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On Spectral Minimal Partitions: A Survey

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Abstract. Given a bounded open set Ω in $\mathbb{R}^n$ (or a Riemannian manifold) and a partition of Ω by k open sets D_j, we can consider the quantity $\max_j \lambda(D_j)$ where $\lambda(D_j)$ is the groundstate energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(D_j)$, a minimal (spectral) k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

In this survey, we mainly consider the two-dimensional case and discuss the properties of minimal spectral partitions. We illustrate the difficulties to determine them explicitly by considering simple cases like the disk, the rectangle or the sphere ($k = 3$) and will also exhibit the possible role of the hexagon in the asymptotic behavior as $k \to +\infty$ of $\mathfrak{L}_k(\Omega)$.

We also discuss the role of some Aharonov-Bohm Schrödinger operator for producing candidates for minimal partitions. Finally we compare different notions of minimal partitions and propose a few open problems.

This work has started in collaboration with T. Hoffmann-Ostenhof and has been continued in the last years with the coauthors V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini and G. Vial.

1. Introduction

We consider mainly the Dirichlet realization of the Laplacian operator $H(\Omega)$ in Ω, when Ω is a bounded domain in $\mathbb{R}^2$ with piecewise-C^1 boundary (corners or cracks permitted). We would like to analyze the relations between the nodal domains of the eigenfunctions of $H(\Omega)$ and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the groundstate energy of the Dirichlet realization of the Laplacian $H(D_i)$ in D_i is minimal. We can also consider the case of a two-dimensional Riemannian manifold and the Laplacian is then the Laplace Beltrami operator. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of eigenfunctions.

The groundstate u_1 can be chosen to be strictly positive in Ω, but the other excited eigenfunctions u_k must have zerosets. Here we recall that for $u \in C^0_0(\overline{\Omega})$, the nodal set (or zeroset) of u is defined by:

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}\,. \tag{1}$$

In the when u is an eigenfunction of the Laplacian, the $\mu(u)$ components of $\Omega \setminus N(u)$ are called the nodal domains of u and define naturally a partition of Ω by $\mu(u)$ open sets, which will be called a **nodal partition**. Our main goal is to discuss the links between the partitions of Ω associated to these eigenfunctions and the minimal partitions of Ω.

This survey is organized as follows. We first introduce the basic definitions in Section 2. Section 3 gives the fundamental theorems of existence and regularity for minimal partitions and the main criteria for determining and characterizing the minimal partitions which are nodal. In Section 4, we discuss an extended notion of minimal partitions and give a criterion of comparison. Section 5 is devoted to examples and we refer here to what has been obtained by intensive use of numerical analysis mainly for minimal 3-partitions but also for minimal 5-partitions. Section 6 is a short description of the Aharonov-Bohm approach. Section 7 presents a fascinating conjecture exhibiting the role of the hexagonal tilings in the asymptotic behavior of the energy of minimal k-partitions as $k \to +\infty$. In Section 8, we discuss the question of minimal-partition of the sphere in connection with conjectures appearing in harmonic analysis. We conclude our survey by adding a few open questions.

Acknowledgements. This work has started for the author five years ago in collaboration with T. Hoffmann-Ostenhof and has been continued in the last years with the coauthors V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini and G. Vial. We would like to thank all of them for this fruitful collaboration. We also thank the Schrödinger Institute where an important part of this work has been done. We also thank the organizers of this colloquium in Milano for giving us the opportunity to present these results and motivating us for writing this survey.

2. Partitions

2.1. Minimal partitions

Following the presentation of [HHOT1], we first introduce more precisely what we will call a partition.

Definition 2.1. *Let* $1 \leq k \in \mathbb{N}$. *A* **partition** *(or* k*-partition for indicating the cardinal of the partition) of* Ω *is a family* $\mathcal{D} = \{D_i\}_{i=1}^k$ *of mutually disjoint sets such that*

$$\cup_{i=1}^k D_i \subset \Omega\,. \tag{2}$$

We denote by $\mathfrak{O}_k = \mathfrak{O}_k(\Omega)$ the set of partitions of Ω where the D_i's are domains (i.e. open and connected).

