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On the Multiplicity of Zeroes of Polynomials with Quaternionic Coefficients

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Abstract. Regular polynomials with quaternionic coefficients admit only isolated zeroes and spherical zeroes. In this paper we prove a factorization theorem for such polynomials. Specifically, we show that every regular polynomial can be written as a product of degree one binomials and special second degree polynomials with real coefficients. The degree one binomials are determined (but not uniquely) by the knowledge of the isolated zeroes of the original polynomial, while the second degree factors are uniquely determined by the spherical zeroes. We also show that the number of zeroes of a polynomial, counted with their multiplicity as defined in this paper, equals the degree of the polynomial. While some of these results are known in the general setting of an arbitrary division ring, our proofs are based on the theory of regular functions of a quaternionic variable, and as such they are elementary in nature and offer explicit constructions in the quaternionic setting.

1. Introduction

Let $\mathbb{H}$ denote the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_l are real, and i, j, k, are imaginary units (i.e. their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. Note that if we denote by $\mathbb{S}$ the 2-dimensional sphere

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of imaginary units of $\mathbb{H}$, i.e. $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$, then every non-real quaternion q can be written in a unique way as $q = x + yI$, with $I \in \mathbb{S}$ and $x, y \in \mathbb{R}, y > 0$. We will refer to $x = Re(q)$ as the real part of q and $y = Im(q)$ as the imaginary part of q. Quite recently, [2], [3], we developed a new theory of regularity for functions of a quaternionic variable. In those papers we begin the study of the structure of the zero-set of regular functions, and in particular of regular polynomials, i.e. polynomials of the form

$$f(q) = \sum_{i=0}^{n} q^i a_i$$

with $a_i \in \mathbb{H}$. We prove that these are the only polynomials to satisfy the regularity conditions, and therefore their behavior resembles very closely that of holomorphic functions of a complex variable. In what follows, we will simply say polynomials when referring to regular polynomials. Subsequent papers, [1], [4], deepened our understanding of the structure of such polynomials. To begin with, we recall that the product of two regular functions is not regular in general. So, for example, even the simple product $(q - \alpha)(q - \beta) = q^2 - \alpha q - q\beta + \alpha\beta$ is not regular when α is not real. Thus, in accordance with the theory of polynomials over skew-fields, one defines a different product (we will use the symbol $*$ to denote such a product) which guarantees that the product of regular functions is regular. For polynomials, for example, this product is defined as follows:

Definition 1.1. Let $f(q) = \sum_{i=0}^{n} q^i a_i$ and $g(q) = \sum_{j=0}^{m} q^j b_j$ be two polynomials. We define the regular product of f and g as the polynomial $f * g(q) = \sum_{k=0}^{mn} q^k c_k$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ for all k.

Remark 1.2. Note that this definition, see e.g. [5], has the effect that multiplication of polynomials is performed as if the coefficients were chosen in a commutative field; as a consequence, the resulting polynomial is still a regular polynomial with all the coefficients on the right of the powers.

To understand the flavor of this paper, we will begin by analyzing three simple examples which, however, contain all the features which differentiate the theory of polynomials in $\mathbb{H}$ from the standard theory of complex polynomials.

Example 1.3. Consider the polynomial $P_1(q) = (q-\alpha)*(q-\beta) = q^2 - q(\alpha + \beta) + \alpha\beta$, where α and β are non-real quaternions with $Re(\alpha) \neq Re(\beta)$ or $|Im(\alpha)| \neq |Im(\beta)|$. It is immediate to verify, by direct substitution, that α

is a solution of $P_1(q) = 0$, while β is not a root of the polynomial. In fact, one can prove (see Theorem 1.6 below), that P_1 has a second root given by $(\overline{\beta} - \alpha)\beta(\overline{\beta} - \alpha)^{-1}$. Thus, as one would expect, the polynomial has two roots (and in fact only two roots), though they are not what one would expect from a first look at the polynomial (this is a consequence of the fact that the valuation is not a homomorphism of rings).

