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Curvatures and Austere Property of Orbits of Path Group Actions Induced by Hermann Actions

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Abstract

It is known that an isometric action of a Lie group on a compact symmetric space gives rise to a proper Fredholm action of a path group on a path space via the gauge transformations. In this paper, supposing that the isometric action is a Hermann action (i.e., an isometric action of a symmetric subgroup of the isometry group), we give an explicit formula for the principal curvatures of orbits of the path group action and study the condition for those orbits to be austere, that is, the set of principal curvatures in the direction of each normal vector is invariant under the multiplication by -1. To prove the results, we essentially use the facts that Hermann actions are hyperpolar and all orbits of Hermann actions are curvature-adapted submanifolds. The results greatly extend the author's previous result in the case of the standard sphere and show that there exist a larger number of infinite dimensional austere submanifolds in Hilbert spaces.

Keywords Hermann action · Hyperpolar action · Proper Fredholm action · Principal curvature · Austere submanifold · Proper Fredholm submanifold

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1 Introduction

In [24] and [29], Palais and Terng introduced a class of *proper Fredholm* (PF) submanifolds in Hilbert spaces. These are submanifolds in Hilbert spaces which have

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finite codimensions and generalize properly immersed submanifolds in Euclidean spaces. By definition, the shape operators of PF submanifolds are compact self-adjoint operators. Moreover, the infinite dimensional differential topology and Morse theory [23, 26, 27] can be applied to PF submanifolds. Typical examples of PF submanifolds are orbits of the gauge transformations. More precisely, let G be a connected compact Lie group with a bi-invariant Riemannian metric, $\mathfrak{g}$ its Lie algebra, and P the trivial principal G-bundle over the circle S^1. The loop group $H^1(S^1, G)$ of Sobolev H^1-loops is the gauge group of P and it acts on the Hilbert space $L^2(S^1, \mathfrak{g})$ of L^2-connections by the gauge transformations:

$$g * u := gug^{-1} - g'g^{-1}.$$

They showed that this action is isometric, proper and Fredholm (PF) [24]. Thus, every orbit of this action is a PF submanifold of $L^2(S^1, \mathfrak{g})$. Moreover, they essentially considered an equivariant map $\Phi : L^2(S^1, \mathfrak{g}) \to G$, which is nowadays called the *parallel transport map*, and showed that the above gauge group action is hyperpolar and the principal orbits are isoparametric PF submanifolds. Here, the action is called hyperpolar if there exists a closed affine subspace Σ of $L^2(S^1, \mathfrak{g})$ which meets every orbit orthogonally.

The above examples were extended by Pinkall and Thorbergsson [25] and eventually reformulated by Terng [30] as follows. Let $\mathcal{G} := H^1([0, 1], G)$ denote the path group of all H^1-paths from [0, 1] to G and $V_{\mathfrak{g}} := L^2([0, 1], \mathfrak{g})$ the Hilbert space of all L^2-paths from [0, 1] to $\mathfrak{g}$. For any closed subgroup L of $G \times G$, the subgroup

$$P(G, L) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in L\}$$

acts on $V_{\mathfrak{g}}$ isometrically by the gauge transformations. If L is the diagonal ΔG, then the $P(G, L)$-action is identified with the action given by Palais and Terng above. If $L = K \times K$ for a symmetric subgroup K of G, then it is just the one given by Pinkall and Thorbergsson [25]. Terng [30] generally proved that the $P(G, L)$-action is PF and showed that the parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ is equivariant with respect to the L- and $P(G, L)$-actions where L acts on G by $(b, c) \cdot a := bac^{-1}$, and each $P(G, L)$-orbit is expressed as the inverse image of an L-orbit under Φ. Using these results, she showed that if the L-action is hyperpolar then the $P(G, L)$-action is also hyperpolar and the principal orbits are isoparametric. Here, the L-action (or more generally, a proper isometric action on a Riemannian manifold) is called *hyperpolar* if there exists a closed connected totally geodesic submanifold Σ which is flat in the induced metric and meets every orbit orthogonally [6]. In this way, the structure of the $P(G, L)$-action is understood through the parallel transport map.

Terng and Thorbergsson [31] investigated the parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ and showed that it is a Riemannian submersion. Moreover, they gave an interesting application to the submanifold geometry in symmetric spaces. Let G/K be a symmetric space of compact type with projection $\pi : G \to G/K$. They considered the composition $\Phi_K := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ and proved that if N is a closed submanifold of G/K then the inverse image $\Phi_K^{-1}(N)$ is a PF submanifold of $V_{\mathfrak{g}}$. Moreover, they showed that if N is an equifocal submanifold of G/K then each component of $\Phi_K^{-1}(N)$ is an isoparametric PF submanifold of $V_{\mathfrak{g}}$ and the converse is also true. Although $\Phi_K^{-1}(N)$ is infinite dimensional, many techniques and results in the finite dimensional Euclidean case are still valid in the case of Hilbert space $V_{\mathfrak{g}}$. Applying techniques for isoparametric submanifolds to $\Phi_K^{-1}(N)$, they studied equifocal submanifolds in G/K. In this way, the parallel transport map is also known as a tool to study the submanifold geometry in symmetric spaces. It is a fundamental problem to show the geometrical relation between N and $\Phi_K^{-1}(N)$.

Afterwards, Koike [13] gave a formula for the principal curvatures of $\Phi_K^{-1}(N)$ under the assumption that N is *curvature-adapted*, that is, for each normal vector v at each $p \in N$ the Jacobi operator R_v leaves the tangent space T_pN invariant and the restriction $R_v|_{T_pN}$ commutes with the shape operator A_v. Recently, the author [18] corrected inaccuracies in that formula and studied the relation between two conditions:

(A) N is an austere submanifold of G/K,
(B) $\Phi_K^{-1}(N)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

Here, a submanifold is called *austere* [4] if the set of principal curvatures with multiplicities in the direction of each normal vector is invariant under the multiplication by (-1). The author showed [18, Theorem 4.1]:

Theorem [18] *If G/K is the standard sphere then* (A) *and* (B) *are equivalent.*

The purpose of this paper is to extend this theorem to the case that G/K is not necessarily the standard sphere. However there are two difficulties to do this. The first one is that N must be curvature-adapted in order to use the formula for the principal curvatures of $\Phi_K^{-1}(N)$, otherwise there is no way to compute those curvatures. The second one is that even if N is curvature-adapted the principal curvatures of $\Phi_K^{-1}(N)$ and their multiplicities are complicated in general and it is not clear whether the austere properties of N and $\Phi_K^{-1}(N)$ are equivalent or not.

