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Global Existence in Reaction-Diffusion Systems with Control of Mass: a Survey

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Abstract. The goal of this paper is to describe the state of the art on the question of global existence of solutions to reaction-diffusion systems for which two main properties hold: on one hand, the positivity of the solutions is preserved for all time; on the other hand, the total mass of the components is uniformly controlled in time. This uniform control on the mass (or – in mathematical terms- on the L^1-norm of the solution) suggests that no blow up should occur in finite time. It turns out that the situation is not so simple. This explains why so many partial results in different directions are found in the literature on this topic, and why also the general question of global existence is still open, while lots of systems arise in applications with these two natural properties. We recall here the main positive and negative results on global existence, together with many references, a description of the still open problems and a few new results as well.

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1. Introduction: dissipation or control of mass, structure (P)+(M)

Let us consider the elementary 2×2 system of ordinary differential equations (ODE) where $h : [0, +\infty) \to [0, +\infty)$ is a given regular function and $u, v : [0, T) \to I\!R$ are the unknown functions:

$$(E)\begin{cases} \partial_t u = -u\, h(v) \\ \partial_t v = u\, h(v) \\ u(0) = u_0 \geq 0,\ v(0) = v_0 \geq 0. \end{cases}$$

It is classical that a local solution exists and may be extended on a maximal interval $[0, T^*)$. If we assume, $h(0) \geq 0$, this solution is nonnegative. Moreover, adding the two equations and integrating lead to

$$\forall t \geq 0, \quad u(t) + v(t) = u_0 + v_0.$$

Together with the nonnegativity, this implies that $u(t), v(t)$ stay uniformly bounded on $[0, T^*)$. It follows that $T^* = +\infty$ and the solution is global in time.

Now, **what happens when space diffusion occurs?**

Let us consider for example the following 2×2 system of reaction-diffusion equations where $u = u(t,x), v = v(t,x), (t,x) \in [0,\infty) \times \Omega$ are the unknown functions:

$$(E) \begin{cases} \partial_t u - d_1 \Delta u = -u\, h(v) & on\ (0,\infty) \times \Omega \\ \partial_t v - d_2 \Delta v = u\, h(v) & on\ (0,\infty) \times \Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = v_0(\cdot) \geq 0 & \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & on\ (0,\infty) \times \partial\Omega. \end{cases} \tag{1.1}$$

Here, $d_1, d_2 > 0,\ \Omega \subset I\!R^N$ is open, bounded and regular and h is as before.

As for the ODE, if $u_0, v_0 \in L^\infty(\Omega)$, local existence of a nonnegative classical solution holds, and the solution may be extended on a maximal interval $[0, T^*)$ (see Lemma 1.1). If the L^∞-norm of the solution $(u(t), v(t))$ is itself uniformly bounded on $[0, T^*)$, then $T^* = +\infty$.

If $d_1 = d_2 = d$, then

$$\partial_t(u+v) - d\Delta(u+v) = 0. \tag{1.2}$$

In particular, we deduce by maximum principle that

$$\|u(t) + v(t)\|_{L^\infty(\Omega)} \leq \|u_0 + v_0\|_{L^\infty(\Omega)}. \tag{1.3}$$

Together with the nonnegativity, this implies that $u(t)$ and $v(t)$ stay uniformly bounded in $L^\infty(\Omega)$ and therefore $T^* = +\infty$. Thus, the situation is the same as the O.D.E. case with respect to global existence.

Question: What happens when $d_1 \neq d_2$? It is known that different diffusions can cause the loss of stability properties of equilibrium solutions (see e.g. [65, 52]**). But, can different diffusions destroy global existence?**

We will see later that the situation is then very different. However, as for the O.D.E., we may add up the two equations:

$$\partial_t(u+v) - \Delta(d_1 u + d_2 v) = 0.$$

Integrate this in space and time. Taking into account the boundary conditions (namely $\int_\Omega \Delta(d_1 u + d_2 v) = 0$), this leads to:

$$\int_\Omega u(t) + v(t) = \int_\Omega u_0 + v_0.$$

Again, together with the nonnegativity of u, v, this implies that

$$\forall t \in [0, T^*), \quad \|u(t)\|_{L^1(\Omega)}, \|v(t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}.$$

In other words, the total mass of the two components does not blow up, and $u(t), v(t)$ stay bounded in $L^1(\Omega)$ *uniformly in time*. Whence another way to ask the global existence question:

Question: how do L^1-estimates on the solution, which are uniform in time, help to provide global existence?

Going back to the first equation in system (1.1), and integrating in space and time, give the following, where $Q_T = (0,T) \times \Omega$:

$$\int_{Q_T} uh(v) \leq \int_\Omega u_0.$$

This implies that the *nonlinearity of* (1.1) *is a priori bounded in* $L^1(Q_T)$. Whence another natural and interesting question:

Question: What can be said of a reaction-diffusion system whose nonlinear reactive terms are bounded in $L^1(Q_T)$?

We will address all the above questions in this paper. More generally, we will review most of the main results on global existence in time for the family of $m \times m$ reaction-diffusion systems satisfying the two main following properties:

- *the nonnegativity of the solutions is preserved for all time*
- *the total mass of the components is a priori bounded on all finite intervals.*

More precisely, let us introduce the general system

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, \ldots, u_m) \text{ on } (0,T) \times \Omega, \\ \alpha_i \frac{\partial u_i}{\partial n} + (1-\alpha_i) u_i = \beta_i \text{ on } (0,T) \times \partial\Omega, \\ u_i(0,\cdot) = u_{i0}, \end{cases} \tag{1.4}$$

where the $f_i : I\!R^m \to I\!R$ are C^1 functions of $u = (u_1, \ldots, u_m)$, and for all $i = 1, \ldots, m,\ d_i \in (0,\infty), \alpha_i \in [0,1], \beta_i \in C^2([0,T] \times \overline{\Omega}), \beta_i \geq 0$. We denote $\Sigma_T = (0,T) \times \partial\Omega$.

By *classical solution* to (1.4) on $[0,T)$, we mean that, at least

$$\begin{cases} u \in C\left([0,T); L^1(\Omega)^m\right) \cap L^\infty([0,T-\tau] \times \Omega)^m,\ \forall \tau \in (0,T), \\ \forall k,l = 1, \ldots, N, \forall p \in [1,\infty),\ \partial_t u, \partial_{x_k} u, \partial_{x_k x_l} u \in L^p\left((\tau, T-\tau) \times \Omega\right)^m, \\ \text{and equations in (1.4) are satisfied a.e..} \end{cases} \tag{1.5}$$

Note that this regularity of u implies that $u, \partial_{x_k} u$ have traces in $L^p_{loc}(\Sigma_T)^m$ (see e.g. [41]). Most of the time, due to more regularity of f, the solutions will be regular enough so that derivatives may be understood in the usual sense (e.g. $u \in C^2\left((0,T) \times \overline{\Omega}\right)$ if f is C^2 itself).

Let us first recall the classical local existence result under the above assumptions (see e.g. [29], [63], [4]):

Lemma 1.1. *Assume $u_0 \in L^\infty(\Omega)^m$. Then, there exist $T > 0$ and a unique* classical solution *of* (1.4) *on* $[0, T)$. *If T^* denotes the greatest of these T's, then*

$$\left[\sup_{t\in[0,T^*),1\le i\le m} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty\right] \Rightarrow [T^* = +\infty]. \tag{1.6}$$

If, moreover, the nonlinearity $(f_i)_{1\le i\le m}$ is **quasi-positive** *(see* (1.7) *below), then*

$$[\forall i = 1, \dots, m,\ u_{i0} \ge 0] \Rightarrow [\forall i = 1, \dots, m,\ \forall t \in [0, T^*),\ u_i(t) \ge 0] .$$

Nonnegativity of the solutions is preserved if (and only if) the nonlinearity $f = (f_1, \dots, f_m)$ is *quasi-positive* which means that

$$(\mathbf{P})\ \ \forall r \in [0, +\infty)^m,\ \ \forall i = 1, \dots, m,\ \ f_i(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_m) \ge 0, \tag{1.7}$$

where we denote $r = (r_1, \cdots, r_m)$. This will be assumed for all nonlinearities throughout in this paper.

Remark. According to (1.6), in order to prove global existence of classical solutions for system (1.4), it is sufficient to prove that, if $T^* < +\infty$, then the solutions u_i are uniformly bounded on $[0, T^*)$. *Thus, a priori L^∞-bounds imply global existence.* As already noticed (see (1.2), (1.3)), it is the case if all the d_i's are equal, and global existence then holds for bounded initial data. The situation is quite more complicated if the diffusion coefficients are different from each other and this will be analyzed all along this paper.

Without any extra assumption on f, blows up generally occurs in finite time ($T^* < +\infty$). Here, we will assume that f satisfies a *"mass-control structure"*

$$(\mathbf{M})\ \ \forall r \in [0, +\infty)^m,\quad \sum_{1\le i\le m} f_i(r) \le C[1 + \sum_{1\le i\le m} r_i]. \tag{1.8}$$

Note that this was satisfied in example (1.1) with $C = 0$ and even with equality instead of inequality. With "correct" boundary conditions, (1.8) implies that *the total mass of the solution is bounded on each interval.* Indeed, let us set $W = \sum_{1\le i\le m} u_i$. Integrating the sum of the m equations of (1.4) leads to

$$\partial_t \int_\Omega W(t) - \int_{\partial\Omega} \nabla[\sum_i d_i u_i] \cdot n \le C \int_\Omega [1 + W(t)].$$

Assume for instance that: $\forall i, \alpha_i \in (0, 1]$. Then, using boundary conditions

$$-\int_{\partial\Omega} \nabla[\sum_i d_i u_i] \cdot n = \int_{\partial\Omega} \sum_i d_i \frac{(1-\alpha_i)u_i - \beta_i}{\alpha_i} \ge -\int_{\partial\Omega} \sum_i \frac{d_i \beta_i}{\alpha_i} := -c.$$

Thus, we have the Gronwall's inequality

$$\partial_t \int_\Omega W(t) \le c + C \int_\Omega 1 + W(t),$$

which implies that, for each t in the interval of existence

$$\int_\Omega W(t) \le e^{tC} \int_\Omega W(0) + k(e^{tC} - 1),\ \ k = \left(c + \int_\Omega C\right)/C. \tag{1.9}$$

It follows that the total mass $\int_\Omega W(t)$ is bounded on any interval. Whence **the question: how does this L^1-estimate help to provide global existence?**

Instead of **(M)**, we could assume that, for some $a = (a_i)_{1\le i\le m}$ with $\forall i, a_i > 0$

$$(\mathbf{M'})\ \ \forall r \in [0, +\infty)^m, \quad \sum_{1\le i\le m} a_i f_i(r) \le C[1 + \sum_{1\le i\le m} r_i]. \tag{1.10}$$

Obviously, **(M')** may be reduced to **(M)**, after multiplying each i-th equation of (1.4) by a_i and upon replacing u_i by $a_i u_i$. For simplicity, we will mainly use **(M)** in the following, although examples may arise with **(M')**.

As shown above when $\alpha_i > 0$ for all i, conditions **(P)+(M)** [i.e. (1.7) + (1.8)] imply for most boundary conditions that the total mass is controlled. However, one has to be careful: if $\alpha_i = 0, \beta_i \neq 0$ for some i and $\alpha_i > 0$ for others, then, L^1-estimates may fail (see Subsection 3.4). This explains why we will generally restrict the values of α_i, β_i in the next sections (see (5.5), (5.6)).

Lots of systems come naturally with the two properties **(P)** and **(M)** (or **(M')**) in applications. This is the purpose of the next section to give several examples of this kind. Therefore, it is worth asking the question of global existence in time with these only two properties. We will also consider systems, where not only the total mass is bounded on any interval, but *where even the nonlinearities are bounded in $L^1(Q_T)$ for all $T < +\infty$* (as it is the case for the example (1.1) above).

If the nonlinearities f_i are bounded in $L^1(Q_T)$ for all $T > 0$ and if their growth is less than $|u|^{\frac{N+2}{N}}$ as $|u| \to +\infty$, then by bootstrap arguments, it is classical to show that they are actually bounded in $L^\infty(Q_T)$ (see e.g. [3]) and therefore the solutions exist globally. Here, we are interested in the other situations, namely, when the growth of the nonlinearities is not small, or, given a nonlinearity, when the dimension is high enough so that we are not in the "bootstrap situation" just mentioned. Note that, even for quadratic nonlinearities, the bootstrap argument is not valid as soon as $N \ge 2$.

This paper describes the state of the art and gives a survey of the wide literature published in the last years on these global existence questions. Due to the increasing need for modeling in biology, chemistry, environment problems, etc, they even have gained a higher interest recently. Besides being a survey, this paper provides most proofs of the results (except for those of Section 4), so that it is rather self-contained. Some results are actually quite new, like the L^2-compactness proved in Proposition 6.3 or the global existence result of Theorem 5.14. Some are stated in a more general situation than those found in the literature like Theorems 5.5 and 5.9, or proved in a new way like for Theorem 3.5. Many interesting questions are still unsolved and, in the last section, we indicate several open problems. We give a rather long list of references: it is not exhaustive but hopefully rich enough to track most connected results.

2. Some examples of reaction-diffusion systems with properties (P)+(M)

Here, we give a few examples of reaction-diffusion systems found here and there in the literature as models for very different applications and for which the two properties **(P)+(M)** hold.

- **The Brussellator.** Let us start with the classical so-called "Brussellator" appearing in the modeling of *chemical morphogenetic processes* ([60, 61, 65]):

$$\begin{cases} \partial_t u - d_1\Delta u = -uv^2 + b\,v \\ \partial_t v - d_2\Delta v = uv^2 - (b+1)\,v + a \\ u_{|\partial\Omega} = b/a,\ v_{|\partial\Omega} = a, \\ a, b, d_1, d_2 > 0. \end{cases} \tag{2.1}$$

If we denote

$$f(u,v) = -uv^2 + bv,\quad g(u,v) = uv^2 - (b+1)v + a,$$

then for all $u, v \geq 0$,

$$f(0,v) = b\,v \geq 0,\ g(u,0) = a \geq 0,\ f(u,v) + g(u,v) \leq a,$$

so that **(P)+(M)** holds. Global existence of classical solutions may be proved in small dimension using bootstrap arguments (see [63]). More sophisticated techniques are required in general, like those explained in the next section.

