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A Survey on the Differential and Symplectic Geometry of Linking Numbers

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Abstract. The aim of the present survey mainly consists in illustrating some recently emerged differential and symplectic geometric aspects of the ordinary and higher order linking numbers of knot theory, within the modern geometrical and topological framework, constantly referring to their multifaceted physical origins and interpretations.

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1. Introduction

The celebrated 1833 Gauß' note "Zur Elektrodynamik"([43]), first published in 1867, furnished a startling integral formula for the (algebraic) *linking number* of two wires wherein an electric current flows, and can be regarded as a milestone in the development of topology, and knot theory in particular. Gauß' formula, which can be recovered via Ampere's theorem and the Biot-Savart law (cf. e.g. [46] and Section 2 below), is emblematic in illustrating the intriguing *mélange* of topology, geometry and physics rooted in fluid dynamics and electrodynamics since their very birth through the enquiries of Helmholtz, Kelvin, Maxwell, Tait (see e.g. [37, 89] for stimulating historical overviews) which provided a continuing source of fascination for many scientists since then, up to — just to mention a single topic — the modern tantalizing topological mysteries of superfluidity, and to the current mathematical approaches. The latter are best illustrated in the monumental treatise by Arnol'd and Khesin ([8]) on topological methods in hydrodynamics, presenting an extremely general and powerful unifying geometrical point of view (see also [54]). Also, we signal the recent Ricca's surveys [87, 88] for a synthetical but clear description of some specific techniques of topological fluid mechanics, of wide-ranging applications and of a carefully reconstructed historical background.

The (highly differential geometrically flavoured) theory of Kontsevich ([57, 12, 77]) can be viewed as the farthest reaching generalization of Gauß' ideas and at the same time perhaps the closest in spirit thereto. However we are going to comment only briefly thereupon in the sequel, since the scope of this note is, by contrast, rather modest: we are going to review, keeping technical details to a minimum and concentrating on the basic ideas, some newly discovered symplectic and differential geometric interpretations of ordinary and higher order linking numbers ([20, 84]), within the general geometric framework — manufactured, among others, by Arnol'd, Marsden and Weinstein, and Brylinski — trying and properly place them within the existing research lines in topological fluid mechanics, quantum field theory and knot theory *per se*. Nevertheless, we shall often intermingle the flow of the exposition with short digressions (which can be skipped by expert readers) providing some background material on the various topics involved, in order to improve readability for a larger audience. A possibly new application of Arnol'd's "Helicity Bounds Energy Theorem" (see [7, 8]) in the context of higher order linking will be also presented, together with the construction of a possibly novel representation of the pure braid group P_3 via a nilpotent flat connection, in the spirit of [84], making contact with Berger's approach ([14, 15, 16, 17, 18, 38]; see [3, 2] as well).

The paper is organized as follows. Section 2 is somewhat preparatory and it is centred around the concept of *helicity*, which allows for a natural introduction of the Gauß linking number in a manner tailored to our future purposes. We both discuss the classical vector analytical and the modern differential form theoretic formulation (*abelian Chern-Simons action*), concluding with a short digression on topological quantum field theory which will lend motivation to our subsequent constructions.

In Section 3, after recalling equivariant moment maps, we discuss the manifold of (mildly -in a sense to be specified) smooth singular knots ("closed vortex filaments") in a three-fold (sticking to the $\mathbf{R}^3$ case) introduced by Brylinski ([25]), which possesses natural symplectic and Riemannian structures (formally) combining into a Kähler one, proceeding subsequently to elucidate its hydrodynamical content, in the framework of the geometrical interpretation of Euler's equation for a perfect (i.e. incompressible, inviscid) fluid in terms of coadjoint orbits of volume preserving diffeomorphisms.

In the following section (4), after a detour on Lagrangian submanifold theory, we discuss the Morse family interpretation of the abelian Chern-Simons action (with knot insertion) set forth in [20, 19], leading to a Maslov theoretical interpretation of the writhe of a knot (after a choice of a plane projection thereof).

We then pass (in Section 5) to geometric quantization issues, reviewing the Bohr-Sommerfeld interpretation of the Feynman-Onsager condition arising in quantum vortex theory, again developed in [20, 19], in which the Gauß linking number plays a pivotal role.

Next, we discuss higher order linking phenomena via the differential geometric apparatus of [84], in terms of Chen-Hain-Tavares nilpotent "topological" connections, focussing on the basic steps of the costruction (which is strongly reminiscent of Chern-Weil theory). This leads to a holonomy interpretation of Massey higher order linking numbers, and to a short proof of a weak version of the Turaev-Porter theorem, stating equality with the so called Milnor higher order linking numbers, defined group combinatorially.

Then we address magnetic relaxation and its topological bounds, a field which has recently witnessed a massive flurry of activity, starting from the seminal work of Arnold, Moffatt and Freedman ([7, 70, 71, 41, 42]). We prove a possibly new result in this direction, in the context of higher order linking, for almost trivial (i.e. Brunnian) links which involves Arnold's "Helicity Bounds Energy Theorem" together with the intepretation, originating in [84], of higher order linking numbers in terms of suitable ordinary linking numbers.

The following section is devoted to pointing out some possible further fruitful connections of the above theory with the work of Berger on higher order braiding and the Kontsevich integral ([17, 18]) and with the theory developed in the final section of [20] aiming at a geometric quantization interpretation of Laughlin's wave functions employed in the theory of the Fractional Quantum Hall Effect. As a new application of the previous Chen integral theoretic techniques, we recover Berger's 3-braid invariant via parallel transport of a nilpotent flat connection manufactured from the Arnol'd identity, defined on the configuration space X_3 of distinct points on the complex plane, thereby yielding a (Heisenberg group) representation of its fundamental group, that is, the pure braid group P_3. The nontrivial entries of the parallel transport matrix will give second and third order (pure) braid invariants.

A short final section is devoted to some concluding remarks and open problems. The paper is accompanied by a certain amount of hand-drawn pictures, mostly taken from [84] and [20], but depicted anew, again in view of clarity enhancement.

2. Prologue

In this preliminary section we collect, in a compact manner, some basic classical electromagnetic and fluidodynamical notions that are needed throughout the paper, both in standard vector calculus terminology and in the modern differential geometric one.

2.1. Solenoidal fields, vector potentials, and abelian connections

The group *sDiff* $(\mathbf{R}^3)$ of (Lebesgue) measure preserving diffeomorphisms of $\mathbf{R}^3$ ("rapidly converging" at the identity at infinity) is an infinite dimensional Lie group (in a suitable sense, see e.g. [36, 59]), with "Lie algebra" *sdiff* $(\mathbf{R}^3)$ consisting of the (rapidly vanishing at infinity) divergence-free (or solenoidal) vector fields (generating volume-preserving flows). Given a divergence-free vector field $\mathbf{B}$, it follows from classical Helmholtz-Hodge theory the existence of a unique *vector potential* $\mathbf{A}$ for $\mathbf{B}$, fulfilling $curl\,\mathbf{A} = \mathbf{B}$ and $div\,\mathbf{A} = 0$ (Coulomb gauge condition), explicitly given by the following expression

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{(\mathbf{r} - \mathbf{r}') \times \mathbf{B}(\mathbf{r}')}{\| \mathbf{r} - \mathbf{r}' \|^3} \, d^3\mathbf{r}' \tag{2.1}$$

(where $\times$ denotes the ordinary vector product in $\mathbf{R}^3$, and $d^3\mathbf{r}'$ the volume element). The integral operator appearing in the r.h.s. is often called the *Biot-Savart* operator, often denoted as $curl^{\,-1}$.

The singular version, for a magnetic or vorticity field concentrated (i.e. δ-like) on a (smooth) knot K, [our knots will be assumed smooth (hence tame) throughout the paper], reads, in standard physicists' notation (see for instance [83])

$$\mathbf{B}_K(\mathbf{r}) = \int_K d\mathbf{r}(s)\,\delta^3(\mathbf{r} - \mathbf{r}(s)) \tag{2.2}$$

with vector potential

$$\mathbf{A}_K(\mathbf{r}) = curl\,\frac{1}{4\pi}\int_K \frac{d\mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|}. \tag{2.3}$$

In the last formula $d\mathbf{r}'$ should be viewed as an "infinitesimal vector" (so that the integral represents a vector field). These expressions can be rigorously interpreted in the sense of *currents* (see e.g. [34, 45], and below).

In differential geometric terms the vector potential $\mathbf{A}$ becomes a real 1-form A, and can be looked upon as a $U(1)$-*connection* on a trivial complex line bundle over $\mathbf{R}^3$ (or, equivalently, on a trivial $U(1)$-principal bundle $\mathbf{R}^3$. The 2-form $B := dA$ is its *curvature* 2-form (and corresponds to $\mathbf{B}$, see e.g. [8]): explicitly, one has, if ν denotes the standard volume form on $\mathbf{R}^3$, $B = i_{\mathbf{B}}\nu$. Furthermore, resorting to Hodge theory, if we impose the (Coulomb) gauge condition $\delta A = 0$ (corresponding to $div\ \mathbf{A} = 0$) one finds, given B, the following formula for A

$$A = \Delta^{-1}\delta B. \tag{2.4}$$

Passing to the singular case, one looks for a connection (viewed as a current) whose curvature 2-form is concentrated (i.e. δ-like) on an ordinary knot K. We introduce the (de Rham) current

$$T_K(A) = \int_K A = \int_{\mathbf{R}^3} A \wedge \eta_K \tag{2.5}$$

upon using a *singular* Poincaré dual η_K form, and we get

$$A_K = \Delta^{-1}\delta\, T_K \tag{2.6}$$

where Δ is the Hodge Laplacian on 1-forms, acting componentwise as the ordinary Laplacian (up to a negative constant), since we are in flat space. Existence, in the sense of currents, follows e.g. from the Hörmander-Łojasiewicz theorem, see e.g. [20, 99].

2.2. Knot framings

Recall that a *framing* of a knot K consists in the choice of a homotopy class of sections in the normal bundle $\mathcal{N}_K$ to K. Concretely, this amounts to specifying a nearby knot K', with linking number $\ell(K, K')$, called the *framing number* of K. The latter can be defined combinatorially, after a plane projection. (see e.g. [90, 53]): one counts, with appropriate signs, the overcrossings of K (over K'); one has ± 1 if the tangent vectors to K and K', in this order, induce the same or the opposite orientation of the given plane, respectively (the latter oriented in the standard counterclockwise manner). One has $\ell(K', K) = \ell(K, K')$. In particular, given a (regular) plane projection of K, still denoted by the same letter, one can define the so called *blackboard framing*, and the ensuing framing number is the *writhe* of K, denoted by $w(K)$ (see, in particular, [53]): one simply draws a knot K' "sufficiently close" to K in the direction of the normal to K at every point, the latter chosen in such a way that the tangent and the normal vector at any point give the same orientation of the given plane. The knot K' is oriented consistently with K, and their linking number will yield $w(K)$. See Figure 1. The integral form of the linking number originally established by Gauß will be recalled below.

2.3. Helicity, the Chern-Simons action and the Gauß linking number

A fundamental hydrodynamical concept is that of *helicity* (cf. [70, 7, 8, 72]). Given a solenoidal field $\mathbf{B}$, its helicity $\mathcal{H}(\mathbf{B})$ is defined as

$$\mathcal{H}(\mathbf{B}) = \int_{\mathbf{R}^3} \langle \mathbf{A} | \mathbf{B} \rangle = \frac{1}{4\pi} \int\!\!\int \frac{\langle \mathbf{r} - \mathbf{r}' | \mathbf{B}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}') \rangle}{\| \mathbf{r} - \mathbf{r}' \|^3} \, d^3\mathbf{r} \, d^3\mathbf{r}' \tag{2.7}$$

($\langle \cdot | \cdot \rangle$ denoting the standard scalar product in $\mathbf{R}^3$) and it is a gauge invariant quantity (i.e. it does not change under the gauge transformation $\mathbf{A} \mapsto \mathbf{A} + \nabla\phi$) and it also enjoys diffeomorphism invariance (Helicity Invariance Theorem).

A short calculation shows that, as a quadratic form, the helicity $\mathcal{H}$ is *non degenerate* (cf. also [8]), a crucial fact.

In terms of differential forms, helicity is nothing but the abelian *Chern-Simons action*

$$CS(A) = \int_{\mathbf{R}^3} A \wedge dA = \mathcal{H}(\mathbf{B}) \tag{2.8}$$

Furthermore, if we again impose the (Coulomb) gauge condition $\delta A = 0$, we may also write

$$CS(A) = \int_{\mathbf{R}^3} A \wedge dA = \int_{\mathbf{R}^3} d^{-1}B \wedge B. \tag{2.9}$$

Now, given a solenoidal vector field $\mathbf{B}$ concentrated in a tube around a knot K, the helicity can be interpreted (using suitable units) as the linking number of two generic flux lines (which run "parallel" to K but at the same time are subject to a uniformly distributed $2\pi n$-Dehn twist, with $n \in \mathbf{Z}$, thus giving a framing number of K), see e.g. [72] for a detailed discussion:

$$\mathcal{H}(\mathbf{B}) = \ell(K, K')\Phi^2 \tag{2.10}$$

with Φ denoting the flux of $\mathbf{B}$ along any section of the tube. See Figure 1 as well. This observation will be crucial in what follows. Clearly, direct insertion of singular fields in the expression for the helicity is not possible. The whole discussion can also be formulated in terms of Poincaré dual forms. For a generic solenoidal field, Arnol'd's Helicity Theorem states that the helicity equals a suitably defined *average linking number* of the field trajectories ([7, 8]). In this respect, we recall the Gauß integral formula for $\ell(K_1, K_2)$, which will be needed in the sequel (in terms of any two parametrizations of the knots involved)

$$\ell(K_1, K_2) = \frac{1}{4\pi}\int_0^1 dt \int_0^1 ds\, \frac{\langle \gamma_2(t) - \gamma_1(s) | \dot{\gamma_1}(s) \times \dot{\gamma_2}(t)\rangle}{\|\gamma_2(t) - \gamma_1(s)\|^3}. \tag{2.11}$$

Notice that the Gauß formula can be also portrayed as follows:

$$\ell(K_1, K_2) = \mathcal{H}(\mathbf{B}_{K_1}, \mathbf{B}_{K_2}) = \int_{\mathbf{R}^3} \langle \mathbf{A}_{K_1} | \mathbf{B}_{K_2}\rangle = \int_{K_2} \langle \mathbf{A}_{K_1} | d\mathbf{r}_2\rangle \tag{2.12}$$

in terms of the bilinear extension of the helicity quadratic form (all expressions are symmetric with respect to K_1 and K_2). The following differential form interpretation of the linking number is useful, and can be generalized to higher order linking numbers ([84], and below):

$$\ell(K_1, K_2) = \int_{K_1} A_{K_2} \qquad \left(= \int_{K_2} A_{K_1}\right) \tag{2.13}$$

(obvious notation).

Amongst the innumerable applications of Gauß' linking number we also mention [10].

2.4. A topological quantum field theoretic intermezzo

A crucial motivation for the whole theory is provided by the Witten Chern-Simons partition function (in the abelian, i.e. $U(1)$ case, with a knot (K) insertion)

$$Z(K) = \int_{\mathcal{A}} e^{i\left(\frac{k}{8\pi}\int_{S^3} A\wedge dA + \int_K A\right)}\, \mathcal{D}A \tag{2.14}$$

given as a path integral over the space of connections $\mathcal{A}$ (identified with real vector potentials), see e.g. [101, 11, 53, 46]. (Indeed, the above expression formally descends to a path integral over $\mathcal{A}/\mathcal{G}$, namely the space of gauge equivalence classes of connections). We can work indifferently with $\mathbf{R}^3$ or its compactification S^3. Let us notice that in the abelian case the requirement of gauge invariance does not force, a priori, quantization of the real constant k, unlike the non-abelian one. We may formally view Witten's partition function as an oscillatory integral over $\widehat{Y}_{\mathbf{R}^3}$, the manifold of mildly singular knots introduced by Brylinski ([25]) and reviewed in Section 3 below.

