

Next we notice that

$$u\Delta u = \frac{1}{2}\Delta(u^2) - |\nabla u|^2 \ , \quad u_{rr} = \Delta u - 4r^{-1}u_r \ , \quad (\Delta u)_r = \Delta u_r - 4r^{-2}u_r,$$

and we test (A.1) with $r^3 u_r$. Using integration by parts, (A.4) and (A.2) we get

$$\begin{aligned} \int_B (\Delta^2 u)\,(r^3 u_r) &= -\omega_4|u_{rr}(1)|^2 + \int_B (\Delta u)(r^3\Delta u_r + u_r\Delta(r^3) + 6r^2 u_{rr}) \\ &= -2\lambda\int_B |\nabla u|^2 + \int_B r^3(\Delta u)(\Delta u)_r + 6\int_B r^2|\Delta u|^2 \\ &\quad - 18\int_B r u_r \Delta u \\ &= -\lambda\int_B |\nabla u|^2 + \frac{5}{2}\int_B r^2|\Delta u|^2 - 3\int_B |\nabla u|^2. \end{aligned}$$

We compute also

$$\lambda\int_B \Delta u(r^3 u_r) = \frac{\lambda}{2}\int_B r^2|\nabla u|^2 \,,$$

$$\int_B u^9(r^3 u_r) = \frac{1}{10}\int_B r^3\left(|u|^{10}\right)_r = -\frac{7}{10}\int_B r^2|u|^{10} \,.$$

Thus, from (A.1) we readily get (A.3). $\square$

A natural approach for studying (A.1) consists in looking for minimizers for

$$S^{**}_\lambda := \inf_{\substack{u\in H^2_{0,\mathrm{rad}}(B) \\ u\neq 0}} \frac{\displaystyle\int_B |\Delta u|^2 dx - \lambda\int_B |\nabla u|^2 dx}{\left(\displaystyle\int_B |u|^{2^{**}} dx\right)^{2/2^{**}}} \,. \tag{A.5}$$

Clearly, the infimum S^{**}_λ is positive provided that $\lambda < \lambda_{2,1}$, where

$$\lambda_{2,1} := \inf_{\substack{u\in H^2_{0,\mathrm{rad}}(B) \\ u\neq 0}} \frac{\displaystyle\int_B |\Delta u|^2 dx}{\displaystyle\int_B |\nabla u|^2 dx} \geq \frac{n^2}{4}$$

by Lemma B.6 in Appendix B.1. Moreover, minimizers for S^{**}_λ give rise to solutions to problem (A.1). Arguing for instance as in Proposition 2.4 one can check that

$S^{**}_\lambda \le S^{**}$ for any $\lambda \in \mathbb{R}$. In particular, by monotonicity, it turns out that $S^{**}_\lambda = S^{**}$ and is not attained if $\lambda \le 0$, accordingly with Theorem A.1. As in [3] or [16], a crucial point in finding an existence result for the minimization problem (A.5) consists in giving sufficient conditions for the validity of the strict inequality $S^{**}_\lambda < S^{**}$. First of all we notice that $S^{**}_\lambda < S^{**}$ provided that λ is close enough to $\lambda_{2,1}$. Notice indeed that for any $\lambda > 0$ it results

$$S^{**}_\lambda \le \inf_{\substack{u \in H^2_{0,\mathrm{rad}}(B) \\ u \ge 0\,,\, u \ne 0}} \frac{\displaystyle\int_B |\Delta u|^2 dx - \lambda\lambda_1^{-1} \int_B |u|^2 dx}{\left(\displaystyle\int_B |u|^{2^{**}} dx\right)^{2/2^{**}}},$$

where λ_1 is the Poincaré constant of the unit ball in $\mathbb{R}^n$. Then one concludes by using known results for problem

$$\begin{cases} \Delta^2 u - \lambda u = |u|^{2^{**}-2} u & \text{in } B \\ u = |\nabla u| = 0 & \text{on } \partial B \end{cases}$$

(see for instance [12] or [13]). The same argument shows that $S^{**}_\lambda < S^{**}$ if $n \ge 8$. The next lemma, that was crucially used in Section 2.2, covers also the case $n \in \{6, 7\}$ and shows that $n = 5$ is the only critical dimension for problem (A.1).

