



Hard Lefschetz Property for Isometric Flows

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Received: 6 April 2021 / Accepted: 18 May 2022 / Published online: 8 July 2022

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Abstract

The hard Lefschetz property (HLP) is an important property which has been studied in several categories of the symplectic world. For Sasakian manifolds, this duality is satisfied by the basic cohomology (so, it is a transverse property), but a new version of the HLP has been recently given in terms of duality of the cohomology of the manifold itself in [1]. Both properties were proved to be equivalent (see [2]) in the case of K-contact flows. In this paper, we extend both versions of the HLP (transverse and not) to the more general category of isometric flows, and show that they are equivalent. We also give some explicit examples which illustrate the categories where the HLP could be considered.

Keywords Lefschetz hard property · Contact manifolds · Isometric flow

Mathematics Subject Classification (2010) 53C12 · 53D10 · 53C25

Introduction

The origins of the hard Lefschetz property (HLP in the sequel) go back to Lefschetz's study of topological properties of algebraic real projective varieties [3], where he

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proved that the repeated cup product by the cohomology class of a hyperplane gives an isomorphism in the cohomology of the variety. Later, a version of that theorem was proved by Hodge (see [4]) for general compact Kähler manifolds, stating isomorphisms between de Rham cohomology groups of complementary degrees given by multiplication by a power of the symplectic form. This property was considered to be one of the most important of this class of manifolds. Compact Kähler manifolds have very strong and particular cohomological properties. A lot of effort was put into distinguishing such properties which could characterize Kähler manifolds within the category of compact symplectic manifolds. Now we know among other things that

- There are compact symplectic manifolds which are not Kähler, cf. [5] for the first such an example;
- The torus is the only nilmanifold which is Kähler, cf. [6];
- There are compact symplectic manifolds whose cohomology ring is formal which are not Kähler, cf. [7];
- There are compact Hermitian manifolds with collapsing Frölicher spectral sequence which are not Kähler, cf. [8];
- There are compact symplectic manifolds satisfying the HLP which are not Kähler, cf. [9]. However, a nilmanifold having the HLP is diffeomorphic to a torus, cf. [6], thus a Kähler manifold.

Over the years, many examples have been discussed and published. Among other publications, let us mention the papers [10–15] and the book [16].

Within the realm of foliations, the foundations of the theory of transversely Kähler foliations were presented by El Kacimi in [17]. Cordero and Wolak in two papers presented a series of examples showing that the corresponding transverse properties of the basic cohomology do not characterize transversely Kähler foliations, cf. [18, 19].

These results proved to be of particular importance in the study of the odd-dimensional counterpart of Kähler manifolds, i.e., Sasakian manifolds. In particular, by [17, par. 3.4.7], the basic cohomology of a compact Sasakian manifold satisfies the HLP. In recent years, a lot of research has been done to distinguish Sasakian manifolds within the class of contact metric manifolds and K-contact manifolds in particular, e.g., cf. [1, 20, 21].

One of the properties used in these considerations was a new version of the HLP for Sasakian manifolds demonstrated in [22] which stated Lefschetz-type isomorphisms not for the basic cohomology groups, but for the de Rham groups of the manifold itself. The authors of that paper extended the scope of the property by giving a definition of Lefschetz contact manifold in the same global terms. Examples of non-Sasakian Lefschetz contact manifolds have been given in [22] and [23], all of them within the category of isometric flows (i.e., the Reeb field associated to the contact structure is a Killing vector field).

So, a priori there are two different properties (global and basic) that a contact manifold may or may not satisfy, both of them generalizing the HLP satisfied by Sasakian manifolds. In [2], the author proves that both properties are equivalent for compact K-contact manifolds. To define the Lefschetz map, the author uses the symplectic Hodge theory. We do not know whether that equivalence is held for all contact flows.

Lefschetz-type isomorphisms also exist in the realm of isometric flows, where the role of the class of the symplectic form is played by the Euler class. In this work, we define two duality properties for isometric flows which resemble the HLP: a transversal one *THL* and a global one *HL*. Although our definition is essentially topological and no symplectic structure is needed, in the case of K-contact flows, our new definitions agree with the previous versions of the HLP introduced above. In Section 1, we prove that both properties are equivalent for isometric flows. So, we can call Lefschetz isometric flows the isometric flows satisfying *THL* or *HL*.

