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On Schrödinger-Poisson Systems

Antonio Ambrosetti

Abstract. We discuss some recent results dealing with the existence of bound states of the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi(x)u = |u|^{p-1}u, \\ -\Delta\phi = K(x)u^2, \end{cases}$$

as well as of the corresponding semiclassical limits. The proofs are based upon Critical Point theory and Perturbation Methods.

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1. Introduction

In the last years there has been a great deal of work dealing with equations arising in Quantum Mechanics studied by means of variational tools. In this paper we will focus on the following class of systems on $\mathbb{R}^3$, see [5],

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi(x)u = |u|^{p-1}u, \\ -\Delta\phi = K(x)u^2. \end{cases} \tag{SP}$$

In (SP) the first equation is a nonlinear Schrödinger equation in which the potential ϕ satisfies a nonlinear Poisson equation. For this reason, (SP) is refereed to as a nonlinear Schrödinger-Poisson system.

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Here and in the sequel $1 < p < 5$, $\lambda > 0$ and we will assume that the following conditions hold

$$\begin{cases} V, K \in L^\infty(\mathbb{R}^3), \\ \inf_{\mathbb{R}^3} V(x) > 0, \\ K(x) \geq 0. \end{cases} \tag{A}$$

The solutions (u, ϕ) will be searched in $W^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

If $K \equiv 0$, (SP) becomes the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = |u|^{p-1}u, \tag{NLS}$$

which has been broadly investigated, see e.g. the monograph [1] and references therein. A natural strategy is to see which one, among the results obtained on (NLS), can be extended to (SP).

First, in Section 3, we carry out this program by reporting some recent results from [16] and [4], dealing with the existence of multiple solutions of (SP) in the autonomous case, namely when, say, $V(x) \equiv K(x) \equiv 1$. We shall see that for $2 < p < 5$, (SP) possesses infinitely many pairs of radial solutions, for all $\lambda > 0$. On the other hand, for $1 < p \leq 2$ the presence of the Poisson equation modifies greatly the structure of the solution set of (NLS) and (SP) has multiple solutions (but not infinitely many) for small values of $\lambda > 0$, only.

The proofs are based on critical point theory, though not in a standard manner. The main new difficulty which has to be overcome relies on the fact that for $2 \leq p < 3$ the boundedness of the Palais-Smale sequences cannot be proved directly, but requires an indirect approach, by means of a suitable approximation procedure.

Section 4 deals with the semiclassical counterpart of (3.1), namely

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + K(x)\phi(x)u = |u|^{p-1}u, \\ -\varepsilon\Delta\phi = K(x)u^2. \end{cases}$$

The existence of spike-like solutions will be discussed, following the recent paper [11]. The results extend the ones obtained for (NLS). Moreover, when the potentials are radial, a general theorem dealing with semiclassical states concentrating at a sphere can be proved. As a particular case, we find both the results in [2] dealing with (NLS), i.e. when $K \equiv 0$, as well as the case of Schrödinger-Poisson systems with $V \equiv K \equiv 1$, considered in [9, 17]. The approach is based upon perturbation methods in critical point theory.

Finally, in the last Section 5, we will prove some new results dealing with the system

$$\begin{cases} -\Delta u + u + \varepsilon K(x)\phi(x)u = (1+\varepsilon h(x))|u|^{p-1}u, \\ -\Delta\phi = K(x)u^2, \end{cases}$$

where ε is sufficiently small.

In the next Section we discuss the variational setting of (SP).

2. The variational setting

Hereafter, the Sobolev space $E := W^{1,2}(\mathbb{R}^3)$ is endowed with the standard norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx,$$

and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect the Dirichlet norm

$$\|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

It is easy to reduce (SP) to a single equation with a non-local term. Actually, since K is bounded and $u \in L^q(\mathbb{R}^3)$ for all $q \in [2,6]$ then $Ku^2 \in L^{6/5}(\mathbb{R}^3)$, for all $u \in E$, and there holds (hereafter c, c_1, c_2 etc. denote positive constants)

$$\begin{aligned} \int_{\mathbb{R}^3} Ku^2 v\, dx &\leq \left(\int_{\mathbb{R}^3} (Ku^2)^{6/5} dx\right)^{5/6} \left(\int_{\mathbb{R}^3} |v|^6\, dx\right)^{1/6} \\ &\leq c\left(\int_{\mathbb{R}^3} (Ku^2)^{6/5} dx\right)^{5/6} \|v\|_{\mathcal{D}}, \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{aligned}$$

Hence there exists a unique $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla v\, dx = \int_{\mathbb{R}^3} Ku^2 v\, dx, \qquad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^3). \tag{2.1}$$

It follows that ϕ satisfies the Poisson equation

$$-\Delta\phi = K(x)u^2$$

and there holds

$$\phi(x) = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy = \frac{1}{|x|} * Ku^2.$$

Moreover, $\phi \geq 0$ because K does and one has

$$\|\phi\|_{\mathcal{D}} \leq c_1 \|u\|^2. \tag{2.2}$$

Substituting ϕ in (SP), we are lead to the equation

$$-\Delta u + V(x)u + \lambda K(x)\left(\frac{1}{|x|} * Ku^2\right) u = |u|^{p-1}u, \qquad u \in W^{1,2}(\mathbb{R}^3). \tag{2.3}$$

Remark 2.1. This equation with V and K positive constants arises, for example, when one deals with the Hartree-Fock equation and makes a local approximation of the exchange potential

$$\sum u_i(x) \int_{\mathbb{R}^3} \frac{\overline{u_i} u_j}{|x-y|} dy \approx Q|u|^{p-1}u.$$

In particular, the value $p = 5/3$ corresponds to the so called "Slater correction", which is frequently used in the Quantum Mechanics. See e.g. [6, 14, 18].