Lemma A.2. *Let B be the unit ball in $\mathbb{R}^n$ and $\lambda > 0$. If $n \ge 6$, then there exists a nonnegative radially symmetric function $u \in C^\infty_c(B)$ such that*

$$\frac{\displaystyle\int_B |\Delta u|^2 dx - \lambda \int_B |\nabla u|^2 dx}{\left(\displaystyle\int_B |u|^{2^{**}} dx\right)^{2/2^{**}}} < S^{**}.$$

Proof. Let U be the non-negative radial mapping defined in (2.15) and let $\xi \in C^\infty_c(B)$ be a radial function with $0 \le \xi \le 1$ and $\xi(x) = 1$ as $|x| \le \frac{1}{2}$. Define

$$u_\varepsilon(x) = \varepsilon^{\frac{4-n}{2}} \xi(x) U(\varepsilon^{-1} x).$$

Hence $u_\varepsilon \in C^2_c(B)$ and

$$\int |\Delta u_\varepsilon|^2 = \int |\Delta U|^2 + O(\varepsilon^{n-4}) \quad \text{and} \quad \int u_\varepsilon^{2^{**}} = \int U^{2^{**}} + O(\varepsilon^n) \quad \text{as } \varepsilon \to 0 \quad \text{(A.6)}$$

(see, e.g., [12]). Thanks to (2.16) and (A.6) we have that

$$\int |\Delta u_\varepsilon|^2 = \left(S^{**} + O(\varepsilon^{n-4})\right) \left(\int U^{2^{**}}\right)^{2/2^{**}} \quad \text{as } \varepsilon \to 0. \qquad \text{(A.7)}$$

If $n \ge 7$, then $U \in D^{1,2}(\mathbb{R}^n)$ and one can easily check that

$$\int |\nabla u_\varepsilon|^2 = \varepsilon^2 \int |\nabla U|^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Therefore

$$\frac{S_\lambda^{**}-S^{**}}{\varepsilon^2} \le \frac{1}{\varepsilon^2}\left[\frac{\int |\Delta u_\varepsilon|^2}{\left(\int u_\varepsilon^{2^{**}}\right)^{2/2^{**}}} - S^{**}\right] - \lambda \frac{\int |\nabla U|^2}{\left(\int U^{2^{**}}\right)^{2/2^{**}}} + o(1) \quad \text{as } \varepsilon \to 0$$

and then $S_\lambda^{**} < S^{**}$, because of (A.7). If $n = 6$, then

$$\frac{1}{\varepsilon^2}\int |\nabla u_\varepsilon|^2 \ge \int_{B_{\frac{1}{2\varepsilon}}} |\nabla U|^2 = C\int_0^{\frac{1}{2\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{n-2}}\, dr \ge C|\log\varepsilon|$$

for some constant $C > 0$. Then

$$\begin{aligned}\frac{S_\lambda^{**}-S^{**}}{\varepsilon^2} &\le \frac{1}{\varepsilon^2}\left[\frac{\int |\Delta u_\varepsilon|^2}{\left(\int u_\varepsilon^{2^{**}}\right)^{2/2^{**}}} - S^{**}\right] - \lambda \frac{\varepsilon^{-2}\int |\nabla u_\varepsilon|^2}{\left(\int u_\varepsilon^{2^{**}}\right)^{2/2^{**}}} \\ &\le O(1) - \lambda \frac{C|\log\varepsilon|}{\left(\int u_\varepsilon^{2^{**}}\right)^{2/2^{**}}}.\end{aligned}$$

Hence also in this case we can conclude that $S_\lambda^{**} < S^{**}$. □

We conclude this section with an existence result, whose proof can be obtained by using the above remarks and standard arguments.