Very similar systems are also used in *models of Glycolysis or in the so-called Gray-Scott models* (see [44] together with more general systems with "telescoping" nonlinearities arising in chemical kinetics).

- **Combustion models.** Exothermic combustion in a gas may be modeled by a system of the following type (see e.g. [30])

$$\begin{cases} \partial_t Y - \mu\Delta Y = -H(Y,T) \\ \partial_t T - \lambda\Delta T = q\,H(Y,T), \end{cases} \tag{2.2}$$

where Y is the concentration of a single reactant, T is the temperature and $H(0,T)=0$, $H(Y,0)\geq 0$. Moreover, if $f(Y,T) = -H(Y,T)$, $g(Y,T) = qH(Y,T)$, we see that $q\,f(Y,T) + g(H,T) = 0$ so that **(P)+(M)** is satisfied for the system in $(q\,Y, T)$. A typical function H is given by $H(Y,T) = Y^m e^T$. Similar equations appear for different applications in [44, 53].

- **Lotka-Volterra systems.** A general class of Lotka-Volterra Systems may be written (see for instance in [43], [24])

$$\forall i = 1,\ldots,m,\ \ \partial_t u_i - d_i\Delta u_i = e_i u_i + u_i \sum_{1\leq j\leq m} p_{ij} u_j, \tag{2.3}$$

with $e_i, p_{ij} \in I\!R$ and various boundary conditions. Condition **(P)** is always satisfied, and so is **(M')** -see (1.10)- if for instance for some $a_i > 0$ (see e.g. [43])

$$\forall w \in R^m,\ \sum_{i,j=1}^{m} a_i p_{ij} w_i w_j \leq 0,$$

- **Quadratic chemical reactions.** Many chemical reactions, when modeled through the mass action law, lead to reaction-diffusion systems with the above **(P)+(M)** structure. Let us first take a typical example that we will discuss later in this paper. We consider the reversible reaction

$$A + B \rightleftharpoons C + D.$$

Then according to the mass action law, the evolution of the concentrations a, b, c, d of A, B, C, D is governed by the following reaction-diffusion system:

$$\begin{cases} \partial_t a - d_1 \Delta a = -k_1 a\, b + k_2 c\, d \\ \partial_t b - d_2 \Delta b = -k_1 a\, b + k_2 c\, d \\ \partial_t c - d_3 \Delta c = k_1 a\, b - k_2 c\, d \\ \partial_t d - d_4 \Delta d = k_1 a\, b - k_2 c\, d. \end{cases} \tag{2.4}$$

with $k_1, k_2 > 0$. Our two conditions are obviously satisfied here. We may also exploit that the entropy is decreasing : see the remark around (6.14) (this is actually the case in reversible reactions).

- **Superquadratic reaction-diffusion systems.** We consider a general reversible chemical reaction of the form

$$p_1 A_1 + p_2 A_2 + \ldots + p_m A_m \rightleftharpoons q_1 A_1 + q_2 A_2 + \ldots + q_m A_m, \tag{2.5}$$

where p_i, q_i are nonnegative integers. According to the usual mass action kinetics and with classical diffusion operators, we model the evolution of the concentrations a_i of A_i by the following system of reaction-diffusion

$$\partial_t a_i - d_i \Delta a_i = (p_i - q_i)\left(k_2 \Pi_{j=1}^m a_j^{q_j} - k_1 \Pi_{j=1}^m a_j^{p_j}\right), \forall i = 1, \ldots, m,$$

where d_i are positive diffusion coefficients. A classical conservation property states that $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0, \infty), i = 1, \ldots, m$. Denoting by f_i the nonlinearity in the i-th equation, this implies $\sum_{i=1}^m m_i\, f_i = 0$, whence the condition $(\mathbf{M}')$. The quasipositivity **(P)** is satisfied as well.

- **Another quadratic model.** Another model for diffusive calcium dynamics with quadratic terms may also be found in H.G. Othmer [53] (see also more comments on it in [44]):

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + \lambda(\gamma_0 + \gamma_1 u_4)(1 - u_1) - \frac{p_1 u_1^4}{p_2^4 + u_1^4} \\ \partial_t u_2 - d_2 \Delta u_2 = -k_1 u_2 + k_1' u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = -k_1' u_3 - k_2 u_1 u_3 + k_1 u_2 + k_2' u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k_2 u_1 u_3 + k_3' u_5 - k_2' u_4 - k_3 u_1 u_4 \\ \partial_t u_5 - d_5 \Delta u_5 = k_3 u_1 u_4 - k_3' u_5. \end{cases} \tag{2.6}$$

- **Electrodynamics.** The following 6×6 system for the electro-deposition of nickel-iron alloy is studied for instance in [1]: it offers a similar semi-linear structure, but now coupled with extra terms: $\forall i = 1, \ldots, 5$

$$\begin{cases} \partial_t w_i - d_i (w_i)_{xx} + b(x)(w_i)_x - [w_i \Phi_x]_x = S_i(w) \\ S_1 = S_2 = 0,\ S_3(w) = S_4(w) = -S_5(w) \\ -[\Phi]_{xx} = \sum_{i=1}^5 z_i w_i, \quad z_i \in I\!R,\ +\text{bdy cond.} \end{cases} \tag{2.7}$$

Here the functions S_i are nonlinearities which preserve nonnegativity and their structure implies **(M)**. Two extra terms are present: a convection term $b(x)(w_i)_x$: if b is a regular enough function, then, for the question of global existence, this perturbation may essentially be treated as if $b \equiv 0$ and may be 'included' in the linear p.d.e. part. The second convection terms $[w_i \Phi_x]_x$ is different since the regularity of the transport coefficient Φ_x depends itself on estimates on the $w_i's$. Therefore, it is important to obtain a priori estimates on w_i from the only **(P)+(M)** structure.

Let us also refer to [17] for the study of a model with similar features which is used in cardiac electrophysiology.

- **Diffusion of pollutants in atmosphere.** Another interesting example comes from the modeling of pollutants transfer in atmosphere (here $N = 3$): this system of 20 equations is studied in [25] and, more recently in [59] (we refer to these two papers for more references):

$$\begin{cases} \partial_t \phi_i = d_i \, \partial^2_{zz} \phi_i + \omega \cdot \nabla \phi + f_i(\phi) + g_i, \ \forall i = 1, \ldots, 20, \\ + \text{Bdy and initial conditions} \end{cases} \tag{2.8}$$

Here the nonlinearities f_i are given by

$$\begin{cases} f_1(\phi) &=& -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\ && +k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\ f_2(\phi) &=& k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2 \\ f_3(\phi) &=& k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3 \\ f_4(\phi) &=& -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 \\ f_5(\phi) &=& 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6 \\ f_6(\phi) &=& 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6 \\ f_7(\phi) &=& -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6 \\ f_8(\phi) &=& k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6 \\ f_9(\phi) &=& -k_7\phi_9 - k_8\phi_9\phi_6 \\ f_{10}(\phi) &=& k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2 \\ f_{11}(\phi) &=& k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1 \\ f_{12}(\phi) &=& k_9\phi_{11}\phi_2 \\ f_{13}(\phi) &=& -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1 \\ f_{14}(\phi) &=& -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2 \\ f_{15}(\phi) &=& k_{14}\phi_1\phi_6 \\ f_{16}(\phi) &=& -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4 \\ f_{17}(\phi) &=& -k_{20}\phi_{17}\phi_6 \\ f_{18}(\phi) &=& k_{20}\phi_{17}\phi_6 \\ f_{19}(\phi) &=& -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1 \\ f_{20}(\phi) &=& -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1. \end{cases} \tag{2.9}$$

where the k_i's are positive real numbers. These nonlinearities may seem complicated, but they are quadratic and, obviously satisfy **(P)+(M)**. The main new point in this system is that diffusion occurs only in the vertical direction. As a consequence, many of the tools, which are based on the regularizing effects of the diffusion, need to be revisited. Even the transport term may cause new difficulties due to the lack of diffusion in two directions. However, the general methods described in the next sections may be used to obtain some global existence results for this degenerate system.

We could go on with more and more examples arising in applications with **(P)+(M)**. We refer for instance to the books [29, 63, 43, 22, 23, 52, 54].

3. Existence of global classical solutions

3.1. A typical result on 2×2 systems

Let us consider the following 2×2 system

$$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) \\ \partial_t v - d_2 \Delta v = g(u,v) \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = v_0(\cdot) \geq 0, \\ \textbf{with either} : \frac{\partial u}{\partial n} = \beta_1, \frac{\partial v}{\partial n} = \beta_2 \ on \ (0,+\infty) \times \partial\Omega, \\ \textbf{or} :\ u = \beta_1, \ \ v = \beta_2 \ on \ (0,+\infty) \times \partial\Omega, \end{cases} \tag{3.1}$$

where $d_1, d_2 \in (0,+\infty), \beta_1, \beta_2 \in [0,+\infty)$ and $f, g : [0,+\infty)^2 \to I\!R$ are C^1.

For $u_0, v_0 \in L^\infty(\Omega)$ with $u_0, v_0 \geq 0$, existence of classical nonnegative bounded solutions holds on some maximal interval $[0, T^*)$ (see Lemma 1.1). Then, we have the first following global existence result (see [31], [46]):

Theorem 3.1. *Assume* **(P)+(M)** *holds for* (3.1) (see (1.7), (1.8)). *Assume moreover that* $u_0, v_0 \in L^\infty(\Omega), u_0, v_0 \geq 0$ *and, for some* $U, C \geq 0$

$$\forall u \geq U, \forall v \geq 0,\ f(u,v) \leq C[1 + u + v], \tag{3.2}$$

$$\exists r \geq 1; \quad \forall u,v \geq 0, |g(u,v)| \leq C[1 + u^r + v^r]. \tag{3.3}$$

Then, $T^* = +\infty$.

Comments. Condition (3.3) means that the growth of $g(u,v)$ as $u, v \to +\infty$ is at most polynomial. The first condition (3.2) means that the first equation is “good”. A typical case is for instance when $f \leq 0$ in which case u is uniformly bounded on the interval of existence by maximum principle. It is more generally the case when $f \leq C(1+u)$. Actually, in the statement of Theorem 3.1, we may replace (3.2) by the a priori knowledge that u is uniformly bounded, no matter the reason of this bound (see [31]). In this case, the second condition (3.3) may be replaced by the weaker condition $g(u,v) \leq \varphi(u)(1+v^r)$ where $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing. The general idea of this theorem is that, for systems with the structure **(P)+(M)**, if moreover one of u or v is uniformly bounded on $[0, T^*)$, then, so is the other; whence global existence.

Before giving the main idea of the proof of this theorem, let us apply it to some of the previous examples.

Application 1. Let us start with the very first example given in the introduction, namely

$$\begin{cases} \partial_t u - d_1 \Delta u = -u\, h(v) \\ \partial_t v - d_2 \Delta v = u\, h(v), \end{cases} \tag{3.4}$$

with $h \geq 0$ and homogeneous Neumann boundary conditions. Then, (3.2) is satisfied with $C = 0$. By maximum principle we have

$$\forall t \in [0, T^*), \quad \|u(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}.$$

Now, it is not obvious that v is bounded. Theorem 3.1 claims that this is the case if moreover h grows at most like a polynomial as $v \to +\infty$.

Open problem. If h grows faster than a polynomial, the problem is in general open. Some positive results with $h(v) = e^{v^\gamma}$ with $\gamma < 1$ or even with $h(v) = e^{k\,v}$ may be found: see [28], [5], [30] and the comments in Subsection 3.4 and in Section 7. But, nothing seems to be known for instance if $h(v) = e^{v^2}$. For this example, the conjecture is that $L^\infty(\Omega)$-blow up occurs in finite time (see [30] for some indication in this direction), but some slight perturbations may lead to global existence as noticed in [9] (see Problem 3 in Section 7). We also refer to Section 5 where existence of global, but non necessarily bounded, weak solutions is proved.

Application 2. The Brusselator system in (2.1). According to Theorem 3.1, there is global existence of classical solutions for this system in any dimension (this was proved in [31]; see also [46]).

Application 3. If $H \geq 0$ and grows at most like a polynomial in the combustion model (2.2), then global existence of a classical solution holds.

Application 4. Let us consider the family of systems which are typical 2×2 models for our purpose: we choose the nonlinearities

$$f(u,v) = k_1 u^p v^q - k_2 u^\alpha v^\beta, \;\; g(u,v) = k_3 u^\alpha v^\beta - k_4 u^p v^q, \tag{3.5}$$

with $k_i > 0$ $p, q, \alpha, \beta \geq 1$. Then, **(M')** holds when $k_1 k_3 \leq k_2 k_4$. Indeed,

$$[k_1 k_3 \leq k_2 k_4] \;\;\Rightarrow\;\; [\forall k \in [k_1 k_4^{-1}, k_2 k_3^{-1}], \;\; f + kg \leq 0]. \tag{3.6}$$

Let us assume this condition. Then, global existence of classical solutions follows from Theorem 3.1 if

$$\begin{cases} [\,\beta > q \textit{ and } \beta p - \alpha q \leq \beta - q\,] \;\; or \;\; [\,\beta = q \textit{ and } p < \alpha\,] \\ Or \;\; [\,p > \alpha \textit{ and } \beta p - \alpha q \leq p - \alpha\,] \;\; or \;\; [\,p = \alpha \textit{ and } \beta < q\,]. \end{cases} \tag{3.7}$$

Indeed, let us assume for instance that $\beta > q, \beta p - \alpha q \leq \beta - q$. Then, we may write

$$f(u,v) = k_2 u^p v^q [\frac{k_1}{k_2} - u^{\alpha - p} v^{\beta - q}].$$

On the set $\{(u,v) \in (0,\infty)^2; u^{\alpha-p} v^{\beta-q} \geq k_1/k_2\}$, we have $f \leq 0$. On the complement K of this set, we write: $\forall (u,v) \in K$

$$u^{p-1} v^q = \left[u^{\alpha-p} v^{\beta-q}\right]^{\frac{q}{\beta-q}} u^{p-1-\frac{(\alpha-p)q}{\beta-q}} \leq \left[\frac{k_1}{k_2}\right]^{\frac{q}{\beta-q}} u^{\frac{\beta p - \alpha q}{\beta - q} - 1}.$$

By assumption, the exponent of u is nonpositive in the above last term: it follows that

$$\forall (u,v) \in K \textit{ with } u \geq 1, \;\; f(u,v) \leq k_1 u [u^{p-1} v^q] \leq k_1 u \left[k_1 k_2^{-1}\right]^{\frac{q}{\beta-q}} =: u\,C.$$

Thus, (3.2) is satisfied with $U = 1$ and C as defined just above.

