Transformation Groups
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ORIGINAL PAPER

On Equivariantly Formal 2-Torus Manifolds

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Abstract
A 2-torus manifold is a closed connected smooth n-manifold with a non-free effective smooth $\mathbb{Z}_2^n$-action. In this paper, we prove that a 2-torus manifold is equivariantly formal if and only if the $\mathbb{Z}_2^n$-action is locally standard and every face of its orbit space (including the whole orbit space) is mod 2 acyclic. Our study is parallel to the study of torus manifolds with vanishing odd-degree cohomology by M. Masuda and T. Panov in (2006). As an application, we determine when such kind of 2-torus manifolds can have regular m-involutions (i.e., involutions with only isolated fixed points of the maximum possible number).

Keywords 2-torus manifold · Equivariantly formal · m-involution

Mathematics Subject Classification (2010) 57S12 · 57R91 · 55N91 · 57S17 · 57S25

1 Introduction

Let G be a compact Lie group and BG be the classifying space of G. For a G-space X, the *G-equivariant cohomology* of X with coefficients in a field $\mathbf{k}$ is the singular cohomology of the Borel construction X_G (see [6])

$$H_G^*(X;\mathbf{k}) := H^*(X_G;\mathbf{k}).$$

There is a natural fibration $X \to X_G \to BG$ associated with X_G called the *Borel fibration*. If the inclusion of the fiber $\iota_X : X \to X_G$ induces a surjection on cohomology $\iota_X^* : H_G^*(X;\mathbf{k}) \to H^*(X;\mathbf{k})$, X is called (cohomologically) *equivariantly formal* over $\mathbf{k}$. This term was coined in 1998 in Goresky-Kottwitz-MacPherson [18]. But this condition had already been studied by A. Borel in [5, § 4] and [6, Ch. XII] where X is called *totally non-homologous to zero in X_G* (also, see [7, Ch. VII]).

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For some special groups G shown below, the equivariant formality of a G-action can be interpreted in some other ways (see [5, § 4], [1, Ch. 3], and [2, Sec. 4]).

- When BG is simply connected (e.g., G is a torus $T^r = (S^1)^r$), X is equivariantly formal if and only if the Serre spectral sequence of the Borel fibration of X degenerates at the E_2 stage.
- When G is the p-torus $\mathbb{Z}_p^r$ (p is prime), X being equivariantly formal is equivalent to either one of the following conditions.

  (i) The Serre spectral sequence with $\mathbb{Z}_p$-coefficients of the Borel fibration of X degenerates at the E_2 stage and the induced action of $\mathbb{Z}_p^r$ on $H^*(X; \mathbb{Z}_p)$ is trivial.

  (ii) $H^*_{\mathbb{Z}_p^r}(X; \mathbb{Z}_p) \cong H^*(X; \mathbb{Z}_p) \otimes H^*(B\mathbb{Z}_p^r; \mathbb{Z}_p)$ is a free $H^*(B\mathbb{Z}_p^r; \mathbb{Z}_p)$-module.

Due to the above fact, we call a $\mathbb{Z}_p^r$-action on X *weakly equivariantly formal* if we only assume that the Serre spectral sequence (with $\mathbb{Z}_p$-coefficients) of the Borel fibration of X degenerates at the E_2 stage. So an equivariantly formal $\mathbb{Z}_p^r$-action is always weakly equivariantly formal.

When $G = T^r$ or $\mathbb{Z}_2^r$ and $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}_2$ respectively, there is another equivalent description of equivariantly formal G-actions given by the so called "Atiyah-Bredon sequence" (see Bredon [8] and Franz-Puppe [16] for the T^r case, and Allday-Franz-Puppe [2] for the $\mathbb{Z}_2^r$ case). In addition, there are many sufficient conditions for a T^r-action to be equivariantly formal (for example: vanishing of odd-degree cohomology, all homology classes being representable by T^r-invariant cycles, etc.).

Equivariantly formal G-spaces provide many nice examples in geometry and topology. Some of them are as follows:

- Smooth compact toric varieties.
- Hamiltonian G-actions on symplectic manifolds which have moment maps (see Atiyah-Bott [3] and Jeffrey-Kirwan [22]).
- Quasitoric manifolds and small covers defined in Davis-Januszkiewicz [14].
- Torus manifolds with vanishing odd degree cohomology (see Masuda-Panov [27]).

In addition, when $G = T^r$ or $(\mathbb{Z}_p)^r$, the following theorem gives us an easy way to recognize equivariantly formal G-actions.

Theorem 1.1 (see Theorem (3.10.4) in Allday-Puppe [1]) *Let $G = T^r$ or $(\mathbb{Z}_p)^r$ where p is a prime and $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively. Let X be a paracompact G-space with only finitely many orbit types and* $\dim_{\mathbf{k}} H^*(X; \mathbf{k}) < \infty$*. Then, the fixed point set X^G always satisfies*

$$\dim_{\mathbf{k}} H^*(X^G; \mathbf{k}) \leq \dim_{\mathbf{k}} H^*(X; \mathbf{k})$$

where the equality holds if and only if X is equivariantly formal over $\mathbf{k}$*. Here* $\dim_{\mathbf{k}} H^*(X; \mathbf{k})$ *denotes the sum of the rank of the cohomology groups of X in all dimensions over* $\mathbf{k}$.

A very special case is when $G = \mathbb{Z}_2$ and $X^{\mathbb{Z}_2}$ consists only of isolated points. By Theorem 1.1, we have

$$|X^{\mathbb{Z}_2}| = \dim_{\mathbb{Z}_2} H^*(X^{\mathbb{Z}_2}; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2) \tag{1}$$

Such a $\mathbb{Z}_2$-action on X is equivariantly formal if and only if the number of the fixed points reaches the maximum, i.e., $|X^{\mathbb{Z}_2}| = \dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2)$. In this case, the involution determined by the $\mathbb{Z}_2$-action is called an m-*involution* on X (this term was named by Puppe [28]).

There is an interesting relation between m-involutions on closed manifolds and binary codes. It was shown in [28] that one can obtain a self-dual binary code from any m-involution on an odd-dimensional closed manifold. This motivates the study in Chen-Lü-Yu [12] on the m-involutions on a special kind of closed manifolds called *small covers* (see [14]). In this paper, we want to study a more general type of closed manifolds with 2-torus actions defined below.

Definition 1.2 (see Lü-Masuda [25]) A 2-*torus manifold* is a closed connected smooth n-manifold W with a non-free effective smooth action of $\mathbb{Z}_2^n$. For such a manifold W, since $\dim(W) = n = \text{rank}(\mathbb{Z}_2^n)$ and the $\mathbb{Z}_2^n$-action is effective, the fixed point set $W^{\mathbb{Z}_2^n}$ must be discrete. Then, since W is compact, $W^{\mathbb{Z}_2^n}$ is a finite set of isolated points (if not empty). Note that we require all 2-torus manifolds to be connected in this paper.

- For brevity, we call a 2-torus manifold W *equivariantly formal* or *weakly equivariantly formal* if the $\mathbb{Z}_2^n$-action on W is so, respectively.
- We call W *locally standard* if for every point $x \in W$, there is a $\mathbb{Z}_2^n$-invariant neighborhood V_x of x such that V_x is equivariantly homeomorphic to an invariant open subset of a real n-dimensional faithful linear representation space of $\mathbb{Z}_2^n$. An equivalently way to describe such a neighborhood V_x is: V_x is weakly equivariantly homeomorphic to an invariant open subset of $\mathbb{R}^n$ under the standard $\mathbb{Z}_2^n$-action defined by: for any $(x_1, \cdots, x_n) \in \mathbb{R}^n$ and $(g_1, \cdots, g_n) \in \mathbb{Z}_2^n$,

$$(g_1, \cdots, g_n) \cdot (x_1, \cdots, x_n) \longmapsto \big((-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n\big).$$

- Every non-zero element $g \in \mathbb{Z}_2^n$ determines a nontrivial involution τ_g on W, called a *regular involution* on W.

We will prove in Theorem 3.3 that if a 2-torus manifold is equivariantly formal, then it must be locally standard.

For an n-dimensional locally standard 2-torus manifold W, the orbit space $Q = W/\mathbb{Z}_2^n$ naturally becomes a connected smooth nice n-manifold with corners and with non-empty boundary (since the $\mathbb{Z}_2^n$-action is non-free). Moreover,

- The $\mathbb{Z}_2^n$-action on W determines a *characteristic function*

$$\lambda_W : \{F_1, \cdots, F_m\} \to \mathbb{Z}_2^n$$

  where $F_1, \cdots, F_m$ are all the facets (codimension-one faces) of Q.
- The free part of the $\mathbb{Z}_2^n$-action on W determines a principal $\mathbb{Z}_2^n$-bundle ξ_W over Q.

It is shown in Lü-Masuda [25, Lemma 3.1] that W can be recovered from the data (Q, λ_W, ξ_W) up to equivariant homeomorphism. In addition, let $\pi : W \to Q$ denote the orbit map. If f is a codimension-k face of Q, then $W_f := \pi^{-1}(f)$ is a codimension-k submanifold of W called a *facial submanifold of* W. Let G_f denote the isotropy subgroup of W_f. Then, W_f is also a 2-torus manifold with respect to the induced action of $\mathbb{Z}_2^n/G_f$. In the following, when we say W_f is equivariantly formal, we always consider W_f being equipped with the induced $\mathbb{Z}_2^n/G_f$-action from W.

The main purpose of this paper is to answer the following two questions:

Question-1: What kind of 2-torus manifolds are equivariantly formal?

Question-2: What kind of locally standard 2-torus manifolds have regular m-involutions?

Generally speaking, it is very hard to compute the equivariant cohomology of a locally standard 2-torus manifold W directly from its orbit space Q and the data (λ_W, ξ_W). So it is difficult to judge whether W is equivariantly formal by directly verifying the condition in the definition. Meanwhile, it was proved by Masuda-Panov [27] that a smooth T^n-action on a connected smooth $2n$-manifold with non-empty fixed points is equivariantly formal if and only if the T^n-action is locally standard and every face of its orbit space is acyclic (also see Goertsches-Töben [19, Theorem 10.19] for a reformulation of this result). This result is also implied by Franz [15, Theorem 1.3]. The arguments in [27] inspire us to prove the following parallel result for 2-torus manifolds.

Theorem 1.3 *Let W be a 2-torus manifold with orbit space Q.*

(i) W is equivariantly formal if and only if W is locally standard and Q is mod 2 face-acyclic.

(ii) W is equivariantly formal and $H^*(W;\mathbb{Z}_2)$ is generated by its degree-one part as a ring if and only if W is locally standard and Q is a mod 2 homology polytope.

The definitions of "mod 2 face-acyclic" and "mod 2 homology polytope" are given in Definition 2.1.

The main strategy in our proof of Theorem 1.3 is very similar to the strategy used in [27] for equivariantly formal torus manifolds. Besides, our proof uses the mod 2 GKM theory introduced in Biss-Guillemin-Holm [4] which allows us to observe the equivariant cohomology of an equivariantly formal 2-torus manifold by restricting to its fixed point set (see Sect. 2.3).

Remark 1.4 If a 2-torus manifold W is assumed to be locally standard in the first place, Theorem 1.3 (i) can also be derived from Chaves [11, Theorem 1.1] whose proof uses the theory of syzygies in the mod 2 equivariant cohomology (see Allday-Franz-Puppe [2, Theorem 10.2]) and the mod 2 "Atiyah-Bredon sequence". But we will use a completely different approach in our proof here.