In Fig. 1, we show the categories where the HLP has been defined. The HLP is satisfied in the rectangular region. We do not know whether the shaded area is nonempty (that is, whether there exist Lefschetz contact flows which are not K-contact), but all other regions are, as we illustrate with some examples in Section 2. In Example 2.2, we provide a Lefschetz isometric flow which does not admit a contact structure. In order to find an example of a flow which is contact, Lefschetz but not isometric, we have to look for a flow which is not Riemannian as the Lefschetz condition ensures tautness in the Riemannian realm; thus, our flow would be isometric. In the case of transversely symplectic but not Riemannian foliations, we can encounter infinite dimensional basic cohomology which makes the Lefschetz condition problematic, as happens in Example 2.9. We do not know whether the transversal and the global definitions of the HLP are equivalent if the contact flow is not isometric. In the referred example, neither of them is satisfied, but the problems appear at non-corresponding degrees, differently as in the isometric case.

1 Lefschetz Duality and Transverse Duality for Isometric Flows

1.1 Preliminaries

Throughout this section, (M, g) denotes a closed Riemannian manifold endowed with an isometric flow \mathcal{F} , that is, a 1-dimensional foliation defined by the orbits of a locally free \mathbb{R} -action by isometries. Let X be the unit vector field defining the

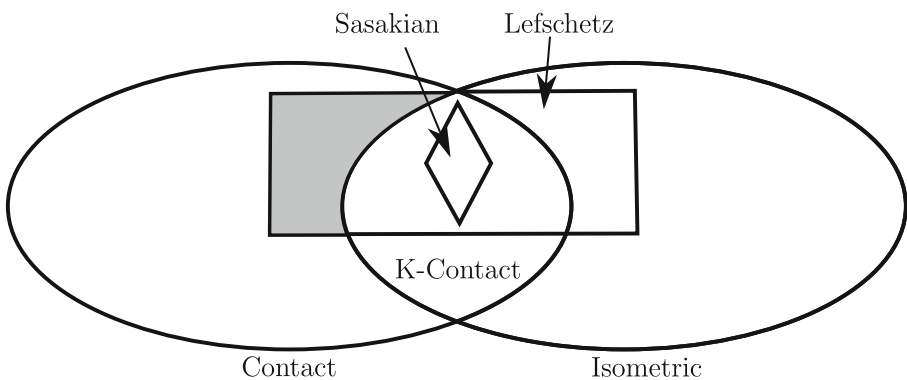


Fig. 1 Some categories where the HLP has been considered

flow. H_M^* and H_B^* stand for the de Rham cohomology of M and the basic cohomology of the flow, respectively. The latter is the cohomology of the complex of *basic forms* $\{\omega \in \Omega(M) \mid i_X \omega = i_X d\omega = 0\}$. The closure of \mathbb{R} in the group of isometries $\text{Iso}(M, g)$ is an abelian compact and connected group, and hence, a torus G . We have an isomorphism $\Psi^* : H_M^* \rightarrow H^*(\Omega(M)^G)$ between the de Rham and the G -invariant cohomology groups (see [24, Th. 1 in p. 151]).

We have the *Gysin exact sequence* (see [25, Th. 6.13]):

$$\dots \longrightarrow H_B^{k-2} \xrightarrow{\varepsilon_{k-1}} H_B^k \xrightarrow{\iota_k} H_M^k \xrightarrow{\rho_k} H_B^{k-1} \xrightarrow{\varepsilon_k} H_B^{k+1} \longrightarrow \dots, \tag{1.1}$$

where ι_k is induced by the natural inclusion of the basic complex into the de Rham complex, ε_k is the multiplication by the Euler Class $[e] = [d\chi] \in H_B^2$ of \mathcal{F} , being $\chi = i_X g$ the characteristic form of \mathcal{F} , and $\rho_k = i_X \circ \Psi^*$, being i_X the contraction operator (notice that $\rho_k([\omega]) = [i_X \omega]$ when ω is X -invariant). In the literature, the multiplication by the Euler class is also known as the *Lefschetz operator*, and is denoted by $L : H_B^* \xrightarrow{\wedge [e]} H_B^{*+2}$.

1.2 Hard Lefschetz Duality Properties

Definition 1.1 Let \mathcal{F} be an isometric flow on the closed manifold M , where $\dim M = 2n + 1$, and let $[e] \in H_B^2$ denote its Euler class. We will say that \mathcal{F} satisfies the *transversal hard Lefschetz property at degree $k \in \mathbb{Z}$* if the following property holds:

$$(THL)_k : L^{n-k} : H_B^k \longrightarrow H_B^{2n-k} \text{ is an isomorphism,}$$

where $L^{n-k}([\beta]) = [\beta \wedge e^{n-k}]$. We also define the following properties:

$$(THL)_{\leq k} : (THL)_j \text{ holds for every } j \leq k$$

$$(THL) : (THL)_j \text{ holds for every } j \in \mathbb{Z}.$$

In this last case, we will say that \mathcal{F} satisfies the *transversal hard Lefschetz property*.