Assume now $\beta = q, p < \alpha$. Then, we write

$$f(u,v) = k_2 u^p v^q [\frac{k_1}{k_2} - u^{\alpha-p}],$$

and it is obvious that $f \leq 0$ for $u \geq U = [k_1 k_2^{-1}]^{(\alpha-p)^{-1}}$.

The second line of (3.7) is exactly the same condition as in the first line when we exchange the roles of u and v.

Remark 3.2. Conditions of type (3.7) may be found in [39], the proof being quite different (see Subsection 3.4). However, conditions (3.7) are weaker: for instance, they allow $f(u,v) = uv^2 - u^2v^{5/2} = -g(u,v)$ while those of [39] do not.

Note also that system (3.5) may be a model for the chemical reaction

$$\alpha U + \beta V \rightleftharpoons pU + qV.$$

More precisely, the following is obtained for the concentrations u, v of U, V

$$f(u,v) = (\alpha - p)[\lambda_2 u^p v^q - \lambda_1 u^\alpha v^\beta], g(u,v) = (q - \beta)[\lambda_1 u^\alpha v^\beta - \lambda_2 u^p v^q],$$

where λ_1, λ_2 are the reaction constants, and $\alpha > p, q > \beta$ to get the same signs as in (3.5). Here, $k_1k_3 = k_2k_4 = (\alpha-p)(q-\beta)\lambda_1\lambda_2$, but (3.7) is not satisfied. Actually, global existence of solutions (even weak) is open (see Section 7) in this limit case.

On the other hand, if $k_1k_3 < k_2k_4$ in (3.5), then global *weak* solutions will be shown to exist in Section 5.

Open problems. *A main extra assumption in Theorem* 3.1 *besides* **(P)+(M)** *is that one of the equation is "good". What happens in system* (3.1) *when none of the equations is "good" (in the sense of* (3.2)*)?; what happens when neither* u *nor* v *is a priori bounded or at least bounded in some* L^p*-space for* p *large?* A typical example is obtained as a perturbation of the previous model example (3.4) (with $h(v) = v^\beta$ to keep the polynomial growth):

$$\begin{cases} \partial_t u - d_1 \Delta u = \lambda u^p v^q - uv^\beta \\ \partial_t v - d_2 \Delta v = -u^p v^q + uv^\beta. \end{cases} \tag{3.8}$$

Note that here

$$f(u,v) + g(u,v) = (\lambda - 1)u^p v^q \leq 0 \text{ if } \lambda \in [0,1]. \tag{3.9}$$

But, except for good values of the exponents, namely except if $\beta p - q < p - 1$ as deduced from (3.7), neither u nor v is a priori controlled! One cannot conclude global existence of classical solutions. The conjecture is that it is false in general (see the counterexamples in Section 4 for similar polynomial nonlinearities). However, we refer to Section 5 for global existence of weak solutions.

The situation is the same for systems like

$$\begin{cases} \partial_t u - d_1 \Delta u = -c(t,x) u^\alpha v^\beta \\ \partial_t v - d_2 \Delta v = c(t,x) u^\alpha v^\beta. \end{cases} \tag{3.10}$$

Here, if $\forall (t,x), c(t,x) \geq 0$ or $\forall (t,x), c(t,x) \leq 0$, then we may apply Theorem 3.1 (the dependence of f, g in t, x does not matter). But, the problem is open otherwise. Nevertheless, we can say a little more as indicated in the following remark.

Remark 3.3. Localization of the result of Theorem 3.1: It is interesting to notice that the L^∞-estimates obtained to deduce global existence in Theorem 1 may be localized. For instance in example (3.10), we may obtain local a priori L^∞-estimates in the subregions Q_1 where $c > 0$ and Q_2 where $c < 0$. Indeed, on Q_1, the first equation is "good", and u is obviously locally bounded: then, so is v by application of a local version of the proof of Theorem 3.1 (see [64, 58, 32, 33]). The same remark is valid for Q_2 if we exchange the roles of u and v. Therefore, blow up may occur only around the region where the sign of $c(t, x)$ changes!

3.2. The main ingredient of the proof: an L^p-estimate obtained by duality

The main tool in the proof of Theorem 3.1 is contained in the following lemma.

Lemma 3.4. *Let w, z be regular functions (in the sense of (1.5)) satisfying*

$$\partial_t w - d_1 \Delta w = \partial_t z - d_2 \Delta z \ \ on \ \ Q_T, \tag{3.11}$$

together with the same constant Neumann or Dirichlet boundary conditions for w, z and with bounded initial data. Then, for all $1 < p < +\infty$, there exist C_1, C_2 such that, for all $t \in (0, T]$

$$C_1 \|w\|_{L^p(Q_t)} \leq \|z\|_{L^p(Q_t)} + 1 \leq C_2[\|w\|_{L^p(Q_t)} + 1].$$

More generally, if (3.11) is replaced by

$$\partial_t w - d_1 \Delta w + \theta_1 w \leq \theta_2 \partial_t z + \theta_3 \Delta z + \theta_4 z + H \ \ on \ \ Q_T, \tag{3.12}$$

where $\theta_i \in I\!R$ and $H \in L^p(Q_T), H \geq 0$, then there exists C_1 such that, for all $t \in (0, T]$

$$C_1 \|w^+\|_{L^p(Q_t)} \leq \|z\|_{L^p(Q_t)} + 1 + \int_0^t \|H(s)\|_{L^p(\Omega)} ds. \tag{3.13}$$

This lemma is a consequence of the *L^p-regularity theory for parabolic operators.* Indeed, very roughly speaking, we may rewrite (3.11) as $w = A_{d_1}^{-1} A_{d_2} z$ where $A_{d_i} = \partial_t - d_i \Delta$. Thus, the question is the continuity of the operator $A_{d_1}^{-1} A_{d_2}$ from $L^p(Q_T)$ into itself. Reducing to zero boundary and initial data, the operator is linear and the question is equivalent to the continuity of the dual operator from the dual space $L^q(Q_T), q = p/(p-1)$ into itself. This continuity holds for $1 < q < +\infty$ according to the L^q-regularity theory.

Proof of Lemma 3.4. It is sufficient to prove (3.13). Let us do it in the Neumann case with data β_1, β_2 (it is similar in the Dirichlet case). For $t \in (0, T]$, let ϕ be the solution of the following dual problem where $\Theta \in \mathcal{C}_0^\infty(Q_t), \Theta \geq 0$.

$$-[\phi_t + d_1 \Delta \phi] + \theta_1 \phi = \Theta \ on \ Q_t, \ \phi(t) = 0, \ \frac{\partial \phi}{\partial n} = 0 \ on \ \Sigma_t.$$

It satisfies $\phi \geq 0$ and the following estimates for all $p \in (1, \infty)$, $q = p/(p-1)$ and all $t \in [0, T]$ (see [41]):

$$\|\phi_t\|_{L^q(Q_t)} + \|\Delta \phi\|_{L^q(Q_t)} + \sup_{s \in [0,t]} \|\phi(s)\|_{L^q(\Omega)} + \|\phi\|_{L^q(\Sigma_t)} \leq C_{q,T} \|\Theta\|_{L^q(Q_t)}.$$

Multiplying the inequality (3.12) by $\phi \geq 0$ and integrating by parts on Q_t give

$$\int_{Q_t} w \Theta \leq \int_\Omega \phi(0)(w(0) \ - \theta_2 z(0)) - \int_{\Sigma_t} \phi(d_1 \beta_1 + \theta_3 \beta_2) + \int_{Q_t} z(-\theta_2 \phi_t + \theta_3 \Delta \phi + \theta_4 \phi) + H \phi.$$

Using Hölder inequality and the above estimates on ϕ, we deduce with a constant C depending on the data and on T

$$\int_{Q_t} w\Theta \leq C\left[1+\|z\|_{L^p(Q_t)}+\int_0^t \|H(s)\|_{L^p(\Omega)}ds\right]\|\Theta\|_{L^q(Q_t)}.$$

Whence (3.13) by duality. □

Proof of Theorem 3.1. We denote by C_i's various positive numbers depending only on the data. We set $z(t)=(u(t)-U)^+, W=u+v$. From the equation in u and from (3.2), we have

$$\partial_t z - d_1\Delta z \leq sign^+(z)f \leq C(1+W), \tag{3.14}$$

from which we deduce for all $p\in(1,\infty)$

$$\|z(t)\|_{L^p(\Omega)} \leq \|z(0)\|_{L^p(\Omega)}+C_1\int_0^t [1+\|W(s)\|_{L^p(\Omega)}]ds,$$

which also implies

$$\|u(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)}+C_2+C_1\int_0^t \|W(s)\|_{L^p(\Omega)}]ds. \tag{3.15}$$

Condition **(M)** implies

$$\partial_t v - d_2\Delta v \leq -(\partial_t u - d_1\Delta u)+C(1+u+v).$$

Using **(P)** and Lemma 3.4 with $H=C, T=T^*$, we have

$$\forall t\in[0,T^*),\ \|v\|_{L^p(Q_t)} \leq C_3[1+\|u\|_{L^p(Q_t)}],$$

which implies

$$\|W\|_{L^p(Q_t)} \leq C_4[1+\|u\|_{L^p(Q_t)}]. \tag{3.16}$$

We use Hölder's inequality and the p-th power of (3.16) to obtain

$$\left[\int_0^t \|W(s)\|_{L^p(\Omega)}ds\right]^p \leq t^{(p-1)}\int_{Q_t}|W|^p \leq C_5[1+T^{*(p-1)}\int_{Q_t}|u|^p].$$

Together with the p-th power of inequality (3.15), this gives

$$\|u(t)\|^p_{L^p(\Omega)} \leq C_6+C_7\int_0^t \|u(s)\|^p_{L^p(\Omega)}ds.$$

Then, we integrate this Gronwall's inequality to deduce that u is bounded in $L^p(Q_{T^*})$ for all $p<+\infty$. Going back to (3.16), it follows that W and therefore v are also bounded in $L^p(Q_{T^*})$ for all $p<+\infty$.

Let us now deduce L^∞-bounds. Since the growth of g is at most polynomial at infinity, we obtain that $g(u,v)$ is also bounded in $L^p(Q_{T^*})$ for all $p<+\infty$. Since v is solution of the heat equation with right-hand side g, this implies that v is actually bounded in $L^\infty(Q_{T^*})$ (see e.g. [41]). Using also the equation (3.14) in z, we deduce similarly that z is bounded in $L^\infty(Q_{T^*})$, and so is u. These uniform bounds on u,v imply $T^*=+\infty$. □

3.3. Extension to $m \times m$ systems

This L^p approach has been extended (see [49, 50, 24] and the references herein) to $m \times m$ systems with a so-called *triangular* structure, which means essentially that $f_1, f_1 + f_2, f_1 + f_2 + f_3, \ldots$ are all bounded above by a linear function of the u_i.

We may for instance state the following

Theorem 3.5. *Let $f \in C^1([0,\infty)^m, I\!R^m)$ with at most polynomial growth and satisfying the quasipositivity* **(P)**. *Assume moreover that there exist $C \geq 0, \mathbf{b} \in I\!R^m$ and a lower triangular invertible $m \times m$ matrix P with nonnegative entries such that*

$$\forall r \in [0,\infty)^m,\ Pf(r) \leq [1 + \sum_{1 \leq i \leq m} r_i]\,\mathbf{b} \tag{3.17}$$

where the usual order in $I\!R^m$ is used. Then, the system

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, \ldots, u_m) \text{ on } (0,T) \times \Omega, \\ \forall i, \frac{\partial u_i}{\partial n} = \beta_i \geq 0 \text{ (or } \forall i, u_i = \beta_i \geq 0) \text{ on } (0,T) \times \partial\Omega, \\ u_i(0,\cdot) = u_{i0} \in L^\infty(\Omega), u_{0i} \geq 0, \end{cases} \tag{3.18}$$

has a global classical solution.

Proof. A proof may be found in [49, 50]. We may extend as well the "elementary" proof of Theorem 3.1 as follows : we denote by $C \geq 0$ any constant depending only the data, including T. We treat the Neumann case (the Dirichlet case is similar) and we set $W = \sum u_i$. By assumption

$$p_{ii}[\partial_t - d_i \Delta](u_i - z_i) \leq \sum_{j \leq i-1} p_{ij}[\partial_t - d_j \Delta] u_j,$$

where z_i is the solution of

$$\partial_t z_i - d_i \Delta z_i = [1 + W]\mathbf{b}_i / p_{ii},\ z_i(0) = 0,\ \frac{\partial z_i}{\partial n} = 0.$$

By an obvious extension of Lemma 3.4, we have for all $t \in (0, T^*)$

$$\|(u_i - z_i)^+\|_{L^p(Q_t)} \leq C[\sum_{j \leq i-1} \|u_j\|_{L^p(Q_t)} + 1].$$

We deduce by induction on i that

$$\forall i, \quad \|u_i\|_{L^p(Q_t)} \leq C[1 + \sum_{j \leq i} \|z_j\|_{L^p(Q_t)}],$$

and therefore, summing over i and taking the pth-power

$$\|W\|^p_{L^p(Q_t)} \leq C[1 + \sum_i \|z_i\|^p_{L^p(Q_t)}]. \tag{3.19}$$

Going back to the definition of z_i, we have

$$\forall t \in [0, T^*), \|z_i(t)\|^p_{L^p(\Omega)} \leq C[1 + \int_0^t \|W(s)\|^p_{L^p(\Omega)} ds] = C[1 + \|W\|^p_{L^p(Q_t)}], \tag{3.20}$$

which, combined with (3.19), gives

$$\sum_i \|z_i(t)\|^p_{L^p(\Omega)} \leq C[1 + \int_0^t \sum_i \|z_i(s)\|^p_{L^p(\Omega)} ds].$$

By integration of this Gronwall's inequality, it implies that the z_i's are bounded in $L^p(Q_T)$, and so is W by (3.19).