Example 1.4. Consider the polynomial $P_2(q) = (q - \alpha) * (q - \overline{\alpha}) = q^2 - q(2Re(\alpha)) + |\alpha|^2$. In this case α is called a spherical root, see [3], and it is easy to verify that every point on the 2−sphere $S_\alpha = Re(\alpha) + Im(\alpha)\mathbb{S}$ is a root for P_2. More precisely we will say that α is a generator of the spherical root S_α.

Example 1.5. Consider now the polynomial $P_3(q) = (q - \alpha) * (q - \beta) = q^2 - q(\alpha + \beta) + \alpha\beta$, where α and β are non-real quaternions with $\beta \in S_\alpha$, and $\beta \neq \overline{\alpha}$. In this case, as it is shown in [1], the only root of the polynomial P_3 is α.

We note that these three examples exhibit a behavior that is distinctively different from the one we are used to in the complex case. To begin with, even when the polynomial is factored as a $*$ product of monomials, we cannot guarantee that each monomial contributes a zero. Even in the case of P_1, when in fact both monomials contribute a zero, the contribution of the second monomial depends explicitly from the first monomial. This is a direct consequence of Theorem 3.3 in [1], which we repeat here for the sake of completeness (but see also [5] for this same statement in the case of polynomials).

Theorem 1.6 (Zeroes of a regular product for power series). *Let f, g be given quaternionic power series with radii greater than R and let $p \in B(0, R)$. Then $f * g(p) = 0$ if and only if $f(p) = 0$ or $f(p) \neq 0$ and $g(f(p)^{-1}pf(p)) = 0$.*

The second fundamental difference, which was already clarified in Theorem 5.1 of [3], is the fact that some polynomials admit spherical zeroes, i.e. entire 2−spheres of the form $x + y\mathbb{S}$ for some real values x, y.

Finally, we come to the peculiarity described by Example 1.5. In this case, the polynomial P_3 is a polynomial of degree two, and therefore one would expect either two solutions, or at least one solution with multiplicity two. In some earlier works, [1], the multiplicity of a root α of a quaternionic polynomial $P(q)$ was defined (in analogy with what one does in the complex case) as the largest n for which $P(q)$ can be written as $P(q) = (q-\alpha)*(q-\alpha)*\cdots*(q-\alpha)*R(q) = (q-\alpha)^{*n}*R(q)$, for some polynomial $R(q)$. Under this definition, it is therefore clear that Example 1.5 shows that the degree of a polynomial can exceed the sum of the multiplicities of its roots, see [1]. This is not surprising since, as pointed out in [6], the problem of defining a good notion of multiplicity for zeroes of quaternionic polynomials is a rather complicated question.

In this paper, we tackle the problem of expressing a polynomial without spherical zeroes as a $*$ product of monomials, and we analyze the relationship between these monomials and the zeroes of the polynomial (the addition of spherical roots does not alter significantly the problem). More specifically, given a polynomial and its roots, we show how to construct a factorization in monomials; conversely we will also show how to find the roots of a polynomial if we have one of its factorizations (note that in the complex case, this is clearly a trivial problem: not so in the quaternionic case). These problems were only partially discussed in [1] and in [4]. In this paper, we provide an explicit algorithm that produces the required factorization. In the process, a new natural definition of multiplicity emerges, and we will be able to show that the sum of the multiplicities of the zeroes of a polynomial coincides with its degree. We note that some of these results (specifically the factorization for general polynomials) are a consequence of classical general results on division algebras due to Wedderburn (see [8]). Nevertheless, our approach is different and interesting because it is elementary in nature, and based on our new theory of regular functions. Moreover, the results we obtain are quite explicit, and at least in some cases (see for example Proposition 2.5) offer a new perspective on how to deal with the uniqueness of factorization, when such uniqueness is available.

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2. Factorization and multiplicity of roots

We begin this section by stating and proving our factorization theorem.