Using Theorem 1.3, we can easily derive the following theorem which gives an answer to Question-2.

Theorem 1.5 *Let W be an n-dimensional locally standard 2-torus manifold with orbit space Q. Then, there exists a regular* m*-involution on W if and only if Q is mod 2 face-acyclic (or equivalently W is equivariantly formal) and the values of the characteristic function λ_W on all the facets of Q consist exactly of a linear basis of $\mathbb{Z}_2^n$.*

A nice manifold with corners Q is called *k-colorable* if we can assign k different colors to all the facets of Q so that no two adjacent facets are of the same color. Clearly, there exists a 2-torus manifold over Q whose characteristic function takes value in a linear basis of $\mathbb{Z}_2^n$ if and only if Q is n-colorable.

Remark 1.6 By Theorem 1.5 and the construction in Puppe [28], we can obtain a self-dual binary code $\mathcal{C}_Q$ from an n-colorable mod 2 face-acyclic nice smooth n-manifold with corners Q when n is odd. This generalizes the self-dual binary codes from n-colorable simple convex n-polytopes in Chen-Lü-Yu [12]. Moreover, we can write down $\mathcal{C}_Q$ explicitly in the same way as the self-dual binary code obtained in [12, Corollary 4.5].

The paper is organized as follows. In Sect. 2, we review the definitions and some basic facts of locally standard 2-torus manifolds and quote some well known results that are useful for our proof. In Sect. 3, we study various properties of equivariantly formal 2-torus manifolds. Since the philosophy of our study is very similar to the study of torus manifolds with vanishing odd degree cohomology in Masuda-Panov [27], many lemmas in this paper are parallel to those in [27]. In Sect. 4, we prove some special properties of equivariantly formal 2-torus manifolds whose mod 2 cohomology rings are generated by their degree-one part. Then finally, in Sect. 5, we prove Theorem 1.3 and Theorem 1.5.

2 Preliminaries

2.1 Manifolds with Corners and Locally Standard 2-Torus Manifolds

Recall a (smooth) *n-dimensional manifold with corners* Q is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the transition functions are (diffeomorphisms) homeomorphisms which preserve the codimension of each point. Here, the *codimension* $c(x)$ of a point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n_{\geq 0}$ is the number of x_i that is 0. So we have a well defined map $c : Q \to \mathbb{Z}_{\geq 0}$ where $c(q)$ is the codimension of a point $q \in Q$. An *open face* of Q of codimension k is a connected component of $c^{-1}(k)$. A (closed) *face* is the closure of an open face. A face of codimension one is called a *facet* of Q. When Q is connected, we also consider Q itself as a face (of codimension zero).

- For any $k \in \mathbb{Z}_{\geq 0}$, the *$k$-skeleton* of Q is the union of all the faces of Q with dimension $\leq k$.
- The *face poset* of Q, denoted by $\mathcal{P}_Q$, is the set of faces of Q ordered by reversed inclusion (so Q is the initial element).

A manifold with corners Q is said to be *nice* if either its boundary ∂Q is empty or ∂Q is non-empty and any codimension-k face of Q is a component of the intersection of k different facets in Q. If Q is nice, $\mathcal{P}_Q$ is a simplicial poset. But in general $\mathcal{P}_Q$ may not be the face poset of a simplicial complex. Indeed, $\mathcal{P}_Q$ is the face poset of a simplicial complex if and only if all non-empty multiple intersections of facets of Q are connected (see [27, Sec. 5.2]).

Definition 2.1 Let Q be a nice manifold with corners.

- We call Q *mod* 2 *face-acyclic* if every face of Q (including Q itself) is a mod 2 acyclic space.
- We call Q a *mod* 2 *homology polytope* if Q is mod 2 face-acyclic and $\mathcal{P}_Q$ is the face poset of a simplicial complex.

A topological space B is called *mod* 2 *acyclic if* $H^*(B; \mathbb{Z}_2) \cong H^*(pt; \mathbb{Z}_2)$.
It is not difficult to prove the following lemma (see [27, p.743 Remark] for a short argument).

Lemma 2.2 *If Q is mod 2 face-acyclic, then every face of Q has a vertex and the* 1*-skeleton of Q is connected.*

In the following, let W be an n-dimensional locally standard 2-torus manifold with orbit space Q. Then, Q is a smooth nice manifold with corners with $\partial Qeq\varnothing$. Let $\pi : W \to Q$ denote the projection, and let the set of facets of Q be

$$\mathcal{F}(Q) = \{F_1, \cdots, F_m\}.$$

Then, $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_m)$ are embedded codimension-one closed connected submanifolds of W, called the *characteristic submanifolds* of W. Moreover, the $\mathbb{Z}_2^n$-action on W determines a *characteristic function* on Q which is a map

$$\lambda_W : \mathcal{F}(Q) \to \mathbb{Z}_2^n \tag{2}$$

where $\lambda_W(F_i) \in \mathbb{Z}_2^n$ is the generator of the $\mathbb{Z}_2$ subgroup that pointwise fixes the submanifold $\pi^{-1}(F_i)$, $1 \leq i \leq m$. Since the $\mathbb{Z}_2^n$-action is locally standard, the function λ_W satisfies the following *linear independence condition*:

> whenever the intersection of k different facets $F_{i_1}, \cdots, F_{i_k}$ is non-empty, the elements $\lambda_W(F_{i_1}), \cdots, \lambda_W(F_{i_k})$ are linearly independent when viewed as vectors of $\mathbb{Z}_2^n$ over the field $\mathbb{Z}_2$.

For a codimension-k face f of Q, let $F_{i_1}, \cdots, F_{i_k}$ be all the facets containing f. Then, the isotropy subgroup of the facial submanifold W_f is

$$G_f = \text{the subgroup generated by } \{\lambda_W(F_{i_1}), \cdots, \lambda_W(F_{i_k})\} \subseteq \mathbb{Z}_2^n. \tag{3}$$

By the linear independence condition of λ_W, $G_f \cong \mathbb{Z}_2^k$. Hence W_f is also a 2-torus manifold with respect to the induced action of $\mathbb{Z}_2^n / G_f \cong \mathbb{Z}_2^{n-k}$.

In addition, W determines a *principal* $\mathbb{Z}_2^n$*-bundle* over Q as follows. We take a small invariant open tubular neighborhood for each characteristic submanifold of W and remove their union from W. Then, the $\mathbb{Z}_2^n$-action on the resulting space is free, and its orbit space can naturally be identified with Q, which gives a principal $\mathbb{Z}_2^n$-bundle over Q, denoted by ξ_W. It is shown in Lü-Masuda [25] that W can be recovered (up to equivariant homeomorphism) from (Q, ξ_W, λ_W). For example, when ξ_W is a trivial

$\mathbb{Z}_2^n$-bundle, W is equivariantly homeomorphic to the following "*canonical model*" determined by (Q, λ_W).

$$M_Q(\lambda_W) := Q \times \mathbb{Z}_2^n / \sim \tag{4}$$

where $(q, g) \sim (q', g')$ if and only if $q = q'$ and $g - g' \in G_{f(q)}$ where $f(q)$ is the unique face of Q that contains q in its relative interior. This canonical model is a generalization of a result of Davis-Januszkiewicz [14, Prop. 1.8]. We will see that the canonical model plays an important role in our proof of Theorem 1.3 in Sect. 5.

2.2 Borel Construction and Equivariant Cohomology

For a topological group G, there exists a contractible free right G-space EG called the *universal G-space*. The quotient $BG = EG/G$ is called the *classifying space* for free G-actions. For example, when $G = \mathbb{Z}_2^n$, we can choose

$$E\mathbb{Z}_2^n = (E\mathbb{Z}_2)^n = (S^\infty)^n, \quad B\mathbb{Z}_2^n = (B\mathbb{Z}_2)^n = (\mathbb{R}P^\infty)^n.$$

Let X be a topological space with a left G-action (we call X a *G-space* for brevity). The *Borel construction* of X is denoted by

$$EG \times_G X = EG \times X / \sim$$

where $(e, x) \sim (eg, g^{-1}x)$ for any $e \in EG$, $x \in X$ and $g \in G$.

The *equivariant cohomology* of X with coefficients in a field $\mathbf{k}$ is defined as

$$H_G^*(X; \mathbf{k}) := H^*(EG \times_G X; \mathbf{k}).$$

Convention: The term "cohomology" of a space in this paper, always mean singular cohomology if not specified otherwise.

The Borel construction determines a canonical fibration called *Borel fibration*:

$$X \to EG \times_G X \to BG. \tag{5}$$

The map ρ collapsing X to a point induces a homomorphism

$$\rho^* : H_G^*(pt; \mathbf{k}) = H^*(BG; \mathbf{k}) \to H_G^*(X; \mathbf{k}) \tag{6}$$

which defines a canonical $H^*(BG; \mathbf{k})$-module structure on $H_G^*(X; \mathbf{k})$. A useful fact is: when X is a paracompact space with finite cohomology dimension, and $G = T^r$ or $(\mathbb{Z}_p)^r$ where p is a prime and $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively, ρ^* is injective if and only if the fixed point set X^G is non-empty (see [21, Ch. IV]).

In general, $H_G^*(X; \mathbf{k})$ may not be a free $H^*(BG; \mathbf{k})$-module. The following *localization theorem* due to A. Borel (see [21, p. 45]) says that we can compute the free $H^*(BG; \mathbf{k})$-module part of $H_G^*(X; \mathbf{k})$ by restricting to the fixed point set.

Theorem 2.3 (Localization Theorem) *Let $G = T^r$ or $(\mathbb{Z}_p)^r$ where p is a prime and $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively. For a paracompact G-space X with finite cohomology dimension, the following localized restriction homomorphism is an isomorphism:*

$$S^{-1}H_G^*(X; \mathbf{k}) \to S^{-1}H_G^*(X^G; \mathbf{k}) = H^*(X^G; \mathbf{k}) \otimes_{\mathbf{k}} (S^{-1}H^*(BG; \mathbf{k}))$$

where $S = R - \{0\}$ where R is the polynomial subring of $H^*(BG; \mathbf{k})$. So the kernel of the restriction $H_G^*(X; \mathbf{k}) \to H_G^*(X^G; \mathbf{k})$ lies in the $H^*(BG; \mathbf{k})$-torsion of $H_G^*(X; \mathbf{k})$. In particular if X is equivariantly formal, $H_G^*(X; \mathbf{k}) \to H_G^*(X^G; \mathbf{k})$ is injective.

The Borel construction can also be applied to a G-vector bundle $\pi : E \to X$ (i.e., both E and X are G-spaces and the projection π is G-equivariant). In this case, the Borel construction E_G of E is a vector bundle over X_G whose mod 2 Euler class, denoted by $e^G(E)$, lies in $H_G^*(X; \mathbb{Z}_2)$. Note that using $\mathbb{Z}_2$-coefficients allows us to ignore the orientation of a vector bundle.

2.3 Mod 2 GKM-Theory

Let W be an n-dimensional equivariantly formal 2-torus manifold. Then, the fixed point set $W^{\mathbb{Z}_2^n}$ is a finite non-empty set (by Theorem 1.1), and $H^*_{\mathbb{Z}_2^n}(W; \mathbb{Z}_2)$ is a free module over $H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2)$. Moreover, $H^*_{\mathbb{Z}_2^n}(W; \mathbb{Z}_2)$ can be computed by the so called Mod 2 GKM-theory (see Biss-Guillemin-Holm [4]) which is an extension of the GKM-theory in [18] to 2-torus actions. In this section, we briefly review some results related to our study. The reader is referred to [4] and [24] for more details.

