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# A Survey of the Selberg Class of $L$-Functions, Part I

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This is the first part of a survey of the axiomatic class of $L$-functions introduced by Selberg [32]. Our aim is to give a rather complete overview of the results, conjectures and problems which are related to the Selberg class. Apart from very few exceptions, we do not give proofs of the results. However, we provide very brief indications of the arguments, as well as motivations for conjectures and problems. While the conjectures are generally expected to be difficult, the level of the problems is not at all homogeneous: some of them should not be difficult, while others are probably very hard.

Of course, this survey benefited from the previous surveys by Kaczorowski-Perelli [15] and Kaczorowski [11]. It is my pleasure to thank Jurek Kaczorowski for his help and suggestions, and for a careful reading of the manuscript. My sincere thanks go to Giuseppe Molteni as well, who went through the manuscript and suggested several improvements. Further, I wish to thank Alessandro Zaccagnini for pointing out several misprints.

The contents of the entire survey is as follows; part I contains the first four sections.

## 1. Classical $L$-functions

The first example of a complex variable $L$-function is the famous *Riemann zeta function* $\zeta(s)$, introduced by Riemann in 1859 and defined for $s=\sigma+it$ with $\sigma>1$ by the absolutely convergent Dirichlet series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The Riemann zeta function was introduced to study the distribution of prime numbers, and in particular to detect the asymptotic behaviour as $x\to\infty$ of the prime numbers counting function

$$\pi(x)=\sum_{p\le x}1.$$

The basic connection between $\zeta(s)$ and the primes is given by the Euler product

$$\zeta(s)=\prod_p\Big(1-\frac{1}{p^s}\Big)^{-1}\qquad\qquad \sigma>1,$$

a simple but very interesting identity since the primes appear explicitly only on the right hand side. Thanks to the Euler product, the relation between $\zeta(s)$ and $\pi(x)$ can be made explicit by a classical Fourier transform argument, thus getting

$$\pi(x)\log x\sim\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\mathrm{d}s. \tag{1.1}$$

Clearly, in order to deduce the asymptotic behaviour of $\pi(x)$ from (1.1), we need to know some analytic properties of $-\frac{\zeta'}{\zeta}(s)$. In particular, we require some information on the polar structure of $-\frac{\zeta'}{\zeta}(s)$ or, equivalently, on the distribution of poles and zeros of $\zeta(s)$.

The fundamental analytic properties of $\zeta(s)$ are as follows.

- $\zeta(s)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$ and its only singularity is a simple pole at $s=1$.

- Writing
$$\Phi(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$
($\Gamma(s)$ is the Euler $\Gamma$-function), $\zeta(s)$ satisfies the functional equation
$$\Phi(s) = \Phi(1-s).$$
- $\zeta(s)$ has polynomial growth on vertical strips, that is
$$\zeta(\sigma+it) = O(|t|^c) \qquad |t| \to \infty$$
uniformly for $a \le \sigma \le b$, where $c = c(a,b)$.
- $\zeta(s) \neq 0$ for $\sigma > 1$ by the Euler product, and hence by the functional equation the zeros of $\zeta(s)$ in the half-plane $\sigma < 0$ are simple and located at the points $s = -2, -4, -6, ...$; such zeros are called the trivial zeros. The other zeros of $\zeta(s)$ are called the non-trivial zeros, are located inside the critical strip $0 \le \sigma \le 1$ and are symmetric with respect to the critical line $\sigma = \frac{1}{2}$ and to the real axis.
- The non-trivial zeros counting function
$$N(T) = \sharp\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 \le \beta \le 1, 0 \le \gamma \le T\}$$
satisfies
$$N(T) \sim \frac{T\log T}{2\pi}.$$
- $\zeta(s) \neq 0$ on the line $\sigma = 1$. Moreover, $\zeta(s)$ has zero-free regions to the left of $\sigma = 1$, the simplest being of the following form: $\zeta(\sigma+it) \neq 0$ for
$$\sigma > 1 - \frac{c}{\log(|t|+2)} \tag{1.2}$$
for some $c > 0$. Better zero-free regions are known at present, but all are asymptotic to the line $\sigma = 1$ as $|t| \to \infty$.

From the integral representation (1.1) and the above analytic properties we can deduce the famous Prime Number Theorem, proved independently by Hadamard and de la Vallée-Poussin in 1896:
$$\pi(x) \sim \frac{x}{\log x}.$$
Stronger forms of the Prime Number Theorem are known; for instance, from the zero-free region (1.2) we can get
$$\pi(x) = \mathrm{li}(x) + O\big(xe^{-c\sqrt{\log x}}\big)$$

for some $c > 0$, where

$$\mathrm{li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$$

is the logarithmic integral function. However, due to the shape of the known zero-free regions, no error term of type $O(x^\theta)$ with $\theta < 1$ is available at present.

The famous Riemann Hypothesis, probably the most important open problem of contemporary mathematics, states that all non-trivial zeros lie on the critical line. Hence the Riemann Hypothesis gives the best possible zero-free region $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$, from which the essentially best possible form of the Prime Number Theorem

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2} \log x)$$

follows.

We refer to the classical book of H.Davenport [5] for an excellent exposition of the basic theory of the Riemann zeta function and its applications to the distribution of primes. We also refer to Weil [37] for a beautiful account of the prehistory of the zeta functions.

Since the appearance of the Riemann zeta function, many other $L$-functions have been introduced in the theory of numbers, and in other branches of mathematics as well. Here is a very synthetic list.

- The *Dirichlet L-functions*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi$ is a character of the multiplicative group $\mathbb{Z}_q^*$ (the coprime residue classes modulo a positive integer $q$), were introduced by Dirichlet in 1837, hence about twenty years before Riemann's work. However, Dirichlet dealt with the $L(s, \chi)$'s as real variable functions, and the basic complex variable theory of the Dirichlet $L$-functions was established after Riemann's fundamental paper. The analytic properties of the Dirichlet $L$-functions are quite similar to those of the Riemann zeta function, and in fact $\zeta(s)$ is the special case corresponding to the character (mod 1). The Dirichlet $L$-functions were originally introduced to prove that the prime numbers are equidistributed in the arithmetic progressions $a \pmod q$ with $(a, q) = 1$, for any fixed modulus $q$. Clearly, the functions $L(s, \chi)$ are of *arithmetic* nature. We refer to Davenport [5] for the basic theory of the Dirichlet $L$-functions.

- The *Hecke L-functions* are defined for $\sigma > 1$ by

$$L_K(s,\chi) = \sum_I \frac{\chi(I)}{N(I)^s},$$

where $K$ is an algebraic number field, $I$ runs over the non-zero ideals of the ring of integers of $K$, $N(I)$ denotes the norm of $I$ and $\chi$ is a Hecke character of finite or infinite order. The functions $L_K(s,\chi)$ are a far reaching generalization of the Dirichlet $L$-functions. In fact, when $K = \mathbb{Q}$ the Hecke $L$-functions reduce to the Dirichlet $L$-functions. Moreover, when $\chi$ is trivial the function $L_K(s,\chi)$ reduces to the important special case of the *Dedekind zeta function*

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}.$$

The analytic behaviour of the Hecke $L$-functions is similar to the Dirichlet $L$-functions, although the functional equation has a more complicated shape and is definitely more difficult to prove.

A different type of $L$-functions associated with algebraic number fields is provided by the *Artin L-functions* $L(s,K/k,\rho)$. Here $K/k$ is a Galois extension of number fields with Galois group $G$, and $\rho$ is a finite dimensional representation of $G$. The Artin $L$-functions are defined for $\sigma > 1$ by certain Euler products, and their analytic properties are eventually deduced from the analytic properties of the Hecke $L$-functions. In fact, the Artin reciprocity law states if $K/k$ is abelian, then $L(s,K/k,\rho)$ coincides with a suitable Hecke $L$-function. Moreover, the Artin-Brauer theory of group characters implies that every function $L(s,K/k,\rho)$ is a product of integer powers of abelian Artin $L$-functions. As a consequence, the Artin $L$-functions can be expressed as products of integer powers of Hecke $L$-functions, hence they have meromorphic continuation to $\mathbb{C}$, possibly with infinitely many poles. However, the famous *Artin conjecture* predicts that every function $L(s,K/k,\rho)$ is holomorphic on $\mathbb{C}$, apart possibly for a pole at $s = 1$. The other analytic properties of the Artin $L$-functions are similar to those of the Hecke $L$-functions.

The Hecke and Artin $L$-functions, clearly of *algebraic* nature, provide quite a lot of information on the structure of algebraic number fields. We refer to Heilbronn [8] for the basic theory of Hecke and Artin $L$-functions.