Theorem 2.1. *Let $P(q)$ be a regular polynomial of degree m. Then there exist $p, m_1, \dots, m_p \in \mathbb{N}$, and $w_1, \dots, w_p \in \mathbb{H}$, generators of the spherical roots of P, so that*

$$P(q) = (q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p} Q(q), \quad (2.1)$$

where $Re(w_i)$ denotes the real part of w_i and Q is a regular polynomial with coefficients in $\mathbb{H}$ having only non-spherical zeroes. Moreover, if $n = m - 2(m_1 + \cdots + m_p)$ there exist a constant $c \in \mathbb{H}$, t distinct $2-$spheres $S_1 = x_1 + y_1\mathbb{S}, \dots, S_t = x_t + y_t\mathbb{S}$, t integers $n_1, \dots, n_t$ with $n_1 + \cdots + n_t = n$, and (for any $i = 1, \dots, t$) n_i quaternions $\alpha_{ij} \in S_i$, $j = 1, \dots, n_i$, such that

$$Q(q) = [\Pi_{i=1}^{*t} \Pi_{j=1}^{*n_i} (q - \alpha_{ij})]c. \quad (2.2)$$

Proof. The first part of the theorem, namely the decomposition (2.1), is Theorem 3.4 in [4]. Thus we only have to prove the decomposition for the quaternionic polynomial Q which has no spherical roots. We can assume Q to be a monic polynomial. If not, there is a constant c, coefficient of the highest degree term, and the process below must be preceded by multiplication by c^{-1} and then followed by multiplication by c. By the fundamental theorem of algebra for quaternionic polynomials, see [4], if $deg(Q) = n > 0$, there is at least one root, say γ_1. Let us add the root $\overline{\gamma_1}$ to the polynomial $Q(q)$; this can be accomplished by a simple multiplication and we can now consider the new polynomial

$$\widetilde{Q}(q) = Q(q) * [q - Q(\overline{\gamma_1})^{-1} \overline{\gamma_1} Q(\overline{\gamma_1})].$$

The fact that $\widetilde{Q}(\overline{\gamma_1}) = 0$ is an immediate consequence of Theorem 1.6. We now note that $\widetilde{Q}(q)$ has a spherical zero on S_{γ_1} because it has two roots (γ_1 and $\overline{\gamma_1}$) on that sphere (see [3]). Thus, by Theorem 3.4 in [4], one can factor a spherical root

$$q^2 - 2qRe(\gamma_1) + |\gamma_1|^2,$$

and therefore obtain a new polynomial

$$Q_{11}(q) = Q(q) * \frac{q - Q(\overline{\gamma_1})^{-1} \overline{\gamma_1} Q(\overline{\gamma_1})}{q^2 - 2qRe(\gamma_1) + |\gamma_1|^2}.$$

We set $\delta_{11} = Q(\overline{\gamma_1})^{-1} \overline{\gamma_1} Q(\overline{\gamma_1})$ and we note that $\delta_{11} \in S_{\gamma_1}$. By repeating this same procedure for Q_{11}, if $Q_{11}(\gamma_1) = 0$, we obtain

$$Q_{12}(q) = Q_{11}(q) * \frac{q - Q_{11}(\overline{\gamma_1})^{-1} \overline{\gamma_1} Q_{11}(\overline{\gamma_1})}{q^2 - 2qRe(\gamma_1) + |\gamma_1|^2},$$

and we set $\delta_{12} = Q_{11}(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q_{11}(\overline{\gamma_1})$. We now continue for n_1 steps, generating the quaternion δ_{1j} at step j, until we finally obtain that $Q_{1n_1}(\gamma_1) \neq 0$. If the degree of Q_{1n_1} is still positive, we can find a new isolated root $\gamma_2 \notin S_{\gamma_1}$ (since γ_1 is an isolated root), and we repeat the process once again. Since at each step we decrease the degree of the polynomial, the process necessarily ends after a finite number of steps. We therefore obtain that

$$Q(q) * \Pi^{*t}_{i=1}\Pi^{*n_i}_{j=1}\frac{q - \delta_{ij}}{q^2 - 2qRe(\gamma_i) + |\gamma_i|^2}$$

is a constant $c \in \mathbb{H}$. To conclude the proof we simply multiply this expression on the right by

$$\Pi^{*t}_{i=1}\Pi^{*n_i}_{j=1}(q - \overline{\delta_{t-i+1,n_i-j+1}}).$$

This immediately gives the result with $\alpha_{ij} = \overline{\delta_{t-i+1,n_i-j+1}}$. □

Note that, unlike what happens in the complex case, the factorization which we have just described is not unique. It strictly depends on the order in which the points α_{ij} are taken. What is unique in the factorization is the set of spheres, as well as the numbers n_t. To illustrate this phenomenon, we first recall a result from [1] (but see also [5]).