For each $1 \le i \le n$, let $\rho_i \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ be the homomorphism defined by

$$\rho_i((g_1, \cdots, g_n)) = g_i, \ \forall (g_1, \cdots, g_n) \in \mathbb{Z}_2^n.$$

By a canonical isomorphism $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \cong H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2)$, we can identify $H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2)$ with the graded polynomial ring $\mathbb{Z}_2[\rho_1, \cdots, \rho_n]$ where $\deg(\rho_i) = 1$, $1 \le i \le n$.

Let $Q = W/\mathbb{Z}_2^n$ be the orbit space of W. By our Theorem 3.3 proved later, a 2-torus manifold W being equivariantly formal implies that it is locally standard. Hence Q is a nice manifold with corners. Then, the 1-skeleton of Q, consisting of vertices (0-faces) and edges (1-faces) of Q, is an n-valent graph denoted by $\Gamma(Q)$. Let $V(Q)$ and $E(Q)$ denote the set of vertices and edges of Q, respectively.

Convention: We will not distinguish a vertex of Q and the corresponding fixed point in $W^{\mathbb{Z}_2^n}$ in the rest of the paper.

- Let $\pi : W \to Q$ be the quotient map.
- For each edge $e \in E(Q)$, $\pi^{-1}(e)$ is a circle whose isotropy subgroup G_e is a rank $n-1$ subgroup of $\mathbb{Z}_2^n$. Then, we obtain a map

$$\alpha : E(Q) \to \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \cong H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2)$$

  where for each edge $e \in E(Q)$, $\ker(\alpha(e)) = G_e$.

- For each vertex $p \in V(Q)$, let $\alpha_p = \{\alpha(e) \mid p \in e\} \subset \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

Such a map α is called an *axial function* which has the following properties:

(i) For every vertex $p \in V(Q)$, α_p is a linear basis of $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.
(ii) For every edge $e \in E(Q)$, $\alpha_p \equiv \alpha_{p'} \bmod \alpha(e)$ where p, p' are the two vertices of e.

By [4, Theorem C] and [4, Remark 5.9], we have the following theorem which is a consequence of the $\mathbb{Z}_2$-version Chang-Skjelbred theorem (see [4, Theorem 4.1] and [10]).

Theorem 2.4 (see [4]) *Let W be an n-dimensional equivariantly formal 2-torus manifold. If we choose an element $\eta_p \in H^*_{\mathbb{Z}_2^n}(W^{\mathbb{Z}_2^n}; \mathbb{Z}_2)$ for each $p \in W^{\mathbb{Z}_2^n}$, then*

$$(\eta_p) \in \bigoplus_{p \in W^{\mathbb{Z}_2^n}} H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2) \cong H^*_{\mathbb{Z}_2^n}(W^{\mathbb{Z}_2^n}; \mathbb{Z}_2)$$

is in the image of the restriction homomorphism $r : H^*_{\mathbb{Z}_2^n}(W; \mathbb{Z}_2) \to H^*_{\mathbb{Z}_2^n}(W^{\mathbb{Z}_2^n}; \mathbb{Z}_2)$ if and only if for every edge $e \in E(Q)$ with vertices p and p', $\eta_p - \eta_{p'}$ is divisible by $\alpha(e)$.

Moreover, we can understand the above axial function α in the following way. For brevity, we use the following notations for an n-dimensional locally standard 2-torus manifold W in the rest of this section.

- Let $G = \mathbb{Z}_2^n$.
- Let $W_i := W_{F_i} = \pi^{-1}(F_i)$, $1 \leq i \leq m$, be all the characteristic submanifolds of W where $F_1, \cdots, F_m$ are all the facets of Q.
- Let $G_i := \langle \lambda_W(F_i) \rangle \cong \mathbb{Z}_2$ be subgroup of G that fixes W_i pointwise.
- Let ν_i be the (equivariant) normal bundle of W_i in W. So we have the *equivariant Euler class* of ν_i, denoted by $e^G(\nu_i) \in H^1_G(W_i; \mathbb{Z}_2)$.
- For any fixed point $p \in W^{\mathbb{Z}_2^n}$, let $I(p) := \{i \mid p \in W_i\}$. We have the decomposition of tangent space T_pW as
$$T_pW = \bigoplus_{i \in I(p)} \nu_i|_p.$$
where $\nu_i|_p$ denotes the restriction of ν_i to p. So $\nu_i|_p$ is a 1-dimensional linear representation of G whose equivariant Euler class
$$e^G(\nu_i|_p) = e^G(\nu_i)|_p \in H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2).$$

The inclusion map $\psi_i : W_i \hookrightarrow W$ defines an equivariant Gysin homomorphism $\psi_{i_!} : H^*_G(W_i; \mathbb{Z}_2) \to H^{*+1}_G(W; \mathbb{Z}_2)$ (see [1, §5.3] for example). For brevity, let

$$\tau_i = \tau_{F_i} = \psi_{i_!}(1) \in H^1_G(W; \mathbb{Z}_2)$$

be the image of the identity $1 \in H^0_G(W_i;\mathbb{Z}_2)$. The element τ_i can be thought of as the Poincaré dual of the Borel construction of W_i in $H^*_G(W;\mathbb{Z}_2)$ and is called the *equivariant Thom class* of ν_i. A standard fact is

$$\tau_i|_p \text{ agrees with the equivariant Euler class of } \nu_i|_p.$$

Note that the elements of $\mathrm{Hom}(\mathbb{Z}_2^n,\mathbb{Z}_2)$ are in one-to-one correspondence with all the 1-dimensional linear representations of $\mathbb{Z}_2^n$. So the canonical isomorphism between $\mathrm{Hom}(\mathbb{Z}_2^n,\mathbb{Z}_2)$ and $H^1(B\mathbb{Z}_2^n;\mathbb{Z}_2)$ is given by the equivariant Euler class of a 1-dimensional representations of $\mathbb{Z}_2^n$. Then, we have the following identification:

$$\alpha_p = \{\alpha(e) \mid p \in e\} \longleftrightarrow \{e^G(\nu_i)|_p = \tau_i|_p\,;\ i \in I(p)\}. \tag{7}$$

where an edge e containing p corresponds to the unique index $i \in I(p)$ so that the facet F_i intersects e transversely (or equivalently $e \nsubseteq F_i$).

- For a codimension-k face f of Q, let ν_f denote the (equivariant) normal bundle of W_f in W. Denote by $\tau_f \in H^k_G(W;\mathbb{Z}_2)$ the equivariant Thom class of ν_f. Then, the restriction of τ_f to $H^k_G(W_f;\mathbb{Z}_2)$ is the equivariant Euler class of ν_f, denoted by $e^G(\nu_f)$. In particular, if $f = Q$, $W_f = W$ and so τ_f is the identity element of $H^0_G(W_f;\mathbb{Z}_2)$.

Let $r_p : H^*_G(W;\mathbb{Z}_2) \to H^*_G(p;\mathbb{Z}_2) \cong H^*(BG;\mathbb{Z}_2)$ denote the restriction map at a fixed point $p \in W^G$. Then,

$$r = \bigoplus_{p\in W^G} r_p : H^*_G(W;\mathbb{Z}_2) \to H^*_G(W^G;\mathbb{Z}_2) = \bigoplus_{p\in W^G} H^*(BG;\mathbb{Z}_2). \tag{8}$$

By Theorem 2.3, the kernel of r is the $H^*(BG;\mathbb{Z}_2)$-torsion subgroup of $H^*_G(W;\mathbb{Z}_2)$.

Clearly, $r_p(\tau_f) = 0$ unless $p \in (W_f)^G$ (i.e., p is a vertex of f). It follows from (7) that for any $p \in W^G$,

$$r_p(\tau_f) = \begin{cases} \prod\limits_{p\in e,\, e\nsubseteq f} \alpha(e), & \text{if } p \in f; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

In addition, define

$$\widehat{H}^*_G(W;\mathbb{Z}_2) := H^*_G(W;\mathbb{Z}_2)\big/ H^*(BG;\mathbb{Z}_2)\text{-torsion}. \tag{10}$$

By the localization theorem (Theorem 2.3), the restriction homomorphism r induces a monomorphism $\widehat{H}^*_G(W;\mathbb{Z}_2) \to H^*_G(W^G;\mathbb{Z}_2)$, still denoted by r.

The following proposition is parallel to [27, Proposition 3.3].

Proposition 2.5 *Let W be an n-dimensional locally standard 2-torus manifold.*

(i) For each characteristic submanifold W_i with $(W_i)^G \neq \varnothing$ where $G = \mathbb{Z}_2^n$, there is a unique element $a_i \in H_1(BG; \mathbb{Z}_2)$ such that

$$\rho^*(t) = \sum_i \langle t, a_i \rangle \tau_i \ \ \textit{modulo} \ \ H^*(BG; \mathbb{Z}_2)\textit{-torsion}$$

for any element $t \in H^1(BG; \mathbb{Z}_2)$. Here the sum is taken over all the characteristic submanifolds W_i with $(W_i)^G \neq \varnothing$ and ρ^* is defined in (6).

(ii) For each W_i with $(W_i)^G \neq \varnothing$, the subgroup G_i fixing W_i coincides with the subgroup determined by $a_i \in H_1(BG; \mathbb{Z}_2)$ through the identification $H_1(BG; \mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2, G)$.

(iii) If n different characteristic submanifolds $W_{i_1}, \cdots, W_{i_n}$ have a G-fixed point in their intersection, then the elements $a_{i_1}, \cdots, a_{i_n}$ form a linear basis of $H_1(BG; \mathbb{Z}_2)$ over $\mathbb{Z}_2$.

Proof The argument is completely parallel to the arguments for torus manifolds in the proof of [26, Lemma 1,3, Lemma. 1.5, Lemma 1.7]. Indeed, we can just replace the torus manifold M in [26] by our 2-torus manifold W and replace T^n by $\mathbb{Z}_2^n$ and $H^2(M; \mathbb{Z})$ by $H^1(W; \mathbb{Z}_2)$ to obtain our proof here. The details of the proof are left to the reader. □

In addition, the following lemma is completely parallel to the torus manifold case [27, Lemma 6.2]. Its proof is also parallel to [27], hence omitted.

Lemma 2.6 *Let W be a locally standard 2-torus manifold with orbit space Q. For any $\eta \in H_G^*(W; \mathbb{Z}_2)$ and any edge $e \in E(Q)$, $r_p(\eta) - r_{p'}(\eta)$ is divisible by $\alpha(e)$ where p and p' are the endpoints of e.*

2.4 Face Ring

A poset (partially ordered set) $\mathcal{P}$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $x \in \mathcal{P}$ the lower segment $[\hat{0}, x]$ is a boolean lattice (the face lattice of a simplex).

Let $\mathcal{P}$ be a simplicial poset. For each $x \in \overline{\mathcal{P}} := \mathcal{P} - \{\hat{0}\}$, we assign a geometrical simplex whose face poset is $[\hat{0}, x]$ and glue these geometrical simplices together according to the order relation in $\mathcal{P}$. The cell complex we obtained is called the *geometrical realization* of $\mathcal{P}$, denoted by $|\mathcal{P}|$. We may also say that $|\mathcal{P}|$ is a *simplicial cell complex*.

For any two elements $x, x' \in \mathcal{P}$, denote by $x \vee x'$ the set of their least common upper bounds, and by $x \wedge x'$ their greatest common lower bounds. Since $\mathcal{P}$ is simplicial, $x \wedge x'$ consists of a single element if $x \vee x'$ is non-empty.