• The *Hecke L-functions associated with modular forms* are defined for $\sigma$ sufficiently large by the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $f(z)$ is a holomorphic modular form and $a(n)$ are its Fourier coefficients. Under suitable restrictions and normalizations, the functions $L_f(s)$ satisfy analytic properties similar to those of the Riemann zeta function. For suitable choices of $f(z)$ (the Eisenstein series), such normalized $L$-functions give raise the Dedekind zeta functions of imaginary quadratic fields. There is an interesting "operation" between $L$-functions associated with modular forms, namely the *Rankin-Selberg convolution.* Roughly speaking, given two modular forms $f(z)$ and $g(z)$ with Fourier coefficients $a(n)$ and $b(n)$ respectively, the Rankin-Selberg convolution is defined by the Dirichlet series

$$L_{f\times\overline{g}}(s) = \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

Under suitable restrictions and normalizations, and modulo a certain "fudge factor", the Rankin-Selberg convolution has analytic properties similar to the Riemann zeta function. A similar, and in a way more fundamental, "operation" is the *m-symmetric product L-function* of two modular forms. In this case, the analytic properties are known at present only for small values of the integer $m$. A related class of $L$-functions are the *Maass L-functions* associated with non-holomorphic modular forms. The definition of such functions is quite complicated, hence we skip it. We only remark that the known analytic properties of the Maass $L$-functions are similar to the Hecke $L$-functions, but the state of the art is more rudimentary in this case.

Around the mid of the last century, a deep interpretation of the Hecke and Maass $L$-functions in terms of representations was established. Roughly speaking, such $L$-functions were associated with automorphic representations of GL(2) over the rational field. This theory then evolved into the theory of *automorphic L-functions*, associated with automorphic representations of GL($n$) over number fields. The theory of automorphic $L$-functions is very deep both from technical and conceptual viewpoints, and is not fully understood at present. For instance, analytic continuation and functional equation of the automorphic $L$-functions have been established, and the above mentioned Rankin-Selberg convolution and $m$-symmetric power $L$-functions are now interpreted as the $L$-functions associated with the tensor product and the $m$-symmetric power of representations, respectively. However, many deep conjectures remain open, and in particular the amazing *Langlands program.* The Langlands program is a very deep unifying program which, roughly speaking, predicts that the $L$-functions of arithmetic, algebraic and geometric (see below) nature are in fact members of the class of automorphic $L$-functions. An important "special case" of the Langlands program in the Shimura-Taniyama conjecture, asserting that the $L$-functions associated with elliptic curves correspond to suitable $L$-functions associated with modular forms. Such a conjecture has been first proved in important special cases by A.Wiles (see Wiles [38] and Taylor-Wiles [35]) as a key step in the proof of Fermat Last Theorem, and then in full generality by Wiles' followers.

The nature of the above $L$-functions is of course *automorphic*, and we refer to Hecke [7], Iwaniec [9] and Bump [2] for the basic theory of such functions (see also the recent survey by Gelbart-Miller [6]). We conclude the synthetic list of $L$-functions by remarking that $L$-functions of *geometric* nature, *i.e.* attached to geometric objects like elliptic curves and varieties, have been introduced as well, and we refer to Silverman [33] for an introductory presentation. Moreover, we refer to Chapter 5 of the recent book by Iwaniec-Kowalski [10] for an excellent introduction to the classical $L$-functions presented in this section.

Although the nature of the above $L$-functions is apparently *different*, once suitably normalized they share the following important *common* properties (in some cases only *conjecturally*):

- ordinary Dirichlet series, absolutely convergent for $\sigma > 1$;
- meromorphic continuation to $\mathbb{C}$, with at most a pole at $s = 1$;
- functional equation of Riemann type with multiple $\Gamma$ factors, relating $s$ with $1 - s$;
- coefficients are $O(n^{\varepsilon})$ for every $\varepsilon > 0$;
- Euler product.

We will see that there is probably a very deep unifying theory behind such common properties which, in a sense, represents an analytic counterpart of the Langlands program.

## 2. What is an $L$-function?

The following two natural questions arise at this point:

- what is *in general* an $L$-function?
- are *all* $L$-functions already known?

Clearly, the second question depends on the first one. In a way, an answer to the first question was given by Selberg [32], defining the **Selberg class** $\mathcal{S}$ of $L$-functions. Writing $\overline{f}(s) = \overline{f(\overline{s})}$ and assuming, as usual, that *an empty product equals* 1, the Selberg class is axiomatically defined as follows: $F \in \mathcal{S}$ if

(i) (*ordinary Dirichlet series*) $F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s}$, absolutely convergent for $\sigma > 1$;

(ii) (*analytic continuation*) there exists an integer $m \geq 0$ such that $(s-1)^m \cdot F(s)$ is an entire function of finite order;

(iii) (*functional equation*) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega \overline{\Phi}(1-s)$, where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

say, with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$ and $|\omega| = 1$;

(iv) (*Ramanujan conjecture*) for every $\varepsilon > 0$, $a_F(n) \ll n^{\varepsilon}$;

(v) (*Euler product*) $\log F(s) = \sum_{n=1}^{\infty} b_F(n) n^{-s}$, where $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_F(n) \ll n^{\vartheta}$ for some $\vartheta < \frac{1}{2}$.

Other axiomatic classes of $L$-functions have been proposed in the literature, see *e.g.* Piatetski-Shapiro [30] and Carletti-Monti Bragadin-Perelli [3]; however, the axioms of the Selberg class appear to be more satisfactory. Moreover, the problems raised by Selberg are definitely very interesting. As we shall see, such problems are of a rather different nature with respect to the classical problems on $L$-functions, in the sense that they deal with the $L$-functions *as a class*.

Examples of members of $\mathcal{S}$ are the Riemann zeta function, the Dirichlet $L$-functions, the Hecke $L$-functions associated with algebraic number fields and, under suitable restrictions and normalizations, the Hecke $L$-functions associated with holomorphic modular forms. The other $L$-functions listed in Section 1 are also in $\mathcal{S}$, provided certain classical conjectures hold. In particular, the Artin $L$-functions belong to $\mathcal{S}$ if the Artin conjecture holds, while the automorphic $L$-functions are in $\mathcal{S}$ provided the Ramanujan conjecture holds true.

Here are few comments on the five axioms defining the Selberg class. By axiom (i), the functions in $\mathcal{S}$ are ordinary Dirichlet series. This is an important point since, as we shall see, the picture would change if *general Dirichlet series* are allowed. We recall that the general Dirichlet series are of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{l_n^s}$$

where $l_n$ is an increasing sequence of positive real numbers tending to $\infty$. Restricting the frequences $l_n$ to be integers, as in axiom (i), carries some arithmetical information.

Axiom (ii) allows $s = 1$ to be the only pole of functions in $\mathcal{S}$, but most probably the picture does not change much if finitely many poles on the line $\sigma = 1$ are allowed.

The function $\gamma(s)$ in axiom (iii) is called $\gamma$-**factor**, and its factors $\Gamma(\lambda_j s + \mu_j)$ are the $\Gamma$-**factors**. The form of the $\gamma$-factor of a given $F \in \mathcal{S}$ is clearly *not unique*. For instance, application of the Legendre duplication formula for the $\Gamma$-function changes its shape, as the following example shows:

$$\begin{aligned} &\left(\frac{\pi}{2}\right)^{-s/2} \Gamma\left(\frac{s}{4}\right) \Gamma\left(\frac{s}{4} + \frac{1}{2}\right) \zeta(s) \\ &= \left(\frac{\pi}{2}\right)^{-(1-s)/2} \Gamma\left(\frac{1-s}{4}\right) \Gamma\left(\frac{1-s}{4} + \frac{1}{2}\right) \zeta(1-s). \end{aligned}$$

In other words, writing $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r)$ and $\boldsymbol{\mu} = (\mu_1, ..., \mu_r)$, the **data** $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ of $F \in \mathcal{S}$ are *not uniquely defined* by $F(s)$. This gives rise to the notion of **invariant**, *i.e.* an expression defined in terms of the data of $F(s)$ which is uniquely determined by $F(s)$ itself. We will soon see an important example of invariant.

Probably, axiom (iv) can be weakened to "for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $a_F(n) \ll n^\varepsilon$ for $(n, M) = 1$" without changing much the picture. The advantage of this form of axiom (iv) rests on the fact that a similar bound can be proved for the coefficients $a_F^{-1}(n)$ and $b_F(n)$. In other words, assuming this form of axiom (iv) and denoting by $c(n)$ any of the coefficients $a_F(n)$, $a_F^{-1}(n)$ and $b_F(n)$, one has that for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $c(n) \ll n^\varepsilon$ for $(n, M) = 1$, and $c(n) \ll n^\vartheta$ for some $\vartheta < \frac{1}{2}$. Moreover, it is interesting to note that axiom (iv) is *crucial* for the Riemann Hypothesis.

In fact, Jurek Kaczorowski constructed the following simple example of $L$-function satisfying axioms (i), (ii), (iii) and (v) but violating the Riemann Hypothesis. Let $\chi$ be a primitive Dirichlet character with $\chi(-1) = -1$ and write $G(s) = L(2s - 1/2, \chi)$. $G(s)$ is absolutely convergent for $\sigma > 3/4$, satisfies a functional equation with $\lambda = 1$ and $\mu = 1/4$, and has an Euler product allowing the choice $\vartheta = 1/4$. Taking $0 < \delta < 1/4$ and writing $F(s) = G(s - \delta)G(s + \delta)$, thanks to the above properties it is easy to see that $F(s)$ satisfies all axioms but the Ramanujan conjecture, and has no zeros on the critical line for suitable choices of $\delta$.