In this paper, we let G/K be a symmetric space of compact type and suppose that N is an orbit of a *Hermann action* [7], that is, an isometric action of a symmetric subgroup H of G on G/K. We know that all orbits of Hermann actions are curvature-adapted submanifolds [2]. Moreover, we can explicitly describe the principal curvatures of orbits of Hermann actions [22]. Furthermore, $\Phi_K^{-1}(N)$ is expressed as an orbit of the $P(G, H \times K)$-action and the conditions (A) and (B) are restated as follows:

(A) The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of G/K,
(B) The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$,

where $w \in \mathfrak{g}$ and $\hat{w}$ denotes the constant path with value w. Note that since the Hermann action is hyperpolar [8], the $P(G, H \times K)$-action is also hyperpolar. This implies that we only have to consider normal vectors which are tangent to a fixed Σ when studying the above conditions (see Lemma 7.2).

In this paper, we first derive an explicit formula for the principal curvatures of $P(G, H \times K)$-orbits (Theorem 6.1), which unifies and generalizes some results by Terng [29], Pinkall-Thorbergsson [25], and Koike [14] (see Remark 6.4). Then, using this explicit formula, we study the relation between (A) and (B). To explain the results, we write σ and τ for the involutions of G associated to the symmetric subgroups K and H respectively. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (resp. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$) the decomposition into the (± 1)-eigenspaces of the differential of σ (resp. τ). Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ and denote by $\Delta = \Delta(\sigma, \tau)$ the associated root system of $\mathfrak{t}$. We will prove the following theorem (Theorem 7.1):

Theorem I *If* Δ *is a reduced root system, then* (A) *and* (B) *are equivalent.*

Then, without supposing that Δ is a reduced root system, we will prove the following theorem (Theorems 8.2, 8.5, and 8.8):

Theorem II (i) *Suppose that* $\sigma = \tau$. *Then* (A) *and* (B) *are equivalent.*
(ii) *Suppose that* σ *and* τ *commute. Then* (A) *implies* (B).
(iii) *Suppose that* G *is simple. Then* (A) *implies* (B).

Note that (B) does not imply (A) in the cases (ii) and (iii). In fact, we will show a counterexample of a minimal H-orbit which is *not* austere but the corresponding $P(G, H \times K)$-orbit is austere (cf. Section 9). Without the assumption of (ii) or (iii), we do not know whether (A) implies (B) or not because in the non-simple case there exist many non-commutative pairs of involutive automorphisms of G [17]. However, the above theorems greatly extend the previous theorem in the spherical case and cover all known examples of austere orbits of Hermann actions [9, 22]. Applying the above theorems to those examples, we obtain a larger number of infinite dimensional austere PF submanifolds in Hilbert spaces. Notice that so obtained austere PF submanifolds are not totally geodesic due to [19].

This paper is organized as follows. In Section 2, we review basic knowledge on $P(G, H \times K)$-actions and the parallel transport map. In Section 3, we review fundamental results on the submanifold geometry of orbits of Hermann actions. In Section 4, we introduce a hierarchy of curvature-adapted submanifolds in symmetric spaces and formulate the curvature-adapted property of orbits of Hermann actions. In Section 5, we refine the formula for the principal curvatures of $\Phi_K^{-1}(N)$ [13, 18] so that it can be applied to orbits of Hermann actions. In Section 6, by applying the refined formula to orbits of Hermann actions, we derive an explicit formula for the principal curvatures of $P(G, H \times K)$-orbits. In Section 7, using this explicit formula, we formulate the conditions (A) and (B) in terms of roots in Δ and prove Theorem I. In Section 8, we show an inequality between the multiplicities of roots α and 2α in Δ and prove Theorem II. Finally, in Section 9, we show a counterexample to the converse of (ii) and (iii) of Theorem II and mention further remarks on the converse.

2 Preliminaries

Let G be a connected compact semisimple Lie group and K a closed subgroup of G. Suppose that K is a symmetric subgroup of G, that is, there exists an involutive automorphism σ of G satisfying the condition $G_0^\sigma \subset K \subset G^\sigma$, where G^σ denotes the fixed point subgroup of G and G_0^σ its identity component. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The differential of σ induces an involutive automorphism of $\mathfrak{g}$, which is still denoted by σ. The direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ into the (± 1)-eigenspaces of σ is called the *canonical decomposition*. We fix an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ which is a negative multiple of the Killing form of $\mathfrak{g}$. Then, it is invariant under all automorphisms of $\mathfrak{g}$ and the canonical decomposition is orthogonal. We equip G with the corresponding bi-invariant Riemannian metric and the homogeneous space G/K with the corresponding G-invariant metric. Then $M := G/K$ is a symmetric space of compact type and the projection $\pi : G \to M$ is a Riemannian submersion with totally geodesic fiber.

We denote by $\mathcal{G} := H^1([0,1], G)$ the path group of all Sobolev H^1-paths from $[0,1]$ to G and by $V_\mathfrak{g} := L^2([0,1], \mathfrak{g})$ the path space of all L^2-paths from $[0,1]$ to $\mathfrak{g}$. Then $\mathcal{G}$ is a Hilbert Lie group and $V_\mathfrak{g}$ a separable Hilbert space. We consider the isometric action of $\mathcal{G}$ on $V_\mathfrak{g}$ given by the gauge transformations

$$g * u := gug^{-1} - g'g^{-1},$$

where $g \in \mathcal{G}$ and $u \in V_\mathfrak{g}$. We know that this action is proper and Fredholm [24, Theorem 5.8.1], [29, Section 4]. For any closed subgroup L of $G \times G$ the subgroup

$$P(G, L) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in L\}$$

acts on $V_\mathfrak{g}$ by the same formula. It follows that the $P(G, L)$-action is also proper and Fredholm [30, p. 132]. Thus every orbit of the $P(G, L)$-action is a proper Fredholm (PF) submanifold of $V_\mathfrak{g}$ [24, Theorem 7.1.6]. We know that the $P(G, \{e\} \times G)$-action on $V_\mathfrak{g}$ is simply transitive [31, Corollary 4.2] and that the $P(G, G \times \{e\})$-action on $V_\mathfrak{g}$ is also simply transitive [19, Section 5].