Sometimes (at least for the proofs) we have to relax this definition by considering quasi-open or measurable sets for the partitions. We will not discuss this point in detail (see [HHOT1]).

We now introduce the notion of spectral minimal partitions.

Definition 2.2. *For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we introduce the energy of the partition:*

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{3}$$

Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k(\Omega)} \Lambda(\mathcal{D}). \tag{4}$$

and call $\mathcal{D} \in \mathfrak{O}_k(\Omega)$ minimal if $\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})$.

Remark 2.3. *If $k = 2$, it is rather well known (see* [HH1] *or* [CTV3]*) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition, i.e. a partition for which the elements are the nodal domains of some eigenfunction corresponding to λ_2.*

2.2. Strong and regular partitions

The analysis of the properties of minimal partitions leads us to introduce two notions of regularity that we present briefly.

Definition 2.4. *A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called* **strong** *if*

$$Int\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{5}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 2.5 (Boundary set).

$$N(\mathcal{D}) = \overline{\cup_i\,(\partial D_i \cap \Omega)}\,. \tag{6}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This leads us to introduce the set $\mathcal{R}(\Omega)$ of **regular** partitions through the properties of its associated boundary set N, which should satisfy the following properties:

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the neigborhood of which N is the union of $\nu_i(x_i)$ smooth curves ($\nu_i \geq 2$) with one end at x_i, N is locally diffeomorphic to a regular curve.

(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.

(iii) N has the equal angle meeting property, that is the half curves cross with equal angle at each singular interior point of N and also at the boundary together with the tangent to the boundary.

Hence $N(\mathcal{D})$ admits a natural decomposition:

$$N(\mathcal{D}) = N^{reg}(\mathcal{D}) \cup N^{sing}(\mathcal{D})\,, \tag{7}$$

with

$$N^{sing}(\mathcal{D}) = N^{sing}_{bnd}(\mathcal{D}) \cup N^{sing}_{int}(\mathcal{D})\,,$$

$N_{bnd}^{sing}(\mathcal{D})$ corresponding to the points z_i introduced in (ii) and $N_{int}^{sing}(\mathcal{D})$ corresponding to the points x_i introduced in (i). These points will be called the singular points of the partition.

2.3. Strong partitions and associated graphs

We say that two sets D_i, D_j of the partition $\mathcal{D}$ are neighbors and write $D_i \sim D_j$, if $D_{i,j} := \operatorname{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected. We can then associate with each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating with each D_i a vertex and to each pair of neighbors (D_i, D_j) an edge. We say that the graph is **bipartite** if it can be colored by two colors (two neighbors having different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite. In short we will call bipartite partition a strong partition whose graph is bipartite.

Here are two "abstract" examples of partitions with associated graph. The first one is a bipartite 10-partition of some open set Ω and the second one is a nonbipartite 7-partition of another open set Ω with two interior singular points x_i $(i = 1, 2)$, where $\nu(x_i)$ is odd, and one interior singular point x_3 with $\nu(x_3) = 6$. The boundary set has in addition 6 singular points in $\partial\Omega$.

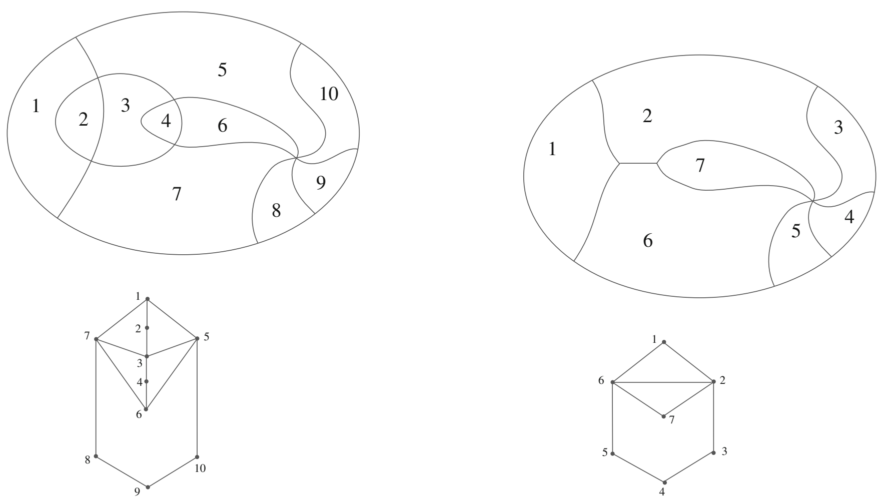

FIGURE 1

3. Main results

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] (existence) and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] (regularity) the following theorem:

Theorem 3.1. *For any k, there exists a minimal regular k-partition. Moreover any minimal k-partition has a regular representative*[1].