Axiom (v) implies in particular that the coefficients $a_F(n)$ are *multiplicative*. Hence the standard Euler product

$$F(s) = \prod_p F_p(s) \qquad\qquad F_p(s) = \sum_{m=0}^{\infty} a_F(p^m)p^{-ms}$$

holds; $F_p(s)$ is the $p$-**Euler factor** of $F(s)$. Moreover, the seemingly harmless condition $\vartheta < \frac{1}{2}$ has in fact a relevant role. For instance it implies that $F_p(s) \neq 0$ for $\sigma > \vartheta$ for every prime $p$, and this will be crucial at several places. Moreover, if such a condition is relaxed and values of $\vartheta$ greater than $\frac{1}{2}$ are allowed, then examples of functions satisfying axioms (i),...,(v) and violating the Riemann Hypothesis are easily constructed. A simple example is

$$f(s) = (1 - 2^{a-s})(1 - 2^{b-s}) \quad \text{with } a + b = 1 \text{ and } a > \tfrac{1}{2}.$$

Note that the five axioms of the Selberg class are not completely independent (for example, axiom (v) implies that $F(s)$ is an ordinary Dirichlet series). We refer to Molteni [26] for further pathological examples arising if parts of the axioms are dropped.

We finally remark that axioms (i), (ii) and (iii) are more of *analytic* nature, while axioms (iv) and (v) are more of *arithmetic* nature. Therefore, we define the **extended Selberg class** $\mathcal{S}^\sharp$ to be the class of the non identically vanishing functions satisfying axioms (i), (ii) and (iii). Clearly, $\mathcal{S}^\sharp \supset \mathcal{S}$, and we shall see that $\mathcal{S}^\sharp$ still carries some of the properties of $\mathcal{S}$. We also remark that many of the definitions involving $\mathcal{S}$ carry over in an obvious way to the case of $\mathcal{S}^\sharp$.

The standard analytic properties of the functions $F \in \mathcal{S}$ are easily obtained by means of the classical arguments used to study the Riemann zeta function. Let $F \in \mathcal{S}$. We define the **polar order** $m_F$ of $F(s)$ to be the least value of $m$ in axiom (ii), and

$$d_F = 2\sum_{j=1}^{r}\lambda_j$$

is the **degree** of $F(s)$. It is easy to see that

$$d_\zeta = 1, \quad d_{L(\cdot,\chi)} = 1, \quad d_{\zeta_K} = [K:\mathbb{Q}], \quad d_{L_K(\cdot,\chi)} = [K:\mathbb{Q}], \quad d_{L_f} = 2 \tag{2.1}$$

and similarly for the other classical $L$-functions. The function $\Psi(s) = s^{m_F}(1-s)^{m_F}\Phi(s)$ is an entire function of order 1, and the Lindelöf $\mu$-function $\mu_F(\sigma)$ satisfies $\mu_F(\sigma) = 0$ for $\sigma \geq 1$ and, by the functional equation, $\mu_F(\sigma) = d_F(\frac{1}{2}-\sigma)$ for $\sigma \leq 0$. This shows in particular that the degree in an *invariant*, and hence $\mathcal{S}$ can be partitioned as

$$\mathcal{S} = \bigcup_{d\geq 0}\mathcal{S}_d,$$

where

$$\mathcal{S}_d = \{F \in \mathcal{S} : d_F = d\}.$$

From the Euler product we have that $F(s) \neq 0$ for $\sigma > 1$, hence by the functional equation we have the familiar notions of **critical strip** and **critical line**, *i.e.* the strip $0 \leq \sigma \leq 1$ and the line $\sigma = \frac{1}{2}$, respectively. The zeros of $F(s)$ located at the poles of the $\gamma$-factor $\gamma(s)$, *i.e.* at $\rho = -\frac{\mu_j+k}{\lambda_j}$ with $k = 0,1,2,\ldots$ and $j = 1,\ldots,r$, are called the **trivial zeros**, and are the only zeros of $F(s)$ in the half plane $\sigma < 0$. The case $\rho = 0$, if present, requires special attention in view of the possible pole of $F(s)$ at $s = 1$. The other zeros, located inside the critical strip, are called the **non-trivial zeros**. We cannot *a priori* exclude the possibility that $F(s)$ has a trivial and a non-trivial zero at the same point, on the line $\sigma = 0$. Moreover, writing

$$N_F(T) = |\{\rho = \beta + i\gamma : F(\rho) = 0,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma \leq T\}|,$$

the analog of the Riemann-von Mangoldt formula holds in the form

$$N_F(T) = \frac{d_F}{2\pi}T\log T + c_F T + O(\log T),$$

where $c_F$ is a certain constant depending on $F(s)$. This shows once again that the degree $d_F$ is an invariant (as well as $c_F$).

For details and further discussions on the matters above we refer to Selberg [32], Conrey-Ghosh [4], Murty [27] and the survey papers Kaczorowski-Perelli [15] and Kaczorowski [11].

Roughly speaking, the problems about the Selberg class are of two types.

• **Classical problems**: these are the extension to $\mathcal{S}$ of the problems about the classical $L$-functions, the most important being the Riemann Hypothesis. In fact, Selberg [32] conjectured that the Riemann Hypothesis holds for every function $F \in \mathcal{S}$, *i.e.*

**Conjecture 2.1 (GRH).** *Let $F \in \mathcal{S}$. Then $F(s) \neq 0$ for $\sigma > \frac{1}{2}$.*

We remark at this point that the knowledge about the distribution of zeros of the functions in $\mathcal{S}$ is definitely poorer than in the case of the classical $L$-functions. For example, it is not yet known in general if $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.

• **Structural problems**: these are the problems on the stucture of $\mathcal{S}$ *as a class*. The classification of the functions in $\mathcal{S}$, the independence properties of the functions in $\mathcal{S}$, the study of the invariants in $\mathcal{S}$, the countability and rigidity conjectures for $\mathcal{S}$ are important examples of structural problems.

In this survey we focus on the structural problems for the Selberg class. Such problems, in part raised by Selberg himself, deal with $L$-functions from a somewhat diffferent perspective with respect to the classical problems, and their solution will eventually lead to a deeper understanding of the nature of $L$-functions.

## 3. Basic theory of the Selberg class

We start with the classification of the functions in the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ with degree smaller than 1, since such results are needed later in this section. The basic result, Theorem 3.1 below, has apparently been proved first by Richert [31] and then independently by Bochner [1] and Conrey-Ghosh [4]. Further proofs have been given by Molteni [23] and Kaczorowski-Perelli [18], [21].

**Theorem 3.1** ([31, 1, 4])**.** $\mathcal{S}^\sharp_d = \emptyset$ *for* $0 < d < 1$.

A key point in the proof of Theorem 3.1 (common to several of the above proofs) is showing that the Dirichlet series of every function in $\mathcal{S}^\sharp_d$ with $0 \leq d < 1$ is absolutely convergent over $\mathbb{C}$. This contradicts $\mu_F(\sigma) > 0$

for $\sigma \leq 0$, provided $0 < d < 1$. For $d = 0$, the functional equation then shows that

$$F(s) = \sum_{n|q_F} \frac{a_F(n)}{n^s} \tag{3.1}$$

with $q_F = Q^2 \in \mathbb{N}$. Thus, in particular, the functions in $\mathcal{S}_0^\sharp$ are Dirichlet polynomials. For $q \in \mathbb{N}$ and $|\omega| = 1$, let $\mathcal{S}_0^\sharp(q,\omega)$ be the set of $F \in \mathcal{S}_0^\sharp$ with given $\omega$ and $q_F = q$, and let

$$V_0^\sharp(q,\omega) = \mathcal{S}_0^\sharp(q,\omega) \cup \{0\}.$$

Moreover, let $d(n)$ denote the divisor function. The above simple argument leads to

**Theorem 3.2** ([14]). *Let $F \in \mathcal{S}_0^\sharp$. Then $q_F \in \mathbb{N}$ and $F(s)$ has the form* (3.1)*. Moreover, $q_F$ and $\omega$ are invariants, thus $\mathcal{S}_0^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_0^\sharp(q,\omega)$ with $q \in \mathbb{N}$ and $|\omega| = 1$. Further, for any such $q$ and $\omega$, $V_0^\sharp(q,\omega)$ is a real vector space of dimension $d(q)$.*

We refer to Steuding [34] for a different characterization of the functions $F \in \mathcal{S}_0^\sharp$. Starting from (3.1), a simple argument based on the Euler product further shows

**Theorem 3.3** ([4]). $\mathcal{S}_0 = \{1\}$.

We already noticed that every function in the Selberg class has a standard Euler product, *i.e.* it can be expressed as a product of its $p$-Euler factors. It may happen that two distinct functions $F, G \in \mathcal{S}$ have equal $p$-Euler factors for certain primes $p$. Denote by $E_{F,G}$ the set of such primes. The "exceptional set" $E_{F,G}$ can be pretty large, as the following example shows. Let $\chi_1$ and $\chi_2$ be distinct primitive Dirichlet characters (mod $q$) such that $\chi_1(a) = \chi_2(a)$ for some $a$ coprime to $q$. Then the corresponding exceptional set contains the primes $p \equiv a \pmod q$. Hence, in particular, $E_{F,G}$ can have positive density.