Remark 1.2 $(THL)_k$ holds trivially if $k < 0$ or $k > 2n$. For $k = 0$, on one hand, if \mathcal{F} is transversally symplectic, then $(THL)_0$ is satisfied. On the other hand, one can easily construct an \mathbb{S}^1 -principal bundle over $B = \mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ with a nontrivial Euler class (say, $[e] = [dx_1 \wedge dx_2] \in H_B^2$). We have that $[e] \neq 0$, but $[e^2] = 0$, and thus $(THL)_0$ does not hold.

Definition 1.3 We define the *k -th basic primitive cohomology group* as the kernel of the map $H_B^k \xrightarrow{\wedge [e]^{n-k+1}} H_B^{2n-k+2}$, that is, $PH_B^k = \left\{ [\beta] \in H_B^k \mid [\beta \wedge e^{n-k+1}] = 0 \text{ in } H_B^{2n-k+2} \right\}$.

Definition 1.4 We will say that \mathcal{F} satisfies the *k -th primitive condition* (and denote it by P_k) if the inclusion of forms induces the following two isomorphisms:

$$(P_1)_k: \quad i_k = \iota_k|_{PH_B^k}: PH_B^k \xrightarrow{\cong} H_M^k;$$

$$(P_2)_k: \quad H_B^k = PH_B^k \oplus L(H_B^{k-2}).$$

Remark 1.5 Notice that as $PH_B^0 = H_B^0 = H_M^0$, then P_0 is always true. $(P_1)_1$ is not always true, but for every $\beta \in \Omega_B^1$ we have $\beta \wedge e^n \in \Omega_B^{2n+1} = 0$, so we have $PH_B^1 = H_B^1$, and then $(P_2)_1$ always holds.

Lemma 1.6 For every $k \leq n$,
 $(THL)_{k-1} \Rightarrow \iota_k: H_B^k \longrightarrow H_M^k$ is an epimorphism.

Proof By Remark 1.5, the result holds trivially for $k = 0$. If $k \geq 1$, by $(THL)_{k-1}$, $L^{n-k+1} = L \circ L \circ \dots \circ L$ is an isomorphism, and thus, a monomorphism. So, the first map in that composition $L = \varepsilon_k: H_B^{k-1} \longrightarrow H_B^{k+1}$ must be a monomorphism, too. The exactness of the Gysin sequence (1.1) implies that ι_k is an epimorphism. \square

Remark 1.7 By degree reasons $\iota_1: H_B^1 \rightarrow H_M^1$ is always a monomorphism (this holds for any foliation). So, by Remark 1.5 and Lemma 1.6, we have that $(THL)_0$ implies P_1 .

Proposition 1.8 For every $k \leq n$,

$$\left. \begin{array}{l} (THL)_{k-1} \\ \text{and} \\ (THL)_{k-2} \end{array} \right\} \implies P_k$$

Proof By Remarks 1.5 and 1.7, we have P_0 and P_1 . Consider $k \geq 2$. On one hand, the Gysin sequence (1.1) gives

$$H_B^k \cong [im \iota_k \oplus \ker \iota_k \cong H_M^k \oplus [im \varepsilon_{k-1} \cong H_M^k \oplus L(H_B^{k-2})], \tag{1.2}$$

where we have used that ι_k is an epimorphism, which holds by $(THL)_{k-1}$ and Lemma 1.6. On the other hand, we consider the sum $PH_B^k + L(H_B^{k-2}) \leq H_B^k$. We now show that the sum is a direct one: take $[\beta] \in PH_B^k \cap L(H_B^{k-2})$. Then, there exists $[\gamma] \in H_B^{k-2}$ such that $[\beta] = [\gamma \wedge e] \in PH_B^k$, which implies

$$0 = [\beta \wedge e^{n-k+1}] = [\gamma \wedge e^{n-k+2}] = L^{n-k+2}([\gamma]),$$

and by $(THL)_{k-2}$ we have $[\gamma] = 0$. Thus, $[\beta] = 0$, and the sum is direct. From Eq. 1.2, we get

$$PH_B^k \oplus L(H_B^{k-2}) \leq H_B^k = H_M^k \oplus L(H_B^{k-2}), \tag{1.3}$$

which implies that $\dim PH_B^k \leq \dim H_M^k$. Hence, if we prove that $i_k : PH_B^k \longrightarrow H_M^k$ is an epimorphism, it would be an isomorphism, yielding $(P_1)_k$ and, by Eq. 1.3, $(P_2)_k$.

