Transformation Groups
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Compatibility of Balanced and SKT Metrics on Two-Step Solvable Lie Groups

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Abstract
It has been conjectured by Fino and Vezzoni that a compact complex manifold admitting both a compatible SKT and a compatible balanced metric also admits a compatible Kähler metric. Using the shear construction and classification results for two-step solvable SKT Lie algebras from our previous work, we prove this conjecture for compact two-step solvmanifolds endowed with an invariant complex structure which is either (a) of pure type or (b) of dimension six. In contrast, we provide two counterexamples for a natural generalisation of this conjecture in the homogeneous invariant setting. As part of the work, we obtain further classification results for invariant SKT, balanced and Kähler structures on two-step solvable Lie groups. In particular, we give the full classification of left-invariant SKT structures on two-step solvable Lie groups in dimension six.

1 Introduction

Hermitian geometry has been a very active field of study for several decades. Historically, most focus has been on Kähler manifolds and many important results have been obtained for these manifolds, including several severe topological restrictions in the compact case. These topological restrictions show that most compact complex manifolds do not admit a Kähler structure and this has led to rising interest in generalisations of Kähler manifolds. In particular, two types of non-Kähler Hermitian manifolds (M, g, J, σ) have intensively been investigated, namely *strong Kähler*

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with torsion (SKT) and *balanced* manifolds, which are characterised by $dJ^*d\sigma = 0$ or $\delta_g\sigma = 0$, respectively. If M has real dimension $2n$, then the balanced condition is equivalent to $d(\sigma^{n-1}) = 0$.

Interest in SKT manifolds stems from various sources. First of all, these manifolds are precisely those almost Hermitian manifolds for which there exists a compatible connection ∇^B with totally skew-symmetric torsion T^B that, when considered as a three-form, is closed. It is because of this property that they occur in physics in the context of supersymmetric theories, see for example [19, 22, 32]. Secondly, any Hermitian conformal class on a compact complex surface contains an SKT metric [20], a property which is no longer true in higher dimensions. Moreover, any compact even-dimensional Lie group admits a left-invariant SKT structure [23, 31].

Balanced metrics are of interest since they occur naturally on several types of complex manifolds: for example, any unimodular complex Lie group [1] admits a compatible left-invariant balanced metric and any compact complex manifold which is bimeromorphic to a compact Kähler manifold admits a compatible balanced metric [2]. Furthermore, they are an important ingredient in the Strominger system from physics [9, 18].

The SKT and balanced conditions for a fixed Hermitian structure are known to be mutually exclusive in the sense that any compatible metric g on a complex manifold (M, J) which is both SKT and balanced has to be Kähler [3]. More generally, Fino and Vezzoni conjectured

Conjecture 1.1 ([14, Problem 3], [15, Conjecture]) *Any compact complex manifold (M, J) which admits a compatible SKT metric and admits a compatible balanced metric also admits a compatible Kähler metric.*

This conjecture has been confirmed in some special cases, including twistor spaces of compact anti-self-dual Riemannian manifolds [35], non-Kähler manifolds belonging to the Fujiki class $\mathcal{C}$ [7] and for left-invariant complex structures on compact semi-simple Lie groups [11, 28]. We will extend the latter result to all compact even-dimensional Lie groups in Theorem 4.1.

We define nil- and solvmanifolds to be manifolds of the form $M = \Gamma\backslash G$, with G nilpotent or solvable, respectively, and Γ a discrete subgroup. This definition includes non-compact examples such as G itself and by [24] is broad enough to cover all compact manifolds with a transitive action of a nilpotent group, but excludes certain examples with a solvable group action such as the non-orientable Klein bottle. One says that a complex structure J on $\Gamma\backslash G$ is invariant if it pulls-back to a left-invariant complex structure on G. Now Conjecture 1.1 has been proved for all compact nilmanifolds with invariant complex structure (by [15] combined with [5]), and for the following classes of compact solvmanifolds with invariant complex structure: six-dimensional solvmanifolds with holomorphically trivial canonical bundle [14], almost Abelian solvmanifolds [12], Oeljeklaus-Toma manifolds [27], regular complex structures on non-compact semi-simple Lie groups [21], and special types of invariant metrics on almost nilpotent solvmanifolds for which the associated Lie algebra has a nilradical with one-dimensional commutator [13].

Actually, compactness of $\Gamma\backslash G$ implies that G is unimodular, and in all of the above cases of $\Gamma\backslash G$ with invariant J the results are proved by just considering all left-invariant structures on G. This is possible since [10, 34] showed that once J is left-invariant, any compatible SKT or balanced metric on compact $\Gamma\backslash G$ may be averaged to a left-invariant metric of the same type. Hence, the following question has a positive answer for these Lie groups G.

Question 1.2 *Let G be a unimodular Lie group with a left-invariant complex structure J. If (G, J) admits a left-invariant compatible SKT metric and admits a left-invariant compatible balanced metric, does (G, J) also admit a left-invariant compatible Kähler metric?*

In this paper, we consider Question 1.2 for G two-step solvable. We have imposed that all the metrics considered be left-invariant, since any solvable group admits a Kähler metric, see Proposition 4.2, however this metric is not necessarily invariant.

We will give two examples of two-step solvable Lie groups to show the condition of unimodularity is necessary. In particular, Examples 4.4 and 4.16 are the first known examples of Lie groups with left-invariant complex structure admitting both left-invariant SKT metrics and left-invariant balanced metrics, without admitting left-invariant Kähler metrics. As these examples are not unimodular, they do not give counterexamples to Conjecture 1.1.

In contrast, we give a positive answer to Question 1.2 for all two-step solvable Lie groups G endowed with an invariant complex structure J in the following situations: (a) J is of *pure type*, (b) G is of dimension 6. Here "pure type" means one of three summands in a natural decomposition of the Lie algebra $\mathfrak{g}$ vanishes. Writing $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ for the derived algebra, we have the following pure types: (I) $\mathfrak{g}' \cap J\mathfrak{g}' = 0$, (II) $\mathfrak{g}' = J\mathfrak{g}'$, (III) $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$.

Our approach, is to build on our study [17] of SKT structures on two-step solvable Lie groups. The individual cases are proved in Theorems 4.3, 4.10 and 4.14 for the three pure types. For the six-dimensional case we first complete the classification of SKT structures on two-step solvable Lie groups in Theorem 4.7 and then apply this to Question 1.2 in Theorem 4.17.

From the remarks on averaging above, we then get

Main result *Let $\Gamma\backslash G$ be a compact two-step solvmanifold and with an invariant complex structure J. Then Conjecture 1.1 holds if either G is six-dimensional, or $(\Gamma\backslash G, J)$ is of pure type.*

The paper is organised as follows. Definitions from Hermitian geometry, the notions of pure type, our approach to two-step solvable Lie algebras via the shear construction, and notation for concrete Lie algebras are summarised in Section 2. We then derive some general results for two-step solvable Lie algebras, in the case that they have either a balanced structure Section 3.1, or a Kähler structure Section 3.2, in the latter case obtaining more detailed information than the general structural results of [8]. Then in Section 4, we prove the main theorems of the paper, as described above. One consequence is an explicit list of two-step solvable Kähler Lie algebras in dimension six, see Corollary 4.21.

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2 Preliminaries

2.1 Non-Kähler Hermitian Geometry

First, we recall the basic definitions. We write (M, J) for a complex manifold, so J is an *integrable* complex structure. A metric g is compatible with J if $g(J\,\cdot\,, J\,\cdot\,) = g(\,\cdot\,, \,\cdot\,)$, and then the triple (M, g, J) is called a Hermitian structure. We write $\sigma := g(J\,\cdot\,, \,\cdot\,)$ for the fundamental two-form.

Definition 2.1 A Hermitian manifold (M, g, J) of dimension $2n$ is

(i) *Kähler* if $d\sigma = 0$,
(ii) *balanced* if $d(\sigma^{n-1}) = 0$,
(iii) *strong Kähler with torsion (SKT)* manifold if $dJ^* d\sigma = 0$.

The following result is given in [3, Remark 1].

Proposition 2.2 *Let (M, g, J) be a Hermitian manifold which is both balanced and SKT. Then (M, g, J) is a Kähler manifold.*

Next, consider a simply connected Lie group G which admits a cocompact discrete subgroup Γ and consider the compact manifold $M := \Gamma\backslash G$. Any left-invariant tensor field on G may be pushed down to a tensor field on M. The resulting tensor fields on M are said to be *invariant*.

By using averaging one has the following result of [10] and [34].

Proposition 2.3 *Suppose $M := \Gamma\backslash G$ is a compact manifold that is the quotient of a simply connected Lie group G by a discrete subgroup Γ.*

Suppose J is an invariant complex structure on M. If (M, J) admits a compatible metric that is balanced or SKT, then it also admits a compatible invariant balanced or SKT metric, respectively.

Now consider left-invariant structures on the simply connected Lie group G directly and identify them with the corresponding structures on the associated Lie algebra $\mathfrak{g}$. We may then also speak of *Hermitian*, *balanced* or *SKT* Lie algebras. Throughout we will assume $\mathfrak{g} \neq 0$ and write $\dim \mathfrak{g} = 2n$ for the dimension of $\mathfrak{g}$ over $\mathbb{R}$.

Definition 2.4 We say a complex Lie algebra $(\mathfrak{g}, J)$ with derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is of

(1) *pure type I* if $\mathfrak{g}'$ is totally real, i.e. $\mathfrak{g}' \cap J\mathfrak{g}' = 0$,

(2) *pure type II* if $\mathfrak{g}'$ is complex, i.e. $\mathfrak{g}' = J\mathfrak{g}'$, and
(3) *pure type III* if $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$.

If one of these conditions hold, we simply say that $(\mathfrak{g}, J)$ is of *pure type*.

When there is a compatible Hermitian metric g, we introduce the following vector subspaces. We set ${\mathfrak{g}'}_J = \mathfrak{g}' \cap J\mathfrak{g}'$ to be the maximal complex subspace of the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Moreover, we let ${\mathfrak{g}'}_r$ be the orthogonal complement of ${\mathfrak{g}'}_J$ in $\mathfrak{g}'$ and then define $V_r := {\mathfrak{g}'}_r \oplus J{\mathfrak{g}'}_r$. Note that this direct sum is not orthogonal in general. Observe that now

$$\mathfrak{g}' + J\mathfrak{g}' = {\mathfrak{g}'}_J \oplus V_r$$

and define V_J to be the orthogonal complement of $\mathfrak{g}' + J\mathfrak{g}'$ in $\mathfrak{g}$. We then have a vector space orthogonal direct sum decomposition

$$\mathfrak{g} = {\mathfrak{g}'}_J \oplus V_r \oplus V_J. \tag{2.1}$$

of $\mathfrak{g}$ by spaces that are preserved by J. We define $s, r, \ell \in \mathbb{N}$ by

$$2s := \dim({\mathfrak{g}'}_J), \qquad 2r := \dim(V_r), \qquad 2\ell := \dim(V_J)$$

and use this notation throughout the article. Note that so $s + r + \ell = n$ and that these numbers depend only on $(\mathfrak{g}, J)$, and not on the metric g, since $2r = \dim(V_r) = \dim(\mathfrak{g}' + J\mathfrak{g}') - \dim({\mathfrak{g}'}_J) = \dim(\mathfrak{g}' + J\mathfrak{g}') - 2s$ and $\ell = n - r - s$.

Then we have $(\mathfrak{g}, J)$ is of pure type I, II or III, if ${\mathfrak{g}'}_J = 0$, $V_r = 0$ or $V_J = 0$, respectively. This is equivalent to there being at most two non-zero summands in (2.1). Note that non-zero Abelian algebras can not be of pure type III.

2.2 Complex Shears of $\mathbb{R}^{2n}$

One can construct every two-step solvable algebra as a shear of the Abelian Lie algebra $\mathbb{R}^{2n}$. Although the shear construction is defined for arbitrary manifolds, we will just need the version for Lie groups and algebras as presented in [17]. The motivating example is as follows. Let H, P, K be simply connected Lie groups with $\dim(H) = \dim(K)$ and whose associated Lie algebras are related by surjective Lie algebra homomorphisms $\mathfrak{h} \leftarrow \mathfrak{p} \rightarrow \mathfrak{k}$ with Abelian kernels. We then have two Abelian Lie algebras $\hat{\mathfrak{a}}_P, \mathfrak{a}_P$ of the same dimension, and two Lie algebra extensions of the form

$$\mathfrak{h} \twoheadleftarrow \mathfrak{p} \hookleftarrow \mathfrak{a}_P \quad \text{and} \quad \hat{\mathfrak{a}}_P \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{k}.$$

The Lie algebra $\mathfrak{p}$ and the *shear algebra* $\mathfrak{k}$ may constructed from certain "shear data" on $\mathfrak{h}$: two Abelian Lie algebras $\mathfrak{a}_H$ and $\mathfrak{a}_P$, a Lie algebra monomorphism $\xi\colon \mathfrak{a}_H \to \mathfrak{h}$, a two-form $\omega \in \Lambda^2\mathfrak{h}^* \otimes \mathfrak{a}_P$ on $\mathfrak{h}$ with values in $\mathfrak{a}_P$, a representation $\eta \in \mathfrak{h}^* \otimes \mathrm{Hom}(\mathfrak{a}_P)$ and a Lie algebra isomorphism $a\colon \mathfrak{a}_H \to \mathfrak{a}_P$, with certain compatibility. The vector space $\mathfrak{p}$ is then $\mathfrak{h} \oplus \mathfrak{a}_P$ and the Lie bracket is specified by $[X, Y]_{\mathfrak{p}} := [X, Y]_{\mathfrak{h}} - \omega(X, Y)$, $[X, Z]_{\mathfrak{p}} := \eta(X)(Z)$, for $X, Y \in \mathfrak{h}$ and $Z \in \mathfrak{a}_P$, and that $\mathfrak{a}_P$ is an Abelian ideal. Writing $\rho\colon \mathfrak{a}_P \to \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{a}_P$ for the natural inclusion, the map

$$\mathring{\xi}\colon \mathfrak{a}_H \to \mathfrak{p}, \qquad \mathring{\xi} := \xi + \rho \circ a$$

is a Lie algebra monomorphism and $\mathring{\xi}(\mathfrak{a}_H)$ is an ideal in $\mathfrak{p}$. The shear algebra $\mathfrak{k}$ is then the quotient $\mathfrak{p}/\mathring{\xi}(\mathfrak{a}_H)$. As vector spaces, $\mathfrak{h}$ is a summand of $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{a}_P$ and is isomorphic the shear algebra $\mathfrak{k} = \mathfrak{p}/\mathring{\xi}(\mathfrak{a}_H)$ via the projection. In this way, tensors on $\mathfrak{h}$ may be transferred to tensors on $\mathfrak{k}$.