Denote by Z the same function above, without the knot term. A short (formal) path-integral calculation (see e.g. [46]) yields for the "vacuum expectation value" for the holonomy $hol_K(A) := e^{i\int_K A}$ (called Wilson line, in physicists' terminology), via a preliminary choice of framing for K:

$$\langle hol_K(\cdot)\rangle := Z^{-1} \cdot Z(K) = e^{-i\frac{2\pi}{k}\ell(K,K')}. \tag{2.15}$$

If we fix a plane and its ensuing blackboard framing, we get the quantity $e^{-i\frac{2\pi}{k}w(K)}$, which we term (for any choice of the constant in front of w) *regular isotopy abelian Witten invariant*; observe, nevertheless, that it also plays a fundamental role in Kauffman's construction of polynomial invariants for links, for transforming any regular isotopy invariant into a true isotopy invariant (Kauffman's principle); generally it appears in the form $(-\alpha)^{w(K)}$, with α a formal parameter.

The above path integral computation can be rendered rigorous, even in the non-abelian case (after axial gauge fixing, see e.g. [4]), however, this will not important for what follows, since we shall not use functional integration at all. The pattern of the calculation can nevertheless easily be grasped by a 1-dimensional Gauss integral computation, via "completion of the square". The essential point is to find d^{-1} (in an appropriate sense), see the preceding subsection. (Compare with [8, 53, 56]).

We also have the "diffuse" version of Witten's partition function, due to Verjovsky and Vila-Freyer ([98]), involving the so-called *average holonomy* of the divergence-free vector field ξ, given by $hol_\xi(A) := e^{i\int_{S^3}(A,\xi)}$:

$$Z(\xi) = \int_{\mathcal{A}} e^{i(\frac{k}{8\pi}\int_{S^3} A\wedge dA + \int_{S^3}(A,\xi)}\ \mathcal{D}A \tag{2.16}$$

(here $(\cdot,\cdot)$ denotes the standard duality pairing between forms and vector fields). A similar path integral calculation, see [98], yields, in turn

$$\langle hol_\xi(\cdot)\rangle := Z^{-1} \cdot Z(\xi) = e^{-i\frac{2\pi}{k}\mathcal{H}(\xi)} \tag{2.17}$$

(with suitable normalizations). So the two vacuum expectation values coincide as long as ξ is a solenoidal field confined in a flux tube around a knot K, as in the preceding subsection. This simple observation plays an important role in our Lagrangian submanifold theory interpretation ([20, 19]).

3. Brylinski's manifold and the symplectic geometry of Euler's equation

3.1. Some basic symplectic geometric terminology

The present subsection recalls some basic notions of symplectic geometry which will prove useful in the sequel. Strictly speaking, the context is finite dimensional, nevertheless we shall eventually work in an infinite dimensional context at different levels of mathematical rigour.

A symplectic manifold (M, ω) is a smooth manifold equipped with a closed non degenerate 2-form ω. Important examples are provided e.g. by the cotangent space T^*X associated to a manifold X, by Kähler manifolds, by coadjoint orbits of a Lie group G (see e.g. [55, 58, 92] for details, and also [80] for a condensed treatment tailored to the purposes of fluid mechanics); the latter live in the dual $Lie(G)^*$ of the Lie algebra $Lie(G)$ of G and take the form $O_{f_0} \cong G/G_{f_0}$, with $f_0 \in Lie(G)^*$ and G_{f_0} denoting the stabilizer of f_0 with respect to the group coadjoint action Ad^*. The (Kirillov) symplectic form B on O_{f_0}, evaluated on two generic fundamental vector fields induced by $u, v \in Lie(G)$ reads, at $f \in O_{f_0}$

$$B_f(ad_u^* f, ad_v^* f) := \langle f, [u, v] \rangle \tag{3.1}$$

(here ad^* denotes (Lie algebra) coadjoint action, which dualizes the standard adjoint action $ad_u v = [u, v]$). If the symplectic manifold (M, ω) is acted upon (symplectically) by a Lie group G, with Lie algebra $Lie(G)$, a G-equivariant moment map $\mu : M \to Lie(G)^*$ (existing under mild topological assumptions on M and G) is characterized by the property

$$\mu(g \cdot x) = Ad^*(g)\, \mu(x), \qquad x \in M,\ g \in G. \tag{3.2}$$

Such a map yields, for each $u \in Lie(G)$, a Hamiltonian $\lambda_u = \lambda_u(x) := \langle \mu(x), u \rangle$ (duality pairing), and the set of such functions yields indeed a Lie algebra isomorphic to $Lie(G)$, via the Poisson bracket $\{\cdot, \cdot\}$ induced by the symplectic form:

$$\{\lambda_u, \lambda_v\}(x) := \omega(u^\sharp, v^\sharp)(x) = \lambda_{[u,v]}(x) \tag{3.3}$$

(for all $x \in M$, with $u^\sharp$ denoting the fundamental vector field induced by $u \in Lie(G)$).

3.2. Brylinski's manifold

We are now going to sketch the basic steps of Brylinski's construction of the manifold $\widehat{Y}_M$ of ("mildly", in the sense to be specified below) singular knots in a manifold M ([25], see also [20]). One begins with the (free) loop space $LM := C^\infty(S^1, M)$ associated to a smooth manifold M of dimension n: it is an infinite dimensional paracompact smooth Fréchet manifold modelled on $C^\infty(S^1, \mathbf{R}^n)$. Then one considers the submanifold $\widehat{X}_M \subset LM$ consisting of smooth loops which are embeddings but for a finite set $A \subset S^1$, and such that the branches of the loop at any two distinct points in A have finite order tangencies. The manifold of all *bona fide* embeddings will be denoted by X_M. The group $Diff^+(S^1)$ of all orientation preserving diffeomorphisms of the circle naturally acts on $\widehat{X}_M$ and the quotient $\widehat{Y}_M := \widehat{X}_M/Diff^+(S^1)$ becomes a smooth paracompact Fréchet manifold modelled on $C^\infty(S^1, \mathbf{R}^{n-1})$, and $\widehat{X}_M \to \widehat{Y}_M$ becomes in turn a principal $Diff^+(S^1)$-bundle. Accordingly, one can define $Y_M := X_M/Diff^+(S^1)$. We shall mostly deal with the case $M = \mathbf{R}^3$; the ensuing manifold $\widehat{Y}_{\mathbf{R}^3}$ is called the manifold of *oriented singular knots* in $\mathbf{R}^3$, whereas $Y_{\mathbf{R}^3}$ is called the manifold of *oriented knots* in $\mathbf{R}^3$. Recall that the tangent space $T_K\widehat{Y}_M$ to $K \in \widehat{Y}_M$ is intrinsically the space of smooth sections of the normal bundle to the normalization $\tilde{K}$ of K, namely, a separation of the branches of K (see [25] for details). Given a volume form ν on a three-fold M, one gets by transgression a 2-form β on LM via the formula

$$\beta = \int_{S^1} ev^*(\nu) \tag{3.4}$$

where $ev : S^1 \times LM \to M$ given by $ev(x, \gamma) := \gamma(x)$ is the evaluation map (of a loop $\gamma \in LM$ at a point $x \in S^1$). More explicitly, given tangent vectors u and v at γ, it reads

$$\beta_\gamma(u, v) = \int_0^1 \nu(\frac{d\gamma}{dx}(x), u(x), v(x)). \tag{3.5}$$

The above formulae can be also written in Chen integral form ([28])

$$\beta|_K = \int_K \nu \qquad \text{or, shortly} \qquad \beta = \int \nu \tag{3.6}$$

(see also Sections 6 and 8 for further applications of Chen integrals).

The 2-form β is basic with respect to the $Diff^+(S^1)$-principal bundle $\widehat{X}_M \to \widehat{Y}_M$, namely $i_\xi\beta = i_\xi d\beta = 0$, with ξ any vertical vector field (i.e.

generating an orientation preserving reparametrization of the loop), therefore it descends to a closed, non degenerate 2-form on $\widehat{Y}_M$, i.e. a (weak) symplectic form. Also recall that, in general, the above transgression gives rise to a (degree shifting) morphism of complexes $\Lambda^\bullet(M) \to \Lambda^{\bullet-1}(LM)$, mapping closed (resp. exact) forms to closed (resp. exact) ones in view of the general formula (direct calculation, or see [28, 33])

$$d\int\omega = -\int d\,\omega \tag{3.7}$$

where, of course, the l.h.s. differential pertains to LM and the r.h.s. one pertains to M.

Consequently, integral cohomology classes on M are mapped to integral cohomology classes on LM. Therefore, if $[\nu]$ is integral, then $[\beta]$ is integral as well, this ensuring, via the Weil-Kostant theorem, the existence of a prequantum bundle $L \to LM$ (Brylinski's line bundle, descending to a line bundle over $\widehat{Y}_M$). This will be further discussed in Section 5. A somewhat sophisticated but at the same time explicit construction can be given via the integral class $[\nu] \in H^3(M, \mathbf{Z})$, defining a *gerbe*, see [25].

We also recall that the the weak symplectic manifold $(\widehat{Y}_M, \beta)$ can be naturally equipped with a (formally) integrable compatible almost complex structure making it a Kähler manifold in an appropriate sense, see e.g. [25, 66, 80, 8]. Basically (working with a background metric), the almost complex structure is given, pointwise on a fixed knot, by taking the vector product against the (unit) tangent vector: this leaves the normal plane invariant, and indeed this operation squares to minus identity.

We are now prepared to comment on the hydrodynamical interpretation of Y_M (see the above references): each connected component thereof is (up to technical subtleties, see [25]) a coadjoint orbit of the group G of unimodular diffeomorphisms of M, i.e. those preserving a volume form, via a natural moment map, roughly consisting in regarding a knot K as an element of the dual of the Lie algebra of G — the latter given (in a suitable technical sense) by divergence free vector fields — by associating to K a vorticity field concentrated thereon. Explicitly we have, resorting to the above notation, the following expression for the associated Hamiltonian algebra (also called *Rasetti-Regge current algebra*, see [80] and references therein, since it has been introduced in the quantum field theoretic context in [86] for discussing quantized vortices), cf. [66, 80, 81, 25]:

$$\lambda_{\mathbf{B}} : \widehat{Y}_{\mathbf{R}^3} \;\to\; \mathbf{R}, \qquad\qquad K \;\mapsto\; \int_K A. \tag{3.8}$$

This is a special case of the general construction briefly portrayed below.

3.3. Coadjoint orbit description of the Euler flow

Let us now put the final considerations of the preceding section into a more general perspective, again modulo (serious!) technical problems. The Hamiltonian for a perfect fluid in $\mathbf{R}^3$ with (divergence-free) velocity $\mathbf{V}$ and vorticity $\mathbf{W} = curl\,\mathbf{V}$ reads

$$H = \frac{1}{2}\int_{\mathbf{R}^3} \langle \mathbf{V}|\mathbf{V}\rangle \equiv \frac{1}{2}(\mathbf{V}, \mathbf{V}) \qquad (< \infty). \tag{3.9}$$

The fluid motion is governed by Euler's equation

$$\partial_t \mathbf{V} = -(\mathbf{V}\cdot\nabla)\mathbf{V} - \nabla p \tag{3.10}$$

(p is the pressure). The latter can be cast in the *vorticity form*

$$\partial_t \mathbf{W} = -[\mathbf{W}, \mathbf{V}]. \tag{3.11}$$

One has the Kelvin circulation theorem (differential form description, obvious notation):

$$\int_{C(t)} v(t) = \int_{C(0)} v(0) \tag{3.12}$$

for a curve $C = C(t)$ transported by the fluid, and the Helmholtz theorem, for curves C_1, C_2 enclosing a vortex tube:

$$\int_{C_1} v = \int_{C_2} v \tag{3.13}$$

(see e.g. [94]).

All this can be vividly formulated ([66, 80, 81]) in symplectic terms: the phase space for the fluid motion is a coadjoint orbit $\mathcal{O}_W$ of the group *sDiff* $(\mathbf{R}^3)$ labelled by the vorticity (or equivalently velocity) field. The ensuing Kirillov-Kostant-Souriau (KKS) symplectic form reads (after standard vector calculus computations)

$$B_{\mathbf{V}}(ad^*_{\mathbf{a}}\,\mathbf{V}, ad^*_{\mathbf{b}}\,\mathbf{V}) := (\mathbf{V}, [\mathbf{a}, \mathbf{b}]) = (\mathbf{W}, \mathbf{a}\times\mathbf{b}) \tag{3.14}$$

upon using the following identities fulfilled by solenoidal fields:

$$[\mathbf{a}, \mathbf{b}] = curl\,(\mathbf{a}\times\mathbf{b}) \tag{3.15}$$

$$div\,(\mathbf{a}\times\mathbf{b}) = \langle \mathbf{b}|\,curl\,\mathbf{a}\rangle - \langle \mathbf{a}|\,curl\,\mathbf{b}\rangle. \tag{3.16}$$

The Hamiltonian algebra (which has been again termed Rasetti-Regge current algebra in [81]) reads

$$\lambda_{\mathbf{b}}(\mathbf{V}) = (\mathbf{V}, \mathbf{b}) \tag{3.17}$$

and allows for a simple reformulation of Hamilton's equations

$$\partial_t \lambda_{\mathbf{b}} = -\{H, \lambda_{\mathbf{b}}\} \tag{3.18}$$

which may be used to get helicity conservation (here denoted by Q, and defined in an obvious manner)

$$\partial_t Q = 0. \tag{3.19}$$

In passing, we also mention the Clebsch variable description ([61, 60, 66, 81]), which has been extensively reviewed elsewhere (see e.g. [79]):

$$\mathbf{V} = \alpha \nabla \beta + \nabla \phi \tag{3.20}$$

then

$$\mathbf{W} = \nabla \alpha \times \nabla \beta \tag{3.21}$$

(with α, β and ϕ local smooth functions of $x \in \mathbf{R}^3$, or more precisely, and under suitable conditions, on its compactification S^3). The intrinsic picture is as follows: The Clebsch variable Hamilton equations (see [60, 61, 81])

$$\partial_t \alpha = -\mathbf{V} \cdot \nabla \alpha, \qquad \partial_t \beta = -\mathbf{V} \cdot \nabla \beta, \tag{3.22}$$

may be written in a more compact form via an *order parameter* $\mathbf{n} : S^3 \to S^2$, leading to the following form of Euler's equation

$$\partial_t \mathbf{n} = -\mathbf{V} \cdot \nabla \mathbf{n}. \tag{3.23}$$

If σ is the normalized area form on S^2, the vorticity form W reads $W = \mathbf{n}^* \sigma$, then one has $W = dV$, since the cohomology group $H^2(S^3) = 0$, and $Q = \mathcal{H}(\mathbf{n}) = \int_{S^3} V \wedge dV$ where $\mathcal{H}(\mathbf{n})$ is the *Hopf invariant* of the map $\mathbf{n}$ (the Chern-Simons action again!).

Geometrically, the overall picture is the following: the Clebsch variables describe, for each $x \in S^3$, a point on the sphere S^2. The preimage (fibre) of a generic point is a circle S^1 (described by the parameter ϕ), and the Hopf invariant equals the linking number of two generic fibres. The maps $\mathbf{n}$ with a fixed Hopf invariant give rise to a Kähler manifold and the map $\mu : \mathbf{n} \mapsto \mathbf{W}$ turns out to be a *sDiff* $(\mathbf{R}^3)$ equivariant moment map ([81]). We close this section by pointing out that the original Arnol'd's approach consisted in interpreting Euler's equation as a geodesic equation for the Lie group *sDiff* $(\mathbf{R}^3)$ equipped with a suitable right-invariant metric (a far reaching generalization of rigid body theory, see [6, 8]); this beautiful portrait has been put on a fully rigorous basis by Ebin and Marsden ([36]).