Theorem 2.2. *Let $f(q) = (q - a) * (q - b)$ with a and b lying on different $2-$spheres. Then $f(q) = (q - b') * (q - a')$ if and only if $a' = cac^{-1}$ and $b' = cbc^{-1}$ for $c = \overline{b} - a \neq 0$.*

Using this result, it is easy to show that the polynomial $P(q) = (q-I)*(q-2J)$ can also be factored as $P(q) = (q - \frac{8I+6J}{5}) * (q - \frac{4J-3I}{5})$. Note that this different factorization still has one representative from the sphere $\mathbb{S}$, and one from the sphere $2\mathbb{S}$. This example also shows (as we would expect) that the quaternions α_{ij} are not, in general, roots of the polynomial. In this case, for example, I is a root, and so is $\frac{8I+6J}{5}$, while neither $2J$ nor $\frac{4J-3I}{5}$ are roots of the polynomial P.

For the purpose of the next theorem, we apply our factorization result to the polynomial Q in the previous theorem, and we reindex the quaternions α_{ij} which appear in its factorization in lexicographical order so to have a single-index sequence β_k for $k = 1, \ldots, n$, so that the factorization can now be written as

$$Q(q) = \Pi^{*n}_{k=1}(q - \beta_k).$$

A repeated application of Theorem 2.2 immediately demonstrates the next result.

Theorem 2.3. *Let $Q(q)$ be a regular polynomial without spherical zeroes, and let*

$$Q(q) = \Pi_{k=1}^{*n}(q - \beta_k)$$

be one of its factorizations. Then the roots of Q can be obtained from the quaternions β_k as follows: β_1 is a root, β_2 is not a root, but it yields the root $\beta_2^{(1)} = (\overline{\beta_2} - \beta_1)\beta_2(\overline{\beta_2} - \beta_1)^{-1}$. In general if we set, for $r = 1, \ldots, n$ and $j = 1, \ldots, r-1$,

$$\beta_r^{(j)} = (\overline{\beta_r^{(j-1)}} - \beta_{r-j})\beta_r^{(j-1)}(\overline{\beta_r^{(j-1)}} - \beta_{r-j})^{-1}$$

we obtain that the roots of Q are given by

$$\beta_r^{(r-1)} = (\overline{\beta_r^{(r-2)}} - \beta_1)\beta_r^{(r-2)}(\overline{\beta_r^{(r-2)}} - \beta_1)^{-1}.$$

Our final result tells us how to build the factorization if we know the roots of the polynomial. To do so, we need to introduce the notion of multiplicity of roots. To explore attentively some aspects of the proof of Theorem 2.1, which will lead us to the definition of multiplicity, we will study in detail a few features of the simple polynomial with quaternionic coefficients

$$P(q) = (q - \alpha_1) * (q - \alpha_2) * \cdots * (q - \alpha_m) \tag{2.3}$$

where $\alpha_i \in S_{\alpha_1}$ for all $i = 1, \ldots, m$ and where $\alpha_{i+1} \neq \overline{\alpha}_i$ for $i = 1, \ldots, m-1$. To begin with, the following (surprising) technical lemma is needed.

Lemma 2.4. *For any two quaternions $\alpha \neq \beta$ belonging to a same sphere S_α, we have*

$$(\beta - \alpha)\beta(\beta - \alpha)^{-1} = \overline{\alpha} = (\beta - \alpha)^{-1}\beta(\beta - \alpha) \tag{2.4}$$

Proof. If $\beta \in S_\alpha$, then β must be a zero of the polynomial

$$q^2 - 2Re(\alpha)q + |\alpha|^2 = q^2 - \alpha q - \overline{\alpha}q + \overline{\alpha}\alpha = (q - \alpha)[q - (q - \alpha)^{-1}\overline{\alpha}(q - \alpha)],$$

which defines the sphere S_α. Thus

$$(\beta - \alpha)[\beta - (\beta - \alpha)^{-1}\overline{\alpha}(\beta - \alpha)] = 0$$

i.e.