For studying two-step solvable algebras, we may take $\mathfrak{h} = \mathbb{R}^N$ Abelian, $\mathfrak{a} = \mathfrak{a}_H = \mathfrak{a}_P$ a subalgebra of $\mathfrak{h} = \mathbb{R}^N$, $\xi = \mathrm{inc}$, the inclusion map, and $a = \mathrm{id}_{\mathfrak{a}}$, see [17]. The remaining shear data is $\omega \in \Lambda^2\mathfrak{h}^* \otimes \mathfrak{a}$ and $\eta \in \mathfrak{h}^* \otimes \mathrm{Hom}(\mathfrak{a})$. Compatibility gives $\eta(X)(Z) = -\omega(X, Z)$ for $X \in \mathfrak{h}$, $Z \in \mathfrak{a}$ and $\omega|_{\Lambda^2\mathfrak{a}^*} = 0$. Thus $\mathfrak{a}$ and ω determine the entire shear data.

Definition 2.5 A pair $(\mathfrak{a}, \omega)$ consisting of a subspace $\mathfrak{a}$ of $\mathbb{R}^N$ and a two-form $\omega \in \Lambda^2(\mathbb{R}^N)^* \otimes \mathfrak{a}$ with $\omega|_{\Lambda^2\mathfrak{a}} = 0$ is called *pre-shear data* (on $\mathbb{R}^N$).

Now suppose that $N = 2n$. As $\mathbb{R}^{2n}$ is Abelian any $J \in \mathrm{End}(\mathbb{R}^{2n})$ with $J^2 = -\mathrm{id}$ defines an (integrable) complex structure. Combining [17, Lemmas 2.1 and 3.1] and [16, Proposition 2.5] gives

Proposition 2.6 *Let J be a complex structure on the Lie algebra $\mathbb{R}^{2n}$ and let $(\mathfrak{a}, \omega)$ be pre-shear data on $\mathbb{R}^{2n}$. Then the shear $(\mathfrak{g}, J_{\mathfrak{g}})$ of $(\mathbb{R}^{2n}, J)$ is a Lie algebra $\mathfrak{g}$ endowed with a complex structure $J_{\mathfrak{g}}$ if and only if*

$$\mathcal{A}(\omega(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,)) = 0, \qquad J^*\omega = \omega - J \circ J.\omega, \tag{2.2}$$

where $J.\omega = -\omega(J\,\cdot\,,\,\cdot\,) - \omega(\,\cdot\,, J\,\cdot\,)$ and $\mathcal{A}$ is anti-symmetrisation. Moreover, every two-step solvable Lie algebra $\mathfrak{g}$ with a complex structure $J_{\mathfrak{g}}$ may be obtained in this way.

Definition 2.7 Pre-shear data $(\mathfrak{a}, \omega)$ on $(\mathbb{R}^{2n}, J)$ that satisfies (2.2) will be called *complex shear data*.

To describe certain consequences of (2.2), we need some further notation. On $\mathbb{R}^{2n}$, let $(\mathfrak{a}, \omega)$ be pre-shear data and let J be a complex structure. Any compatible metric g on $\mathbb{R}^{2n}$ then makes $(\mathbb{R}^{2n}, g, J)$ into a Kähler Lie algebra.

Set $\mathfrak{a}_J := \mathfrak{a} \cap J\mathfrak{a}$, and let $\mathfrak{a}_r$ be the orthogonal complement of $\mathfrak{a}_J$ in $\mathfrak{a}$. Put $U_r := \mathfrak{a}_r \oplus J\mathfrak{a}_r$ and let U_J be the orthogonal complement of $\mathfrak{a} + J\mathfrak{a} = \mathfrak{a}_J \oplus U_r$ in $\mathbb{R}^{2n}$.

For each $X \in \mathfrak{a}$, we define

$$A_X := \omega(JX,\,\cdot\,)|_{\mathfrak{a}} \in \mathrm{End}(\mathfrak{a})$$

and decompose A_X according to the splitting $\mathfrak{a} = \mathfrak{a}_J \oplus \mathfrak{a}_r$ as

$$\begin{aligned} A_X &= K_X + G_X + H_X + F_X \\ &\in \mathrm{End}(\mathfrak{a}_J) \oplus \mathrm{Hom}(\mathfrak{a}_J, \mathfrak{a}_r) \oplus \mathrm{Hom}(\mathfrak{a}_r, \mathfrak{a}_J) \oplus \mathrm{End}(\mathfrak{a}_r). \end{aligned}$$

We have associated bilinear maps $f\colon \mathfrak{a}_r \otimes \mathfrak{a}_r \to \mathfrak{a}_r$ and $h\colon \mathfrak{a}_r \otimes \mathfrak{a}_r \to \mathfrak{a}_J$ given by

$$f(X, \hat{X}) := F_X(\hat{X}), \qquad h(X, \hat{X}) := H_X(\hat{X}).$$

Moreover, for each $Z \in U_J$, we set

$$B_Z := \omega(Z, \cdot)|_{\mathfrak{a}} \in \mathrm{End}(\mathfrak{a}).$$

Lemma 2.8 *On the Abelian Lie algebra $\mathbb{R}^{2n}$, let (g, J) be a Kähler structure and let $(\mathfrak{a}, \omega)$ be complex shear data. Then*

(i) $G_X = 0$ *for all* $X \in \mathfrak{a}_r$,
(ii) $[J, K_X] = 0$ *for all* $X \in \mathfrak{a}_r$,
(iii) f *is symmetric,*
(iv) *for all* $X, \hat{X} \in \mathfrak{a}_r$, *we have*

$$\omega(JX, J\hat{X}) \in \mathfrak{a}_J \quad \text{and} \quad \omega(JX, J\hat{X}) = J(h(X, \hat{X}) - h(\hat{X}, X)),$$

(v) *for all* $\tilde{X}, \hat{X} \in \mathfrak{a}_r$ *and all* $\tilde{Z}, \hat{Z} \in U_J$, *we have*

$$[A_{\tilde{X}}, A_{\hat{X}}] = 0 = [A_{\tilde{X}}, B_{\tilde{Z}}] = [B_{\tilde{Z}}, B_{\hat{Z}}] \quad \text{and} \quad [K_{\tilde{X}}, K_{\hat{X}}] = 0.$$

Moreover,

$$\omega^r(JZ, JX) = \omega^r(Z, X),$$

where ω^r is the part of ω that takes values in $\mathfrak{a}_r$.

Proof Parts (i)–(iv) and the final part of (v) may be found in [17, Lemma 3.3]. The rest of part (v) follows from the first equation in (2.2) evaluated on one element of $\mathfrak{a}$ and two elements of $J\mathfrak{a}_r \oplus U_J$, and from the fact that the endomorphism A_X is block upper triangular by part (i). □

Finally, we recall the following result from [17, Lemma 3.1].

Lemma 2.9 *On the Abelian Lie algebra $\mathbb{R}^{2n}$, let (g, J) be a Kähler structure and let $(\mathfrak{a}, \omega)$ be complex shear data. Then the shear $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$ of $(\mathbb{R}^{2n}, g, J)$ is an SKT Lie algebra if and only if*

$$\mathcal{A}\left(g(J^*\omega(\cdot, \cdot), \omega(\cdot, \cdot)) + 2g(J^*\omega(\omega(\cdot, \cdot), \cdot), \cdot)\right) = 0, \tag{2.3}$$

where $\mathcal{A}$ is anti-symmetrisation.

Remark 2.10 In the rest of the article, when we consider a two-step solvable (almost) Hermitian Lie algebra $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$, we will regard it as being obtained by appropriate pre-shear data $(\mathfrak{a}, \omega)$ from a flat Kähler structure (g, J) on $\mathbb{R}^{2n}$. The identification of $\mathfrak{g} = \mathbb{R}^{2n}$ as vector spaces, identifies $g_{\mathfrak{g}}$ with g and $J_{\mathfrak{g}}$ with J. Note that this also gives $-\omega = [\cdot, \cdot]_{\mathfrak{g}}$ and that $\mathfrak{g}' = \mathrm{Im}\omega$. We can then without loss of generality identify $\mathfrak{a}$ with $\mathfrak{g}'$.

2.3 Lie Algebra Notation

When we need to specify concrete Lie algebras, we will often use Salamon's notation [29]. If $e_1, \ldots, e_n$ is a basis for $\mathfrak{g}$ with dual basis $e^1, \ldots, e^n$, then $de^i(e_j, e_k) = -e^i([e_j, e_k])$ and the algebra is specified by listing the differentials $(de^1, \ldots, de^n)$, but writing for example $3(e^1 \wedge e^2 - e^4 \wedge e^6) = 3(e^{12} - e^{46})$ as $3.(12 - 46)$.

Table 1 Notation for certain Lie algebras

$\mathfrak{g}$	dim	differentials	
$\mathfrak{aff}_{\mathbb{R}}$	2	$(0, 21)$	
$\mathfrak{h}_3$	3	$(0, 0, 21)$	
$\mathfrak{r}'_{3,\lambda}$	3	$(0, \lambda.21 + 31, -21 + \lambda.31)$	$\lambda \geqslant 0$
$\mathfrak{r}_{4,\mu,\lambda}$	4	$(0, 21, \mu.31, \lambda.41)$	$0 < \lvert\lambda \leqslant \lvert\mu \leqslant 1$
$\mathfrak{r}'_{4,\mu,\lambda}$	4	$(0, \mu.21, \lambda.31 + 41, -31 + \lambda.41)$	$\mu > 0$
$\mathfrak{g}_{5,17}^{\alpha,\beta,\gamma}$	5	$(0, \alpha.21 + 31, -21 + \alpha.31,$ $\beta.41 + \gamma.51, -\gamma.41 + \alpha.51)$	$\alpha \geqslant 0, \gamma \neq 0$
$\mathfrak{g}_{6,11}^{\alpha,\beta,\gamma,\delta}$	6	$(0, \alpha.21, \beta.31 + 41, -31 + \beta.41,$ $\gamma.51 + \delta.61, -\delta.51 + \gamma.61)$	$\alpha\delta \neq 0$
$N_{6,1}^{\alpha,\beta,\gamma,\delta}$	6	$(\alpha.15 + \beta.16, \gamma.25 + \delta.26,$ $35, 46, 0, 0)$	$\alpha\beta \neq 0, (\gamma, \delta) \neq (0, 0)$
$N_{6,14}^{\alpha,\beta,\gamma}$	6	$(\alpha.15 + \beta.16, 26,$ $\gamma.35 - 45, \gamma.45 + 35, 0, 0)$	$\alpha\beta \neq 0$

In Table 1, we use this notation to list the Lie algebras of this article that have standard names, with notation coming from [4, 6, 25, 26, 33].

3 Balanced and Kähler Geometry

3.1 Balanced Lie Algebras

Lemma 3.1 *Let (g, J) be a flat Kähler structure on the Lie algebra $\mathbb{R}^{2n}$ with associated Kähler form σ. Then the shear $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$ of $(\mathbb{R}^{2n}, g, J)$ by complex shear data $(\mathfrak{a}, \omega)$ is balanced if and only if*

$$\mathcal{A}\left(\sigma(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,) \wedge \sigma^{n-2}\right) = 0. \tag{3.1}$$

Proof Since $\xi = \mathrm{inc}\colon \mathfrak{a} \to \mathbb{R}^{2n}$ and $a = \mathrm{id}_{\mathfrak{a}}$, [16, Corollary 3.11] implies

$$\begin{aligned} d_{\mathfrak{g}}\sigma_{\mathfrak{g}}^{n-1} &= d_{\mathbb{R}^{2n}}\sigma^{n-1} - (\xi \circ a^{-1} \lrcorner\, \sigma^{n-1}) \wedge \omega = -(n-1)\,(\mathrm{inc} \lrcorner\, \sigma) \wedge \omega \wedge \sigma^{n-2} \\ &= -(n-1)\,\mathcal{A}\left(\sigma(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,) \wedge \sigma^{n-2}\right). \end{aligned}$$

So $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$ is balanced, i.e. $d_{\mathfrak{g}}\sigma_{\mathfrak{g}}^{n-1} = 0$, if and only if (3.1) holds. □

Proposition 3.2 *Let $(\mathfrak{g}, g, J)$ be a Hermitian two-step solvable Lie algebra and write $\mathfrak{g} = \mathfrak{g}'_J \oplus V_r \oplus V_J$ as in (2.1). Then (g, J) is balanced if and only if*

$$\mathrm{tr}(\mathrm{ad}(Z)) = 0$$

for all $Z \in V_J$, and there exist unitary bases $X_1, \ldots, X_{2r}$ of (V_r, g, J) and $Z_1, \ldots, Z_{2\ell}$ of (V_J, g, J) such that the element

$$C = \sum_{i=1}^{r} [X_{2i-1}, X_{2i}] + \sum_{j=1}^{\ell} [Z_{2j-1}, Z_{2j}] \tag{3.2}$$

is orthogonal to $\mathfrak{g}'_J$ and satisfies

$$\mathrm{tr}(\mathrm{ad}(X)) = -\sigma\,(C, X)\,,$$

for any $X \in V_r$.

Proof Fix unitary bases $Y_1, Y_2 = JY_1, \ldots, Y_{2s}$ of $(\mathfrak{g}'_J, g, J)$, $X_1, \ldots, X_{2r}$ of (V_r, g, J) and $Z_1, \ldots, Z_{2\ell}$ of (V_J, g, J). We have to check (3.1) for all combinations of $2n - 1$ vectors from the basis $Y_1, \ldots, Y_{2s}, X_1, \ldots, X_{2r}, Z_1, \ldots, Z_{2\ell}$ of $\mathfrak{g}$. There are thus three cases, corresponding to omitting one basis vector W from $\mathfrak{g}'_J$, V_r or V_J.