Theorem A.3. *Let B be the unit ball in $\mathbb{R}^n$ and $0 < \lambda < \lambda_{2,1}$.*

(i) *If $n \ge 6$, then problem* (A.1) *admits a ground state solution, i.e., a function $u \in H_0^2(B)$ solving* (A.1) *and minimizing S_λ^{**}.*
(ii) *If $n = 5$, then there exists $\lambda^* \in (0, \lambda_{2,1})$ such that S_λ^{**} is not achieved if $\lambda < \lambda^*$ and achieved if $\lambda^* < \lambda < \lambda_{2,1}$.*

Appendix B. Auxiliary results and open problems

This Appendix contains some technical results used in the previous sections. In particular we prove of some estimates that were used in the proof of Theorem 2.5 and a couple of continuity lemmas. Finally we write a list of open problems.

Lemma B.1. *Let $a \in \mathbb{R}$ and $e \in \mathbb{S}^{n-1}$. Then there exists a constant $K_a > 0$ such that for every radial mapping $u \in C_c^2(B)$ and for every $t \in [0,1]$ one has*

$$\int |tx+e|^{-2a} \left|\Delta\left(|tx+e|^a u\right)\right|^2 \le \int |\Delta u|^2 - 2C_a t^2 \int |\nabla u|^2 + K_a t^3 \int |\nabla u|^2$$

where $C_a = a(a+2)(n-2)/n$.

Proof. One computes

$$\begin{aligned}&\Delta\left(|tx+e|^a u\right)\\&\quad=|tx+e|^{a-2}\left[|tx+e|^2\Delta u+2at\nabla u\cdot(tx+e)+a(n-2+a)t^2u\right]\end{aligned}$$

and then

$$\begin{aligned}&\int|tx+e|^{-2a}\left|\Delta\left(|tx+e|^a u\right)\right|^2\\&=\int|\Delta u|^2+4at\underbrace{\int|tx+e|^{-2}(\nabla u\cdot(tx+e))\Delta u}_{I_1}\\&+4a^2t^2\underbrace{\int|tx+e|^{-4}|\nabla u\cdot(tx+e)|^2}_{I_2}+2a(n-2+a)t^2\underbrace{\int|tx+e|^{-2}u\Delta u}_{I_3}\\&+4a^2(n-2+a)t^3\underbrace{\int|tx+e|^{-4}u(\nabla u\cdot(tx+e))}_{I_4}\\&+a^2(n-2+a)^2t^4\underbrace{\int|tx+e|^{-4}|u|^2}_{I_5}.\end{aligned}$$

Since u is radial one has that

$$\begin{gathered}\int|\nabla u\cdot e|^2=\int\frac{(e\cdot x)^2}{|x|^2}|\nabla u|^2=\frac{1}{n}\int|\nabla u|^2\\\int\frac{e\cdot x}{|x|^2}|\nabla u|^2=\int(x\cdot\nabla u)(e\cdot\nabla u)=0.\end{gathered}\tag{B.1}$$

For future convenience we also point out that since

$$\begin{gathered}\nabla|tx+e|^{-2}=-2t|tx+e|^{-4}(e+x)\\1-|tx+e|^{-2}=t|tx+e|^{-2}(2x\cdot e+t|x|^2)\end{gathered}\tag{B.2}$$

and since $|tx+e|\ge 1-t$ for $x\in B$, we have the following estimates:

$$|tx+e|^{-4}\le C\ ,\quad\left|\nabla|tx+e|^{-2}\right|\le Ct\ ,\quad\left|1-|tx+e|^{-2}\right|\le Ct\tag{B.3}$$

for all $x\in B$ and for every $t\ge 0$ small enough.