Definition 2.7 (see Stanley [29]) The *face ring* of a simplicial poset $\mathcal{P}$ over a field $\mathbf{k}$ is the quotient

$$\mathbf{k}[\mathcal{P}] := \mathbf{k}[v_x : x \in \mathcal{P}] \big/ \mathcal{I}_{\mathcal{P}}$$

where $\mathcal{I}_{\mathcal{P}}$ is the ideal generated by all the elements of the form

$$v_x v_{x'} - v_{x \wedge x'} \cdot \sum_{x'' \in x \vee x'} v_{x''}.$$

Let Q be a nice manifold with corners. It is easy to see that the face poset of Q is a simplicial poset, denoted by $\mathcal{P}_Q$. We call $|\mathcal{P}_Q|$ the simplicial cell complex *dual* to Q.

We define the *face ring* of Q to be the face ring of $\mathcal{P}_Q$. Equivalently, we can write the face ring of Q as

$$\mathbf{k}[Q] := \mathbf{k}[v_f : f \text{ a face of } Q] \big/ \mathcal{I}_Q.$$

where $\mathcal{I}_Q$ is the ideal generated by all the elements of the form

$$v_f v_{f'} - v_{f \vee f'} \cdot \sum_{f'' \in f \cap f'} v_{f''}.$$

where $f \vee f'$ denotes the unique minimal face of Q containing both f and f'.
Convention: For any face f of Q, define the degree of v_f to be the codimension of f. Then, $\mathbf{k}[Q] = \mathbf{k}[\mathcal{P}_Q]$ becomes a graded ring. Note that in the discussion of torus manifolds in [27], the degree of v_f is defined to be twice the codimension of f to fit the study there.

The *f-vector* of Q is defined as $\mathbf{f}(Q) = (f_0, \cdots, f_{n-1})$ where $n = \dim(Q)$ and f_i is the number of faces of codimension $i + 1$. The equivalent information is contained in the *h-vector* $\mathbf{h}(Q) = (h_0, \cdots, h_n)$ determined by the equation:

$$h_0 t^n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}. \tag{11}$$

The *Hilbert series* of $\mathbf{k}[Q]$ is $F(\mathbf{k}[Q]) := \sum_i \dim_{\mathbf{k}} \mathbf{k}[Q]_i \cdot t^i$ where $\mathbf{k}[Q]_i$ denotes the homogeneous degree i part of $\mathbf{k}[Q]$. By [29, Proposition 3.8],

$$F(\mathbf{k}[Q]; t) = \frac{h_0 + h_1 t + \cdots + h_n t^n}{(1-t)^n}. \tag{12}$$

The following construction is taken from [27, Sect. 5]. For any vertex (0-face) $p \in Q$, we define a map

$$s_p : \mathbf{k}[Q] \to \mathbf{k}[Q] \big/ (v_f : p \notin f). \tag{13}$$

If p is the intersection of n different facets $F_1, \cdots, F_n$, then $\mathbf{k}[Q] \big/ (v_f : p \notin f)$ can be identified with the polynomial ring $\mathbf{k}[v_{F_1}, \cdots, v_{F_n}]$.

Lemma 2.8 (Lemma 5.6 in [27]) *If every face of Q has a vertex, then the direct sum $s = \bigoplus_p s_p$ over all vertices $p \in Q$ is a monomorphism from $\mathbf{k}[Q]$ to the sum of polynomial rings $\mathbf{k}[Q] \big/ (v_f : p \notin f)$.*

A finitely generated graded commutative ring R over $\mathbf{k}$ is called *Cohen-Macaulay* if there exists an *h.s.o.p* (homogeneous system of parameters) $\theta_1, \cdots, \theta_n$ such that R is a free $\mathbf{k}[\theta_1, \cdots, \theta_n]$-module. Clearly, if $\mathbf{k}[Q] = \mathbf{k}[\mathcal{P}_Q]$ is Cohen-Macaulay, then it has a *l.s.o.p* (linear system of parameters).

A simplicial complex K is called a *Gorenstein* complex* over $\mathbf{k}$ if its face ring $\mathbf{k}[K]$ is Cohen-Macaulay and $H^*(K; \mathbf{k}) \cong H^*(S^d; \mathbf{k})$ where $d = \dim(K)$. The reader is referred to Bruns-Herzog [9] and Stanley [30] for more information of Cohen-Macaulay rings and Gorenstein* complexes.

The following proposition is parallel to [27, Lemma 8.2 (1)].

Proposition 2.9 *If Q is an n-dimensional mod 2 homology polytope, then the geometrical realization $|\mathcal{P}_Q|$ of $\mathcal{P}_Q$ is a Gorenstein* simplicial complex over $\mathbb{Z}_2$. In particular, $\mathbb{Z}_2[\mathcal{P}_Q]$ is Cohen-Macaulay and $H^*(|\mathcal{P}_Q|; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2)$.*

Proof The proof is almost identical to the proof in [27, Lemma 8.2] except that we use $\mathbb{Z}_2$-coefficients instead of $\mathbb{Z}$-coefficients when applying [30, II 5.1] in the argument. □

3 Equivariantly Formal 2-Torus Manifolds

In this section, we study various properties of equivariantly formal 2-torus manifolds. One may find that many discussions on 2-torus manifolds here are parallel to the discussions in [27] on torus manifolds. The condition "vanishing of odd degree cohomology" on a torus manifold in [27] is now replaced by the equivariant formality condition on a 2-torus manifold and, the coefficients $\mathbb{Z}$ is replaced by $\mathbb{Z}_2$. Many arguments in [27] are transplanted into our proof here while some of them actually become simpler.

In Sect. 3.1, we prove some general results of equivariantly formal $\mathbb{Z}_2^r$-actions on compact manifolds. In particular, we prove that any equivariantly formal 2-torus manifold is locally standard, and the equivariant formality of a 2-torus manifold is inherited by all its facial submanifolds.

In Sect. 3.2, we explore the relations between the equivariant cohomology of a locally standard 2-torus manifold and the face ring of its orbit space.

In Sect. 3.3, we prove that the equivariant formality of a 2-torus manifold is preserved under real blow-ups along its facial submanifolds. Our proof uses a result from Gitler [17].

3.1 Equivariantly Formal ⇒ Locally Standard

Lemma 3.1 *Suppose M is a compact manifold whose connected components are $M_1, \cdots, M_k$. A $\mathbb{Z}_2^r$-action on M is equivariantly formal if and only if each M_i is $\mathbb{Z}_2^r$-invariant and the restricted $\mathbb{Z}_2^r$-action on M_i is equivariantly formal.*

Proof The "if" part is obvious. For the "only if" part, assume that $M_1, \cdots, M_s$, $s \le k$, are all the components each of which is preserved under the $\mathbb{Z}_2^r$-action. Since the $\mathbb{Z}_2^r$-

action on M is equivariantly formal, by Theorem 1.1 we have

$$\dim_{\mathbb{Z}_2} H^*(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2).$$

So in particular, $M^{\mathbb{Z}_2^r}$ is not empty. Clearly, $M^{\mathbb{Z}_2^r}$ must lie in $M_1 \cup \cdots \cup M_s$, so $s > 0$ and $M^{\mathbb{Z}_2^r}$ is the disjoint union of $M_1^{\mathbb{Z}_2^r}, \cdots, M_s^{\mathbb{Z}_2^r}$. Then, by Theorem 1.1,

$$\begin{aligned}\dim_{\mathbb{Z}_2} H^*(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2) &= \sum_{i=1}^{s} \dim_{\mathbb{Z}_2} H^*(M_i^{\mathbb{Z}_2^r}; \mathbb{Z}_2) \le \sum_{i=1}^{s} \dim_{\mathbb{Z}_2} H^*(M_i; \mathbb{Z}_2) \\ &\le \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2).\end{aligned}$$

By comparing this inequality with the previous equation, we can deduce that $s = k$ and on every component M_i, $\dim_{\mathbb{Z}_2} H^*(M_i^{\mathbb{Z}_2^r}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M_i; \mathbb{Z}_2)$. So by Theorem 1.1 again, the $\mathbb{Z}_2^r$-action on M_i is equivariantly formal. □

Lemma 3.2 *If a $\mathbb{Z}_2^r$-action on a compact manifold M is equivariantly formal, then for every subgroup H of $\mathbb{Z}_2^r$,*

(i) The action of H on M is equivariantly formal.
(ii) The induced action of $\mathbb{Z}_2^r$ on M^H and $\mathbb{Z}_2^r/H$ on M^H are both equivariantly formal.
(iii) The induced action of $\mathbb{Z}_2^r$ (or $\mathbb{Z}_2^r/H$) on every connected component N of M^H is equivariantly formal, hence N has a $\mathbb{Z}_2^r$-fixed point.

Proof (i) By Theorem 1.1, it is equivalent to prove

$$\dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2). \tag{14}$$

Otherwise, assume $\dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) < \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2)$. Observe that the $\mathbb{Z}_2^r$-action on M induces an action of $\mathbb{Z}_2^r/H$ on M^H and we have

$$M^{\mathbb{Z}_2^r} = (M^H)^{\mathbb{Z}_2^r/H}. \tag{15}$$

So by Theorem 1.1, $\dim_{\mathbb{Z}_2} H^*(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2) \le \dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) < \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2)$, which contradicts the assumption that the $\mathbb{Z}_2^r$-action on M is equivariantly formal. This proves (i).

(ii) By (15) and the assumption that the $\mathbb{Z}_2^r$-action is equivariantly formal,

$$\begin{aligned}\dim_{\mathbb{Z}_2} H^*\big((M^H)^{\mathbb{Z}_2^r/H}; \mathbb{Z}_2\big) &= \dim_{\mathbb{Z}_2} H^*\big(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2\big) \\ &= \dim_{\mathbb{Z}_2} H^*\big(M; \mathbb{Z}_2\big) \overset{14}{=} \dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2).\end{aligned}$$

Then, by Theorem 1.1, the action of $\mathbb{Z}_2^r/H$ on M^H is equivariantly formal, so is the action of $\mathbb{Z}_2^r$ on M^H.

(iii) By the conclusion in (ii) and Lemma 3.1, the induced action of $\mathbb{Z}_2^r$ (or $\mathbb{Z}_2^r/H$) on every connected component N of M^H is equivariantly formal. So by Theorem 1.1, N must have a $\mathbb{Z}_2^r$-fixed point.

□

Next, we prove a theorem that is parallel to [27, Theorem 4.1].

Theorem 3.3 *If a* 2*-torus manifold W is equivariantly formal, then W must be locally standard.*

Proof Suppose $\dim(W) = n$. For a point $x \in W$, denote by G_x the isotropy group of x.

- If G_x is trivial, then x is in a free orbit of the $\mathbb{Z}_2^n$-action. So W is locally standard near x.
- Otherwise, let N be the connected component of W^{G_x} containing x. By Lemma 3.2 (iii), the induced $\mathbb{Z}_2^n$-action on N has a fixed point, say x_0. Since $W^{\mathbb{Z}_2^n}$ is discrete, the tangential $\mathbb{Z}_2^n$-representation $T_{x_0}W$ is faithful. Then, since x and x_0 are in the same connected component fixed pointwise by G_x, the G_x-representation on T_xW agrees with the restriction of the tangential $\mathbb{Z}_2^n$-representation $T_{x_0}W$ to G_x. This implies that W is locally standard near x.

