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FAITHFUL ACTIONS OF AUTOMORPHISM GROUPS OF FREE GROUPS ON ALGEBRAIC VARIETIES

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To the memory of J. E. Humphreys

Abstract. Considering a certain construction of algebraic varieties X endowed with an algebraic action of the group $\mathrm{Aut}(F_n)$, $n < \infty$, we obtain a criterion for the faithfulness of this action. It gives an infinite family $\mathscr{F}$ of Xs such that $\mathrm{Aut}(F_n)$ embeds into $\mathrm{Aut}(X)$. For $n \geqslant 3$, this implies nonlinearity, and for $n \geqslant 2$, the existence of F_2 in $\mathrm{Aut}(X)$ (hence nonamenability of the latter) for $X \in \mathscr{F}$. We find in $\mathscr{F}$ two infinite subfamilies $\mathscr{N}$ and $\mathscr{R}$ consisting of irreducible affine varieties such that every $X \in \mathscr{N}$ is nonrational (and even not stably rational), while every $X \in \mathscr{R}$ is rational and $3n$-dimensional. As an application, we show that the minimal dimension of affine algebraic varieties Z, for which $\mathrm{Aut}(Z)$ contains the braid group B_n on n strands, does not exceed $3n$. This upper bound significantly strengthens the one following from the paper by D. Krammer [Kr02], where the linearity of B_n was proved (this latter bound is quadratic in n). The same upper bound also holds for $\mathrm{Aut}(F_n)$. In particular, it shows that the minimal rank of the Cremona groups containing $\mathrm{Aut}(F_n)$, does not exceed $3n$, and the same is true for B_n.

1. Introduction

The exploration of abstract-algebraic, topological, algebro-geometric and dynamical properties of biregular automorphism groups and groups of birational self-maps of algebraic varieties has become the trend of the last decade. In terms of popularity, the Cremona groups are probably the leaders among the studied groups.

Below, algebraic varieties and algebraic groups are understood in the same sense as in [Se55], [Sh07], [Bo91], [Hu75] and are taken over an algebraically closed field k.

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The subject of this paper is the following questions on the group embeddability related to automorphism groups of algebraic varieties.

(Q1) For a given group S, is there an algebraic variety Z such that S embeds in the group $\mathrm{Aut}(Z)$ of its biregular automorphisms?

(Q2) If yes, what are the properties of such Z? Are there such Z in some distinguished classes of varieties (e.g., rational, nonrational, affine, complete, etc.)? What are the "extreme" values of the parameters of such Z (e.g., the minimum of their dimensions)? Etc.

(Q3) Conversely, in which groups can automorphism groups of algebraic varieties of some type be embedded (e.g., are these groups linear)?

Similar questions are also formulated in the context of groups of birational self-maps of algebraic varieties.

It is clear that question (Q1) (but not (Q2)) stands only for "large" groups S, in particular nonlinear ones. Generally speaking, the answer to it is no.[1] Finding for a given S the varieties Z such that the answer is "yes" serves not only as a source of information about $\mathrm{Aut}(Z)$, but also as the method of obtaining essential information about the structure of S (see [BL83], [Ma81], [CX18]).

In this paper, we explore the case of $S = \mathrm{Aut}(F_n)$, where F_n is a free group of rank $n < \infty$. To this end, we consider a general construction that assigns to any finitely generated group Σ a family of algebraic varieties Z endowed with an action of $\mathrm{Aut}(\Sigma)$ by biregular automorphisms. Our results concern each of questions (Q1), (Q2), and (Q3). The main question for us is (Q1), i.e., that of faithfulness of the action of $\mathrm{Aut}(F_n)$ on Z which means that the homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(Z)$ defining the action is an embedding.

Here is this construction. Let Σ and G be groups, and let

$$X := \mathrm{Hom}(\Sigma, G). \tag{1}$$

For any $\sigma \in \mathrm{End}(\Sigma)$, $\gamma \in \mathrm{End}(G)$, put

$$\sigma_X \colon X \to X, x \mapsto x \circ \sigma, \qquad \gamma_X \colon X \to X, x \mapsto \gamma \circ x. \tag{2}$$

If $\sigma \in \mathrm{Aut}(\Sigma)$ and $\gamma \in \mathrm{Aut}(G)$, then σ_X and γ_X are invertible (their inverses σ_X^{-1} and γ_X^{-1} are respectively $(\sigma^{-1})_X$ and $(\gamma^{-1})_X$), and the mapping

$$\big(\mathrm{Aut}(\Sigma) \times \mathrm{Aut}(G)\big) \times X \to X, \quad (\sigma\gamma, x) \mapsto (\sigma_X^{-1} \circ \gamma_X)(x) \tag{3}$$

is an action on X of the group $\mathrm{Aut}(\Sigma) \times \mathrm{Aut}(G)$ (whose factors are naturally identified with its subgroups).

Below, when considering the action on X of a subgroup of this group, the restriction of the action (3) on it is always meant. The actions of $\mathrm{Aut}(\Sigma)$ and $\mathrm{Aut}(G)$ commute with each other.

[1] E.g., in view of [CX18, Thm. C], even in the context of groups of birational self-maps, the answer is negative if S is an infinite simple torsion group with Kazhdan's property (T) (such a group exists; see [Ki94, Sect. 5]).

If Σ is a finitely-generated group, and G is an algebraic group, then X is endowed with the structure of an algebraic variety so that all σ_X and γ_X lie in $\mathrm{Aut}(X)$. Let R be an algebraic subgroup of $\mathrm{Aut}(G)$, for whose action on X there is a categorical quotient

$$\pi_{X/\!\!/R}\colon X \to X/\!\!/R \tag{4}$$

in the sense of geometric invariant theory (see [MF82, Def. 05], [PV94, Def. 4.5]). The following two cases are the main examples when this quotient exists (see Proposition 4.1 below):

(F) R is finite;
(R) G is affine and R is reductive.

As the actions of $\mathrm{Aut}(\Sigma)$ and R on X commute, it follows from the definition of categorical quotient that for every $\sigma \in \mathrm{Aut}(\Sigma)$, the authomorphism σ_X of the variety X descends to a uniquely defined automorphism $\sigma_{X/\!\!/R}$ of the variety $X/\!\!/R$ having the property

$$\pi_{X/\!\!/R} \circ \sigma_X = \sigma_{X/\!\!/R} \circ \pi_{X/\!\!/R}. \tag{5}$$

The map

$$\mathrm{Aut}(\Sigma) \to \mathrm{Aut}(X/\!\!/R), \quad \sigma \mapsto \sigma_{X/\!\!/R}^{-1} \tag{6}$$

is a group homomorphism. It determines an action of $\mathrm{Aut}(\Sigma)$ on $X/\!\!/R$ by biregular automorphisms. In view of (5), the morphism $\pi_{X/\!\!/R}$ is $\mathrm{Aut}(\Sigma)$-equivariant.

In the present paper, for $\Sigma = F_n$, we consider the problem of classifying pairs (G, R) such that the action of $\mathrm{Aut}(\Sigma)$ on $X/\!\!/R$ is *faithful*. Our main results concern case (F).[2] This problem is related to question (Q1). We apply our results to questions (Q2) and (Q3) as well. These results consist of the following.

The first main result is the faithfulness criterion for the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ in case (F).

Theorem 1.1. *Let G be an algebraic group (not necessarily connected or affine), $X = \mathrm{Hom}(F_n, G)$, $n \geqslant 2$, and let R be a finite subgroup of $\mathrm{Aut}(G)$. The following properties are equivalent:*

(a) *the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ is faithful;*
(b) *the connected component of the identity of the group G is nonsolvable.*

Corollary 1.2 describes the applications of Theorem 1.1 to questions (Q1)–(Q3); namely, to the problem of linearity of automorphism groups of algebraic varieties (considered in [CD13, Prop. 5.1], [Co17], [Ca12]), and to the problem of describing subgroups of the Cremona groups.