We complete the proof by showing that i_k is an epimorphism. Let $[\alpha] \in H_M^k$. By $(THL)_{k-1}$ and Lemma 1.6, there exists $[\beta] \in H_B^k$ such that $\iota_k([\beta]) = [\alpha]$. From $(THL)_{k-2}$, there exists $[\gamma] \in H_B^{k-2}$ such that

$$[\beta \wedge e^{n-k+1}] = L^{n-k+2}([\gamma]) = [\gamma \wedge e^{n-k+2}] \in H_B^{2n-k+2},$$

which leads to $[(\beta - \gamma \wedge e) \wedge e^{n-k+1}] = 0 \in H_B^{2n-k+2}$ and thus, $[\beta - \gamma \wedge e] \in PH_B^k$. Finally, we have

$$i_k([\beta - \gamma \wedge e]) = [\beta - d(\chi \wedge \gamma)] = [\beta] \in H_M^k.$$

□

Definition 1.9 Let \mathcal{F} be an isometric flow on the closed manifold M , where $\dim(M) = 2n + 1$, and let $[e] \in H_B^2$ denote its Euler class. We say that \mathcal{F} satisfies the *hard Lefschetz property at degree k* (and denote it by $(HL)_k$) if there exists an isomorphism $\mathcal{L}^{n-k} : H_M^k \longrightarrow H_M^{2n-k+1}$ making the following diagram commutative:

$$\begin{array}{ccc}
 H_M^k & \xrightarrow{\mathcal{L}^{n-k}} & H_M^{2n-k+1} \\
 \uparrow i_k & & \downarrow \rho_{2n-k+1} \\
 PH_B^k \hookrightarrow H_B^k & \xrightarrow{L^{n-k}} & H_B^{2n-k}
 \end{array} \tag{1.4}$$

We shall also use the following notations:

$$\begin{aligned}
 (HL)_{\leq k} : & \quad (HL)_j \text{ holds for every } j \leq k \\
 (HL) : & \quad (HL)_j \text{ holds for every } j \in \mathbb{Z}.
 \end{aligned}$$

In this last case, we will say that \mathcal{F} satisfies the *hard Lefschetz property*.

Theorem 1.10 *Let \mathcal{F} be an isometric flow on a closed oriented manifold M of dimension $2n + 1$. Then, for every $k \leq n$, we have $(THL)_{\leq k} \iff (HL)_{\leq k}$.*

Proof \Rightarrow Assume $(THL)_{\leq k}$. By Proposition 1.8, P_k is true, and so i_k is an isomorphism. To define \mathcal{L}^{n-k} , we fix a basis $\{[\beta_i]\}_i$ of PH_B^k . As $[\beta_i \wedge e^{n-k+1}] = 0 \in H_B^{2n-k+2}$, there exists a basic form $\gamma_i \in \Omega_B^{2n-k+1}$ such that $\beta_i \wedge e^{n-k+1} = d\gamma_i$, and so, $\chi \wedge \beta_i \wedge e^{n-k} - \gamma_i \in \Omega_M^{2n-k+1}$ is a closed form. Thus, we can define

$$\mathcal{L}^{n-k}([\beta_i]) = [\chi \wedge \beta_i \wedge e^{n-k} - \gamma_i] \tag{1.5}$$

and extend it by linearity. As γ_i is basic and $\chi \wedge \beta_i \wedge e^{n-k} - \gamma_i$ is X -invariant, we have

$$\rho_{2n-k+1}([\chi \wedge \beta_i \wedge e^{n-k} - \gamma_i]) = [i_X(\chi \wedge \beta_i \wedge e^{n-k} - \gamma_i)] = [\beta_i \wedge e^{n-k}],$$

and the diagram (1.4) is commutative. Finally, from Eq. 1.4, we have

$$\ker \mathcal{L}^{n-k} \leq \ker(\rho \circ \mathcal{L}^{n-k}) \leq \ker(L^{n-k}|_{PH_B^k} \circ (i_k)^{-1}) = \{0\},$$

because i_k and L^{n-k} are isomorphisms. So, \mathcal{L}^{n-k} is a monomorphism between H_M^k and H_M^{2n-k+1} , who have the same dimension by Poincaré duality. Hence, an isomorphism.

⊞ First notice that \mathcal{F} is transversally orientable, and by [26, Th. A] and [27, Th. 4.10], H_B^* satisfies the Poincaré duality. In particular, H_B^k and H_B^{2n-k} have the same dimension. So, in order to prove that L^{n-k} is an isomorphism, it suffices to show that L^{n-k} is an epimorphism.