Next, we finish as in Theorem 3.1: as f has at most polynomial growth, all right-hand sides of (3.18) are bounded in $L^p(Q_{T^*})$ for all $p < \infty$, and this implies that all u_i's are bounded in $L^\infty(Q_{T^*})$. Whence $T^* = +\infty$. □

Remark 3.6. The condition (3.17) may still be weakened by allowing a nonlinear dependence in the upper bound of $P\,f$, namely

$$P\,f(r) \leq \mathbf{b}(1 + \sum |u_i|^r), \;\; r < 1 + 2a/(N+2),$$

where $a \geq 1$ is such that an a priori bound on $\max_i \|u_i\|_{L^a(Q_{T^*})}$ is known. We refer to [49, 50, 24] for results in this direction. We could as well adapt the above proof: the linear estimate (3.20) of z_i in terms of W should be replaced by an estimate in terms on W^r. Then, an adequate interpolation between L^a and L^p allows to conclude. Note that the dependence in H in Lemma 3.4 could be improved by using Sobolev embedding (by duality, we may estimate $\|\phi\|_{L^r(Q_T)}$ for some $r > q$).

3.4. More remarks on global classical solutions

Use of Lyapunov functions. Global existence for specific 2×2 systems with the above kind of structure has also been proved by using suitable Lyapunov functions. A first result in this direction was obtained by Masuda [48] in the case

$$f(u,v) \leq 0, \; g(u,v) \leq -\varphi(u)f(u,v), \;\; g(u,v) \leq \varphi(u)(v + v^r), \; r \geq 1.$$

It was extended by Haraux-Youkana [28] who could handle growth of type $e^{\alpha v^\beta}$ with $\alpha > 0$ and $\beta < 1$ and in the case $g = -f$. This method could even reach an exponential growth (that is $\beta = 1$), but, curiously, only with restrictions on the size of $\|u_0\|_\infty$ (see [5]). This approach was recently coupled with a nice change of function in [9]: they prove global existence for new specific systems in this family, and with possible quite higher growth (see Problem 3 in Section 7). However, the problem is still open for instance for

$$-f(u,v) = g(u,v) = ue^{v^\beta}, \beta \geq 1. \tag{3.21}$$

(See however below for the case $\beta = 1$ in $\mathbb{R}^N$). Let us also mention the use of Lyapunov functions in [39] to treat more elaborate polynomial systems like (3.5).

Exponential growth. The exponential growth is a limit to most methods for system (3.21). In [30], it is proved that global existence holds in the case of $\mathbb{R}^N$ for problem (3.21) with $\beta = 1$. The method is quite different from those already described: it is based on a careful analysis of the heat kernel. We may also refer to [37, 38, 40] for other results in this limit case. It is proved in [30] that their own method is optimal and cannot absorb higher growth. However, no example of blow up is given. Actually, the L^p-duality method described above could also be extended to those

Orlicz spaces for which regularity results hold as for L^p-spaces. This has not been done yet, but a rough analysis suggests that exponential growth should probably be handled also, but not more. We refer to Section 7 for a more detailed discussion on these open problems.

More results. A curious result may be found in the survey [46] for the 2×2 system with $0 \leq g = -f$ and $\Omega = I\!\!R^N$: it says that, if the diffusion coefficients satisfy $d_1 \geq d_2$, then global existence holds. The proof is based on properties of the heat kernel and it has been exploited and extended in [30]. A similar result for the same system, but set on a bounded domain and with various boundary conditions, may also be found in [36]: it is based on precise estimates for the Green function.

It is interesting to mention the global existence results obtained in any dimension for systems with **(P)+(M)** and with strictly sub-quadratic growth ([34, 35, 36, 19]). The quadratic case is also handled for some systems in dimension $N \leq 2$ (see [26, 20, 33]). Let us finally mention the case of coupled or cross-diffusions where a few techniques have been developed to prove global existence of classical solutions (see [15, 16, 40]).

Other boundary conditions. One must be careful with boundary conditions. Indeed, it is a consequence of the results in [8] that *blow up may occur in finite time* for the following system

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^\gamma \\ \partial_t v - d_2 \Delta v = uv^\gamma \\ u = 1, \ \ \frac{\partial v}{\partial n} = 0 \ \ on \ \ \Sigma_T, \end{cases}$$

where $\gamma > 2$. Here, we check that $\int_\Omega u(t) + v(t)$ is not bounded. This means that this system does not actually belong to our class, whence the question: what are the "acceptable" boundary conditions? A general rule for 2×2 systems is that most techniques carry over to cases where *boundary conditions are of the same kind for the two equations*. When they are of different kind (like Dirichlet/Neumann) and moreover non homogeneous, then difficulties might occur. This explains the comments we made on this point in the Introduction. We refer to [47] where this is carefully analyzed.

Other diffusion operators. Most results of this paper are stated with simple diffusion operators of the form $-d_i\Delta$. This is mainly for simplicity, in order to concentrate on the main difficulty due the fact that the diffusion operators involved in the various equations are different from each other: this simple case turns out to be highly significant of this difficulty. We emphasize that in several of the references that we gave, general diffusion operators are considered. We refer, among other papers, to the survey [46] where the L^p-technique is developed for general parabolic operators and to [51] and its references where nonlinear diffusions are considered.

4. The structure (P)+(M) does not keep from blowing up in L^∞ !

The results of the previous section may seem unsatisfactory, since, even for a 2×2 polynomial system satisfying **(P)+(M)**, a strong extra assumption is required to get global classical solutions, namely (3.2). This leaves for instance open the question of global existence in apparently "simple" systems like (3.8), (3.10).

It turns out that this restriction makes sense: indeed, in general, $L^\infty(\Omega)$-blow up may occur in finite time for polynomial 2×2 systems satisfying **(P)+(M)** as proved in [56, 57] where the two following theorems may be found (B denotes the open unit ball in $\mathbb{R}^N$ and $Q_T = (0,T) \times B$):

Theorem 4.1. *One can find C^∞ functions f, g, with polynomial growth and satisfying* **(P)+(M)**, *together with $d_1, d_2 > 0, u_0, v_0 \in C^\infty(\overline{B}), \beta_1, \beta_2 \in C^\infty([0,T])$ and u, v nonnegative classical solutions on $(0,T)$ of*

$$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) \\ \partial_t v - d_2 \Delta v = g(u,v) \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = v_0(\cdot) \geq 0, \\ u = \beta_1, \ \ v = \beta_2 \text{ on } (0,T) \times \partial\Omega \end{cases} \tag{4.1}$$

with $T < +\infty$ and

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

Theorem 4.2. *One can find $\alpha, \beta > 1, d_1, d_2 > 0, u_0, v_0 \in C^\infty(\overline{B}), \beta_1, \beta_2 \in C^\infty([0,T])$, $c_1, c_2 \in C^k(\overline{Q}_T)$ with $k \geq 0$, $c_1(t,x) + c_2(t,x) \leq 0$ and u, v nonnegative classical solutions on $(0,T)$ of*

$$\begin{cases} \partial_t u - d_1 \Delta u = c_1(t,x) u^\alpha v^\beta \\ \partial_t v - d_2 \Delta v = c_2(t,x) u^\alpha v^\beta \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = v_0(\cdot) \geq 0, \\ u = \alpha_1, \ \ v = \alpha_2 \text{ on } (0,T) \times \partial\Omega \end{cases} \tag{4.2}$$

with $T < +\infty$ and

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

Remark 4.3. Theorems 4.1 and 4.2 provides blowing up solutions to systems of the form (4.1) with $f + g \leq 0$ or of the form (4.2) with $c_1 + c_2 \leq 0$. These (counter)examples are actually more surprising than expected since we even have

$$\exists \lambda \in (0,1); \forall \mu \in [\lambda, 1], \ f + \mu g \leq 0, \ c_1 + \mu c_2 \leq 0.$$

We will see in next Section that this richer structure allows global existence of weak solutions. In other words, the solutions constructed in the two above theorems blow up at T^*, but continue to live "in a weak sense".

Remark 4.4. About the proof of the two above theorems: the proof is obtained by building explicitely the solutions u, v so that they are solutions of this kind of systems and nevertheless blow up at time t. They are of the form

$$u(t,x) = \frac{a(T-t)+b|x|^2}{(T-t+|x|^2)^\gamma}, \quad v(t,x) = \frac{c(T-t)+d|x|^2}{(T-t+|x|^2)^\gamma},$$

where $a, b, c, d > 0, \gamma > 1$. We see that blow up occurs in $L^\infty(\Omega)$ at time $t = T$. Actually, the behavior of the solutions is to be seen as the behavior of $w(t,x) = \frac{1}{(T^*-t)^2+|x|^2}$ which is, for $N > 4$, weak solution of a "good" reaction-diffusion equation of the form

$$\partial_t w - \Delta w = c(t,x)\, u^2,$$

with $c(t,x)$ bounded. Thus, this solution is no longer in $L^\infty(\Omega)$ at time T^*, but then comes back in $L^\infty(\Omega)$. It is even possible to write down similar weak solutions which blow up in $L^\infty(\Omega)$ for infinitely many times: just replace $(T^*-t)^2$ by $(T^*-t)^4 \sin\left((T^*-t)^{-1}\right)$. This is sometimes called *incomplete blow up* in the literature (see e. g. [62] for more comments in this direction).

Conclusion at this stage. When looking for global solution in time for reaction-diffusion systems, it is more convenient to look for *weak* solutions that are allowed to leave $L^\infty(\Omega)$… but continue to live…

5. Systems with nonlinearities bounded in L^1: weak global solutions

5.1. Introduction: an example

Recall the examples of the form

$$\begin{cases} \partial_t u - d_1 \Delta u = \lambda u^p v^q - u^\alpha v^\beta (= f(u,v)) \\ \partial_t v - d_2 \Delta v = -u^p v^q + u^\alpha v^\beta (= g(u,v)). \end{cases}$$

We noticed that

$$(\mathbf{M}) \quad f(u,v) + g(u,v) = (\lambda - 1) u^p v^q \leq 0 \text{ if } \lambda \in [0,1].$$

But, we also have

$$(\mathbf{M}_\lambda) \quad f(u,v) + \lambda g(u,v) = (\lambda - 1) u^\alpha v^\beta \leq 0,$$

which gives one more relation between f and g if $\lambda \leq 1$. It turns out that $(\mathbf{P}) + (\mathbf{M}) + (\mathbf{M}_\lambda)$ with $\lambda \neq 1$ imply that the nonlinearities $f(u,v), g(u,v)$ are bounded in $L^1(Q_T)$ for all T.

Let us state this result for the more general system

$$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) & \text{on } Q_T \\ \partial_t v - d_2 \Delta v = g(u,v) & \text{on } Q_T \\ \alpha_0 \frac{\partial u}{\partial n} + (1-\alpha_0) u = \alpha_0 \frac{\partial v}{\partial n} + (1-\alpha_0) v = 0 & \text{on } \Sigma_T \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = v_0(\cdot) \geq 0\ . \end{cases} \tag{5.1}$$

Proposition 5.1. *Assume* **(P)**, $\alpha_0 \in [0,1]$, *and* $\exists\, C \geq 0, \exists\, \lambda \in [0,+\infty), \lambda \neq 1$ *with*

$$f + g \leq C(1+u+v) \quad \text{and} \quad f + \lambda g \leq C(1+u+v). \tag{5.2}$$

Then, if u, v *are solutions of* (5.1) *on* $(0,T)$,

$$\int_{Q_T} [|f(u,v))| + |g(u,v)|]\, dt\, dx \leq M = M(data) < +\infty.$$

Remark 5.2. Note that the assumptions of Proposition 5.1 imply

$$\forall \mu \in [\lambda, 1], \quad f + \mu\, g \leq C(1+u+v).$$

Proof of Proposition 5.1. We denote by C_0 any constant depending only on the data and on T. We know that, when $\alpha_0 \in (0,1]$ (see (1.9)) then, for all $t \in [0,T]$, $\int_\Omega u(t), \int_\Omega v(t) \leq C_0$. This estimate can easily be extended to the case $\alpha_0 = 0$.

For $\mu = 1$ and $\mu = \lambda$, we have (no matter the value of $\alpha_0 \in [0,1]$),

$$-\int_{\partial\Omega} d_1 \frac{\partial u}{\partial n} + d_2 \mu \frac{\partial v}{\partial n} \geq 0.$$

We deduce

$$\partial_t \int_\Omega (u + \mu v)(t) + \int_\Omega -[f(u,v) + \mu g(u,v)] \leq 0. \tag{5.3}$$

Now, we use $-[f(u,v) + \mu g(u,v)] + C(1+u+v) \geq 0$ in (5.3). After integration in time on $(0,T)$, this leads to

$$\int_\Omega (u+\mu v)(T) + \int_{Q_T} h(u,v) \leq \int_\Omega u_0 + \mu v_0 + \int_{Q_T} C(1+u+v),$$

where

$$h(u,v) = \big| -[f(u,v) + \mu g(u,v)] + C(1+u+v) \big|.$$

It follows that, for $\mu = 1$ and for $\mu = \lambda \neq 1$

$$\|f(u,v) + \mu g(u,v)\|_{L^1(Q_T)} \leq C_0,$$

and therefore

$$\|f(u,v)\|_{L^1(Q_T)} \leq C_0 \text{ and } \|g(u,v)\|_{L^1(Q_T)} \leq C_0. \qquad \square$$

Remark 5.3. Many systems come naturally with an extra structure which makes the nonlinearities to be bounded in $L^1(Q_T)$. Recall that it was the case for the counterexamples built in the previous section: see Remark 4.3. Therefore, it is interesting to ask the question:

What can be said of a system which preserves positivity and for which the nonlinearities are bounded in $L^1(Q_T)$ (without even assuming (M))?