[1]possibly after a modification of the open sets of the partition by capacity 0 subsets.

Other proofs of a somewhat weaker version of the existence statement have been given by Bucur-Buttazzo-Henrot [BBH], Caffarelli-F.H. Lin [CL]. Note that in these references these minimal partitions are also called **optimal** partitions.

A natural question is whether a minimal partition of Ω is a nodal partition. We have first the following converse theorem ([HH1], [HHOT1]):

Theorem 3.2. *If the graph of the minimal partition is bipartite this is a nodal partition.*

The next question is then to determine how general is the previous situation. Surprisingly this only occurs in the so called Courant-sharp situation. Let us start with two classical theorems and a few definitions.

The Courant nodal theorem says that if for $k \geq 1$ λ_k denotes the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with λ_k, then $\forall u \in E(\lambda_k)\setminus\{0\}\ ,\ \mu(u) \leq k$.

Then we say that u is **Courant-sharp** if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k\ .$$

The Pleijel Theorem (see [Pl]) says that, for a given open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, then there are no Courant-sharp eigenfunctions of the Dirichlet Laplacian in Ω for k large.

For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.

In general, one can show, as an easy consequence of the max-min characterization of the eigenvalues, that

$$\lambda_k \leq \mathfrak{L}_k \leq L_k\ . \tag{8}$$

The last result gives the full picture of the equality cases:

Theorem 3.3. *Suppose $\Omega \subset \mathbb{R}^2$ is regular. If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$, then*

$$\lambda_k = \mathfrak{L}_k = L_k\ .$$

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

This answers a question in [BHIM] (Section 7).

4. Looking at p-minimal k-partitions

More generally (see in [HHOT1]) we can consider for a k-partition $\mathcal{D}$ and for $p \in [1, +\infty[$

$$\Lambda_p(\mathcal{D}) = \Big(\frac{1}{k}\sum_{i=1}^{k}\lambda(D_i)^p\Big)^{\frac{1}{p}}\,, \tag{9}$$

and

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda_p(\mathcal{D})\,. \tag{10}$$

We write $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$ and recall the monotonicity property

$$\mathfrak{L}_{k,p}(\Omega) \leq \mathfrak{L}_{k,q}(\Omega) \text{ if } p \leq q\,. \tag{11}$$

The notion of p-minimal k-partition can be extended accordingly, by minimizing $\Lambda_p(\mathcal{D})$. Note that the inequalities can be strict! Take a disjoint union of two disks (possibly related by a thin channel) (see [BBH, HHOT2] for details).

The inequality

$$\lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega)\,. \tag{12}$$

is replaced by the inequality (see [HHOT2] for $p = 1$ and [HH3] for general p)

$$\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Omega)^p\right)^{\frac{1}{p}} \leq \mathfrak{L}_{k,p}(\Omega)\,. \tag{13}$$

This is optimal for the disjoint union of k-disks with different (but close) radius.

Note that we do not know if, for the disk D, $\mathfrak{L}_{2,1}(D) = \mathfrak{L}_{2,\infty}(D)$. But this equality has been proved for the sphere $\mathbb{S}^2$ (see [Bis, FrHa] and the references in [HHOT2]). We will discuss other aspects of this question in Section 7.

Coming back to open sets in $\mathbb{R}^2$, we have proved recently [HH3] that the inequality

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

is "generically" satisfied. Moreover, we can give explicit examples (equilateral triangle) of convex domains for which this is true. This answers (by the negative) some question in [BBH].