The *parallel transport map* [30, 31] $\Phi : V_\mathfrak{g} \to G$ is defined by

$$\Phi(u) := g_u(1),$$

where $g_u \in \mathcal{G}$ is the unique solution to the linear ordinary differential equation

$$\begin{cases} g_u^{-1} g_u' = u, \\ g_u(0) = e. \end{cases}$$

We know that Φ is a Riemannian submersion and a principal $\Omega_e(G)$-bundle, where $\Omega_e(G) = P(G, \{e\} \times \{e\})$ denotes the based loop group [31, Corollary 4.4, Theorem 4.5]. The normal space of the fiber $\Phi^{-1}(e)$ at $\hat{0} \in V_\mathfrak{g}$ is identified with the subspace $\hat{\mathfrak{g}} = \{\hat{x} \mid x \in \mathfrak{g}\}$ of $V_\mathfrak{g}$, where $\hat{x}$ denotes the constant path with value x. It follows that $\Phi(\hat{x}) = \exp x$. The composition

$$\Phi_K := \pi \circ \Phi : V_\mathfrak{g} \to G \to M$$

is a Riemannian submersion which is also called the parallel transport map.

We consider the isometric action of G on M defined by

$$b \cdot (aK) := (ba)K,$$

where $b \in G$ and $aK \in M$, and the isometric action of $G \times G$ on G defined by

$$(b, c) \cdot a := bac^{-1},$$

where $(b, c) \in G \times G$ and $a \in G$. Then π and Φ have the following equivariant properties [30, Proposition 1.1 (i)]:

$$\pi((b, c) \cdot a) = b \cdot \pi(a) \qquad \text{for } (b, c) \in G \times K \text{ and } a \in G, \tag{2.1}$$

$$\Phi(g * u) = (g(0), g(1)) \cdot \Phi(u) \qquad \text{for } g \in \mathcal{G} \text{ and } u \in V_{\mathfrak{g}}. \tag{2.2}$$

From these, we have

$$\Phi_K(g * u) = g(0)\Phi_K(u) \qquad \text{for } g \in P(G, G \times K) \text{ and } u \in V_{\mathfrak{g}}. \tag{2.3}$$

Let H be a closed subgroup of G; in later sections, we will suppose that it is a symmetric subgroup of G. We denote by $\mathfrak{h}$ the Lie algebra of H and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ the orthogonal direct sum decomposition. Then H acts on M, the subgroup $H \times K$ acts on G and the subgroup $P(G, H \times K)$ acts on $V_{\mathfrak{g}}$. We know the following relations for orbits [30, Proposition 1.1 (ii)]:

$$(H \times K) \cdot a = \pi^{-1}(H \cdot aK) \quad \text{and} \quad P(G, H \times K) * u = \Phi^{-1}((H \times K) \cdot \Phi(u)).$$

Thus, we have

$$P(G, H \times K) * u = \Phi_K^{-1}(H \cdot \Phi_K(u)). \tag{2.4}$$

Then, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{G} & \supset & P(G, H \times K) & \curvearrowright & V_{\mathfrak{g}} & \supset & P(G, H \times K) * u & = & \Phi^{-1}((H \times K) \cdot a) \\
\psi \downarrow & & \psi \downarrow & & \Phi \downarrow & & \Phi \downarrow & & \Phi \downarrow \\
G \times G & \supset & H \times K & \curvearrowright & G & \supset & (H \times K) \cdot a & = & \pi^{-1}(H \cdot aK) \\
p \downarrow & & p \downarrow & & \pi \downarrow & & \pi \downarrow & & \\
G & \supset & H & \curvearrowright & M & \supset & H \cdot aK & & (\Phi(u) = a),
\end{array}$$

where p denotes the projection onto the first component and ψ the submersion defined by $\psi(g) := (g(0), g(1))$ for $g \in \mathcal{G}$. We know that the following conditions are equivalent: the orbit $H \cdot aK$ is a minimal submanifold of M, the orbit $(H \times K) \cdot a$ is a minimal submanifold of G, and the orbit $P(G, H \times K) * u$ through $u \in \Phi^{-1}(a)$ is a minimal PF submanifold of $V_{\mathfrak{g}}$ [12, Theorem, 4.12], [5, Lemma 5.2].

Recall that an isometric action of a compact Lie group A on a Riemannian manifold X is called *polar* if there exists a closed connected submanifold Σ of X which meets every A-orbit and is orthogonal to the A-orbits at every point of intersection. Such a Σ is called a *section*, which is automatically totally geodesic in X. If Σ is flat in the induced metric, then the action is called *hyperpolar* [6]. For a proper Fredholm action on a Hilbert space, we can define it to be hyperpolar in the similar way. We know that the following conditions are equivalent [6, Proposition 2.11], [30, Theorem 1.2], [3, Lemma 4]:

(i) The H-action on M is hyperpolar,
(ii) The $H \times K$-action on G is hyperpolar,
(iii) The $P(G, H \times K)$-action on $V_{\mathfrak{g}}$ is hyperpolar.

Since we fixed a bi-invariant Riemannian metric on G induced by a negative multiple of the Killing form of $\mathfrak{g}$, the condition (ii) is equivalent to the existence of a c-dimensional abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ where c is the cohomogeneity of the $H \times K$-action [6, Theorem 2.1]. Then $\pi(\exp \mathfrak{t})$, $\exp \mathfrak{t}$, and $\hat{\mathfrak{t}} = \{\hat{x} \mid x \in \mathfrak{t}\}$ are sections of the H-action, the $H \times K$-action, and the $P(G, H \times K)$-action respectively.