4. Knot framings and Maslov theory

4.1. Lagrangian submanifolds and Maslov index

A Lagrangian submanifold of a symplectic manifold is defined by the property that the symplectic form vanishes thereupon, and it is of maximal dimension (namely, the tangent space at any point is a maximal isotropic subspace with respect to the symplectic form, i.e. it coincides with its symplectic complement). If M be a smooth manifold of dimension n, then its cotangent space T^*M is a symplectic manifold (equipped with a canonical symplectic form). A Lagrangian submanifold $\Lambda \subset T^*M$ in general position can be described in the following way (Maslov-Hörmander Morse family theorem, see e.g. [63, 51, 48, 65]): there exists (locally) a smooth function $\phi = \phi(q, a), (q, a) \in M \times \mathbf{R}^k$ (for some k: $\mathbf{R}^k$ is a space of auxiliary parameters) and a submanifold

$$C_\phi = \{(q, a) \in M \times \mathbf{R}^k \mid d_a \phi = 0\} \tag{4.1}$$

with $d(d_a)$ of maximal rank thereon (here $d = d_q + d_a$) such that the map

$$\begin{array}{ccc} C_\phi & \to & T^*M \\ (q, a) & \mapsto & (q, d_q \phi) \end{array} \tag{4.2}$$

is an immersion with image Λ. If the Hessian H_a (with respect to the auxiliary variables a) is non degenerate, one can write $a = a(q)$ and define the *phase function* $F = F(q) := \phi(q, a(q))$, with $(q, dF(q)) \in \Lambda$. The covector $dF(q) =: p(q)$ is the momentum at q.

This fails at the singular points of the obvious projection $\Lambda \to M$, but the singular locus Z (the *Maslov cycle*) turns out to be orientable and of codimension 1 in Λ with ∂Z of codimension ≥ 3.

Taking a good open cover $\{V_i\}_{i \in I}$ of Λ, and letting σ_i be the signature of the Hessian H_a on $V_i \setminus Z$, for a curve γ crossing (once) Z transversally, starting in V_i and ending in V_j, one arrives at the *Maslov formula*

$$\frac{1}{2}(\sigma_j - \sigma_i) = \pm 1 = \gamma \circ Z \tag{4.3}$$

(the r.h.s. denoting intersection index) which is readily generalized to

$$m(\gamma) = \gamma \circ Z \tag{4.4}$$

upon summing over all intersections points $\gamma \cap Z$, with appropriate signs. The singular Poincaré dual of the Maslov cycle Z is represented by a closed 1-current η_Z, and one finds

$$m(\gamma) = \gamma \circ Z = \int_\gamma \eta_Z. \tag{4.5}$$

An analogous portrait can be set up in knot theory ([20, 19]). Though this context is intrinsically infinite dimensional, the construction described above can be rigorously carried out *ad hoc.*

4.2. A Morse family interpretation of the Chern-Simons action

In the present section we are going to review our knot theoretical version of Maslov theory ([20, 19]). The first step is the definition of an appropriate Morse family. We consider the weak symplectic manifold $T^*\widehat{Y}_{\mathbf{R}^3}$ the cotangent space associated to $\widehat{Y}_{\mathbf{R}^3}$. The space of $U(1)$-connections $\mathcal{A}$ is treated as a set of auxiliary parameters. It may be identified with $\mathcal{D}_{\mathbf{R}}(\mathbf{R}^3) \otimes \mathbf{R}^3$, the space of compactly supported (real) vector fields on $\mathbf{R}^3$, and standardly topologized accordingly. We regard it as an infinite dimensional manifold modelled on itself (cf. e.g. [59, p.439])

The function

$$\Phi(K, A) := \frac{k}{8\pi} \int_{\mathbf{R}^3} A \wedge dA + \int_K A =: \frac{k}{8\pi} \int_{\mathbf{R}^3} A \wedge dA + T_K A \tag{4.6}$$

with T_K denoting the *current* pertaining to K can be formally interpreted as a Morse family in the sense of Hörmander (see e.g. [63, 51]). As such, it defines locally a Lagrangian submanifold Λ of the cotangent space $T^*\widehat{Y}_{\mathbf{R}^3}$ via the position

$$d_{\mathcal{A}}\Phi \mid_{(K,A)} = \frac{k}{4\pi} F_A + T_K = \frac{k}{4\pi} dA + T_K = 0 \tag{4.7}$$

i.e. it coincides with the Euler-Lagrange equation for the Chern-Simons action plus a source term. This means that we are looking for a connection (viewed as a current) whose curvature is concentrated (i.e. δ-like) on K. Referring to the discussion of Section 2, one has

$$A_K = -\frac{4\pi}{k} \Delta^{-1} \delta T_K. \tag{4.8}$$

Notice that if we want to insert A_K into Φ, we are forced to consider *ordinary* knots. In this case the current T_K may be written in terms of a *singular* Poincaré dual η_K form (which is, nevertheless, cohomologically trivial, since a knot K is always the boundary of a Seifert surface), and represented by a 2-form concentrated on K:

$$T_K(A) = \int_K A = \int_{\mathbf{R}^3} A \wedge \eta_K. \tag{4.9}$$

Also notice that, in view of the vector representation, even a regularized A_K is not compactly supported, but nevertheless lies in the domain of Φ. So, though no strict adherence to the finite dimensional case is possible, the basic pattern persists. This being the case, upon substitution, and regularization (i.e. framing) via employment of a nearby knot K', we find a phase function

$$\phi(K) = -\frac{2\pi}{k}\int_{K'} \eta_K = \int_K \eta_{K'} = -\frac{2\pi}{k}\ell(K, K'). \tag{4.10}$$

The above procedure can be implemented by approximating T_K by means of *bona fide* Poincaré dual forms via the Localization Principle, see e.g. [24]; alternatively, one resorts to the techniques of [72], hinted at in Section 2, yielding solenoidal fields localized in a tube around a knot K possessing a prescribed helicity.

Also notice that the momentum $p\mid_K = d_{\widehat{Y}_{\mathbf{R}^3}}\phi \mid_K = 0$ (for $K \in Y_{\mathbf{R}^3}$). This gives the local description $(K, d_{\widehat{Y}_{\mathbf{R}^3}}\phi|_K = 0)$ of Λ (or rather, its non singular knot part) and gives rise precisely to the (exponent of) the (regular isotopy) Witten invariant ([101, 46, 53], and Subsection 2.4). We summarize the above discussion by means of the following

Theorem 4.1 ([20, 19])**.**

(i) *A framing number can be interpreted as a phase function pertaining to* $Y_{\mathbf{R}^3}$*, looked upon as a Lagrangian submanifold of* $T^*Y_{\mathbf{R}^3}$*, described by the Chern-Simons action, with source term* $\Phi = \Phi(K, A)$*. Moreover, after considering a plane projection and the resulting blackboard framing, and applying the above interpretation to the writhe* w *of a knot, one obtains the (regular isotopy)* Witten invariant *for an* abelian Chern-Simons theory.

(ii) *The local constancy of* $w = w(K)$ *translates into the* eikonal (Hamilton-Jacobi) equation *for this phase function (with zero Hamiltonian)*

$$\|dw|_K\| = 0 \qquad K \in Y_{\mathbf{R}^3}. \tag{4.11}$$

Thus the Witten invariant may be interpreted as a sort of "WKB-wave function".

Remark. One often finds the constant $\frac{k}{4\pi}$ in front of the Chern-Simons term; this being the case, the Witten invariant then becomes $e^{-\frac{\pi i}{k}w(K)}$.

In order to pursue our symplectic interpretation, we need to compute the Hessian of the Morse family $\Phi = \Phi(K, A)$ with respect to the $\mathcal{A}$-variables. Now, since the abelian Chern-Simons action is quadratic, and the source term is linear in A, the Hessian $H(\cdot,\cdot)$ is simply the operator $d = d_{\mathcal{A}}$. As an infinite dimensional, non degenerate quadratic form, it should be regularized via the η-function and replaced by a "relative" version thereof (see e.g [75]); however, inspired by Atiyah's analysis ([11]: indeed he discusses the general case), we bypassed the whole procedure and took directly $H(A_K, A_K)$ as a *regularized signature*, modulo solving the Euler-Lagrange equation for A and removal of the auxiliary parameters. As we have seen before, this can be done at the cost of using singular connections A_K, but substitution in Φ is then possible for ordinary knots only, and $H(A_K, A_K)$ *becomes the* helicity $\mathcal{H}(K)$ *of a solenoidal field associated to K.*

4.3. The cone construction

Let us now retrieve the symplectic manifold $(\hat{Y}_{\mathbf{R}^3}, \beta_0)$ with

$$\beta_0 \mid_K = \int_K \nu_0 \tag{4.12}$$

with ν_0 equal to $(\frac{3}{2})$ the standard volume form on $\mathbf{R}^3$. This particular choice is needed in the discussion of the Bohr-Sommerfeld condition in Section 5. Also, let $\mathbf{r} = \mathbf{r}(t)$, $t \in [0,1]$ denote the vector representation of a generic loop γ. recall the following lemma from [20]:

Lemma 4.2 ([20]**).** *The* 1*-form*

$$\vartheta_0 \mid_\gamma (\cdot) := -\frac{1}{2}\int_\gamma \langle \mathbf{r} | \dot{\mathbf{r}} \times \cdot \rangle \tag{4.13}$$

furnishes a (global) symplectic potential for the form β_0*, i.e.* $\beta_0 = d\vartheta_0$.

The explicit formula for the symplectic potential given above motivates the following geometric construction (*cone construction*, see Figure 2).

Given a singular knot K_0 over the plane $\pi : z = c$ in $\mathbf{R}^3$, $c > 0$, in order to fix ideas. Form a (semi)-cone C_0 over K_0 with vertex at the origin O. It is a ruled, singular surface (with singular lines the generating (half-)lines connecting the vertex with the singular points of K_0. Now define

$$\Lambda_{K_0} := \{K \subset C_0 \mid K \text{ lies strictly above } \pi \text{ and projects regularly onto } K_0\}.$$

Thus any K intersects once all generating lines transversally and is obviously projected from the origin onto K_0 along the generating (half-)lines of C_0. Then Λ_{K_0} is clearly a submanifold of $\hat{Y}_{\mathbf{R}^3}$ and it is immediate to verify that

Theorem 4.3 **(**[20, 19]**).**
(i) *The submanifold* $\Lambda_{K_0} \subset \hat{Y}_{\mathbf{R}^3}$ *is indeed Lagrangian and it is modelled on* $C^\infty(S^1, \mathbf{R})$.
(ii) *The Chern-Simons Lagrangian with knot insertion, i.e.* $\Phi = \Phi(K, A)$ *can be taken as a Morse family for* Λ_{K_0} *as well.*

Remarks.

1. The condition on regular projection (implying transversal intersection with all the generating lines) is crucial for enforcing the Lagrangian condition (indeed, taking for instance a knot running in part along a singular line, the symplectic form β does not vanish thereon. Part (i) of Theorem 4.2 extends slightly Brylinski's basic observation ([25]) that knots on a smooth surface in a threefold M yield a Lagrangian submanifold of $\widehat{Y}_M$. The isotropy condition for the tangent space at any K is clear, the coisotropy one also readily follows from the geometric interpretation of the symplectic form.
2. Consider a non singular knot K on Λ_{K_0}: a sufficiently "small" ribbon around it lying on Λ_{K_0} gives rise to a framing of K, by taking a nearby knot K', which may be naturally assimilated to a blackboard framing (see Figure 2).
3. The second assertion of the above proposition may be interpreted as an infinite dimensional instance of Weinstein's Lagrangian neighbourhood theorem ([100, 65]), roughly asserting that any Lagrangian submanifold L of a symplectic manifold can be viewed locally as a Lagrangian submanifold L' of the cotangent space T^*L'.

4.4. The writhing number as a Maslov index

We now come to our Maslov theoretic interpretation of the writhing number ([20, 19]). The present subsection follows [20] verbatim (see also Figure 2). Consider the submanifold Z'_1 of Λ_{K_0} consisting of all knots on Λ_{K_0} possessing at least one double point (lying necessarily on a singular line of the cone). Its tangent space at any point is modelled on $W_{a,b} = \{f \in C^\infty(S^1, \mathbf{R}) \mid f(a) = f(b)\}$, with $a, b \in S^1$, $a \neq b$, i.e. the kernel of the linear map $f \mapsto f(a) - f(b)$. Thus the quotient space $C^\infty(S^1, \mathbf{R})/W_{a,b} \cong \mathbf{R}$), whence Z'_1 has codimension 1 in Λ_{K_0}. One similarly defines Z'_k (knots on

Λ_{K_0} having at least k double points). Let $Z := Z_1 := Z'_1 \setminus Z'_2$, i.e. Z consists of all knots on Λ_{K_0} possessing exactly one double point: this submanifold is boundary free and can be given a natural co-orientation, induced by the the initial choice of a reference frame and by the orientation on the knots: the "positive" side corresponding to "raising" — after arbitrarily numbering the two oriented branches of the knot at the double point by 1 and 2, respectively, and choosing corresponding (unit) tangent vectors, v_1 and v_2 — the first branch over the other in the direction of $v_1 \times v_2$. This does not depend on the numbering, and a moment's reflection shows that this just corresponds to a passage to writhe $+1$ for the crossing (cf. also [77]). Actually, this arrangement takes place along the cone, but the overall portrait is clear. Similarly, we may define Z_k, $k = 1, 2, \ldots, n$, which have codimension k in Λ_{K_0}. This is essentially, in a simplified context, Vassiliev's point of view ([97]).

If one now takes a path of (ordinary) knots in Λ_{K_0}, starting from a certain desingularization of K_0 to another, one can arrange in such a way that it crosses transversally just Z. At each step, the writhe jumps by ± 2. Thus, for a path Γ from a knot K_- to a knot K_+ subject to a single transversal crossing of Z corresponding to a writhe switch of $+2$, one has:

Theorem 4.4 (+ Definitions, [20, 19]**).**

(i) *The following intersection theoretic formula holds*

$$\frac{1}{2}(w(K_+) - w(K_-)) = 1 = \Gamma \circ Z \tag{4.14}$$

with the r.h.s. denoting the intersection index of Γ *and* Z. *In terms of helicity (looked upon as a regularized signature), the above formula reads*

$$\frac{1}{2}(\mathcal{H}(K_+) - \mathcal{H}(K_-)) = 1 = \Gamma \circ Z. \tag{4.15}$$

Similarly, if for a path of knots Γ *leading from one desingularization to another (still requiring crossing* Z *only), one defines the* Maslov index $m(\Gamma)$ *as (one-half) of the sum of the individual writhe differences, one has*

$$m(\Gamma) = \Gamma \circ Z \tag{4.16}$$

*(*Maslov theorem *for knots).*

(ii) *The following "enhanced" eikonal formula holds (with w regarded as a distribution, i.e. a 0-current):*

$$dw =: 2\,\eta_Z \quad (= p : \ \textit{generalized momentum}) \tag{4.17}$$

where the singular Poincaré dual η_Z *of* Z *is defined by the l.h.s.*

Remarks.

1. Notice that (ii) is just a generalization of the 1-variable formula $d\,sgn = 2\,\delta$, with sgn the sign function.
2. Reversal of the orientation of the knot path (but leaving the orientation of the single knots unchanged) switches the intersection index.
3. We also observe that, since Λ_{K_0} is contractible (clear), no nontrivial global Maslov class can arise. Equivalently, it is clear that the $m(\Gamma) = 0$ for any closed path Γ.
4. The singular Poincaré dual form η_Z admits also regular versions, obtained by regularizing dw, since Λ_{K_0} admits smooth partitions of unity.
5. Clearly, upon changing the origin and/or the projection plane, $w = w(K)$ may change (discontinuously). This phenomenon is well known both in knot theory and, on the other side, in geometric optics and in geometric quantization. This stresses the relevance of Maslov type phenomena in knot theory.

5. Geometric quantization and the Feynman-Onsager condition

5.1. Review of geometric quantization

Let us briefly review the basics of geometric quantization; we refer to e.g. [55, 58, 92, 102, 25] for a thorough treatment). We employ the conventions of [102], with $\hbar = 1$. Recall that the Weil-Kostant theorem states that, given a symplectic manifold (M, ω), with $[\frac{1}{2\pi}\omega] \in H^2(M, \mathbf{Z})$ — $[\,\cdot\,]$ denoting Čech or de Rham cohomology classes — then there exists a complex line bundle (L, ∇, h) over M equipped with a metric $h = (\cdot\,,\,\cdot)$ and a compatible connection ∇ such that its curvature F_∇ equals ω (hence $[\omega] = c_1(L)$, the (first) Chern class of $L \to M$). Call ∇ a prequantum connection, and $L \to M$ a prequantum line bundle. The different choices of $L \to M$ are parametrized by $H^1(M, S^1)$.