The proof is based [HH3] on

Proposition 4.1. *Let Ω be a domain in $\mathbb{R}^2$ and $k \geq 2$. Let $\mathcal{D}$ be a minimal k-partition for $\mathfrak{L}_k(\Omega)$ and suppose that there is a pair of neighbors (D_i, D_j) such that for the second eigenfunction ϕ_{ij} of $H(D_{ij})$ having D_i and D_j as nodal domains we have*

$$\int_{D_i} |\phi_{ij}(x,y)|^2 dxdy \neq \int_{D_j} |\phi_{ij}(x,y)|^2 dxdy\,. \tag{14}$$

Then

$$\mathfrak{L}_{k,1}(\Omega) < \mathfrak{L}_k(\Omega)\,. \tag{15}$$

The proof is a consequence of the Hadamard Formula concerning the variation of the eigenvalue of the Dirichlet problem by deformation of the boundary.

5. Examples of minimal k-partitions

5.1. On minimal 3-partitions

Of course, using Theorem 3.3, the first examples of minimal 3-partitions corrrespond to the case when we are in the Courant-sharp situation. This should be typically the case for thin domains like the rectangles $]0,a[\times]0,b[$ with $b/a > \sqrt{\frac{8}{3}}$ (see [HHOT1, BHHO] and below). This can also be explored numerically. One could expect for example that the same situation occurs for domains delimited by ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$ with $\frac{b}{a}$ large enough.

Here we describe following [HH2] the possible "topological" types of minimal nonbipartite 3-partitions for a general domain Ω in $\mathbb{R}^2$.

Proposition 5.1. *Let Ω be simply-connected and consider a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ associated with $\mathfrak{L}_3$ and suppose that it is not bipartite. Then the boundary set $N(\mathcal{D})$ has one of the following properties :*

- **Type [a].** *one singular point x_0 with $\nu(x_0) = 3$, three points z_i $(i = 1, \dots, 3)$ on the boundary with $\rho(z_i) = 1$;*
- **Type [b].** *two singular points x_0 and x_1 with $\nu(x_0) = 3 = \nu(x_1)$, two points z_1 and z_2 on the boundary with $\rho(z_1) = 1 = \rho(z_2)$;*
- **Type [c].** *two singular points x_0 and x_1 with $\nu(x_0) = 3 = \nu(x_1)$, no point on the boundary.*

The three types are described in Figure 3 (with an additional symmetry assumption with respect to the y-axis).

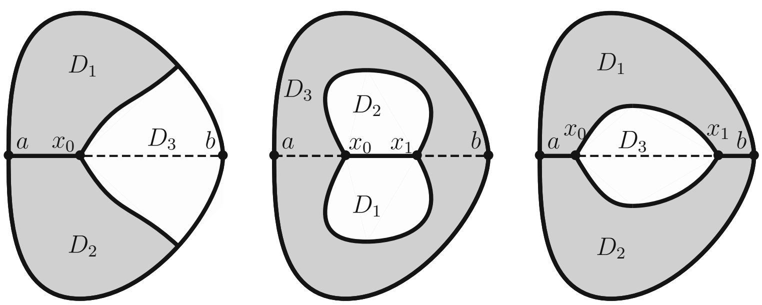

FIGURE 2. Three topological types : [a], [b] and [c]

The proof of Proposition 5.1 relies essentially on the Euler formula together with the property that the associated graph should be a triangle.

This leads (with some success) to analyze the minimal 3-partitions with some topological type. If in addition, we introduce some symmetries, this leads to guess some candidates for minimal 3-partitions. We actually do not know any example where the minimal 3-partitions are of type [*b*] and [*c*]. Numerical computations for the rectangle ([BHV]) were discouraging.

Note also that we do not know about results claiming that the minimal 3-partition of a domain with symmetry should keep some of these symmetries. We actually know in the case of the disk (see Proposition 1.6 in [HH2]) that a minimal 3-partition cannot keep all the symmetries.

5.2. Double covering argument

This double covering idea was initially developed in [HHOO] in connection with superconductivity. In the context of minimal partitions, it is first described in [HH2] and then further explored in [NT] and [BH]. If we have a partition of a domain where only one singular point x_i has an odd $\nu(x_i)$ (typically a type [a] 3-partition has this structure), one can puncture this domain at this singular point and consider the double covering of this punctured domain. Then the projection of a nodal 6-partition of this covering which is antisymmetric with respect to the deck map could be a good candidate (by projection) for a minimal 3-partition.