We consider the functional $I_\lambda : E \mapsto \mathbb{R}$ given by

$$I_\lambda(u) = \tfrac{1}{2}\|u\|^2 + \lambda F(u) - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx$$

where

$$F(u) = \tfrac{1}{4}\int_{\mathbb{R}^3} K(x)\phi(x)u^2(x)dx, \qquad \phi(x) = \frac{1}{|x|} * Ku^2.$$

Remark 2.2. Since K and ϕ are non-negative, then $F(u) \geq 0$. Moreover, the Hölder inequality, the Sobolev inequality $\|\phi_\xi\|_{L^6} \leq c_1\|\phi_\xi\|_{\mathcal{D}}$ and (2.2) imply

$$F(u) \leq c_2\|u\|^2_{L^{6/5}}\|\phi\|_{L^6} \leq c_3\|u\|^4.$$

Therefore, I_λ is a well defined C^1 functional and if $u \in W^{1,2}(\mathbb{R}^3)$ is a critical point of it, then the pair (u, ϕ), with $\phi = \frac{1}{|x|} * Ku^2$, is a classical solution of (3.1).

3. The Autonomous Case

In this section we consider the system (3.1) when V and K are positive constants, say $V \equiv K \equiv 1$. In such a case, we can look for radial solutions working in the subspace E_r of radial functions of E.

Let us start by studying the geometry of I_λ. First of all, from the fact that $F(u) \geq 0$ (actually, $F(u) > 0$ in the present case) it follows immediately that $u = 0$ is a strict local minimum of I_λ, $\forall\, p \in (1,5)$, $\forall\, \lambda \geq 0$. Next, let us consider the behavior as $\|u\| \to \infty$. Roughly, in the functional I_λ there are two competing parts: the coercive functional $F(u)$ and the anti-coercive functional $\int |u|^{p+1}$. As a consequence it follows that

- $1 < p < 2 \implies \inf I_\lambda > -\infty$
- $3 < p < 5 \implies \inf I_\lambda = -\infty.$

In order to handle the case in which $2 < p \leq 3$ let us consider the curve $t \mapsto v_t = t^2 u(t\,\cdot)$. Then

$$\begin{aligned} F(v_t) &= \tfrac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v_t^2(x)v_t^2(y)}{|x-y|}dxdy = \tfrac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{t^8 u^2(tx)u^2(ty)}{|x-y|}dxdy \\ &= \tfrac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{t^3 u^2(x')u^2(y')}{|x'-y'|}dx'dy' = t^3 F(u), \end{aligned}$$

whence

$$I_\lambda(v_t) = \frac{t^3}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{t}{2}\int_{\mathbb{R}^3} u^2 + \lambda t^3 F(u) - \frac{t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

It follows that

- $2 < p \leq 3 \implies \inf I_\lambda = -\infty.$

If $p = 2$, we take the curve $t \mapsto av_t = a\,t^2 u(t\,\cdot)$. Since $F(v_t) = \frac{1}{4}\, a^4\, t^3 F(u)$, then

$$I_\lambda(av_t) = \frac{a^2t^3}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{a^2 t}{2}\int_{\mathbb{R}^3} u^2 + \lambda a^4 t^3 F(u) - \frac{a^3t^3}{3}\int_{\mathbb{R}^3} |u|^3.$$

Hence, taking $a \gg 1$ and $\lambda \sim 0$ we get

- $p = 2$ and $\lambda \sim 0 \implies \inf I_\lambda = -\infty$

Moreover, one could also show that

- $p = 2$ and $\lambda \geq \frac{1}{4} \implies \inf I_\lambda = 0.$

Let us now investigate the validity of the Palais-Smale condition (PS). Since we are working in the radial space E_r, (PS) holds provided the (PS) sequences are bounded. This is easily verified if $p \in (1,2) \cup [3,5)$, while it is not known if $2 \leq p < 3$.

To overcome this problem, we extend an argument used in specific cases in [19] and [13], see also [20].

Consider a Hilbert space E and $\Phi_\mu : E \to \mathbb{R}$ a family of functionals in the form

$$\Phi_\mu(u) = \alpha(u) - \mu\beta(u), \qquad \mu > 0$$

where $\alpha \in C^1$ is coercive (that is, $\lim_{\|u\|\to+\infty} \alpha(u) = +\infty$), $\beta \in C^1$, $\beta(u) \geq 0$, and β, β' map bounded sets into bounded sets.

Let $\mathcal{F}$ be a class of compact sets in E such that

- ($\mathcal{F}$.1). $\exists\, K \subset E$ s.t. $K \subset A\ \forall\, A \in \mathcal{F}$ and $\sup_K \Phi_\mu(u) < c_\mu$,
- ($\mathcal{F}$.2). For any homotopy $\eta(t,x)$ such that $\eta(t,x) = x$ for all $x \in K$, there holds $\eta(1,A) \in \mathcal{F}$, $\quad \forall\, A \in \mathcal{F}$.