[2] In [Po23], we consider the situation of case (R) where G is connected and semisimple and R is the image in $\mathrm{Int}(G)$ of a closed subgroup of a maximal torus of G. We prove the faithfulness of the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ in this case.

Corollary 1.2. *In the notation of Theorem* 1.1*, let the connected component of the identity of the group G be nonsolvable. Then*

(a) $\mathrm{Aut}(X/\!\!/R)$ *contains the following groups:*
- $\mathrm{Aut}(F_n)$,
- F_2,
- *the braid group B_n on n strands;*

(b) $\mathrm{Aut}(X/\!\!/R)$ *is nonamenable and, if $n \geqslant 3$, nonlinear.*

Among the varieties $X/\!\!/R$ from Corollary 1.2 whose automorphism group contains $\mathrm{Aut}(F_n)$, F_2 and B_n, there are both rational and nonrational (and even not stably rational), namely, the following.

Proposition 1.3. *Let, in the notation of Theorem* 1.1*, the group G be connected and the group R be trivial. Then the variety $X/\!\!/R$ is rational if G is affine, and nonunirational if G is nonaffine.*

In the general case, the rationality of the variety $X/\!\!/R$ for a connected reductive G and $R = \mathrm{Int}(G)$ is an old problem, open even for $n = 2$ and $G = \mathrm{GL}_d$ with $d \geqslant 5$ (see [Po94, (1.5.2)], [DF04, pp. 190–191]).

In view of Proposition 1.3, if G is nonaffine, then the variety $X/\!\!/R$ with trivial R is not stably rational. For nontrivial finite R, the variety $X/\!\!/R$ with the faithful action of $\mathrm{Aut}(F_n)$ may be not stably rational even if G is affine. Our second main result is Theorem 1.4 giving the construction of such affine $X/\!\!/R$ with a connected reductive G. Its proof also uses Theorem 1.1.

Theorem 1.4. *For every prime number $p \neq \mathrm{char}(k)$, there is a finite p-group K, having the following property. Let V be a finite-dimensional vector space over k, and let $\iota\colon K \hookrightarrow \mathrm{GL}(V)$ be a group embedding for which $\iota(K)$ contains no nontrivial center elements of the group $\mathrm{GL}(V)$ (such pairs (V, ι) exist for any finite group K). Let $X = \mathrm{Hom}(F_n, \mathrm{GL}(V))$, $n \geqslant 2$, and let R be the image of the group $\iota(K)$ under the canonical homomorphism $\mathrm{GL}(V) \to \mathrm{Int}\big(\mathrm{GL}(V)\big)$. Then $X/\!\!/R$ is nonrational (and even not stably rational) affine algebraic variety, on which the group $\mathrm{Aut}(F_n)$ acts faithfully.*

Examples of groups K from Theorem 1.4 can be explicitly specified using generators and relations (see Remark 10.2 below).

Our third main result concerns question (Q2) and, in particular, gives upper bounds for "extremal" parameter values for embeddings of $\mathrm{Aut}(F_n)$ and B_n into automorphism groups of algebraic varieties.

Theorem 1.5. *Keep the notation of Theorem* 1.1*. Let $n \geqslant 2$, $G = \mathrm{SL}_2$ or PSL_2, and let R be finite. Then $X/\!\!/R$ is an irreducible rational affine $3n$-dimensional algebraic variety, whose automorphism group contains $\mathrm{Aut}(F_n)$.*

Note that the variety $X/\!\!/R$ from Theorem 1.5 in the case of trivial R and $G = \mathrm{SL}_2$ (respectively, PSL_2) is the product of n copies of the smooth affine quadric Q in $\mathbb{A}^4$ given by the equation $x_1x_2+x_3x_4 = 1$ (respectively, n copies of Q/I, where I is the group, generated by the automorphism $(a, b, c, d) \mapsto (-a, -b, -c, -d)$).

Definition 1.6. Let S be a group. If there are (defined over k) irreducible algebraic varieties Z (respectively, Cremona groups $\mathcal{C}$) such that S embeds into $\mathrm{Aut}(Z)$ (respectively, $\mathcal{C}$), then we denote the minimum of dimensions of such varieties (respectively, of ranks of such Cremona groups) by

$$\mathrm{var}_k(S) \quad \text{(respectively, } \mathrm{Crem}_k(S)).$$

If there are no such varieties (respectively, Cremona groups), then we set $\mathrm{var}_k(S) = \infty$ (respectively, $\mathrm{Crem}_k(S) = \infty$).

Groups S with $\mathrm{var}_k(S) = \mathrm{Crem}_k(S) = \infty$ exist (see footnote [1]).

Corollary 1.7. *Let $S = \mathrm{Aut}(F_n)$ or B_n. Then*

$$\mathrm{var}_k(S) \leqslant 3n \quad \textit{and} \quad \mathrm{Crem}_k(S) \leqslant 3n. \tag{7}$$

The upper bounds (7) for $S = B_n$ significantly strengthen the ones following from the paper by D. Krammer [Kr02], where embeddability of B_n into $\mathrm{GL}_{n(n-1)/2}$ was proved (which yields the upper bound $n(n-1)/2$).[3]

A special case of the described construction, where $R = \mathrm{Int}(G)$ with reductive G, is explored in many publications, starting essentially with the paper by Vogt of 1889. The subjects of these studies are:

(a) applications to the theory of deformations of hyperbolic structures on topological surfaces; see [Go09] (in this case, Σ is the fundamental group of a surface, $G = \mathrm{SL}_2(\mathbb{C})$, and $X/\!\!/R$ is the "variety of characters" of Σ);

(b) dynamic properties of the action of $\mathrm{Aut}(\Sigma)$ on $X/\!\!/R$; see [Go97], [Go06]; [Ca13$_2$];

(c) for $\Sigma = F_n$, finding the equations of the "variety of characters" and describing the kernel of the action of $\mathrm{Aut}(F_n)$ on it; see [Ho72], [Ho75]; [Ma80].

For the purposes of the present paper, this special case is of little interest, since the group $\mathrm{Int}(\Sigma)$ is always contained in the kernel of the action of $\mathrm{Aut}(\Sigma)$ on $X/\!\!/\mathrm{Int}(G)$, and therefore, for $\Sigma = F_n$, the faithfulness of this action is only possible for $n = 1$ (in which case $\mathrm{Aut}(F_n)$ is a group of order 2). For $n = 1$ and connected G, the rare cases when this action is faithful are described in the following theorem.

Theorem 1.8. *Let G be a connected reductive algebraic group, $X = \mathrm{Hom}(F_n, G)$ and $R = \mathrm{Int}(G)$. The action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ is faithful if and only if $n = 1$ and G contains a connected simple normal subgroup of any of the following types:*

$$\mathsf{A}_\ell \textit{ with } \ell \geqslant 2,\ \mathsf{D}_\ell \textit{ with odd } \ell,\ \mathsf{E}_6. \tag{8}$$

The proof of Theorem 1.1 is given in Sections 7 and 8, of Corollary 1.2 in Section 9, of Theorem 1.4 in Section 10, of Theorem 1.5 and Corollary 1.7 in Section 11, and of Theorem 1.8 in Section 12.

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[3] As the referee noted, a lower bound for $\mathrm{var}_k(\mathrm{Aut}(F_n))$ can be obtained using the methods of [CX18].