In the coadjoint orbit case the integrality of O_{f_0} amounts (Kostant) to the possibility of lifting the stabilizer Lie algebra representation

$$g_{f_0} \ni u \to \langle f_0, u \rangle \in i\,\mathbf{R} \tag{5.1}$$

to a character of the stabilizer G_{f_0}. Then Mackey's inducing procedure yields the desired prequantization bundle.

Now, given a Lagrangian submanifold Λ of M, the two-form ω vanishes upon restriction to Λ by definition, and any (local) symplectic potential ϑ (i.e. a 1-form such that (locally, in general) $d\vartheta = \omega$) becomes a closed form thereon, giving a (local) connection form pertaining to the restriction of the prequantum connection ∇, denoted by the same symbol. The latter is a *flat* connection, and a global covariantly constant section of the (restriction of) the prequantum line bundle) exists if and only if it has trivial holonomy, that is, otherwise stated, the induced character $\chi : \pi_1(\Lambda) \to U(1)$ (with a base point tacitly understood) is trivial (see e.g. [96]). A covariantly constant section (which we call WKB wave function) takes the form

$$s(m) := hol_\gamma(\nabla) \cdot s(m_0) = e^{i \int_\gamma \vartheta} s(m_0) \tag{5.2}$$

with γ denoting any path connecting a chosen point m_0 in a (connected) symplectic manifold M with a generic point $m \in M$, $hol_\gamma(\nabla)$ being the holonomy of the (restriction to Λ of the) prequantum connection ∇ along γ. The r.h.s. tacitly assumes a trivialization of $L \to M$ around m_0, and m in a corresponding local chart.

Remarks.

1. Our definition of WKB-wave function is slightly different (and rougher) from the conventional one (see e.g. [102, Ch. 9]), Indeed we do not require square-integrability, and we do not twist the prequantization bundle with Δ_Λ (whose smooth sections, in the finite dimensional case, consist of the complex n-forms on Λ), thus neglecting the "amplitude-squared" (accordingly, we do not consider the ensuing transport equation).

   Necessary and sufficient conditions for the existence of a covariantly constant section are provided by the *Bohr-Sommerfeld conditions*:

$$\int_\gamma \vartheta \in 2\pi\mathbf{Z} \tag{5.3}$$

   for any closed loop γ in Λ. Clearly, they only depend on the classes $[\gamma] \in \pi_1(\Lambda)$.

2. There is a version of the Bohr-Sommerfeld conditions incorporating the Maslov class, but we shall not need this refinement in what follows. Also, we are not going further in the actual completion of the quantization procedure, in this real environment, since it requires more ingredients (*in primis* Lagrangian fibrations) than those present in our context.

We also wish to comment briefly on *holomorphic quantization*, which is possible as soon as the (integral) symplectic manifold is also a Kähler manifold, and one takes the space of holomorphic sections of a holomorphic prequantum bundle, provided it is nontrivial, as the Hilbert space of the theory. In this case one has a naturally defined connection, called the Chern or Chern-Bott connection, compatible with both the hermitian and the holomorphic structure (see [45]). Working out the basic case given by $\mathbf{C}^n$ with its standard Kähler form yields the well known Bargmann-Fock representation. We are going to delve a bit further into this in Section 8, where we shall examine a slightly more refined example which is important in the FQHE theory ([27]).

5.2. The regular isotopy Witten invariant as a covariantly constant section

In this subsection we are going to restrict Brylinski's prequantum bundle $L \to \hat{Y}_{\mathbf{R}^3}$ (it is, in this case, unique up to isomorphism) to two different types of Lagrangian submanifolds:

1. the submanifold Π of plane knots with a finite number of crossings wherein transversal intersections occur.
2. the submanifold Λ_{K_0},

and discuss the corresponding Bohr-Sommerfeld conditions.

In the following subsection we will do the same for:

3. the submanifold $\hat{Y}_{S^2}$ of singular knots lying on the unit sphere.

Case 1. In this case, given a knot $K_0 \in \Pi$, and choosing a desingularization K_0' thereof (if K_0 has n double points with transversal crossings, there are 2^n choices), we find, after trivializing, a natural covariantly constant section s which is global on the connected component of Π containing K_0:

$$s(K) = e^{i\,\alpha\, w(K)} \tag{5.4}$$

(with α any real constant)

The connected component in question is clearly contractible, therefore no BS-condition arises.

Case 2. The previous discussion can be carried over verbatim to the submanifold Λ_{K_0}, and we allow for discontinuities coming from crossing the Maslov cycle Z. A "natural" choice would be to set $\alpha = \frac{\pi}{4}$, which yields a phase shift of $\frac{\pi}{2}$ upon traversal of Z, in view of the enhanced eikonal equation, in accordance with geometric optics. This condition can be viewed as a sort of corrected Bohr-Sommerfeld condition in which there is the potential term vanishes and a purely Maslov part (the Poincaré dual η_Z) survives, i.e. we have a sort of Parmenidean classical mechanics, with no dynamics at all.

The preceding discussion can be summarized via the following:

Theorem 5.1 ([20, 19]).

(i) *The abelian (regular isotopy) Witten invariant can be interpreted as a WKB wave function in Brylinski's framework, i.e. as a covariantly constant section of the restriction of the prequantum line bundle to the submanifold* Π *of singular knots on a plane, looked upon as a Lagrangian submanifold of the classical phase space* $\hat{Y}_{\mathbf{R}^3}$.

(ii) *The same holds for the Lagrangian submanifold* Λ_{K_0} *defined above via the cone construction.*

5.3. The Feynman-Onsager condition

Here we discuss the Bohr-Sommerfeld origin of the Feynman-Onsager (FO) condition arising in quantum vortex theory (see e.g. [76, 40, 86, 47]) and interpreted in terms of integrality of knot coadjont orbits in [80] (see also similar remarks in [8, Ch. 6]). It states that the flux is quantized in multiples of $\Phi = \frac{h}{m_4}$, with h being Planck's constant and m_4 the mass of the ^4He atom.

We consider the following Lagrangian submanifold of $\hat{Y}_{\mathbf{R}^3}$:

Let $\hat{Y}_{S^2} = \{C \in \hat{Y}_{\mathbf{R}^3} \mid C \subset S^2\}$. Its non-singular counterpart is Lagrangian as well.

Resuming the 2-form μ_0 defined above, we immediately see that μ_0 restricts to one-half the standard area form on S^2. The 2-form β_0 vanishes upon restriction to LS^2 (hence to $\hat{Y}_{S^2}$), and ϑ_0 becomes a closed 1-form thereupon, giving a (local) connection form pertaining to the restriction of the prequantum connection ∇, which is thence *flat*. Now, the Gauß integral yielding the linking number $\ell(\gamma_1, \gamma_2)$ pertaining to two non intersecting parametrized knots γ_1 and γ_2, namely

$$\ell(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^1 dt \int_0^1 ds \, \frac{\langle \gamma_2(t) - \gamma_1(s) | \dot{\gamma_1}(s) \times \dot{\gamma_2}(t) \rangle}{\|\gamma_2(t) - \gamma_1(s)\|^3} \tag{5.5}$$

can be naturally interpreted as $\frac{1}{2\pi}\int_\Gamma \vartheta$ along a suitable loop Γ in LS^2, (i.e. $\Gamma \in LLS^2$) inducing, generically, an obvious loop in $\hat{Y}_{S^2}$ (denoted in the same way):

$$\Gamma : t \mapsto \Gamma_t : s \mapsto \Gamma(t,s) := \frac{\gamma_2(t) - \gamma_1(s)}{\|\gamma_2(t) - \gamma_1(s)\|}. \tag{5.6}$$

This is checked as in [8]. Also recall, again e.g. from [8], that $\ell(\gamma_1, \gamma_2) = deg\,\Gamma$, the degree of Γ, looked upon as a map from the torus $S^1 \times S^1$ to S^2. This is a manifestation of the fact that $\pi_1(LS^2) \cong \pi_2(S^2) \cong \mathbf{Z}$. Notice, in passing, that in view of the latter, any loop in LS^2 is homotopic to a loop induced by the above construction: in particular the knots γ_i can be chosen to be very special.

Now set $\vartheta := \alpha\vartheta_0$. The Bohr-Sommerfeld condition now requires

$$\int_\Gamma \vartheta \in 2\pi\mathbf{Z} \tag{5.7}$$

which entails $\alpha = k$, $k \in \mathbf{Z}$. That is, the Bohr-Sommerfeld condition selects those symplectic forms which are an integer multiple of a fixed one, i.e. β_0, and positivity can be imposed upon requiring that the corresponding Kähler form is positive. This is, in essence, the Feynman-Onsager quantization condition arising in quantum vortex theory. We use the same name for the present mathematical version. The above discussion provides a rigorous derivation of the latter. The various powers give rise to a covariantly constant section of the tensor powers $\mathcal{L}^{\otimes k} \to \widehat{Y}_{\mathbf{R}^3}$ of the prequantum line bundle. The upshot of the preceding discussion is following:

Theorem 5.2 (FO = BS, [20, 19]**).** *The FO-quantization condition is tantamount to the BS-quantization condition applied to the Lagrangian submanifold $\hat{Y}_{S^2}$ of $\hat{Y}_{\mathbf{R}^3}$, ensuring the existence of a covariantly constant section of the restriction of Brylinski's prequantum bundle (and its tensor powers) thereon.*

Remark. The full quantization of the theory remains problematic. In particular it is not clear whether the above covariantly constant section could be usefully employed in quantum vortex theory: we just observe that it is entirely governed by the Biot-Savart kernel and that, since the symplectic form depends on the knot γ just up to a rigid motion by its very definition, we may use $\dot{\gamma}$ (derivation with respect to the arc length) in the actual description of the knot. But the evolution of $\dot{\gamma}$ takes place, by definition, on a unit sphere (the length of γ may vary, in general). The explicit formulae derived in [80] might prove useful in this context.

6. A differential geometric approach to higher order linking

In this section we condense the content of [84], sometimes improving the presentation given therein. As before, we need to make some digressions, freely delving into that paper, again for the sake of clarity.

6.1. Chen's iterated path integrals and nilpotent connections

Chen's iterated path integrals provide an extremely general and flexible technical tool usefully employed throughout mathematics (see [31] for a comprehensive account). Here we just follow our own exposition of some basic facts concerning the simplest of them, extracted from [84].

Let M be a smooth manifold. Let $\gamma : [0,1] \to M$ be any smooth path with velocity field $\dot\gamma$. Let $\omega_1, \ldots, \omega_m$ be 1-forms, with $\omega_i(t_i) := \omega_i(\gamma(t_i), \dot\gamma(t_i))$, $i = 1, \ldots, m$, and denote by $\triangle^m$ the standard m-simplex in $\mathbf{R}^m$:

$$\triangle^m := \{(t_1, \ldots, t_m) \in \mathbf{R}^m \mid 0 \le t_1 \le t_2 \le \cdots \le t_m \le 1\}.$$

Then define the (*Chen*) *iterated path integral*

$$\int_\gamma \omega_1 \cdots \omega_m := \int_{\triangle^m} \omega_1(t_1) \cdots \omega_m(t_m)\ dt_1 \cdots dt_m. \tag{6.1}$$

Equivalently, setting $\gamma^t : [0,1] \ni s \mapsto \gamma(ts) \in M$, we may also write down, recursively:

$$\int_\gamma \omega_1 \cdots \omega_m \quad = \quad \int_\gamma \left(\int_{\gamma^t} \omega_1 \cdots \omega_{m-1} \right) \omega_m. \tag{6.2}$$

These kinds of Chen integrals allow us to express Hain's and Tavares' formulae for parallel transport of nilpotent connections, (see [28, 50, 93]). We consider connections (on the appropriate trivial vector bundle over M) of the form

$$\mathbf{v} = \begin{bmatrix} 0 & v_1 & v_{12} & \ldots & v_{12\ldots n} \\ 0 & 0 & v_2 & \ldots & v_{2\ldots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & v_n \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{6.3}$$

where the v's are 1-forms. The curvature form Ω of $\mathbf{v}$ reads

$$\Omega = d\mathbf{v} + \mathbf{v} \wedge \mathbf{v} = \begin{bmatrix} 0 & w_1 & w_{12} & \dots & w_{12\dots n} \\ 0 & 0 & w_2 & \dots & w_{2\dots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & w_n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{6.4}$$

where

$$\begin{aligned} w_1 &= dv_1, \\ w_{12} &= v_1 \wedge v_2 + dv_{12}, \\ &\dots \\ w_{12\dots n} &= v_1 \wedge v_{2\dots n} + v_{12} \wedge v_{3\dots n} + \dots dv_{12\dots n}, \end{aligned} \tag{6.5}$$

and analogous formulae for the other terms.

The parallel transport (holonomy) operator reads:

$$U(\gamma) = \begin{bmatrix} 1 & \int_\gamma u_1 & \int_\gamma u_{12} & \dots & \int_\gamma u_{12\dots n} \\ 0 & 1 & \int_\gamma u_2 & \dots & \int_\gamma u_{2\dots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \int_\gamma u_n \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{6.6}$$

where

$$\begin{aligned} \int_\gamma u_1 &:= \int_\gamma v_1, \\ \int_\gamma u_{12} &:= \int_\gamma v_1 v_2 + v_{12}, \\ \int_\gamma u_{123} &:= \int_\gamma v_1 v_2 v_3 + v_{12} v_3 + v_1 v_{23} + v_{123} \\ &\dots \\ \int_\gamma u_{12\dots n} &:= \int_\gamma v_1 v_2 \dots v_n + \dots + v_{12\dots n} \end{aligned} \tag{6.7}$$

with Chen integrals appearing in the r.h.s. (see [50, 93]). Formula 6.6 can be also written in what we call *Magnus form* (familiar from combinatorial group theory, see [62] and below):

$$U(\gamma) = I + K_\gamma \tag{6.8}$$

where $I = I_{n+1}$ is the identity matrix in $n + 1$ dimensions, and with an obvious definition of K_γ. This will be useful in the sequel. Moreover, for any permutation $\sigma \in \Sigma_n$, one can define a connection $\mathbf{v}_\sigma$ with curvature Ω_σ as above. Furthermore, in the sequel we shall employ the Dirac bracket notation (which requires the introduction of the standard Euclidean structure on the appropriate numerical vector space acted upon by the Hain-Tavares matrices) . We also notice the following

Proposition 6.1 (Bianchi identity).

$$d\Omega = \Omega \wedge \mathbf{v} - \mathbf{v} \wedge \Omega. \tag{6.9}$$

whose proof is immediate. In our specific applications of iterated integrals we shall work on S^3 (and deal with currents thereon), or with $S^3 \setminus \mathcal{L}$ (the complement of a link $\mathcal{L}$, see Subsection 6.2 below), possibly with a background metric on S^3 (this has clearly no influence in the final outcome, which is topological in nature), or we may resort to the topology considered by Tavares ([93]).

We recall the following crucial

Theorem 6.2.

Approximate Stokes Formula. *Let γ_ϵ be a "small" loop (cf.* [93] *and Figure* 3*) bounding a "surface"* (γ_ϵ). *With the above notations, we have*

$$\int_{\gamma_\epsilon} u_{12\ldots r} = \int_{(\gamma_\epsilon)} w_{12\ldots r} + o(\epsilon^2). \tag{6.10}$$

Topological Stokes Formula ([84]**).** [we work in 3-d, see also below]. *Assume that the curvature form $w_{12\ldots r}$ is delta-like and supported by a knot $L_{12\ldots r}$, i.e. it is the vorticity form pertaining to $L_{12\ldots r}$ (i.e. it is looked upon as a current). Then Stokes' formula becomes exact:*

$$\int_{\gamma_\epsilon} u_{12\ldots r} = \int_{(\gamma_\epsilon)} w_{12\ldots r}. \tag{6.11}$$

Sketch of the proof. The proof of the first part is folklore, see e.g. [26] or [93]. As for the second assertion, observe that for $\epsilon > 0$ sufficiently small the parallel transport operator $U(\gamma_\epsilon)$ is constant by virtue of Schlesinger's theorem ([91], and also [93, 26, 35]), so is the integral in the r.h.s., whence the correction term must vanish. □

6.2. Link homology

We need to gather together some basic notions concerning links (cf. [90, 39, 73]). An n-link $\mathcal{L} = (L_1, L_2, \ldots, L_n)$ (with n components L_i, $i = 1, 2, \ldots, n$) is a set of (smooth) closed non intersecting curves in the three-sphere S^3, viewed as the one point-compactification of $\mathbf{R}^3$). Let $C(n)$ denote the space consisting of n disjoint circles. Two links $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be *homotopic* if there exists a 1-parameter family h_t, $t \in [0, 1]$ of smooth embeddings of $C(n)$ into S^3 such that $h_0(C(n)) = \mathcal{L}_1$, $h_1(C(n)) = \mathcal{L}_2$ and such that disjoint circles in $C(n)$ have disjoint images. Two links $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isotopic* if in addition to the above, $h_t(C(n))$ is a link for every t. The latter notion is stronger than the former. One can give a slightly different but equivalent definition by defining links to be the (smooth) embeddings themselves and changing things accordingly ([85]). A link is said to be *(homotopically) trivial* if it is homotopic to a link consisting of n points. It is said to be *isotopically trivial* if every component is unlinked with the others, i. e. it can be separated from the other components by a homeomorphic image of the two-sphere S^2. For example, the link consisting of the *Borromean rings* is not homotopically trivial, whereas the *Whitehead link* — already appearing in Maxwell's investigations — is homotopically trivial but not isotopically trivial (Figure 4).