Setting

$$c_\mu := \inf_{A\in\mathcal{F}} \max_{u\in A} \Phi_\mu(u),$$

it is easy to see that the map $\mu \mapsto c_\mu$ is non-increasing and left-continuous. Hence $\mu \mapsto c_\mu$ is almost everywhere differentiable. Let $\mathcal{J} \subset (0,+\infty)$ denote the set of values of μ so that c_μ is differentiable. The key point is the following result.

Lemma 3.1. *For any $\mu \in \mathcal{J}$ there exists a* <u>*bounded*</u> *(PS) sequence at the level c_μ, that is, there exists a* <u>*bounded*</u> *sequence $u_n \in E$ such that:*

$$\Phi_\mu(u_n) \to c_\mu,\ \ \Phi'_\mu(u_n) \to 0.$$

Let us apply the preceding procedure in the case $2 < p < 3$, taking $E = W^{1,2}_r(\mathbb{R}^3)$

$$\alpha(u) = \alpha_\lambda(u) = \tfrac{1}{2}\|u\|^2 + \lambda F(u), \qquad \beta(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx$$

and looking for the critical points of $I_{\lambda,\mu} = \alpha_\lambda(u) - \mu\beta(u)$.

Let B the unit ball in E, $S = \partial B$. Since $2 < p < 3$, from

$$I_{\lambda,\mu}(t^2u_t) = \frac{t^3}{2}\int_{\mathbb{R}^3} |\nabla u|^2dx + \frac{t}{2}\int_{\mathbb{R}^3} u^2dx + \frac{\lambda\, t^3}{4}F(u) - \frac{\mu\, t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx.$$

it follows that $\forall\, z \in S\ \exists!\ T = T(z) > 0$ such that for $\mu = \frac{1}{2}$ there holds

$$I_{\lambda,\frac{1}{2}}(T^2z_T) = 0,$$
$$I_{\lambda,\frac{1}{2}}(t^2z_t) < 0,\ \forall\, t > T(z),$$
$$I_{\lambda,\frac{1}{2}}(t^2z_t) > 0,\ \forall\, t < T(z).$$

For $k \in \mathbb{N}$, let E_k be a k-dimensional subspaces such that $E_k \subset E_{k+1}$, set $S_k = S \cap E_k$ and

$$V_k = \{v \in E : v = t^2 z_t,\ t \geq 0,\ z \in S_k\} \approx E_k.$$

From the preceding considerations $T_k := \sup\{T(z) : z \in S_k\} < +\infty$ and thus the set

$$A_k = \{v \in E : v = t^2 z_t,\ t \in [0, T_k],\ z \in S_k\}$$

is compact (and symmetric). Furthermore, $T_k \geq T(z)$, $\forall\, z \in S_k$ implies

$$I_{\lambda,\frac{1}{2}}(v) \leq 0, \qquad \forall\, v \in \partial A_k.$$

Next, let us set

$$H = \{g : E \mapsto E \text{ odd homeomorphism s.t. } g(v) = v,\ \forall\, v \in \partial A_k,\ \forall\, k \in \mathbb{N}\}$$

consider

$$G_k = \{g(A_k) : g \in H\}$$

and define

$$c_{k,\mu} = \inf_{A \in G_k} \max\{I_{\lambda,\mu}(u) : u \in A\}.$$

The class G_k can be taken as $\mathcal{F}$, with $K = \partial A_k$. Actually,

$$I_{\lambda,\mu}(v) \leq I_{\lambda,\frac{1}{2}}(v) \leq 0,\ \forall\, \mu \in [\tfrac{1}{2}, 1], \forall\, v \in \partial A_k.$$

Then Lemma 3.1 implies that for almost every $\mu \in [\frac{1}{2}, 1]$ and all $k \in \mathbb{N}$, there is a bounded (PS) sequence for $I_{\lambda,\mu}$ at level $c_{k,\mu}$ and this implies that $c_{k,\mu}$ are critical levels of $I_{\lambda,\mu}$.

In order to find the Critical points of I_λ we take $\mu_n \uparrow 1$ and choose $u_n \in E$ such that $I_{\lambda,\mu_n}(u_n) = c_{k,\mu_n}$, $I'_{\lambda,\mu_n}(u_n) = 0$, namely

$$\int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}u_n^2 + \frac{\lambda}{4}\phi_{u_n} u_n^2 - \frac{\mu_n}{p+1}|u_n|^{p+1}\right] dx = c_{k,\mu_n} \tag{3.1}$$

and

$$\int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + u_n^2 + \lambda\phi_{u_n} u_n^2 - \mu_n|u_n|^{p+1}\right] dx = 0. \tag{3.2}$$

Moreover, we can take advantage from the fact that u_n is a sequence of solutions of $I'_{\lambda,\mu_n}(u_n) = 0$. Actually, it has been shown in [8] that these u_n satisfy the following Pohozaev type identity:

$$\int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u_n|^2 + \frac{3}{2}u_n^2 + \frac{5\lambda}{4}\phi_{u_n} u_n^2 - \frac{3\mu_n}{p+1}|u_n|^{p+1}\right] dx = 0. \tag{3.3}$$