5.3. Special domains

5.3.1. The case of the disk. In the case of the disk, we have no proof that the minimal 3-partition is the "Mercedes star". But if we assume that the minimal 3-partition is of type (a), then by going on the double covering of the punctured disk, one can show that it is indeed the Mercedes star.

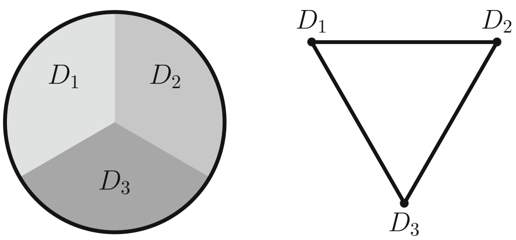

FIGURE 3. The logo Mercedes and associated graph.

5.3.2. The case of the square. In the case of the square, it is not too difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$, and there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).

Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with one of the perpendicular bisectors of one side of the square, one is reduced to analyze a family of Dirichlet-Neumann problems. Numerical computations performed by V. Bonnaillie-Noël and G. Vial lead to a natural candidate for a symmetric minimal partition.

See http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

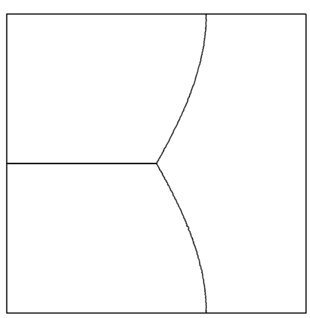

FIGURE 4

In the case of the square, we have no proof that the candidate described by Figure 4 is a minimal 3-partition. But if we assume that the minimal partition is of type [a] and has the symmetry, then numerical computations lead to the Figure 4. Numerics suggests more : the center is the unique singular point of the partition inside the square. Once this property is accepted, one can go to the double covering of the punctured square at the center and observe numerically that this partition is the projection by the covering map of a nodal partition of an antisymmetric (with respect to the deck map) eigenfunction of the Laplacian on this covering. This point of view is explored numerically in a rather systematic way by Bonnaillie-Helffer [BH] and theoretically by Noris-Terracini [NT].

One can also look for a possible minimal 3-partition having the symmetry with respect to the diagonal (see Figure 5). THIS LEADS TO THE SAME VALUE OF $\Lambda(\mathcal{D})$.

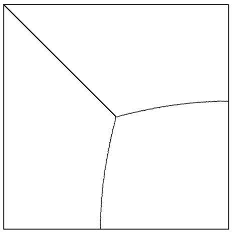

FIGURE 5. Another candidate

So this strongly suggests that there is a continuous family of minimal 3-partitions of the square. This is done indeed numerically in [BH].

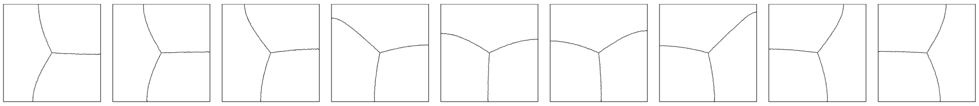

FIGURE 6. Continuous family of 3-partitions with the same energy.

This can be explained by a double covering argument, which is analogous to the argument of isospectrality of Jakobson-Levitin-Nadirashvili-Polterovich [JNLP] and Levitin-Parnovski-Polterovich [LPP], see also old papers by Bérard [Be], Sunada [Su] and the more recent paper by O. Parzanchevski and R. Band [PB]. We refer to [BHHO] for this discussion and to [BH] from which we took the pictures. We will come back to this isospectrality by using another point of view in Section 6.

5.4. Minimal 5-partitions

Using the covering approach, we were able [BH] to produce the following candidate for a minimal 5-partition of a specific topological type.

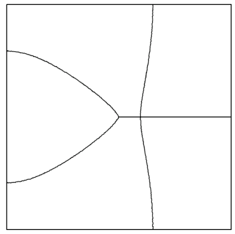

FIGURE 7. First candidate for the 5-partition of the square.