On the other hand, a well known result in representation theory, called the *Strong Multiplicity One Theorem* (see Piatetski-Shapiro [29]), implies that if the $p$-Euler factors of two automorphic $L$-functions are equal for all but finitely many primes, then the two $L$-functions are equal. The analog of such a result for the Selberg class is called the **multiplicity one** property, and has been proved by Murty-Murty [28].

**Theorem 3.4** ([28]). *Let $F, G \in \mathcal{S}$. If $F_p(s) = G_p(s)$ for all but finitely many primes $p$, then $F(s) = G(s)$.*

The proof amounts to the observation that by the functional equation $F(s)/\ G(s)$ is entire and non-vanishing, hence the result follows by Hadamard's theory. The same argument shows that the assumption $F_p(s) = G_p(s)$ can be replaced by the weaker requirement that $a_F(p^m) = a_G(p^m)$ for $m = 1, 2$. It would be desirable to remove the condition involving the squares, as suggested by the following conjecture.

**Conjecture 3.1 (strong multiplicity one).** *Let $F, G \in \mathcal{S}$. If $a_F(p) = a_G(p)$ for all but finitely many primes $p$, then $F(s) = G(s)$.*

We will describe a rather sharp unconditional result in this direction in Section 9.

Clearly, the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ are *multiplicative semigroups* and the degree is *additive*, in the sense that $d_{F_1F_2} = d_{F_1} + d_{F_2}$. Moreover, given an *entire* $F \in \mathcal{S}^\sharp$ and $\theta \in \mathbb{R}$ we define the **shift** $F_\theta(s)$ as $F_\theta(s) = F(s + i\theta)$. Clearly, $F_\theta \in \mathcal{S}$ if $F \in \mathcal{S}$, and the same holds for $\mathcal{S}^\sharp$. Further, 1 is the only constant function in $\mathcal{S}$, and also the only invertible element of $\mathcal{S}$. A function $F \in \mathcal{S} \setminus \{1\}$ is **primitive** if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1(s) = 1$ or $F_2(s) = 1$; in other words, primitive functions are the irreducible elements of the semigroup $\mathcal{S}$. In view of Theorems 3.1 and 3.3, every function of degree $< 2$ is primitive, hence $\zeta(s)$ and the $L(s, \chi)$'s with primitive $\chi$ are primitive. Other examples of primitive functions are provided by a suitable class of normalized $L$-functions associated with holomorphic modular forms. These are degree 2 functions, and the proof requires a deeper knowledge of the structure of $\mathcal{S}_1$, see Section 6.

Primitive functions play an important role in the theory of the Selberg class. As a first result, Theorems 3.1, 3.3 and a simple induction on the degree give

**Theorem 3.5** ([4, 27]). *Every $F \in \mathcal{S}$ can be factored as a product of primitive functions.*

In other words, every $F \in \mathcal{S}$ has a factorization of type

$$F(s) = \prod_{j=1}^{k} F_j(s)^{e_j} \tag{3.2}$$

with $e_j \in \mathbb{N}$ and $F_j(s)$ primitive and distinct. A related natural conjecture is

**Conjecture 3.2 (unique factorization, UF).** *Factorization into primitive functions is unique.*

If the UF conjecture holds, then (3.2) is called the **standard form** of $F(s)$.

The Rankin-Selberg convolution method shows that the $L$-functions associated with holomorphic modular forms satisfy a kind of orthogonality relation. Precisely, under suitable normalizations and restrictions, the function $L_{f\times\overline{g}}(s)$ defined in Section 1 has a simple pole at $s = 1$ if $f(z) = g(z)$ and is entire otherwise. A similar result holds in general for the irreducible automorphic $L$-functions. Motivated by such properties, Selberg formulated the following fundamental conjecture.

**Conjecture 3.3 (Selberg orthonormality conjecture, SOC).** *Let $F, G \in \mathcal{S}$ be primitive functions and $\delta_{F,G} = 1$ if $F(s) = G(s)$, $\delta_{F,G} = 0$ otherwise. Then as $x \to \infty$*

$$\sum_{p\le x} \frac{a_F(p)\overline{a_G(p)}}{p} = (\delta_{F,G} + o(1)) \log\log x.$$

In order to appreciate the depth of the Selberg orthonormality conjecture, we list few simple but interesting consequences in Theorem 3.6 below. We first recall the the *Dedekind conjecture* asserts that $\zeta(s)$ divides $\zeta_K(s)$ for every algebraic number field $K/\mathbb{Q}$. This is well known in the case of normal extensions by the Aramata-Brauer theorem (see [8]). Moreover, given $F \in \mathcal{S}$ we define the real number $n_F$, if it exists, by

$$\sum_{p\le x} \frac{|a_F(p)|^2}{p} = (n_F + o(1)) \log\log x. \tag{3.3}$$

Further, we denote as usual by $\sigma_a(F)$ the abscissa of absolute convergence of $F \in \mathcal{S}^\sharp$. We have

**Theorem 3.6** ([4, 27, 28])**.** *Assume* SOC *and let $e_j$ be as in* (3.2)*. Then*

- (i) *the* UF *conjecture holds;*
- (ii) $n_F = \sum_{j=1}^k e_j^2$*, and hence $F \in \mathcal{S}$ is primitive if and only if $n_F = 1$;*
- (iii) $\zeta(s)$ *is the only primitive function in $\mathcal{S}$ with a pole at $s = 1$, and hence the Dedekind conjecture holds;*
- (iv) $F(1 + it) \ne 0$ *for every $t \in \mathbb{R}$, for every $F \in \mathcal{S}$;*
- (v) *the strong multiplicity one conjecture holds;*
- (vi) $\sigma_a(F) = 1$ *for every $F \in \mathcal{S} \setminus \{1\}$.*

We already remarked that at present is not unconditionally known if $F(1+it) \neq 0$, $t \in \mathbb{R}$, for every $F \in \mathcal{S}$. However, it is not surprising that this follows from SOC. In fact, the standard proofs of the non-vanishing of $L$-functions on the 1-line are based on the properties of the Rankin-Selberg convolution. We also remark that under SOC (in fact, under UF) the usual notions of *coprimality* and of *greatest common divisor* are easily defined in $\mathcal{S}$. From (ii) of Theorem 3.6 it is quite clear that in the case of *any* $F, G \in \mathcal{S}$, SOC becomes

$$\sum_{p\le x} \frac{a_F(p)\overline{a_G(p)}}{p} = \Big(\sum_{j=1}^{l} f_j g_j + o(1)\Big) \log\log x,$$

where

$$F(s) = H(s) \prod_{j=1}^{l} F_j(s)^{f_j} \qquad\qquad G(s) = K(s) \prod_{j=1}^{l} F_j(s)^{g_j},$$

the functions $F_j(s)$ are primitive and distinct, and $H(s)$, $K(s)$ are coprime and not divisible by the $F_j(s)$'s.

We remark here that the proof of the assertion on page 6 of Murty [27] that UF implies the Dedekind conjecture (unfortunately reported as Proposition 4.2 in Kaczorowski-Perelli [15]) appears to be incorrect. In fact (using the notation in [27]) assuming only UF we do not see how to exclude, for example, that $\zeta_K(s)$ is primitive and $F(s) = \zeta(s)H(s)$ with a primitive $H \in \mathcal{S}$ vanishing at $s = 1$.

Another interesting consequence of SOC, based on the Artin-Brauer theory and on Chebotarev density theorem, is

**Theorem 3.7 ([27]).** SOC *implies the Artin conjecture.*

We recall that the Artin conjecture states that the Artin $L$-functions $L(s,\ K/k,\rho)$ are entire if $\rho$ is irreducible and non-trivial. Moreover, the argument in the proof of Theorem 3.7 shows also that such functions are *primitive*. It is interesting to note how a conjecture concerning an axiomatic class of $L$-functions has a strong consequence on a classical conjecture. The argument in the proof of Theorem 3.7, coupled with work of Arthur-Clozel on solvable extensions, can be suitably adapted to show that SOC implies the Langlands reciprocity conjecture for solvable extensions of $\mathbb{Q}$, see Murty [27].

One may ask if primitivity can be characterized by the functional equation. Apparently this is not the case, as shown by an example in Molteni [26] of a degree 2 functional equation with a non-primitive solution in $\mathcal{S}$ (a Dedekind zeta function of a real quadratic field) and, assuming the Takhtajan-Vinogradov conjecture on the dimension of the space of Maass forms, a primitive solution as well. We state here a problem about the shifts of primitive functions.

**Problem 3.1.** *Show that $F_\theta(s)$ is primitive for all $\theta \in \mathbb{R}$ if $F \in \mathcal{S}$ is primitive.*

There is an easy proof of this statement if axiom (ii) of the Selberg class is weakened to allow a finite number of poles on the line $\sigma = 1$ (note that *every* function in such a larger class can be shifted). In fact, suppose that $F(s)$ is primitive, while $F_\theta(s) = F_1(s)F_2(s)$ is a non-trivial factorization for some $\theta \in \mathbb{R}$. Then $F(s) = F_\theta(s-i\theta) = F_1(s-i\theta)F_2(s-i\theta)$, a contradiction. In the framework of the Selberg class $\mathcal{S}$, the problem arises from the situation, which we cannot *a priori* exclude, that $F_\theta(s)$ is entire while $F_1(s)$ has a pole and $F_2(s)$ has a zero at $s = 1$. This situation is of course impossible under SOC.