Setting

$$A_n = \int_{\mathbb{R}^3} |\nabla u_n|^2,\ B_n = \int_{\mathbb{R}^3} u_n^2,\ C_n = \lambda \int_{\mathbb{R}^3} \phi_{u_n} u_n^2,\ D_n = \mu_n \int_{\mathbb{R}^3} |u_n|^{p+1},$$

equations (3.1), (3.2), (3.3) become

$$\begin{cases} \frac{1}{2}A_n & + & \frac{1}{2}B_n & + & \frac{1}{4}C_n & - & \frac{1}{p+1}D_n & = & c_{k,\mu_n}, \\ A_n & + & B_n & + & C_n & - & D_n & = & 0, \\ \frac{1}{2}A_n & + & \frac{3}{2}B_n & + & \frac{5}{4}C_n & - & \frac{3}{p+1}D_n & = & 0. \end{cases}$$

Solving the above system, we get that

$$B_n = \frac{(4-2p)A_n + c_{k,\mu_n}(5p-7)}{p-1}.$$

Since c_{k,μ_n} is bounded $p > 2$ and $B_n > 0$ then $A_n \le c$. Thus also $B_n \le c$ and hence $\|u_n\|^2 = A_n + B_n$ is bounded.

Since we are working in the space of radial functions, we can now conclude as usual: $u_n \to u$, $c_{k,\mu_n} \to c_k (= c_{k,1})$ and

$$I_\lambda(u) = c_k, \qquad I'_\lambda(u) = 0.$$

Therefore, for every $k \in \mathbb{N}$ we have found a critical point u_k of I_λ.

The final step consists in showing that $c_{\lambda,k} := I_\lambda(u_k) \to \infty$ (let us point out that we cannot carry out a "Lusternik-Schnirelman" type argument). For this we shall use a comparison argument. From

$$I_\lambda(u) \ge I_0(u),\ \forall\, u \in E,$$

it follows that $c_{\lambda,k} \ge c_{0,k}$ for all $\lambda \ge 0$. Moreover, the arguments carried out in [3] show that $c_{0,k} \to \infty$ and hence $c_{\lambda,k} \to \infty$.

The preceding arguments have been outlined for $2 < p < 3$. The case $p \in [3,5)$ can be handled in a more direct manner, by means of symmetric versions of the mountain-pass theorem. When $1 < p < 2$ a different approach is needed, to take into account the geometry of the corresponding functionals. Finally, in the case $p = 2$ the geometry of I_λ is as for $1 < p < 2$ while the lack of (PS) has to be overcome as for $2 < p < 3$.

In conclusion we can state the following multiplicity results, proved in [4], that extend previous theorems in [16].

Theorem 3.2. *For any $2 < p < 5$ and any $\lambda > 0$, I_λ has infinitely many pairs of critical points $\pm u_k$, $k \in \mathbb{N}$, such that $I_\lambda(\pm u_k) \to +\infty$, as $k \to \infty$.*

Theorem 3.3. *If $1 < p < 2$ then for any $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that for all $\lambda \in (0, \Lambda_k)$, I_λ has at least k pairs of critical points $\pm u_{k,\lambda}$ such that $I_\lambda(\pm u_{k,\lambda}) > 0$ and k pairs of critical points $\pm v_{k,\lambda}$ such that $I_\lambda(\pm v_{k,\lambda}) < 0$. Moreover one has that $\Lambda_k \le \Lambda_{k-1} \le \cdots \le \Lambda_1 < \frac{1}{4}$.*

Theorem 3.4. *If $p = 2$, for any $k \in \mathbb{N}$ there exists $\Lambda'_k > 0$ such that for all $\lambda \in (0, \Lambda'_k)$, I_λ has at least k pairs of critical points $\pm u_{k,\lambda}$ such that $I_\lambda(\pm u_{k,\lambda}) > 0$. Moreover, $\Lambda'_k \le \Lambda'_{k-1} \le \cdots \le \Lambda'_1 < \frac{1}{4}$.*

Remark 3.5. The preceding existence results are completed by pointing out that for $1 < p \le 2$ and $\lambda \ge \frac{1}{4}$ the only critical point of I_λ is $u = 0$.

Remark 3.6. It is clear that the same results hold if we consider (SP) with *radial* potentials V, K satisfying (A). Moreover, it is also possible to deal with radial potentials which decay to zero at infinity in a suitable manner, see [15].

Remark 3.7. If we use (3.2) and the Pohozaev identity (3.3), with $p = 5$ (critical exponent), $\mu_n \equiv 1$ and $u_n = u$, we get, respectively,

$$\int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2 + \lambda \phi_u u^2 - |u|^6 \right] dx = 0, \tag{3.4}$$

and

$$\int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u|^2 + \frac{3}{2}u^2 + \frac{5\lambda}{4}\phi_u u^2 - \frac{1}{2}|u|^6 \right] dx = 0. \tag{3.5}$$

From (3.4) we find

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = -\int_{\mathbb{R}^3} \left[u^2 + \lambda \phi_u u^2 - |u|^6 \right] dx.$$

Substituting in (3.5) equation we deduce

$$4 \int_{\mathbb{R}^3} u^2 dx = -3\lambda \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

Since $\lambda \ge 0$ and $\phi_u \ge 0$, it follows that $u = 0$. This shows that in the case of critical exponent, (SP) has no non-trivial solution, for all $\lambda \ge 0$, see [8].