$$(\beta - \alpha)\beta(\beta - \alpha)^{-1} = \overline{\alpha}.$$

Since the following equality holds

$$q^2 - 2Re(\alpha)q + |\alpha|^2 = q^2 - q\alpha - q\bar{\alpha} + \alpha\bar{\alpha} = \left[q - (q - \alpha)\bar{\alpha}(q - \alpha)^{-1}\right](q - \alpha)$$

we also get

$$\left[\beta - (\beta - \alpha)\bar{\alpha}(\beta - \alpha)^{-1}\right](\beta - \alpha) = 0$$

and hence

$$(\beta - \alpha)^{-1}\beta(\beta - \alpha) = \bar{\alpha}$$

which completes the proof. □

The polynomial (2.3) has several nice features, as the following statements explain:

Proposition 2.5. *The polynomial with quaternionic coefficients*

$$P(q) = (q - \alpha_1) * (q - \alpha_2) * \cdots * (q - \alpha_m) \tag{2.5}$$

where $\alpha_i \in S_{\alpha_1}$ *for all* $i = 1, \dots, m$ *and where* $\alpha_{i+1} \neq \overline{\alpha}_i$ *for* $i = 1, \dots, m-1$, *has a unique root, equal to* α_1. *Moreover the factorization* (2.5) *is the only factorization of the polynomial* $P(q)$. *Finally the following equality holds*

$$P(q) * \frac{(q - [P(\overline{\alpha}_1)^{-1}]\overline{\alpha}_1[P(\overline{\alpha}_1)])}{q^2 - 2qRe(\alpha_1) + |\alpha_1|^2} = (q-\alpha_1)*(q-\alpha_2)*\cdots*(q-\alpha_{m-1}) \tag{2.6}$$

Proof. We will prove the first two assertions by induction on the number m of terms of the factorization. If we set $f(q) = (q - \alpha_1)$ and $g(q) = (q - \alpha_2) * \cdots * (q - \alpha_m)$, then Theorem 1.6 establishes that $P(\alpha_1) = 0$. Theorem 1.6 establishes also that $\beta \neq \alpha_1$ is a root of $P(q)$ if and only if $f(\beta)^{-1}\beta f(\beta) = (\beta - \alpha_1)^{-1}\beta(\beta - \alpha_1) \in S_\beta$ is a root of $g(q)$, i.e., by Lemma 2.4, if and only if $\overline{\alpha}_1$ is a root of $g(q) = (q - \alpha_2) * \cdots * (q - \alpha_m)$. Since we have that $\alpha_2 \neq \overline{\alpha}_1$, the induction hypothesis leads to the conclusion that no $\beta \neq \alpha_1$ can be a root of $P(q)$.
Suppose now that

$$P(q) = (q - \alpha_1) * (q - \alpha_2) * \cdots * (q - \alpha_m) = (q - \alpha_1') * (q - \alpha_2') \cdots * (q - \alpha_m')$$

are two factorizations of $P(q)$. The fact that α_1 is the only root of $P(q)$ implies that $\alpha_1' = \alpha_1$, which directly yields the equality

$$(q - \alpha_2) * \cdots * (q - \alpha_m) = (q - \alpha_2') \cdots * (q - \alpha_m')$$

and, by the induction hypothesis, the uniqueness of the factorization follows.

To prove equality (2.6), notice at first that Theorem 1.6 and the fact that α_1 is the only root of $P(q)$ imply that $P(q) * (q - [P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)])$ has two roots on S_{α_1}, namely α_1 and $\overline{\alpha}_1$. The first assertion of this same theorem now forces the equality $[P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)] = \overline{\alpha}_m$. Therefore

$$P(q)*(q-[P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)]) = (q-\alpha_1)*(q-\alpha_2)*\cdots*(q-\alpha_m)*(q-\overline{\alpha}_m)$$
$$= (q - \alpha_1) * (q - \alpha_2) * \cdots * (q - \alpha_{m-1})(q^2 - 2qRe(\alpha_m) + |\alpha_m|^2).$$

Since $\alpha_m \in S_{\alpha_1}$, the last assertion of our statement follows. □

The equality (2.6) means, roughly speaking, that we can "extract" m times the root α_1 from the polynomial $P(q)$. This explains the meaning of the following

Definition 2.6. Let $P(q)$ be a polynomial with quaternionic coefficients. If $x+y\mathbb{S}$ is a spherical root of $P(q)$, its multiplicity is defined as two times the largest integer m for which it is possible to factor $(q^2 - 2qx + (x^2 + y^2))^m$ from $P(q)$. On the other hand, we say that $\alpha \in \mathbb{H}$ has multiplicity k as an (isolated) root for $P(q)$ if, in the factorization (2.2), there are exactly k quaternions α_{ij} which lie on the sphere S_α.