A very general result in this direction may be found in [55]: we state it below in Theorem 5.5 in a more general setting.

5.2. Existence of global weak supersolutions for bounded L^1-nonlinearities

We consider a general $m \times m$ system of type (1.4), namely

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, \ldots, u_m) \; on \; (0,T) \times \Omega, \\ \alpha_i \frac{\partial u_i}{\partial n} + (1-\alpha_i) u_i = 0 \; on \; (0,T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i0} \geq 0, \end{cases} \tag{5.4}$$

where $\forall i = 1, \ldots, m, \; d_i > 0, \alpha_i \in [0,1]$. As explained in the introduction (see also Subsection 3.4), we will restrict the choice of the α_i and of the corresponding family of test functions to the following situations: either

$$\forall i = 1, \ldots, m, \; \alpha_i \in (0,1], \; \mathcal{D} = \{\psi \in \mathcal{C}^\infty(\overline{Q}_T); \psi \geq 0, \; \psi(\cdot, T) = 0\}. \tag{5.5}$$

or (all Dirichlet conditions)

$$\forall i = 1, \ldots, m, \; \alpha_i = 0, \;\; \mathcal{D} = \{\psi \in \mathcal{C}^\infty(\overline{Q}_T); \psi \geq 0, \; \psi(\cdot, T) = 0, \psi = 0 \; on \; \Sigma_T\}. \tag{5.6}$$

We choose homogeneous boundary conditions for simplicity; the nonhomogeneous case can be reduced to this one after an adequate change of unknown function, since we allow the f_i's to depend on (t,x): let us assume that, for all $i = 1, \ldots, m$

$$\begin{cases} f_i : Q_T \times [0, +\infty)^m \to I\!R \; is \; measurable; \;\; f_i(\cdot, 0) \in L^1(Q_T), \\ \exists K : Q_T \times [0, +\infty) \to [0, +\infty) \; with \; \forall M > 0, K(\cdot, M) \in L^1(Q_T) \; and \\ a.e. \; (t,x) \in Q_T, \forall r, \hat{r} \in [0, +\infty)^m \; with \; |r|, |\hat{r}| \leq M, \\ |f_i(t,x,r) - f_i(t,x,\hat{r})| \leq K(t,x,M)|r - \hat{r}|, \; and \; \forall r \in [0, +\infty)^m \\ f_i(t, x, r_1, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_m) \geq 0 \; (quasi-positivity) \; (\mathbf{P}). \end{cases} \tag{5.7}$$

Approximate problem: We consider approximations of this system, namely classical solutions $u^n = (u_1^n, \ldots, u_m^n)$ of

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i^n - d_i \Delta u_i^n = f_i^n(t, x, u_1^n, \ldots, u_m^n) \; on \; Q_T, \\ \alpha_i \frac{\partial u_i}{\partial n} + (1-\alpha_i) u_i = 0 \; on \; \Sigma_T, \\ u_i^n(0, \cdot) = u_{i0}^n, \end{cases} \tag{5.8}$$

where the f_i^n are essentially "truncations" of the f_i's. More precisely, we will assume that the f_i^n have the same properties (5.7) as the f_i, that $|f_i^n|$ is uniformly bounded for each n, and that, for all $M > 0$, ϵ_M^n tends to zero in $L^1(Q_T)$ and $a.e.$ where

$$\epsilon_M^n(t,x) = \sup_{0 \leq |r| \leq M, 1 \leq i \leq m} |f_i^n(t,x,r) - f_i(t,x,r)|. \tag{5.9}$$

Since f^n is uniformly bounded for fixed n, there exists a global classical solution u^n to (5.8) (see (1.5)): this may be deduced from classical results as Lemma 1.1.

Remark 5.4. Note that property (5.9) is satisfied by $f_i^n = T_n \circ f_i$ where $T_n : I\!R \to I\!R$ is the truncation defined as follows, where $\sigma_n = m\,n$

$$T_n(\sigma) = \sigma \; if \; \sigma \in (-\sigma_n, n), \; T_n(\sigma) = -\sigma_n \; if \; \sigma < -\sigma_n, \; T_n(\sigma) = n \; if \; \sigma > n. \tag{5.10}$$

Indeed, in this case,

$$\epsilon_M^n \leq \sup_{|r| \leq M, 1 \leq i \leq m} \chi_{[-\sigma_n < f_i < n]} |f_i(t,x,r)|.$$

Coupled with the following inequality coming from the Lipschitz property (5.7)

$$|r| \leq M \Rightarrow |f_i(t,x,r)] \leq |f_i(t,x,0)| + K(t,x,M)M,$$

where $f_i(\cdot,0), K(\cdot,M) \in L^1(Q_T)$, this implies that $\epsilon_M^n \to 0$ in $L^1(Q_T)$ and a.e..

Note also for future reference that, if f satisfies **(P)** (resp. **(M)**), then so does $f^n = T_n \circ f$; the choice of σ_n is made to keep **(M)** exactly.

Theorem 5.5. [55] *Let $u^n = (u_1^n, \ldots, u_m^n)$ be a nonnegative solution to the approximate system* (5.8) *satisfying*

$$\sup_{n\geq 1, 1\leq i\leq m} \int_{Q_T} |f_i^n(u^n)| < +\infty. \tag{5.11}$$

Assume that, for $i = 1,\ldots,m$, u_{i0}^n tends to u_{i0} in $L^1(\Omega)$. Then, up to a subsequence, u^n converges in $L^1(Q_T)$ and a.e. to a super-solution *of system* (5.4) *which means*

$$\begin{cases} \forall i = 1,\ldots,m,\ f_i(u) \in L^1(Q_T), \nabla u_i \in L^1(Q_T),\ and\ \forall \psi \in \mathcal{D}, \\ -\int_\Omega \psi(0)u_{i0} + \int_{Q_T}[-\psi_t u_i + d_i \nabla\psi\nabla u_i] + \beta(\alpha_i)d_i \int_{\Sigma_T} \psi u_i \geq \int_{Q_T} \psi f_i(u), \end{cases} \tag{5.12}$$

where $\forall \alpha \in (0,1], \beta(\alpha) = (1-\alpha)/\alpha, \beta(0) = 0$. Moreover, $u_i \in L^1(0,T;W_0^{1,1}(\Omega))$ in case (5.6)*.*

5.3. Proof of the existence of weak supersolutions

We start with the following compactness lemma (see e.g. [7], or Appendix of [13]; for the compactness of the trace, we use the continuity of the trace operator from from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$).

Lemma 5.6. *Let $d > 0, \alpha \in [0,1]$. The mapping $(w_0, \Theta) \to w$ where w is the solution of*

$$w_t - d\Delta w = \Theta,\ \ \alpha\partial_n w + (1-\alpha)w = 0 \text{ on } \partial\Omega,\ \ w(0,\cdot) = w_0, \tag{5.13}$$

is compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1(Q_T)$, and even into $L^1((0,T);W^{1,1}(\Omega))$, and the trace mapping $(w_0,\Theta) \to w_{|\Sigma_T} \in L^1(\Sigma_T)$ is also compact.

Starting the proof of Theorem 5.5. According to the a priori estimate (5.11) and to Lemma 5.6, there exists $u \in L^1(Q_T)^m$ with $\nabla u \in [L^1(Q_T)^N]^m$ such that, up to a subsequence, one may assume that

$$\begin{cases} u^n \to u \text{ in } L^1(Q_T)^m \text{ and a.e. in } Q_T,\ \ \nabla u^n \to \nabla u \in [L^1(Q_T)^N]^m, \\ u^n_{|\Sigma_T} \to u_{|\Sigma_T} \text{ in } L^1(\Sigma_T) \text{ and a.e. in } \Sigma_T \\ u_i \in L^1(0,T;W_0^{1,1}(\Omega) \text{ in case } (5.6)\,. \end{cases} \tag{5.14}$$

Thanks to the choice of the f^n (convergence in L^1 and a.e. to zero of ϵ_M^n – see (5.9)-) and thanks to the continuity of $f(\cdot,\cdot,r)$ with respect to r, we also have

$$f_n(u_n) \to f(u) \text{ a.e. in } Q_T.$$

By Fatou's Lemma

$$\int_{Q_T} |f_i(u)| \leq \liminf_{n\to+\infty} \int_{Q_T} |f_i^n(u^n)|,$$

and, in particular, $f(u) \in [L^1(Q_T)]^m$. One would also need the convergence in $[L^1(Q_T)]^m$ of $f^n(u^n)$ in order to be able to pass to the limit in the equation! And this is not true in general. We will prove however that we can keep at least one inequality at the limit: this is the content of Theorem 5.5.

Lemma 5.7. *Let w be a solution of* (5.13). *Then, for all $k > 0$*

$$d \int_{[|w| \leq k]} |\nabla w|^2 \leq k \left[\int_{Q_T} |\Theta| + \int_\Omega |w_0| \right]. \tag{5.15}$$

Proof. We may assume Θ is regular. The estimate (5.15) is easily obtained by multiplying the equation (5.13) in w by $T_k(w)$ where $T_k(r)$ is the projection of r onto $[-k, k]$. We denote $j_k(r) = \int_0^r T_k(s) ds$. We obtain after integration

$$\int_\Omega j_k(w(t)) + \int_{Q_T} d\, T_k'(w) |\nabla w|^2 + d\beta(\alpha) \int_{\Sigma_T} T_k(w) w = \int_{Q_T} T_k(w)\Theta + \int_\Omega j_k(w_0).$$

We use the following estimates to deduce (5.15):

$$|T_k(w)| \leq k,\ j_k(w_0) \leq k \int_\Omega |w_0|,\ j_k(w(t)) \geq 0,\ \beta(\alpha) \int_{\Sigma_T} T_k(w) w \geq 0. \qquad \square$$

Continuing the proof of Theorem 5.5. Now, we fix $\eta \in (0, 1)$ and for all $i = 1, \dots, m$, we denote

$$U_i^n = \sum_{j \neq i} u_j^n,\ w_i^n = u_i^n + \eta U_i^n,\ v_{i,k}^n = T_k(w_i^n).$$

Since we will have to differentiate twice T_k, we replace T_k by a C^2-regularized version, still denoted T_k so that on $[0, \infty)$

$$0 \leq T_k' \leq 1, -1 \leq T_k'' \leq 0,\ \forall r \in [0, k-1], T_k(r) = r, \forall r \geq k, T_k'(r) = 0.$$

A main point is to use the inequality satisfied by $v^n := v_{i,k}^n$. Thanks to the concavity of T_k, we have in the sense of distributions:

$$-\Delta v^n = -\Delta T_k(u_i^n + \eta U_i^n) \geq -T_k'(u_i^n + \eta U_i^n)[\Delta u_i^n + \eta \Delta U_i^n].$$

This implies

$$v_t^n - d_i \Delta v^n \geq F_i^n + \eta S_i^n,$$

where

$$F_i^n = T_k'(u_i^n + \eta U_i^n)[f_i^n(u^n) + \eta \sum_{j \neq i} f_j^n(u^n)],$$

$$S_i^n = T_k'(u_i^n + \eta U_i^n) \sum_{j \neq i} (d_j - d_i) \Delta u_j^n.$$

We may write for $\psi \in \mathcal{D}$:

$$\begin{cases} -\int_\Omega \psi(0) v^n(0) + \int_{Q_T} [-\psi_t v^n + d_i \nabla\psi \nabla v^n] + d_i \int_{\Sigma_T} \psi T_k'(w_i^n) V_i^n \\ \geq \int_{Q_T} \psi [F_i^n + \eta S_i^n] \end{cases} \tag{5.16}$$

where $V_i^n = \beta(\alpha_i) u_i^n + \eta \sum_{j \neq i} \beta(\alpha_j) u_j^n$ (here we use the fact that we are in one of the situations (5.5) or (5.6); note that $\beta(\alpha_j) = 0$ for all j in the second case (5.6)).

We keep k and η fixed. We know that v^n converges in $L^1(Q_T)$ and a.e. to $v_{i,k}$ where

$$v_{i,k} = T_k(w_i),\ w_i = u_i + \eta U_i,\ U_i = \sum_{j \neq i} u_j. \tag{5.17}$$

Now the convergence of the nonlinearities holds only a.e. The point is that, since $T_k'(r) = 0$ for $r > k$ then, $F_i^n = 0$ on the set $[u_i^n + \eta U_i^n > k]$. But, on the complement of this set, all u_j^n are uniformly bounded:

$$u_i^n \leq k,\ \forall j \neq i,\ u_j^n \leq k/\eta. \tag{5.18}$$

By the dominated convergence theorem, we may claim that, as $n \to \infty$

$$F_i^n \to F_{i,k} := T_k'(u_i + \eta U_i)[f_i(u) + \eta \sum_{j \neq i} f_j(u)] \ \ in\ L^1(Q_T). \tag{5.19}$$

On the other hand, ∇v^n converges in $L^1(Q_T)$ and the trace of v^n converges to the trace of v in $L^1(\Sigma_T)$ so that V_i^n converges in $L^1(\Sigma_T)$ to $V_i = \beta(\alpha_i)u_i + \eta \sum_{j \neq i} \beta(u_j)u_j$.

Thererefore, to pass to the limit as $n \to +\infty$ in (5.16), we only need to control the term S_i^n. This is the main point of the proof.