4. Semiclassical states

An important feature of NLS equations as

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-1}u \tag{4.1}$$

is that there exist solutions concentrating at non-degenerate critical points of the potential V. This kind of results can be extended, without any restriction on K, to systems like

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)\phi u = |u|^{p-1}u, \\ -\Delta\phi = K(x)u^2. \end{cases} \tag{SP_ε}$$

Theorem 4.1. [11] *Let* (A) *holds and let* x_0 *be a non-degenerate critical point of* V. *Then* (SP_ε) *has for* ε *small, a solution* u_ε *such that*

$$u_\varepsilon(x) \sim U_\lambda\left(\frac{x-x_0}{\varepsilon}\right), \qquad \lambda^2 = V(x_0),$$

where U_λ *is the positive radial solution of* $-\Delta u + \lambda u = u^p,\ u \in W^{1,2}(\mathbb{R}^3)$.

On the other hand, the potential K plays a role when x_0 is degenerate. Suppose that $\exists$ an even integer k such that $D^jV(x_0) = 0$, $\forall\, j = 1, 2, ..., k-1$ and $D^kV(x_0)[\xi] = \sum a_i\xi_i^k$, with $a_i > 0$ or < 0 for all $i = 1, 2, 3$. Furthermore, suppose that $\exists$ an even integer m such that $D^jK(x_0) = 0$, $\forall\, j = 1, 2, ..., m-1$ and $D^mK(x_0)[\xi] = \sum b_i\xi_i^m$, with $b_i \geq 0$, $(b_1, b_2, b_3) \neq (0, 0, 0)$. Then the previous concentration result holds provided one makes suitable assumptions on b_i. Referring to [11] for complete results, we limit ourselves to some specific examples:

- if $k < 2m + 2$, for all $b_i \geq 0$;
- if $k = 2m + 2$ and $b_i \geq 0$ are small;
- if $k > 2m + 2$ and $(b_1, b_2, b_3) = (1, \delta, \delta)$ with δ small.

Remark 4.2. If $K \equiv 0$ we recover the known results on (4.1). Moreover, if $V \equiv Const. > 0$, concentration occurs, roughly, for almost every (b_1, b_2, b_3).

When V is radial, solutions of (4.1) which concentrate at a sphere of radius $r = R$ has been proved in [2]. It is shown that for every non-degenerate minimum or maximum of

$$M(r) = r^2V^\theta(r), \qquad \theta = \frac{p+3}{2(p-1)},$$

there exists a radial solution of (4.1) with asymptotic profile

$$u_\varepsilon(r) \sim U_\lambda\left(\frac{r-R}{\varepsilon}\right), \qquad \lambda^2 = V(R). \tag{4.2}$$

In order to extend this result to Schrödiger-Poisson systems, let us the system

$$\begin{cases} -\varepsilon^2\Delta u + V(|x|)u + K(|x|)\phi u = |u|^{p-1}u, \\ -\varepsilon\Delta\phi = K(|x|)u^2, \end{cases} \tag{SP'_ε}$$

where V, K satisfy (A).

Theorem 4.3. [10] *If R is a non-degenerate minimum or maximum of $M = r^2V^\theta$ and $K(R) = 0$, then (SP_ε) has a radial solution u_ε concentrating at the sphere of radius R, whose asymptotic profile is given by* (4.2)*.*

In particular, if $K \equiv 0$, we recover the result proved in [2].

If $V \equiv K \equiv 1$, solutions of (SP_ε) concentrating on the sphere of radius R,

$$R = c\,\frac{\overline{a}}{(1+\overline{a})^{\frac{5-p}{2(p-1)}}}, \quad \overline{a} = \frac{8(p-1)}{11-7p}, \quad c = \int_{\mathbb{R}^3} U^2 dx,$$

has been established in [9, 17], provided $1 < p < \frac{11}{7}$.

This turns out to be a specific case of the following result, proved in [10], see also [12] for the necessary conditions. Referring to the aforementioned papers for precise statements, let us give an idea of the results. First of all, the radius R of the concentration is the possible solution of the equation

$$a(r) = c\,rK(r)[V(r) + K(r)a(r)]^\gamma, \qquad \gamma = \frac{5-p}{2(p-1)}, \tag{$\star$}$$

where

$$a(r) = -\frac{rV'(r) + \nu V(r)}{rK'(r) + \kappa K(r)}, \quad \nu = \frac{4(p-1)}{p+3},\ \kappa = \frac{7p-11}{2(p+3)}.$$

Then, if some suitable non-degeneracy conditions are satisfied, then (SP'_ε) has a radial solution u_ε concentrating at $r = R$. Furthermore, the asymptotic profile of u_ε is given by

$$u_\varepsilon(r) \sim U_\lambda\left(\frac{r-R}{\varepsilon}\right), \qquad \lambda^2 = V(R) + a(R)K(R).$$

The proof of this result relies on a perturbation approach which requires the overcoming of several technical difficulties, and cannot be reported here. Instead, we discuss a couple of specific cases, to illustrate the nature of the results.