If the actions are hyperpolar then the following conditions are equivalent [30, Theorem 1.2]: $aK \in M$ is a regular point of the H-action, $a \in G$ is a regular point of the $H \times K$-action, and $u \in \Phi^{-1}(a)$ is a regular point of the $P(G, H \times K)$-action. Here, a point is called *regular* if the orbit though it is principal.

3 Submanifold Geometry of Orbits of Hermann Actions

In this section, we review fundamental results on the submanifold geometry of orbits of Hermann actions. For details, see Ohno [22] (see also Goertsches-Thorbergsson [2] and Ikawa [9]). Throughout this section, $M = G/K$ denotes a symmetric space of compact type and H a symmetric subgroup of G. We denote by σ and τ the involutions of G associated to K and H respectively and by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ the canonical decompositions. We choose and fix a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ so that $\Sigma := \pi(\exp \mathfrak{t})$ is a section of the Hermann action.

Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}(0) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}(\alpha),$$

where

$$\begin{aligned} \mathfrak{g}(0) &= \{z \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}(\eta)z = 0 \text{ for all } \eta \in \mathfrak{t}\}, \\ \mathfrak{g}(\alpha) &= \{z \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}(\eta)z = \sqrt{-1}\langle \alpha, \eta \rangle z \text{ for all } \eta \in \mathfrak{t}\}. \end{aligned}$$

Here $\Delta = \{\alpha \in \mathfrak{t} \backslash \{0\} \mid \mathfrak{g}(\alpha) \neq \{0\}\}$ is a root system of $\mathfrak{t}$ [9, Lemma 4.12]. Since $\overline{\mathfrak{g}(\alpha)} = \mathfrak{g}(-\alpha)$, where the bar denotes the complex conjugation, the real form is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_0 = \mathfrak{g}(0) \cap \mathfrak{g}, \qquad \mathfrak{g}_\alpha = (\mathfrak{g}(\alpha) + \mathfrak{g}(-\alpha)) \cap \mathfrak{g}.$$

Note that

$$\begin{aligned} \mathfrak{g}_0 &= \{x \in \mathfrak{g} \mid \mathrm{ad}(\eta)x = 0 \text{ for all } \eta \in \mathfrak{t}\}, \\ \mathfrak{g}_\alpha &= \{x \in \mathfrak{g} \mid \mathrm{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x \text{ for all } \eta \in \mathfrak{t}\}. \end{aligned}$$

Since σ commutes with $\mathrm{ad}(\eta)^2$ for all $\eta \in \mathfrak{t}$, we have

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_\alpha, \qquad \mathfrak{m} = \mathfrak{m}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha,$$

where

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{g}_0 \cap \mathfrak{k}, \quad \mathfrak{m}_0 = \mathfrak{g}_0 \cap \mathfrak{m}, \\ \mathfrak{k}_\alpha &= \mathfrak{g}_\alpha \cap \mathfrak{k}, \quad \mathfrak{m}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{m}. \end{aligned}$$

We define a linear orthogonal transformation ψ_α of $\mathfrak{g}_\alpha$ by

$$\psi_\alpha(x) := \frac{1}{\langle \alpha, \alpha \rangle} \mathrm{ad}(\alpha)x \qquad \text{for } x \in \mathfrak{g}_\alpha. \tag{3.1}$$

An equivalent definition is that

$$\psi_\alpha(z+\bar{z}) := \sqrt{-1}(z-\bar{z}) \qquad \text{for } z \in \mathfrak{g}(\alpha).$$

Since $\sigma \circ \psi_\alpha = -(\psi_\alpha \circ \sigma)$, we have a linear isometry $\psi_\alpha : \mathfrak{m}_\alpha \to \mathfrak{k}_\alpha$. Set

$$m(\alpha) := \dim \mathfrak{k}_\alpha = \dim \mathfrak{m}_\alpha.$$

By setting $x_i^\alpha := \psi_\alpha\left(y_i^\alpha\right)$, we can take bases $\left\{x_i^\alpha\right\}_{i=1}^{m(\alpha)}$ of $\mathfrak{k}_\alpha$ and $\left\{y_i^\alpha\right\}_{i=1}^{m(\alpha)}$ of $\mathfrak{m}_\alpha$ satisfying

$$\left[\eta, x_i^\alpha\right] = -\langle\alpha, \eta\rangle y_i^\alpha \quad \text{and} \quad \left[\eta, y_i^\alpha\right] = \langle\alpha, \eta\rangle x_i^\alpha \tag{3.2}$$

for any $\eta \in \mathfrak{t}$.

The root space decompositions above are refined by combining a decomposition derived from the involutions σ and τ. More precisely, we consider the composition

$$\sigma \circ \tau : \mathfrak{g} \to \mathfrak{g}$$

and the eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\epsilon \in U(1)} \mathfrak{g}(\epsilon),$$

where

$$\mathfrak{g}(\epsilon) = \left\{z \in \mathfrak{g}^{\mathbb{C}} \mid (\sigma \circ \tau)(z) = \epsilon z\right\}.$$

Here, the eigenvalues belong to $U(1) = \{\epsilon \in \mathbb{C} \mid |\epsilon| = 1\}$. For each $\epsilon \in U(1)$, we denote by $\arg \epsilon$ its argument satisfying $-\pi < \arg \epsilon \leq \pi$. Since $\sigma \circ \tau$ commutes with $\mathrm{ad}(\eta)$ for all $\eta \in \mathfrak{t}$, we have

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\epsilon \in U(1)} \mathfrak{g}(0, \epsilon) \oplus \bigoplus_{\alpha \in \Delta} \bigoplus_{\epsilon \in U(1)} \mathfrak{g}(\alpha, \epsilon),$$

where

$$\mathfrak{g}(0, \epsilon) = \mathfrak{g}(0) \cap \mathfrak{g}(\epsilon), \qquad \mathfrak{g}(\alpha, \epsilon) = \mathfrak{g}(\alpha) \cap \mathfrak{g}(\epsilon).$$