So let us consider a smooth oriented n-link $\mathcal{L}$ in S^3 with components L_j, $j = 1, 2, \ldots, n$. We take $n \geq 2$ and require the single components to be *trivial* knots, whereby the Seifert surfaces of their components are discs; nevertheless, any link is homotopically equivalent to a link with unknotted components.

Recall the isomorphisms

$$\begin{array}{ccccc} H^1(S^3 \setminus \mathcal{L}) & \cong & H_2(S^3, \mathcal{L}) & \cong & \mathbf{R}^n \\ H^2(S^3 \setminus \mathcal{L}) & \cong & H_1(S^3, \mathcal{L}) & \cong & \mathbf{R}^{n-1} \end{array} \qquad (6.12)$$

where (real) singular (or de Rham) cohomology and relative homology groups are involved (see also Figure 5). It is crucial for the sequel to bear in mind their explicit hydrodynamical realization ([78, 49, 84], and cf. Section 2 as well). As natural representatives for a basis of $H^1(S^3 \setminus \mathcal{L})$ one may take v_j equal to the velocity fields of a perfect fluid (viewed as 1-forms) pertaining to the link components, thought of as vortex lines: one has $dv_j = 0$, in $S^3 \setminus \mathcal{L}$, or otherwise work in S^3, so $dv_j = \eta_j$ is a δ-like distribution (current) supported by L_j (singular Poincaré dual). In terms of singular homology (i.e. in $H_1(S^3 \setminus \mathcal{L})$) they correspond to suitable loops γ_i encircling the corresponding components with linking number equal to one . The homological counterpart (in $H_2(S^3, \mathcal{L})$) of the v_j are discs $\mathbf{a}_j$ with boundary L_j. One has, similarly, $v_j = \eta_{\mathbf{a}_j}$. As for $H_1(S^3, \mathcal{L})$, explicit representatives are provided by (smooth) oriented paths connecting different link components, call them γ_{ij}. The $\zeta_{ij} := [\gamma_{ij}]$ are subject to the relation $\zeta_{ik} + \zeta_{kj} = \zeta_{ij}$. Now fix the link component L_n. The (singular) *homological* version of ζ_{kn} in $H_2(S^3 \setminus \mathcal{L})$, for $k = 1, 2 \ldots n-1$, which we denote by α_{kn} can be represented by ∂T_k, the boundary of a toroidal neighbourhood T_k of L_k. Its de Rham representative is, in turn, the "electric field" 2-form f_{kn} generated by the link components L_k and L_n, carrying opposite unit charges (see [78, Section 4.2]).

Notice that (for $h, k = 1, 2, \ldots, n-1$)

$$< [f_{kn}], \alpha_{hn} >= \int_{\partial T_h} f_{kn} = i(\gamma_{kn}, \partial T_h) = \delta_{kh} \tag{6.13}$$

and (for $h, k = 1, 2, \ldots, n$)

$$< [v_k], [\gamma_h] >= \int_{\gamma_h} v_k = i(v_k, \gamma_h) = \delta_{kh} \tag{6.14}$$

(with $<,>$ and i denoting here de Rham pairings and intersection numbers, respectively) since the above duality intertwines wedge product of forms in de Rham groups (or, equivalently, cup product for singular cohomology) and cap product in relative homology groups. We are going to elaborate on this in the following reformulation of the Gauß linking number.

Consider, to be specific, two link components L_1 and L_2. The curvature form

$$w_{12} = v_1 \wedge v_2 \tag{6.15}$$

is *closed* (on $S^3 \setminus \mathcal{L}$) so it represents an element $< v_1, v_2 >$ (*Massey product* notation) in $H^2(S^3 \setminus \mathcal{L})$. Duality yields an element

$$\ell_{12} \cdot \zeta_{12} \in H_1(S^3, \mathcal{L}) \tag{6.16}$$

which can be represented by a (multiple of an) oriented path connecting L_2 to L_1, which, in turn, appears as the (oriented, and with appropriate multiplicity) intersection of discs $\mathbf{a}_1$ and $\mathbf{a}_2$ bounding the corresponding components L_1 and L_2 (self-intersection is allowed). The number ℓ_{12} is an *integer* and is precisely the *Gauss linking number* pertaining to the oriented components L_j, $j = 1, 2$. It can be readily computed (distributionally or via a Poincaré dual "regularization", cf. [24]) as

$$\ell_{12} = \int_{S^3} \eta_1 \wedge v_2 = \int_{S^3} \eta_2 \wedge v_1 = \ell_{21}. \tag{6.17}$$

Indeed (by Stokes' formula, and resorting to currents on S^3 — see also Section 2 — upon recalling that $dv_j = \eta_j$), evaluating
$< v_1, v_2 >$ on $\alpha_{12} = [\partial T_1]$ yields

$$< v_1, v_2 > [\partial T_1] = \int_{\partial T_1} w_{12} = \int_{T_1} dw_{12} = \int_{T_1} \eta_1 \wedge v_2 = \int_{S^3} \eta_1 \wedge v_2 = \ell_{12} \tag{6.18}$$

and similarly $\int_{\partial T_2} w_{12} = \ell_{12}$, yielding the desired assertion. We can rephrase this by saying that $w_{12} = v_1 \wedge v_2$ is cohomologous to $\ell_{12} f_{12}$. We also use the notation $m(1, 2)$ for ℓ_{12} (Massey linking number, see below). This is a curvature interpretation of the Gauss linking number.

Also observe that, regarding, say v_2 as a *connection* form on $S^3 \setminus \mathcal{L}$, one realizes ℓ_{12} as the *holonomy* (parallel transport) — abuse of language — of v_2 around L_1 (see below for the definition of Γ_1):

$$\ell_{12} = m(1,2) = \int_{L_1} v_2 = \int_{\Gamma_1} v_2. \tag{6.19}$$

Thus we have a second differential geometric interpretation of the Gauss linking number, in terms of holonomy of a subconnection. We are going to set up a similar portrait for higher order linking numbers later on.

Now, the vanishing of ℓ_{12} entails the exactness of $v_1 \wedge v_2$, whence there exists a 1-form v_{12} fulfilling

$$v_1 \wedge v_2 + dv_{12} = 0. \tag{6.20}$$

The following homological interpretation of v_{12} is crucial: it corresponds to a *new* disc $\mathbf{a}_{12}$ formed with part of the boundaries of $\mathbf{a}_1$ and $\mathbf{a}_2$ and paths γ_{12} and γ_{21} connecting the two components with opposite orientations. Thus v_{12} can also be viewed as the velocity 1-form pertaining to a knot L_{12} (in S^3) bounding $\mathbf{a}_{12}$, with vorticity $dv_{12} = \eta_{L_{12}}$. Eventually, the homological counterpart or the above equation is (see Figure 5)

$$\mathbf{a}_1 \cap \mathbf{a}_2 + \partial \mathbf{a}_{12} = 0. \tag{6.21}$$

We shall resume this discussion later on, after a necessary detour on Milnor invariants.

6.3. Combinatorial group theory and Milnor invariants

In this Subsection we gather some basic notions from combinatorial group theory. We refer to [62] for full details. Let F be a free group on r generators a_1,a_2,…,a_r and let F_n be its nth lower central subgroup, i.e. setting $F_1 =: F$, define, recursively, $F_k := [F, F_{k-1}]$ ($[.,.]$ denoting group commutator. Furthermore, let $\mathbf{Z}$ be the ring of integers and denote the freely generated associative ($\mathbf{Z}$-)algebra of rank r on generators K_1,K_2,…,K_r by $A_0(\mathbf{Z}, r)$, and the free ($\mathbf{Z}$-)Lie algebra of rank r over $\mathbf{Z}$ on generators ξ_1,ξ_2,…,ξ_r by $\Lambda_0(\mathbf{Z}, r)$. Recall that there is a natural (injective) map $\mu : \Lambda_0(\mathbf{Z}, r) \to A_0(\mathbf{Z}, r)$ (such that $\mu(\xi_j) = K_j$) which is $\mathbf{Z} - linear$ and preserves brackets (in $A_0(\mathbf{Z}, r)$ one sets $[a, b] := ab - ba$). The image $Im\, \mu(\Lambda_0(\mathbf{Z}, r)) \subset A_0(\mathbf{Z}, r)$ consists of the so called Lie elements and turns out to be a Lie algebra isomorphic with $\Lambda_0(\mathbf{Z}, r)$.

Let $\delta : F \to A_0(\mathbf{Z}, r)$ be induced by the position $a_j \mapsto 1 + K_j$ (Magnus' trick, cf. Subsection 6.1). We need the following theorem (which is part of Corollary 5.12 in [62]).

Theorem 6.3 ([62]). *With the above notations, if u_n is any homogeneous Lie element of degree n in $A_0(\mathbf{Z}, r)$, then there exists an element W_n in F_n, uniquely determined* $\bmod F_{n+1}$ *such that*

$$\delta(W_n) = u_n.$$

Let us consider a link $\mathcal{L}$ as above. The fundamental group of the link is by definition $G := \pi_1(S^3 \backslash \mathcal{L})$. Set $G_1 = G$ and let $G_{q+1} := [G, G_q]$ be the qth lower central subgroup of G. The factor groups G/G_q are isotopy invariants of $\mathcal{L}$ ([29, 31, 68]). They can be given the so-called *Milnor presentation* (see [68, 85]) which we briefly recall, closely following [85] (see Figure 6 as well). Explicitly, consider disjoint tubular neighbourhoods of the link components. Then for each tube T, choose subtubes $T \supseteq T^1 \cdots \supseteq T^q$. Further, one chooses the base point $*$ in the complement of the union of all tubes. Then one takes a path p_i from $*$ to L_i for each $i = 1, 2, \ldots, n$. A *meridian* γ_i is obtained by following p_i up to a point in $T^q \setminus L_i$, then encirling L_i via a loop (homotopically trivial in T^q) having linking number $+1$ with L_i and finally tracing p in the opposite direction to reach $*$ again. In this way one gets an element of G/G_q, still denoted by γ_i ([68]). A *parallel* Γ_i is obtained by reaching a point in $T^q \setminus L_i$ from $*$, then by traversing a loop homotopic to L_i within T^q having zero linking number with L_i and then returning to $*$ along p_i. Again one gets an element in G/G_q, still denoted by Γ_i. Upon choosing another path p', the above meridians and parallels are replaced by suitable conjugates. Then the so called *Milnor presentation* of G/G_q reads as follows:

$$G/G_q = \{\gamma_i, i = 1, 2, \ldots, n; [\gamma_i, \Gamma_i] = \mathbf{1}, i = 1, 2, \ldots, n, F_q\} \tag{6.22}$$

where $\mathbf{1}$ is the identity of G, F_q is the qth lower central subgroup of the free group generated by the γ's, and Γ_i is expressed as a word in the γ's.

Now we come to the definition of the Milnor invariants. Upon viewing Γ_i as a word in the γ's, we cast it in the Magnus form starting from the positions

$$\gamma_i = 1 + K_i, \qquad {\gamma_i}^{-1} = 1 - K_i + K_i^2 - K_i^3 + \cdots . \tag{6.23}$$

Here 1 denotes the unit in $A_0(\mathbf{Z}, n)$. For $p < q$, let $(i_1, i_2, \ldots, i_p)$ be a sequence of integers with $1 \leq i_j \leq n$. Let $\mu(i_1, i_2, \ldots, i_p)$ be the coefficient, in the Magnus expansion of $\Gamma_{i_1,}$, of the term $K_{i_2} K_{i_3} .. K_{i_p}$ (we are adopting a slight different definition from [85], which however does not alter the nature of things). Setting $\Delta(i_1, i_2, \ldots, i_p)$ equal to the g.c.d. of the numbers $\mu(j_1, j_2, \ldots, j_s)$, with $s \geq 2$, where $(j_1, j_2, \ldots, j_s)$ ranges over all cyclic permutations of proper subsequences of $(i_1, i_2, \ldots, i_p)$, we define the Milnor invariant $\overline{\mu}(i_1, i_2, \ldots, i_p)$ as the residue class modulo $\Delta(i_1, i_2, \ldots, i_p)$ of $\mu(i_1, i_2, \ldots, i_p)$. This yields a homotopy invariant of the link for distinct indices, and an isotopy invariant in general. The vanishing of all $\overline{\mu}$'s with distinct indices is a necessary and sufficient condition for the homotopical triviality of the link. Here we shall be concerned with μ-invariants only.

6.4. Brunnian links and the Borromean algorithm

An n-link $\mathcal{L}$ is called *almost trivial*, or *Brunnian* if every sublink extracted therefrom is trivial ([67, 68, 90]). We now recall the algebraic procedure for constructing such links, based on combinatorial group theory, devised in [84] (Borromean Algorithm), see also Figure 6.

Let $\mathcal{L}'$ be a trivial $(n-1)$-link with components $L_2, L_3, \ldots, L_n$. The fundamental group of its complement, let it be H, is the free group on $n-1$ generators, which may be represented, modulo H_q (we take $q > n$), by γ_i, $i = 2, \ldots, n$. Now starting from a homogeneous Lie element u_{n-1} in $A_0(\mathbf{Z}, n-1)$ of degree $n-1$ in the K's, an element Γ_1, determined modulo H_n, is obtained via the "inverse" of the Magnus map: this element may be taken as L_1, up to homotopy. As a concrete example take, say,

$$\Gamma_1 = [\gamma_2, [\gamma_3, [\ldots[\gamma_{n-1}, \gamma_n]]\ldots]. \tag{6.24}$$

In this case, a straightforward computation yields

$$\mu(1, 2, \ldots, n) = 1 \tag{6.25}$$

and one obtains an almost trivial, but not trivial n-link. The other Γ's are trivial.

The number of possibilities is governed by Witt's theorem, namely, we have the following result:

Theorem 6.4 (Borromean algorithm, [84]**).**

(i) *Given a trivial link $\mathcal{L}'$ with components $L_2, L_3, \ldots, L_n$. If L_1 is an extra component such that its corresponding parallel Γ_1 belongs to the $(n-1)$-th lower central subgroup of $\mathcal{L}'$, then the new link $\mathcal{L}$ obtained by adjoining L_1 is almost trivial (but not trivial).*

(ii) *Specifically, any choice of a basis element in $\Lambda_0(\mathbf{Z}, n-1)$ of degree $n-1$ yields an almost trivial n-link with $\mu(1, 2, \ldots, n) = \pm 1$. The number ν of such basis elements is*

$$\nu = \frac{1}{n-1} \sum_{d|(n-1)} \mu(d)(n-1)^{\frac{n-1}{d}}. \tag{6.26}$$

Different basis elements yield non homotopic links.

Here μ denotes the Möbius function. For the sake of completeness, we record its definition: $\mu(1) = 1$, $\mu(p) = -1$ for p prime, $\mu(p^k) = 0$, for $k > 1$, and $\mu(bc) = \mu(b)\mu(c)$ whenever b and c are coprime.

An example of Brunnian 4-component link is given in Figure 4 (see [73, 84], and also [38]). It will be again used in the next Section for the demonstration of the differential geometric construction described therein.