As the statement is trivial for $k < 0$, we shall proceed by induction on k starting at $k = -2$. Assume that $(HL)_{\leq k}$ holds and assume $(HL)_{\leq k-1} \Rightarrow (THL)_{\leq k-1}$. By Proposition 1.8, P_k holds, giving that i_k is an isomorphism. To show that L^{n-k} is onto, we now take $[\varphi] \in H_B^{2n-k}$. By $(THL)_{k-2}$, L^{n-k+2} is an isomorphism, and so, there exists $[\gamma] \in H_B^{k-2}$ such that

$$[\varphi \wedge e] = L^{n-k+2}[\gamma] = [\gamma \wedge e^{n-k+2}] \in H_B^{2n-k+2}.$$

We have the following commutative diagram,

$$\begin{array}{ccccc}
 H_M^k & \xrightarrow[\cong]{\mathcal{L}^{n-k}} & H_M^{2n-k+1} & & \\
 \uparrow \cong & & \downarrow \rho_{2n-k+1} & & \\
 PH_B^k \subset H_B^k & \xrightarrow{L^{n-k}} & H_B^{2n-k} & & \\
 & & \downarrow \varepsilon_{2n-k+1} & & \\
 & & H_B^{k-2} & \xrightarrow[\cong]{L^{n-k+2}} & H_B^{2n-k+2}
 \end{array} \tag{1.6}$$

whose right column is part of the Gysin sequence (1.1). We have

$$[\varphi - \gamma \wedge e^{n-k+1}] \in \ker \varepsilon_{2n-k+1} = \text{im} \rho_{2n-k+1},$$

and so, by Eq. 1.6, there exists $[\beta] \in H_B^k$ such that

$$[\beta \wedge e^{n-k}] = L^{n-k}[\beta] = [\varphi - \gamma \wedge e^{n-k+1}],$$

which implies

$$[\varphi] = [\beta \wedge e^{n-k} + \gamma \wedge e^{n-k+1}] = [(\beta + \gamma \wedge e) \wedge e^{n-k}] = L^{n-k}([\beta + \gamma \wedge e]),$$

which concludes the proof. □

Now, the following definition makes sense.

Definition 1.11 We say that an isometric flow \mathcal{F} on a closed manifold M is an *isometric Lefschetz flow* if it satisfies (THL) or (HL) .

Remark 1.12 Given an isometric flow \mathcal{F} on a compact manifold, in [28, Section 3.2], it is proved that the Euler classes associated to two invariant metrics are the same up to a multiplicative nonzero constant. As a result, whether an isometric flow is

Lefschetz or not is a topological property in the sense that it does not depend on the chosen invariant metric, but only on the foliation \mathcal{F} itself.

In [1], the authors define a $(2n + 1)$ -dimensional contact manifold (M, η) with Reeb vector field ξ to be a *Lefschetz contact manifold* if for every $k \leq n$, the relation between H_M^k and H_M^{2n+1-k} defined by

$$\mathcal{R}^k = \left\{ \left([\beta], \left[\eta \wedge (d\eta)^{n-k} \beta \right] \right) \mid \beta \in \Omega_M^k, d\beta = 0, i_\xi \beta = 0, (d\eta)^{n-k+1} \wedge \beta = 0 \right\} \tag{1.7}$$

is the graph of an isomorphism $H_M^k \cong H_M^{2n-k+1}$.

Remark 1.13 Notice that \mathcal{R}^0 is always the graph of the isomorphism $H_M^0 \cong H_M^{2n+1}$ because $(d\eta)^{n+1} = 0$.

We now see that if ξ is Killing (i.e., we have a K-contact flow), then this notion is equivalent to (HL).

Proposition 1.14 *Let (M, η) be a $(2n + 1)$ -dimensional K-contact manifold. Then, the isometric flow defined by its Reeb vector field is an isometric Lefschetz flow if and only if (M, η) is a Lefschetz contact manifold.*

Proof We have that $X = \xi$ is a Killing vector field, $\chi = \eta$ and $e = d\eta$. If (M, η) is a Lefschetz contact manifold, then for every $k \leq n$, the isomorphism \mathcal{L}^{n-k} whose graph is the relation (1.7) clearly makes the diagram (1.4) commutative and so, X defines an isometric Lefschetz flow. Conversely, if X is a Lefschetz isometric flow, it satisfies (THL) and we can construct an isomorphism $\mathcal{L}^{n-k} : H_M^k \rightarrow H_M^{2n-k+1}$ as in the \Rightarrow part of Theorem 1.10. As (M, η) is contact, by [29, Th. 11(1)], each basic cohomology class has a harmonic representative, and thus, each primitive basic cohomology class admits a primitive basic representative¹. So, as $H_M^k \cong PH_B^k$, we can find a basis $\{[\beta_i]\}_i$ of H_M^k , where β_i are primitive closed basic forms. In the proof of Theorem 1.10, we have to add the forms γ_i to get closed forms, but as the β_i are primitive, we can choose $\gamma_i = 0$ and so $\mathcal{L}^{n-k}([\beta]) = [\chi \wedge \beta \wedge e^{n-k}]$ defines an isomorphism whose graph is the relation (1.7). Thus, (M, η) is a Lefschetz contact manifold. □

In [1], the authors prove that the small odd Betti numbers (up to the middle dimension) of a Lefschetz Contact flow are even. As we show now, the same algebraic proof works to prove the corresponding result for isometric Lefschetz flows.