It is interesting to compare with other possible topological types of minimal 5-partitions. They can be classified by using the Euler formula (see [HH2]). Inspired by numerical computations in [CyBaHo], one looks for a configuration which has the symmetries of the square and four singular points. We get two types of models that we can reduce to a Dirichlet-Neumann problem on a triangle corresponding to the eighth of the square. Moving the Neumann boundary on one side like in [BHV] leads to two other candidates. One has a lower energy for one of the pictures which coincides with the picture given in [CyBaHo].

$\lambda_{11}(\dot{\Omega}_X^{\mathcal{R}}) = 111.910$ $\lambda_2^{DN} = 104.294$ $\lambda_2^{DN} = 131.666$

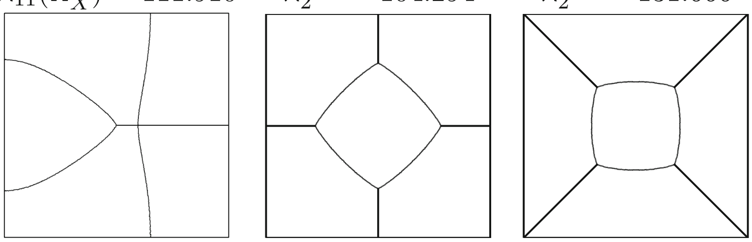

FIGURE 8. Three candidates for the 5-partition of the square.

Note that in the case of the disk a similar analysis leads to a different answer. The partition of the disk by five halfrays with equal angle has a lower energy that the minimal 5-partition with four singular points.

104.367 110.832

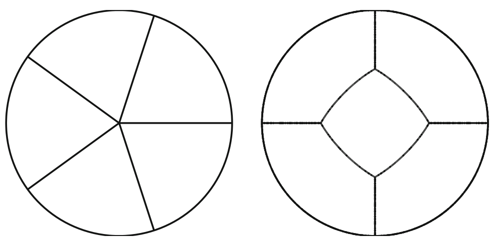

FIGURE 9. Two candidates for the 5-partition of the disk.

6. An Aharonov-Bohm approach

Here is an alternative (but equivalent) approach to the double covering approach. One considers the Aharonov-Bohm Laplacian in the square minus its center $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized flux $\frac{1}{2}$. We can also puncture with any point of the square (see [BH, NT]).

The magnetic potential takes the form

$$\mathbf{A}(x,y) = (A_1, A_2) = \alpha \left(-\frac{y}{r^2}, \frac{x}{r^2}\right) . \tag{16}$$

We know that the magnetic field vanishes in $\dot{\Omega}$ and in any cutted domain (such that it becomes simply connected) one has

$$\mathbf{A}(x,y) = \alpha\, d\theta\,, \tag{17}$$

where

$$z = x + iy = r \exp i\theta\ . \tag{18}$$

Then the flux condition reads

$$\alpha = \frac{1}{2}\ . \tag{19}$$

So the Aharonov-Bohm operator in any open set $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ is the Friedrichs extension starting from $C_0^\infty(\Omega)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}} := (D_x - A_1)^2 + (D_y - A_2)^2\ . \tag{20}$$

In the case of the square, the operator commutes with the $\frac{\pi}{2}$ rotation. In the case of rectangles, it commutes with the symmetries with respect to the main axes but these symmetries should be quantized by antilinear operators,

$$\Sigma_1 u(x,y) = i\ \overline{u(-x,y)}\ .$$

and

$$\Sigma_2 u(x,y) = \overline{u(x,-y)}\ .$$

This operator is preserving "real" functions in the following sense. We say (cf Helffer–M. and T. Hoffmann-Ostenhof–Owen [HHOO]) that a function u is K-real, if it satisfies

$$Ku = u\,, \tag{21}$$

where K is an anti-linear operator in the form

$$K = \exp i\theta\, \Gamma\ , \tag{22}$$

where

$$\Gamma u = \bar{u}\ . \tag{23}$$

The fact that $(-\Delta_{\mathbf{A}})$ preserves K-real eigenfunctions is an immediate consequence of

$$K \circ (-\Delta_{\mathbf{A}}) = (-\Delta_{\mathbf{A}}) \circ K\ . \tag{24}$$

It is easy to find a basis of K-real eigenfunctions. These eigenfunctions (which can be identified to real antisymmetric eigenfunctions of the Laplacian on the double covering of the punctured square) have a nice nodal structure,

- which is locally the same inside the punctured square as the nodal set of real eigenfunctions of the Laplacian,
- with the specific property that the number of lines arriving at the origin should be odd.