The Selberg orthonormality conjecture can be regarded as a rather strong form of independence of the functions in $\mathcal{S}$. The unique factorization conjecture, which follows from SOC, is also an independence statement in $\mathcal{S}$. We may therefore ask if the simplest form of independence, namely the *linear independence*, holds in $\mathcal{S}$. We recall that a Dirichlet series $D(s)$, absolutely convergent in some right half-plane, is called *p-finite* if there exists a positive integer $M$ such that the coefficients $c(n)$ of $D(s)$ vanish whenever $n$ has a prime factor not dividing $M$. In this case, the arithmetic function $c(n)$ is called $p$-finite as well. We denote by $\mathcal{F}$ both the ring of $p$-finite Dirichlet series and the ring of $p$-finite arithmetic functions; note that $\mathcal{F}$ contains all Dirichlet polynomials.

**Theorem 3.8** ([12]). *Distinct functions in $\mathcal{S}$ are linearly independent over $\mathcal{F}$.*

In particular, distinct functions of $\mathcal{S}$ are linearly independent over $\mathbb{C}$. We remark that Theorem 3.8 is basically a result on multiplicative arithmetic functions. We call *equivalent* two multiplicative functions $f(n)$ and $g(n)$ if $f(p^m) = g(p^m)$ for all integer $m \geq 1$ and all but finitely many primes

$p$. The main step in the proof of Theorem 3.8 is showing that pairwise non-equivalent multiplicative functions are linearly independent over $\mathcal{F}$. This is in fact an analogue of Artin's well known result that distinct characters are linearly independent, and the proof is similar. Theorem 3.8 follows then by Theorem 3.4, which ensures that the coefficients of distinct functions in $\mathcal{S}$ are pairwise non-equivalent multiplicative functions.

We remark that Theorem 3.8 is a special case of a more general result, see Kaczorowski-Molteni-Perelli [13]. In fact, its proof can be suitably modified to show the linear independence of functions in a larger class, including the derivatives of all orders and the inverses of the functions in $\mathcal{S}$. Moreover, such a class also contains the Artin and the automorphic $L$-functions, which are not yet known to belong to $\mathcal{S}$. See also Molteni [25] for further results.

It is well known that the Prime Number Theorem is equivalent to $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$. Although the non-vanishing on the 1-line is at present a conditional result in the general setting of the Selberg class, the analog of the above-mentioned equivalence can be proved unconditionally in $\mathcal{S}$. Let $\Lambda_F(n)$ be the **generalized von Mangoldt function**, defined by

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s},$$

*i.e.* $\Lambda_F(n) = b_F(n)\log n$, and let

$$\psi_F(x) = \sum_{n\leq x} \Lambda_F(n).$$

It is expected that the prime number theorem (PNT) holds in the form

$$\psi_F(x) = m_F x + o(x)$$

for every $F \in \mathcal{S}$, where $m_F$ is the polar order of $F(s)$ defined in Section 2. Writing

$$\psi_{F\times\overline{F}}(x) = \sum_{n\leq x} |\Lambda_F(n)|^2,$$

a simple consequence of axioms (iv) and (v) is that $\psi_{F\times\overline{F}}(x) \ll x^{1+\varepsilon}$, and hence the bound $\psi_F(x) \ll x^{1+\varepsilon}$ holds unconditionally.

**Theorem 3.9** ([20])**.** *Let $F \in \mathcal{S}$. Then* PNT *holds if and only if $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.*

The proof is based on a weak density estimate for the zeros of $F(s)$ close to the 1-line and on a simple almost periodicity argument. From Theorems 3.6 and 3.9 we see that SOC *implies* PNT. However, the argument in the proof of Theorem 3.9 allows to obtain a sharper result. To this end we introduce the following much weaker version of SOC.

**Conjecture 3.4 (normality conjecture, NC).** *Let* $F \in \mathcal{S} \setminus \{1\}$*. Then* (3.3) *holds with* $n_F > 0$*, and* $n_F \leq 1$ *if* $F(s)$ *is primitive.*

We have

**Theorem 3.10** ([20]). *Assume* NC *and let* $F \in \mathcal{S}$*. Then* $F(1+it) \neq 0$ *for every* $t \in \mathbb{R}$*.*

In view of Theorems 3.9 and 3.10, NC *implies* PNT. We recall that

$$\sum_{p \leq x} \frac{1}{p} \sim \log\log x$$

is a weaker statement than the Prime Number Theorem, and an analogous assertion holds for other classical $L$-functions as well. Hence, NC for a *given* $F \in \mathcal{S}$ is, in general, weaker than PNT for the *same* function. Therefore, Theorem 3.10 is a simple example of the philosophy that general properties of a *family* of $L$-functions can be used to derive stronger consequences for *individual* members of the family.

Now we turn to a discussion of the factorization problem in $\mathcal{S}^\sharp$. In order to extend the notion of primitive function to the class $\mathcal{S}^\sharp$, we need to know the invertible functions in $\mathcal{S}^\sharp$. Clearly, the non-zero complex constants belong to $\mathcal{S}^\sharp$, and it is easy to see that these are the only invertible elements of $\mathcal{S}^\sharp$. Hence we say that a non-constant $F \in \mathcal{S}^\sharp$ is **$\sharp$-primitive** if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}^\sharp$ implies that $F_1(s)$ or $F_2(s)$ is constant. The problem of the factorization into primitive functions can therefore be raised for $\mathcal{S}^\sharp$ as well. The analogous property for $\mathcal{S}$ depends on the following three facts: the degree is additive, there are no functions with degree $0 < d_F < 1$ and $\mathcal{S}_0 = \{1\}$. The first two facts hold for $\mathcal{S}^\sharp$ as well, but $\mathcal{S}_0^\sharp$ is definitely more complicated than $\mathcal{S}_0$. Therefore, the proof of Theorem 3.5 does not carry over to the case of $\mathcal{S}^\sharp$. However, the argument can be suitably modified to prove

**Theorem 3.11** ([19]). *Every* $F \in \mathcal{S}^\sharp$ *can be factored as a product of* $\sharp$*-primitive functions.*

The proof is based on the notion of **almost-primitive** function, that is a function $F \in \mathcal{S}^\sharp$ such that $F(s) = F_1(s)F_2(s)$ implies $d_{F_1} = 0$ or $d_{F_2} = 0$. The main part of the proof of Theorem 3.11 is devoted to the following characterization of almost-primitive functions: *if* $F \in \mathcal{S}^\sharp$ *is almost-primitive, then* $F(s) = P(s)G(s)$ *with* $P(s)$ *$\sharp$-primitive and* $d_G = 0$. In turn, such a characterization is based on a uniform estimate for the number of zeros of the Dirichlet polynomials of $\mathcal{S}_0^\sharp$. Theorem 3.11 follows then from the above characterization by a double induction, first on the degree (giving the factorization into almost-primitive functions) and then on the integer $q_F$ in Theorem 3.2 (giving the factorization of the functions of $\mathcal{S}_0^\sharp$ into $\sharp$-primitive functions). We will see in the next section that such an integer $q_F$ is a special instance of the general notion of *conductor* in $\mathcal{S}^\sharp$.

We remark that *the analog of* SOC *does not hold for* $\mathcal{S}^\sharp$. Indeed, let $\chi_1, \chi_2$ be two primitive Dirichlet characters with the same modulus and parity, and let $F(s) = L(s,\chi_1) + L(s,\chi_2)$ and $G(s) = L(s,\chi_1)$. Thanks to Theorems 3.1 and 3.2 and to the description of $\mathcal{S}_1^\sharp$ in Section 6, in view of Theorem 3.8 we have that $F(s)$ and $G(s)$ are $\sharp$-primitive, but it is easily checked that SOC does not hold for $F(s)$ and $G(s)$. In view of this, we conclude the section with two problems.

**Problem 3.2.** *Does UF hold for* $\mathcal{S}^\sharp$*?*

**Problem 3.3.** *Is it true that a primitive* $F \in \mathcal{S}$ *is also $\sharp$-primitive?*

We conclude this section with a problem on the characterization of divisibility in $\mathcal{S}$. In view of the Hadamard product, a function in $\mathcal{S}$ is essentially determined by its zeros. Denoting by $Z_F$ the **set of zeros** of $F \in \mathcal{S}$ *counted with multiplicity*, we raise the following

**Problem 3.4.** *Let* $F, G \in \mathcal{S}$. *Show that* $F(s)$ *divides* $G(s)$ *in* $\mathcal{S}$ *if and only if* $Z_F \subset Z_G$.

We refer to Molteni [26] and [24] for closely related results.

## 4. Invariants

We already pointed out in Section 2 that, due to the application of suitable identities satisfied by the $\Gamma$-function, the shape of the $\gamma$-factor $\gamma(s)$ of $F \in \mathcal{S}^\sharp$ is not uniquely determined by $F(s)$. We also remarked that this fact gives rise to the notion of *invariant, i.e.* an expression defined in terms of the data of $F(s)$ which is uniquely determined by $F(s)$ itself. Moreover, we already met an important invariant, namely the degree $d_F$.