Theorem 1.15 *Let \mathcal{F} be an isometric Lefschetz flow on the compact manifold M , being $\dim(M) = 2n + 1$. Then, the Betti number $b_k(M)$ is even for every odd $k \leq n$.*

¹This is [2, Lemma 2.11], which can also be proved by the last paragraph of the proof of Theorem 0.1 of [30], which applies verbatim to the complex of basic forms of the contact manifold.

Proof Consider a basis of primitive basic classes $\{[\beta_i]\}_i$ of H_M^k , and construct an isomorphism $\mathcal{L}^{n-k}: H_M^k \rightarrow H_M^{2n-k+1}$ as in the proof of Theorem 1.10. Consider the non-degenerate bilinear form B on H_M^k defined as the composition

$$H_M^k \otimes H_M^k \xrightarrow{1_M \otimes \mathcal{L}^{n-k}} H_M^k \otimes H_M^{2n-k+1} \xrightarrow{P} \mathbb{R},$$

being $P([\omega_1], [\omega_2]) = \int_M \omega_1 \wedge \omega_2$ the usual non-degenerate pairing. Now, we have

$$\begin{aligned} B([\beta_i], [\beta_j]) &= P([\beta_i], \mathcal{L}^{n-k}[\beta_j]) \\ &= \int_M (\beta_i \wedge \chi \wedge \beta_j \wedge e^{n-k} - \beta_i \wedge \gamma_i) \\ &= \int_M \beta_i \wedge \chi \wedge \beta_j \wedge e^{n-k} \end{aligned}$$

where we have used that $\beta_i \wedge \gamma_i = 0$ because it is a basic form of degree $2n + 1$. As $\beta_i \chi \beta_j = (-1)^k \beta_j \chi \beta_i$, it follows that $B([\beta_i], [\beta_j]) = (-1)^k B([\beta_j], [\beta_i])$. So, B is a non-degenerate skew-symmetrical bilinear form, and the dimension of H_M^k must be even. □

2 Examples

Example 2.1 (Sasakian manifolds) Consider a Sasakian manifold M of dimension $2n + 1$ (for the essentials of Sasakian geometry, we refer the reader to [31]). Recall that its associated Reeb vector field X defines an isometric flow with respect to the metric g of the Sasakian structure, being the associated contact form $\chi = i_X g$ the characteristic form of the isometric flow. As any Sasakian manifold is transversally Kähler, it satisfies (THL) (cf.[17, 3.4.7]), and by Theorem 1.10, it satisfies (HL), which has been proved in [1, Section 4].

In [32], Boothby and Wang use the construction described by Kobayashi in [33, Th. 2]) to get examples of contact manifolds out of integral symplectic forms. The same construction can also be applied to get isometric flows with a prescribed integral Euler form as follows: given an integral closed form $\omega \in \Omega^2(B)$, Kobayashi’s construction gives an S^1 -principal bundle $\pi: M \rightarrow B$ whose connection form $\chi \in \Omega^1(M)^{S^1}$ satisfies $d\chi = \pi^*\omega$. Let \mathcal{F} be the foliation on M defined by the orbits of the principal S^1 -action and consider on $TM = T\mathcal{F} \oplus \ker \chi$ the Riemannian metric $g = \chi \otimes \chi + \pi_b^* g_B$, being g_B any metric on B and π_b the restriction of $\pi_*: TM \rightarrow TB$ to $\ker \chi$, which is an isomorphism by degree reasons. Then, \mathcal{F} is an isometric flow on M whose Euler form is $d\chi = \pi^*\omega$. We shall use this construction in the following examples. First, we show that Theorem 1.10 applies to new cases outside the contact category:

Example 2.2 Let $B = \mathbb{C}P^2 \# \mathbb{C}P^2$. Recall that its cohomology is given by:

$$\begin{aligned} H_B^0 &= \mathbb{R} & H_B^2 &= \langle [a], [b] \rangle = \mathbb{R} \oplus \mathbb{R} \\ H_B^1 &= H_B^3 = 0 & H_B^4 &= \langle [a]^2 \rangle = \langle [b]^2 \rangle = \mathbb{R}, \end{aligned}$$

where a and b can be assumed to be integral. Choose $e = a$ (we could also take $e = b$). Using Kobayashi’s construction, we construct a 5-manifold M with an isometric flow \mathcal{F} whose orbit space is B and with Euler class $[e] = [a] \in H_B^2$. As B does not admit almost-complex structures [34, Prop. 1.3.1], then it does not admit a symplectic structure, and M cannot be a contact flow. On the other hand, as the Lefschetz maps:

$$L^2: H^0(B) \xrightarrow{\wedge [a]^2} H^4(B) \quad L^1: H^1(B) \xrightarrow{\wedge [a]} H^3(B) \quad L^0: H^2(B) \xrightarrow{Id} H^2(B)$$

are isomorphisms, the flow satisfies (THL) and thus (HL) . So, \mathcal{F} is a Lefschetz isometric flow that cannot be generated by the Reeb vector field of any contact structure.