2. Conventions and notation

If X is an algebraic variety (respectively, a differentiable manifold), then $\mathrm{Aut}(X)$ denotes the group of its regular automorphisms (respectively, diffemorphisms).

Groups are considered in multiplicative notation. The identity element of a group is denoted by e.

The claim that the group G contains the group H means the existence of a group embedding $\iota\colon H \hookrightarrow G$, by which H is identified with $\iota(H)$.

$\mathscr{C}(G)$ is the center of the group G.

$\mathscr{C}_G(g)$ is the centralizer in G of an element $g \in G$.

int_g is the inner group automorphism determined by an element g.

$\langle g_1, \dots, g_m\rangle$ is the group generated by the elements $g_1, \dots, g_m$.

G^0 is the connected component of the identity of an algebraic group or a real Lie group G.

The Lie algebra of an algebraic group is denoted by the lowercase Gothic version of the letter denoting that group.

$\underline{G}$ is the underlying variety (or manifold) of an algebraic group (or real Lie group) G.

$\mathrm{Aut}(G)$, $\mathrm{Int}(G)$, $\mathrm{Out}(G)$, and $\mathrm{End}(G)$ are respectively the group of automorphisms, inner automorphisms, outer automorphisms, and the monoid of endomorphisms of a group G. If G is an algebraic group or a real Lie group, then by its automorphisms we mean automorphisms in the category of algebraic groups or real Lie groups, so that $\mathrm{Aut}(G)$ denotes the intersection of $\mathrm{Aut}(\underline{G})$ with the automorphism group of the abstract group G. If an algebraic group H faithfully acts by automorphisms of an algebraic group G and the mapping $H \times G \to G$ defining this action is a morphism of algebraic varieties, then H is called an algebraic subgroup of $\mathrm{Aut}(G)$.

The reductivity of an affine algebraic group G does not assume its connectedness and is understood in the sense of [MF82], i.e., as the triviality of the unipotent radical of the group G^0.

3. Fixing a system of generators of Σ

Consider a group G and a finitely generated group Σ. Let $s_1, \dots, s_n$ be a system of generators of Σ. Consider a free group F_n with a basis $f_1, \dots, f_n$ and let $\varphi\colon F_n \to \Sigma$ be the epimorphism defined by the equalities $\varphi(f_j) = s_j$ for every j.

For any group H, any $w \in F_n$, and any $h = (h_1, \dots, h_n) \in H^n$, denote by $w(h) = w(h_1, \dots, h_n)$ the image of w under the (unique) homomorphism $F_n \to H$ mapping f_j to h_j for every j. In other words, if we write w as a word

$$f_{i_1}^{\varepsilon_1} \cdots f_{i_d}^{\varepsilon_d}, \quad \text{where } \varepsilon_j \in \mathbb{Z}, \tag{9}$$

(a noncommutative Laurent monomial in $f_1, \dots, f_n$), then $w(h)$ is obtained by replacing f_j with h_j in (9) for each j.

The map

$$X := \mathrm{Hom}(\Sigma, G) \to G^n, \quad x \mapsto \big(x(s_1), \dots, x(s_n)\big) \in G^n \tag{10}$$

is an injection. Its image is the set

$$\{g \in G^n \mid w(g) = e \text{ for all } w \in \operatorname{Ker}(\varphi)\}. \tag{11}$$

In the rest of this paper, we *identify X with the set* (11) using the injection (10). For $\Sigma = F_n$ and $s_j = f_j$ for all j, we have $X = G^n$, so in this case X is the group (with the componentwise multiplication).

Let $g = (g_1, \ldots, g_n) \in X \subseteq G^n$ and $t \in \Sigma$. It follows from (11) that the element $w(g) \in G$ is the same for all $w \in \varphi^{-1}(t)$. Denote it by $t(g)$. In other words, writing t as a noncommutative Laurent monomial in $s_1, \ldots, s_n$ and replacing s_j in this monomial by g_j for each j, we obtain, regardless of the chosen monomial, $t(g)$. In this notation, for any $\sigma \in \operatorname{End}(\Sigma)$, $\gamma \in \operatorname{End}(G)$, formulas (2) are rewritten as follows:

$$\begin{aligned}
&\sigma_X\colon X \to X, \quad g = (g_1, \ldots, g_n) \mapsto \big(\sigma(s_1)(g), \ldots, \sigma(s_1)(g)\big),\\
&\gamma_X\colon X \to X, \quad (g_1, \ldots, g_n) \mapsto \big(\gamma(g_1), \ldots, \gamma(g_n)\big).
\end{aligned} \tag{12}$$

If G is an algebraic group, then X is closed in G^n and therefore endowed with the structure of an algebraic variety. This structure does not depend on the choice of systems of generators, and the maps (12) are morphisms. If $\Sigma = F_n$ and G is a real Lie group, then (12) are differentiable mappings $G^n \to G^n$.

Some properties, selectively used below and in [Po21], [Po23], [Po22], are brought together in Proposition 3.1 for ease of reference.

Proposition 3.1. *We maintain the notation introduced above in Section* 3. *Let σ and $\tau \in \operatorname{End}(F_n)$. Then the following hold:*

(a) $(\sigma \circ \tau)_X = \tau_X \circ \sigma_X$.
(b) $e_X = \mathrm{id}$.
(c) $\sigma_X(X \cap S^n) \subseteq X \cap S^n$ *for any subgroup S of the group G.*
(d) *Let $\theta\colon G \to H$ be a group homomorphism and let $Y := \operatorname{Hom}(\Sigma, H) \subseteq H^n$. Then the map*

$$\theta_n\colon X \to Y, \quad (g_1, \ldots, g_n) \mapsto (\theta(g_1), \ldots, \theta(g_n))$$

is $\operatorname{End}(\Sigma)$*-equivariant, i.e.,* $\theta_n \circ \sigma_X = \sigma_Y \circ \theta_n$.
(e) *If $\sigma = \operatorname{int}_t$ for $t \in \Sigma$, then the following properties of an element*

$$x = (g_1, \ldots, g_n) \in X \subseteq G^n \tag{13}$$

are equivalent:
(e_1) $\sigma_X(x) = x$;
(e_1) $t(x) \in \bigcap_{i=1}^n \mathscr{C}_G(g_i)$.

In statements (f) *and* (g), *it is assumed that* $\Sigma = F_n$.

(f) *The following properties of element* (13) *are equivalent:*
(f_1) $\sigma_X(x) = x$ *for each* $\sigma \in \operatorname{Aut}(F_n)$;
(f_2) *if $n > 1$, then $g_1 = \cdots = g_n = e$, and if $n = 1$, then $g_1^2 = e$.*
(g) *The multiplication in $X = G^n$ has the property:*

$$\sigma_X(xz) = \sigma_X(x)\sigma_X(z) \text{ for all } x \in X = G^n,\ z \in \mathscr{C}(X).$$

In particular, the restriction of σ_X to the group $\mathscr{C}(X)$ is its endomorphism.

Proof. Statement (f) follows from the fact that for $n = 1$, the only nonidentity element of $\operatorname{Aut}(F_n)$ maps f_1 to f_1^{-1}, and for $n \geqslant 2$, for any $i, j \in \{1, \dots, n\}$, $i \neq j$, the element $\sigma_{ij} \in \operatorname{End}(F_n)$ defined by the formula

$$\sigma_{ij}(f_l) = \begin{cases} f_l & \text{for } l \neq i, \\ f_i f_j & \text{for } l = i, \end{cases}$$

lies in $\operatorname{Aut}(F_n)$.