Estimate of I_1. Firstly we integrate by parts, obtaining that

$$\begin{aligned}
I_1 &= \int |tx+e|^{-2}(\nabla u\cdot(tx+e))\Delta u = -\int \nabla\left(|tx+e|^{-2}(tx+e)\cdot\nabla u\right)\cdot\nabla u \\
&= 2t^2\int |tx+e|^{-4}\left[(x\cdot\nabla u)^2+(e\cdot\nabla u)(x\cdot\nabla u)\right] \\
&\quad + 2t\int |tx+e|^{-4}\left[(e\cdot\nabla u)^2+(e\cdot\nabla u)(x\cdot\nabla u)\right] \\
&\quad + \frac{n-2}{2}t\int |tx+e|^{-2}|\nabla u|^2 - t^4\int |tx+e|^{-4}\left(e\cdot x+|x|^2\right)|\nabla u|^2 \\
&\quad - \int |tx+e|^{-2}\left[\frac{e\cdot x}{|x|}u_{rr}u_r+\frac{e\cdot\nabla u}{|x|}u_r-\frac{e\cdot x}{|x|^2}u_r^2\right]
\end{aligned}$$

where, as in the proof of Theorem A.1, u_r and u_{rr} denote the first and second radial derivatives of u, respectively. Then we use the fact that u is radial, in particular the identity $e\cdot\nabla u=\frac{e\cdot x}{|x|}u_r$, and (B.1), getting that

$$\begin{aligned}
I_1 &= \frac{n-1}{2}\underbrace{\int |tx+e|^{-2}\frac{e\cdot x}{|x|^2}|\nabla u|^2}_{J_1}+\left(\frac{n}{2}-1\right)t\underbrace{\int |tx+e|^{-2}|\nabla u|^2}_{J_2} \\
&\quad + t\underbrace{\int |tx+e|^{-4}\left[e\cdot x+\frac{(e\cdot x)^2}{|x|^2}\right]|\nabla u|^2}_{J_3} \\
&\quad + t^2\underbrace{\int |tx+e|^{-4}\left[e\cdot x+|x|^2\right]|\nabla u|^2}_{J_4}.
\end{aligned}$$

We need to compute the terms of I_1 of order 0 and 1 in t. Therefore, in view of (B.1) and (B.2), we can write

$$\begin{aligned}
J_1 &= \int\left[|tx+e|^{-2}-1\right]\frac{e\cdot x}{|x|^2}|\nabla u|^2 \\
&= -t^2\int |tx+e|^{-2}(e\cdot x)|\nabla u|^2-2t\int |tx+e|^{-2}\frac{(e\cdot x)^2}{|x|^2}|\nabla u|^2.
\end{aligned}$$

Next, using again (B.1) and (B.2), we estimate

$$\begin{aligned}
&\int |tx+e|^{-2}\frac{(e\cdot x)^2}{|x|^2}|\nabla u|^2 \\
&\qquad = \int |tx+e|^{-2}\left[\frac{(e\cdot x)^2}{|x|^2}-1\right]|\nabla u|^2+\int\frac{(e\cdot x)^2}{|x|^2}|\nabla u|^2 \\
&\qquad = -t\int |tx+e|^{-2}\left[2e\cdot x+t|x|^2\right]\frac{(e\cdot x)^2}{|x|^2}|\nabla u|^2+\frac{1}{n}\int|\nabla u|^2.
\end{aligned}$$

Hence, by (B.3),

$$J_1\le -\frac{2t}{n}\int|\nabla u|^2+Ct^2\int|\nabla u|^2.$$

Then, using (B.3) too, we estimate

$$J_2 = \int |\nabla u|^2 + \int \left[|tx+e|^{-2} - 1\right] |\nabla u|^2 \le \int |\nabla u|^2 + Ct \int |\nabla u|^2.$$

Moreover, again by (B.1), (B.2) and (B.3), we have

$$\begin{aligned} J_3 &= \int \left[|tx+e|^{-4} - 1\right] \left[e\cdot x + \frac{(e\cdot x)^2}{|x|^2}\right] |\nabla u|^2 + \int \left[e\cdot x + \frac{(e\cdot x)^2}{|x|^2}\right] |\nabla u|^2 \\ &= -t \int |tx+e|^{-2} \left[|tx+e|^{-2} + 1\right] \left[e\cdot x + \frac{(e\cdot x)^2}{|x|^2}\right] \left[2e\cdot x + t|x|^2\right] |\nabla u|^2 \\ &\quad + \frac{1}{n} \int |\nabla u|^2 \le \frac{1}{n} \int |\nabla u|^2 + Ct \int |\nabla u|^2 \end{aligned}$$

and

$$J_4 \le C \int |\nabla u|^2.$$

In conclusion

$$I_1 \le \left(\frac{2}{n} + \frac{n}{2} - 2\right) t \int |\nabla u|^2 + Ct^2 \int |\nabla u|^2$$

for $t \ge 0$ small enough.