Example 1. If $V \equiv K \equiv 1$ the function $a(r)$ becomes the constant $\overline{a}$ given by

$$\overline{a} = -\frac{\nu}{\kappa} = \frac{8(p-1)}{11-7p} \qquad (> 0 \Leftrightarrow 1 < p < \frac{11}{7}).$$

Then the equation $(\star)$ has always a solution given, for $1 < p < \frac{11}{7}$, by

$$\overline{a} = c\, r[1+\overline{a}]^\gamma, \quad i.e. \quad R = \frac{1}{c}\, \frac{\overline{a}}{(1+\overline{a})^\gamma}.$$

Moreover, in this specific case, the non-degeneracy conditions are automatically satisfied and we recover exactly the result found in [9, 16]. Let us remark that the result sketched before is not confined to the case in which $V, K \equiv 1$, but it can be clearly extended, for example, to potentials $V, K \approx 1$. Actually, if this is the case then the function $a(r) \approx \overline{a}$, $(\star)$ has a solution close to the preceding R and the non-degeneracy conditions still hold true.

Example 2. The purpose of this example is to show that there could be cases in which there exist radial solutions of (SP'_ε) concentrating at a sphere, even if $\frac{11}{7} \le p < 5$.

Let us take $p \approx 5$ and $K \equiv 1$, then

$$a(r) \sim \frac{2}{3}(2V(r) + rV'(r))$$

and $(\star)$ becomes

$$2V(r) + rV'(r) = c\, r, \qquad c > 0.$$

If this equation has a solution $r = R$, the non-degeneracy conditions referred to in the preceding discussion become merely

$$V''(R) \neq \frac{3c}{R} + \frac{6V(R)}{R^2}.$$

If this condition holds, then there exists a radial solution of (SP'_ε) concentrating at $r = R$.

5. The Non-Autonomous Case: Perturbation Results

Here we consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \varepsilon K(x)\phi(x)u = (1+\varepsilon h(x))|u|^{p-1}u, \\ -\Delta \phi = K(x)u^2, \end{cases} \tag{SP_ε}$$

where $\varepsilon > 0$ is sufficiently small. We shall make the following assumptions on K and h:

$$K > 0, \qquad K \in L^2(\mathbb{R}^3), \tag{K}$$

$$h \in L^{6/(5-p)}(\mathbb{R}^3). \tag{h}$$

Remark 5.1. If K holds we still have that $Ku^2 \in L^{6/5}(\mathbb{R}^3)$ for all $u \in E$ and hence the discussion about the solvability of the Poisson equation $-\Delta\phi = Ku^2$ carried out in Section 2, as well as the variational setting, can be repeated without changes.

The corresponding Euler functional I_ε is given by

$$I_\varepsilon(u) = I_0(u) + \varepsilon F(u) - \varepsilon G(u),$$

where

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx$$

and

$$G(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} h(x)|u|^{p+1} dx.$$

For $\varepsilon = 0$ the unperturbed functional I_0 is translation invariant and has the following $3D$-manifold of critical points

$$Z = \{U_\xi(x) := U(x-\xi) : \xi \in \mathbb{R}^3\},$$

where U is the positive radially symmetric solution of

$$-\Delta u + u = u^p, \qquad u \in E_r.$$

It is well known that Z is non-degenerate, in the sense that the $Ker I_0''(z)$ coincides with the tangent space $T_z Z$, for all $z \in Z$, cfr. [1, Lemma 4.1]. Then, according to the abstract Theorem 2.16 of [1], to find a critical point of I_ε, for $\varepsilon \ll 1$, it suffices to find a strict maximum or minimum of

$$\Gamma(\xi) = F(U_\xi) - G(U_\xi).$$

In order to study the behavior of Γ as $|\xi| \to \infty$ we will use the integrability conditions on K and h.

Lemma 5.2. *Suppose that* (K) *and* (h) *hold. Then*

$$\lim_{|\xi|\to\infty} \Gamma(\xi) = 0.$$

Proof. Let ϕ_ξ be the solution of (2.1) with $u = U_\xi$. Then

$$\|\phi_\xi\|^2_{\mathcal{D}} = \int_{\mathbb{R}^3} K U_\xi^2 \phi_\xi dx.$$

Since $U(x) \sim e^{-|x|}$ as $|x| \to \infty$, and using the Hölder inequality,

$$\|\phi_\xi\|^2_{\mathcal{D}} \le \|\phi_\xi\|_{L^6} \left(\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx\right)^{5/6}.$$

This and $\|\phi_\xi\|_{L^6} \le c_1 \|\phi_\xi\|_{\mathcal{D}}$ yield

$$\|\phi_\xi\|_{L^6} \le c_1 \left(\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx \right)^{5/6}. \tag{5.1}$$

Then

$$F(U_\xi) = \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_\xi(x) U_\xi^2(x) dx \le c_2 \|\phi_\xi\|_{L^6} \left(\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx \right)^{5/6},$$

and (5.1) implies

$$F(U_\xi) \le c_3 \left(\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx \right)^{5/3}. \tag{5.2}$$