6.5. Topological nilpotent connections, Massey invariants and the Turaev-Porter theorem

We shall now review the recursive procedure for constructing connections associated to a link set up in [84]. Here we shall confine ourselves to the case of distinct indices, and refer to the original paper for the general case. We start by considering the following nilpotent connection

$$\mathbf{v}^{(1)} = \begin{bmatrix} 0 & v_1 & 0 & \dots & 0 \\ 0 & 0 & v_2 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & v_n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{6.27}$$

Computing its curvature yields

$$\begin{array}{rcl} w_1 & = & dv_1 = 0, \\ w_{12} & = & v_1 \wedge v_2, \\ & \dots & \\ w_{123} & = & 0 \end{array} \tag{6.28}$$

etc. Now, the vanishing *all* linking numbers $\ell_{i,i+1}$ entails the exactness of all forms $v_i \wedge v_j$, $j = i + 1$, whence there exists 1-forms v_{ij} fulfilling

$$v_i \wedge v_j + dv_{ij} = 0 \tag{6.29}$$

and, homologically, discs $\mathbf{a}_{ij}$ such that

$$\mathbf{a}_i \cap \mathbf{a}_j + \partial \mathbf{a}_{ij} = 0. \tag{6.30}$$

Using v_{ij} we can manufacture a *new* connection $\mathbf{v}^{(2)}$, whose curvature partially vanishes (the w_{ij} terms, by construction), but yielding a priori non zero terms w_{ijk}. Consider, to fix ideas, the term w_{123}:

$$w_{123} = v_1 \wedge v_{23} + v_{12} \wedge v_3. \tag{6.31}$$

A direct check (or the use of Bianchi identity) shows that w_{123} is *closed*, thereby yielding and element in $H^2(S^3 \setminus \mathcal{L})$ corresponding, homologically, to $N \cdot \zeta_{13}$, with

$$N =: \ell_{123} = \textstyle\int_{T_1} w_1 \wedge v_{23} = \int_{S^3} w_1 \wedge v_{23} = \ell_{1,23} \tag{6.32}$$

where $\ell_{1,23}$ is an *ordinary* linking number (this is clear in view of the previous discussion). The *integer* ℓ_{123} is, by definition, a *third order linking number* (in short, 3-linking number). The class $[w_{123}] =:< v_1, v_2, v_3 >$ is a *Massey product* ([64, 73, 85]). Recall, from Section 2, that the original linking number (which may be called, accordingly, 2-linking number) was associated with $< v_1, v_2 >$. The class $[w_{123}]$ may well be not trivial (*Borromean rings*). The whole scheme can be readily extended inductively so, provided all $(k-1)$-linking numbers vanish, one can define k-linking numbers. the whole process can, in principle, be pursued up to the connection $\mathbf{v} := \mathbf{v}^{(n-1)}$ and to the (Massey) n-linking number $\ell_I := \ell_{12\ldots n} = m(1, 2, \ldots, n)$. A similar reasoning can be applied to any permutation σ, abutting at the n-linking number ℓ_σ defined accordingly. The n-links for which n-linking numbers can be defined (for all permutations) are the almost trivial links previously defined. In summary, the following is true:

Theorem 6.5 ([84])**.** *Under the preceding assumptions, and with the above notation, the following assertions hold:*

(i) *If all connections $\mathbf{v}^{(k)}$, $k = 1, 2, \ldots, n-2$ are flat, the n-linking number ℓ_I, (i.e. the Massey invariant $m(1, 2, \ldots, n) =< v_1, v_2, \ldots, v_n > [\partial T_1]$), can be detected by the cohomology class in $H^2(S^3 \setminus \mathcal{L})$ determined by the only a priori non vanishing curvature form $w_{12\ldots n}$ in the curvature matrix of the connection $\mathbf{v}$, namely*

$$w_{12\ldots n} = v_1 \wedge v_{23\ldots n} + v_{12} \wedge v_{3\ldots n} + \cdots + v_{12\ldots n-1} \wedge v_n, \tag{6.33}$$

and reads:

$$\begin{aligned} \ell_I = m(1, 2, \ldots, n) &= \int_{\partial T_1} w_{12\ldots n} \\ &= \int_{T_1} dw_{12\ldots n} = \int_{S^3} \eta_{L_1} \wedge v_{2\ldots n}. \end{aligned} \tag{6.34}$$

The 1-form $v_{2\ldots n}$ can be chosen to be closed on $S^3 \setminus L_{23\ldots n}$, with $dv_{23\ldots n} = \eta_{L_{23\ldots n}}$ (a Poincaré dual form pertaining to $\eta_{L_{23\ldots n}}$). Thus we also have

$$\ell_I = m(1, 2, \ldots, n) = \ell(L_1, L_{23\ldots n}) \tag{6.35}$$

i.e. ℓ_I can be interpreted as an ordinary *linking number.*

(ii) *The parallel transport operator for the subconnection* $\mathbf{v}'$ *(which is flat on* $S^3 \setminus (\mathcal{L}' \cup L_{23\ldots n})$*) corresponding to the link* $\mathcal{L}'$ *obtained by removing the first component of* $\mathcal{L}$ *(in the r.h.s.* I *denotes the* $n \times n$ *identity matrix) reads*

$$\begin{aligned} U'(L_1) &= I + m(1,2,\ldots,n)|1><n| \\ &= I + \mu(1,2,\ldots,n)|1><n|. \end{aligned} \tag{6.36}$$

The last equality expresses a special case of the Turaev-Porter theorem.

The preceding formula is obtained by expressing L_1 (or, rather, Γ_1) as a word in γ_i, $i = 2, 3, \ldots, n$, and by exploiting flatness and the very definition of Milnor numbers. In the course of the proof one uses the following differential geometric Magnus type formula

$$U'(\gamma_i) = I + K_i = I + |i-1><i|, \qquad i = 2, 3, \ldots, n \tag{6.37}$$

which again follows easily from the topological Stokes formula.

See Figure 7 for two instances of the preceding techniques. We just mention the fact that Whitehead link is accounted for by means of two fourth-order invariants with repeated indices, see [85, 84].

For a clear account of the (standard proof of the) Turaev-Porter theorem, the reader can consult, besides the original papers, Fenn's monograph [39]. We shall draw some further conclusions from the above theory in the following sections.

Brunnian links have found some applications in the study of quantum entanglement, see e.g. Aravind ([5]) and [13] as well.

7. Magnetic energy relaxation and topological bounds

We would like to add a few words about the issue of magnetic fields frozen in a fluid, when their flux lines exhibit a prescribed topological pattern (e.g. they are modelled on a knot or link) which is preserved under the motion. The magnetic energy is, on the other hand, dissipated (*relaxation*, with a consequent "shortening" and "fattening" of the flux tubes) up to an insurmountable limit dictated by topology ([70, 41, 42, 8]). A prototype of this kind of results is Arnol'd's "Helicity Bounds Energy Theorem" (HBET), see e.g. [7, 8, 54], which will be employed in the course of the proof of Theorem 7.1 below. We substantiate the preceding remarks with a few computations (cf. [71]); the frozen (magnetic field $\mathbf{B} = \mathbf{B}(x,t)$) equation reads

$$\frac{\partial \mathbf{B}}{\partial t} = curl\,(\mathbf{V} \times \mathbf{B}) \tag{7.1}$$

together with

$$k\mathbf{V} = -\nabla p + curl\,\mathbf{B} \times \mathbf{B} \tag{7.2}$$

(k is a positive constant) An easy computation shows that the magnetic energy

$$M(t) = \frac{1}{2}\int_{\mathbf{R}^3} \langle \mathbf{B} | \mathbf{B} \rangle \tag{7.3}$$

evolves in time according to

$$\dot{M}(t) = -k \int_{\mathbf{R}^3} \langle \mathbf{V} | \mathbf{V} \rangle \tag{7.4}$$

whence $M^E := \lim_{t\to\infty} M(t)$ exists and is positive if there is essential linking (see [41, 42]).

We set aside the question of whether different end states may occur if one starts from different geometrical configurations of the same knot (see [71]). We state and sketch the proof of the following (possibly new) result, naturally stemming from the interpretation of higher order linking previously discussed, but first recall that a divergence-free vector field ξ on $\mathbf{R}^3$ is *strongly modelled* on an n-component link L if there is a volume preserving embedding carrying the vector field $\frac{\partial}{\partial \theta}$ directed along the circles in $\cup_{i=1}^{n} D^2 \times S^1$ into ξ within a tubular neighbourhood of L (then consisting of solid tori), see [42, 8].

Theorem 7.1. *Let ξ be a divergence-free vector field strongly modelled on an n-component Brunnian link possessing n-linking numbers ℓ_σ, $\sigma \in S_n$ (symmetric group), and contained in a union of tubes V, of volume V (abuse of notation). Then we have the following energy estimate*

$$E(\xi) = \int_V \langle \xi, \xi \rangle \, dv \geq C \max_{\sigma \in S_n} |\ell_\sigma| \tag{7.5}$$

where the (positive) costant C only depends ultimately on the shape and size of the domain V .

Proof. Let, for any permutation σ, ξ'_σ be a natural solenoidal vector field corresponding to ξ, strongly modelled on the two-component link formed by L_1 and L_σ, the latter being obtained by the recursive procedure outlined in Section 6, and contained in a tube of volume V'_σ (one can arrange things in such a way that $V'_\sigma \leq V$). Then, applying HBET to ξ'_σ we have:

$$E(\xi) = \int_V \langle \xi, \xi \rangle \, dv \geq \int_{V'_\sigma} \langle \xi'_\sigma, \xi'_\sigma \rangle \, dv \geq C'_\sigma |\mathcal{H}(\xi'_\sigma)| \equiv C_\sigma |\ell_\sigma| \geq C |\ell_\sigma| \quad (7.6)$$

(obvious notations, and with $C := \min_{\sigma \in S_n} C_\sigma > 0$), wherefrom our conclusion follows. □

8. Some braid group considerations

In this section, besides reviewing the geometric interpretation of anyon type wave functions devised in [20], we shall look at higher order linking through a (pure) braid theoretic viewpoint, making contact with Berger's work ([14, 15, 16, 17, 18, 38]).

8.1. Braid groups. The abelian ZKZ-connection

First recall that the braid group B_n can be presented via generators b_i, $i = 1, 2, \ldots, n-1$ subject to relations $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $b_i b_j = b_j b_i$ for $|i - j| \geq 2$. In view of Alexander's theorem, all links can be obtained by closing a braid (determined up to Markov moves ([21, 53])).

We realize, as usual, its elements (up to suitable equivalence) via geometric strands $z_j = z_j(t)$, $j = 1, 2, \ldots, n$, $t \in [0, 1]$ in $\mathbf{R}^3$ with coordinates (z, t), see Figure 8. At fixed t_0 the braid punctures the plane $t = t_0$ in n distinct points. B_n is isomorphic to the fundamental group $\pi_1(Y_n, *)$ of the space $Y_n := Conf(\mathbf{C}, n)/S_n$ consisting of all collections of n different but indistinguishable points on the complex plane $\mathbf{C}$ (so it is the quotient of the configuration space $X_n := Conf(\mathbf{C}, n)$ by the obvious action of the permutation group S_n), where $*$ denotes a base point (an initial configuration of the points). On the other hand, the fundamental group $\pi_1(X_n, *)$ concides with the normal subgroup P_n of B_n consisting of the *pure* (or coloured) braid group, i.e. the kernel of the obvious surjection $B_n \to S_n$. A set of generators for the latter group (see e.g. [9, 74]) is provided by the braids $A_{ij} = A_{ji}$ wherein the ith strand winds up around the jth strand avoiding the others, see again Figure 8; neither their expression in terms of the b's nor the ensuing relations among them, which are a bit involved (cf. [9, 74]) will be needed here. Set, for $\mu \in \mathbf{R}$,

$$\omega := \mu \sum_{i<j}^{n} d \log(z_i - z_j). \quad (8.1)$$

It can be regarded as a differential 1-form on the space Y_n, and it can also be written in the form

$$\omega = i\mu\, d\sum_{i<j}^{n} arg(z_i - z_j) + \mu\, d\sum_{i<j}^{n} \log \mid z_i - z_j \mid =: i\mu\, d\sum_{i<j}^{n} arg(z_i - z_j) + dh \tag{8.2}$$

where h is, up to a factor, the vortex assembly (Kirchhoff) Hamiltonian (see e.g. [8]). It is easy to see that the above formula gives rise to a flat connection thereon (Knizhnik-Zamolodchikov-Kohno, see e.g. [56]) and to a scalar parallel transport (monodromy) $hol_b(\omega)$ along b, viewed as a loop in $\pi_1(Y_n, *) = B_n$. It is straightforward to check that (notice that h plays no role in the calculation)

$$hol_b(\omega) := e^{\int_b \omega} = e^{-i\pi\mu w(b)} \tag{8.3}$$

with $w(b)$ denoting the writhe of the braid b (coinciding of course with the writhe $w(L_b)$ if L_b is the link obtained by closing b).

This is just the simplest case of the (Kohno-) Kontsevich construction involving an abstract flat connection on $Conf(\mathbf{C}, n)$:

$$\omega := \mu \sum_{i\neq j} t_{ij} d\log(z_i - z_j) \tag{8.4}$$

with the t_{ij}'s ($i \neq j$) generating the so-called (braid) holonomy algebra, fulfilling the *infinitesimal braid relations*:

$$t_{ji} = t_{ij}; \qquad [t_{ij} + t_{jk}, t_{ik}] = 0; \qquad [t_{ij}, t_{kh}] = 0 \tag{8.5}$$

(distinct indices throughout). The flatness of ω arises from the above relations, in conjunction with the Arnol'd relations involving the logarithmic forms $\omega_{ij} := \frac{1}{2\pi i} d\log(z_i - z_j)$:

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0 \tag{8.6}$$

with $i < j < k$, which are easily proved. Recall that the cohomology ring $H^*(Y_n, \mathbf{Z})$ is given precisely by the exterior algebra generated by the 1-forms ω_{jk} modulo the ideal generated by the Arnol'd relations.

In our case, the t_{jk} are all given by a constant. The parallel transport (holonomy) pertaining to the connection yields a scalar representation of the braid group.

It is necessary to observe at this point that Kontsevich's approach is also profoundly differential geometric in spirit; without entering in a detailed exposition, we just point out that its universal invariant (for knots and, with simplified features, for braids) comes from computing a parallel transport operator for a flat connection built upon abstractly defined infinitesimal braid relations (or Gauß chord diagrams subject to the 4-term relation of Vassiliev's theory) together with the Arnol'd relations. We recommend in particular [77] (and the appropriate references therein) for a detailed and comprehensive exposition.

In principle, an explicit (i.e. matricial) "approximation" to the Kontsevich integral, via e.g. nilpotency, this being suggested by the very nature of Vassiliev invariants) could be feasible. Especially intriguing is Berger's Hamiltonian interpretation of Vassiliev's invariants ([15, 16]).

8.2. A geometric interpretation of anyon wave functions

It is interesting to relate the preceding constructions with the scalar representations of the braid group arising in quantum vortex and anyon theories (see e.g. [44] and references therein, [103]). We closely follow [20].

Resorting to the manifold Y_n, we observe that emerges as a coadjoint orbit of the area preserving diffeomorphism group $G := sDiff\,(\mathbf{R}^2)$ (with suitable behaviour at infinity) pertaining to (singular) vorticity 2-form, concentrated on the plane punctures $c_1, c_2, \ldots, c_n$,

$$w = \lambda \sum_{j=1}^{n} 2\pi i\, \delta(z - c_j) dz \wedge d\overline{z}. \tag{8.7}$$

Specifically

$$Y_n \cong G/G_{\mathbf{c}} \tag{8.8}$$

with $G_{\mathbf{c}}$ the isotropy group (stabilizer) of $\mathbf{c} := (c_1, c_2, \ldots, c_n)$. Moreover

$$B_n = \pi_1(Y_n) \cong G_{\mathbf{c}}/G^0_{\mathbf{c}} \tag{8.9}$$

(^{0} denoting connected component). The above representation of $B_n \cong G_{\mathbf{c}}/G^0_{\mathbf{c}}$ is induced by a character, yielding, in turn, a character of $G_{\mathbf{c}}$. Application of Mackey's induction in the framework of Kirillov's orbit method gives a geometric prequantization bundle on the coadjoint orbit Y_n. However, as it has been pointed out by Wu ([103]), in this way only the motion of slow variables (the vortex or anyon locations) is taken into account, that of the fast variables (the superfluid particles) being averaged out. The higher genus Riemann surface case has been discussed in [82] (for physical motivation see e.g. [69]). In that paper a singular Hamiltonian, generalizing Kirchhoff's one, has been devised in terms of Riemann's theta function, and the "true" arena of the vortex motion (if we insist on describing it through an order parameter, i.e. an appropriate meromorphic function) is an abstract projective space coming from Riemann-Roch theory. We refer to it for full details, since even a short discussion here would lead us definitely too far afield. An interesting related work is [22].