Estimate of I_2. Thanks to (B.1) and (B.3), we have

$$\begin{aligned} I_2 &= \int |tx+e|^{-4} |\nabla u \cdot (tx+e)|^2 \\ &\le (1-t)^{-4} \left[t^2 \int |x\cdot \nabla u|^2 + 2t \int (x\cdot\nabla u)(e\cdot\nabla u) + \int |e\cdot\nabla u|^2\right] \\ &\le Ct \int |\nabla u|^2 + \frac{1}{n} \int |\nabla u|^2 \end{aligned}$$

for $t \ge 0$ small enough.

Estimate of I_3. Firstly we integrate by parts, then we use Cauchy-Schwarz and Poincaré inequalities and the estimates (B.3) getting that

$$\begin{aligned} I_3 &= \int |tx+e|^{-2} u \Delta u = -\int \nabla\left(|tx+e|^{-2} u\right) \cdot \nabla u \\ &= 2t \int |tx+e|^{-4} u\,(x+e)\cdot\nabla u + \int \left(1 - |tx+e|^{-2}\right) |\nabla u|^2 - \int |\nabla u|^2 \\ &\le 4t(1-t)^{-4} \int |u|\,|\nabla u| + Ct \int |\nabla u|^2 - \int |\nabla u|^2 \\ &\le Ct \int |\nabla u|^2 - \int |\nabla u|^2 \end{aligned}$$

for $t \ge 0$ small enough.

Estimate of I_4. Using Cauchy-Schwarz and Poincaré inequalities we have that

$$I_4 = \int |tx+e|^{-4} u(\nabla u \cdot (tx+e)) \le (1-t)^{-4} \int |u|\,|\nabla u| \le C \int |\nabla u|^2$$

for $t \ge 0$ small enough.

Estimate of I_5. Thanks to the Poincaré inequality we have

$$I_5 = \int |tx+e|^{-4}|u|^2 \le (1-t)^{-4} \int |u|^2 \le C \int |\nabla u|^2$$

for $t \ge 0$ small enough.

The conclusion easily follows from the previous estimates of I_1,...,I_5. □

B.1. Continuity lemmas

Here we discuss the following two auxiliary results.

Lemma B.2. *If $q \to 2^+$, then $S_q(\mathcal{C}_\Sigma;\alpha) \to S_2(\mathcal{C}_\Sigma;\alpha)$ and $S_q^+(\mathcal{C}_\Sigma;\alpha) \to S_2^+(\mathcal{C}_\Sigma;\alpha)$.*

Lemma B.3. *If $\alpha \to \alpha_0$, then $S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha) \to S_q^{\mathrm{rad}}(\mathbb{R}^n;\alpha_0)$.*

Let us point out the following general fact.

Remark B.4. Let X be a nonempty set, let I be an interval in $\mathbb{R}$ and let $F\colon X \times I \to [0,+\infty)$ be a given mapping. For every $a \in I$ set $S(a) = \inf_{u\in X} F(u,a)$. If F is continuous with respect to the parameter $a \in I$, then $S(a) \ge \limsup S(a_k)$ when $a_k \to a$. This fact can be proved in an elementary way.

In view of Remark B.4, in order to prove Lemmas B.2 and B.3 we need to show just the lower semicontinuity inequalities.

Proof of Lemma B.2. It is a consequence of Remark B.4 and of the following general result.