The rest of the statements follow directly from the definitions and the fact that each element of Σ is written as a noncommutative Laurent monomial in $s_1, \dots, s_n$. $\square$

4. The existence of categorical quotient

Proposition 4.1. *Let Σ be a finitely generated group, let G be an algebraic group (not necessarily connected or affine), let R be an algebraic subgroup of* $\operatorname{Aut}(G)$*, and let $X = \operatorname{Hom}(\Sigma, G)$. The categorical quotient* (4) *exists in each of the following two cases:*

(F) *R is finite;*
(R) *G is affine and R is reductive.*

If (F) *holds, then the categorical quotient* (4) *is the geometric quotient. If* (R) *holds, then the variety $X /\!\!/ R$ is affine. In both cases, the morphism $\pi_{X/\!\!/R}$ is surjective.*

Proof. According to [Ba54], the variety $\underline{G}$ is quasi-projective. Hence X, being closed in the product of several copies of $\underline{G}$, is quasi-projective as well. This implies the existence of the geometric quotient in case (F) (see [Se97, Chap. III, Sect. 12, Prop. 19, Ex. 2]). This quotient is automatically categorical, and $\pi_{X/\!\!/R}$ is a surjective morphism (see [PV94, 4.3], [Bo91, Sect. II, §6]).

In case (R), the variety $\underline{G}$ is affine. In view of the remark on closedness made in the previous paragraph, X is affine as well. According to [MF82, Chap. 1, §2], from this and the reductivity of R it follows the existence of the categorical quotient (4), the affineness of $X/\!\!/R$, and the surjectivity of $\pi_{X/\!\!/R}$. $\square$

5. The kernel of the action of $\operatorname{Aut}(\Sigma)$ on $\operatorname{Hom}(\Sigma, G)/\!\!/R$ in cases (F) and (R): geometric description

Let Σ be a finitely generated group and let G be an algebraic group (not necessarily connected or affine). Having fixed a system of n generators in Σ, we identify $X = \operatorname{Hom}(\Sigma, G)$ with a closed subset of G^n as described in Section 3. For any $w \in \Sigma$, $\gamma \in \operatorname{Aut}(G)$, and $i \in \{1, \dots, n\}$, the closed set

$$X_{w,\gamma,i} := \{x = (g_1, \dots, g_n) \in X \mid w(x) = \gamma(g_i)\} \tag{14}$$

is the fiber over e of the morphism

$$X \to G, \quad x = (g_1, \dots, g_n) \mapsto w(x)\gamma(g_i)^{-1}.$$

As it contains $(e, \dots, e)$, it is nonempty.

From (12) and (14) it follows that for any $\sigma \in \operatorname{Aut}(\Sigma)$ we have

$$\bigcap_{i=1}^{n} X_{\sigma(f_i),\gamma,i} = \{x \in X \mid \sigma_X(x) = \gamma(x)\}. \tag{15}$$

The following Lemmas 5.1 and 5.2 describe the kernel of the action of $\operatorname{Aut}(\Sigma)$ on $\operatorname{Hom}(\Sigma, G)/\!\!/R$ respectively in cases (F) and (R).

Lemma 5.1. *Let R be a finite subgroup of $\operatorname{Aut}(G)$. The following properties of an element $\sigma \in \operatorname{Aut}(\Sigma)$ are equivalent:*

(a) *σ lies in the kernel of the action of $\operatorname{Aut}(\Sigma)$ on $X/\!\!/R$;*
(b) *$\sigma_X(\mathcal{O}) = \mathcal{O}$ for every R-orbit $\mathcal{O}$ in X;*
(c) *for every irreducible component Y of the variety X there is an element $\gamma \in R$ such that*

$$Y \subseteq \bigcap_{i=1}^{n} X_{\sigma(f_i),\gamma,i}. \tag{16}$$

Proof. In view of Proposition 4.1, each fiber of the morphism $\pi_{X/\!\!/R}$ is an R-orbit in X and vice versa. As the actions of $\operatorname{Aut}(\Sigma)$ and R on X commute, it follows from (5) that for each point $b \in X/\!\!/R$, the restriction of the morphism σ_X to the orbit $\pi_{X/\!\!/R}^{-1}(b)$ is its R-equivariant isomorphism with the orbit $\pi_{X/\!\!/R}^{-1}(\sigma_{X/\!\!/R}(b))$. This proves the equivalence of the conditions (a) and (b) and, given (15), their equivalence to the equality

$$X = \bigcup_{\gamma\in R} \Big(\bigcap_{i=1}^{n} X_{\sigma(f_i),\gamma,i}\Big). \tag{17}$$

(a)$\Rightarrow$(c) If the equality (17) holds, then each irreducible component Y of X is the union of closed subsets of the form

$$Y \cap \Big(\bigcap_{i=1}^{n} X_{\sigma(f_i),\gamma,i}\Big), \text{where } \gamma \in R. \tag{18}$$

As the group R is finite, there are finitely many of these subsets. The irreducibility of Y therefore implies that Y coincides with one of them. Hence, (16) holds for some $\gamma \in R$.

(c)$\Rightarrow$(a) If (c) holds, then the union of all irreducible components of X lies on the right-hand side of the equality (17), i.e., this right-hand side contains X. The reverse inclusion is obvious. Hence, the equality (17) holds. $\square$

Lemma 5.2. *Assume that the group G is affine. Let R be a reductive algebraic subgroup of $\operatorname{Aut}(G)$. The following properties of an element $\sigma \in \operatorname{Aut}(\Sigma)$ are equivalent:*

(a) *σ lies in the kernel of the action of $\operatorname{Aut}(\Sigma)$ on $X/\!\!/R$;*
(b) *$\sigma_X(\mathcal{O}) = \mathcal{O}$ for every closed R-orbit $\mathcal{O}$ in X;*
(c) *each closed R-orbit in X belongs to the set*

$$\bigcup_{\gamma\in R} \Big(\bigcap_{i=1}^{n} X_{\sigma(f_i),\gamma,i}\Big). \tag{19}$$

Proof. For every $b \in X/\!\!/R$, the fiber $\pi_{X/\!\!/R}^{-1}(b)$ of the surjective (see Proposition 4.1) morphism $\pi_{X/\!\!/R}$ is an R-invariant closed subset of X, which contains a unique closed R-orbit $\mathcal{O}_b$ (see [MF82, §2 and Append. 1B]. The restriction of σ_X to $\pi_{X/\!\!/R}^{-1}(b)$ is an R-equivariant isomorphism with the fiber $\pi_{X/\!\!/R}^{-1}(\sigma_{X/\!\!/R}(b))$. In view of the uniqueness of closed orbits in the fibers, this means that $\sigma_X(\mathcal{O}_b) = \mathcal{O}_{\sigma_{X/\!\!/S}(b)}$. Therefore, the equalities $\sigma_{X/\!\!/S}(b) = b$ and $\sigma_X(\mathcal{O}_b) = \mathcal{O}_b$ are equivalent. This proves (a)$\Leftrightarrow$(b). In turn, this and (15) imply (a)$\Leftrightarrow$(c). $\square$

Corollary 5.3. *If the conditions of Lemma* 5.2 *hold and* $R = \operatorname{Int}(G)$, *then* $\operatorname{Int}(\Sigma)$ *lies in the kernel of the action of* $\operatorname{Aut}(\Sigma)$ *on* $X/\!\!/R$.