The previous braid group representations give rise to a so called *topological phase* ([103]). A FO-quantization condition forces exclusion of non-trivial (i.e. anyonic) representations of the braid group. We now present the FQHE-related geometric observations made in [20].

The space Y_n is a Kähler manifold with Kähler form induced from the standard one in $\mathbf{C}^n$, and its prequantization leads to a family of line bundles $L_\mu \to Y_n$ parametrized by $H^1(Y_n, S^1)$ or, equivalently, by characters of $\pi_1(Y_n) = B_n$ (holonomies of (hermitian) flat connections). More precisely, one has $L_\mu = F_\mu \otimes L$; that is, F_μ is the flat line bundle attached to μ, possessing the following holomorphic section (becoming an ordinary holomorphic function on the universal covering space $\widetilde{Y}_n$)

$$\psi_\mu = \prod_{i<j}(z_i - z_j)^\mu \tag{8.10}$$

equipped with the hermitian metric induced by the position

$$(\psi_\mu, \psi_\mu) \, := | \, \psi_\mu \, |^2 =: e^{-f_\mu}. \tag{8.11}$$

Notice that monodromy is encoded via the hermitian frame $s_0 := e^{i \arg \psi_\mu}$.

Furthermore, L is the standard trivial line bundle on Y_n induced by the canonically trivialized line bundle over $\mathbf{C}^n$, endowed with the metric (obvious notation)

$$(1, 1) \, := e^{-\sum_{j=1}^n |z_j|^2} =: e^{-f}. \tag{8.12}$$

Thus, the following holomorphic frame and hermitian metric is defined on L_μ:

$$(\Psi_\mu \, \Psi_\mu) := (\psi_\mu \otimes 1, \psi_\mu \otimes 1) := e^{-(f_\mu + f)} =: e^{-\mathcal{F}_\mu} \tag{8.13}$$

One recovers the (abelian) Kohno representations of the braid group parameterized by $\mu \in \mathbf{R}$. Integrality entails, equivalently: triviality of the representation, existence of a covariantly constant section on the flat line bundle and again a FO condition (quantization of "vorticity"). The following ("multivalued") trial wave function ([27]) plays a pivotal role in the theory of the Fractional Quantum Hall Effect (FQHE) ([27]):

$$\widetilde{\Psi}_\mu = \prod_{i<j}(z_i - z_j)^\mu \cdot e^{-\sum_{j=1}^n |z_j|^2}. \tag{8.14}$$

One has the following

Theorem 8.1 ([20]). *The wave function $\Psi_\mu = \psi_\mu \otimes 1$ can be looked upon as a holomorphic section of the geometric quantization line bundle $L_\mu \to Y_n$ equipped with the canonical holomorphic hermitian (Chern-Bott) connection (slight notational abuses) $\nabla_\mu^{(1,0)} = \partial - \partial\mathcal{F}_\mu$ (and $\nabla_\mu^{(0,1)} = \overline{\partial}$), whose curvature equals the standard Kähler form on Y_n. The Kähler potential $\mathcal{F}_\mu$ is essentially the Hamiltonian pertaining to a 2-dimensional classical plasma. Moreover, parallel transport with respect to ∇_μ, i.e. adiabatic evolution, yields a geometric phase, governed by the Kähler form, plus a purely topological phase* (topological Berry's phase).

(See e.g. the recent monograph [32] for a comprehensive discussion of geometric phases). The plasma Hamiltonian is basically the sum of a vortex type Hamiltonian and a harmonic n-oscillator term. In the integral odd case recovers Laughlin's Jastrow-type electron function (in suitable units) and in general case generalized Laughlin and Halperin quasiparticle (anyon) wave functions used in FQHE. We refer to [27] and references therein for details and for a thorough physical discussion of this issue.

8.3. A nilpotent group representation of P_3

As a final application of the preceding techniques, we prove the following statement:

Theorem 8.2. *Consider the following nilpotent connection on the configuration space X_3:*

$$\mathbf{v} = \begin{bmatrix} 0 & \omega_{12} - \omega_{23} & 0 \\ 0 & 0 & \omega_{13} - \omega_{23} \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.15}$$

Then

(i) *its curvature $\Omega = d\mathbf{v} + \mathbf{v} \wedge \mathbf{v}$ vanishes, i.e. $\mathbf{v}$ is flat.*

(ii) *thus, its associated parallel transport operator gives rise to a nilpotent group representation (call it ρ) of the pure (or colored) braid group P_3 (the fundamental group of X_3), reading*

$$\rho(b) = \begin{bmatrix} 1 & \int_b(\omega_{12} - \omega_{23}) & \int_b(\omega_{12} - \omega_{23})(\omega_{13} - \omega_{23}) \\ 0 & 1 & \int_b(\omega_{13} - \omega_{23}) \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.16}$$

(iii) *In particular one has, with appropriate conventions,*

$$\rho(A_{12}) = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix},\ \rho(A_{13}) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix},\ \rho(A_{23}) = \begin{bmatrix}1 & -1 & \frac{1}{2}\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}. \tag{8.17}$$

Proof. The first assertion follows from the closure of the ω's and by the Arnold relation $\omega_{21} \wedge \omega_{13} + \omega_{13} \wedge \omega_{32} + \omega_{32} \wedge \omega_{21} = 0$ (also recall that $\omega_{ji} = \omega_{ij}$). The second one is an immediate consequence of the first and of the general parallel transport operator formulae (6.6), (6.7). As for the third one, we first recall the identity, valid for any 1-forms ω_1, ω_2 (see e.g. [93] for additional ones)

$$\int \omega_1\omega_2 + \omega_1\omega_2 = \int \omega_1 \int \omega_2. \tag{8.18}$$

This can be used in the calculation of the iterated integrals

$$\int_b (\omega_{12} - \omega_{23})(\omega_{13} - \omega_{23}) = \int_b \omega_{12}\omega_{13} - \int_b \omega_{23}\omega_{13} - \int_b \omega_{12}\omega_{23} + \int_b \omega_{23}\omega_{23} \tag{8.19}$$

by setting, successively, $b = A_{12}, A_{13}, A_{23}$. In the first case ($b = A_{12}$), in view of homotopy invariance, one can take A_{12} in such a way that strand 1 first goes down vertically, then, still keeping a small slope, winds up around strand 2 and then again proceeds vertically (Figure 8 again). Then is easily seen that all the iterated logarithmic integrals (after taking a limit) vanish, and the same is easily seen to occur for $b = A_{13}$. The last case is handled similarly, but now the only surviving integral is $\int_{A_{23}} \omega_{23}\omega_{23} = \frac{1}{2}(\int_{A_{23}} \omega_{23})^2 = \frac{1}{2}$. □

Remarks.

1. The pig-tail braid (whose closure yields the Borromean rings, cf. Figure 8) can be represented as $b' := A_{12}A_{13}A_{12}^{-1}A_{13}^{-1}$, whence

$$\rho(b') = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \tag{8.20}$$

   showing that the typical Borromean weaving is indeed accounted for.
2. Notice the strict analogy with the link group representation previously discussed as far as the generators A_{12} and A_{13} are concerned. This suggests the possibility of extending this kind of ideas to tackle higher braiding phenomena and establishing a closer contact with the

Kontsevich integrals. The task appears to be, as far as we superficially see, not so straightforward, since we heavily exploited the particular structure exhibited by the Arnold identity. Also notice that taking the ordinary trace yield the constant value 3, so no nontrivial Markov type link invariants are produced in this manner.

Notice that P_3 is actually mapped to the *Heisenberg group* H, represented by all 3×3 real matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.21}$$

Equivalently, H is given by all real triples (x, y, z), with group operation

$$(x, y, z) \circ (x', y', z') := (x + x', y + y', z + z' + xy'). \tag{8.22}$$

Upon setting $\alpha := \omega_{12} - \omega_{23}$, $\beta := \omega_{13} - \omega_{23}$, we get, in view of the above remarks, the following

Corollary 8.3. *With obvious notation, the following Chen integral identities hold, for all* b_1*,* b_2 *in* P_3

$$\begin{aligned} \int_{b_1 b_2} \alpha &= \int_{b_1} \alpha + \int_{b_2} \alpha \\ \int_{b_1 b_2} \beta &= \int_{b_1} \beta + \int_{b_2} \beta \\ \int_{b_1 b_2} \alpha\beta &= \int_{b_1} \alpha\beta + \int_{b_2} \alpha\beta + \int_{b_1} \alpha \int_{b_2} \beta. \end{aligned} \tag{8.23}$$

Remarks.

1. We point out that generalized Heisenberg groups naturally arise in braid group theory (see e.g. [1]). A closer comparison between the general purely algebraic approach of that paper and our differential geometric one could be fruitful.
2. The centre Z of H consists of the elements $(0, 0, z)$, $z \in \mathbf{R}$ (direct inspection), and the pigtail braid corresponds to $(0, 0, 1)$.

8.4. On the Evans-Berger fourth order invariant

We mention at this point the following relationship with the work of Evans and Berger ([38]) on higher order linking and braiding numbers, also based on Massey products but less general than the one described above. Their Brunnian 4-link (also appearing in [73], see [84] and Figures 4 and 7) can be again represented as a closed coloured braid , with one of the strands (the first one, say) winding around the others, depicted as straight line segments ([38]). Now, observing that, in general *the Borromean algorithm can be naturally implemented in a pure braid context (we insist on using pure n-braids since their closures are n-links), via the correspondence* $\gamma_j \mapsto A_{1j}$, $j = 2, \ldots, n$

and resorting therefore to the Massey picture (or equivalently, to the Milnor one, by Turaev-Porter) one immediately computes the fourth-order invariant (equal to ± 1) avoiding their involved logarithmic form computation altogether (they have 40 (!) summands). This is a circle of ideas deserving further scrutiny, see also [3] and [2].

9. Conclusions and outlook

Among the topics touched upon in the present survey, many fascinating problems are still open. Among them we mention a generalized Arnold's Helicity theorem for higher order linking numbers (our result applies only to quite special divergence-free fields) — see also [52, 23] — the construction of nilpotent flat connection representations of P_n and their relationship with the Kontsevich approach, and the establishment of a bridge between the (geometric) quantization of Brylinski's manifold and the representation theory of the special diffeomorphism group (possibly via the Chen calculus). For a survey of related questions one may also consult [79, 25], together with the references already given throughout the paper. We hope to be able to report on progress on (some of) them elsewhere.

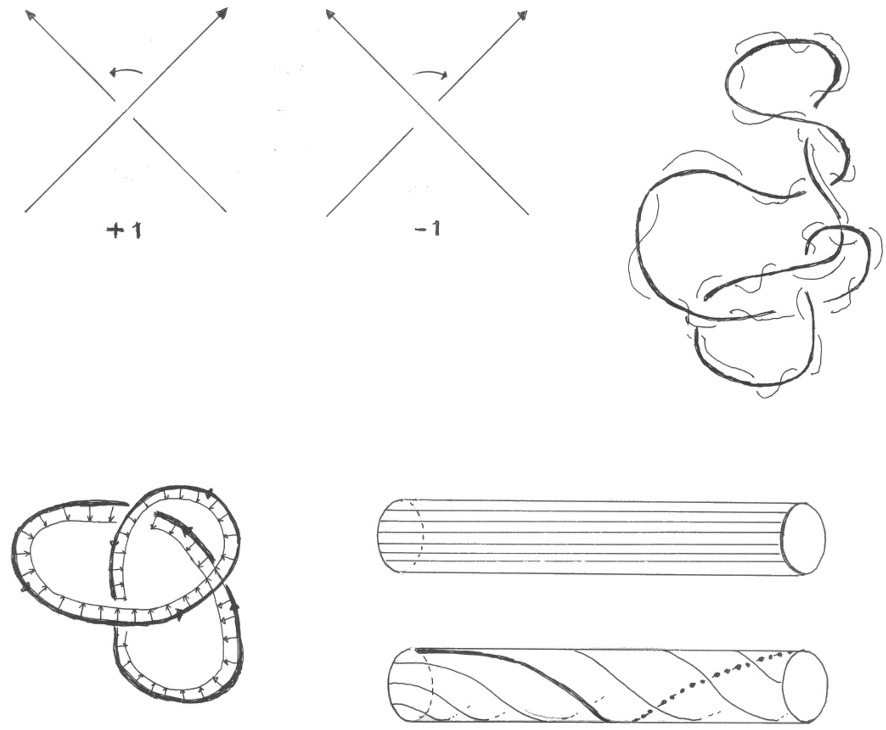

FIGURE 1. Crossing signs. Framing of a knot. Blackboard framing. Helicity.

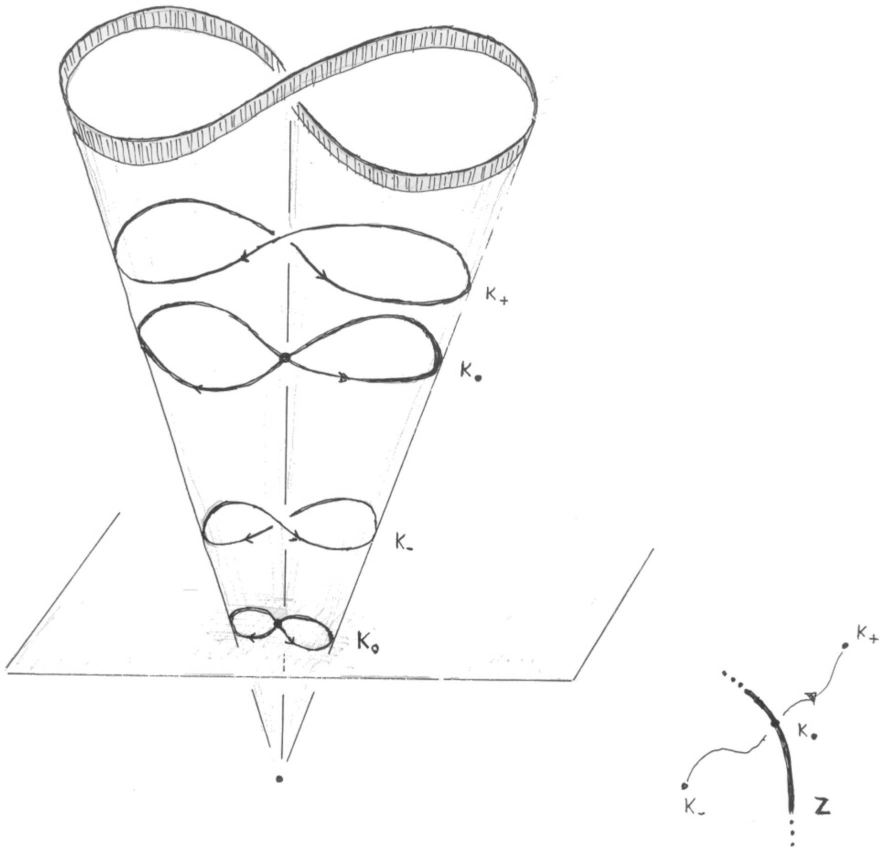

FIGURE 2. The cone construction. Crossing the Maslov cycle.