Lemma B.5. *Let Ω be a domain in $\mathbb{R}^n$ and let μ be a positive measure on Ω. Let X be a space of measurable functions from Ω into $\mathbb{R}$, endowed with some norm $\|\cdot\|$. Assume that X is continuously embedded into $L^p(\mu)$ for all p in some compact interval $I \subset [1,\infty)$. Then the mapping*

$$p \mapsto S(p) := \inf_{\substack{u\in X\\ u\neq 0}} \frac{\|u\|}{|u|_p} \quad \textit{where } |u|_p = \left(\int_\Omega |u|^p \, d\mu\right)^{\frac{1}{p}} \quad \textit{and } p \in I$$

is lower semicontinuous in I, i.e., if $(p_k) \subset I$ and $p_k \to p$, then $S(p) \le \liminf S(p_k)$.

Proof. Let $I = [p_0, p_1]$. By Hölder's inequality, for every $u \in X$ and $\theta \in [0,1]$ one has that

$$|u|_{p_\theta}^{p_\theta} \le |u|_{p_0}^{(1-\theta)p_0} |u|_{p_1}^{\theta p_1}$$

where $p_\theta = \theta p_1 + (1-\theta)p_0$. This readily implies that

$$S(p_\theta)^{p_\theta} \ge S(p_0)^{(1-\theta)p_0} S(p_1)^{\theta p_1}. \tag{B.4}$$

Setting $f(p) = p \log S(p)$, (B.4) reads

$$(1-\theta)f(p_0) + \theta f(p_1) \le f(\theta p_1 + (1-\theta)p_0)$$

namely f is concave in $[p_0, p_1]$. This implies that f is continuous in (p_0, p_1) and lower semicontinuous in $[p_0, p_1]$. Clearly the same holds for $S(p)$, too. $\square$

Proof of Lemma B.3. It is trivial if $n = 2$, by Corollary 2.11. Thus assume $n \geq 3$. If $\alpha_0 \in \{n, 4-n\}$, then $S_q^{\mathrm{rad}}(\mathbb{R}^n; \alpha_0) = 0$ (see Theorem 2.7) and the result is a consequence of Remark B.4. If $\alpha_0 \in \mathbb{R} \setminus \{n, 4-n\}$, then the proof of Lemma B.3 can be accomplished according to the following argument. First of all let us introduce the infimum

$$\mu_{2,1}^{\mathrm{rad}}(\mathbb{R}^n; \alpha) := \inf_{\substack{u \in C_c^2(\mathbb{R}^n \setminus \{0\}) \\ u = u(|x|),\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2 \, dx}{\displaystyle\int_{\mathbb{R}^n} |x|^{\alpha-2} |\nabla u|^2 \, dx}.$$

Lemma B.6. *There results* $\mu_{2,1}^{\mathrm{rad}}(\mathbb{R}^n; \alpha) = \left(\frac{n-\alpha}{2}\right)^2$.

Proof. Proceeding as in the proof of Theorem 2.7, by means of the Emden-Fowler transform we have that

$$\begin{aligned} &\mu_{2,1}^{\mathrm{rad}}(\mathbb{R}^n; \alpha) - \left(\frac{n-\alpha}{2}\right)^2 \\ &\qquad = \inf_{\substack{w \in C_c^2(\mathbb{R}) \\ w \neq 0}} \frac{\displaystyle\int_{-\infty}^{\infty} |w''|^2 ds + \left[2\overline{\gamma}_\alpha - \left(\frac{n-\alpha}{2}\right)^2\right] \int_{-\infty}^{\infty} |w'|^2 ds}{\displaystyle\int_{-\infty}^{\infty} |w'|^2 ds + \left(\frac{n-4+\alpha}{2}\right)^2 \int_{-\infty}^{\infty} |w|^2 ds} \end{aligned}$$

where $\overline{\gamma}_\alpha$ is defined in (2.5). We notice that

$$2\overline{\gamma}_\alpha - \left(\frac{n-\alpha}{2}\right)^2 = \left(\frac{n-4+\alpha}{2}\right)^2$$