Remark 2.3 The manifold $\mathbb{C}P^2 \# \mathbb{C}P^2$ is an example of a compact c-symplectic manifold (cohomologically symplectic manifold) which is not symplectic, as defined by Lupton and Oprea in [35]. The authors also suggest a method of construction of other compact c-symplectic manifolds in [36]. Therefore using such examples, we can construct more Lefschetz flows which are not transversely symplectic, and thus not contact.

Example 2.4 It is very easy to construct isometric flows that are not Lefschetz flows. For example, let the circle \mathbb{S}^1 act by multiplication on the first factor of $M = \mathbb{S}^1 \times B$ for any B and call \mathcal{F} the flow associated to that free action. Let π_1 and π_2 stand for the projections of M onto \mathbb{S}^1 and B , respectively. Consider the metric $g = \pi_1^* g_{\mathbb{S}^1} + \pi_2^* g_B$ induced by any invariant metric in \mathbb{S}^1 and any metric in B . Recall the characteristic form $\chi = i_X g$, being X the unit vector field. It is straightforward to check locally that $d\chi = 0$, i.e., the Euler form (hence, the Euler class) vanishes. So, \mathcal{F} is not a Lefschetz flow.

The following two lemmata will be useful to find integral closed forms within a family.

Lemma 2.5 *Let $n \in \mathbb{N}$. The only polynomial with real coefficients in n variables that vanishes in a lattice of \mathbb{R}^n is the zero polynomial.*

Proof Let q be a real polynomial vanishing in a lattice Λ of \mathbb{R}^n . For any nonzero $z \in \Lambda$, the restriction of q to the line joining the origin and z is a polynomial in one variable with an infinite amount of zeroes, hence zero. In particular, q vanishes in any point of \mathbb{R}^n with rational coefficients with respect to any basis of Λ , and by density, in all \mathbb{R}^n . □

Lemma 2.6 *Let $H_M^2 = \langle [\omega_1], \dots, [\omega_n] \rangle$ and let p be a nontrivial polynomial with real coefficients in n variables. Then, the following subset contains an integral closed form:*

$$W_p = \{c_1\omega_1 + \dots + c_n\omega_n \mid c_i \in \mathbb{R} \text{ and } p(c_1, \dots, c_n) \neq 0\} \subset \Omega^2(M).$$

Proof Denote by $H_b^2(M, \mathbb{Z})$ the Betti part of $H^2(M, \mathbb{R})$, i.e., the natural inclusion of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. Notice that $H_b^2(M, \mathbb{Z}) \cong H_M^2$. Consider $\beta = \{[\alpha_1], \dots, [\alpha_n]\}$ the image of a basis of $H_b^2(M, \mathbb{Z})$ via the de Rham isomorphism. So, $\alpha_1, \dots, \alpha_n$ are integral cycles. As β is a basis of H_M^2 , there exist $a_{ij} \in \mathbb{R}$ and forms $\gamma_i \in \Omega^1(M)$ such that

$$\sum_{j=1}^n a_{ij}\omega_j = \alpha_i + d\gamma_i, \quad \text{for all } i \in \{1, \dots, n\}. \tag{2.1}$$

Consider the polynomial in n variables

$$q(x_1, \dots, x_n) = p\left(\sum_{i=1}^n a_{i1}x_i, \dots, \sum_{i=1}^n a_{in}x_i\right). \tag{2.2}$$

By Lemma 2.5, there exist integers z_1, \dots, z_n such that $q(z_1, \dots, z_n) \neq 0$. Now we define

$$\omega = \sum_{i=1}^n z_i \sum_{j=1}^n a_{ij}\omega_j = \sum_{j=1}^n \sum_{i=1}^n z_i a_{ij}\omega_j = \sum_{j=1}^n c_j\omega_j$$

where $c_j = \sum_{i=1}^n a_{ij}z_i$ for $j \in \{1, \dots, n\}$. On one hand, by Eq. 2.2, we have $p(c_1, \dots, c_n) = q(z_1, \dots, z_n) \neq 0$. Thus, $\omega \in W_p$. On the other hand, by Eq. 2.1, we have

$$\omega = \sum_{i=1}^n z_i \sum_{j=1}^n a_{ij}\omega_j = \sum_{i=1}^n z_i\alpha_i + d\left(\sum_{i=1}^n z_i\gamma_i\right)$$