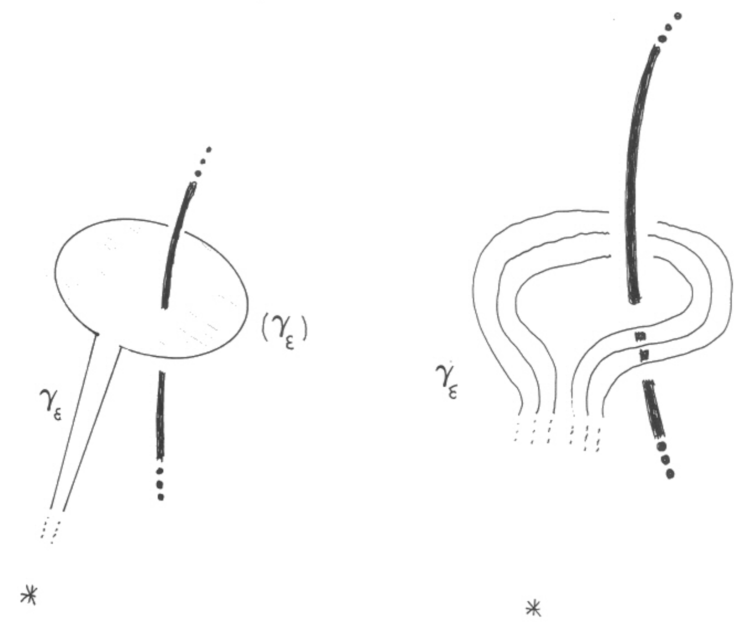

FIGURE 3. A small loop. The topological Stokes formula.

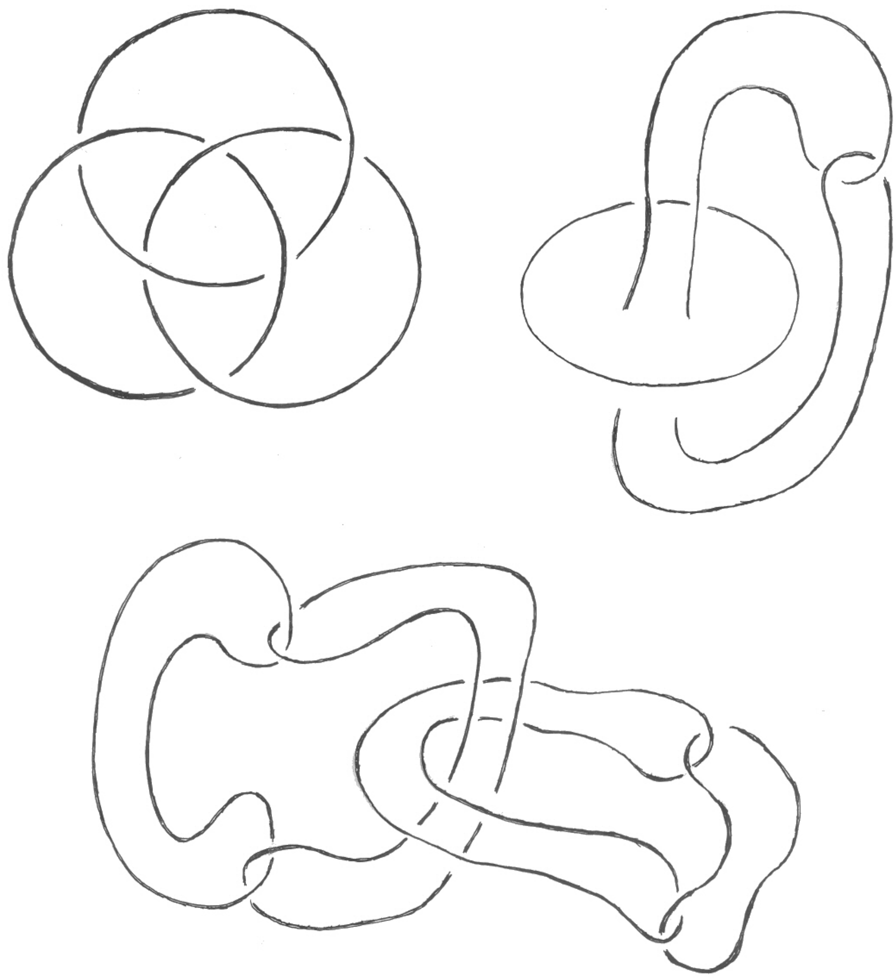

FIGURE 4. Borromean rings. The Whitehead link. A Brunnian 4-component link.

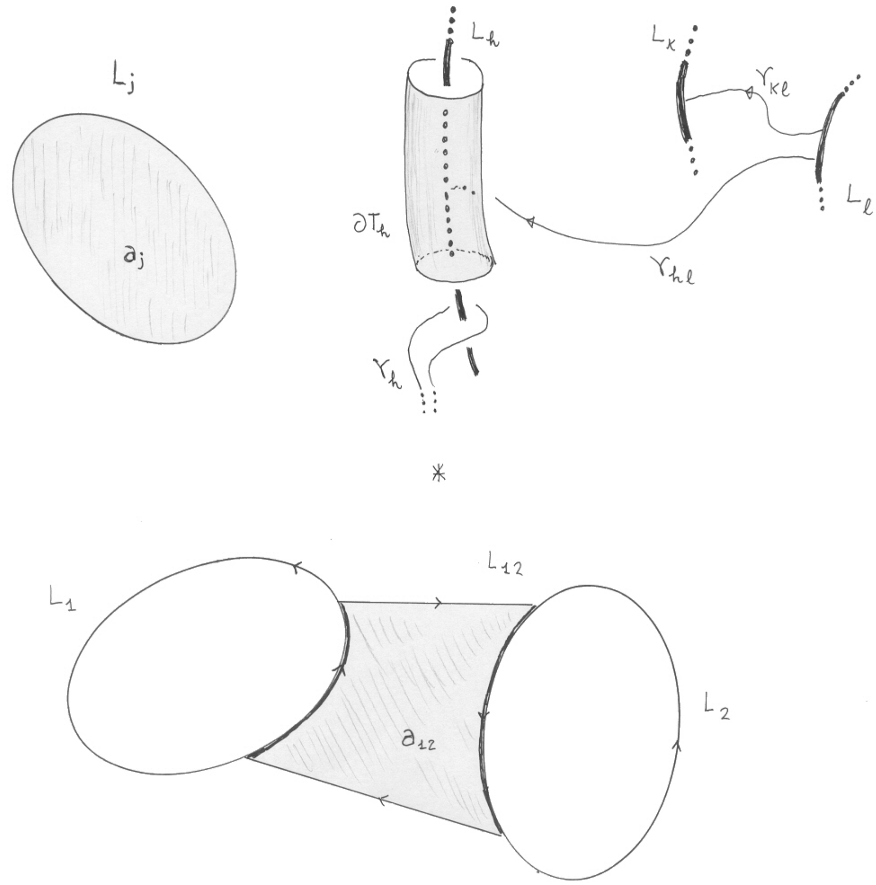

FIGURE 5. Link homology. Construction of a new disc.

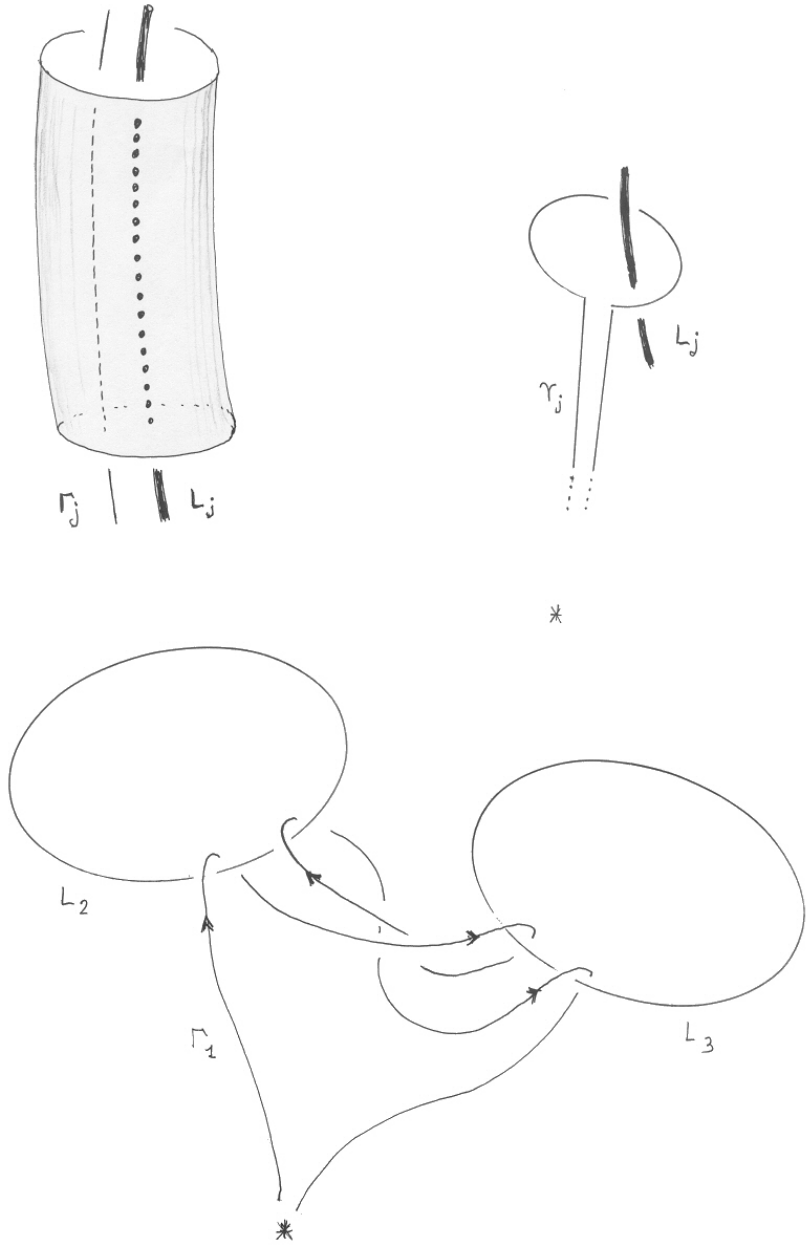

FIGURE 6. Milnor presentation. The Borromean algorithm.

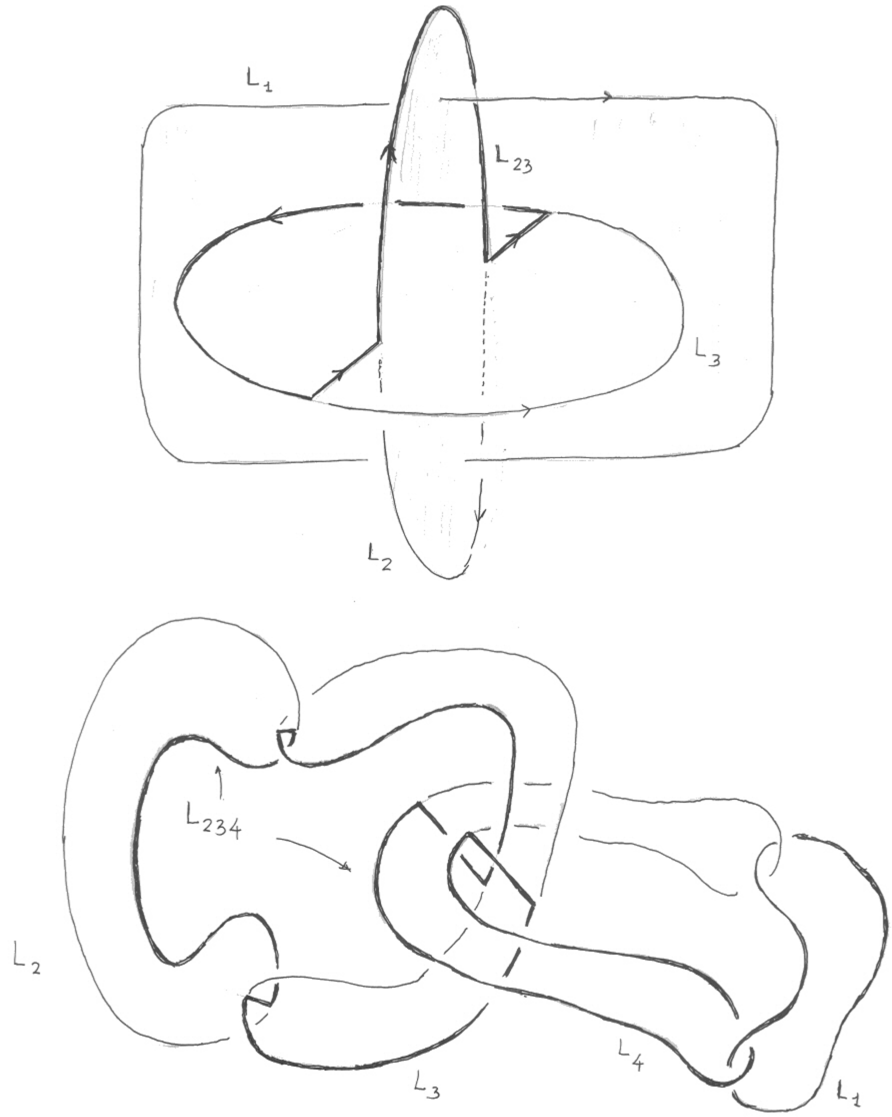

Figure 7. Borromean rings and a Brunnian 4-component link: higher order linking number computation.

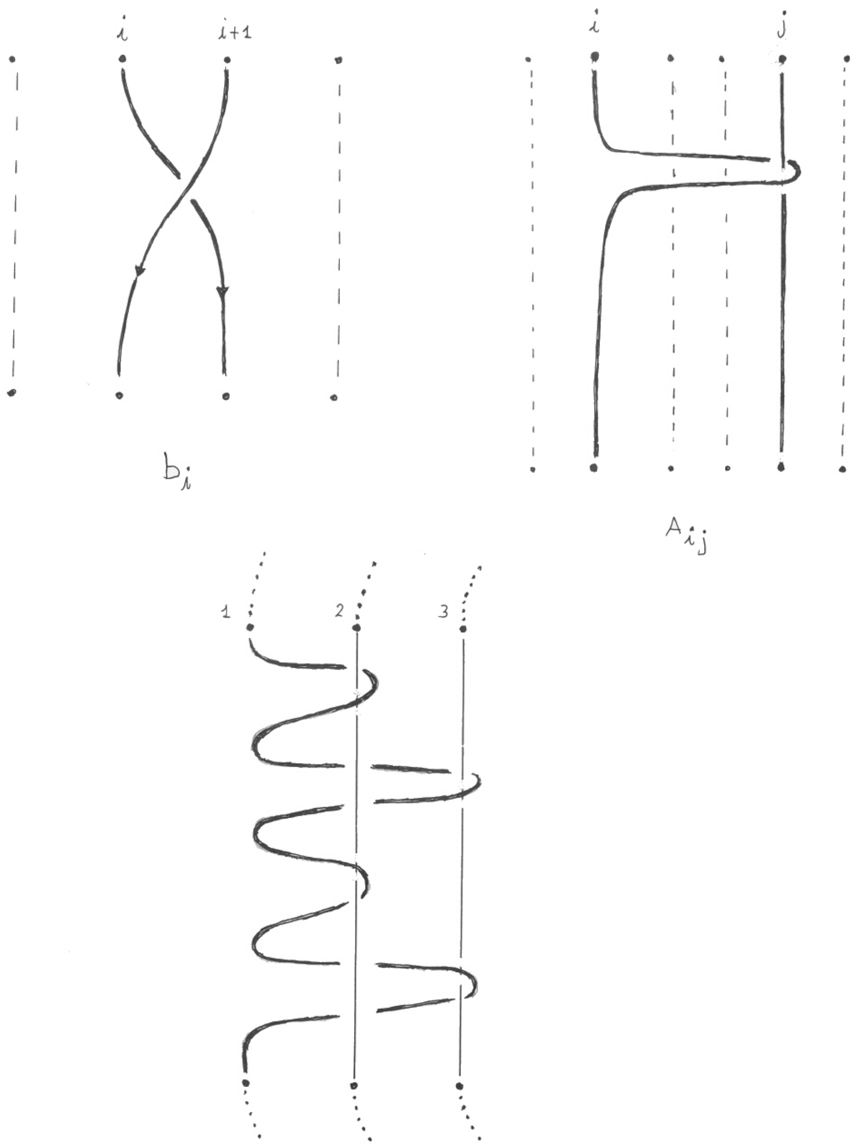

Figure 8. Braid groups. The pig-tail braid.

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