Proof. If $\sigma \in \operatorname{Int}(\Sigma)$, then (12) implies that x and $\sigma_X(x)$ lie in the same R-orbit for each $x \in X$. The assertion therefore follows from the equivalence of conditions (a) and (b) in Lemma 5.2. □

6. The faithfulness of the actions of Aut(F_n) and Int(F_n) on Hom(F_n,G): algebraic criteria

Theorem 6.1. *Let* G *be a group and let* $X = \operatorname{Hom}(F_n, G)$. *Consider the following properties:*

(a) *the action of* $\operatorname{Aut}(F_n)$ *on* X *is faithful;*
(b) *the action of* $\operatorname{Int}(F_n)$ *on* X *is faithful;*
(c) $n \geqslant 2$ *and there is no nonempty reduced word in an alphabet consisting of* n *letters that is an identity relation in* G*;*
(d) $n = 1$ *and* G *contains an element of order* $\geqslant 3$.

Then (a)⇔(b)⇔(c) *for each* $n \geqslant 2$, *and* (a)⇔(d) *for* $n = 1$.

Proof. We use the notation of Section 3 with $\Sigma = F_n$ and $s_j = f_j$ for all j. For $n = 1$, the statement follows from Proposition 3.1(f). Consider the case $n \geqslant 2$.

(a)⇒(b) This is clear.

(b)⇒(c) Suppose, arguing by contradiction, that (b) holds, but there exists a nonempty reduced word in an alphabet consisting of n letters that is an identity relation in G. So there is a nonidentity element $w \in F_n$ such that

$$w(x) = e \quad \text{for each } x \in X. \tag{20}$$

The element $\sigma := \operatorname{int}_w \in \operatorname{Aut}(F_n)$ is different from the identity because the group $\mathscr{C}(F_n)$ is trivial for $n \geqslant 2$ (cf. [LS77, Chap. I, Prop. 2.19]) and $w \neq e$. However, from (20) it follows that $\sigma_X(x) = x$ for each $x \in X$, i.e., that σ lies in the kernel of the action of $\operatorname{Aut}(F_n)$ on X. This contradicts (b).

(c)⇒(a) Suppose, arguing by contradiction, that (c) holds, but the kernel of the action of $\operatorname{Aut}(F_n)$ on X contains a nonidentity element $\sigma \in \operatorname{Aut}(F_n)$, so that we have (see (12))

$$\sigma(f_i)(x) = f_i(x) \quad \text{for all } x \in X \text{ and } i. \tag{21}$$

In view of $\sigma \neq e$, there exists f_j for which $\sigma(f_j) \neq f_j$, i.e., $w := \sigma(f_j)f_j^{-1}$ is a nonidentity element of the group F_n. At the same time, (21) implies that this w satisfies condition (20). Therefore, there is a nonempty reduced word in the alphabet $f_1, \ldots, f_n$ that is an identity relation in G. This contradicts (c). □

Corollary 6.2. *For each virtually solvable group* G, *the action of* $\operatorname{Aut}(F_n)$ *on* $X := \operatorname{Hom}(F_n, G)$, $n \geqslant 2$, *is nonfaithful.*

Proof. By the definition of virtual solvable group, G has a solvable subgroup S of a finite index d. We can (and shall) assume that S is normal, replacing it with the intersection of all subgroups conjugate to it. As S is solvable, there exists a nonempty reduced word in the alphabet x, y that is an identity relation in S (see

[Ne67, 14.65]). Denote this word by $r(x, y)$. It follows from the normality of S that $g^d \in S$ for each $g \in G$. Hence the nonempty reduced word $r(x^d, y^d)$ is an identity relation in G. The claim now follows from Theorem 6.1. □

7. Proof of Theorem 1.1: the case of trivial subgroup *R*

To prove Theorem 1.1, we first need to consider the special case of $R = \{e\}$. We will prove a more general statement concerning not only algebraic groups, but also real Lie groups.

Theorem 7.1. *Let* $X = \mathrm{Hom}(F_n, G)$, $n \geqslant 2$, *and let* G *be either an algebraic group* (*not necessarily connected or affine*) *or a real Lie group with a finite number of connected components. Then the following properties are equivalent:*

(a) *the action of* $\mathrm{Aut}(F_n)$ *on* X *is faithful;*
(b) *the group* G^0 *is nonsolvable.*

If G *is a real Lie group, then the implication* (b)⇒(a) *is true even without the condition that the number of its connected components is finite.*

Proof. If G is a real Lie group, then

$$[G : G^0] < \infty \tag{22}$$

by the condition. If G is an algebraic group, then (22) is satisfied automatically. It follows from (22) that if G^0 is solvable, then G is virtually solvable. Together with Corollary 6.2, this proves implication (a)⇒(b).

(b)⇒(a) Let the group G^0 be nonsolvable. In view of Theorem 6.1, it is required to prove that there is no nonempty reduced word in an alphabet consisting of n letters that is an identity relation in G. Arguing by contradiction, suppose that such a word exists. Hence, there is a nontrivial element $w \in F_n$ with the property (20).

Let G be a connected real Lie group. Then, due to nonsolvability, G^0 contains a free subgroup of rank n (see [Ep71, Thm.]). Let $g_1, \dots, g_n$ be its free system of generators. Then $w(g_1, \dots, g_n) = e$ due to (20), which contradicts the absence of nontrivial relations between $g_1, \dots, g_n$.

Let now G be an algebraic group. By Chevalley's theorem, the algebraic group G^0 contains the largest connected affine normal subgroup G^0_{aff}, and G^0/G^0_{aff} is an Abelian variety. As the group G^0 is nonsolvable and the group G^0/G^0_{aff} is commutative (and therefore solvable), the group G^0_{aff} is nonsolvable. Hence, G^0_{aff} does not coincide with its radical $\mathrm{Rad}(G^0_{\mathrm{aff}})$, and therefore, $G^0_{\mathrm{aff}}/\mathrm{Rad}(G^0_{\mathrm{aff}})$ is a nontrivial connected semisimple algebraic group. This reduces the proof to the case where G is a nontrivial connected semisimple algebraic group. We will therefore further assume that this condition is met. In view of [Bo83, Thm. B], from it and the inequality $n \geqslant 2$ it follows that the morphism

$$X \to G, \quad x \mapsto w(x)$$

is dominant. In view of (20), this means that the group G is trivial, which is a contradiction. □

Remark 7.2. If G is a nonsolvable algebraic group and the field k is uncountable, then G contains a free subgroup of any finite rank (see [BGGT12, Thm. 1.1], [BGGT15, App. D]), which means that the same proof of the implication (b)⇒(a) in Theorem 7.1 as in the case of a real Lie group goes through. This proof is given in the first version [Po21] of the present paper. However, in the general case, G may not contain a free subgroup (for example, this is the case for $G = \mathrm{SL}_d$ if k is the algebraic closure of a finite field, since then the order of every element of SL_d is finite).

Remark 7.3. Without the condition that the number of connected components is finite, the implication (a)⇒(b) in Theorem 7.1 is false. Indeed, take as G the group F_n considered as a real Lie group with $G^0 = \{e\}$. Then $\mathrm{id}_{F_n} \in \mathrm{Hom}(F_n, F_n) = X$ and, for any $\sigma \in \mathrm{Aut}(F_n)$, we have $\sigma_X(\mathrm{id}_{F_n}) = \sigma$ (see (2)). Therefore, (a) holds, but (b) does not.

8. Proof of Theorem 1.1: general case

In view of the surjectivity and $\mathrm{Aut}(F_n)$-equivariance of the morphism $\pi_{X/\!\!/R}$ (see (4)), the implication (a)⇒(b) follows from Theorem 7.1.