First, if we omit $W \in \{Z_1, \ldots, Z_{2\ell}\} \subset V_J$, we can relabel our bases, so $W = Z_{2\ell}$ and put $Z = Z_{2\ell-1}$. Then (3.1) only has non-zero contributions when each of the pure σ factors in σ^{n-1} is evaluated on pairs A, JA. This implies that contributions from Z are only from terms where it is one of the arguments of the factor $\sigma(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,)$. But $\mathrm{Im}(\omega) \perp V_J$, so Z is an argument of ω and the only contributions to (3.1) are

$$\begin{aligned} 0 &= \sum_{j=1}^{s} \sigma(\omega(Z, Y_{2j-1}), Y_{2j}) - \sigma(\omega(Z, Y_{2j}), Y_{2j-1}) \\ &\quad + \sum_{k=1}^{r} \sigma(\omega(Z, X_{2k-1}), X_{2k}) - \sigma(\omega(Z, X_{2k}), X_{2k-1}) \\ &= \sum_{j=1}^{s} g(\omega(Z, Y_{2j-1}), Y_{2j-1}) + g(\omega(Z, Y_{2j}), Y_{2j}) \\ &\quad + \sum_{k=1}^{r} g(\omega(Z, X_{2k-1}), X_{2k-1}) + g(\omega(Z, X_{2k}), X_{2k}) \\ &= \mathrm{tr}(\omega(Z,\,\cdot\,)) = -\mathrm{tr}(\mathrm{ad}(Z)). \end{aligned}$$

For the next case, $W = X_{2r}$ is omitted and we put $X = X_{2r-1}$. As before, the only contributions are from inserting X and two J-linearly dependent vectors into $\sigma(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,)$. Noting that $\omega|_{\Lambda^2\mathfrak{a}} = 0$ and $\mathrm{Im}(\omega) \perp V_J$, one computes

$$\begin{aligned} 0 &= \sum_{j=1}^{s} \sigma(\omega(X, Y_{2j-1}), Y_{2j}) - \sigma(\omega(X, Y_{2j}), Y_{2j-1}) \\ &\quad + \sum_{k=1}^{r-1} \sigma(\omega(X, X_{2k-1}), X_{2k}) - \sigma(\omega(X, X_{2k}), X_{2k-1}) \\ &\quad + \sigma\left(\sum_{k=1}^{r-1} \omega(X_{2k-1}, X_{2k}) + \sum_{k=1}^{\ell} \omega(Z_{2k-1}, Z_{2k}), X\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{2s} g(\omega(X, Y_p), Y_p) + \sum_{q=1}^{2r-2} g(\omega(X, X_q), X_q) + \sigma(-C + [X, W], X) \\
&= -\mathrm{tr}(\mathrm{ad}(X)) + g([X, W], W) - \sigma\,(C, X) + g(J[X, W], X) \\
&= -\mathrm{tr}(\mathrm{ad}(X)) - \sigma(C, X).
\end{aligned}
$$

Finally, for $W = Y_{2s}$, $Y = Y_{2s-1}$, we need to have Y as the final argument of $\sigma(\omega(\cdot, \cdot), \cdot)$, and the first two arguments need to be J-dependent vectors from $V_r \oplus V_J$. So (3.1) on these vectors reduces to

$$0 = -\sigma(C, Y) = g(C, JY),$$

giving C is orthogonal to $\mathfrak{g}'_J$. □

Remark 3.3 The proof shows that if the conditions of Proposition 3.2 hold for one pair $X_1, \dots, X_{2r}$ and $Z_1, \dots, Z_{2\ell}$ of unitary bases, then they hold automatically for all such pairs.

Remark 3.4 Proposition 3.2 allows us to recover the classification of certain balanced Hermitian Lie algebras in [12]. For this, let $(\mathfrak{g}, g, J)$ be a $2n$-dimensional almost Abelian Hermitian Lie algebra, meaning that there is a codimension 1 Abelian ideal. For simplicity, let us just consider the generic case when $\dim(\mathfrak{g}') = 2n - 1$. With more effort the methods also apply to $\dim(\mathfrak{g}') < 2n - 1$. Under this additional assumption, $(\mathfrak{g}, g, J)$ is of pure type III with $\dim(\mathfrak{g}'_J) = 2n - 2$ and $\dim(\mathfrak{g}'_r) = 1$. Choose $X \in \mathfrak{g}'_r$ of norm one and consider $A_X = \omega(JX, \cdot) \in \mathrm{End}(\mathfrak{g}')$. By Lemma 2.8, there exist $a \in \mathbb{R}$, $v \in \mathfrak{g}'_J$ and $A \in \mathfrak{gl}(\mathfrak{g}'_J, J)$ such that

$$A_X = \begin{pmatrix} a & v \\ 0 & A \end{pmatrix}$$

with respect to the splitting $\mathfrak{g}' = \mathfrak{g}'_r \oplus \mathfrak{g}'_J$. By Proposition 3.2, $(\mathfrak{g}, g, J)$ is balanced if and only if

$$a + \mathrm{tr}(A) = \mathrm{tr}(\mathrm{ad}(JX)) = -\sigma([X, JX], JX) = g(aX, X) = a$$

and $[X, JX] = v$ is orthogonal to $\mathfrak{g}'_J$. But $v \in \mathfrak{g}'_J$, so we have balanced if and only if $\mathrm{tr}(A) = 0$ and $v = 0$, which coincides with [12].

For unimodular Lie algebras $\mathrm{tr}(\mathrm{ad}(X)) = 0$ for all $X \in \mathfrak{g}$, so Proposition 3.2 simplifies as below and Remark 3.3 holds in this context.

Corollary 3.5 *Let $(\mathfrak{g}, g, J)$ be a unimodular Hermitian two-step solvable Lie algebra. Then (g, J) is balanced if and only if $C = 0$ in (3.2).*

Note that in Proposition 3.2, the restriction of g to $\mathfrak{g}'_J$ plays no role.

Corollary 3.6 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable balanced Lie algebra. If $\tilde{g}$ is another metric compatible with $(\mathfrak{g}, J)$ satisfying $(\mathfrak{g}'_J)^{\perp_g} = (\mathfrak{g}'_J)^{\perp_{\tilde{g}}}$ and $g|_{(\mathfrak{g}'_J)^{\perp_g}} = \tilde{g}|_{(\mathfrak{g}'_J)^{\perp_{\tilde{g}}}}$, then $\tilde{g}$ is balanced too.*

3.2 Kähler Lie Algebras

Lemma 3.7 *Let (g, J) be a flat Kähler structure on the Lie algebra $\mathbb{R}^{2n}$ with associated Kähler form σ. Then the shear $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$ of $(\mathbb{R}^{2n}, g, J)$ by complex shear data $(\mathfrak{a}, \omega)$ is Kähler if and only if*

$$\mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot)) = 0. \tag{3.3}$$

Proof Similar to the proof of Lemma 3.1, [16, Corollary 3.11] implies

$$d_{\mathfrak{g}}\sigma_{\mathfrak{g}} = -\mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot))$$

and so that $(\mathfrak{g}, g_{\mathfrak{g}}, J_{\mathfrak{g}})$ is Kähler, i.e. $d_{\mathfrak{g}}\sigma_{\mathfrak{g}} = 0$, if and only if (3.3) holds. □

We first note some general consequences of (3.3).

Lemma 3.8 *When the shear in Lemma 3.7 is Kähler, we have*

$$\omega(U_J, U_J) = 0, \qquad \omega(J\mathfrak{a}_r, J\mathfrak{a}_r) = 0 \quad \textit{and} \quad h = 0.$$

Furthermore, there exist a complex unitary basis $Y_1, \ldots, Y_s$ of $\mathfrak{a}_J$, a real orthonormal basis $X_1, \ldots, X_r$ of $\mathfrak{a}_r$, one-forms $\alpha_1, \ldots, \alpha_s \in \mathfrak{a}_r^*$ on $\mathfrak{a}_r$, and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

$$K_X(Y_j) = -\alpha_j(X)JY_j, \qquad f(X_k, X_m) = -\delta_{km}\lambda_k X_k$$

for all $X \in \mathfrak{a}_r$, $Z \in U_J$, $j \in \{1, \ldots, s\}$ and $k, m \in \{1, \ldots, r\}$.

Proof Inserting $Z_1, Z_2 \in U_J$, which is orthogonal to $\mathfrak{a}$, and $W \in \mathfrak{a} + J\mathfrak{a}$ into (3.3) yields

$$0 = \sigma(\omega(Z_1, Z_2), W),$$

and so $\omega(Z_1, Z_2) = 0$, giving $\omega(U_J, U_J) = 0$. Next, observe that for $\hat{X}, \tilde{X} \in \mathfrak{a}_r$, the endomorphisms $K_{\tilde{X}}, K_{\hat{X}} \in \mathrm{End}(\mathfrak{a}_J)$ are complex and commute by Lemma 2.8. Inserting $\tilde{Y}, \hat{Y} \in \mathfrak{a}_J$ and $JX \in J\mathfrak{a}_r$ into (3.3) yields

$$0 = \sigma(K_X(\tilde{Y}), \hat{Y}) + \sigma(\tilde{Y}, K_X(\hat{Y})),$$

meaning that $K_X \in \mathfrak{sp}(\mathfrak{a}_J, \sigma)$. But K_X is complex, so $K_X \in \mathfrak{u}(\mathfrak{a}_J, g, J)$. It follows, that the K_X are simultaneously complex diagonalisable with imaginary eigenvalues. In particular, there exists a complex unitary basis $Y_1, \ldots, Y_s$ of $(\mathfrak{a}_J, g)$ and one-forms $\alpha_1, \ldots, \alpha_s \in \mathfrak{a}_r^*$ on $\mathfrak{a}_r$ such that

$$K_X(Y_j) = -\alpha_j(X)JY_j$$

for all $X \in \mathfrak{a}$ and all $j \in \{1, \ldots, s\}$. Next, (3.3) evaluated on $Y \in \mathfrak{a}_J$, $\tilde{X} \in \mathfrak{a}_r$ and $J\hat{X} \in J\mathfrak{a}_r$ shows

$$0 = \sigma(Y, H_{\hat{X}}(\tilde{X})) = \sigma(Y, h(\hat{X}, \tilde{X})),$$

so $h = 0$. Lemma 2.8 gives $\omega(J\tilde{X}, J\hat{X}) = J(h(\tilde{X}, \hat{X}) - h(\hat{X}, \tilde{X})) = 0$, and hence $\omega(J\mathfrak{a}_r, J\mathfrak{a}_r) = 0$. Finally, putting $\bar{X}, J\tilde{X}, J\hat{X} \in \mathfrak{a}_r$ in (3.3) gives

$$0 = -\sigma(f(\tilde{X}, \bar{X}), J\hat{X}) + \sigma(f(\hat{X}, \bar{X}), J\tilde{X}) = -g(f(\tilde{X}, \bar{X}), \hat{X}) + g(f(\hat{X}, \bar{X}), \tilde{X}).$$

However f is symmetric by Lemma 2.8, we get that $g(f(\cdot,\cdot),\cdot)$ is totally symmetric. In particular, all endomorphisms F_X are symmetric. Since these endomorphisms commute by Lemma 2.8, we have a common orthonormal basis $X_1,\dots,X_r$ of $(\mathfrak{a}_r, g)$ of eigenvectors for all F_X. Now

$$\operatorname{span}(X_j) \ni F_{X_i}(X_j) = f(X_i, X_j) = f(X_j, X_i) = F_{X_j}(X_i) \in \operatorname{span}(X_i)$$

for all $i, j \in \{1,\dots,r\}$ implies the existence of $\lambda_1,\dots,\lambda_r \in \mathbb{R}$ such that

$$f(X_i, X_j) = -\delta_{ij}\lambda_j X_j$$

for all $i, j \in \{1,\dots,r\}$. □

It seems hard to solve (3.3) in full generality, so we now restrict to a certain subclass, namely those two-step solvable Kähler Lie algebras $(\mathfrak{g}, g, J)$ with $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$. Note that this class includes those of pure type I. Moreover, it also includes algebras of pure type III, i.e. with $V_J = 0$, since by Lemma 3.8 we have $[J\mathfrak{g}'_r, J\mathfrak{g}'_r] = 0$ and $h = 0$ and so must have $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$ in order for $\mathfrak{g}'$ to be the commutator ideal.

When $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$, the form of B_Z, $Z \in V_J$, simplifies and allows for a classification. The condition $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$ is equivalent to $\alpha_j \neq 0$ for all $j = 1,\dots,r$. Moreover, for any $Z \in U_J$, the endomorphisms B_Z commute with A_X for all $X \in \mathfrak{a}_r$ by Lemma 2.8. As A_X has imaginary non-zero eigenvalues on $\mathfrak{a}_J$ and real eigenvalues on $\mathfrak{a}_r$, we get that B_Z preserves the splitting $\mathfrak{a} = \mathfrak{a}_r \oplus \mathfrak{a}_J$. Using this property, we will obtain:

Theorem 3.9 *Let $(\mathfrak{g}, g, J)$ be an almost Hermitian Lie algebra. Then $(\mathfrak{g}, g, J)$ is a two-step solvable Kähler Lie algebra $(\mathfrak{g}, g, J)$ with $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$ if and only if $J\mathfrak{g}'_r \perp \mathfrak{g}'_r$ and there exist a complex unitary basis $Y_1,\dots,Y_s$ of $\mathfrak{g}'_J$, an orthonormal basis $X_1,\dots,X_r$ of $\mathfrak{g}'_r$, non-zero one forms $\alpha_1,\dots,\alpha_s \in (\mathfrak{g}'_r)^* \setminus \{0\}$, one-forms $\beta_1,\dots,\beta_s \in V_J^*$ and non-zero real numbers $\lambda_1,\dots,\lambda_r \in \mathbb{R}\setminus\{0\}$ such that the only non-zero Lie brackets (up to anti-symmetry and complex linear extension on $\mathfrak{g}'_J$) are given by*

$$[JX, Y_j] = \alpha_j(X) JY_j, \qquad [Z, Y_j] = \beta_j(Z) JY_j, \qquad [JX_k, X_k] = \lambda_k X_k$$

for $j \in \{1,\dots,s\}$, $k \in \{1,\dots,r\}$, $X \in \mathfrak{g}'_r$ and $Z \in V_J$.

For the proof, we first need the following result.