More generally a path of index one around the origin should always meet an odd number of nodal lines. One can also show that the multiplicity of any eigenvalue is at least 2.

Proposition 6.1. *The following problems are isospectral:*

- *The Dirichlet problem for the Aharonov-Bohm operator on the punctured square.*
- *The Dirichlet-Neumann problem on the upper-half square.*
- *The Dirichlet-Neumann problem on the left-half square.*
- *The Dirichlet-Neumann problem on the upper diagonal-half square.*

This proposition is proved in [BHHO].

Remarks 6.2.

- *The guess for the punctured square (at the center) is that any nodal partition of a third K-real eigenfunction gives a minimal* 3*-partition.*
- *Numerics performed with Virginie Bonnaillie shows that this is only true if the square is punctured at the center (see* [BH] *for a systematic study). Moreover the third eigenvalue is maximal there and has multiplicity two.*

All the results or observations around the square and the rectangle arise from discussions or common work with V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Verzini or G. Vial [HHOT1, BHV, BHHO].

7. The problem for k large.

We mention two conjectures. The first one is that

Conjecture 7.1. *The limit of $\mathfrak{L}_k(\Omega)/k$ as $k \to +\infty$ exists.*

The second one is that this limit is more explicitly given by

Conjecture 7.2.

$$|\Omega| \lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda_1(Hexa_1)\,.$$

This last conjecture, that we learn from M. Van den Berg, says in particular that the limit is independent of Ω if Ω is a regular domain.

Of course the optimality of the regular hexagonal tiling appears in various contexts in Physics. It is easy to show the upper bound in Conjecture 7.2 and the Faber-Krahn inequality gives a weaker lower bound involving the first eigenvalue of the Laplacian on the unit disk. But we have at the moment no idea of any approach for proving this in our context. We have explored in [BHV] numerically why this conjecture looks numerically reasonable. We have indeed controlled many non trivial consequences of this conjecture. Very roughly, if we start from a minimal k-partition $\mathcal{D}$ and look at its boundary set $N(\mathcal{D})$, one first observes that the set of the regular points $N^{reg}(\mathcal{D})$ of $N(\mathcal{D})$ introduced in (7) is a finite union of arcs γ_j ($j \in \mathcal{J}$), which are either closed or joining two points of $N^{sing}(\mathcal{D})$. We now observe, as a consequence of the minimality of the partition and of Theorem 3.3, that if we consider any connected Ω^{red} obtained by substracting from Ω a subset of (closures of) γ_j such that $\mathcal{D}$ becomes bipartite in Ω^{red}, then $\mathcal{D}$ should be the nodal partition of an eigenfunction associated with the k-th eigenvalue of the Dirichlet Laplacian in Ω^{red}.

Other recent numerical computations devoted to $\lim_{k\to+\infty} \frac{1}{k}\mathfrak{L}_{k,1}(\Omega)$ and to the asymptotic structure of the minimal partitions by Bourdin-Bucur-Oudet [BBO] are very enlightening. The conjecture is also mentioned in Caffarelli-Lin [CL].

8. The problem on the sphere

Let us mention one interesting conjecture on $\mathbb{S}^2$ due to [Bis] and a quite recent theorem of Helffer–Hoffmann-Ostenhof–Terracini [HHOT2]. We parametrize $\mathbb{S}^2$ by the spherical coordinates $(\theta,\phi) \in [0,\pi]\times[-\pi,\pi]$ with $\theta = 0$ corresponding to the north pole, $\theta = \frac{\pi}{2}$ corresponding to the equator and $\theta = \pi$ corresponding to the south pole.