(b)⇒(a) Let the group G^0 be nonsolvable. Arguing by contradiction, suppose that a nonidentity element $\sigma \in \mathrm{Aut}(F_n)$ lies in the kernel of the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$. The variety X is isomorphic to $\underline{G}^n$. It is clear that $(\underline{G}^0)^n$ is one of the irreducible components of the variety $\underline{G}^n$. By virtue of what was said in Section 3, this implies that $X^0 := \mathrm{Hom}(F_n, G^0)$ is an $\mathrm{Aut}(F_n)$-invariant irreducible component of the variety X. In turn, in view of Lemma 5.1 and formulas (15), (12), this implies the existence of an element $\gamma \in R$ such that for every $i \in \{1, \ldots, n\}$, the following identity relation holds in G^0:

$$\sigma(f_i)(g_1, \ldots, g_n) = \gamma(g_i) \quad \text{for any } g_1, \ldots, g_n \in G^0. \tag{23}$$

In particular, for every $g \in G^0$, the equality obtained by substituting $g_1 = \cdots = g_n = g$ in (23) holds. As $\sigma(f_i)$ has the form (9), this means the existence of an integer d such that the following identity relation holds:

$$g^d = \gamma(g) \quad \text{for each } g \in G^0. \tag{24}$$

Notice that $d \neq 0$ because $\gamma \in \mathrm{Aut}(G)$, and that

$$d \neq 1 \quad \text{and} \quad d \neq -1. \tag{25}$$

Indeed, if $d=1$ then from (24) and (23) it follows that $\sigma_{X^0} = \mathrm{id}_{X^0}$, i.e., σ lies in the kernel of the action of $\mathrm{Aut}(F_n)$ on X^0. As σ is a nonidentity element, and the group G^0 is nonsolvable, this contradicts Theorem 7.1.

If $d = -1$, then for any $g, h \in G^0$, the equality

$$h^{-1}g^{-1} = (gh)^{-1} \overset{(24)}{=} \gamma(gh) = \gamma(g)\gamma(h) \overset{(24)}{=} g^{-1}h^{-1}$$

holds, meaning that the group G^0 is commutative contrary to its nonsolvability.

Further, for any positive integer m, we obtain from (24) by induction the following identity relation:

$$g^{d^m} = \gamma^m(g) \quad \text{for each } g \in G^0. \tag{26}$$

As the group R is finite, the order of γ is finite. Let m in (26) be equal to this order. Then (26) becomes the identity relation

$$g^{d^m-1} = e \quad \text{for every } g \in G^0. \tag{27}$$

As $d^m - 1 \neq 0$ due to (25), from (27) we infer that G^0 is a torsion group whose element orders are bounded from above. Let us show that this contradicts the properties of the group G^0.

Indeed, as in the proof of Theorem 7.1 (whose notation we retain), the affine algebraic group $G^0_{\rm aff}$ is nonsolvable. Hence, it contains a nontrivial semisimple element, and therefore a torus of positive dimension (see [Bo91, Thms. 4.8, 11.10]). But the set of orders of elements of the torsion subgroup of any torus of positive dimension is not bounded (see [Bo91, Prop. 8.9]). This gives the required contradiction. □

9. Proofs of Corollary 1.2 and Proposition 1.3

Proof of Corollary 1.2. Statements (a) and (b) follow from Theorem 1.1 and the next Proposition 9.1. □

Proposition 9.1. *Assume that a group H contains* $\operatorname{Aut}(F_n)$. *Then*

(i) *H contains F_2 if $n \geqslant 2$;*
(ii) *H contains B_n;*
(iii) *H is not amenable if $n \geqslant 2$;*
(iv) *H is nonlinear if $n \geqslant 3$.*

Proof. If $n \geqslant 2$, then $\mathscr{C}(F_n)$ is trivial and therefore, $\operatorname{Int}(F_n)$ is isomorphic to F_n. This gives (i).

As $\operatorname{Aut}(F_n)$ contains B_n (see [KT08, Sect. 1.5]), we have (ii).

If $n \geqslant 3$, then $\operatorname{Aut}(F_n)$ is nonlinear (see [FP92]); whence (iv).

(i) implies (iii). □

Proof of Proposition 1.3. If G is affine, then the rationality of $X = G^n$ follows from the rationality of $\underline{G}$ (see [Bo91, Cor. 14.14]).

Let G be nonaffine. Arguing by contradiction, assume that $X = G^n$ is unirational, i.e., there is a dominant rational map of a rational variety to X. Let us use the notation of the proof of Theorem 7.1. Our assumption implies that the variety $G/G_{\rm aff}$ is unirational in view of the surjectivity of the composition of the following morphisms

$$X = G^n \xrightarrow{\alpha} G \xrightarrow{\beta} G/G_{\rm aff},$$

where α is a projection onto some factor, and β is the canonical projection. By the condition, $G_{\rm aff} \neq G$, therefore, the Abelian variety $G/G_{\rm aff}$ is nontrivial. As such varieties are nonunirational (see [Sh07, Chap. 3, Sect. 6.2, 6.4]), we get a contradiction. □

10. Proof of Theorem 1.4

We use in the proof of Theorem 1.4 the following known statement (see, e.g., [Po13, Thm. 1]).

Lemma 10.1. *If the field of invariant rational functions of some faithful linear action of a finite group on a finite-dimensional vector space over k is stably rational over k, then the same property holds for any other such action of this group.*

Proof of Theorem 1.4. Consider one of the pairs (K, ι) found in [Sa84], where K is a finite group, and

$$\iota\colon K \hookrightarrow \mathrm{GL}(V)$$

is a group embedding, where V is a finite-dimensional vector space over k, for which the field of $\iota(K)$-invariant rational functions on V is not stably rational over k.

In view of Lemma 10.1, replacing V and ι if necessary, we can (and shall) assume that

$$\iota(K) \cap \mathscr{C}\big(\mathrm{GL}(V)\big) = \{\mathrm{id}_V\}. \tag{28}$$

Indeed, let L be a one-dimensional vector space over k. As we have $\mathscr{C}\big(\mathrm{GL}(V \oplus L)\big) = \{c \cdot \mathrm{id}_{V\oplus L} \mid c \in k, c \neq 0\}$, the group embedding

$$\iota'\colon K \hookrightarrow \mathrm{GL}(V \oplus L), \quad f \mapsto \iota(f) \oplus \mathrm{id}_L,$$

has the property $\iota'(K) \cap \mathscr{C}\big(\mathrm{GL}(V \oplus L)\big) = \{\mathrm{id}_{V\oplus L}\}$.

It follows from (28) that the diagonal linear action of $\iota(K)$ on the vector space $\mathrm{End}(V)^{\oplus n}$ by conjugation is faithful. Therefore, in view of Lemma 10.1, the field of $\iota(K)$-invariant rational functions on $\mathrm{End}(V)^{\oplus n}$ is not stably rational over k. But $\mathrm{GL}(V)^n$ is a $\iota(K)$-invariant open subset of $\mathrm{End}(V)^{\oplus n}$. It is $\iota(K)$-equivariantly isomorphic to the algebraic variety $\mathrm{Hom}(F_n, \mathrm{GL}(V))$. Hence the field of R-invariant rational functions on X is not stably rational over k. But this field is isomorphic to the field of rational functions on $X/\!\!/R$ because, by Proposition 4.1, the categorical quotient (4) is geometric. Hence the variety $X/\!\!/R$ is not stably rational. It is affine due to the affinness of $\mathrm{GL}(V)$ (see Proposition 4.1). Finally, by Theorem 1.1, from the nonsolvability of $\mathrm{GL}(V)$ with $n \geqslant 2$ it follows that the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ is faithful. □

Remark 10.2. The first examples of groups, which can be taken as K in Theorem 1.4, have been obtained in [Sa84]; they have order p^9. At present, all groups of order p^5 with the specified property have been found (see details and references in [Po13, p. 414, Rem.]). For example, for $p \geqslant 5$, one of them is the group $K = \langle g_1, g_2, g_3, g_4, g_5\rangle$ of order p^5 given by the following conditions (in which $[a, b] := a^{-1}b^{-1}ab$):

$$\mathscr{C}(F) = \langle g_5\rangle, g_i^p = e \quad \text{for each } i,$$
$$[g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = [g_3, g_2] = g_5, [g_4, g_2] = [g_4, g_3] = e.$$

11. Proofs of Theorem 1.5 and Corollary 1.7

Proof of Theorem 1.5. In view of the finiteness of R, it follows from Proposition 4.1 that (4) is the geometric quotient, and the variety $X/\!\!/R$ is affine (and irreducible due to the connectedness of G). In particular, the fibers of the surjective morphism (4) are R-orbits and therefore zero-dimensional. This implies the claim about the dimension, since $\dim(G) = 3$ and $X = G^n$. It remains to prove the rationality.