Although their *shape* may change considerably, $\gamma$-factors are essentially *uniquely determined as functions*. In fact we have

**Theorem 4.1** ([4]). *Let $\gamma(s)$ and $\gamma'(s)$ be two $\gamma$-factors of $F \in \mathcal{S}^\sharp$. Then there exists a constant $c_0 = c_0(\gamma, \gamma') \in \mathbb{C}$ such that $\gamma(s) = c_0\gamma'(s)$.*

The proof follows by Hadamard's theory, observing that $h(s) = \gamma(s)/\gamma'(s)$ is entire and non-vanishing thanks to the functional equation.

In view of Theorem 4.1, in order to study the invariants we need to detect the operations which transform a $\gamma$-factor $\gamma(s)$ of a function $F \in \mathcal{S}^\sharp$ into another $\gamma$-factor of the same function. It turns out that such a transformation can be performed by repeated applications to $\gamma(s)$ of two basic formulae in the theory of the $\Gamma$-function, namely the *Legendre-Gauss multiplication formula*

$$\Gamma(s) = m^{s-\frac{1}{2}}(2\pi)^{\frac{1-m}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{s+k}{m}\right) \qquad m = 2, 3, ... \tag{4.1}$$

and the *factorial formula*

$$\Gamma(s+1) = s\Gamma(s).$$

We also need some definitions. Two positive real numbers $\alpha, \beta$ are $\mathbb{Q}$-*equivalent* if $\alpha/\beta \in \mathbb{Q}$. We denote by $h_F$, the $\gamma$-**class number**, the number of $\mathbb{Q}$-equivalence classes arising from the $\lambda$-coefficients $\lambda_1, ..., \lambda_r$ of a $\gamma$-factor of $F \in \mathcal{S}^\sharp$. Moreover, we say that $F(s)$ is **reduced** if it has a $\gamma$-factor with $0 \leq \Re\mu_j < 1$ for $j = 1, ..., r$; such a $\gamma$-factor is also called *reduced*. It turns out that $h_F$ *is an invariant*, and that $F(s)$ *is reduced if and only if all its $\gamma$-factors are reduced*, so these are reasonable definitions.

Recalling that $c_0$ is the constant in Theorem 4.1 we have

**Theorem 4.2** ([16]). *Let $\gamma(s)$ and $\gamma'(s)$ be two $\gamma$-factors of $F \in \mathcal{S}^\sharp$. Then $\gamma(s)$ can be transformed into $c_0\gamma'(s)$ by repeated applications of the multiplication and factorial formulae. Moreover, the factorial formula can be avoided if $h_F = 1$ or if $F(s)$ is reduced.*

We refer to Section 4 of Vignéras [36], as well as to Serre's appendix there, for related results. It is clear that applications of the multiplication formula to a $\gamma$-factor give rise to another $\gamma$-factor, and do not change the $\mathbb{Q}$-equivalence classes. Applications of the factorial formula are a bit more involved. Basically, such a formula is used to *reduce* a $\gamma$-factor, *i.e.* to write it as the product of a reduced $\gamma$-factor, called the *reduced part*, times a product of suitable linear factors. Such linear factors are then re-absorbed into the $\Gamma$-factors by further applications of the factorial formula, provided suitable consistency conditions hold. Although examples of non-reduced $\gamma$-factors are easily produced, see for instance the case of $L$-functions associated with holomorphic modular forms (suitably nomalized to meet the axioms of $\mathcal{S}$), according to the following conjecture we expect $h_F = 1$ to be the general case.

**Conjecture 4.1 ($\gamma$-class number conjecture).** *Every $F \in \mathcal{S}^\sharp$ has $h_F = 1$.*

We will see in Section 6 motivations for this and for the following *much stronger* conjecture.

**Conjecture 4.2 ($\boldsymbol{\lambda}$-conjecture).** *Every $F \in \mathcal{S}^\sharp$ has a $\gamma$-factor with $\lambda_j = \frac{1}{2}$ for $j = 1, ..., r$.*

Therefore, we expect that the factorial formula is not necessary in the transformation of the $\gamma$-factors. However, at the present state of the knowledge, we cannot in general avoid using it, and here is an example:

$$\Gamma(s)\Gamma\left(\sqrt{2}s+1\right) = \sqrt{\frac{2}{\pi}}2^s\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\sqrt{2}s\right).$$

Note that there are two $\mathbb{Q}$-equivalence classes, and that the pole at $s = 0$ comes, in the two sides of the identity, from $\Gamma$-factors belonging to different classes. This is the typical situation requiring application of the factorial formula. However, we expect that no $F \in \mathcal{S}^\sharp$ has such $\Gamma$-factors in its functional equation.

In view of the identity $\gamma(s) = c_0\gamma'(s)$ in Theorem 4.1, the proof of Theorem 4.2 rests on a detailed analysis of the structure of the following general $\Gamma$-identity

$$\prod_{j=1}^{N}\Gamma(\lambda_j s+\mu_j) = e^{as}R(s)\prod_{j=1}^{M}\Gamma(\lambda_j' s+\mu_j'), \tag{4.2}$$

where $a \in \mathbb{C}$ and $R(s)$ is a rational function. Clearly, $R(s)$ arises from applications of the factorial formula. The structure of (4.2) is studied by means of the analysis of the poles of both sides. This leads to a *transformation algorithm* for $\gamma$-factors, which we briefly outline. Let $\gamma(s)$ and $\gamma'(s)$ be as in Theorem 4.2. Then $\gamma(s)$ is transformed into $c_0\gamma'(s)$ as follows.

**Step 1** (*reducing*). Apply the factorial formula to reduce $\gamma(s)$ and $\gamma'(s)$.

**Step 2** (*grouping*). Group the $\Gamma$-factors of the reduced parts and the corresponding linear factors according to $\mathbb{Q}$-equivalence classes. The $\mathbb{Q}$-equivalence classes arising from $\gamma(s)$ and $\gamma'(s)$ are the same, and identity $\gamma(s) = c_0\gamma'(s)$ induces suitable sub-identities of type (4.2) between the pairs of groups with the same $\mathbb{Q}$-equivalence class.

**Step 3** (*equating*). Apply the multiplication formula to each pair of groups, to obtain new pairs of groups with the property that all the $\Gamma$-factors in the same pair of groups have the same $\lambda$-coefficient. In such a situation, in each pair of groups the $\Gamma$-factors coming from $\gamma(s)$ are a permutation of those coming from $\gamma'(s)$.

**Step 4** (*transforming*). Perform on the $\Gamma$-factors coming from $\gamma(s)$ the inverse of the operations performed in steps 3, 2 and 1 on the $\Gamma$-factors coming from $\gamma'(s)$, thus transforming $\gamma(s)$ into $c_0\gamma'(s)$.

A more combinatorial argument leading to a simple proof of Theorem 4.2 is provided by Wirsing [39].

The proof of Theorem 4.2 involves also the notion of *exact covering system*, *i.e.* a family $(M, l_j, m_j), j = 1, ..., M$, with the property that for every integer $n$ there exists a unique $j$ such that $n \equiv l_j \pmod{m_j}$. As a byproduct of the arguments in the proof, we can get the following complete description of all $\gamma$-factors of the Dirichlet $L$-functions. Of course, other known $L$-functions can be treated analogously. Let $\chi \pmod q$ be a primitive Dirichlet character. Then *all $\gamma$-factors of $L(s,\chi)$ are of the form*

$$Q^s \prod_{j=1}^{M} \Gamma\left(\frac{s}{2m_j} + \frac{2l_j + a(\chi)}{2m_j}\right),$$

*where $(M, l_j, m_j)$ is any exact covering system,*

$$Q = \left(\frac{q}{\pi} \prod_{j=1}^{M} m_j^{1/m_j}\right)^{1/2}$$

*and* $a(\chi) = \frac{1+\chi(-1)}{2}$.

In order to give a characterization of the invariants by means of Theorem 4.2, we need to give a more formal definition of invariant. An expression depending on the "variables" $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ is called a *parameter*. An

**invariant** is a parameter depending only on $F(s)$ and not on the particular choice of the data of $F(s)$, for every $F \in \mathcal{S}^\sharp$. In other words, a parameter $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ is an invariant if $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) = I(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ for any pair of data $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega), (Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ of $F(s)$, for every $F \in \mathcal{S}^\sharp$. Parameters and invariants will sometimes be denoted by $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$. A generic invariant will be denoted by $I$, and when referred to a function $F(s)$ will be denoted by $I_F$ or $I(F)$. An invariant $I$ is called **numerical** if $I_F \in \mathbb{C}$ for every $F \in \mathcal{S}^\sharp$.

We say that a parameter is *stable by multiplication formula* if $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) = I(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$, where $(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ are the new data obtained by application of the multiplication formula to a $\Gamma$-factor. Similarly we say that a parameter is *stable by factorial formula*, although this case is a bit more subtle since we always apply the factorial formula to a pair of $\Gamma$-factors satisfying a consistency condition. In fact, by the factorial formula we have

$$\Gamma(\lambda s + \mu)\Gamma(\lambda' s + \mu') = \frac{\lambda}{\lambda'}\left(\lambda' s + \frac{(\mu - 1)\lambda'}{\lambda}\right)\Gamma(\lambda s + \mu - 1)\Gamma(\lambda' s + \mu'),$$

and assuming the *consistency condition*

$$\frac{\mu - 1}{\lambda} = \frac{\mu'}{\lambda'} \tag{4.3}$$

we get

$$\Gamma(\lambda s + \mu)\Gamma(\lambda' s + \mu') = \frac{\lambda}{\lambda'}\Gamma(\lambda s + \mu - 1)\Gamma(\lambda' s + \mu' + 1). \tag{4.4}$$

The above notions of stability will be clarified below, where we will list several important examples of invariants. From Theorem 4.2 we immediately obtain

**Corollary 4.1** ([16]). *A parameter is an invariant if and only if it is stable by multiplication and factorial formulae.*

Here is a short list of important invariants of $\mathcal{S}^\sharp$, as well as some remarks; see Kaczorowski-Perelli [16], [17].