The theorem is proved. □

Proposition 3.4 *Let W be an equivariantly formal* 2*-torus manifold with orbit space Q. For any face f of Q, the facial submanifold* W_f *is also an equivariantly formal* 2*-torus manifold.*

Proof Suppose $\dim(W) = n$ and f is a codimension-k face of Q. By Theorem 3.3, W is locally standard. Then, W_f is a connected $(n-k)$-dimensional embedded submanifold of W fixed pointwise by $G_f \cong \mathbb{Z}_2^k$ (see (3)). By Lemma 3.2 (iii), the induced action of $\mathbb{Z}_2^n/G_f \cong \mathbb{Z}_2^{n-k}$ on W_f is equivariantly formal. □

3.2 Equivariant Cohomology of Locally Standard 2-Torus Manifolds

Let W be an n-dimensional locally standard 2-torus manifold with orbit space Q. We explore the relation between $H_G^*(W;\mathbb{Z}_2)$ where $G = \mathbb{Z}_2^n$ and the face ring $\mathbb{Z}_2[Q]$ under some conditions on Q. In the following, we use the notations from Sect. 2.3.

First of all, we have a lemma that is parallel to [27, Lemma 6.3].

Lemma 3.5 *For any faces f and f′ of Q, the relation below holds in* $\widehat{H}_G^*(W;\mathbb{Z}_2)$*:*

$$\tau_f \tau_{f'} = \tau_{f\vee f'} \cdot \sum_{f''\in f\cap f'} \tau_{f''}.$$

Here we define $\tau_\varnothing = 0$.

Proof The proof is parallel to the proof of [27, Lemma 6.3]. The idea is to use the monomorphism $r : \widehat{H}_G^*(W;\mathbb{Z}_2) \to H_G^*(W^G;\mathbb{Z}_2)$ to map both sides of the identity to the fixed points and then use the formula (9) to check that they are equal. □

By Lemma 3.5, we obtain a well-defined homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}_2[Q] &\longrightarrow \widehat{H}_G^*(W;\mathbb{Z}_2). \\ v_f &\longmapsto \tau_f \end{aligned}$$

The following lemma and its proof are parallel to [27, Lemma 6.4].

Lemma 3.6 *The homomorphism φ is injective if every face Q has a vertex.*

Proof According to the definitions of r and s (see (8) and (13)), we have $s = r \circ \varphi$ by identifying $H^*_G(p, \mathbb{Z}_2)$ with $\mathbb{Z}_2[Q]\big/(v_f : p \notin f)$ for every vertex p of Q. Then, by Lemma 2.8, s is injective if every face of Q has a vertex, so is φ. □

The following lemma is parallel to [27, Proposition 7.4].

Lemma 3.7 *If the 1-skeleton of every face of Q (including Q itself) is connected, then $\widehat{H}^*_G(W; \mathbb{Z}_2)$ is generated by the elements $\tau_{F_1}, \cdots, \tau_{F_m} \in H^1_G(W; \mathbb{Z}_2)$ as an $H^*(BG; \mathbb{Z}_2)$-module, where $F_1, \cdots, F_m$ are all the facets of Q.*

Proof The argument is a bit technical, but it is completely parallel to the proof of [27, Proposition 7.4]. The main idea of the proof is to consider the restriction of an element $\eta \in H^*_G(W; \mathbb{Z}_2)$ to the fixed point set W^G via $r : H^*_G(W; \mathbb{Z}_2) \to H^*_G(W^G; \mathbb{Z}_2)$, and then use $\tau_{F_1}, \cdots, \tau_{F_m}$ and elements in $H^*(BG; \mathbb{Z}_2)$ to spell out $r(\eta)$ at each fixed point $p \in W^G$ (see Proposition 2.5). The details of the proof are left to the reader. □

The following theorem is parallel to [27, Theorem 7.5].

Theorem 3.8 *Let W be a locally standard 2-torus manifold with orbit space Q. If every face f of Q has a vertex and the 1-skeleton of f is connected, then the map $\varphi : \mathbb{Z}_2[Q] \to \widehat{H}^*_G(W; \mathbb{Z}_2)$ is an isomorphism of graded rings.*

Proof By Lemma 3.6, φ is injective and, by Lemma 3.7, φ is surjective. □

Lemma 3.9 *Let W be an equivariantly formal 2-torus manifold with orbit space Q. Then the 1-skeleton of every face of Q (including Q itself) is connected.*

Proof Since W is equivariantly formal, the localization theorem (Theorem 2.3) implies that the restriction homomorphism $r : H^*_G(W; \mathbb{Z}_2) \to H^*_G(W^G; \mathbb{Z}_2)$ is injective. In addition, since W is connected, the image of $H^0_G(W; \mathbb{Z}_2)$ under the restriction homomorphism is isomorphic to $\mathbb{Z}_2$. So the “if” part of Theorem 2.4 implies that the 1-skeleton of Q must be connected.

For any proper face f of Q, the facial submanifold W_f is also an equivariantly formal 2-torus manifold by Proposition 3.4. Then, by applying the above argument to W_f, we obtain that the 1-skeleton of f is also connected. □

Corollary 3.10 *If W is an equivariantly formal 2-torus manifold, then the map $\varphi : \mathbb{Z}_2[Q] \to H^*_G(W; \mathbb{Z}_2)$ is an isomorphism of graded rings.*

Proof Since W is equivariantly formal, its equivariant cohomology $H^*_G(W; \mathbb{Z}_2)$ is a free module over $H^*(BG; \mathbb{Z}_2)$. So by definition, $\widehat{H}^*_G(W; \mathbb{Z}_2) = H^*_G(W; \mathbb{Z}_2)$. For any face f of Q, the facial submanifold W_f is also an equivariantly formal 2-torus manifold by Proposition 3.4. This implies that f has a vertex. Moreover, the 1-skeleton of f is connected by Lemma 3.9. Then, the corollary follows from Theorem 3.8. □

When a 2-torus manifold W is equivariantly formal, Corollary 3.10 tells us that the equivariant cohomology ring of W is completely determined by the face poset of its orbit space (so independent on the characteristic function λ_W or the principal bundle ξ_W). This suggests that the orbit space of W should be rather special.

The following corollary is parallel to [27, Corollary 7.8]. It generalizes the calculation of the mod 2 cohomology ring of a small cover in [14].

Corollary 3.11 *If a 2-torus manifold W is equivariantly formal, then*

$$H^*(W;\mathbb{Z}_2) \cong \mathbb{Z}_2[v_f : f \text{ a face of } Q]\big/ I$$

where I is the ideal generated by the following two types of elements: (a) $v_f v_{f'} - v_{f\vee f'} \sum_{f''\in f\cap f'} v_{f''}$, (b) $\sum_{i=1}^{m} \langle t, a_i\rangle v_{F_i}$, $t \in H^1(BG;\mathbb{Z}_2)$.

Here, $F_1, \cdots, F_m$ are all the facets of Q, and the elements $a_i \in H_1(BG;\mathbb{Z}_2)$ are defined in Proposition 2.5.

Proof Since W is equivariantly formal, $\iota_W^* : H_G^*(W;\mathbb{Z}_2) \to H^*(W;\mathbb{Z}_2)$ is surjective and its kernel is generated by all $\rho^*(t)$ with $t \in H^1(BG;\mathbb{Z}_2)$ (see (6)). Then, the statement follows from Corollary 3.10 and Proposition 2.5. □

3.3 Real Blow-up of a Locally Standard 2-Torus Manifold Along a Facial Submanifold

Let W be a locally standard 2-torus manifold with orbit space Q. For a codimension-k face f of Q, the facial submanifold W_f is an embedded connected codimension-k submanifold of W. So the equivariant normal bundle ν_f of W_f in W is a real vector bundle of rank k. If we replace $W_f \subset W$ by the real projective bundle $P(\nu_f)$, we obtain a new 2-torus manifold denoted by $\widetilde{W}^f$ called the *real blow-up* of W along W_f. This is analogous to the blow-up of a torus manifold in [27, Sec. 9] (also see [20, p. 605] and [15, Sec. 4]).

The orbit space of $\widetilde{W}^f$, denoted by Q^f, is the result of "cutting off" the face f from Q (see Fig. 1). So $\widetilde{W}^f$ is also locally standard. Correspondingly, the simplicial cell complex $|\mathcal{P}_{Q^f}|$ is obtained from $|\mathcal{P}_Q|$ by a stellar subdivision of the face dual to f.

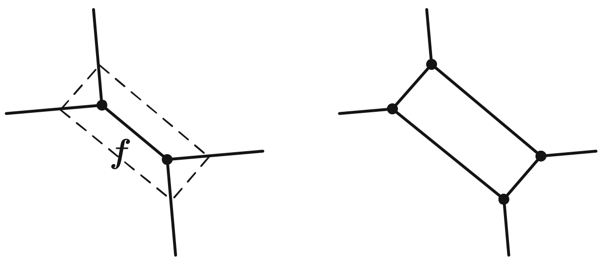

Fig. 1 Cutting off a face from a nice manifold with corners

Proposition 3.12 *Let W be a locally standard 2-torus manifold with orbit space Q and f be a proper face of Q with codimension-k. Then, $\widetilde{W}^f$ is equivariantly formal if and only if so is W.*

Proof (a) Let $\widetilde{\nu}_f$ denote the equivariant normal bundle of $P(\nu_f)$ in $\widetilde{W}^f$. Besides, let $\mathrm{Th}(\nu_f)$ and $\mathrm{Th}(\widetilde{\nu}_f)$ be the Thom space of ν_f and $\widetilde{\nu}_f$, respectively. Then, we have a natural commutative diagram of continuous maps:

$$\begin{array}{ccccc}
P(\nu_f) & \xrightarrow{\widetilde{i}} & \widetilde{W}^f & \xrightarrow{\widetilde{t}} & \mathrm{Th}(\widetilde{\nu}_f) \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
W_f & \xrightarrow{i} & W & \xrightarrow{t} & \mathrm{Th}(\nu_f)
\end{array}$$

where i and $\widetilde{i}$ are the inclusions; t and $\widetilde{t}$ are the Thom-Pontryagin maps; $p : \widetilde{W}^f \to W$ is the blow-down map; p_0 is the restriction of p to $P(\nu_f)$; and q is the induced map by p in the Thom spaces.