Note that in the case of isolated zeroes, this definition does not imply that one can factor $(q-\alpha)^{*j}$, and therefore this definition is essentially different from the one suggested in [1].

Remark 2.7. Our definition of multiplicity for roots of a polynomial with quaternionic coefficients reduces to the classical definition when applied to the case of a complex polynomial. In fact the sphere of imaginary units of $\mathbb{C}$ reduces to the set $\{i, -i\}$, and hence for any complex number α the sphere S_α is the set $\{\alpha, \overline{\alpha}\}$. Therefore, given a complex polynomial $P(z)$, a spherical root of $P(z)$ consists of a couple $\{w, \overline{w}\}$ of roots of $P(z)$. If, according to Definition 2.6, the spherical root $\{w, \overline{w}\}$ of $P(z)$ has multiplicity $2s$ and w has multiplicity r as an isolated root of $P(z)$, then $\overline{w}$ will have classical multiplicity s and w will have classical multiplicity $r+s$. Notice that in the complex case the simple polynomial (2.5) $P(q) = (q-\alpha_1)*(q-\alpha_2)*\cdots*(q-\alpha_m)$ (where $\alpha_i \in S_{\alpha_1}$ for all $i = 1, \ldots, m$ and where $\alpha_{i+1} \neq \overline{\alpha}_i$ for $i = 1, \ldots, m-1$) reduces to the classical degree m binomial $(q-\alpha_1)^m$.

We end the paper with the announced result.

Theorem 2.8. *The family of all regular polynomials with quaternionic coefficients with assigned spherical roots $x_1 + y_1\mathbb{S}, \ldots, x_p + y_p\mathbb{S}$ with multiplicities $2m_1, \ldots, 2m_p$, and assigned isolated roots $\gamma_1, \ldots, \gamma_t$ with multiplicities $n_1, \ldots, n_t$ consists of all polynomials P which can be written as*

$$P(q) = [q^2 - 2qx_1 + (x_1^2 + y_1^2)]^{m_1} \cdots [q^2 - 2qx_p + (x_p^2 + y_p^2)]^{m_p} Q(q)c \quad (2.7)$$

with

$$Q(q) = \Pi_{i=1}^{*t} \Pi_{j=1}^{*n_i} (q - \alpha_{ij}) = \Pi_{i=1}^{*t} Q_i(q)$$

where $c \in \mathbb{H}$ is an arbitrary non-zero constant and where $\alpha_{11} = \gamma_1$, the quaternions α_{1j} are arbitrarily chosen in S_{γ_1} for $j = 2, \ldots, n_1$ in such a *way that* $\alpha_{1j+1} \neq \overline{\alpha}_{1j}$ *(for* $j = 1, \ldots, n_1 - 1$*), and in general, for* $i = 2, \ldots, t$*,*

$$\alpha_{i1} = [(\Pi^{*(i-1)}_{k=1} Q_k)(\gamma_i)]^{-1} \gamma_i [(\Pi^{*(i-1)}_{k=1} Q_k)(\gamma_i)]$$

while the remaining α_{ij} *are arbitrarily chosen in* S_{γ_i}*, for* $j = 2, \ldots, n_i$ *in such a way that* $\alpha_{ij+1} \neq \overline{\alpha}_{ij}$ *(for* $j = 1, \ldots, n_i - 1$*)*

Proof. The fact that every polynomial with the assigned roots can be represented in the form (2.7) is an immediate consequence of the proof of Theorem 2.1. Conversely, the fact that if a polynomial can be written as in (2.7) then it has the required zeroes and multiplicities, is an immediate consequence of Theorem 1.6 and of Proposition 2.5. □

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