and we conclude by a standard scaling argument. $\square$

Lemma B.7. *If* $\alpha, \tilde{\alpha} \in \mathbb{R} \setminus \{n, 4-n\}$, *then*

$$\left[1 - \frac{4|g(\alpha, \tilde{\alpha})|}{(n-\tilde{\alpha})^2}\right] S_q^{\mathrm{rad}}(\tilde{\alpha}) \leq |\tau(\tilde{\alpha}, \alpha)|^{3+\frac{2}{q}} S_q^{\mathrm{rad}}(\alpha) \leq \left[1 + \frac{4|g(\alpha, \tilde{\alpha})|}{(n-\alpha)^2}\right] S_q^{\mathrm{rad}}(\tilde{\alpha}) \quad \text{(B.5)}$$

where $\tau(\tilde{\alpha}, \alpha)$ *and* $g(\alpha, \tilde{\alpha})$ *are defined in* (2.17).

Proof. As in the proof of Lemma 2.10, for every $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ radially symmetric let $\widetilde{u} \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ be the radial function defined by means of the transformation (2.18). Thanks to the identities (2.19)–(2.20) and using the definition of $\mu_{2,1}^{\mathrm{rad}}(\mathbb{R}^n; \alpha)$ and Lemma B.6, the conclusion readily follows. $\square$

Completion of the proof of Lemma B.3. Let $\alpha_0 \in \mathbb{R} \setminus \{n, 4-n\}$ and $\alpha_k \to \alpha_0$. We can apply Lemma B.7 and, since $\tau(\alpha_0, \alpha_k) \to 1$ and $g(\alpha_k, \alpha_0) \to 0$, the conclusion follows from (B.5). □

B.2. Remarks and open problems

The arguments in Section 3 can be used to get existence results of minimizers for

$$S_{q,\lambda}(\mathcal{C}_\Sigma;\alpha) = \inf_{\substack{u\in\mathcal{N}^2_\lambda(\mathcal{C}_\Sigma;\alpha)\\ u\neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^\alpha |\Delta u|^2 dx - \lambda \int_{\mathcal{C}_\Sigma} |x|^{\alpha-4}|u|^2 dx}{\left(\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-\beta}|u|^q dx\right)^{2/q}}.$$

Here α is any real parameter, $q > 2$ and $q \le 2^{**}$ if $n \ge 5$, $\lambda < \operatorname{dist}(-\gamma_\alpha, \Lambda(\Sigma))^2$ (compare with [7]), and $\mathcal{N}^2_\lambda(\mathcal{C}_\Sigma;\alpha)$ is a suitably defined function space. We refer to the forthcoming paper [2] for the case $\alpha = 0$ and $\Sigma = \mathbb{S}^{n-1}$.

The approach in Section 3 can be plainly applied also to prove existence results for extremals of Lin's inequality in [15] and for more general dilation-invariant inequalities.

The present paper raises several open questions. We list few of them.

i) It might be interesting to generalize the results of this paper when $|\Delta u|^2$ is replaced by $|\Delta u|^p$ with $p > 1$. Some partial results can be found in [23].

ii) Our results about breaking positivity and breaking symmetry hold only for some restricted ranges of α and/or q. Is it possible to give a sharper description of the region of parameters α, q where breaking positivity/symmetry occur? In particular, is it true that breaking positivity occurs for any α large enough?

iii) Is it true that for $n \ge 3$ and $\alpha \in (4-n, n)$ extremals for $S_q(\mathbb{R}^n \setminus \{0\};\alpha)$ are radially symmetric and/or positive?

iv) Let Σ be properly contained in $\mathbb{S}^{n-1}$, and take $n \ge 5$, $\alpha = 0$. Then

$$0 < S_{2^{**}}(\mathcal{C}_\Sigma;0) \le S^D_{2^{**}}(\mathcal{C}_\Sigma;0) = S^{**}.$$

Is it true that $S_{2^{**}}(\mathcal{C}_\Sigma;0) = S^{**}$? This question is related to [14].

v) When $n \ge 6$ we showed that $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) < S^{**}$ if $|\alpha - 2| > 2$. Indeed we suspect that if $\alpha \in (0,4)$, then $S_{2^{**}}(\mathcal{C}_\Sigma;\alpha) = S^{**}$ and is not achieved.