According to [17, §5] and [17, Theorem 3.7], there is a short exact sequence:

$$0 \longrightarrow H^*(\mathrm{Th}(\nu_f);\mathbb{Z}_2) \xrightarrow{\alpha} H^*(W;\mathbb{Z}_2) \oplus H^*(\mathrm{Th}(\widetilde{\nu}_f);\mathbb{Z}_2) \xrightarrow{\beta} H^*(\widetilde{W}^f;\mathbb{Z}_2) \longrightarrow 0. \tag{16}$$

where $\alpha = (t^*, q^*)$ and $\beta = p^* - \widetilde{t}^*$. This implies:

$$\dim_{\mathbb{Z}_2} H^*(\widetilde{W}^f;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2) + \dim_{\mathbb{Z}_2} H^*(\mathrm{Th}(\widetilde{\nu}_f);\mathbb{Z}_2) - \dim_{\mathbb{Z}_2} H^*(\mathrm{Th}(\nu_f);\mathbb{Z}_2).$$

By the Thom isomorphism, we have

$$\begin{aligned}
\dim_{\mathbb{Z}_2} H^*(\mathrm{Th}(\nu_f);\mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^*(W_f;\mathbb{Z}_2),\\
\dim_{\mathbb{Z}_2} H^*(\mathrm{Th}(\widetilde{\nu}_f);\mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^*(P(\nu_f);\mathbb{Z}_2).
\end{aligned}$$

By Leray-Hirsch theorem, $H^*(P(\nu_f);\mathbb{Z}_2) \cong H^*(W_f;\mathbb{Z}_2) \otimes H^*(\mathbb{R}P^{k-1};\mathbb{Z}_2)$ (as $\mathbb{Z}_2$-vector spaces), which implies $\dim_{\mathbb{Z}_2} H^*(P(\nu_f);\mathbb{Z}_2) = k \cdot \dim_{\mathbb{Z}_2} H^*(W_f;\mathbb{Z}_2)$. So

$$\dim_{\mathbb{Z}_2} H^*(\widetilde{W}^f;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2) + (k-1) \cdot \dim_{\mathbb{Z}_2} H^*(W_f;\mathbb{Z}_2). \tag{17}$$

If W is equivariantly formal, then W is locally standard and so Q is a nice manifold with corners. It is easy to see

$$\#\text{vertices of } Q^f = \#\text{vertices of } Q + (k-1) \cdot \#\text{vertices of } f.$$

Since the fixed point set W^G ($G = \mathbb{Z}_2^n$) corresponds to the vertex set of Q which is discrete, the number of fixed points of the G-action satisfies

$$|(\widetilde{W}^f)^G| = |W^G| + (k-1) \cdot |(W_f)^G|. \tag{18}$$

By Proposition 3.4, W_f is also equivariantly formal. So by Theorem 1.1,

$$\dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2) = |W^G|, \quad \dim_{\mathbb{Z}_2} H^*(W_f;\mathbb{Z}_2) = |(W_f)^G|.$$

It follows from (17) and (18) that $|(\widetilde{W}^f)^G| = \dim_{\mathbb{Z}_2} H^*(\widetilde{W}^f;\mathbb{Z}_2)$. So we deduce from Theorem 1.1 that $\widetilde{W}^f$ is equivariantly formal.

Conversely, if $\widetilde{W}^f$ is equivariantly formal, we have

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2) &\overset{17}{=} \dim_{\mathbb{Z}_2} H^*(\widetilde{W}^f;\mathbb{Z}_2) - (k-1)\cdot \dim_{\mathbb{Z}_2} H^*(W_f;\mathbb{Z}_2) \\ \text{(by Theorem 1.1)} &\leq |(\widetilde{W}^f)^G| - (k-1)\cdot |(W_f)^G| \overset{17}{=} |W^G| = \dim_{\mathbb{Z}_2} H^*(W^G;\mathbb{Z}_2). \end{aligned}$$

But by Theorem 1.1, $\dim_{\mathbb{Z}_2} H^*(W^G;\mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2)$. So we must have $\dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W^G;\mathbb{Z}_2)$, which implies that W is equivariantly formal. The proposition is proved. □

The following lemma is parallel to [27, Lemma 9.1]. Its proof is almost identical to the proof in [27], hence omitted.

Lemma 3.13 *Let Q be a nice manifold with corners and f be a proper face of Q. Then, Q^f is mod 2 face-acyclic if and only if so is Q.*

4 Equivariantly Formal 2-Torus Manifolds with Mod 2 Cohomology Generated by Degree-One Part

In our study of equivariantly formal 2-torus manifolds, those manifolds whose mod 2 cohomology rings are generated by their degree-one part are of special importance. We will see in Sect. 5 that the study of general equivariantly formal 2-torus manifolds can be reduced to the study of these special 2-torus manifolds by a sequence of real blow-ups along facial submanifolds.

The following lemma is parallel to [27, Lemma 2.3].

Lemma 4.1 *Suppose there is an equivariantly formal $\mathbb{Z}_2^r$-action on a compact manifold M where the cohomology ring $H^*(M;\mathbb{Z}_2)$ is generated by its degree-one part. Then, for any subgroup H of $\mathbb{Z}_2^r$ and every connected component N of M^H, the homomorphism $i^*: H^*(M;\mathbb{Z}_2) \to H^*(N;\mathbb{Z}_2)$ is surjective where $i: N \hookrightarrow M$ is the inclusion. In particular, $H^*(N;\mathbb{Z}_2)$ is also generated by its degree-one part.*

Proof First, we assume $H \cong \mathbb{Z}_2$. We have a commutative diagram as follows:

$$\begin{array}{ccc} H^*_H(M;\mathbb{Z}_2) & \xrightarrow{\widehat{i}^*_H} & H^*_H(N;\mathbb{Z}_2) \\ \downarrow{\scriptstyle \iota^*_M} & & \downarrow{\scriptstyle \iota^*_N} \\ H^*(M;\mathbb{Z}_2) & \xrightarrow{i^*} & H^*(N;\mathbb{Z}_2) \end{array} \tag{19}$$

where $H_H^*(N;\mathbb{Z}_2) \cong H^*(N;\mathbb{Z}_2) \otimes H^*(BH;\mathbb{Z}_2)$ and $\widehat{i}_H^*$ is the homomorphism on equivariant cohomology induced by i. By our assumption, both ι_M^* and ι_N^* are surjective. The following argument is parallel to the proof of [27, Lemma 2.3].

By [7, Theorem VII.1.5], the inclusion $M^H \hookrightarrow M$ induces an isomorphism $H_H^k(M;\mathbb{Z}_2) \to H_H^k(M^H;\mathbb{Z}_2)$ for sufficiently large k, which implies that

$$\widehat{i}_H^* : H_H^k(M;\mathbb{Z}_2) \to H_H^k(N;\mathbb{Z}_2)$$

is surjective if k is sufficiently large.

Let $v_1, \cdots, v_d \in H^1(M;\mathbb{Z}_2)$ be a set of multiplicative generators of $H^*(M;\mathbb{Z}_2)$. For each $1 \le l \le d$, let $\widehat{v}_l$ be a lift of v_l in $H_H^*(M;\mathbb{Z}_2)$ and $w_l := i^*(v_l) \in H^1(N;\mathbb{Z}_2)$. Let t be a generator of $H^1(BH;\mathbb{Z}_2) \cong \mathbb{Z}_2$. By the commutativity of the above diagram (19),

$$\widehat{i}^*(\widehat{v}_l) = b_l t + w_l \ \text{ for some } \ b_l \in \mathbb{Z}_2.$$

Then, for an arbitrary element $\zeta \in H^*(N;\mathbb{Z}_2)$, there exists a large enough integer $q \in \mathbb{Z}$ and a polynomial $P(x_1, \cdots, x_d)$ such that

$$\widehat{i}^*\big(P(\widehat{v}_1, \cdots, \widehat{v}_d)\big) = \zeta \otimes t^q.$$

On the other hand, we have

$$\widehat{i}^*\big(P(\widehat{v}_1, \cdots, \widehat{v}_d)\big) = P(b_1 t + w_1, \cdots, b_d t + w_d) = \sum_{k \ge 0} P_k(w_1, \cdots, w_d) \otimes t^k$$

for some polynomials P_k, $k \ge 0$. Hence $\zeta = P_q(w_1, \cdots, w_d) = i^*(P(v_1, \cdots, v_d))$. Therefore, i^* is surjective and $H^*(N;\mathbb{Z}_2)$ is generated by $w_1, \cdots, w_d \in H^1(N;\mathbb{Z}_2)$.

For the general case, suppose $H \cong \mathbb{Z}_2^s$, $1 \le s \le r$. Then, we have a sequence:

$$\{0\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_s = H$$

where $H_l \cong \mathbb{Z}_2^l$ for each $0 \le l \le s$. Moreover, we have

$$M^H = \big((M^{H_1})^{H_2/H_1}) \cdots\big)^{H_s/H_{s-1}}, \ H_l/H_{l-1} \cong \mathbb{Z}_2, \ l = 1, \cdots, s.$$

Repeating the above argument for each H_l/H_{l-1} proves the lemma. □

The following lemma is parallel to [27, Lemma 3.4].

Lemma 4.2 *Let W be an equivariantly formal 2-torus manifold whose cohomology ring $H^*(W;\mathbb{Z}_2)$ is generated by its degree-one part. Then, all non-empty multiple intersections of the characteristic submanifolds of W are equivariantly formal 2-torus manifolds whose mod 2 cohomology rings are generated by their degree-one part as well.*

Proof Let $F_1, \cdots, F_m$ be all the facets of Q and $G = \mathbb{Z}_2^n$ where $n = \dim(W)$. In the following, we use the notations defined in Sect. 2.3. First of all, since the characteristic submanifold W_i is a connected component of the fixed point set X^{G_i}, Lemma 4.1 implies that the restriction $H^*(W;\mathbb{Z}_2) \to H^*(W_i;\mathbb{Z}_2)$ is surjective. So the G-action on W_i is equivariantly formal (by Proposition 3.4). Then, we have

$$H^*_G(W;\mathbb{Z}_2) \cong H^*(W;\mathbb{Z}_2) \otimes H^*(BG;\mathbb{Z}_2),$$
$$H^*_G(W_i;\mathbb{Z}_2) \cong H^*(W_i;\mathbb{Z}_2) \otimes H^*(BG;\mathbb{Z}_2).$$

It follows that the restriction $H^*_G(W;\mathbb{Z}_2) \to H^*_G(W_i;\mathbb{Z}_2)$ is also surjective. In addition, by using Proposition 2.5 (i) and a completely parallel argument to the proof of [26, Prop. 3.4(2)], we can prove the following claim:

Claim: $H^*_G(W;\mathbb{Z}_2)$ is generated as a ring by all the equivariant Thom classes $\tau_1, \cdots, \tau_m$ of the normal bundles of the characteristic submanifolds $W_1, \cdots, W_m$.

When $W_{j_1} \cap \cdots \cap W_{j_s} = \varnothing$, $\tau_{j_1} \cdots \tau_{j_s}$ clearly vanishes. So the above claim implies that for any $k \geq 0$, $H^k_G(W;\mathbb{Z}_2)$ is additively generated by the monomials $\tau^{k_1}_{j_1} \cdots \tau^{k_s}_{j_s}$ such that $W_{j_1} \cap \cdots \cap W_{j_s} \neq \varnothing$ and $k_1 + \cdots + k_s = k$.

Let N be a connected component of $W_{i_1} \cap \cdots \cap W_{i_k}$, $1 \leq k \leq n$. Then, N is the facial submanifold W_f over some codimension-k face f of Q. So by Lemma 4.1, N is an equivariantly formal 2-torus manifold whose cohomology ring $H^*(N;\mathbb{Z}_2)$ is generated by its degree-one part. Moreover, by a completely parallel argument to the proof of [27, Lemma 3.4], we can show that N is the only connected component of $W_{i_1} \cap \cdots \cap W_{i_k}$ from the above discussion of $H^k_G(W;\mathbb{Z}_2)$. The lemma is proved. □

The following proposition is parallel to [27, Lemma 8.2 (2)].

Proposition 4.3 *Suppose W is an n-dimensional equivariantly formal 2-torus manifold with orbit space Q and the cohomology ring $H^*(W;\mathbb{Z}_2)$ is generated by its degree-one part. Then, the geometrical realization $|\mathcal{P}_Q|$ of the face poset $\mathcal{P}_Q$ of Q is a Gorenstein* simplicial complex over $\mathbb{Z}_2$. In particular, $\mathbb{Z}_2[\mathcal{P}_Q] = \mathbb{Z}[Q]$ is Cohen-Macaulay and $H^*(|\mathcal{P}_Q|;\mathbb{Z}_2) \cong H^*(S^{n-1};\mathbb{Z}_2)$.*

Proof By Lemma 4.2, all non-empty multiple intersections of the characteristic submanifolds of W are connected. This implies that $|\mathcal{P}_Q|$ is a simplicial complex. Moreover, by [30, II 5.1(d)], it is enough to verify the following three conditions to prove that $|\mathcal{P}_Q|$ is Gorenstein* over $\mathbb{Z}_2$:

(a) $\mathbb{Z}_2[\mathcal{P}_Q]$ is Cohen-Macaulay;
(b) Every $(n-2)$-simplex in $\mathcal{P}_Q$ is contained in exactly two $(n-1)$-simplices;
(c) $\chi(\mathcal{P}_Q) = \chi(S^{n-1})$.