Lemma 5.8. *There exists C depending on k, ψ and the data, but not on n, η ($\eta \leq 1$) such that*

$$\left| \int_{Q_T} \psi S_i^n \right| \leq C \eta^{-\frac{1}{2}}.$$

Proof. We have $S_i^n = T_k'(w_i^n) \sum_{j \neq i}(d_j - d_i)\Delta u_j^n$ and

$$-\int_{Q_T} \psi T_k'(w_i^n)\Delta u_j^n = \int_{Q_T} \nabla u_j^n [\nabla \psi T_k'(w_i^n) + \psi T_k''(w_i^n)\nabla w_i^n] + \beta(\alpha_j) \int_{\Sigma_T} \psi T_k'(w_i^n) u_j^n.$$

We have the following bound where C denotes any constant independent of n, η, but which may depend on k, ψ and the data:

$$|\int_{\Sigma_T} \psi T_k'(w_i^n) u_j^n| \leq C, \left| \int_{Q_T} \nabla u_j^n \nabla \psi T_k'(w_i^n) \right| \leq C,$$

$$\left| \int_{Q_T} \psi T_k''(w_i^n) \nabla u_j^n \nabla w_i^n \right| \leq C \left(\int_{[w_i^n \leq k]} |\nabla u_j^n|^2 \right)^{1/2} \left(\int_{[w_i^n \leq k]} |\nabla w_i^n|^2 \right)^{1/2}.$$

On the set $A_i := [w_i^n \leq k]$, we have the estimates (5.18). From Lemma 5.7 and thanks to the main assumption (5.11), we have

$$\int_{A_i} |\nabla u_i^n|^2 \leq C,\ \forall j \neq i,\ \int_{A_i} |\nabla u_j^n|^2 \leq C\,\eta^{-1},\ \int_{A_i} |\nabla w_i^n|^2 \leq C.$$

This implies that

$$\left| \int_{Q_T} \psi T_k''(w_i^n) \nabla u_j^n \nabla w_i^n \right| \leq C \eta^{-1/2}.$$

The estimate of Lemma 5.8 follows. □

End of the proof of Theorem 5.5. Passing to the limit as $n \to +\infty$ in (5.16) (η, k being fixed), we obtain (see the notations in (5.17),(5.19))

$$\left\{ \begin{array}{l} -\int_\Omega \psi(0) v_{i,k}(0) + \int_{Q_T}[-\psi_t v_{i,k} + d_i \nabla\psi \nabla v_{i,k}] + d_i \int_{\Sigma_T} \psi T_k'(w_i) V_i \\ \geq \int_{Q_T} \psi F_{i,k} + \epsilon(i,\eta,k,\psi), \end{array} \right. \tag{5.20}$$

where $\epsilon(i,\eta,k,\psi) \geq -C(i,k,\psi)\eta^{1/2}$ so that $\liminf_{\eta\to 0} \epsilon(i,\eta,k,\psi) \geq 0$. We now let η decrease to zero, then k tend to $+\infty$ to obtain the expected inequality (5.12) of Theorem 5.5. □

5.4. Global existence of weak solutions for systems (P)+(M) and L^1-a priori estimates

From Theorem 5.5, we now deduce existence *of global solutions* for a (wide) subfamily of systems with the structure **(P)+(M)**. Let us consider the approximate system where u_{i0} is replaced by $u_{i0}^n = \inf\{u_{i0}, n\}$ and the nonlinearities f_i by $f_i^n = T_n \circ f_i$ where T_n is defined in Remark 5.4. Then, thanks to the uniform bound on $|f_i^n|$, for all n, there exists a global classical solution u^n of the approximate system on $[0,T)$. And we have the following (see [55]):

Theorem 5.9. *Let us consider system* (5.4) *together with* (5.5) *or* (5.6)*, with* (5.7) *and with* $u_0 \in L^1(\Omega)^m, u_0 \geq 0$. *Assume that the structure* **(P)+(M)** *holds together with the following a priori estimate on the solution* u^n *just defined*

$$\sup_{n\geq 1, 1\leq i\leq m} \int_{Q_T} |f_i^n(u^n)| < +\infty. \tag{5.21}$$

Then, system (5.4) *has a weak solution on* $(0,T)$ (*i.e. equality holds in* (5.12)).

Proof of Theorem 5.9. By Theorem 5.5, up to a subsequence, the approximate solution u^n converges to a weak supersolution. Let us prove that it is also a weak subsolution. We use the notations of Theorem 5.5:

$$(u^n, \nabla u^n, u^n_{|\Sigma_T}) \to (u, \nabla u, u_{|\Sigma_T}) \text{ in } [L^1(Q_T)]^m \times [L^1(Q_T)^N]^m \times [L^1(\Sigma_T)]^m$$

where $f_i(u) \in L^1(Q_T)$ and for all $\psi \in \mathcal{D}$ and all $i = 1, \ldots, m$

$$-\int_\Omega \psi(0) u_{i0} + \int_{Q_T} [-\psi_t u_i + d_i \nabla\psi\nabla u_i] + \beta(\alpha_i) d_i \int_{\Sigma_T} \psi u_i \geq \int_{Q_T} \psi f_i(u). \tag{5.22}$$

We will see that the structure **(M)** provides the other inequality. We introduce the notations:

$$W^n = \sum_{1\leq i\leq m} u_i^n,\ Z^n = \sum_{1\leq i\leq m} d_i u_i^n,\ V_n = \sum_{1\leq i\leq m} \beta(\alpha_i) d_i u_i^n,$$

$$W = \sum_{1\leq i\leq m} u_i,\ Z = \sum_{1\leq i\leq m} d_i u_i,\ V = \sum_i \beta(\alpha_i) d_i u_i.$$

Adding up the m equations of the approximate problem, we have

$$-\int_\Omega \psi(0) W^n(0) + \int_{Q_T} [-\psi_t W^n + \nabla\psi\nabla Z^n] + \int_{\Sigma_T} \psi V^n = \int_{Q_T} \psi \sum_{1\leq i\leq m} f_i^n(u^n). \tag{5.23}$$

But

$$-\sum_{1\le i\le m} f_i^n(u^n) + C[1+W^n] \ge 0.$$

By *a.e.* convergence of all functions, by $L^1(Q_T)$-convergence of W^n and by Fatou's Lemma, we have

$$\int_{Q_T} -\psi \sum_{1\le i\le m} f_i(u) \le \liminf_{n\to+\infty} \int_{Q_T} -\psi \sum_{1\le i\le m} f_i^n(u^n).$$

It follows that, at the limit, equation (5.23) gives the inequality

$$-\int_\Omega \psi(0)W(0) + \int_{Q_T} [-\psi_t W + \nabla\psi\nabla Z] + \int_{\Sigma_T} \psi V \le \int_{Q_T} \psi \sum_{1\le i\le m} f_i(u).$$

We deduce that all inequalities in (5.22) are actually equalities for all $i = 1,\ldots,m$; in other words, u is a weak solution on $(0,T)$. □

We deduce for instance a global existence result for the 2×2 system

$$\begin{cases} \partial_t u - d_1\Delta u = f(u,v) & on\ Q_\infty \\ \partial_t v - d_2\Delta v = g(u,v) & on\ Q_\infty \\ \alpha\frac{\partial u}{\partial n} + (1-\alpha)u = 0,\ \alpha\frac{\partial v}{\partial n} + (1-\alpha)v = 0\ on\ \Sigma_\infty & \\ u(0,\cdot) = u_0 \ge 0,\ v(\cdot,0) = v_0 \ge 0, & \end{cases} \tag{5.24}$$

where again f,g are C^1 functions, $d_1,d_2>0, \alpha\in[0,1]$.

Corollary 5.10. *Let $u_0,v_0\in L^1(\Omega)^+$. Assume that f,g satisfy the quasi-positivity* **(P)** *and, for some $C\ge 0$ and some $\lambda\in[0,+\infty), \lambda\ne 1$: $\forall r,s\in[0,+\infty)$*

$$(\mathbf{M})(f+g)(r,s) \le C[1+r+s],\ (\mathbf{M}_\lambda)\ (f+\lambda g)(r,s) \le C[1+r+s]. \tag{5.25}$$

*Then, there exists a global weak solution to system (*5.24*).*

Proof of Corollary 5.10. We know (see Remark 5.4) that $(f^n,g^n) = (T_n\circ f, T_n\circ g)$ satisfies also **(P)+(M)**. By Proposition 5.1, the a priori estimate (5.21) is satisfied. We apply Theorem 5.9 together with a diagonal extraction process on a sequence $T^p\to\infty$ to conclude. □

Some applications of Corollary 5.10**.** This corollary applies to the system

$$\begin{cases} \partial_t u - d_1\Delta u = -u^\alpha\, e^{v^2} \\ \partial_t v - d_2\Delta v = u^\alpha\, e^{v^2}, \end{cases}$$

for which global existence of classical solutions is likely to fail (see Subsection 3.4).

The corollary also applies to the system

$$\begin{cases} \partial_t u - d_1\Delta u = \lambda u^p v^q - u^\alpha v^\beta \\ \partial_t v - d_2\Delta v = u^\alpha v^\beta - u^p v^q, \end{cases}$$

with $\lambda\in[0,1)$ and more generally to all systems (3.5) as soon as $k_1k_3 < k_2k_4$. Indeed, in this case, we have (see (3.6)) $f+kg\le 0$ for all k in the interval $[k_1k_4^{-1}, k_2k_3^{-1}]$ which contains more than two points. Then, we apply the corollary.

It is also interesting to notice that Corollary 5.10 applies as well to the two counterexamples mentioned in Theorems 4.1 and 4.2: to see this, we use Remark 4.3. Thus, they are examples where L^∞-blow up does occur while global existence of weak solutions holds.

Remark 5.11. We can extend Corollary 5.10 to $m \times m$ systems. The condition (5.25) can then be replaced by the following: there exists an invertible $m \times m$ matrix P with nonnegative entries and $\mathbf{b} \in I\!R^m$ such that

$$\forall r \in [0,\infty)^m, \;\; P\, f(r) \leq \mathbf{b}\,[1 + \sum_i r_i],$$

for the usual order in $I\!R^m$. As in Proposition 5.1, we easily check that this condition implies an a priori $L^1(Q_T)$ estimate for $f(u)$. Since it also implies condition **(M')**, we may apply Theorem 5.9. Actually, a somehow stronger result is stated below in Theorem 5.14.

5.5. Global existence for quadratic systems with (P)+(M)

An interesting and very general consequence of Theorem 5.9 and of next Section is that global existence of weak solutions holds for all systems with the structure **(P)+(M)** and *whose nonlinearities are at most quadratic*. Note that this includes all the following systems mentioned in Section 2 (2.1, 2.3, 2.4, 2.6 and 2.8, 2.9 with nondegenerate diffusion) .

Proposition 5.12. *Let us consider system* (5.4) *with* (5.5) *or* (5.6) *and with* (5.7). *We assume* **(P)+(M)** *and for some* $C \geq 0$

$$\forall i = 1,\ldots,m, \; |f_i(u)| \leq C[1 + \sum_{1\leq i\leq m} u_i^2]. \tag{5.26}$$

Assume also $u_0 \in [L^2(\Omega)]^m$. *Then, there exists a global weak solution to* (5.4).

The main new idea for the proof of this proposition is that, strangely enough, the structure **(P)+(M)** implies also **an a priori $L^2(Q_T)$-estimate on the solutions**. If the nonlinearities are at most quadratic, they are consequently bounded in $L^1(Q_T)$ and we are in the situation of Theorem 5.9. The following proposition is proved in next Section.

Proposition 5.13. *Let us consider system* (5.4) *with* (5.5) *or* (5.6) *and with* (5.7). *Assume* **(P)+(M)**. *Let u be a nonnegative classical solution on* $(0,T)$ *of* (5.4) *with* $u_0 \in [L^2(\Omega)]^m, u_0 \geq 0$. *Then, for some C depending on the data*

$$\int_{Q_T} \sum_{1\leq i\leq m} u_i^2(t,x)\, dt\, dx \leq C.$$

Let us see how this proposition implies Proposition 5.12. This approach was described in the Appendix of [21].

Proof of Proposition 5.12. Again, we truncate the initial data into $u_0^n = inf\{n, u_0\}$ and we replace f by $f^n = T_n \circ f$ as in Remark 5.4 so that f^n satisfies the same assumptions as f. By Proposition 5.13, the corresponding solution u^n is bounded in $[L^2(Q_T)]^m$. Thanks to the quadratic growth (5.26) of f, we deduce that $f^n(u^n)$ satisfies (5.21). We may then apply Theorem 5.9. □

Actually, it is possible to prove a quite stronger result than Proposition 5.12. Indeed, a main step in its proof is to deduce from the L^2-bound on u that f_i is bounded in $L^1(Q_T)$. But, this may be deduced for instance from the following weaker unilateral version of (5.26):

$$\forall i = 1, \ldots, m,\ \forall r \in [0, \infty)^m,\ \ f_i(r) \leq C[1 + \sum_i r_i^2].$$

More generally, we can even state

Theorem 5.14. *Let us consider system* (5.4) *with* (5.5) *or* (5.6) *and* (5.7)*. We assume that* **(P)+(M)** *holds and that there exist an invertible $m \times m$ matrix P with nonnegative entries and $\mathbf{b} \in I\!R^m$ such that*

$$\forall r \in [0, \infty)^m,\ P\, f(r) \leq \mathbf{b}[1 + \sum_{1 \leq i \leq m} r_i^2]. \tag{5.27}$$

Assume also $u_0 \in [L^2(\Omega)]^m, u_0 \geq 0$. Then, there exists a global weak solution to (5.4).