Lemma 3.10 *Let V be $2n$-dimensional vector space endowed with a Hermitian structure (g, J) and denote by σ the associated fundamental two-form. Suppose $A_1, A_2 \in \mathfrak{sp}(V,\sigma)$ satisfy $[A_1, A_2] = 0$ and $A_1 + JA_1J + JA_2 - A_2J = 0$. Then we have $A_1, A_2 \in \mathfrak{u}(V, g, J)$.*

Proof For $i = 1, 2$, decompose $A_i = A_i^J + A_i^{J-}$ into the sum of its J-invariant part A_i^J and its J-anti-invariant part A_i^{J-}. Then $A_i^J \in \mathfrak{u}(V, g, J)$, and A_i^{J-} is symmetric with respect to g.

Now

$$0 = A_1 + JA_1J + JA_2 - A_2J = J[A_1, J] - [A_2, J] = 2JA_1^{J-} - 2A_2^{J-},$$

so

$$A_2^{J-} = JA_1^{J-}.$$

Moreover, the J-invariant part of $0 = [A_1, A_2]$ yields

$$0 = [A_1^J, A_2^J] + [A_1^{J-}, A_2^{J-}] = [A_1^J, A_2^J] - 2JA_1^{J-}A_1^{J-}.$$

Since $A_1^J \in \mathfrak{u}(V, g, J)$, there exists a basis of V consisting of vectors $v \in V$ with $A_1^J v = cv$ and $A_1^{J-}Jv = -cv$ for some $c \in \mathbb{R}$. For such a vector v, we obtain

$$\begin{aligned} &g(Jv, [A_1^J, A_2^J]v) \\ &\quad = g(Jv, A_1^J A_2^J v) - g(Jv, A_2^J A_1^J v) = -g(A_1^J Jv, A_2^J v) - c\, g(Jv, A_2^J Jv) \\ &\quad = c\, g(v, A_2^J v) - c\, g(Jv, JA_2^J v) = c\, g(v, A_2^J v) - c\, g(v, A_2^J v) = 0. \end{aligned}$$

Hence,

$$0 = g(Jv, JA_1^{J-}A_1^{J-}v) = g(v, A_1^{J-}A_1^{J-}v) = g(A_1^{J-}v, A_1^{J-}v) = \|A_1^{J-}v\|^2.$$

This shows $A_1^{J-} = 0$ and so also $A_2^{J-} = JA_1^{J-} = 0$, finishing the proof. □

Lemma 3.11 *If the shear in Lemma 3.7 is Kähler and for each $Z \in U_J$, B_Z preserves the splitting $\mathfrak{a} = \mathfrak{a}_J + \mathfrak{a}_r$, then the Y_j in Lemma 3.8 may be chosen so that $B_Z(Y_j) = -\beta_j(Z)JY_j$, for all j, for some $\beta_j \in U_J^*$.*

Proof Write $B_Z = b_Z + c_Z$ with $b_Z \in \mathrm{End}(\mathfrak{a}_J)$ and $c_Z \in \mathrm{End}(\mathfrak{a}_r)$. Then $[b_Z, b_{JZ}] = 0$ and $[b_Z, K_X] = 0$ for all $X \in \mathfrak{a}_r$. Moreover, the second equation in (2.2) yields

$$\begin{aligned} b_{JZ}J(Y) &= J^*\omega(JZ, JY) = \omega(Z, Y) + J(\omega(JZ, Y) + \omega(Z, JY)) \\ &= b_Z(Y) + Jb_{JZ}(Y) + Jb_ZJ(Y) \end{aligned}$$

for all $Y \in \mathfrak{a}_J$, and so

$$b_Z + Jb_ZJ + Jb_{JZ} - b_{JZ}J = 0.$$

Inserting $Z, \tilde{Y}, \hat{Y}$, where $\tilde{Y}, \hat{Y} \in \mathfrak{a}_J$, into (2.2) gives

$$0 = \sigma(b_Z(\tilde{Y}), \hat{Y}) + \sigma(\tilde{Y}, (b_Z(\hat{Y})),$$

so $b_Z \in \mathfrak{sp}(\mathfrak{a}_J, \sigma)$. Thus, we deduced from Lemma 3.10 that $b_Z \in \mathfrak{u}(\mathfrak{a}_J, g, J)$. Lemma 2.8 gives that all b_Z commute pairwise and with all K_X, $X \in \mathfrak{a}_r$, and the result follows. □

Proof of Theorem 3.9 Using shear data, let $Z \in U_J$ be given. As we argued before Theorem 3.9, B_Z preserves the subspaces $\mathfrak{a}_J$ and $\mathfrak{a}_r$, so we may apply Lemma 3.11 to get the Y_j, α_j and β_j. The hypothesis $[J\mathfrak{g}', \mathfrak{g}'_J] = \mathfrak{g}'_J$ implies that each α_j is non-zero.

Next, by Lemma 2.8, we have $\omega^r(JZ, JX) = \omega^r(Z, X) = c_Z(X)$ for any $Z \in U_J$, $X \in \mathfrak{a}_r$. Hence,

$$\begin{aligned} 0 &= \mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot))(JZ, J\tilde{X}, J\hat{X}) = \sigma(c_Z(\tilde{X}), J\hat{X}) - \sigma(c_Z(\hat{X}), J\tilde{X}) \\ &= g(c_Z(\tilde{X}), \hat{X}) - g(c_Z(\hat{X}), \tilde{X}), \end{aligned}$$

and c_Z is symmetric. Since all c_Z commute pairwise and with all F_X, $X \in \mathfrak{a}_r$, by Lemma 2.8, there is a common eigenbasis $X_1, \ldots, X_r$ of $\mathfrak{g}'_r$ for all these operators. Moreover, by Lemma 3.8, there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with

$$f(X_j, X_k) = F_{X_j}(X_k) = -\delta_{jk}\lambda_k X_k.$$

Next,

$$\begin{aligned} 0 &= \mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot))(Z, X_j, JX_j) = \sigma(c_Z(X_j), JX_j) - \sigma(\omega(Z, JX_j), X_j) \\ &= g(c_Z(X_j), X_j) + \sigma(c_{JZ}(X_j), X_j) = g(c_Z(X_j), X_j) \end{aligned}$$

for any $j \in \{1, \ldots, r\}$ since $c_{JZ}(X_j) \in \operatorname{span}(X_j)$. As $c_Z(X_j) \in \operatorname{span}(X_j)$, we get $c_Z(X_j) = 0$ and hence $c_Z = 0$. So $\omega^r(U_J, J\mathfrak{a}_r) = 0$ as well. Hence, $f\colon \mathfrak{a}_r \times \mathfrak{a}_r \to \mathfrak{a}_r$ has to be surjective in order to have $\operatorname{Im}(\omega) = \mathfrak{a}_r$ and thus $\lambda_j \neq 0$ for all $j = 1, \ldots, r$.

Next, inserting X_j, X_k, JX_k for $j, k \in \{1, \ldots, r\}$ with $j \neq k$ into (3.3) yields

$$0 = \mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot))(X_j, X_k, JX_k) = -\sigma(f(X_k, X_k), X_j) = \lambda_k g(JX_k, X_j).$$

Since $\lambda_k \neq 0$ and trivially $g(JX_k, X_k) = 0$ holds, this implies $X_k \perp J\mathfrak{a}_r$, so $\mathfrak{a}_r \perp J\mathfrak{a}_r$.

We now have all the claimed properties except that $\omega(Z, JX) = 0$ for $Z \in U_J$, $X \in \mathfrak{a}_r$. However, we already showed that $\omega(Z, JX) \in \mathfrak{a}_J$. Moreover, inserting Z, JX, Y into Lemma 3.7 for $Y \in \mathfrak{a}_J$, $X \in \mathfrak{a}_r$ and $Z \in U_J$ yields

$$0 = \mathcal{A}(\sigma(\omega(\cdot,\cdot),\cdot))(Z, JX, Y) = \sigma(\omega(Z, JX), Y).$$

Thus, $\omega(Z, JX) = 0$ for all $X \in \mathfrak{a}_r$, $Z \in U_J$, which completes the proof. □

For the three different pure types we arrive at the classification below.

Corollary 3.12 *Let $(\mathfrak{g}, g, J)$ be a $2n$-dimensional almost Hermitian Lie algebra that is two-step solvable. Then we have the following.*

(I) *$(\mathfrak{g}, g, J)$ is Kähler of pure type I if and only if $J\mathfrak{g}' \perp \mathfrak{g}'$, and there exists an orthonormal basis $X_1, \ldots, X_r$ of $\mathfrak{g}'$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R} \setminus \{0\}$ such that the only non-zero Lie brackets (up to anti-symmetry) are given by*

$$[JX_j, X_j] = \lambda_j X_j$$

for $j = 1, \ldots, r$.

(II) *$(\mathfrak{g}, g, J)$ is Kähler of pure type II if and only if there exists a complex unitary basis $Y_1, \ldots, Y_s$ of $\mathfrak{g}'$ and non-zero one-forms $\beta_1, \ldots, \beta_s \in V_J^*$ such that the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension) are given by*

$$[Z, Y_j] = \beta_j(Z) JY_j$$

for $j \in \{1, \ldots, s\}$ and $Z \in V_J$.

(III) $(\mathfrak{g}, g, J)$ *is Kähler of pure type III if and only if* $J\mathfrak{g}'_r \perp \mathfrak{g}'_r$, *and there exist a complex unitary basis* $Y_1, \ldots, Y_s$ *of* $\mathfrak{g}'_J$, *an orthonormal basis* $X_1, \ldots, X_r$ *of* $\mathfrak{g}'_r$, *non-zero one-forms* $\alpha_1, \ldots, \alpha_s \in (\mathfrak{g}'_r)^* \setminus \{0\}$ *on* $\mathfrak{g}'_r$ *and non-zero real numbers* $\lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\}$ *such that the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension) are given by*

$$[JX_k, Y_j] = \alpha_j(X_k)JY_j, \qquad [JX_k, X_k] = \lambda_k X_k.$$

Proof For pure types I and III this is just specialisation of Theorem 3.9. For pure type II, we have $\mathfrak{g}'_r = 0$, so we may use Lemma 3.11. □

Remark 3.13 For pure type I this implies $(\mathfrak{g}, J) \cong r(\mathfrak{aff}_{\mathbb{R}}, J) \oplus (\mathbb{R}^{2(n-r)}, J)$ as Lie algebras with complex structures. Up to change of basis, complex structures on $\mathfrak{aff}_{\mathbb{R}}$ and $\mathbb{R}^{2(n-r)}$ are unique.

4 Compatibility of Balanced and SKT Metrics

We now consider the question of Lie groups or Lie algebras with a complex structure that admit both a compatible balanced metric $\hat{g}$ and a compatible SKT metric $\tilde{g}$. We will say that a complex structure J is *SKT*, *balanced* or *Kähler* if it admits a compatible metric that is SKT, balanced or Kähler, respectively.

We first consider some general results for compact groups and for solvable groups in Section 4.1. Thereafter, we will focus on two-step solvable Lie algebras, considering each pure type in turn, and then specialising to the six-dimensional case.

4.1 General Results

Theorem 4.1 *Let* J *be a left-invariant complex structure on a compact Lie group* G. *Then* J *is SKT, but is balanced only if* G *is Abelian, in which case it is also Kähler.*

The special case when G is also semi-simple was proved in [11, 28].

Proof Existence of the SKT metric was given in [23, 31].

Now suppose that G is not Abelian. It is sufficient to assume G is connected. Note that $\mathfrak{g} = \mathbb{R}^k \oplus \mathfrak{k}$ for some semi-simple Lie algebra $\mathfrak{k}$ of compact type. It follows that there is a finite cover of G by the group $T^k \times K$, where K is compact, connected and simply connected with Lie algebra $\mathfrak{k}$, and it is sufficient to consider the case $G = T^k \times K$. By [30, Theorem in (3.2)] there is a Cartan subgroup N of G and holomorphic fibration $\pi\colon G \to G/N$ with G/N projective. Note that $N = T^k \times N_1$, with N_1 a Cartan subgroup of K. Let Y be a complex submanifold of G/N of complex codimension one, and put $X = \pi^{-1}(Y)$. Then $X = T^k \times (X \cap K)$ and is of real codimension two. In particular, for the fundamental class $[X] \in H_{n-2}(X)$ of X and any generator $c \in H^m(K, \mathbb{Z}) \cong \mathbb{Z}$, where $m = \dim K$, we have $c \frown [X] = 0$.

On the other hand, the Whitehead Theorems imply that $H^1(K) = 0 = H^2(K)$, so by duality $H^{m-1}(K) = 0 = H^{m-2}(K)$. Writing $2n = k + m = \dim G$, the Künneth formula gives $H^{2n-2}(G) = \bigoplus_{s=0}^{2} H^{k-2+s}(T^k) \otimes H^{m-s}(K) = H^{k-2}(T^k) \otimes H^m(K)$. If σ is the two-form of a balanced metric, we have $d(\sigma^{n-1}) = 0$, so $[\sigma^{n-1}] = b \otimes_{\mathbb{R}} c$ for some $b \in H^{k-2}(T^k)$. But now $0 < \int_X \sigma^{n-1} = (b \otimes c) \frown [X] = b \frown (c \frown [X]) = 0$, which is a contradiction. Thus if J is balanced, then $K = \{e\}$ and G is Abelian. But then J is a left-invariant complex structure on a torus and so admits a compatible Kähler metric. □

Now let us show that invariance of the Kähler metric is necessary in Question 1.2.

Proposition 4.2 *Let G be a simply connected solvable Lie group and let J be a left-invariant complex structure on G. Then (G, J) admits a compatible Kähler metric.*

Proof By the theorem in [30, (1.3)], there is a discrete subgroup Γ of G such that $\Gamma \backslash G$ is biholomorphic to an open subset V of $\mathbb{C}^n$. Now $\mathbb{C}^n$, and hence V, carries a compatible Kähler metric that we may pull back under the natural projection $\pi : G \to \Gamma \backslash G \cong V \subseteq \mathbb{C}^n$ to get a compatible Kähler metric on (G, J). □

4.2 Pure Type I

Pure type I gives ${\mathfrak{g}'}_J = 0$, so $\mathfrak{g}'$ is totally real.