Since W is equivariantly formal, $H^*_G(W;\mathbb{Z}_2)$ is a free $H^*(BG;\mathbb{Z}_2)$-module and $\mathbb{Z}_2[\mathcal{P}_Q] = \mathbb{Z}_2[Q]$ is isomorphic to $H^*_G(W;\mathbb{Z}_2)$ (by Corollary 3.10) where $G = \mathbb{Z}^n_2$. This implies (a).

Note that each $(n-2)$-simplex of $\mathcal{P}_Q$ corresponds to a non-empty intersection of $n-1$ characteristic submanifolds of W. The latter intersection is an equivariantly formal 1-manifold by Lemma 4.2, so it is a circle with exactly two G-fixed points. This implies (b).

The proof of (c) is completely parallel to [27, Lemma 8.2 (2)], so we leave it to the reader. The proposition is proved. □

Using the above proposition and the lemmas from Sect. 3, we obtain the following theorem that is parallel to [27, Theorem 7.7].

Theorem 4.4 *Let W be a 2-torus manifold whose orbit space is Q. Then, W is equivariantly formal and the cohomology ring $H^*(W;\mathbb{Z}_2)$ is generated by its degree-one part if and only if the following three conditions are satisfied:*

(a) $H^*_G(W;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[Q]=\mathbb{Z}_2[\mathcal{P}_Q]$ as a graded ring.
(b) $\mathbb{Z}_2[Q]$ is Cohen-Macaulay.
(c) $|\mathcal{P}_Q|$ is a simplicial complex.

Proof The argument is completely parallel to the proof of [27, Theorem 7.7]. We only need to replace T^n by $\mathbb{Z}_n^2$ and $\mathbb{Q}$-coefficients by $\mathbb{Z}_2$-coefficients to obtain our proof here. □

5 Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Our proof follows the proof of [27, Theorem 8.3, Theorem 9.3] almost step by step, while some arguments for 2-torus manifolds here are simpler than those for torus manifolds in [27].

5.1 Equivariant Cohomology of the Canonical Model

Let Q be a connected compact smooth nice n-manifold with corners. We call any function $\lambda:\mathcal{F}(Q)\to\mathbb{Z}_2^n$ that satisfies the linear independence relation in Sect. 2.1 a *characteristic function* on Q. By the same gluing rule in (4), we can obtain a space $M_Q(\lambda)$ from any characteristic function λ on Q, called the *canonical model* determined by (Q,λ). It is easy to see that $M_Q(\lambda)$ is a 2-torus manifold of dimension n.

Let $Q^\vee$ denote the cone of the geometrical realization of the *order complex* $\mathrm{ord}(\overline{\mathcal{P}}_Q)$ of $\overline{\mathcal{P}}_Q=\mathcal{P}_Q-\{\hat{0}\}$. So topologically, $Q^\vee$ is homeomorphic to $\mathrm{Cone}(|\mathcal{P}_Q|)$. Moreover, $Q^\vee$ is a "space with faces" (see Davis [13, Sec. 6]) where each proper face f of Q determines a unique "face" $f^\vee$ of $Q^\vee$ that is the geometrical realization of the order complex of the poset $\{f'\mid f'\subseteq f\}$. More precisely, $f^\vee$ consists of all simplices of the form $f'_k\subsetneq\cdots\subsetneq f'_1\subsetneq f'_0=f$ in $\mathrm{ord}(\overline{\mathcal{P}}_Q)$. The "boundary" of $Q^\vee$, denoted by $\partial Q^\vee$, is $\mathrm{ord}(\overline{\mathcal{P}}_Q)$ which is homeomorphic to $|\mathcal{P}_Q|$. So we have homeomorphisms:

$$\partial Q^\vee\cong|\mathcal{P}_Q|,\quad Q^\vee\cong\mathrm{Cone}(|\mathcal{P}_Q|). \tag{20}$$

Remark 5.1 When $|\mathcal{P}_Q|$ is a simplicial complex, the space $Q^\vee$ with the face decomposition was called in [14, p. 428] a *simple polyhedral complex*.

Suppose $F_1,\cdots,F_m$ are all the facets of Q. Let $\mathcal{F}(Q^\vee)=\{F_1^\vee,\cdots,F_m^\vee\}$. Then, any characteristic function $\lambda:\mathcal{F}(Q)\to\mathbb{Z}_2^n$ induces a map $\lambda^\vee:\mathcal{F}(Q^\vee)\to\mathbb{Z}_2^n$ where $\lambda^\vee(F_i^\vee)=\lambda(F_i)$, $1\le i\le m$. Then, by the same gluing rule in (4), we obtain a space $M_{Q^\vee}(\lambda^\vee)$ with a canonical $\mathbb{Z}_2^n$-action. By the same argument as in the proof of [27, Proposition 5.14], we can prove the following.

Proposition 5.2 *There exists a continuous map $\phi : Q \to Q^\vee$ which preserves the face structure and induces an equivariant continuous map*

$$\Phi : M_Q(\lambda) \to M_{Q^\vee}(\lambda^\vee).$$

Here $\phi : Q \to Q^\vee$ is constructed inductively, starting from an identification of vertices and extending the map on each higher-dimensional face by a degree-one map. Since every face $f^\vee$ of $Q^\vee$ is a cone, there are no obstructions to such extensions.

In addition, by a similar argument to that in [14, Theorem 4.8], we can obtain the following result.

Proposition 5.3 *$H_G^*(M_{Q^\vee}(\lambda^\vee); \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[Q]$ where $G = \mathbb{Z}_2^n$.*

On the other hand, $H_G^*(M_Q(\lambda); \mathbb{Z}_2)$ could be much more complicated. Indeed, it is shown in [31, Theorem 1.7] that $H_G^*(M_Q(\lambda); \mathbb{Z}_2)$ isomorphic to the so called *topological face ring* of Q over $\mathbb{Z}_2$ which involves the mod 2 cohomology rings of all the faces of Q.

5.2 Proof of Theorem 1.3 (ii)

Proof We first prove the "if" part. Let Q be an n-dimensional mod 2 homology polytope and $G = \mathbb{Z}_2^n$. Since $H^1(Q; \mathbb{Z}_2) = 0$ and W is locally standard, the principal G-bundle ξ_W determined by W is a trivial G-bundle over Q. Then, by [25, Lemma 3.1], W is equivariantly homeomorphic to the canonical model $M_Q(\lambda_W)$ (see (4)). So by Proposition 5.3, there exists an equivariant continuous map

$$\Phi : W = M_Q(\lambda_W) \to M_{Q^\vee}(\lambda_W^\vee) := W^\vee.$$

Let $\pi : W \to Q$ and $\pi^\vee : W^\vee \to Q^\vee$ be the projections, respectively. Let $F_1, \cdots, F_m$ be all the facets of Q. Since Q is a mod 2 homology polytope, so are $F_1, \cdots, F_m$. For brevity, let

$$W_i = \pi^{-1}(F_i), \quad W_i^\vee = (\pi^\vee)^{-1}(F_i^\vee), \ 1 \le i \le m.$$

It is easy to see that the $\mathbb{Z}_2^n$-actions on $W \backslash \bigcup_i W_i$ and $W^\vee \backslash \bigcup_i W_i^\vee$ are both free. Then, we have

$$H_G^*\Big(W, \bigcup_i W_i; \mathbb{Z}_2\Big) \cong H^*(Q, \partial Q; \mathbb{Z}_2), \ H_G^*\Big(W^\vee, \bigcup_i W_i^\vee; \mathbb{Z}_2\Big) \cong H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2).$$

So $\Phi : W \to W^\vee$ induces a map between the following two exact sequences:

$$\begin{array}{ccccccc} \longrightarrow H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) & \longrightarrow & H^*_G(W^\vee; \mathbb{Z}_2) & \longrightarrow & H^*_G\big(\bigcup_i W_i^\vee; \mathbb{Z}_2\big) & \longrightarrow \cdots \\ \downarrow \phi^* & & \downarrow \Phi^* & & \downarrow \Phi^* & \\ \longrightarrow H^*(Q, \partial Q; \mathbb{Z}_2) & \longrightarrow & H^*_G(W; \mathbb{Z}_2) & \longrightarrow & H^*_G\big(\bigcup_i W_i; \mathbb{Z}_2\big) & \longrightarrow \cdots \end{array} \tag{21}$$

Each W_i is a 2-torus manifold over the homology polytope F_i. So using induction and a Mayer-Vietoris argument, we may assume that in the diagram (21), $\Phi^* : H^*_G(\bigcup_i W_i^\vee; \mathbb{Z}_2) \to H^*_G(\bigcup_i W_i; \mathbb{Z}_2)$ is an isomorphism.

By Proposition 2.9, $H^*(|\mathcal{P}_Q|; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2)$. Then, by (20), we obtain

$$H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2).$$

We also have $H^*(Q, \partial Q; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2)$ since Q is an n-dimensional mod 2 homology polytope. By the construction of ϕ, it is easy to see that the homomorphism $\phi^* : H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2)$ is an isomorphism. Then, by applying the five-lemma to the diagram (21), we can deduce that $\Phi^* : H^*_G(W^\vee; \mathbb{Z}_2) \to H^*_G(W; \mathbb{Z}_2)$ is also an isomorphism. So by Proposition 5.3, $H^*_G(W; \mathbb{Z}_2) \cong \mathbb{Z}_2[Q]$.

Besides, we also know that $\mathbb{Z}_2[Q]$ is Cohen-Macaulay by Proposition 2.9. Then, since $|\mathcal{P}_Q|$ is a simplicial complex, all the three conditions in Theorem 4.4 are satisfied. Hence W is equivariantly formal and $H^*(W; \mathbb{Z}_2)$ is generated by its degree-one part as a ring. The "if" part is proved.

Next, we prove the "only if" part. By the assumption on W and Lemma 4.2, all non-empty multiple intersections of characteristic submanifolds of W are connected and their cohomology rings are generated by their degree-one elements. So we may assume by induction that all the proper faces of Q are mod 2 homology polytopes. In particular, the proper faces of Q are all mod 2 acyclic. From these assumptions, we need to prove that Q itself is mod 2 acyclic.

By Proposition 4.3, $|\mathcal{P}_Q|$ is a simplicial complex. So $|\mathcal{P}_Q|$ is the nerve simplicial complex of the cover of ∂Q by the facets of Q. By a Mayer-Vietoris sequence argument, we can deduce that $H^*(\partial Q; \mathbb{Z}_2) \cong H^*(|\mathcal{P}_Q|; \mathbb{Z}_2)$. This together with Proposition 4.3 shows that

$$H^*(\partial Q; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2). \tag{22}$$

Claim: $H^1(Q; \mathbb{Z}_2) = 0$.