and therefore, ω is an integral cycle. □

Example 2.7 In [22], the authors construct K-contact flows which are not Lefschetz, and thus, not Sasakian. We can construct another example of the same kind (K-contact and not Lefschetz) as follows: consider B the 6-dimensional solvmanifold defined in [37]. The authors construct a family of closed forms:

$$\omega = a\omega_1 + b\omega_2 + c\omega_3,$$

being $[\omega_i]$ certain generators of H_B^2 . They show that if $ac \neq 0$, then ω is a symplectic form, but B fails to satisfy the hard Lefschetz property. By applying Lemma 2.6 to the polynomial $p(x, y) = xy$, we can suppose ω to be a closed integral form. So, the Kobayashi construction yields a K-contact manifold which is not Lefschetz.

Example 2.8 Let B be one of the manifolds constructed in [38], which satisfy the following properties:

- (i) B is a compact 6-solvmanifold of completely solvable type;
- (ii) B admits a nondefinite Kähler metric, and thus a symplectic structure;
- (iii) B satisfies the strong Lefschetz property;
- (iv) The Poincaré polynomial of B is $P_B(t) = 1 + 2t + 5t^2 + 8t^3 + 5t^4 + 2t^5 + t^6$.

By (ii), B is a symplectic manifold. More precisely, in [38, p. 63], it is shown that any linear combination ω of certain closed forms generating H_B^2 with parameters r, s, t, u, v is a valid symplectic form if the condition $r(sv + tu) \neq 0$ is satisfied. By Lemma 2.6, we can choose ω to be integral and consider M its associated Kobayashi construction. So, M is a K-contact flow, and by (iii) it is a Lefschetz flow.

In [39, Th. 3.3], it is proved that B does not admit a Kähler structure. We show here a shorter proof of this fact using a later result: by the Benson-Gordon's Theorem, proved by Hasegawa in [40], a manifold satisfying (i) admits a Kähler structure if and only if it is a complex torus. So, by (iv), B does not admit a Kähler structure. In particular, M does not admit a Sasakian structure.

So, we have a K-contact Lefschetz flow which does not admit a Sasakian structure. Other examples of isometric flows which are Lefschetz but not transversely Kähler can be constructed using examples of manifolds from [41] which are symplectic and Lefschetz but not Kähler. For another example in dimension 5, see [23].

We finish this paper by showing that the leftmost region of Fig. 1 is not empty.

Example 2.9 As described in [42, Remarque 3], the geodesic flow of the flat torus \mathbb{T}^2 induces a contact flow on its unit tangent bundle whose basic cohomology has infinite dimension (and so, it cannot be isometric because the basic cohomology of a Riemannian foliation is finite-dimensional). More explicitly, it can be described as the usual contact flow on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ with contact form $\eta = \cos 2\pi t dx + \sin 2\pi t dy$. We have a fibration $\pi: M \rightarrow \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ such that $\pi^{-1}(\{t\})$ is foliated diffeomorphic to \mathbb{T}^2 with a linear flow of slope t , and is, thus, foliated by circles if $t \in \mathbb{Q}$ or by dense non-compact orbits otherwise. It is easily checked that all basic 1-forms are written locally as $f(t) dt$. So, $H_B^1 \cong H^1(\mathbb{S}^1) = \mathbb{R}$ and every basic 1-form is closed, which yields $H_B^2 = \Omega_B^2$ by degree reasons. It is straightforward to show that $\Omega_B^2 \cong C^\infty([0, 1], \{0, \frac{1}{2}, 1\})$ and hence, it has infinite dimension and the Lefschetz map between extremal degrees $L: H_B^0 \rightarrow H_B^2$ is not an isomorphism, while by Remark 1.13, \mathcal{R}^0 is the graph of the isomorphism $H_M^0 \cong H_M^3$. In contrast, the remaining Lefschetz map $L^0: H_B^1 \rightarrow H_B^1$ is the identity isomorphism, but the relation \mathcal{R}^1 is not the graph of $H_M^1 \cong H_M^2$, because in this example every closed basic 1-form is a primitive form by degree reasons and $PH_B^1 = H_B^1$ is not isomorphic to H_M^1 . So, we have a contact flow which is not K-contact and fails to satisfy any of the Lefschetz properties, these failures happening at non-corresponding degrees. Notice that for isometric flows, the properties (THL) and (HL) are intimately related at corresponding degrees.

Acknowledgements The authors wish to thank the referee for the useful remarks made in order to improve this paper.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. The first author has been partially supported by the Gobierno Vasco grant IT1094-16; the first and second authors by Ministerio de Ciencia, Spain, grant MTM2016-77642-C2-1-P, and the three authors by Ministerio de Ciencia, Spain, grant PID2019-105621GB-I00.

Declarations

Conflict of Interest The authors declare no competing interests.

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