Theorem 4.3 *Let $(\mathfrak{g}, J)$ be a unimodular two-step solvable Lie algebra $\mathfrak{g}$ with complex structure J of pure type I that is SKT and is balanced. Then $\mathfrak{g}$ is Abelian and so J is Kähler.*

Proof of Theorem 4.3 Without unimodularity, the structure of the SKT algebras is given in [17, Theorem 5.5]: $\mathfrak{g} \cong r\, \mathfrak{aff}_{\mathbb{R}} \oplus \mathfrak{h}$, for some nilpotent Lie algebra $\mathfrak{h}$. But $\mathfrak{h}$ is unimodular and $\mathfrak{aff}_{\mathbb{R}}$ is not, so $\mathfrak{g}$ is unimodular if and only if $r = 0$. [17, Theorem 5.5] now gives that $\mathfrak{g} = \mathfrak{h}$ is two-step nilpotent. As J is SKT, is balanced and $\mathfrak{g}$ is two-step nilpotent, [15, proof of Theorem 1.1] shows that $(\mathfrak{g}, J)$ is Kähler. □

Next, we provide an example that shows that the unimodular condition in Theorem 4.3 is necessary, and hence is also necessary in Question 1.2.

Example 4.4 Let $\mathfrak{g} = \mathfrak{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}$. Then $\mathfrak{g}$ is a non-unimodular two-step solvable Lie algebra with a basis $e_1, \dots, e_6$ for which (up to anti-symmetry) the only non-zero Lie brackets are

$$[e_1, e_2] = e_2 \quad \text{and} \quad [e_3, e_4] = e_5.$$

Let J be the almost complex structure with $Je_{2i-1} = e_{2i}$ for $i = 1, 2, 3$. Thus $(\mathfrak{g}, J)$ is a direct sum of $(\mathfrak{aff}_{\mathbb{R}}, J_1)$ and $(\mathfrak{h}_3 \oplus \mathbb{R}, J_2)$. For the dual basis $e^1, \dots, e^6$ the non-zero differentials are $de^2 = -e^{12}$ and $de^5 = -e^{34}$. Then J_1 is integrable, and for J_2 the $(1, 0)$-forms are spanned by $\alpha_1 = e^3 - ie^4$ and $\alpha_2 = e^5 - ie^6$. As $d\alpha_1 = 0$ and $d\alpha_2 = -e^3 \wedge e^4 = \frac{i}{2} \alpha_1 \wedge \overline{\alpha_1}$ is of type $(1, 1)$, we have that J_2, and hence J, is integrable. Moreover, $\mathfrak{g}' = \operatorname{span}(e_2, e_5)$ is totally real, so $(\mathfrak{g}, J)$ is of type I.

Let $\tilde{g}$ be the metric on $\mathfrak{g}$ for which $e_1, \ldots, e_6$ is an orthonormal basis. Then $\tilde{g}$ is compatible with J, and the associated fundamental two-form is

$$\tilde{\sigma} = e^{12} + e^{34} + e^{56}.$$

We now find that

$$dJ^* d\tilde{\sigma} = -dJ^* e^{346} = de^{345} = 0,$$

so $\tilde{g}$ is an SKT metric.

Next, let $\hat{g}$ be the metric on $\mathfrak{g}$ for which $e_1 - e_6, e_2 + e_5, e_6, -e_5, e_3, e_4$ is an orthonormal basis. Since this basis is unitary, $\hat{g}$ is compatible with J. Moreover, a dual basis is given by $e^1, e^2, e^1+e^6, e^2-e^5, e^3, e^4$ and so the associated fundamental form $\hat{\sigma}$ is given by

$$\hat{\sigma} = e^{12} + (e^1 + e^6) \wedge (e^2 - e^5) + e^{34} = 2e^{12} - e^{15} - e^{26} + e^{34} + e^{56}.$$

We now have

$$\begin{aligned} d(\hat{\sigma}^2) &= 2\hat{\sigma} \wedge d\hat{\sigma} = 2\hat{\sigma} \wedge (-e^{134} + e^{126} - e^{346}) \\ &= 2(e^{12346} - e^{13456} + e^{12346} - 2e^{12346} + e^{13456}) = 0, \end{aligned}$$

and hence that $\hat{g}$ is balanced.

However, $\mathfrak{g} \ncong r\,\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^{6-2r}$ for any $r \in \{1, \ldots, 3\}$, so by Corollary 3.12(I), $(\mathfrak{g}, J)$ does not admit any compatible Kähler metric.

4.3 Pure Type II

Pure type II means that $\mathfrak{g}'$ is complex. We will first classify two-step solvable SKT Lie algebras $(\mathfrak{g}, g, J)$ of pure type II up to some remaining "nilpotent" equations and give a full classification if $\mathfrak{g}'$ is of codimension two. The latter case was the remaining open case in our classification of the six-dimensional two-step solvable Lie algebras admitting an SKT structure in [17, Theorem 7.1] and so we complete this classification here in Theorem 4.7.

We begin by deriving some consequences for the form of the endomorphisms B_Z.

Lemma 4.5 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable SKT Lie algebra of pure type II. Then there exists a complex unitary basis $Y_1, \ldots, Y_r$ of $\mathfrak{g}' = \mathfrak{g}'_J$ and one-forms $\alpha_1, \ldots, \alpha_r \in V_J^*$ such that for any $Z \in V_J$, the endomorphism $\mathrm{ad}(Z)|_{\mathfrak{g}'} \in \mathrm{End}(\mathfrak{g}'_J)$ is complex and satisfies*

$$[Z, Y_j] = \alpha_j(Z) J Y_j$$

for all $j \in \{1, \ldots, r\}$.

Proof Working with shear data, choose $Z \in U_J$ and set $B_1 := B_Z$, $B_2 := B_{JZ}$. Then $[B_1, B_2] = 0$ by the first equation in (2.2). Moreover, the second equation in (2.2) yields

$$\begin{aligned} B_2 J(Y) &= \omega(JZ, JY) = \omega(Z, Y) + J(\omega(Z, JY) + \omega(JZ, Y)) \\ &= B_1(Y) + JB_1J(Y) + JB_2(Y), \end{aligned}$$

for all $Y \in \mathfrak{a}$, so

$$B_1 + JB_1J + JB_2 - B_2J = 0.$$

Next, inserting $\tilde{Y}, \hat{Y} \in \mathfrak{a}$ and Z, JZ into (2.3) yields

$$\begin{aligned}
0 &= -g(\omega(JZ, J\tilde{Y}), \omega(JZ, \hat{Y})) + g(\omega(JZ, J\hat{Y}), \omega(JZ, \tilde{Y})) - g(\omega(Z, J\tilde{Y}), \omega(Z, \hat{Y})) \\
&\quad + g(\omega(Z, J\hat{Y}), \omega(Z, \tilde{Y})) + g(\omega(J\omega(Z, \tilde{Y}), Z), \hat{Y}) - g(\omega(J\omega(Z, \hat{Y}), Z), \tilde{Y}) \\
&\quad + g(\omega(J\omega(JZ, \tilde{Y}), JZ), \hat{Y}) - g(\omega(J\omega(JZ, \hat{Y}), JZ), \tilde{Y}) \\
&= -g(B_2J(\tilde{Y}), B_2(\hat{Y})) + g(B_2J(\hat{Y}), B_2(\tilde{Y})) - g(B_1J(\tilde{Y}), B_1(\hat{Y})) \\
&\quad + g(B_1J(\hat{Y}), B_1(\tilde{Y})) - g(B_1JB_1(\tilde{Y}), \hat{Y}) + g(B_1JB_1(\hat{Y}), \tilde{Y}) \\
&\quad - g(B_2JB_2(\tilde{Y}), \hat{Y}) + g(B_2JB_2(\hat{Y}), \tilde{Y}) \\
&= -g\left((\mathcal{C}(B_1) + \mathcal{C}(B_2))\tilde{Y}, \hat{Y}\right) + g\left(\tilde{Y}, (\mathcal{C}(B_1) + \mathcal{C}(B_2))\hat{Y}\right),
\end{aligned}$$

where $\mathcal{C}(B) := B^T BJ + BJB$. We conclude that $\mathcal{C}(B_1) + \mathcal{C}(B_2)$ is required to be g-symmetric.

For $i = 1, 2$, decompose $B = B^J + B^{J-}$ into its J-invariant part B^J and into its J-anti-invariant part B^{J-} and then for $A \in \{J, J-\}$ decompose $B^A := B^A_+ + B^A_-$ into its g-symmetric part B^A_+ and its g-skew-symmetric part B^A_-. Then the g-skew-symmetric part of $\mathcal{C}(B)$ is

$$\begin{aligned}
\tfrac{1}{2}(\mathcal{C}(B) - \mathcal{C}(B)^T) &= \tfrac{1}{2}\left(B^T BJ + BJB + JB^T B + B^T JB^T\right) \\
&= \tfrac{1}{2}\left(B^T(BJ + JB^T) + (BJ + JB^T)B\right) \\
&= B^T J(B^J_+ - B^{J-}_-) + J(B^J_+ - B^{J-}_-)B \\
&= J\left(2(B^J_+)^2 - 2(B^{J-}_-)^2 + [B^J_+ - B^{J-}_-, B^J_- + B^{J-}_+]\right).
\end{aligned} \tag{4.1}$$

Note that this has trace

$$2\mathrm{tr}((B^J_+)^2) - 2\mathrm{tr}((B^{J-}_-)^2) = 2(\|B^J_+\|^2 + \|B^{J-}_-\|^2),$$

since for $B^T = \varepsilon B$, $\varepsilon = \pm 1$, and any orthonormal basis $E_1, \dots, E_{2s}$ of $\mathfrak{a}$, we have

$$\mathrm{tr}(B^2) = \sum_{j=1}^{2s} g(B^2 E_j, E_j) = \varepsilon \sum_{j=1}^{2s} g(BE_j, BE_j) = \varepsilon \|B\|^2.$$

Thus for $\mathcal{C}(B_1) + \mathcal{C}(B_2)$ to be g-symmetric we must have $B^J_{i,+} = 0 = B^{J-}_{i,-}$ for $i = 1, 2$. But then $B_i = B^J_{i,-} + B^{J-}_{i,+}$ which lies in $\mathfrak{sp}(\mathfrak{a}, \sigma)$. Consequently, we may apply Lemma 3.10 to deduce that we actually have $B_i \in \mathfrak{u}(\mathfrak{a}, g, J)$, so $B^{J-}_{i,+} = 0$. Then (4.1) gives that $\mathcal{C}(B_1) + \mathcal{C}(B_2)$ is g-symmetric.

Thus $B_Z \in \mathfrak{u}(\mathfrak{a}, g, J)$ for $Z \in U_J$. But by Lemma 2.8 gives that all B_Z commute pairwise, so these complex endomorphisms of $(\mathfrak{a}, J)$ are simultaneously diagonalisable with only imaginary eigenvalues. This is the assertion of Lemma 4.5 □

In the case that $\mathfrak{g}'$ is of codimension two, there are no further conditions to be satisfied, cf. [17, §7.2]. Hence

Corollary 4.6 *Let $(\mathfrak{g}, g, J)$ be an almost Hermitian Lie algebra. Then $(\mathfrak{g}, g, J)$ is a two-step solvable SKT Lie algebra of pure type II for which $\mathfrak{g}'$ is of codimension two if and only if there is a complex unitary basis $Y_1, \dots, Y_s$ of $(\mathfrak{g}', g, J)$, elements $Z_1, Z_2 \in \mathfrak{g}$ spanning a two-dimensional complement to $\mathfrak{g}'$,* *and* $(a_{1,1}, a_{1,2}), \dots, (a_{1,s}, a_{2,s}) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *such that (up to anti-symmetry and complex-linear extension) the only non-zero Lie brackets are given by*

$$[Z_k, Y_j] = a_{k,j} J Y_j,$$

for $k = 1, 2$ *and* $j = 1, \dots, s$.

Proof Choose $Z \in V_J \setminus \{0\}$. Then Z, JZ is a basis for V_J and hence the union of the images of $\mathrm{ad}(Z)|_{\mathfrak{g}'}$ and $\mathrm{ad}(JZ)|_{\mathfrak{g}'}$ spans $\mathfrak{g}'$. As $[Z, JZ] \in \mathfrak{g}'$ and $\mathfrak{g}'$ is Abelian, we can find $\tilde{Y}, \hat{Y} \in \mathfrak{g}'$ such that $Z_1 = Z + \tilde{Y}$, $Z_2 = JZ + \hat{Y}$ has $[Z_1, Z_2] = 0$. The result now follows from Lemma 4.5 with $a_{1,j} = \alpha_j(Z)$ and $a_{2,j} = \alpha_j(JZ)$. □

If $\mathfrak{g}$ is six-dimensional, one deduces that $\mathfrak{g}$ admits a dual basis $e^1, \dots, e^6$ whose differentials are given either by

$$(25, -15, 46, -36, 0, 0), \tag{4.2}$$

when $(a_{k,j})$ is of rank two, or by

$$(25, -15, \lambda.45, -\lambda.35, 0, 0) \qquad \text{for some } \lambda \in (0, 1], \tag{4.3}$$

when $(a_{k,j})$ has rank one. In the first case, the Lie algebra is isomorphic to $2\mathfrak{r}'_{3,0}$ and in the second case to $\mathfrak{g}_{5,17}^{0,0,\lambda} \oplus \mathbb{R}$. This covers the remaining equations in [17, Theorem 7.1] and we have

Theorem 4.7 *A six-dimensional two-step solvable Lie algebra* $\mathfrak{g}$ *admits an SKT structure if and only if it is one of the algebras explicitly listed in* [17, Corollary 4.8, Theorems 4.10, 7.5 and 7.1] *or it is one of the algebras in* (4.2) *or* (4.3).

Returning to Corollary 4.6, we see from the fact that $\mathrm{ad}(Z + Y)|_{\mathfrak{g}'} = \mathrm{ad}(Z)|_{\mathfrak{g}'}$ for all $Z \in V_J$ and $Y \in \mathfrak{g}'$, that the SKT condition only depends on $g|_{\mathfrak{g}'}$.