References

[1] Adams, R. A., *Sobolev spaces*, Academic Press, 1970.

[2] Bhakta, M., Musina, R., *Entire solutions for a class of variational problems involving the biharmonic operator and Rellich potentials*, (in progress).

[3] Brézis, H., Nirenberg, L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Commun. Pure Appl. Math. **36** (1983), 437–477.

[4] Caffarelli, L., Kohn, R., Nirenberg, L., *First Order Interpolation Inequalities with Weight*, Compositio Math. **53** (1984), 259–275.

[5] Calanchi, M., Ruf, B., *Radial and non radial solutions for Hardy-Hénon type elliptic systems*, Calc. Var. PDE **38** (2010), 111–133.

[6] Caldiroli, P., Musina, R., *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent*, Calc. Var. PDE **8** (1999), 365–387.

[7] Caldiroli, P., Musina, R., *Rellich inequalities with weights*, Calc. Var. PDE (to appear)

[8] Catrina, F., Wang, Z.-Q., *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math. **54** (2001), 229–258.

[9] de Figueiredo, D.G., Peral, I., Rossi, J D., *The critical hyperbola for a Hamiltonian system with weights*, Ann. Mat. Pura Appl. (4) **187** (2008), 531–545.

[10] Dolbeault, J., Esteban, M., Loss, M., Tarantello, G., *On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities*, Adv. Nonlinear Stud. **9** (2009), 713–726.

[11] Felli, V., Schneider, M., *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Diff. Eq. **191** (2003), 121–142.

[12] Gazzola, F., *Critical growth problems for polyharmonic operators*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 251–263.

[13] Gazzola, F., Grunau, H.-C., Sweers, G., *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics 1991. Berlin: Springer, 2010.

[14] Gazzola, F., Grunau, H.-C., Sweers, G., *Optimal Sobolev and Hardy–Rellich constants under Navier boundary conditions*, Ann. Mat. Pura Appl. (4) **189** (2010), 475–486.

[15] Lin, C.-S., *Interpolation inequalities with weights*, Comm. Part. Diff. Eq. **11** (1986), 1515–1538.

[16] Lions, P.-L., *The concentration-compactness principle in the Calculus of Variations. The locally compact case, parts 1 and 2.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **1** (1984) no. 2, 109–145, no. 4, 223–283.

[17] Liu, F., Yang, J., *Nontrivial solutions of Hardy-Henon type elliptic systems*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 673–688.

[18] Pucci, P., Serrin, J., *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl., **69** (1990), 53–83.

[19] Rellich, F., *Halbbeschränkte Differentialoperatoren höherer Ordnung.* In: J.C.H. Gerretsen, J. de Groot (Eds.): Proceedings of the International Congress of Mathematicians 1954, Volume III (pp. 243–250) Groningen: Noordhoff 1956.

[20] Rellich, F., *Perturbation theory of eigenvalue problems*, Gordon and Breach, New York, 1969.

[21] Struwe, M., *Variational Methods (fourth edition)*, Springer, 2008.

[22] Swanson, C.A., *The best Sobolev constant*, Appl. Anal. **47** (1992), 227–239.

[23] Szulkin, A., Waliullah, S., *Sign-changing and symmetry-breaking solutions to singular problems*, Complex Variables and Elliptic Eq., First published on 02 February 2011.

Paolo Caldiroli
Dipartimento di Matematica
Università di Torino
via Carlo Alberto, 10
10123 Torino
Italy
e-mail: `paolo.caldiroli@unito.it`

Roberta Musina
Dipartimento di Matematica ed Informatica
Università di Udine
via delle Scienze, 206
33100 Udine
Italy
e-mail: `roberta.musina@uniud.it`

Received: June 21, 2011.