Since $\overline{\mathfrak{g}(\alpha, \epsilon)} = \mathfrak{g}(-\alpha, \epsilon^{-1})$ the real form is

$$\mathfrak{g} = \bigoplus_{\epsilon \in U(1)_{\geq 0}} \mathfrak{g}_{0,\epsilon} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)} \mathfrak{g}_{\alpha,\epsilon},$$

where

$$U(1)_{\geq 0} = \{\epsilon \in U(1) \mid \mathrm{Im}(\epsilon) \geq 0\},$$

$$\mathfrak{g}_{0,\epsilon} = (\mathfrak{g}(0, \epsilon) + \mathfrak{g}(0, \epsilon^{-1}) \cap \mathfrak{g},$$
$$\mathfrak{g}_{\alpha,\epsilon} = (\mathfrak{g}(\alpha, \epsilon) + \mathfrak{g}(-\alpha, \epsilon^{-1})) \cap \mathfrak{g}.$$

Setting $\rho^+ = \sigma \circ \tau + \tau \circ \sigma$ and $\rho^- = \sigma \circ \tau - \tau \circ \sigma$ we can write

$$\mathfrak{g}_{0,\epsilon} = \{x \in \mathfrak{g}_0 \mid \rho^+(x) = 2\mathrm{Re}(\epsilon)x\},$$
$$\mathfrak{g}_{\alpha,\epsilon} = \{x \in \mathfrak{g}_\alpha \mid \rho^+(x) = 2\mathrm{Re}(\epsilon)x,\ \rho^-(x) = 2\mathrm{Im}(\epsilon)\psi_\alpha(x)\},$$

where $\mathrm{Re}(\epsilon)$ and $\mathrm{Im}(\epsilon)$ denote the real and imaginary parts of ϵ respectively. Since $\mathfrak{g}_{0,\epsilon}$ and $\mathfrak{g}_{\alpha,\epsilon}$ are invariant under σ, we have

$$\mathfrak{k} = \bigoplus_{\epsilon \in U(1)_{\geq 0}} \mathfrak{k}_{0,\epsilon} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)} \mathfrak{k}_{\alpha,\epsilon},$$
$$\mathfrak{m} = \bigoplus_{\epsilon \in U(1)_{\geq 0}} \mathfrak{m}_{0,\epsilon} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)} \mathfrak{m}_{\alpha,\epsilon},$$

where

$$\mathfrak{k}_{0,\epsilon} = \mathfrak{g}_{0,\epsilon} \cap \mathfrak{k}, \quad \mathfrak{k}_{\alpha,\epsilon} = \mathfrak{g}_{\alpha,\epsilon} \cap \mathfrak{k},$$
$$\mathfrak{m}_{0,\epsilon} = \mathfrak{g}_{0,\epsilon} \cap \mathfrak{m}, \quad \mathfrak{m}_{\alpha,\epsilon} = \mathfrak{g}_{\alpha,\epsilon} \cap \mathfrak{m}.$$

Since $\mathfrak{g}_{\alpha,\epsilon}$ is invariant under ψ_α, we have a linear isometry $\psi_\alpha : \mathfrak{m}_{\alpha,\epsilon} \to \mathfrak{k}_{\alpha,\epsilon}$. Set

$$m(\alpha, \epsilon) := \dim \mathfrak{k}_{\alpha,\epsilon} = \dim \mathfrak{m}_{\alpha,\epsilon}.$$

Then, similarly, we can take bases $\{x_i^{\alpha,\epsilon}\}_{i=1}^{m(\alpha,\epsilon)}$ of $\mathfrak{k}_{\alpha,\epsilon}$ and $\{y_i^{\alpha,\epsilon}\}_{i=1}^{m(\alpha,\epsilon)}$ of $\mathfrak{m}_{\alpha,\epsilon}$ satisfying

$$\left[\eta, x_i^{\alpha,\epsilon}\right] = -\langle \alpha, \eta \rangle y_i^{\alpha,\epsilon} \quad \text{and} \quad \left[\eta, y_i^{\alpha,\epsilon}\right] = \langle \alpha, \eta \rangle x_i^{\alpha,\epsilon} \tag{3.3}$$

for any $\eta \in \mathfrak{t}$.

We now take $w \in \mathfrak{t}$, set $a := \exp w$ and consider the orbit $N := H \cdot aK$ through aK. Denote by L_a the isometry of M defined by $L_a(bK) := (ab)K$. Identifying $T_{eK}M$ with $\mathfrak{m}$, we can describe the tangent space and the normal space of N as follows [22, p. 12]:

$$T_{aK}N = dL_a\Big(\bigoplus_{\substack{\epsilon \in U(1)_{\geq 0} \\ \epsilon \neq 1}} \mathfrak{m}_{0,\epsilon} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\substack{\epsilon \in U(1) \\ \langle \alpha, w \rangle + \frac{1}{2}\arg\epsilon \notin \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big), \tag{3.4}$$

$$T_{aK}^{\perp}N = dL_a\Big(\mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\substack{\epsilon \in U(1) \\ \langle \alpha, w \rangle + \frac{1}{2}\arg\epsilon \in \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big). \tag{3.5}$$

Moreover, the decomposition (3.4) is just the common eigenspace decomposition of the family of shape operators $\{A^N_{dL_a(\xi)}\}_{\xi \in \mathfrak{t}}$. In fact [22, p. 17]:

$dL_a(\mathfrak{m}_{0,\epsilon})$: the eigenspace associated with the eigenvalue 0,

$dL_a(\mathfrak{m}_{\alpha,\epsilon})$: the eigenspace associated with the eigenvalue $-\langle \alpha, \xi \rangle \cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg\epsilon\right)$

for each $\xi \in \mathfrak{t}$. If σ and τ commute then $\epsilon = \pm 1$ and thus we get [2, Theorem 5.3]:

$$T_{aK}N = dL_a\Big(\mathfrak{m}_0 \cap \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w \rangle \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} \oplus \bigoplus_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w \rangle + \pi/2 \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big), \tag{3.6}$$

$$T_{aK}^{\perp}N = dL_a\Big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w \rangle \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} \oplus \bigoplus_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w \rangle + \pi/2 \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big), \tag{3.7}$$

where

$dL_a(\mathfrak{m}_0 \cap \mathfrak{h})$: the eigenspace associated with the eigenvalue 0,

$dL_a(\mathfrak{m}_\alpha \cap \mathfrak{p})$: the eigenspace associated with the eigenvalue $-\langle\alpha, \xi\rangle \cot\langle\alpha, w\rangle$,

$dL_a(\mathfrak{m}_\alpha \cap \mathfrak{h})$: the eigenspace associated with the eigenvalue $\langle\alpha, \xi\rangle \tan\langle\alpha, w\rangle$.