In this case, $\mathrm{Aut}(G) = \mathrm{Int}(G)$. Consider the adjoint action of $\mathrm{Int}(G)$ on $\mathfrak{g} = \mathrm{Mat}_2^0 := \{m \in \mathrm{Mat}_2(k) \mid \mathrm{trace}(m) = 0\}$. According to [LPR06, Examples 1.11 and 1.16], the groups SL_2 and PSL_2 are Cayley, i.e., there is an $\mathrm{Int}(G)$-equivariant birational mapping

$$\lambda\colon G \dashrightarrow \mathfrak{g}.$$

For the readers who prefer to restrict with direct checking, we note that in this case λ and λ^{-1} can be specified quite explicitly. Namely, if $G = \mathrm{SL}_2$, then

$$\begin{aligned} \lambda(g) &= (I_2 - g)(I_2 + g)^{-1} \quad \text{for } g \in \mathrm{SL}_2,\\ \lambda^{-1}(m) &= (I_2 - m)(I_2 + m)^{-1} \quad \text{for } m \in \mathrm{Mat}_2^0. \end{aligned}$$

If $G = \mathrm{PSL}_2$, and $[g]$ denotes the image of $g \in \mathrm{SL}_2$ under the canonical projection $\mathrm{SL}_2 \to \mathrm{PSL}_2$, then

$$\begin{aligned} \lambda([g]) &= 2\,\mathrm{trace}(g)^{-1}g - I_2 \quad \text{for } g \in \mathrm{SL}_2,\\ \lambda^{-1}(m) &= [m + I_2] \quad \text{for } m \in \mathrm{Mat}_2^0. \end{aligned}$$

For the diagonal actions of $\mathrm{Int}(G)$ on $X = G^n$ and $\mathfrak{g}^n := \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ (n summands), the existence of λ implies the existence of an $\mathrm{Int}(G)$-equivariant (hence, R-equivariant) birational mapping

$$X \dashrightarrow \mathfrak{g}^n. \tag{29}$$

Consequently, the fields of R-invariant rational functions on X and $\mathfrak{g}^n$ are isomorphic. Hence the geometric quotients $X/\!\!/R$ and $\mathfrak{g}^n/\!\!/R$ are birationally isomorphic. But the linearity of the action of R on $\mathfrak{g}^n$ and the decomposition $\mathfrak{g}^n = \mathfrak{g} \oplus \mathfrak{g}^{n-1}$ with the R-invariant summands imply, in view of the No-name lemma (see [Po13, Lem. 1], [PV94, Thm. 2.13]), that $\mathfrak{g}^n/\!\!/R$ is birationally isomorphic to $\big(\mathfrak{g}/\!\!/R\big) \times \mathbb{A}^{(n-1)\dim(\mathfrak{g})}$. As $\dim(\mathfrak{g}) = 3$ implies the rationality of $\mathfrak{g}/\!\!/R$ (see [Mi71, Thm. 2]), this shows that $\mathfrak{g}^n/\!\!/R$, and therefore, also $X/\!\!/R$, is rational. □

Proof of Corollary 1.7. This claim follows from Definition 1.6, Theorem 1.5, and Corollary 1.2. □

12. Proof of Theorem 1.8

For $n \geqslant 2$, the group $\mathrm{Int}(F_n)$ is nontrivial, and hence by Corollary 5.3, the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ is nonfaithful.

Now, let $n = 1$. We have $X = G$ and $\mathrm{Aut}(F_1)$ is the group of order two. Let $\sigma \in \mathrm{Aut}(F_1)$, $\sigma \neq e$. Then $\sigma(f_1) = f_1^{-1}$, so $\sigma_X(g) = g^{-1}$ for any $g \in X$. Each fiber of the morphism (4) contains a single orbit consisting of semisimple elements, and it is the only closed orbit in this fiber (see [St65]). As $R = \mathrm{Int}(G)$, from here and Lemma 5.2 the equivalence of the following properties follows:

(i) σ lies in the kernel of the action of $\mathrm{Aut}(F_1)$ on $X/\!\!/R$;
(ii) g and g^{-1} are conjugate for each semisimple element $g \in G$.

As the intersection of any semisimple conjugacy class with a fixed maximal torus T of G is nonempty (see [Bo91, Thm. 11.10]) and is an orbit of the normalizer of this torus (see [St65, 6.1]), property (ii) is equivalent to the fact that the Weyl group W of the group G considered as a subgroup of the group $\mathrm{GL}(\mathfrak{t})$ contains -1. This, in turn, is equivalent to the fact that -1 is contained in the Weil group of every nontrivial connected simple normal subgroup of the group G. Let C be a Weyl chamber in $\mathfrak{t}$. As $-C$ is also a Weyl chamber, the simple transitivity of the action of W on the set of all Weyl chambers implies the existence of a unique element $w_0 \in W$ such that $w_0(C) = -C$. In view of $(-1)(C) = -C$, this means that the inclusion of $-1 \in W$ is equivalent to the equality $w_0 = -1$. In [Bo68, Table I–IX], the explicit description of the element w_0 is given for every connected simple algebraic group. It follows from it that the equality $w_0 = -1$ for such a group is equivalent to the fact that the type of this group is not contained in the list (8). This completes the proof. □

13. Final remarks

(a) Corollary 1.7 concerns, in particular, the subgroups of the Cremona groups. Taking this opportunity, we will supplement it here with a remark on S. Cantat's question about these subgroups.

In [Co17], examples are given of finitely generated (and even finitely presented) groups nonembeddable into any Cremona group, which answers S. Cantat's question about the existence of such groups (see also [Ca13$_1$]). These examples are based on the fact that the word problem ([Co13, Thm. 1.2]) is solvable in every finitely generated subgroup of any Cremona group. Below we indicate another way to answer this question (and even in a stronger form, with the addition of the group simplicity condition). This is not for claiming that the new way uses simpler means, but because adding a new approach always contributes to a better understanding, especially if it yields some answers so far unreachable for the other means. We assume that $\mathrm{char}(k) = 0$.

Namely, we recall [Po14, Def. 1] that a group H is called *Jordan* if there exists a finite set $\mathcal{F}$ of finite groups such that every finite subgroup of H is an extension of an Abelian group by a group taken from $\mathcal{F}$. According to [Po14, p. 188, Exmp. 6], the R. Thompson group V is an example of a non-Jordan finitely presented group. As any Cremona group is Jordan (see [Bi16, Cor. 1.5], [PS16]), the group V is nonembeddable into it. Furthermore, in addition to this property, V is simple, and therefore every homomorphism of V into any Cremona group is trivial (unlike [Co13], this proves [Co13, Cor. 1.4] without using the obtained in [Mi81] amplification of the Boone–Novikov construction).