Since W is equivariantly formal, $H^*_G(W; \mathbb{Z}_2)$ is a free $H^*(BG; \mathbb{Z}_2)$-module. On the other hand, $H^*(Q, \partial Q; \mathbb{Z}_2)$ is finitely generated over $\mathbb{Z}_2$ since Q is compact. So $H^*(Q, \partial Q; \mathbb{Z}_2)$ is a torsion $H^*(BG; \mathbb{Z}_2)$-module. It follows that the whole bottom row in the diagram (21) splits into short exact sequences:

$$0 \to H^k_G(W; \mathbb{Z}_2) \to H^k_G\Big(\bigcup_i W_i; \mathbb{Z}_2\Big) \to H^{k+1}(Q, \partial Q; \mathbb{Z}_2) \to 0, \ k \geq 0. \tag{23}$$

Take $k = 0$ above, we clearly have $H^0_G(W; \mathbb{Z}_2) \cong H^0_G\Big(\bigcup_i W_i; \mathbb{Z}_2\Big) \cong \mathbb{Z}_2$. This implies $H^1(Q, \partial Q; \mathbb{Z}_2) = 0$. So in the following exact sequence,

$$\cdots \to H^1(Q, \partial Q; \mathbb{Z}_2) \to H^1(Q; \mathbb{Z}_2) \to H^1(\partial Q; \mathbb{Z}_2) \to \cdots,$$

$H^1(Q; \mathbb{Z}_2)$ is mapped injectively into $H^1(\partial Q; \mathbb{Z}_2) \cong H^1(S^{n-1}; \mathbb{Z}_2)$. Note that if $n = 1$, the claim is trivial. When $n = 2$, we have $\partial Q = S^1$ and $H^1(Q; \mathbb{Z}_2) = 0$ or $\mathbb{Z}_2$. But by the classification of compact surfaces, the latter case is impossible. When $n \geq 3$, we have $H^1(\partial Q; \mathbb{Z}_2) = 0$, so $H^1(Q; \mathbb{Z}_2) = 0$. The claim is proved.

Now since $H^1(Q; \mathbb{Z}_2) = 0$, by the above proof of the "if" part, there exists an equivariant homeomorphism Φ from W to the canonical model $M_Q(\lambda_W)$. In addition, by (20) and Proposition 4.3, we have

$$H^*(\partial Q^\vee; \mathbb{Z}_2) \cong H^*(|\mathcal{P}_Q|; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2).$$

So we have an isomorphism

$$H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2). \tag{24}$$

Then, by the construction of ϕ, the map $\phi^* : H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2)$ is an isomorphism in degree n (since Q is connected) and thus is injective in all degrees. So by an extended version of the 5-lemma, we can deduce that in the diagram (21) the map $\Phi^* : H^*_G(W^\vee; \mathbb{Z}_2) \to H^*_G(W; \mathbb{Z}_2)$ is injective. Moreover,

- $H^*_G(W^\vee; \mathbb{Z}_2) = H^*_G(M_{Q^\vee}(\lambda^\vee_W); \mathbb{Z}_2) \cong \mathbb{Z}_2[Q]$ by Proposition 5.3, and
- $\mathbb{Z}_2[Q] \cong H^*_G(W; \mathbb{Z}_2)$ by Corollary 3.10.

So $H^*_G(W^\vee; \mathbb{Z}_2)$ and $H^*_G(W; \mathbb{Z}_2)$ have the same dimension over $\mathbb{Z}_2$ in each degree. Therefore, the monomorphism $\Phi^* : H^*_G(W^\vee; \mathbb{Z}_2) \to H^*_G(W; \mathbb{Z}_2)$ is actually an isomorphism. Then, by the 5-lemma again, we can deduce from the diagram (21) that $\phi^* : H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2)$ is an isomorphism. So by (24),

$$H^*(Q, \partial Q; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2)$$

which implies that Q is mod 2 acyclic by Poincaré-Lefschetz duality. This finishes the proof. □

5.3 Proof of Theorem 1.3 (i)

Proof We can reduce Theorem 1.3 (i) to Theorem 1.3 (ii) by real blow-ups of W along sufficient many facial submanifolds, which corresponds to doing some barycentric subdivisions of the face poset $\mathcal{P}_Q$ of Q (see Fig. 2). Indeed, after doing enough barycentric subdivisions to $\mathcal{P}_Q$, we can turn $|\mathcal{P}_Q|$ into a simplicial complex. Let $\widehat{W}$ be the 2-torus manifold obtained after these real blow-ups on W and $\widehat{Q}$ be its orbit space (with $|\mathcal{P}_{\widehat{Q}}|$ being a simplicial complex).

Fact-1: $\widehat{W}$ is equivariantly formal if and only if so is W (by Proposition 3.12).

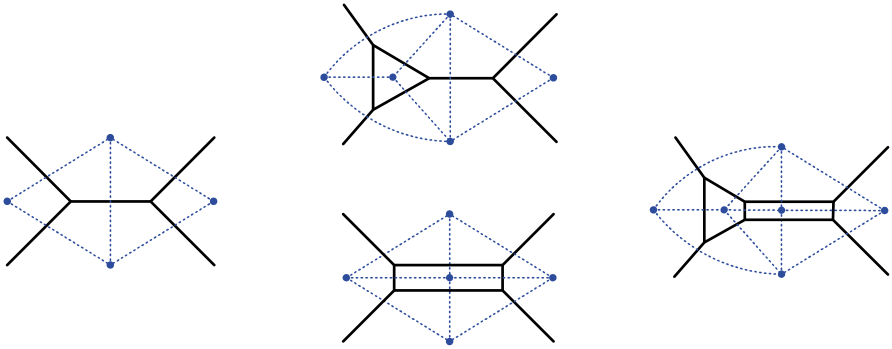

Fig. 2 Cutting a vertex and an edge

Fact-2: $\widehat{Q}$ is mod 2 face-acyclic if and only if so is Q (by Lemma 3.13).

We first prove the "if" part. Suppose W is locally standard and Q is mod 2 face-acyclic. Then, $\widehat{W}$ is also locally standard and $\widehat{Q}$ is a mod 2 homology polytope by Fact-2. So by Theorem 1.3 (ii), $\widehat{W}$ is equivariantly formal, then so is W.

Next, we prove the "only if" part. If W is equivariantly formal, then so is $\widehat{W}$, and W is locally standard by Theorem 3.3. So by Corollary 3.10, we have a graded ring isomorphism $H^*_G(\widehat{W};\mathbb{Z}_2) \cong \mathbb{Z}_2[\widehat{Q}]$. Moreover, since $|\mathcal{P}_{\widehat{Q}}|$ is a simplicial complex, $\mathbb{Z}_2[\widehat{Q}]$ is generated by its degree-one elements, then so is $H^*_G(\widehat{W};\mathbb{Z}_2)$. In addition, since $\iota^*_{\widehat{W}} : H^*_G(\widehat{W};\mathbb{Z}_2) \to H^*(\widehat{W};\mathbb{Z}_2)$ is surjective, $H^*(\widehat{W};\mathbb{Z}_2)$ is also generated by its degree-one elements. Then, by Theorem 1.3 (ii), $\widehat{Q}$ is a mod 2 homology polytope. So by Fact-2, Q is mod 2 face-acyclic. □

5.4 Proof of Theorem 1.5

Proof We first prove the "if" part. Assume that there exists a regular m-involution τ on W. By definition the fixed point set W^τ of τ is discrete, then so is $W^{\mathbb{Z}_2^n} \subseteq W^\tau$. This implies that Q must have vertices. Let p be a vertex of Q and let $F_1, \cdots, F_n$ be all the facets containing p. By the property of λ_W,

$$e_1 = \lambda_W(F_1), \cdots, e_n = \lambda_W(F_n)$$

form a linear basis of $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$. Then, since the $\mathbb{Z}_2^n$-action on W is locally standard, it is easy to see that only when $g = e_1 + \cdots + e_n$ could the fixed point set W^{τ_g} be discrete. So we must have $\tau = \tau_{e_1+\cdots+e_n}$, and in particular

$$W^\tau = W^{\tau_{e_1+\cdots+e_n}} = W^{\mathbb{Z}_2^n}.$$

Hence

$$\dim_{\mathbb{Z}_2} H^*(W^{\mathbb{Z}_2^n};\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W^\tau;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W;\mathbb{Z}_2)$$

where the second "=" is due to the assumption that τ is an m-involution. So by Theorem 1.1, W is equivariantly formal. Then, Q is mod 2 face-acyclic by Theorem 1.3. In particular, every face of Q has a vertex and the 1-skeleton of Q is connected (by Lemma 2.2).

It remains to prove that the image of $\lambda_W : \mathcal{F}(Q) \to \mathbb{Z}_2^n$ is exactly $\{e_1, \cdots, e_n\}$. Indeed, take an edge e of Q whose vertices are p and p'. So the n facets of Q that contain p' are $F_1, \cdots, F_{i-1}, F_i', F_{i+1}, \cdots, F_n$ for some $1 \leq i \leq n$. Then, since $\tau_{e_1+\cdots+e_n}$ is an m-involution, we must have

$$\lambda_W(F_1) + \cdots + \lambda_W(F_{i-1}) + \lambda_W(F_i') + \lambda_W(F_{i+1}) + \cdots + \lambda_W(F_n) = e_1 + \cdots + e_n.$$

This implies $\lambda_W(F_i') = e_i$. Then, since the 1-skeleton of Q is connected and every facet F of Q contains a vertex, we can iterate the above argument to prove that every $\lambda_W(F)$ must take value in $\{e_1, \cdots, e_n\}$.

Next, we prove the "only if" part. Suppose Q is mod 2 face-acyclic and the values of the characteristic function λ_W of Q consist exactly of a linear basis $e_1, \cdots, e_n$ of $\mathbb{Z}_2^n$. By Theorem 1.3 (i), W is equivariantly formal. So we have

$$\dim_{\mathbb{Z}_2} H^*(W^{\mathbb{Z}_2^n}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) \ \text{(by Theorem 1.1)}.$$

On the other hand, our assumption on λ_W implies that the regular involution $\tau = \tau_{e_1+\cdots+e_n}$ satisfies $W^\tau = W^{\mathbb{Z}_2^n}$ which is a discrete set. Then, we have

$$\dim_{\mathbb{Z}_2} H^*(W^\tau; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W^{\mathbb{Z}_2^n}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2).$$

So τ is a regular m-involution on W by definition. The theorem is proved. □

Remark 5.4 If we do not assume a 2-torus manifold W to be locally standard, even if W admits a regular m-involution, W may not be equivariantly formal or locally standard. For example: let

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Define two involutions σ and σ' on S^2 by

$$\sigma(x_1, x_2, x_3) = (-x_1, -x_2, x_3), \quad \sigma'(x_1, x_2, x_3) = (x_1, x_2, -x_3).$$

It is easy to see that σ is an m-involution on S^2 with two isolated fixed points $(0, 0, 1)$ and $(0, 0, -1)$. But since the $\mathbb{Z}_2^2$-action on S^2 determined by σ and σ' has no global fixed point, it is not equivariantly formal. We can also directly check that this $\mathbb{Z}_2^2$-action on S^2 is not locally standard.

Finally, we propose some questions on weakly equivariantly formal 2-torus manifolds:

Question-3: Does there exist a weakly equivariantly formal 2-torus manifold which is not equivariantly formal?

Question-4: If a 2-torus manifold is weakly equivariantly formal, are there any restrictions on the topology and combinatorial structure of its orbit space?

Question-5: Whether or not a 2-torus manifold being weakly equivariantly formal is determined only by the topology and combinatorial structure of its orbit space?

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Declarations

Conflict of Interest The author declares no competing interests.

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