In particular if $\sigma = \tau$ then we have [28, p. 122]

$$T_{aK}N = dL_a\Big(\bigoplus_{\substack{\alpha\in\Delta^+ \\ \langle\alpha,w\rangle\notin\pi\mathbb{Z}}} \mathfrak{m}_\alpha\Big), \tag{3.8}$$

$$T_{aK}^{\perp}N = dL_a\Big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha\in\Delta^+ \\ \langle\alpha,w\rangle\in\pi\mathbb{Z}}} \mathfrak{m}_\alpha\Big), \tag{3.9}$$

where

$dL_a(\mathfrak{m}_\alpha)$: the eigenspace associated with the eigenvalue $-\langle\alpha, \xi\rangle \cot\langle\alpha, w\rangle$.

4 The Curvature-Adapted Property

In this section, we formulate the curvature-adapted property of orbits of Hermann actions.

First, we recall the concept of curvature-adapted submanifolds [1]. Let N be a submanifold of a Riemannian manifold M. For each $v \in T_p^{\perp}N$ at each $p \in N$, the Jacobi operator R_v is a symmetric linear transformation of T_pM defined by

$$R_v(x) = R^M(x, v)v \qquad \text{for } x \in T_pM,$$

where R^M denotes the curvature tensor of M. Then N is called *curvature-adapted* if for every $v \in T_p^{\perp}N$ at each $p \in N$ the Jacobi operator R_v leaves T_pN invariant and the restriction $R_v|_{T_pN}$ commutes with the shape operator A_v^N of N.

We now make the following definition:

Definition 4.1. *Let $M = G/K$ be a symmetric space of compact type and N a submanifold of M. For an integer c satisfying $1 \le c \le \operatorname{codim} N$, we say that N is c-curvature-adapted if for each $aK \in N$ the following two conditions are satisfied:*

(i) *For every $v \in T_{aK}^{\perp}N$, the Jacobi operator R_v leaves $T_{aK}N$ invariant,*

(ii) *For each $v \in T_{aK}^{\perp}N$, there exists a c-dimensional abelian subspace $\mathfrak{t}$ in $\mathfrak{m}$ satisfying $v \in dL_a(\mathfrak{t}) \subset T_{aK}^{\perp}N$ such that the union*

$$\{R_{dL_a(\xi)}|_{T_{aK}N}\}_{\xi\in\mathfrak{t}} \cup \left\{A^N_{dL_a(\xi)}\right\}_{\xi\in\mathfrak{t}}$$

is a commuting family of endomorphisms of $T_{aK}N$.

Note that 1-curvature-adapted submanifolds are just curvature-adapted submanifolds in the original sense. Note also that if $aK = eK$ then R_v is identified with $-\operatorname{ad}(v)^2$ since M is a symmetric space. Typical examples of c-curvature-adapted submanifolds are given by the following proposition, which was essentially shown by Goertsches and Thorbergsson [2, Corollaries 3.3 and 3.4]:

Proposition 4.2 (Goertsches-Thorbergsson [2]) *All orbits of Hermann actions of cohomogeneity c are c-curvature-adapted submanifolds.*

Proof Let N be an orbit of a Hermann action $H \curvearrowright M$ of cohomogeneity c. Take $aK \in N$. Since $L_a^{-1}N = (a^{-1}Ka) \cdot eK$ we can assume $aK = eK$ without loss of generality. Take $v \in T_{eK}^{\perp}N$. Choose a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p} = T_{eK}^{\perp}N$ containing v. Since $\pi(\exp \mathfrak{t})$ is a section of the Hermann action, we have $\dim \mathfrak{t} = c$. Then it follows from the decomposition (3.4) that the Jacobi operator $R_v = -\operatorname{ad}(v)^2$ leaves $T_{eK}N$ invariant and

$$R_v \circ R_w = R_w \circ R_v, \quad R_v|_{T_{eK}N} \circ A_w^N = A_w^N \circ R_v|_{T_{eK}N}, \quad A_v^N \circ A_w^N = A_w^N \circ A_v^N$$

hold for any $v, w \in \mathfrak{t}$. Thus, N is a c-curvature-adapted submanifold of M. □

Remark 4.3 We do not know whether all orbits of hyperpolar actions of cohomogeneity c are c-curvature-adapted submanifolds or not. We know that any indecomposable hyperpolar action of cohomogeneity at least two on M is orbit equivalent to a Hermann action [16]. We also know that any cohomogeneity one action on M is automatically hyperpolar [6, Corollary 2.13]. There exist examples of cohomogeneity one actions on the standard sphere which are different from Hermann actions [6, 15]. Since the standard sphere is of constant sectional curvature, all orbits of such cohomogeneity one actions are 1-curvature-adapted submanifolds.