We note that, after [Co13], many unrelated to the word problem examples of finitely generated (and even finitely presented) groups nonembeddable into any Cremona group were obtained in [CX18]. However, basing on the currently available (July 2022) information, it is impossible to deduce from [CX18, Thms. C and 7.15] that V is nonembeddable into any Cremona group. Indeed, the group V does not have Kazhdan's property (T) (see [BJ19]), and whether it has property (τ^∞) (see [CX18, Sect. 7.1.3]) is unknown [Co22].

(b) Among the irreducible affine varieties $X/\!\!/R$ whose automorphism group contains $\mathrm{Aut}(F_n)$, there are open subsets of affine spaces. Indeed, by Theorem 1.1 for $n \geqslant 2$, such an example is $X/\!\!/R$ with $G = \mathrm{GL}_d$, $d \geqslant 2$, and trivial R. The following construction generalizes this example.

Consider a finite dimensional associative k-algebra A with identity. The group A^* of its invertible elements is a connected affine algebraic group whose underlying variety is open in A. If A^* is nonsolvable, then in view of Theorem 1.1, it can be taken instead of GL_d in the example from the previous paragraph.

(c) In [CX18, pp. 272], the following lower bound is obtained:

$$n - 2 \leqslant \mathrm{var}_{\mathbb{C}}\big(\mathrm{Out}(F_n)\big).$$

The following theorem yields, among other things, an upper bound.

Theorem 13.1. *We retain the notation of Theorem* 1.1. *Let* $\mathrm{char}(k) = 0$, $n \geqslant 3$, $G = \mathrm{SL}_2$ *or* PSL_2, *and* $R = \mathrm{Int}(G)$. *Then* $X/\!\!/R$ *is an irreducible rational affine* $(3n-3)$*-dimensional manifold, whose automorphism group contains* $\mathrm{Out}(F_n)$.

Proof. The affineness of $X/\!\!/R$ follows from the reductivity of R. According to [Ho75], for $n \geqslant 3$, the kernel of the action of $\mathrm{Aut}(F_n)$ on $X/\!\!/R$ is $\mathrm{Int}(F_n)$, so $\mathrm{Out}(F_n)$ is embedded in $\mathrm{Aut}(X/\!\!/R)$. From [Ri88, Lem. 3.3, Thm. 4.1] and $\dim(G) = 3$, we infer the nonemptiness of the open subsets of $X = G^n$ and $\mathfrak{g}^n$ comprised by points whose R-orbits are three-dimensional and closed. This and the existence of the geometric quotients for the suitable open subsets of $X = G^n$ and $\mathfrak{g}^n$ (see [PV94, Thm. 4.4]) imply that the dimensions of the varieties $G^n/\!\!/R$ and $\mathfrak{g}^n/\!\!/R$ are equal to $3n - 3$, and their fields of rational functions coincide under the natural embedding with the fields of R-invariant rational functions on X and $\mathfrak{g}^n$ respectively. As in the proof of Theorem 1.5, there is an R-equivariant birational mapping (29), and therefore, the specified fields of R-invariant rational functions are isomorphic. Hence the algebraic varieties $G^n/\!\!/R$ and $\mathfrak{g}^n/\!\!/R$ are birationally isomorphic. But, according to P. Katsylo, the field of invariant rational functions on any finite dimensional algebraic SL_2-module is purely transcendental over k (see [PV94, Thm. 2.12]). Hence $G^n/\!\!/R = X/\!\!/R$ is a rational algebraic variety. □

Corollary 13.2. *If* $\mathrm{char}(k) = 0$ *and* $n \geqslant 3$, *then*

$$\mathrm{var}_k\big(\mathrm{Out}(F_n)\big) \leqslant 3n - 3 \quad \textit{and} \quad \mathrm{Crem}_k\big(\mathrm{Out}(F_n)\big) \leqslant 3n - 3. \tag{30}$$

As noted in [CX18, pp. 272] (with reference to [MS75]), over $\mathbb{C}$, the minimal dimension in which $\mathrm{Out}(F_n)$ is the group of birational self-maps, does not exceed $6n$. The right-hand side inequality in (30) is the twice stronger upper bound.

(d) Let Y be an irreducible algebraic variety. A subgroup H of the group $\mathrm{Bir}(Y)$ of birational self-maps of Y is called *compressible* (cf. [Re04], [Po19, Sect. 2.1]) if there are an irreducible algebraic variety Z, a group embedding $\iota\colon H \hookrightarrow \mathrm{Bir}(Z)$, and a dominant rational map $\varphi\colon X \dashrightarrow Z$ (called a *compression for* H) such that

(c_1) φ is not birational;

(c_2) φ is H-equivariant, i.e., $\varphi \circ h = \iota(h) \circ \varphi$ for each $h \in H$.

Otherwise, H is called *incompressible*.

In this context, the above results give the following. Let G be a connected nonsolvable algebraic group (not necessarily affine) and let $X = \mathrm{Hom}(F_n, G)$, $n \geqslant 2$. By Theorem 7.1, the homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(X)$, $\sigma \mapsto \sigma_X$, is a group embedding, so we can (and shall) identify $\mathrm{Aut}(F_n)$ with its image.

Theorem 13.3. $\mathrm{Aut}(F_n)$ *is a compressible subgroup of* $\mathrm{Aut}(X)$.

Proof. First, $\mathrm{Aut}(G)$ contains a nontrivial finite subgroup. To prove this, it suffices, in view of $\mathrm{Int}(G) = G/\mathscr{C}(G)$, to show that $G \setminus \mathscr{C}(G)$ contains an element of finite order. In turn, for this, it suffices to show that there exists such an element in $G_{\mathrm{aff}} \setminus \mathscr{C}(G_{\mathrm{aff}})$ (we use the notation of the proof of Theorem 1.1). Arguing by contradiction, suppose it does not exist. As every torus is the closure of its torsion subgroup (see [Bo91, Cor. 8.9], then every torus of G_{aff} lies in $\mathscr{C}(G_{\mathrm{aff}})$. As every semisimple element of G_{aff} lies in a torus (see [Bo91, Thm. 11.10], this means that $\mathscr{C}(G_{\mathrm{aff}})$ contains all semisimple elements of G_{aff}. Using that the canonical projection $G_{\mathrm{aff}} \to G_{\mathrm{aff}}/\mathscr{C}(G_{\mathrm{aff}})$ preserves Jordan decompositions (see [Bo91, Thm. 4.4], from this we infer that every element of $G_{\mathrm{aff}}/\mathscr{C}(G_{\mathrm{aff}})$ is unipotent. Whence $G_{\mathrm{aff}}/\mathscr{C}(G_{\mathrm{aff}})$ is a solvable group (see [Bo91, Cor. 4.8]). Therefore, G_{aff} is solvable as well. This contradicts the fact that, as is explained in the proof of Theorem 1.1, G_{aff} is nonsolvable.

Now take a nontrivial finite subgroup R of $\mathrm{Aut}(G)$. As the categorical quotient

$$\pi_{X/\!\!/R}\colon X \to X/\!\!/R \tag{31}$$

is geometric (see Proposition 4.1), the degree of the finite morphism $\pi_{X/\!\!/R}$ is equal to $|R| > 1$. Therefore, $\pi_{X/\!\!/R}$ is not birational. From this, Theorem 1.1, and conditions (c_1), (c_2) in the definition of compressibility we infer that (31) is a compression for $\mathrm{Aut}(F_n)$. $\square$

Question 13.4. Are there G, n, and a finite subgroup R of $\mathrm{Aut}(G)$ such that $\mathrm{Aut}(F_n)$ is an incompressible subgroup of $\mathrm{Aut}(X/\!\!/R)$?

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