Corollary 4.8 *Let* $(\mathfrak{g}, g, J)$ *be a two-step solvable SKT Lie algebra of pure type II such that* $\mathfrak{g}'$ *is of codimension two in* $\mathfrak{g}$. *If* $\tilde{g}$ *is another Hermitian metric on* $(\mathfrak{g}, J)$ *with* $\tilde{g}|_{\mathfrak{g}'} = g|_{\mathfrak{g}'}$, *then* $\tilde{g}$ *is also SKT.*

Thus, we obtain the desired result in the codimension two case:

Corollary 4.9 *Suppose* $(\mathfrak{g}, J)$ *is a two-step solvable Lie algebra with a complex structure of pure type II such that* $\mathfrak{g}'$ *is of codimension two. If J is SKT and is balanced, the J is also Kähler.*

Proof Let $\tilde{g}$ be the SKT metric and $\hat{g}$ the balanced metric. Write $\hat{V}_J$ for the orthogonal complement of $\mathfrak{g}'$ with respect to $\hat{g}$. Define a new metric g on $\mathfrak{g}$ by declaring $\mathfrak{g}'$ to be g-orthogonal $\hat{V}_J$ and setting

$$g|_{\mathfrak{g}'} = \tilde{g}|_{\mathfrak{g}'}, \quad g|_{\hat{V}_J} = \hat{g}|_{\hat{V}_J}.$$

Then g is compatible with $(\mathfrak{g}, J)$ and both balanced by Corollary 3.6 and SKT by Corollary 4.8. Thus, g is Kähler by Proposition 2.2. □

Next, we consider the general case of two-step solvable SKT Lie algebras and give a full classification of these, up to some "nilpotent terms". This will be sufficient to obtain the generalisation of Corollary 4.9.

Theorem 4.10 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable almost Hermitian Lie algebra of pure type II. Then $(\mathfrak{g}, g, J)$ is SKT if and only if there exists a complex unitary basis of $Y_1, \dots, Y_s$ of $(\mathfrak{g}', g, J)$ and, for some $m \in \{0, \dots, s\}$, one-forms $\alpha_1, \dots, \alpha_m \in V_J^* \setminus \{0\}$, numbers $z_1, \dots, z_m \in \mathbb{C}$, complex $(1, 1)$-forms $\varphi_{m+1}, \dots, \varphi_s \in \Lambda^{1,1} V_J^*$ and complex $(2, 0)$-forms $\psi_{m+1}, \dots, \psi_s \in \Lambda^{2,0} V_J^*$ such that*

$$\sum_{k=m+1}^{s} \varphi_k \wedge \overline{\varphi_k} - \psi_k \wedge \overline{\psi_k} = 0,$$

the two forms $\varphi_k + \psi_k$, $k = m + 1, \dots, s$, are linearly independent, and the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension) are given by

$$[Z, Y_j] = \alpha_j(Z)\, JY_j, \qquad j = 1, \dots, m, \tag{4.4}$$

$$[Z, W] = \sum_{j=1}^{m} z_j\, (\alpha_j \wedge J^*\alpha_j)(Z, W)\, Y_j + \sum_{k=m+1}^{s} (\varphi_k + \psi_k)\, (Z, W)\, Y_k \tag{4.5}$$

for all $Z, W \in V_J$.

In the above, we have used complex notation, so $(x + iy)Z = xZ + yJZ$, etc.

Proof Use Lemma 4.5 to choose a complex unitary basis $Y_1, \dots, Y_s$ so that (4.4) holds, with $\alpha_1, \dots, \alpha_m$ non-zero, and $[Z, Y_k] = 0$, for $k > m$. Using shear data, define

$$\nu := \omega(\,\cdot\,, \,\cdot\,)|_{\Lambda^2 U_J} \in \Lambda^2 U_J^* \otimes \mathfrak{a} = \Lambda^2 U_J^* \otimes \mathfrak{a}_J.$$

In complex notation, we may write $\nu = \sum_{j=1}^{s} \nu_j Y_j$ with $\nu_j \in \Lambda^2 U_J^* \otimes \mathbb{C}$. The first equation of (2.2) yields

$$\begin{aligned} 0 &= \sum_{\text{cyclic}} \omega(\omega(Z_1, Z_2), Z_3) = \sum_{\text{cyclic}} \omega\left(\sum_{j=1}^{s} \nu_j(Z_1, Z_2) Y_j, Z_3\right) \\ &= \sum_{j=1}^{m} \sum_{\text{cyclic}} \alpha_j(Z_3)\nu_j(Z_1, Z_2) JY_j = \sum_{j=1}^{m} (\alpha_j \wedge \nu_j)(Z_1, Z_2, Z_3) JY_j, \end{aligned}$$

for all $Z_k \in U_J$. Hence, there exist complex one-forms $\zeta_j \in U_J^* \otimes \mathbb{C}$, for $j = 1, \dots, m$, such that

$$\nu_j = \alpha_j \wedge \zeta_j. \tag{4.6}$$

Evaluating (2.3) on $Z_1, Z_2, Z_3 \in U_J$ and some $Y \in \operatorname{span}(Y_j, JY_j)$ gives

$$\begin{aligned} 0 &= \sum_{\text{cyclic}} g(\omega(JZ_1, JZ_2), \omega(Z_3, Y)) + g(\omega(JZ_3, JY), \omega(Z_1, Z_2)) \\ &\qquad + g(\omega(J\omega(Z_1, Z_2), JZ_3), Y). \\ &= \sum_{\text{cyclic}} \sum_{k=1}^{s} -g\left(J^*\nu_k(Z_1, Z_2)Y_k, \alpha_j(Z_3)JY\right) + g\left(\alpha_j(JZ_3)Y, \nu_k(Z_1, Z_2)Y_k\right) \\ &\qquad - g\left(\alpha_k(JZ_3)\nu_k(Z_1, Z_2)Y_k, Y\right) \\ &= -\sum_{\text{cyclic}} \alpha_j(Z_3) g\left(J^*\nu_j(Z_1, Z_2)Y_j, JY\right). \end{aligned} \tag{4.7}$$

This holds trivially for $j > m$. For $j \leqslant m$ it gives

$$\left(\alpha_j \wedge J^*\alpha_j \wedge g(J^*\zeta_j(\,\cdot\,)Y_j, JY)\right)(Z_1, Z_2, Z_3) = 0.$$

Taking $Y = Y_j$ and then $Y = JY_j$ we get that $\alpha_j \wedge J^*\alpha_j \wedge J^*\zeta_j = 0$. So $\zeta_j \in \operatorname{span}(\alpha_j, J^*\alpha_j)$ and $\nu_j = -z_j \alpha_j \wedge J^*\alpha_j$ for some $z_j \in \mathbb{C}$.

In complex notation the second equation of (2.2) is $J^*\nu_j = \nu_j - i\,J.\nu_j$. This says that the $(0,2)$-part of ν_j vanishes. For $j \leqslant m$, we already have that ν_j is type $(1,1)$. For $j > m$, we write $\nu_j = -\varphi_j - \psi_j$ with φ_j type $(1,1)$ and ψ_j type $(2,0)$.

Now the only remaining equation to satisfy is (2.3) evaluated on $\Lambda^4 U_J$. In this case, only the first term of (2.3) contributes, since $\operatorname{Im}(J^*\omega) = \mathfrak{a} \perp U_J$, so we have

$$\begin{aligned} 0 &= \mathcal{A}(g(J^*\omega(\,\cdot\,,\,\cdot\,), \omega(\,\cdot\,,\,\cdot\,)))|_{\Lambda^4 U_J} = \sum_{j=1}^{s} \operatorname{Re}(J^*\nu_j \wedge \overline{\nu_j})) \\ &= \sum_{k=m+1}^{s} \operatorname{Re}\left((\varphi_k - \psi_k) \wedge \overline{(\varphi_k + \psi_k)}\right) = \sum_{k=m+1}^{s} \varphi_k \wedge \overline{\varphi_k} - \psi_j \wedge \overline{\psi_j}, \end{aligned}$$

and the claimed result. □

As the metric on V_J plays no role in Theorem 4.10, we get the following version of Corollary 4.8 in arbitrary codimension.

Corollary 4.11 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable SKT Lie algebra of pure type II. If $\tilde{g}$ is another Hermitian metric on $(\mathfrak{g}, J)$ with $\tilde{g}|_{\mathfrak{g}'} = g|_{\mathfrak{g}'}$ and $\mathfrak{g}'^{\perp_{\tilde{g}}} = \mathfrak{g}'^{\perp_g}$, then $\tilde{g}$ is SKT too.*

Moreover, we may also change an SKT metric on a two-step solvable SKT Lie algebra of pure type II in such a way that Theorem 4.10 holds with $z_1 = \cdots = z_k = 0$.

Proposition 4.12 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable SKT Lie algebra of pure type II. Then $(\mathfrak{g}, J)$ admits a compatible SKT metric $\tilde{g}$ with*

$$\mathfrak{g}' = [\tilde{V}_J, \mathfrak{g}'] \oplus [\tilde{V}_J, \tilde{V}_J]$$

as a Hermitian orthogonal direct sum, where $\tilde{V}_J$ is the $\tilde{g}$-orthogonal complement of $\mathfrak{g}'$.

Proof We use the notation from Theorem 4.10. Define an injective $R\colon V_J \to \mathfrak{g}$ by

$$R(Z) := Z + r(Z), \qquad \text{where } r(Z) = \sum_{j=1}^{k} (\alpha_j - iJ^*\alpha_j)(Z) z_j Y_j.$$

As $\alpha_j - iJ^*\alpha_j$ is type $(1,0)$, we have that R is complex linear, so $\tilde{V}_J := R(V_J)$ is a J-invariant complement to $\mathfrak{g}'$ in $\mathfrak{g}$. We get a Hermitian metric $\tilde{g}$ on $\mathfrak{g}$ by declaring $\mathfrak{g}'$ to be to be $\tilde{g}$-orthogonal to $\tilde{V}_J$, letting $\tilde{g}$ be g on $\mathfrak{g}'$ and setting $\tilde{g}|_{\tilde{V}_J} := (R^{-1})^*(g|_{V_J})$.

For $Z \in V_J$ and $Y \in \mathfrak{g}'$, we have $[R(Z), Y] = [Z + r(Z), Y] = [Z, Y]$, so $[R(Z), Y_j] = i\alpha_j(Z) Y_j$ for $j \leqslant m$ and $[R(Z), Y_k] = 0$ for $k > m$. Moreover, we have

$$\begin{aligned}
[r(Z), W] + [Z, r(W)] &= \sum_{j=1}^{m} (\alpha_j - iJ^*\alpha_j)(Z) z_j [Y_j, W] \\
&\qquad + (\alpha_j - iJ^*\alpha_j)(W) z_j [Z, Y_j] \\
&= \sum_{j=1}^{m} ((\alpha_j - i\,J^*\alpha_j) \wedge \alpha_j)(Z, W) z_j JY_j \\
&= -\sum_{j=1}^{m} (z_j\, \alpha_j \wedge J^*\alpha_j)(Z, W)\, Y_j.
\end{aligned}$$

So

$$\begin{aligned}
[R(Z), R(W)] &= [Z, W] + [r(Z), W] + [Z, r(W)] \\
&= \sum_{k=m+1}^{s} (\varphi_k + \psi_k)\,(Z, W)\, Y_k = \sum_{k=m+1}^{s} \left(\tilde{\varphi}_k + \tilde{\psi}_k\right) (R(Z), R(W))\, Y_k,
\end{aligned}$$

where $\tilde{\varphi}_k = (R^{-1})^*\varphi_k$ and $\tilde{\psi}_j = (R^{-1})^*\psi_j$. We may now apply Theorem 4.10, to conclude that $\tilde{g}$ is SKT. The non-vanishing of the α_j, $j = 1, \ldots, m$, gives $[\tilde{V}_J, \mathfrak{g}'] = \operatorname{span}(Y_1, \ldots, Y_m)$, and the linear independence of $\tilde{\varphi}_k + \tilde{\psi}_k$, $k = m+1, \ldots, s$, implies $[\tilde{V}_J, \tilde{V}_J] = \operatorname{span}(Y_{m+1}, \ldots, Y_s)$, so these two spaces are orthogonal. □

These preparations now allow us to prove

Theorem 4.13 *Let $(\mathfrak{g}, J)$ be a unimodular two-step solvable Lie algebra $\mathfrak{g}$ with complex structure J of pure type II that is SKT and is balanced. Then $(\mathfrak{g}, J)$ is Kähler.*

Proof Let $\tilde{g}$ be an SKT metric and $\hat{g}$ be a balanced metric, both compatible with $(\mathfrak{g}, J)$. By Proposition 4.12, we may assume that $\mathfrak{g}'$ splits as an $\tilde{g}$-orthogonal direct sum of the complex spaces $[\tilde{V}_J, \mathfrak{g}']$ and $[\tilde{V}_J, \tilde{V}_J]$. Let $\hat{V}_J$ be the $\hat{g}$-orthogonal complement to $\mathfrak{g}'$. Then there is a complex vector space isomorphism $R\colon \tilde{V}_J \to \hat{V}_J$ of the form $R(Z) = Z + r(Z)$ with $r\colon \tilde{V}_J \to \mathfrak{g}'$ complex linear.

We define a new metric g on $\mathfrak{g}$ by requiring $\mathfrak{g}'$ to be g-orthogonal to $\tilde{V}_J$, putting g to be $\tilde{g}$ on $\mathfrak{g}'$ and letting g on $\tilde{V}_J$ be $R^*(\hat{g}|_{\hat{V}_J})$. This metric g is Hermitian and, by Corollary 4.11, SKT.

To show that g is also balanced, recall Proposition 3.2, which for pure type II implies the existence of a $(\hat{g}, J)$-unitary basis $\hat{Z}_1, \ldots, \hat{Z}_{2\ell}$ of $\hat{V}_J$ with

$$\hat{C} = \sum_{j=1}^{\ell} [\hat{Z}_{2j-1}, \hat{Z}_{2j}] = 0.$$

Defining $Z_j \in \tilde{V}_J$ by $Z_j = R^{-1}(\hat{Z}_k)$, we get a unitary basis for $(\tilde{V}_J, g, J)$. Let $C = \sum_{j=1}^{\ell}[Z_{2j-1}, Z_{2j}]$ which lies in $[\tilde{V}_J, \tilde{V}_J] \subset \mathfrak{g}'$. As $\mathfrak{g}'$ is the g-orthogonal direct sum of $[\tilde{V}_J, \tilde{V}_J]$ and $[\tilde{V}_J, \mathfrak{g}']$, we have for $Y \in [\tilde{V}_J, \tilde{V}_J]$, that

$$\begin{aligned} 0 = g\left(Y, \hat{C}\right) &= \sum_{j=1}^{\ell} g(Y, [R(Z_{2j-1}), R(Z_{2j})]) \\ &= \sum_{j=1}^{\ell} g(Y, [Z_{2j-1}, Z_{2j}]) + g(Y, [Z_{2j-1}, r(Z_{2j})]) + g(Y, [r(Z_{2j-1}), Z_{2j}]) \\ &= \sum_{j=1}^{\ell} g(Y, [Z_{2j-1}, Z_{2j}]) = g(Y, C). \end{aligned}$$

We conclude that $C = 0$. As $\mathfrak{g}$ is unimodular, Corollary 3.5 implies that g is also balanced. By Proposition 2.2, we learn that g is Kähler. □

4.4 Pure Type III

Pure type III means that $V_J = 0$, so $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$.