Let N be a c-curvature-adapted submanifold of a symmetric space $M = G/K$ of compact type. Take $aK \in N$. Choose and fix an arbitrary c-dimensional abelian subspace $\mathfrak{t}$ in $\mathfrak{m}$ satisfying the condition (ii) of Definition 4.1 for some $v \in T_{aK}^{\perp}N$. Consider the root space decomposition

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_\alpha, \qquad \mathfrak{m} = \mathfrak{m}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha,$$

where

$$\begin{aligned} \mathfrak{k}_0 &= \{x \in \mathfrak{k} \mid \operatorname{ad}(\eta)x = 0 \ \text{ for all } \eta \in \mathfrak{t}\}, \\ \mathfrak{k}_\alpha &= \{x \in \mathfrak{k} \mid \operatorname{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x \ \text{ for all } \eta \in \mathfrak{t}\}, \\ \mathfrak{m}_0 &= \{y \in \mathfrak{m} \mid \operatorname{ad}(\eta)y = 0 \ \text{ for all } \eta \in \mathfrak{t}\}, \\ \mathfrak{m}_\alpha &= \{y \in \mathfrak{m} \mid \operatorname{ad}(\eta)^2 y = -\langle \alpha, \eta \rangle^2 y \ \text{ for all } \eta \in \mathfrak{t}\}. \end{aligned}$$

These are just the common eigenspace decompositions of the commuting operators $\left\{\operatorname{ad}(\xi)^2\right\}_{\xi \in \mathfrak{t}}$. On the other hand, the following lemma concerns the common eigenspace decomposition of the commuting operators $\left\{A^N_{dL_a(\xi)}\right\}_{\xi \in \mathfrak{t}}$.

Lemma 4.4 *There exists a unique finite subset* Λ *of* $\mathfrak{t}$ *such that*

$$T_{aK}N = \bigoplus_{\lambda\in\Lambda} S_\lambda,$$

where S_λ *is a nonzero subspace of* $T_{aK}N$ *defined by*

$$S_\lambda = \left\{ x \in T_{aK}N \mid A^N_{dL_a(\eta)}(x) = \langle \lambda, \eta \rangle x \text{ for all } \eta \in \mathfrak{t} \right\}.$$

Proof By left translation, we can assume $aK = eK$ without loss of generality. It is easy to see that such a subset Λ is unique. To see the existence, we take a basis $\{\eta_i\}_{i=1}^c$ of $\mathfrak{t}$. We denote by $\{\lambda(\eta_i)_1, \cdots, \lambda(\eta_i)_{m(i)}\}$ the set of all distinct eigenvalues of the shape operator $A^N_{\eta_i}$ and by $W_{\lambda(\eta_i)_1}, \cdots, W_{\lambda(\eta_i)_{m(i)}}$ their eigenspaces. Since $\{A^N_{\eta_i}\}_{i=1}^c$ is a commuting family, we have the common eigenspace decomposition

$$T_{eK}N = \bigoplus_{j_1=1}^{m(1)} \cdots \bigoplus_{j_c=1}^{m(c)} \left(W_{\lambda(\eta_1)_{j_1}} \cap \cdots \cap W_{\lambda(\eta_c)_{j_c}} \right).$$

Define a linear functional $\lambda_{j_1\cdots j_c} : \mathfrak{t} \to \mathbb{R}$ by

$$\lambda_{j_1\cdots j_c}(a_1\eta_1 + \cdots + a_c\eta_c) = a_1\lambda(\eta_1)_{j_1} + \cdots + a_c\lambda(\eta_c)_{j_c}, \quad \text{where } a_1, ..., a_c \in \mathbb{R}.$$

Then for each $\eta = a_1\eta_1 + \cdots + a_c\eta_c \in \mathfrak{t}$ and $x \in W_{\lambda(\eta_1)_{j_1}} \cap \cdots \cap W_{\lambda(\eta_c)_{j_c}}$ we have

$$A^N_\eta(x) = \left(a_1 A^N_{\eta_1} + \cdots + a_c A^N_{\eta_c} \right)(x) = \lambda_{j_1\cdots j_c}(\eta)x.$$

Set $\Lambda := \{\lambda_{j_1\cdots j_c}\}_{1\le j_1\le m(1), \cdots, 1\le j_c\le m(c)}$ and $S_{\lambda_{j_1\cdots j_c}} := W_{\lambda(\eta_1)_{j_1}} \cap \cdots \cap W_{\lambda(\eta_c)_{j_c}}$. Identifying $\mathfrak{t}$ with the dual space $\mathfrak{t}^*$ we obtain the desired subset $\Lambda \subset \mathfrak{t}$ and the decomposition $T_{eK}N = \bigoplus_{\lambda\in\Lambda} S_\lambda$. This proves the lemma. □

The following proposition concerns the common eigenspace decomposition of the union of commuting operators $\{R_{dL_a(\xi)}\}_{\xi\in\mathfrak{t}} \cup \left\{ A^N_{dL_a(\xi)} \right\}_{\xi\in\mathfrak{t}}$.

Proposition 4.5 *Let* Λ *be as in Lemma* 4.4. *Then the tangent space and the normal space of* N *are decomposed as follows:*

$$T_{aK}N = \bigoplus_{\lambda\in\Lambda_0} (dL_a(\mathfrak{m}_0) \cap S_\lambda) \oplus \bigoplus_{\alpha\in\Delta^+} \bigoplus_{\lambda\in\Lambda_\alpha} (dL_a(\mathfrak{m}_\alpha) \cap S_\lambda), \tag{4.1}$$

$$T^\perp_{aK}N = dL_a(\mathfrak{m}_0) \cap T^\perp_{aK}N \oplus \bigoplus_{\alpha\in\Delta^+} \left(dL_a(\mathfrak{m}_\alpha) \cap T^\perp_{aK}N \right), \tag{4.2}$$

where $\Lambda_0 := \{\lambda \in \Lambda \mid dL_a(\mathfrak{m}_0) \cap S_\lambda \neq \{0\}\}$ *and* $\Lambda_\alpha := \{\lambda \in \Lambda \mid dL_a(\mathfrak{m}_\alpha) \cap S_\lambda \neq \{0\}\}$.