Next, let us show that $\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx \to 0$ as $|\xi| \to \infty$. For $R > 0$, let us first evaluate

$$\begin{aligned}
\int_{|x|>R} K^{6/5} U_\xi^{12/5} dx &\le \left(\int_{|x|>R} K^2 dx \right)^{3/5} \left(\int_{|x|>R} U_\xi^6 dx \right)^{2/5} \\
&\le \left(\int_{|x|>R} K^2 dx \right)^{3/5} \left(\int_{\mathbb{R}^3} U_\xi^6 dx \right)^{2/5} \\
&\le c_4 \left(\int_{|x|>R} K^2 dx \right)^{3/5}.
\end{aligned}$$

Using (K) we infer that, for any $\delta > 0$, there exists $\overline{R} > 0$ such that

$$\int_{|x|>\overline{R}} K^{6/5} U_\xi^{12/5} dx \le \delta. \tag{5.3}$$

Next, one has

$$\begin{aligned}
\int_{|x|<\overline{R}} K^{6/5} U_\xi^{12/5} dx &\le \left(\int_{|x|<\overline{R}} K^2 dx \right)^{3/5} \left(\int_{|x|<\overline{R}} U_\xi^6 dx \right)^{2/5} \\
&\le c_5 \left(\int_{|x+\xi|<\overline{R}} U^6 dx \right)^{2/5}.
\end{aligned}$$

Since U decays exponentially to zero as $|x| \to \infty$, it follows that

$$\lim_{|\xi|\to\infty} \int_{|x|<\overline{R}} K^{6/5} U_\xi^{12/5} dx = 0.$$

This and (5.3) imply that $\int_{\mathbb{R}^3} K^{6/5} U_\xi^{12/5} dx \to 0$ as $|\xi| \to \infty$, as claimed. Finally, since $F(u) > 0$, see Remark 2.2, and using (5.2), we infer that $F(U_\xi) \to 0$ as $|\xi| \to \infty$.

Similarly, using (h), we find that for all $\delta > 0$ there exists $\overline{R} > 0$ such that

$$\begin{aligned}
\int_{|x|>\overline{R}} h U_\xi^{p+1} dx &\leq \left(\int_{|x|>\overline{R}} h^{6/(5-p)} dx \right)^{(5-p)/6} \left(\int_{|x|>\overline{R}} U_\xi^6 dx \right)^{(p+1)/6} \\
&\leq c_6 \left(\int_{|x|>\overline{R}} h^{6/(5-p)} dx \right)^{(5-p)/6} < \delta.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_{|x|<\overline{R}} h U_\xi^{p+1} dx &\leq \left(\int_{|x|<\overline{R}} h^{6/(5-p)} dx \right)^{(5-p)/6} \left(\int_{|x+\xi|<\overline{R}} U^6 dx \right)^{(p+1)/6} \\
&\leq c_7 \left(\int_{|x+\xi|<\overline{R}} U^6 dx \right)^{(p+1)/6} = o(1), \quad \text{as } |\xi| \to \infty.
\end{aligned}$$

It follows that also $G(U_\xi) \to 0$ as $|\xi| \to \infty$ and hence $\Gamma(\xi) = F(U_\xi) + G(U_\xi) \to 0$ as $|\xi| \to \infty$ proving the lemma. □

Lemma 5.2 implies that Γ has a strict maximum or a strict minimum, unless $\Gamma(\xi) \equiv 0$. The simplest way to rule out this possibility, is to require that, say, $\Gamma(0) \neq 0$. Since

$$\begin{aligned}
\Gamma(0) &= F(U) - G(U) \\
&= \tfrac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x)K(y) \frac{U^2(x)U^2(y)}{|x-y|} dxdy - \tfrac{1}{p+1} \int_{\mathbb{R}^3} h(x) U^{p+1}(x) dx,
\end{aligned}$$

we get the following existence result

Theorem 5.3. *Let (K) and (h) hold and suppose that*

$$\tfrac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x)K(y) \frac{U^2(x)U^2(y)}{|x-y|} dxdy \neq \tfrac{1}{p+1} \int_{\mathbb{R}^3} h(x) U^{p+1}(x) dx. \tag{5.4}$$

Then for all ε sufficiently small, (SP_ε) has at least one solution.

Proof. By the general theory, see [1, Theorem 2.16], it follows that any strict maximum or minimum of Γ gives rise to a critical point of I_ε and hence to a solution of (SP_ε). □

Corollary 5.4. *Let* (K) *and* (h) *hold. If* $\int_{\mathbb{R}^3} h(x)U^{p+1}(x)dx \leq 0$, *then for all* ε *sufficiently small,* (SP_ε) *has at least one solution.*

Proof. It suffices to point out that $K > 0$ and $\int_{\mathbb{R}^3} h(x)U^{p+1}(x)dx \leq 0$ immediately imply that (5.4) holds. □

We conclude this section with a result in which we replace (5.4) with an asymptotic assumption on the sign of h. Precisely, we will assume that

$$\exists \rho > 0 : \quad h(x) < 0, \quad \forall\, |x| \geq \rho. \tag{h'}$$

Theorem 5.5. *Let* $3 < p < 5$, *let* (h) *and* (h') *hold and suppose that* K *is continuous and satisfies* (K). *Then for all* ε *sufficiently small,* (SP_ε) *has at least one solution.*

Proof. As before, we need to consider

$$\Gamma(\xi) = F(U_\xi) - G(U_\xi).$$

Let us evaluate $F(U_\xi)$ and $G(U_\xi)$ separately.