Proof. The notations are the same as in the proof of Proposition 5.12. To keep the condition (5.26) valid for f^n, we slightly modify the definition of T_n given in Remark 5.4 as follows

$$\sigma_n = n \max\{m, M\},\ M = \max_i m_i^{-1} \sum_j p_{ij}\ \ m_i = \min\{p_{ij}; p_{ij} > 0, j = 1, \ldots, m\}.$$

It is sufficient to prove that $f_i^n(u^n)$ is bounded in $L^1(Q_T)$ for all i, then we apply Theorem 5.9. For this, using system (5.4) and the condition (5.27) which is also valid for f^n, we write for all $i = 1, \ldots, m$

$$\sum_j p_{ij}[\partial_t u_j^n - d_j \Delta u_j^n] + \left[\sum_j p_{ij} f_j^n(u^n)\right]^- \leq \left[\sum_j p_{ij} f_j^n(u^n)\right]^+ \leq b_i^+[1 + \sum_i (u_i^n)^2].$$

Using first the last inequality and u_i^n bounded in $L^2(Q_T)$, we first deduce that the positive part of $\sum_j p_{ij} f_j^n(u^n)$ is bounded in $L^1(Q_T)$. Next, integrating the first inequality, using the boundary conditions and $p_{ij} \geq 0$, we deduce

$$\forall i,\ \int_{Q_T} \left| \sum_j p_{ij} f_j^n(u^n) \right| \leq C.$$

If $\|\cdot\|$ denotes a norm in $I\!R^m$ and also the norm induced on the $m \times m$ matrices, it follows

$$\int_{Q_T} \|f^n(u^n)\| dtdx = \int_{Q_T} \|P^{-1} P f^n(u^n)\| dtdx \leq \|P^{-1}\| \int_{Q_T} \|P f^n(u^n)\| dtdx \leq C.$$

Whence the expected estimate. □

Remark 5.15. With Theorem 5.14, we may prove global existence in systems with more than quadratic growth, like for instance

$$f_1 = f_2 = -f_3 = u_3^3 - u_1u_2. \tag{5.28}$$

Indeed $f_1 + f_2 + 2f_3 = 0$ and (5.27) is satisfied as follows:

$$f_1 + f_3 = 0,\ f_2 + f_3 = 0,\ f_3 \le u_1u_2 \le u_1^2 + u_2^2.$$

Similarly, we can treat the systems modeling the chemical reactions (2.5) when $\sum_i p_i \le 2$ or $\sum_j q_j \le 2$.

6. A surprising L^2-estimate... and L^2-compactness

As announced at the end of the previous section, it turns out that an L^2-estimate is hidden behind the "L^1-type of structure" **(P)+(M)**. This estimate was first noticed in [56, 57] and then widely exploited in [21, 11, 13, 14, 10, 59]. Let us explain the idea.

Let $u_i, i = 1, \dots, m$ be the solution of system (5.4) (or of an approximate version). We set $W = \sum_i u_i, Z = \sum_i d_iu_i$. Assume for simplicity that $\sum_i f_i \le 0$. Then

$$W_t - \Delta Z \le 0 \ \ or\ \ W_t - \Delta(A\,W) \le 0, \tag{6.1}$$

where we set $A = Z/W$. The point is that, thanks to the nonnegativity of the u_i, we have

$$0 < \min_i d_i \le A \le \max_i d_i < +\infty.$$

Therefore, the operator $W \to \partial_t W - \Delta(A\,W)$ is parabolic. It is not of divergence form and, moreover, no continuity may a priori be expected on A. But the parabolicity is enough to imply the following estimate

$$\int_{Q_T} W^2 \le C \int_\Omega W(0)^2,\ \ C = C(\min d_i, \max d_i, T).$$

Let us state this result in the following more general situation where $W_t - \Delta Z \le H$ is satisfied in a weak sense.

Proposition 6.1. *Let $W_0 \in L^2(\Omega), W, Z \in H^1(Q_T), H \in L^2(Q_T)$ such that for all $\psi \in \mathcal{C}^\infty(\overline{Q}_T)$ with $\psi \ge 0, \psi(T) = 0$*

$$\begin{cases} -\int_\Omega \psi(0)W_0 + \int_{Q_T} -\psi_t W + \nabla\psi\nabla Z \le \int_{Q_T} H\,\psi, \\ 0 \le W, Z,\ \ 0 < a = \inf Z/W \le b = \sup Z/W < +\infty. \end{cases} \tag{6.2}$$

Then, there exists $C = C(a, b, T)$ such that

$$\|W\|_{L^2(Q_T)} \le C[\|W_0\|_{L^2(\Omega)} + \|H\|_{L^2 Q_T)}]. \tag{6.3}$$

The same is valid when "$\psi = 0$ on Σ_T" is also required in (6.2) if moreover $Z = 0$ on Σ_T (case of Dirichlet boundary conditions).

Proof. By density, (6.2) holds for all $\psi \in H^1(Q_T)$ such that $\psi \geq 0$ and $\psi(T) = 0$. We apply it with $\psi(t,x) = \int_t^T Z(s,x)ds$. We note that

$$\int_{Q_T} \nabla Z \cdot \nabla \int_t^T Z(s)ds = \int_{Q_T} -\frac{1}{2}\partial_t \left| \int_t^T \nabla Z(s)ds \right|^2 = \int_\Omega \frac{1}{2} \left| \int_0^T \nabla Z(s)ds \right|^2 \geq 0.$$

Thus (6.2) implies (with C depending on T)

$$-\int_\Omega W_0 \int_0^T Z(s)ds + \int_{Q_T} Z\,W \leq \int_{Q_T} H \int_t^T Z(s)ds \leq C(\int_{Q_T} H^2)^{1/2}(\int_{Q_T} Z^2)^{1/2}.$$

But, using the second relation of (6.2), we obtain

$$a\int_{Q_T} W^2 \leq b\left[\sqrt{T}\left(\int_\Omega W_0^2\right)^{1/2} + \left(\int_{Q_T} H^2\right)^{1/2}\right]\left(\int_{Q_T} W^2\right)^{1/2},$$

whence the estimate (6.3). The case of Dirichlet boundary condition is proved the same way. □

Proof of Proposition 5.13. We set $\hat{u}(t) = e^{-Ct}u(t)$ where C is the constant appearing in the condition **(M)** (see (1.8)). We set $W = \sum_i \hat{u}_i, Z = \sum_i d_i\hat{u}_i$. Summing all equations

$$\partial_t\hat{u}_i - d_i\Delta\hat{u}_i = e^{-Ct}[f_i(u) - Cu_i],$$

we obtain

$$\partial_t W - \Delta Z = e^{-Ct}[\sum_i f_i(u) - C\sum_i u_i] \leq Ce^{-Ct} \leq C,$$

which, together with the boundary conditions, implies for all ψ as in (6.2)

$$-\int_\Omega \psi(0)W_0 + \int_{Q_T} -\psi_t W + \nabla\psi\nabla Z + \int_{\Sigma_T} \psi V \leq C\int_{Q_T} \psi,$$

where $V = \sum_i d_i\beta(\alpha_i)\hat{u}_i \geq 0$. This implies that W, Z satisfy the first line of the condition (6.2) with $H \equiv C$. It also satisfies the second line since

$$0 < \min_i d_i \leq \inf \frac{\sum d_i\hat{u}_i}{\sum \hat{u}_i} \leq \frac{Z}{W} \leq \sup \frac{\sum d_i\hat{u}_i}{\sum \hat{u}_i} \leq \max_i d_i < +\infty. \qquad \square$$

Remark 6.2. The L^2-estimate of Proposition 5.13 is very robust and may be generalized to quite more general diffusion operators (see e.g. [21, 59, 10]), and may even allow some degeneracy in the diffusion coefficients (see e.g. [21]).

We gave here a "direct" proof of the L^2-estimate. We can also obtain it by duality, looking at the regularizing properties of the dual operator $-\partial_t - A\Delta$. As one can easily check, when $H \equiv 0$, inequality (6.3) is equivalent to the dual inequality $\|\phi(0)\|_{L^2(\Omega)} \leq C\|\Theta\|_{L^2(Q_T)}$ for the solution ϕ of the dual problem

$$-[\phi_t + A\Delta\phi] = \Theta \geq 0,\ \phi(T) = 0\ +\ \textit{boundary conditions}. \tag{6.4}$$

This dual inequality is easily obtained by multiplying the equation (6.4) by $-\Delta\phi$ which gives, for instance with $\phi = 0$ on Σ_T, and after integration by parts of the first term:

$$\int_{Q_T} -\frac{1}{2}\partial_t|\nabla\phi|^2 + A(\Delta\phi)^2 = \int_{Q_T} -\Theta\Delta\phi. \tag{6.5}$$

Recalling $0 < a \leq A \leq b < +\infty$ and using

$$\int_{Q_T} -\Theta\Delta\phi \leq \frac{a}{2}\int_{Q_T}(\Delta\phi)^2 + \frac{1}{2a}\int_{Q_T}\Theta^2, \tag{6.6}$$

we obtain

$$\int_\Omega |\nabla\phi(0)|^2 + a\int_{Q_T}(\Delta\phi)^2 \leq a^{-1}\int_{Q_T}\Theta^2. \tag{6.7}$$

This says that, not only $\phi(0)$ is bounded in $L^2(\Omega)$, but it is even bounded in $H_0^1(\Omega)$ so that the mapping $\Theta \in L^2(Q_T) \to \phi(0) \in L^2(\Omega)$ is not only continuous but compact ! By duality, the original operator is also compact. More generally, we have the following *compactness result* where we denote $\mathcal{L} = L^2(\Omega) \times L^2(Q_T)$:

Proposition 6.3. *Let $0 < a \leq b < +\infty, \beta \in [0,\infty)$. For $(W_0, H) \in \mathcal{L}$, let $\mathcal{F}_{a,b,W_0,H}$ denote the family of functions $W \in H^1(Q_T)$ such that for some $Z \in H^1(Q_T)$, for all $\psi \in \mathcal{C}^\infty(\overline{Q}_T), \psi \geq 0, \psi(T) = 0$*

$$\begin{cases} -\int_\Omega \psi(0)W_0 + \int_{Q_T} -\psi_t W + \nabla\psi\nabla Z + \beta\int_{\Sigma_T}\psi Z = \int_{Q_T} H\psi, \\ 0 \leq W, Z;\;\; a \leq Z/W \leq b, \end{cases} \tag{6.8}$$

Then, for all bounded $\mathcal{K} \subset \mathcal{L}$, the family $\{\mathcal{F}_{a,b,W_0,H}; (W_0,H) \in \mathcal{K}\}$ is relatively compact in $L^2(Q_T)$. The same is valid when the test-functions ψ of (6.8) are restricted to satisfy also $\psi = 0$ on Σ_T and if moreover $W = Z = 0$ on Σ_T.

Proof. Let $A \in \mathcal{C}(\overline{Q}_T)$ with $a \leq A \leq b$. For all $\Theta \in L^2(Q_T), \Theta \geq 0$, let ϕ be the solution of the dual problem

$$\begin{cases} \phi \in C([0,T], L^2(\Omega)),\ \phi \geq 0,\ \phi_t, \Delta\phi \in L^2(Q_T), \\ -\phi_t - A\Delta\phi = \Theta,\ \ \phi(T) = 0,\ \ -\frac{\partial\phi}{\partial n} = \beta\phi \text{ on } \Sigma_T. \end{cases} \tag{6.9}$$

Existence of ϕ with these properties uses the continuity of A and may be found in [41]. Let us prove that, for some $C = C(a,b,T)$

$$\|\phi(0)\|_{H^1(\Omega)} + \|\phi_t\|_{L^2(Q_T)} + \|\Delta\phi\|_{L^2(Q_T)} \leq C\|\Theta\|_{L^2(Q_T)}. \tag{6.10}$$

Multiplying the equation in ϕ by $-\Delta\phi$ and integrating by parts the first term lead to

$$-\beta\int_{\Sigma_T}\frac{1}{2}\partial_t\phi^2 + \int_{Q_T} -\frac{1}{2}\partial_t|\nabla\phi|^2 + A(\Delta\phi)^2 = \int_{Q_T} -\Theta\Delta\phi. \tag{6.11}$$

With the same computations as in (6.6), (6.7), we deduce

$$\beta\int_{\partial\Omega}\phi(0)^2 + \int_\Omega|\nabla\phi(0)|^2 + a\int_{Q_T}(\Delta\phi)^2 \leq a^{-1}\int_{Q_T}\Theta^2.$$

Going back to the equation in ϕ and using $A \leq b$, we have also

$$\int_{Q_T}(\phi_t)^2 \leq [b\|\Delta\phi\|_{L^2(Q_T)} + \|\Theta\|_{L^2(Q_T)}]^2 \leq (1 + \frac{b}{a})^2\int_{Q_T}\Theta^2.$$

Using $\phi(0) = \int_T^0 \phi_t$, we deduce

$$\int_\Omega \phi(0)^2 \leq T \int_{Q_T} (\phi_t)^2 \leq T(1 + \frac{b}{a})^2 \int_{Q_T} \Theta^2.$$

The three last inequalities yield (6.10). The case of Dirichlet boundary conditions is done in an even easier way (we use directly the computations (6.5), (6.6), (6.7)).