Theorem 4.14 *Let $(\mathfrak{g}, J)$ be a unimodular two-step solvable Lie algebra $\mathfrak{g}$ with complex structure J of pure type III. Then $(\mathfrak{g}, J)$ cannot be both SKT and balanced.*

For the proof, we first need to recall some facts on two-step solvable SKT Lie algebras from our previous paper. In particular [17, Proposition 3.8 and Corollary 4.4] give:

Lemma 4.15 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable SKT Lie algebra. Then:*

(1) *There exists a complex unitary basis $Y_1, \ldots, Y_s$ of ${\mathfrak{g}'}_J$ and complex-valued one-forms $\xi_j \in ({\mathfrak{g}'}_r)^* \otimes \mathbb{C}$ such that*

$$[JX, Y_j] = \xi_j(X)\, Y_j$$

holds for all $X \in {\mathfrak{g}'}_r$ and all $j \in \{1, \ldots, s\}$.

(2) *If* $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$, *then there exists an* $X_0 \in J\mathfrak{g}'_r$ *such that*

$$[X_0, X] - X \in \mathfrak{g}'_J$$

for all $X \in \mathfrak{g}'_r$.

Proof of Theorem 4.14 Suppose on the contrary that $(\mathfrak{g}, J)$ admits a Hermitian metric $\tilde{g}$ that is SKT and a Hermitian metric g that is balanced. As usual $\mathfrak{g}'_r$ is the g-orthogonal complement of $\mathfrak{g}'_J$ in $\mathfrak{g}'$ and $V_r = \mathfrak{g}'_r \oplus J\mathfrak{g}'_r$. Write $\mathfrak{g}'_{\tilde{r}}$ for the $\tilde{g}$-orthogonal complement of $\mathfrak{g}'_J$ in $\mathfrak{g}'$, and put $V_{\tilde{r}} = \mathfrak{g}'_{\tilde{r}} \oplus J\mathfrak{g}'_{\tilde{r}}$.

Corollary 3.5 gives a (g, J)-unitary basis $X_1, \ldots, X_{2r}$ of V_r with

$$C = \sum_{k=1}^{r} [X_{2k-1}, X_{2k}] = 0. \tag{4.8}$$

As $\mathfrak{g} = \mathfrak{g}'_J + \mathfrak{g}'_{\tilde{r}} + J\mathfrak{g}'_{\tilde{r}}$, we may write each $X \in V_r$ as $X = \tilde{Y} + \tilde{W} + J\tilde{X}$ with $\tilde{Y} \in \mathfrak{g}'_J$ and $\tilde{W}, \tilde{X} \in \mathfrak{g}'_{\tilde{r}}$.

By Lemma 4.15(a) there is a complex unitary basis $Y_1, \ldots, Y_s$ of $(\mathfrak{g}'_J, \tilde{g}, J)$ and $\xi_1, \ldots, \xi_s \in (\mathfrak{g}'_{\tilde{r}})^* \otimes \mathbb{C}$ such that

$$[X, Y_j] = [J\tilde{X}, Y_j] = \xi_j(\tilde{X})\, Y_j$$

for each $j \in \{1, \ldots, s\}$ and $X \in V_r$. Inserting now $X = X_{2k-1}$, $JX = X_{2k}$, $Y = Y_j$, JY into the version of (2.3) for the SKT metric $\tilde{g}$, and writing $z = \xi_j(\tilde{X})$, $w = \xi_j(\tilde{JX})$, yields

$$\begin{aligned}
0 &= -\tilde{g}([JX, JY], [JX, JY]) + \tilde{g}([JX, JJY], [JX, Y]) + \tilde{g}([JJX, JY], [X, JY]) \\
&\quad -\tilde{g}([JJX, JJY], [X, Y]) + \tilde{g}([J[X, JX], JY], JY) \\
&\quad -\tilde{g}([J[X, JX], JJY], Y) - \tilde{g}([J[X, Y], JJX], JY) \\
&\quad +\tilde{g}([J[X, JY], JJX], Y) + \tilde{g}([J[JX, Y], JX], JY) \\
&\quad -\tilde{g}([J[JX, JY], JX], Y) \\
&= -2\tilde{g}(wY, wY) - 2\tilde{g}(zY, zY) + \tilde{g}([J[X, JX], JY], JY) + \tilde{g}([J[X, JX], Y], Y) \\
&\quad -2\tilde{g}(z^2 Y, Y) - 2\tilde{g}(w^2 Y, Y) \\
&= -2(z\overline{z} + \mathrm{Re}(z^2) + w\overline{w} + \mathrm{Re}(w^2)) + \tilde{g}([J[X, JX], JY], JY) \\
&\quad +\tilde{g}([J[X, JX], Y], Y) \\
&= -4(\mathrm{Re}(z)^2 + \mathrm{Re}(w)^2) + \tilde{g}([J[X, JX], JY], JY) + \tilde{g}([J[X, JX], Y], Y) \\
&= -4(\mathrm{Re}(\xi_j(\tilde{X}_{2k-1}))^2 + \mathrm{Re}(\xi_j(\tilde{X}_{2k}))^2) \\
&\quad +\tilde{g}([J[X_{2k-1}, X_{2k}], JY_j], JY_j) + \tilde{g}([J[X_{2k-1}, X_{2k}], Y_j], Y_j).
\end{aligned}$$

Summing over k and using (4.8), we get

$$0 = \sum_{k=1}^{r} \left(\mathrm{Re}(\xi_j(\tilde{X}_{2k-1}))^2 + (\mathrm{Re}(\xi_j(\tilde{X}_{2k}))^2 \right).$$

Thus, $\mathrm{Re}(\xi_j(\tilde{X}_t)) = 0$ for all t, so $\mathrm{Re}(\xi_j(\tilde{X})) = 0$ for all $X \in V_r$ and all j.

Note that

$$\begin{aligned}\operatorname{tr}(\operatorname{ad}(X)|_{\mathfrak{g}'_J}) &= \sum_{j=1}^{s} \tilde{g}([X, Y_j], Y_j) + \tilde{g}([X, JY_j], JY_j) \\ &= \sum_{j=1}^{s} 2\tilde{g}(\xi_j(\tilde{X})Y_j, Y_j) = 2\sum_{j=1}^{s} \operatorname{Re}(\xi_j(\tilde{X})) = 0.\end{aligned}$$

As $\operatorname{Im}(\operatorname{ad}(X)) \subseteq \mathfrak{g}' = \mathfrak{g}'_J \oplus \mathfrak{g}'_{\tilde{r}}$, the unimodularity of $\mathfrak{g}$ gives for $X \in V_r$ that

$$0 = \operatorname{tr}(\operatorname{ad}(X)) = \operatorname{tr}(\operatorname{ad}(X)|_{\mathfrak{g}'_J}) + \operatorname{tr}(\operatorname{ad}(X)|_{\mathfrak{g}'_{\tilde{r}}}) = \operatorname{tr}(\operatorname{ad}(X)|_{\mathfrak{g}'_{\tilde{r}}}).$$

By Lemma 4.15, there exists an $\tilde{X}_0 \in J\mathfrak{g}'_{\tilde{r}}$ with

$$[\tilde{X}_0, \tilde{X}] - \tilde{X} \in \mathfrak{g}'_J$$

for any $\tilde{X} \in \mathfrak{g}'_{\tilde{r}}$. Write

$$\tilde{X}_0 = X_0 + Y_0$$

for $X_0 \in V_r$ and $Y_0 \in \mathfrak{g}'_J$. Then $\operatorname{ad}(Y_0) = 0$ on $\mathfrak{g}'$ and

$$0 = \operatorname{tr}(\operatorname{ad}(X_0)|_{\mathfrak{g}'_{\tilde{r}}}) = \operatorname{tr}(\operatorname{ad}(\tilde{X}_0)|_{\mathfrak{g}'_{\tilde{r}}}).$$

Choosing a $\tilde{g}$-orthonormal basis $\tilde{S}_1, \ldots, \tilde{S}_r$ of $\mathfrak{g}'_{\tilde{r}}$, we have

$$0 = \operatorname{tr}(\operatorname{ad}(\tilde{X}_0)|_{\mathfrak{g}'_{\tilde{r}}}) = \sum_{k=1}^{r} \tilde{g}([\tilde{X}_0, \tilde{S}_k], \tilde{S}_k) = \sum_{k=1}^{r} \tilde{g}(\tilde{S}_k, \tilde{S}_k) = \dim(\mathfrak{g}'_{\tilde{r}}).$$

Thus $\mathfrak{g}'_r = 0$ and $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}'$, contradicting that $\mathfrak{g}$ is solvable. □

We end this section by providing an example that the unimodular condition in Theorem 4.14 is necessary, and which also supports the need for this condition in Question 1.2.

Example 4.16 Let $\mathfrak{g}$ be the six-dimensional Lie algebra with basis $e_1, \ldots, e_6$ whose dual basis $e^1, \ldots, e^6$ has differentials given by

$$(-15 + 16, -25 + 26, 2.(35 + 46), 2.(36 + 45), 0, 0),$$

which is isomorphic to $N_{6,1}^{-1/2,-1/2,0,0}$. Consider the almost complex structure J on $\mathfrak{g}$ given by $Je_1 = e_2$, $Je_3 = e_5$, $Je_4 = e_6$. Then J is integrable, so defines a complex structure on $\mathfrak{g}$. We have $\mathfrak{g}' = \operatorname{span}(e_1, \ldots, e_4)$ and $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$.

Consider the metric $\tilde{g}$ on $\mathfrak{g}$ for which $e_1, \ldots, e_6$ is orthonormal. Then $\tilde{g}$ is compatible with J and the associated fundamental two-form $\tilde{\sigma}$ is

$$\tilde{\sigma} = e^{12} + e^{35} + e^{46}.$$

A direct computation yields $dJ^*d\tilde{\sigma} = 2dJ^*(e^{12} \wedge (e^5 - e^6)) = 2d(e^{12} \wedge (e^3 - e^4)) = 0$, so $\tilde{g}$ is SKT.

Next, consider the metric $\hat{g}$ for which $e_1, e_2, e_3, e_5, e_3 + e_4, e_5 + e_6$ is an orthonormal basis. Since this basis is unitary, $\hat{g}$ is compatible with J. As $e^1, e^2, e^3 - e^4$,

$e^5 - e^6, e^4, e^6$ is the dual of the above basis, the associated fundamental two-form $\hat{\sigma}$ is given by

$$\hat{\sigma} = e^{12} + (e^3 - e^4) \wedge (e^5 - e^6) + e^{46} = e^{12} + e^{35} + 2e^{46} - e^{36} - e^{45}.$$

One computes

$$d(\hat{\sigma}^2) = 2\hat{\sigma} \wedge d\hat{\sigma} = 4\hat{\sigma} \wedge (e^{12} \wedge (e^5 - e^6) + e^{456}) = 0,$$

so $\hat{g}$ is a balanced metric.

Thus, $(\mathfrak{g}, J)$ is a two-step solvable Hermitian Lie algebra of pure type III that is SKT and is balanced.

We claim that $(\mathfrak{g}, J)$ is not Kähler. For contradiction, suppose g is a compatible Kähler metric. Let ${\mathfrak{g}'}_r$ be the orthogonal complement of ${\mathfrak{g}'}_J = \mathrm{span}(e_1, e_2)$ in $\mathfrak{g}'$. Then $J{\mathfrak{g}'}_r$ is a complement of $\mathfrak{g}' = \mathrm{span}(e_1, \dots, e_4)$ in $\mathfrak{g}$ and so it has to contain a vector of the form $e_5 + W$ for some $W \in \mathfrak{g}'$. Moreover, by Corollary 3.12(III), one has $\mathrm{tr}(\mathrm{ad}(V)|_{{\mathfrak{g}'}_J}) = 0$ for any $V \in J{\mathfrak{g}'}_r$. However, $\mathrm{ad}(e_5 + W)(e_i) = [e_5, e_i] = e_i$ for $i = 1, 2$ and so $\mathrm{tr}(\mathrm{ad}(e_5 + W)|_{{\mathfrak{g}'}_J}) = 2$, a contradiction.

4.5 Dimension 6

We can now consider general unimodular six-dimensional two-step solvable Lie algebras $\mathfrak{g}$ endowed with a complex structure J.

Theorem 4.17 *Let $\mathfrak{g}$ be a six-dimensional unimodular two-step solvable Lie algebra endowed with a complex structure J. If $(\mathfrak{g}, J)$ is SKT and is balanced, then it is also Kähler.*

Theorems 4.3, 4.13 and 4.14 give the result when $(\mathfrak{g}, J)$ is of pure type. However, in dimension 6, if $(\mathfrak{g}, J)$ is not of pure type, then we have $\dim({\mathfrak{g}'}_J) = 2$, $\dim({\mathfrak{g}'}_r) = 1$ and $\dim(V_J) = 2$. The SKT Lie algebras of this type are described in detail in [17, Theorem 7.5]. There are three cases, but they share common properties, so that the following holds.