- The $H$**-invariants** $H_F(n)$. For a non-negative integer $n$ let

$$H_F(n) = 2\sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}},$$

where $B_n(z)$ denotes the $n$-th Bernoulli polynomial. Since $B_0(z) = 1$, $B_1(z) = z - \frac{1}{2}$, $B_2(z) = z^2 - z + \frac{1}{6}$, ..., we have for instance

$$H_F(0) = 2\sum_{j=1}^{r} \lambda_j = d_F \qquad \text{(the degree)}$$

$$H_F(1) = 2\sum_{j=1}^{r} (\mu_j - \frac{1}{2}) = \xi_F = \eta_F + i\theta_F \qquad \text{(the } \xi\text{-}\textbf{invariant}\text{)}.$$

We sketch the proof that the $H_F(n)$ are invariants, hence clarifying Corollary 4.1. Let $\Gamma(\lambda s + \mu)$ be one of the $\Gamma$-factors of $F(s)$. After application of the multiplication formula (4.1) to such a $\Gamma$-factor, we have to prove that

$$\frac{B_n(\mu)}{\lambda^{n-1}} = \sum_{j=0}^{m-1} \frac{B_n(\frac{\mu+j}{m})}{(\frac{\lambda}{m})^{n-1}} \qquad n \geq 0,\ m \geq 1,$$

and this follows from the following identity for Bernoulli polynomials

$$B_n(z) = m^{n-1} \sum_{j=0}^{m-1} B_n(\frac{z+j}{m}) \qquad n \geq 0,\ m \geq 1.$$

Therefore the $H_F(n)$ are stable by multiplication formula. In order to check that the $H_F(n)$ are stable by factorial formula as well, let $\Gamma(\lambda s + \mu)$ and $\Gamma(\lambda' s + \mu')$ be two $\Gamma$-factors of $F(s)$ and apply the factorial formula as in (4.4). Consequently, we have to prove that

$$\frac{B_n(\mu)}{\lambda^{n-1}} + \frac{B_n(\mu')}{\lambda'^{n-1}} = \frac{B_n(\mu-1)}{\lambda^{n-1}} + \frac{B_n(\mu'+1)}{\lambda'^{n-1}} \qquad n \geq 0,$$

and this follows from the identity

$$B_n(z+1) = B_n(z) + nz^{n-1} \qquad n \geq 0,$$

under the consistency condition (4.3). Hence the $H_F(n)$ are invariants by Corollary 4.1. Note that the $H$-invariants are *additive*, *i.e.* $H_{FG}(n) = H_F(n) + H_G(n)$.

We already saw in Section 2 the meaning of the degree $d_F$ in terms of the function $F(s)$. Note that the degree of the functions in (2.1) is always an *integer*; in Section 6 we will state a fundamental conjecture about the degree, namely the *degree conjecture*. Concerning the $\xi$-invariant $\xi_F$, its real part $\eta_F$ is called the **parity** of $F(s)$, while its imaginary part $\theta_F$ is the **shift**, not to be confused with the shift $F_\theta(s)$ introduced in Section 3. Observe that the shift $\theta_F$ is usually 0 for the classical $L$-functions ($\mu_j \in \mathbb{R}$ in many cases). Observe also that the Hecke $L$-functions $L_K(s,\chi)$, with $\chi$ character of infinite order, provide non-trivial examples of $\theta_F = 0$, due to the fact that $\chi$ is normalized. We refer again to Section 6 for the meaning of the invariants $\eta_F$ and $\theta_F$, at least for degree 1 functions. For general $n$ we raise the following problem about $H$-invariants:

**Problem 4.1.** *Give a meaning in terms of $F(s)$ to every invariant $H_F(n)$, $n \geq 2$.*

In Kaczorowski-Perelli [17] an asymptotic expansion of $\log\gamma(s)$ is given that involves the $H_F(n)$. However, Problem 4.1 asks for a more explicit meaning for such invariants, possibly without explicit reference to the functional equation.

• The **conductor** $q_F$. We already defined in the previous section the *conductor* in the case of functions of degree 0, and we saw that it is an integer and an invariant. In the general case of $F \in \mathcal{S}^\sharp$ we define

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$

As before, it is easy to show that the conductor is stable by multiplication and factorial formulae, hence it is an invariant by Corollary 4.1. Moreover, it is easy to check that

$$q_\zeta = 1, \quad q_{L(\cdot,\chi)} = q, \quad q_{\zeta_K} = |D_K|, \quad q_{L_K(\cdot,\chi)} = |D_K| N(\mathfrak{f}), \quad q_{L_f} = N,$$

where $q$ is the modulus of the primitive Dirichlet character $\chi$, $D_K$ is the discriminant of $K$, $N(\mathfrak{f})$ is the norm of the conductor $\mathfrak{f}$ of the primitive Hecke character $\chi$ and $N$ is the level of the holomorphic modular form $f(z)$. Hence the conductor $q_F$ appears to be the right extension to $\mathcal{S}^\sharp$ of the various classical notions of conductor. Note that the conductor is *multiplicative*, *i.e.* $q_{FG} = q_F q_G$. Note also that the above functions belong to $\mathcal{S}$, and their conductor is an *integer*. In fact, we have

**Conjecture 4.3 (conductor conjecture).** *Every $F \in \mathcal{S}$ has $q_F \in \mathbb{N}$.*

Probably this conjecture does not hold for $\mathcal{S}^\sharp$, and counterexamples can possibly be found among the $L$-functions associated with the Hecke groups $G(\lambda)$.

- The **root number** $\omega_F^*$. The root number of $F \in \mathcal{S}^\sharp$ is defined by

$$\omega_F^* = \omega e^{-i\frac{\pi}{2}(\eta_F+1)} \left(\frac{q_F}{(2\pi)^{d_F}}\right)^{i\theta_F/d_F} \prod_{j=1}^r \lambda_j^{-2i\Im\mu_j}.$$

Once again, it is easy to show that the root number is stable by multiplication and factorial formulae, hence it is an invariant by Corollary 4.1. The root number $\omega_F^*$ comes out naturally in certain computations, and is of course closely related to $\omega$ for the classical $L$-functions. Here are two problems about $\omega_F^*$.

**Problem 4.2.** *What is the meaning of $\omega_F^*$? Is $\omega_F^*$ the correct definition of "root number"?*

**Problem 4.3.** *Is it true that $\omega_F^*$ is always an algebraic number for $F \in \mathcal{S}$?*

Problem 4.2 is related with the definition of $\omega_F$ below. Moreover, Problem 4.3 has a negative answer in the case of $\mathcal{S}^\sharp$, as we will see in Section 6.

A set $\{I_j\}_{j\in J}$ of *numerical* invariants is called a set of **basic invariants** if the $I_j$ characterize the functional equation of *every* $F \in \mathcal{S}^\sharp$, in the sense that if $I_j(F) = I_j(G)$ for all $j \in J$ then $F(s)$ and $G(s)$ satisfy the same functional equation, for any $F, G \in \mathcal{S}^\sharp$. In principle, such a set should be called a **global** set of basic invariants, since we will also deal with **local** sets of basic invariants, characterizing the functional equation of a *given* function $F \in \mathcal{S}^\sharp$.

**Theorem 4.3** ([17]). *The $H$-invariants $H_F(n)$, $n \geq 0$, the conductor $q_F$ and the root number $\omega_F^*$ are a global set of basic invariants.*

The proof is based on the fact that the function

$$K_F(z) = 2z \sum_{j=1}^r \frac{e^{z\mu_j/\lambda_j}}{e^{z/\lambda_j} - 1} = -2z \sum_\rho e^{\rho z}, \tag{4.5}$$

where the last sum is over the poles of a $\gamma$-factor of $F \in \mathcal{S}^\sharp$, has the power series expansion

$$K_F(z) = \sum_{n=0}^\infty \frac{H_F(n)}{n!} z^n,$$

hence the $\gamma$-factors of $F(s)$ and $G(s)$ differ by a factor $e^{as+b}$ if the $H$-invariants are equal. Assuming further that conductors and root numbers are equal, it is not difficult to show that $F(s)$ and $G(s)$ satisfy the same functional equation.