We denote

$$\mathcal{H} = \{(\phi(0), \phi) \in \mathcal{L}; A \in \mathcal{C}(\overline{Q}_T),\ a \leq A \leq b,\ \|\Theta\|_{L^2(Q_T)} \leq 1\}.$$

The estimate (6.10) implies that $\mathcal{H}$ is relatively compact in $L^2(\Omega) \times L^2(Q_T)$. Let us now go back to W, Z satisfying (6.8). If $A = Z/W$ is continuous on $\overline{Q}_T$, we may solve (6.9) and we have exactly

$$\int_{Q_T} W\,\Theta = \int_\Omega \phi(0) W_0 + \int_{Q_T} \phi\, H.$$

Since, we did not assume A to be continuous, we may approximate A by A^p continuous on $\overline{Q}_T$ and such that

$$a \leq A^p \leq b, \quad A^p \to A\ a.e.$$

We solve (6.9) with A^p instead of A and the solution ϕ^p satisfies

$$\int_{Q_T} W\,\Theta = \int_\Omega \phi^p(0) W_0 + \int_{Q_T} \phi^p H + \int_{Q_T} W(A^p - A)\Delta\phi^p. \tag{6.12}$$

By compactness of $\mathcal{H}$, up to a subsequence, $(\phi^p(0), \phi^p)$ converges in $\mathcal{L}$ to some $(\phi_0, \phi) \in \overline{\mathcal{H}}$ and we check that the last integral in (6.12) tends to 0 (using that $\Delta\phi^p$ is bounded in $L^2(Q_T)$) so that

$$\int_{Q_T} W\Theta = \int \phi_0 W_0 + \int_{Q_T} \phi\, H. \tag{6.13}$$

Now, let us take a sequence W_n, Z_n satisfying (6.8) with data $(W_{0n}, H^n) \in \mathcal{K}$. Let $G_n : \overline{\mathcal{H}} \to I\!R$ defined by

$$\forall (\phi_0, \phi) \in \overline{\mathcal{H}},\ G_n(\phi_0, \phi) = \int_\Omega \phi_0 W_{0n} + \int_{Q_T} \phi\, H_n.$$

The family (G_n) of ("linear") continuous mappings from the compact $\overline{\mathcal{H}}$ into $I\!R$ satisfies the hypotheses of Ascoli-Arzela's Theorem: therefore, up to a subsequence, me may assume that G_n converges uniformly to some continuous function on $\overline{\mathcal{H}}$. Thus

$$\lim_{p,q\to\infty} \sup_{(\phi_0,\phi)\in\overline{\mathcal{H}}} |G_p(\phi_0, \phi) - G_q(\phi_0, \phi)| = 0\ .$$

We use

$$\|W_p - W_q\|_{L^2(Q_T)} = \sup_{\|\Theta\|_{L^2(Q_T)} \leq 1} \left| \int_{Q_T} (W_p - W_q)\Theta \right|,$$

and (6.13) to deduce

$$\limsup_{p,q\to\infty} \|W_p - W_q\|_{L^2(Q_T)} \leq \lim_{p,q\to\infty} \sup_{(\phi_0,\phi)\in\overline{\mathcal{H}}} |G_p(\phi_0, \phi) - G_q(\phi_0, \phi)| = 0.$$

Whence the expected compactness. □

An application: a new proof of Proposition 5.12**.** As a corollary of the compactness result of Proposition 6.3, we may give a proof of Proposition 5.12 which does not use the general Theorems 5.5 and 5.9 on the existence of weak global supersolutions and solutions. A similar proof is also given in [14], but the L^2-compactness is proved differently.

Proof. As before (see the previous proof of Proposition 5.12), we truncate the initial data as $u_{i0}^n = \inf\{n, u_{0i}\}$ for all i and we replace f by $f^n = T_n \circ f$ which satisfy the same assumptions as f. By Proposition 5.13, we know that the approximate solutions are bounded in $L^2(Q_T)$, so that the nonlinearities are bounded in $L^1(Q_T)$. This provides compactness of u^n in $[L^1(Q_T)]^m$ and we may assume (up to a subsequence) that u^n converges to u in $L^1(Q_T)$ and a.e., and that $f^n(u^n)$ converges to $f(u)$ a.e. where $f(u) \in L^1(Q_T)$. We will prove that the convergence of $f^n(u^n)$ holds in $L^1(Q_T)$: thanks to the pointwise estimate (5.26), it is sufficient to prove the compactness of u^n in $L^2(Q_T)$.

Setting, like in the proof of Proposition 5.12,

$$\hat{u}^n(t) = e^{-Ct}u^n(t),\ W^n = \sum_i \hat{u}_i^n,\ Z^n = \sum_i d_i \hat{u}_i^n,$$

we have that

$$W_t^n - \Delta(A^n W^n) \leq e^{-Ct}[\sum_i f_i^n(u^n) - \sum_i u_i^n] \leq C,$$

where $A^n = Z^n/W^n$. By maximum principle, $W^n \leq w^n$ where w^n is the solution of

$$\partial_t w^n - \Delta(A^n w^n) = C,\ w^n(0) = W^n(0),\ -\frac{\partial(A^n w^n)}{\partial n} = 0 \ on\ \Sigma_T.$$

Note that here A^n is regular so that existence and comparison principle hold. But, by Proposition 6.3, the sequence w^n is compact in $L^2(Q_T)$. Since we have $0 \leq \hat{u}_i^n \leq W^n \leq w^n$, and since $\hat{u}^n$ converges a.e., by the dominated convergence theorem, it follows that each $\hat{u}_i^n$ converges in $L^2(Q_T)$. Thanks to the pointwise estimate (5.26), $f^n(u^n)$ converges strongly in $L^1(Q_T)$ and we may pass to the limit in the approximate system to obtain a solution. □

Remark 6.4. It seems that this kind of approach is insufficient to prove the stronger result of Theorem 5.14: we still have the L^2-convergence of u^n, but this is a priori insufficient to control terms which would not be quadratic, like in example (5.28).

More applications of the L^2-estimate. We already mentioned several places where this L^2-estimate has been used or generalized, even in the context of nonlinear diffusions like in [10]. It has also been exploited in [21], [13] together with the entropy inequality which is valid for most reversible systems like (2.4). This entropy inequality may be written as follows, where $w_i = u_i \ln u_i + 1 - u_i \geq 0$:

$$\sum_i [\partial_t - d_i \Delta] w_i \leq 0. \tag{6.14}$$

With adequate boundary conditions, it follows from Proposition 5.13 that w_i is bounded in $L^2(Q_T)$. This implies that, not only u_i^2 is bounded in $L^1(Q_T)$, but it is locally integrable on Q_T. We may take advantage of this extra-property for a more direct proof of global existence of weak solutions for reversible systems (as done in [21]).

These L^2-estimates are also the main ingredient to prove the convergence of the solutions to the system modeling the reaction

$$A + B \rightleftharpoons C \rightleftharpoons P + Q$$

to those of the limit system

$$A + B \rightleftharpoons P + Q$$

when the decay of C into the products P and Q, or back to the educts A and B, is extremely fast (see [11, 13, 14]).

One more curious estimate. Let us consider again the inequality $W_t - \Delta Z \leq 0$ which has been used to deduce the L^2-estimate. After integrating this inequality in time, we deduce from nonnegativity of $W(t)$

$$-\Delta \int_0^t Z(s)\, ds \leq W_0.$$

If $W_0 \in L^\infty(\Omega)$ (or $W_0 \in L^p(\Omega), p > N/2$), and with usual boundary conditions, this implies the curious uniform L^∞ a priori estimate

$$\forall i = 1, \ldots, m, \quad \sup_{t\in(0,T^*), x\in\Omega} \int_0^t u_i(s,x)\, ds < +\infty.$$

This might be useful when studying the asymptotic behavior of global solutions.

7. Open problems

- **Problem 1.** A most challenging problem is to understand whether global solutions exist for a system, even 2×2, for which the structure **(P)+(M)** holds, but for which there is no obvious a priori $L^1(Q_T)$-bound on the nonlinearities. Let us indicate two simple examples of this situation:

$$\begin{cases} \partial_t u - d_1 \Delta u = u^3 v^2 - u^2 v^3 & \textit{on } Q_T, \\ \partial_t v - d_2 \Delta v = u^2 v^3 - u^3 v^2 & \textit{on } Q_T, \\ + \text{ initial and boundary conditions.} \end{cases}$$

$$\begin{cases} \partial_t u - d_1 \Delta u = -c(t,x) u^2 v^2 & \textit{on } Q_T, \\ \partial_t v - d_2 \Delta v = c(t,x) u^2 v^2 & \textit{on } Q_T, \\ + \text{ initial and boundary conditions,} \end{cases}$$

  when $c(t,x)$ is not of constant sign. Here, it could well be that the nonlinearities are not bounded in $L^1(Q_T)$ in general. Therefore, even the definition of weak solution is not obvious in this case. We feel that one should truncate the nonlinearities and introduce some sort of *renormalized solution*: but, this is not available yet for this kind of systems where maximum principle is missing. A direction could be to better understand the proof of Theorem 5.5: it heavily involves truncation operators and the result looks like providing some sort of maximum principle for the system.

The question is open also for the chemical systems (2.5) of Section 2 when they are not quadratic, or not bounded above by a quadratic polynomial (see Remark 5.15).

- **Problem 2.** What about uniqueness of weak solutions? Working with uniformly bounded solutions is satisfactory, since they come with a uniqueness property and the problem is well posed in this class. Unfortunately, as shown in Section 4, even for regular initial data, the solution may leave $L^\infty(\Omega)$ so that we must give up with this confortable framework. But we do not know any more what happens with uniqueness. The question is certainly delicate since it is known that there is not uniqueness of weak solutions even for the simple *equation*:

$$u_t - \Delta u = u^3,\ u(0) = u_0 \geq 0, u = 0 \text{ on } \partial\Omega,$$

and even for $\mathcal{C}^\infty$ initial data (see [6, 27]). The right question to ask is probably: is there a way to select the "good" solution among the possible several ones? But, is there a "good" solution? In the previous example, the smallest one, which is uniformly bounded, seems to be the "good" one. In a system without maximum principle, it is not clear. This question of uniqueness could be generalized to some kind of adequate renormalized solution ('adequate' actually requires that uniqueness holds).

- **Problem 3.** Once we have proved global existence of weak solutions for a system, it remains interesting to decide whether it is uniformy bounded (and therefore classical) or not. Let us take for instance the quadratic system

$$\begin{cases} \partial_t a - d_1\Delta a = -a\,b + c\,d \\ \partial_t b - d_2\Delta b = -a\,b + c\,d \\ \partial_t c - d_3\Delta c = a\,b - c\,d \\ \partial_t d - d_4\Delta d = a\,b - c\,d, \end{cases} \tag{7.1}$$

for which global existence of weak solutions holds in any dimension. It is proved that in dimensions $N = 1, 2$ and for bounded initial data, the solutions are bounded (see [20, 26] and also [33] and its references). What happens in higher dimensions. A result in this direction may be found in [26] where it is proved in dimensions $N = 3, 4$ that the Hausdorff dimension of the possible set of blow up in Q_T is at most $(N^2 - 4)/N$. But, is blow up indeed possible? Same question for

$$\begin{cases} \partial_t u - d_1\Delta u = -u^\alpha\, e^{v^2} \\ \partial_t v - d_2\Delta v = u^\alpha\, e^{v^2}. \end{cases}$$

Note that, as proved in [9], L^∞-bounds hold for the slightly better system

$$f(u, v) = -u^\alpha(1 + v^2)e^{v^2},\ \ g(u, v) = u^\alpha\, e^{v^2}.$$

- **Problem 4.** What happens for initial data in $L^1(\Omega)$ only? The general existence result of Theorem 5.9 is stated for L^1-initial data. However, when applied for instance to system (7.1), it requires that the initial data be in $L^2(\Omega)$. What happens for this system if they are only in $L^1(\Omega)$, or even worse only bounded measures? The same question is of interest for several other systems. It is actually very connected with the global existence question, since we need to extend solutions after an L^∞ blow up at time T^* in situations where merely an L^1-estimate holds on $u(T^*)$. See [12] for some results in this direction.

- **Problem 5.** What happens for degenerate diffusions? We saw that most of the estimates were based on the regularizing effects of the diffusions. We loose them in general when degeneracies appear, and it is the case in many applications of interest: for instance, when nonlinear diffusions occur like in

$$\begin{cases} \partial_t u - \Delta u^m = f(u,v) & \textit{on } Q_T \\ \partial_t v - \Delta v^p = g(u,v) & \textit{on } Q_T. \end{cases}$$

See some results in this direction in [42]. Even the case of linear diffusions is of interest; see some examples in [21] or also system (2.8-2.9) (see [25, 59] for some contributions).

- **Problem 6.** How far is it possible to extend the results recalled in this survey to situations where the nonlinearities depend also on the gradient of the solutions, like they do in several models? A 2×2 model would be of the form

$$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v,\nabla u, \nabla v) \\ \partial_t v - d_2 \Delta v = g(u,v,\nabla u, \nabla v), \end{cases}$$

together with conditions of the kind $f + g \leq 0$. All questions about global existence of classical, or of weak solutions, are of interest. See for instance [2, 18] for some results in this direction and for more references.

- **Problem 7.** How do the known techniques extend to cross-diffusions, namely

$$\begin{cases} \partial_t u - d_1 \Delta u - \nabla \cdot (a_1(u,v)\nabla u + a_2(u,v)\nabla v) = f(u,v) & \textit{on } Q_T \\ \partial_t v - d_2 \Delta v - \nabla \cdot (b_1(u,v)\nabla u + b_2(u,v)\nabla v) = g(u,v) & \textit{on } Q_T? \end{cases}$$

More and more pertinent models require these cross-diffusions. Conditions are required to preserve positivity. Next, global existence remains a natural question (see [40, 15, 16, 10]).

- **Problem 8.** Instead of having an L^1-structure of type **(M)**, namely $f + g \leq 0$ or of type **(M')**: $a\,f + b\,g \leq 0$ with $a, b > 0$, there are systems for which a more general Lyapunov structure holds like

$$h_1'(u)f(u,v) + h_2'(v)g(u,v) \leq 0,$$

where $h_1, h_2 : [0,\infty) \to [0,\infty)$ satisfy $\lim_{r\to\infty} h_i(r) = +\infty$. Global existence would still hold for the associated O.D.E. But, what about the P.D.E. system, even for convex h_i?

- **Problem 9.** We dealt in this paper with evolution problems governed by parabolic operators. But, the same type of questions may be asked for elliptic systems. To progress, it may actually be a good idea to address them in this context where the technicality is sometimes easier. For instance, we may look at existence results for

$$\begin{cases} u - \Delta u - \lambda u_{x_1 x_1} = f(u,v) + F \text{ on } \Omega \\ v - \Delta v = g(u,v) + G \text{ on } \Omega, \end{cases}$$

  where F, G are regular nonnegative given functions on Ω and where f, g satisfy **(P)+(M)**. Here, when $\lambda > 0$ is large, the two diffusion operators are very different from each other, like the two parabolic operators $\partial_t - d_1\Delta$ and $\partial_t - d_2\Delta$ are when d_1/d_2 is away from 1. The difficulties are of the same kind. Some results may be found in [45].

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