Proposition 4.18 *Let $(\mathfrak{g}, g, J)$ be a six-dimensional two-step solvable SKT Lie algebra which is not of pure type. Then there exist $Y \in {\mathfrak{g}'}_J$, $X \in {\mathfrak{g}'}_r$ and $Z \in V_J$, all non-zero, such that $\mathrm{ad}(Z)$, $\mathrm{ad}(JZ)$ preserve ${\mathfrak{g}'}_J$ and are complex-linear on that space. Additionally there exist $(b_0, b_1, b_2, b_3) \in \mathbb{R}^4 \backslash \{0\}$ and $z_0, z_1, z_2, w_0, \dots, w_5 \in \mathbb{C}$, with*

$$b_0 b_3 + b_1^2 + b_2^2 = 0, \qquad \mathrm{Re}(z_i) = -\delta_i b_i/2, \quad \textit{for } i = 0, 1, 2 \textit{ and some } \delta_i \in \{0, 1\},$$

and with $z_0 = 0$ implying $b_0 = b_1 = b_2 = 0$, such that the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension) are given by

$$\begin{aligned}
[JX, Y] &= z_0 Y, \quad [Z, Y] = z_1 Y, \quad [JZ, Y] = z_2 Y,\\
[JX, X] &= b_0 X + w_0 Y, \quad [Z, X] = b_1 X + w_1 Y, \quad [JZ, X] = b_2 X + w_2 Y,\\
[Z, JX] &= -b_2 X + w_3 Y, \quad [JZ, JX] = b_1 X + w_4 Y, \quad [Z, JZ] = b_3 X + w_5 Y.
\end{aligned}$$

Theorem 4.17 now follows from the following result that does not require $\mathfrak{g}$ to be unimodular.

Proposition 4.19 *Let $\mathfrak{g}$ be a six-dimensional Lie algebra endowed with a complex structure J such that $(\mathfrak{g}, J)$ is not of pure type. If $(\mathfrak{g}, J)$ is SKT and is balanced, then it also Kähler.*

Proof We use the notation of Proposition 4.18. Moreover, let $\tilde{g}$ be a compatible balanced metric on $\mathfrak{g}$, write $\tilde{V}_r$ for the $\tilde{g}$-orthogonal complement of $\mathfrak{g}'_J$ in $\mathfrak{g}' + J\mathfrak{g}'$ and $\tilde{V}_J$ for the $\tilde{g}$-orthogonal complement of $\mathfrak{g}' + J\mathfrak{g}'$ in $\mathfrak{g}$.

Choose a $\tilde{g}$-unit vector $\tilde{X} \in \tilde{V}_r \cap \mathfrak{g}'$. Then $\tilde{X}$ has the form

$$\tilde{X} = \mu_0 X + \tilde{Y}$$

for some $\mu_0 \in \mathbb{R} \setminus \{0\}$ and $\tilde{Y} \in \mathfrak{g}'_J$. We may also find a $\tilde{g}$-unit vector $\tilde{Z} \in \tilde{V}_J$ of the form

$$\tilde{Z} = \mu_1 Z + \mu_2 JX + \mu_3 X + \hat{Y}$$

with $\mu_1 \in \mathbb{R} \setminus \{0\}$, $\mu_2, \mu_3 \in \mathbb{R}$ and $\hat{Y} \in \mathfrak{g}'_J$. Proposition 3.2 has $C = [\tilde{X}, J\tilde{X}] + [\tilde{Z}, J\tilde{Z}]$ and implies

$$\operatorname{tr}(\operatorname{ad}(J\tilde{X})) = -\tilde{\sigma}(C, J\tilde{X}) = -\tilde{g}([\tilde{X}, J\tilde{X}] + [\tilde{Z}, J\tilde{Z}], \tilde{X}). \tag{4.9}$$

We have $\operatorname{tr}(\operatorname{ad}(JX)) = 2\operatorname{Re}(z_0) + b_0 = (1 - \delta_0)b_0$ and hence

$$\operatorname{tr}(\operatorname{ad}(J\tilde{X})) = \mu_0 \operatorname{tr}(\operatorname{ad}(JX)) = \mu_0(1 - \delta_0)b_0.$$

On the other hand, $\tilde{X}$ is $\tilde{g}$-orthogonal to $\mathfrak{g}'_J$, so

$$\begin{aligned}
&\tilde{g}([\tilde{X}, J\tilde{X}] + [\tilde{Z}, J\tilde{Z}], \tilde{X}) \\
&\quad = \tilde{g}([\mu_0 X, \mu_0 JX] + [\mu_1 Z + \mu_2 JX + \mu_3 X, \mu_1 JZ - \mu_2 X + \mu_3 JX], \tilde{X}) \\
&\quad = \tilde{g}\left(\left(-b_0(\mu_0^2 + \mu_2^2 + \mu_3^2) + \mu_1^2 b_3 - 2\mu_1\mu_2 b_1 - 2\mu_1\mu_3 b_2\right) X, \tilde{X}\right) \\
&\quad = \tfrac{1}{\mu_0}\left(-b_0(\mu_0^2 + \mu_2^2 + \mu_3^2) + \mu_1^2 b_3 - 2\mu_1(\mu_2 b_1 + \mu_3 b_2)\right).
\end{aligned}$$

Putting these expressions in to (4.9) times μ_0, we get

$$b_0(\delta_0\mu_0^2 + \mu_2^2 + \mu_3^2) - \mu_1^2 b_3 + 2\mu_1(\mu_2 b_1 + \mu_3 b_2) = 0. \tag{4.10}$$

Recall that $b_0 b_3 + b_1^2 + b_2^2 = 0$. If $b_0 = 0$, then $b_1 = b_2 = 0$, so (4.10) gives $b_3 = 0$, which contradicts (b_0, b_1, b_2, b_3) being non-zero. Thus $b_0 \neq 0$. Now multiplying (4.10) by b_0 gives

$$\begin{aligned}
0 &= b_0^2(\delta_0\mu_0^2 + \mu_2^2 + \mu_3^2) + (b_1^2 + b_2^2)\mu_1^2 + 2\mu_1 b_0(\mu_2 b_1 + \mu_3 b_2) \\
&= b_0^2\delta_0\mu_0^2 + (\mu_1 b_1 + \mu_2 b_0)^2 + (\mu_1 b_2 + \mu_3 b_0)^2.
\end{aligned}$$

As $\mu_0 \neq 0$, we thus have

$$\delta_0 = 0, \qquad \mu_1 b_1 + \mu_2 b_0 = 0 \quad \text{and} \quad \mu_1 b_2 + \mu_3 b_0 = 0.$$

Using this, and since the structure equations give $\operatorname{tr}(\operatorname{ad}(Z)) = (1 - \delta_1)b_1$ and $\operatorname{tr}(\operatorname{ad}(JZ)) = (1 - \delta_2)b_2$, one computes

$$\begin{aligned}
\operatorname{tr}(\operatorname{ad}(\tilde{Z})) &= \mu_1 \operatorname{tr}(\operatorname{ad}(Z)) + \mu_2 \operatorname{tr}(\operatorname{ad}(JX)) = \mu_1(1 - \delta_1)b_1 + \mu_2 b_0 \\
&= -\delta_1\mu_1 b_1 = 2\mu_1 \operatorname{Re}(z_1),
\end{aligned}$$

and similarly $\mathrm{tr}(\mathrm{ad}(J\tilde{Z})) = -\delta_2\mu_1 b_2 = 2\mu_1\mathrm{Re}(z_2)$. By Proposition 3.2, we must have $\mathrm{tr}(\mathrm{ad}(\tilde{Z})) = \mathrm{tr}(\mathrm{ad}(J\tilde{Z})) = 0$, so $\mathrm{Re}(z_i) = 0$ for $i = 1, 2$. Write $z_j = ic_j$, for some $c_j \in \mathbb{R}$, $j = 0, 1, 2$.

Putting $\hat{X} := X + (w_0/(1 - ic_0))Y$, one computes that

$$[J\hat{X}, Y] = ic_0 Y, \qquad [J\hat{X}, \hat{X}] = b_0\hat{X}.$$

Note that, since $b_0 \neq 0$, we must have $c_0 = -iz_0 \neq 0$. Noting that

$$[Z, \hat{X}] = b_1\hat{X} + \hat{w}_1 Y \quad \text{and} \quad [JZ, \hat{X}] = b_2\hat{X} + \hat{w}_2 Y,$$

for some $\hat{w}_1, \hat{w}_2 \in \mathbb{C}$, the Jacobi identity yields

$$\begin{aligned} 0 &= [Z, [J\hat{X}, \hat{X}]] + [J\hat{X}, [\hat{X}, Z]] = [Z, b_0\hat{X}] - [J\hat{X}, b_1\hat{X} + \hat{w}_1 Y] \\ &= b_0 b_1\hat{X} + b_0\hat{w}_1 Y - b_0 b_1\hat{X} - ic_0\hat{w}_1 Y = (b_0 - ic_0)\hat{w}_1 Y. \end{aligned}$$

As $b_0 \neq 0$, we conclude that $\hat{w}_1 = 0$. Similarly, we obtain $\hat{w}_2 = 0$. Thus, setting

$$\check{Z} := Z - \frac{b_1}{b_0} J\hat{X} - \frac{b_2}{b_0}\hat{X},$$

one easily checks that

$$[\check{Z}, \hat{X}] = 0, \quad [J\check{Z}, \hat{X}] = 0, \quad [\check{Z}, Y] = i\tilde{c}_1 Y, \quad [J\check{Z}, Y] = i\tilde{c}_2 Y,$$

for some $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$ and that $[\check{Z}, J\check{Z}] \in \mathfrak{g}'_J$. Moreover, we check that

$$[\check{Z}, J\hat{X}] = \tilde{w} Y, \qquad [J\check{Z}, J\hat{X}] = \hat{w} Y,$$

for certain $\tilde{w}, \hat{w} \in \mathbb{C}$. Then, the vanishing of the Nijenhuis tensor N_J on $(\check{Z}, \hat{X})$ yields $\hat{w} = i\tilde{w}$. Hence, setting

$$\hat{Z} := \tilde{Z} + \frac{\tilde{w}}{ic_0} Y,$$

one calculates

$$[\hat{Z}, J\hat{X}] = [J\hat{Z}, J\hat{X}] = 0.$$

Furthermore, $[\hat{Z}, J\hat{Z}] \in \mathfrak{g}'_J$ so the Jacobi identity yields

$$0 = [\hat{Z}, [J\hat{Z}, J\hat{X}]] + [J\hat{Z}, [J\hat{X}, \hat{Z}]] + [\hat{JX}, [\hat{Z}, J\hat{Z}]] = ic_0[\hat{Z}, J\hat{Z}],$$

and we conclude that $[\hat{Z}, J\hat{Z}] = 0$.

Thus, denoting the basis $Y, JY, \hat{X}, J\hat{X}, \hat{Z}, J\hat{Z}$ of $\mathfrak{g}$ by $e_1, \ldots, e_6$, the differentials of the dual basis $e^1, \ldots, e^6$ are given by

$$(-c_0.24 - \tilde{c}_1.25 - \tilde{c}_2.26, c_0.14 + \tilde{c}_1.15 + \tilde{c}_2.16, b_0.34, 0, 0, 0).$$

The metric g for which $e_1, \ldots, e_6$ is orthonormal is Hermitian for J with associated two-form $\sigma = e^{12} + e^{34} + e^{56}$. But $d\sigma = 0$, so $(\mathfrak{g}, g, J)$ is Kähler. □

Remark 4.20 From the proof of Proposition 4.19, one deduces that the 6-dimensional two-step solvable Lie algebras which admit a Kähler structure of non-pure type are given by

$$(-24, 14, a.34, 0, 0, 0), \qquad (-25, 15, 34, 0, 0, 0)$$

for $a > 0$, with the second case occurring exactly when $(\tilde{c}_1, \tilde{c}_2) \neq 0$. The algebras in the family are almost Abelian and isomorphic to $\mathfrak{r}'_{4,a,0} \oplus \mathbb{R}^2$. The second algebra is isomorphic to $\mathfrak{r}'_{3,0} \oplus \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}$.

From Remark 4.20 and Corollary 3.12 we deduce

Corollary 4.21 *The six-dimensional two-step solvable Lie algebras admitting a Kähler structure are the following ones:*

$$N_{6,14}^{\alpha,\beta,0}\ (\alpha\beta \neq 0),\ \mathfrak{g}_{6,11}^{\alpha,0,0,\delta}\ (\alpha\delta \neq 0),\ \mathfrak{g}_{5,17}^{0,0,\lambda} \oplus \mathbb{R}\ (\lambda \in (0,1]),\ \mathfrak{r}'_{4,a,0} \oplus \mathbb{R}^2\ (a > 0),$$
$$\mathfrak{r}'_{3,0} \oplus \mathfrak{r}'_{3,0},\ \mathfrak{r}'_{3,0} \oplus \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R},\ \mathfrak{r}'_{3,0} \oplus \mathbb{R}^3,\ 3\mathfrak{aff}_{\mathbb{R}},\ 2\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^2,\ \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^4,\ \mathbb{R}^6.$$

Proof By Remark 4.20, $\mathfrak{r}'_{4,a,0} \oplus \mathbb{R}^2$ for $a > 0$ and $\mathfrak{r}'_{3,0} \oplus \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}$ are precisely the six-dimensional two-step solvable Lie algebras admitting a Kähler structure which are not of pure type.

For pure type I, Corollary 3.12(I) gives that the algebras are $k\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^{6-2k}$ for $k \in \{0, \ldots, 3\}$.

For pure type II, we use Corollary 3.12(II). This gives (a) for $\dim(\mathfrak{g}'_J) = 2$, the algebra

$$(-23, 13, 0, 0, 0, 0),$$

which is $\mathfrak{r}'_{3,0} \oplus \mathbb{R}^3$ and (b) for $\dim(\mathfrak{g}'_J) = 4$, one of

$$(-25, 15, -46, 36, 0, 0), \qquad (-25, 15, -\lambda.45, \lambda.35, 0, 0)$$

for $\lambda \in (0, 1]$, which are $\mathfrak{r}'_{3,0} \oplus \mathfrak{r}'_{3,0}$ and $\mathfrak{g}_{5,17}^{0,0,\lambda} \oplus \mathbb{R}$, respectively.

Finally, for pure type III, Corollary 3.12(III) gives (a) for $\dim(\mathfrak{g}'_J) = 2$ the algebras

$$(-25 - c.26, 15 + c.16, a_1.35, a_2.46, 0, 0)$$

for some $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$, which are isomorphic to $N_{6,14}^{\alpha,\beta,0}$ for certain $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, and (b) for $\dim(\mathfrak{g}'_J) = 4$, the algebras

$$(-26, 16, -c.46, c.36, a.56, 0)$$

for certain $a, c \in \mathbb{R} \setminus \{0\}$, which are isomorphic to $\mathfrak{g}_{6,11}^{\alpha,0,0,\delta}$ for $\alpha, \delta \in \mathbb{R} \setminus \{0\}$. □

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Declarations

Conflict of Interest The authors declare no competing interests.

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