8.1. Minimal 3-partitions

There is a particular 3-partition of $\mathbb{S}^2$, which will be called the Y-partition and corresponds to cutting $\mathbb{S}^2$ by the half-hyperplanes $\phi = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$. The conjecture due to C. Bishop-Friedland-Hayman [Bis, FrHa] is:

Conjecture 8.1. *The Y-partition gives a minimal 3-partition for $\mathbb{S}^2$ when minimizing $\frac{1}{3}\sum_{j=1}^{3}\lambda(D_j)$ over all the 3-partitions of S^2 .*

Actually one can have the same conjecture for $\max_j \lambda(D_j)$. This conjecture is actually a consequence of the first conjecture but could be easier to prove directly. This is indeed the case (Helffer-T.Hoffmann-Ostenhof-Terracini [HHOT2]).

Theorem 8.2. *The Y-partition gives a minimal 3-partition for $\mathbb{S}^2$ when minimizing $\max_j \lambda(D_j)$ over all the 3-partitions of S^2 .*

The techniques developed in the previous sections give some insight on the second conjecture which has some similarity with the Mercedes star conjecture. One can first observe that a minimal 3-partition of the sphere cannot be bipartite. What makes the analysis easier is then that the Euler formula shows that there exists only one topological type for the minimal 3-partitions of the sphere, corresponding to two singular points x_1 and x_2 with $\nu(x_1) = \nu(x_2) = 3$. A specific role is played by the proof that these two points are two antipodal points. This involves Lyusternik-Shnirelman's theorem. Then we can consider a double covering of the double-punctured sphere and show that the minimal 3-partition is actually the projection by the covering map of an antisymmetric eigenfunction on this double covering which was explicitly known from the beginning of quantum mechanics.

8.2. Minimal 4-partitions

We have seen that for the disk the minimal 4-partition for $\max_j \lambda(D_j)$ consists simply in the complement in the disk of the two perpendicular axes.

One could think that a minimal 4-partition of $\mathbb{S}^2$ could be what is obtained by cutting $\mathbb{S}^2$

- either by the two planes $\phi = 0$ and $\theta = \frac{\pi}{2}$
- or by the two planes $\phi = 0$ and $\phi = \frac{\pi}{2}$.

This is actually excluded by observing that a minimal 4-partition on $\mathbb{S}^2$ cannot be a nodal partition, as a direct consequence of the fact that the second eigenvalue for the sphere has multiplicity 3. Hence, we have probably to look to the spherical tetrahedron which has a relatively low energy (see [HHOT2]).

8.3. Minimal k-partitions for k large

As in Section 7, it is natural to conjecture that:

Conjecture 8.3. *"Hexagonal" Conjecture on* $\mathbb{S}^2$

$$\lim_{k\to+\infty} \frac{\mathfrak{L}_k(\mathbb{S}^2)}{k} = \lim_{k\to+\infty} \frac{\mathfrak{L}_{k,1}(\mathbb{S}^2)}{k} = \frac{1}{Area(\mathbb{S}^2)}\lambda(Hexa^1)\,.$$

The first equality in the conjecture corresponds to the idea, which is well illustrated in the recent paper by Bourdin-Bucur-Oudet [BBO] that, asymptotically as $k \to +\infty$, a minimal k-partition for Λ_p corresponds to D_j's such that the $\lambda(D_j)$ are equal.

9. Other open problems

We conclude this survey by mentioning additional questions.

1. What can we say in the case of the torus $\mathbb{T}^2$?
2. What can we say in higher dimension? The Courant-sharp characterization of the minimal k-partitions has been obtained recently for the $3D$-case [HHOT3]. But as far as we know, it is unproved in this case that a bipartite minimal k-partition is a nodal partition (see Theorem 3.2 for the $2D$-case). One of the basic problems in higher dimension is that the description of the properties of regularity of the minimal partitions is much more difficult and still unachieved.
3. For which Ω (with given area) is $\mathfrak{L}_k(\Omega)$ minimal.
4. Can we characterize the sets Ω such that for a given k:

$$\mathfrak{L}_{k,1}(\Omega) = \mathfrak{L}_{k,\infty}(\Omega)\,.$$

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