Since $K > 0$, we get for any $R > 0$,

$$\begin{aligned} F(U_\xi) &= \tfrac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x)K(y) \frac{U_\xi^2(x)U_\xi^2(y)}{|x-y|} dxdy \\ &\geq \tfrac{1}{4} \int_{|x|<R} \int_{|y|<R} K(x)K(y) \frac{U_\xi^2(x)U_\xi^2(y)}{|x-y|} dxdy \\ &\geq \tfrac{1}{4} \int_{|x|<R} \int_{|y|<R} \frac{K(x)K(y)}{|x-y|} dxdy \cdot \min_{|x|<R} U^4(x-\xi). \end{aligned}$$

Since K is continuous and positive, there exists $C_R > 0$ such that

$$F(U_\xi) \geq C_R \cdot \alpha(\xi), \tag{5.5}$$

where

$$\alpha(\xi) =: \min_{|x|<R} U^4(x-\xi) \sim e^{-4|\xi|}, \quad \text{as } |\xi| \to \infty.$$

Next, let $h^+(x) = \max\{h(x), 0\}$. Using (h') we find

$$\int_{\mathbb{R}^3} h(x)U_\xi^{p+1}(x)dx \leq \int_{|x|<\rho} h^+(x)U^{p+1}(x-\xi)dx \leq c_\rho \cdot \beta(\xi) \tag{5.6}$$

where $c_\rho \geq 0$ and

$$\beta(\xi) := \max_{|x|<\rho} U^{p+1}(x-\xi) \sim e^{-(p+1)|\xi|}, \quad \text{as } |\xi| \to \infty.$$

From (5.6) and (5.5) we infer

$$\Gamma(\xi) = F(U_\xi) - G(U_\xi) \geq C_R \cdot \alpha(\xi) - c_\rho \cdot \beta(\xi).$$

Moreover,

$$\alpha(\xi) - \beta(\xi) \sim e^{-(p-3)|\xi|}, \quad \text{as } |\xi| \to \infty.$$

Then, $p > 3$ and $C_R > 0$ imply that $\Gamma(\xi) > 0$ provided $|\xi| \gg 1$. This shows that $\Gamma(\xi) \not\equiv 0$ and the conclusion follows as before. □

References

[1] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^n$. Progress in Math. Vol. 240, Birkhäuser , 2006.

[2] A. Ambrosetti, A. Malchiodi and W.M. Ni, *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I.* Comm. Math. Phys. **235** (2003), 427–466.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical points theory and applications.* Jour. Funct. Anal. **14** (1973), 349–381.

[4] A. Ambrosetti and D. Ruiz, *Multiple solitons to nonlinear Schrödinger-Poisson systems.* Comm. Cont. Math. **10**-3 (2008), 391–404.

[5] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations.* Top. Math. Nonl. Anal. **11** (1998), 283–293.

[6] O. Bokanowski and N.J. Mauser, *Local approximation of the Hartree-Fock exchange potential: a deformation approach.* M^3AS **9** (1999), 941–961.

[7] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations.* Proc. Royal Soc. Edinburgh A **134** (2004), 893–906.

[8] T. D'Aprile and D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations.* Adv. in Nonl. Studies **4** (2004), 307–322.

[9] T. D'Aprile and J. Wei, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation.* SIAM J. Math. Anal. **4** (2004), 307–322.

[10] I. Ianni, *Solutions of the Schrödinger-Poisson system concentrating on spheres, part II: existence.* M^3AS, to appear.

[11] I. Ianni and G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials.* Adv. Nonlin. Studies, to appear.

[12] I. Ianni and G.Vaira, *Solutions of the Schrödinger-Poisson system concentrating on spheres, part I: necessary conditions.* M^3AS, to appear.

[13] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer type problem set on* $\mathbb{R}^N$. Proc. Roy. Soc. Edinburgh A **129** (1999), 787–809.

[14] N.J. Mauser, *The Schrödinger-Poisson-$X\alpha$ equation.* Applied Math. Letters **14** (2001), 759–763.

[15] C. Mercuri, *Positive solutions of non-linear Schrödinger-Poisson systems with radial potentials vanishing at infinity.* Rend. Acc. Naz. Lincei, to appear.

[16] D. Ruiz, *Semiclassical states for coupled Schrodinger-Maxwell equations: concentration around a sphere.* $\mathrm{M}^3\mathrm{AS}$ **15** (2005), 141–164.

[17] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term.* J. Funct. Anal. **237** (2006), 655–674.

[18] J.C. Slater, *A simplification of the Hartree-Fock method.* Phys. Review **81** (1951), 385–390.

[19] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces.* Comment. Math. Helvetici **60** (1985), 558–581.

[20] M. Struwe, *Variational Methods.* Ergeb. der Math. u. Grenzgeb. Vol. 34, Springer Verlag, Berlin, 1996.

Antonio Ambrosetti
SISSA
Via Beirut 2-4
Trieste 34014
Italy
e-mail: ambr@sissa.it

Lecture held on February 15, 2008, by A. Ambrosetti, recipient of the *Luigi and Wanda Amerio gold medal* awarded by the Istituto Lombardo Accademia di Scienze e Lettere.

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