Clearly, if we drop the condition that basic invariants are numerical invariants, then finite global sets of basic invariants are easily detected, for instance $\{K_F(z), q_F, \omega_F^*\}$. However, Jurek Kaczorowski and Giuseppe Molteni pointed out that there exist global sets of basic invariants formed by a single numerical invariant. The argument is, roughly speaking, as follows. The set of the functional equations of axiom (iii) (modulo the "equivalent" functional equations in the sense of Theorem 4.2) has the cardinality of the continuum, and hence there exists an injective mapping $\phi$ from such functional equations to $\mathbb{R}$. Given $F \in \mathcal{S}^\sharp$, define the numerical invariant $I_F$ as the value of the mapping $\phi$ at the functional equation satisfied by $F(s)$. Clearly, such an invariant forms a global set of basic invariants. Of course, the invariants coming from this argument are not explicit, but more explicit versions can be obtained by refining the argument. However, such invariants are quite artificial, while the invariants in Theorem 4.3 are definitely more interesting.

Another problem related with invariants is determining an **invariant form** of the functional equation, where all data are invariants. Clearly, such an invariant form provides in particular a local set of basic invariants. We deal with this problem by constructing a special (essentially) invariant form of the functional equation, which we call the *canonical functional equation*. The motivation comes from the fact that the $\lambda$-coefficients in the standard functional equation of the classical $L$-functions are all equal to $\frac{1}{2}$ (or easily transformed to $\frac{1}{2}$). Roughly speaking, the canonical functional equation plays this role in the general case of $\mathcal{S}^\sharp$.

To construct the canonical functional equation, we split the function $K_F(z)$ in (4.5) into $\mathbb{Q}$-equivalence classes as

$$K_F(z) = \sum_{j=1}^{h_F} K_j(z)$$

and define the **canonical exponents** $\Lambda_j$ by

$$\Lambda_j = \max\{\Lambda \in \mathbb{R} : (e^{z/\Lambda} - 1)K_j(z) \text{ is entire}\}.$$

The canonical exponents exist, are positive and distinct, and are invariants, see Kaczorowski-Perelli [17]. Moreover, every $F \in \mathcal{S}^\sharp$ has a **balanced** $\gamma$-factor, *i.e.* of the form

$$\gamma(s) = Q^s \prod_{j=1}^{h_F} \prod_k \Gamma(\lambda_j s + \mu_{j,k})$$

with all ratios $\Lambda_j/\lambda_j$ equal. Such ratios are positive integers, and their minimum over all balanced $\gamma$-factors of $F(s)$ is called the **reduction index** $l_F$, clearly an invariant; see [17]. Given positive integers $K_j$ $(j = 1, ..., h_F)$ and complex numbers $\mu_{j,k}$ with $\Re\mu_{j.k} \geq 0$ $(j = 1, ..., h_F,\ k = 1, ..., l_F K_j)$ to be specified later, we write

$$Q_F = \left( q_F (2\pi)^{-d_F} l_F^{d_F} \prod_{j=1}^{h_F} \Lambda_j^{-2K_j\Lambda_j} \right)^{1/2}$$

$$\omega_F = \omega_F^* e^{i\frac{\pi}{2}(\eta_F+1)} \left( \frac{q_F}{(2\pi)^{d_F}} \right)^{-i\theta_F/d_F} l_F^{-i\theta_F} \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Lambda_j^{2i\Im\mu_{j,k}} \qquad (4.6)$$

$$\gamma_F(s) = Q_F^s \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left( \frac{\Lambda_j}{l_F} s + \mu_{j,k} \right).$$

**Theorem 4.4** ([17])**.** *Every $F \in \mathcal{S}^\sharp$ uniquely determines positive integers $K_j$ such that*

$$\gamma_F(s)F(s) = \omega_F \overline{\gamma_F}(1-s)\overline{F}(1-s), \qquad (4.7)$$

*where $\gamma_F(s)$ and $\omega_F$ are given by* (4.6) *and the $\mu_{j,k}$'s are uniquely determined* (mod $\mathbb{Z}$) *by $F(s)$. Moreover, the $\mu_{j,k}$'s are uniquely determined by $F(s)$ if $h_F = 1$ or if $F(s)$ is reduced, and $l_F = 1$ in the latter case.*

The functional equation in (4.7) is called the **canonical functional equation**, and in view of Conjecture 4.1 we expect that (4.7) is in invariant form. The non-uniqueness of the $\mu_{j,k}$ when $h_F > 1$ comes from possible applications of the factorial formula to $\Gamma$-factors belonging to different $\mathbb{Q}$-equivalence classes. The proof of Theorem 4.4 is quite technical; we refer to Kaczorowski-Perelli [17] for the proof and for an algorithm for the computation of the canonical functional equation from a given one.

Assuming that $h_F = 1$, a $\gamma$-factor is balanced if and only if all its $\lambda$-coefficients are equal, hence by the definition of $l_F$ we have that *the canonical functional equation has the minimum number of $\Gamma$-factors among the*

*$\gamma$-factors with all $\lambda$-coefficients equal.* This clarifies somewhat the meaning of the $\Lambda_j$ and of $l_F$ in the case of balanced $\gamma$-factors: the $\Lambda_j$ are the "largest possible" $\lambda$-coefficients and $l_F$ somehow measures the "reduction" of $\gamma$-factors, attaining its minimum ($l_F = 1$) in the reduced case.

The standard functional equation of $\zeta(s)$ and $L(s,\chi)$, $\chi$ primitive Dirichlet character, coincides with the canonical one. This holds for the $L$-functions $L_f(s)$ as well. The canonical functional equation of $\zeta_K(s)$ is obtained from the standard one by applying the Legendre duplication formula to the $\Gamma$-factors with $\lambda$-coefficient equal to 1, in those cases where both $\frac{1}{2}$ and 1 are present as $\lambda$-coefficients. Note that all the classical $L$-functions have a balanced $\gamma$-factor with $\lambda$-coefficient equal to $\frac{1}{2}$ or 1. A related problem is

**Problem 4.4.** *Is it true that the canonical functional equation of the classical $L$-functions has $\lambda$-coefficient always equal to $\frac{1}{2}$ or $1$?*

In other words: is it true that all the balanced $\gamma$-factors of the classical $L$-functions have $\lambda$-coefficient not larger than 1?

Coming to the local sets of basic invariants, with the notation in Theorem 4.4 let

$$r_F = l_F \sum_{j=1}^{h_F} K_j, \qquad g_F = 2^{2h_F-1} r_F - 2h_F + 1.$$

Using the function $K_F(z)$ in (4.5) and the canonical exponents, by Theorems 4.3 and 4.4 we get

**Theorem 4.5 ([17]).** (i) *A local set of basic invariants of $F \in \mathcal{S}^\sharp$ is provided by $h_F$, $r_F$*

$$q_F,\ \omega_F^* \quad \text{and the } H_F(n) \text{ with } n \le g_F. \tag{4.8}$$

(ii) *Assuming the $\boldsymbol{\lambda}$-conjecture, a local set of basic invariants of $F \in \mathcal{S}^\sharp$ is provided by the invariants in* (4.8) *with $g_F$ replaced by $d_F$.*

As a consequence, we expect that $q_F$, $\omega_F^*$ *and the $H$-invariants with $n \le d_F$ characterize the functional equation of $F \in \mathcal{S}^\sharp$.* In Section 6 we will see that this is in fact the case for the degree 1 functions. We remark that (ii) of Theorem 4.5 is best possible, in the sense that for every integer $d \ge 1$ there exist $F, G \in \mathcal{S}_d^\sharp$ for which the invariants in (4.8) with $g_F$ replaced by $d-1$ are equal, but $F(s)$ and $G(s)$ satisfy different functional equations. Examples are provided by suitable products of shifted Dirichlet $L$-functions, see Kaczorowski-Perelli [17].

A fundamental problem in the theory of the Selberg class is describing the admissible values of the numerical invariants, that is the set of values that numerical invariants attain at the functions of $\mathcal{S}$ and $\mathcal{S}^\sharp$:

**Problem 4.5.** *Given a numerical invariant $I : \mathcal{S}^\sharp \to \mathbb{C}$, describe $I(\mathcal{S})$ and $I(\mathcal{S}^\sharp)$.*

For some invariants there are good conjectures about admissible values, see for example Conjectures 4.1 and 4.3, Problem 4.3 and the degree conjecture in Section 6.

We end this section by a first measure theoretic result on Problem 4.5; more precise results of this type will be obtained in Section 6. We denote by $\mathbb{R}^+$ and by $\mathbb{C}^+$ the positive real numbers and the complex numbers with non-negative real part, and by $T^1$ the unit circle. A numerical invariant $I$ is called a **continuous invariant** if for every $r \geq 0$ there exits a continuous function

$$f_r : \mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{C}^+)^r \times T^1 \to \mathbb{C}$$

such that $I(F) = f_r(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$, where $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ are the data of $F \in \mathcal{S}^\sharp$ (remember that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are $r$-dimensional vectors). Examples of continuous numerical invariants are the $H$-invariants $H_F(n)$, the conductor $q_F$ and the root number $\omega_F^*$.

**Theorem 4.6** ([22])**.** *Let $I$ be a continuous invariant. Then the sets $\Re I(\mathcal{S})$, $\Im I(\mathcal{S})$, $\Re I(\mathcal{S}^\sharp)$ and $\Im I(\mathcal{S}^\sharp)$ are Lebesgue measurable.*

Roughly speaking, the proof of Theorem 4.6 is based on the fact that for a given continuous invariant $I$, the extended Selberg class $\mathcal{S}^\sharp$ can be endowed with a suitable metric, thus becoming a metric space with good properties.

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