Proof By left translation, we can assume $aK = eK$ without loss of generality. Since the tangent space is invariant under $\{R_\xi\}_{\xi\in\mathfrak{t}}$ the normal space is also invariant under

$\{R_\xi\}_{\xi \in \mathfrak{t}}$ and we have the decompositions

$$T_{eK}N = \mathfrak{m}_0 \cap T_{eK}N \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{m}_\alpha \cap T_{eK}N),$$
$$T^\perp_{eK}N = \mathfrak{m}_0 \cap T^\perp_{eK}N \oplus \bigoplus_{\alpha \in \Delta^+} \left(\mathfrak{m}_\alpha \cap T^\perp_{eK}N\right).$$

By the curvature-adapted property, $\mathfrak{m}_0 \cap T_{eK}N$ and $\mathfrak{m}_\alpha \cap T_{eK}N$ are invariant under $\{A^N_\xi\}_{\xi \in \mathfrak{t}}$. Thus, by similar arguments as in the proof of Lemma 4.4, we obtain the common eigenspace decompositions

$$\mathfrak{m}_0 \cap T_{eK}N = \bigoplus_{\lambda \in \Lambda_0} (\mathfrak{m}_0 \cap S_\lambda), \quad \mathfrak{m}_\alpha \cap T_{eK}N = \bigoplus_{\lambda \in \Lambda_\alpha} (\mathfrak{m}_\alpha \cap S_\lambda)$$

and the assertion follows. □

Example 4.6 Let $H \curvearrowright M$ be a Hermann action of cohomogeneity c. Choose a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. Take $w \in \mathfrak{t}$, set $a := \exp w$ and consider the orbit $N = H \cdot aK$ through aK. From the decomposition (3.4), it is clear that $\mathfrak{t}$ is a c-dimensional abelian subspace in $\mathfrak{m}$ satisfying the condition (ii) of Definition 4.1 for any $v \in \{dL_a(\xi)\}_{\xi \in \mathfrak{t}}$. We set $U(1)_0 := \{\epsilon \in U(1)_{\geq 0} \mid \mathfrak{m}_{0,\epsilon} \neq \{0\}\}$ and

$$U(1)_0^\top := \{\epsilon \in U(1)_0 \mid \epsilon \neq 1\}.$$

We also set $U(1)_\alpha := \{\epsilon \in U(1) \mid \mathfrak{m}_{\alpha,\epsilon} \neq \{0\}\}$ and

$$U(1)_\alpha^\top := \left\{\epsilon \in U(1)_\alpha \mid \langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon \notin \pi\mathbb{Z}\right\},$$
$$U(1)_\alpha^\perp := \left\{\epsilon \in U(1)_\alpha \mid \langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon \in \pi\mathbb{Z}\right\}.$$

Then, we can rewrite the decompositions (3.4) and (3.5) as follows:

$$T_{aK}N = \bigoplus_{\epsilon \in U(1)_0^\top} dL_a(\mathfrak{m}_{0,\epsilon}) \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^\top} dL_a(\mathfrak{m}_{\alpha,\epsilon}), \tag{4.3}$$
$$T^\perp_{aK}N = dL_a(\mathfrak{t}) \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^\perp} dL_a(\mathfrak{m}_{\alpha,\epsilon}). \tag{4.4}$$

For each $\alpha \in \Delta^+$ and $\epsilon \in U(1)_\alpha^\top$, we set

$$\lambda(\alpha, \epsilon) := -\cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon\right)\alpha \in \mathfrak{t}.$$

Then, Λ_0 and Λ_α in Proposition 4.5 are

$$\Lambda_0 \begin{cases} = \{0\} & (\text{if } U(1)_0^\top \neq \emptyset) \\ = \emptyset & (\text{if } U(1)_0^\top = \emptyset) \end{cases}, \qquad \Lambda_\alpha = \{\lambda(\alpha, \epsilon) \mid \epsilon \in U(1)_\alpha^\top\}.$$

Note that the correspondence $U(1)_\alpha^\top \ni \epsilon \mapsto \lambda(\alpha, \epsilon) \in \Lambda_\alpha$ is one-to-one because $\cot x$ is strictly decreasing on $\mathbb{R}/\pi\mathbb{Z}$. Thus, we have

$$\bigoplus_{\epsilon \in U(1)_0^\top} dL_a(\mathfrak{m}_{0,\epsilon}) = dL_a(\mathfrak{m}_0) \cap S_0, \qquad dL_a(\mathfrak{m}_{\alpha,\epsilon}) = dL_a(\mathfrak{m}_{\alpha,\epsilon}) \cap S_{\lambda(\alpha,\epsilon)},$$
$$dL_a(\mathfrak{t}) = dL_a(\mathfrak{m}_0) \cap T_{aK}^\perp N, \qquad \bigoplus_{\epsilon \in U(1)_\alpha^\perp} dL_a(\mathfrak{m}_{\alpha,\epsilon}) = dL_a(\mathfrak{m}_\alpha) \cap T_{aK}^\perp N$$

and therefore the decompositions (4.3) and (4.4) are expressed as

$$T_{aK}N = dL_a(\mathfrak{m}_0) \cap S_0 \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\lambda \in \Lambda_\alpha} (dL_a(\mathfrak{m}_\alpha) \cap S_\lambda), \tag{4.5}$$

$$T_{aK}^\perp N = dL_a(\mathfrak{m}_0) \cap T_{aK}^\perp N \oplus \bigoplus_{\alpha \in \Delta^+} \left(dL_a(\mathfrak{m}_\alpha) \cap T_{aK}^\perp N \right). \tag{4.6}$$

5 Principal Curvatures via the Parallel Transport Map

Let $M = G/K$ be a symmetric space of compact type and $\Phi_K : V_\mathfrak{g} \to M$ the parallel transport map. In [13] and [18], an explicit formula for the principal curvatures of the PF submanifold $\Phi_K^{-1}(N)$ of $V_\mathfrak{g}$ was given under the assumption that N is a curvature-adapted submanifold of M. In this section, we refine that formula to the case of c-curvature-adapted submanifolds so that it can be applied to orbits of Hermann actions.

Let N be a c-curvature-adapted submanifold of M. To consider the PF submanifold $\Phi_K^{-1}(N)$ of $V_\mathfrak{g}$, we can assume $eK \in N$ without loss of generality due to the equivariant property (2.3) of Φ_K. Choose and fix an arbitrary c-dimensional abelian subspace $\mathfrak{t}$ in $T_{eK}^\perp N$ satisfying the condition (ii) of Definition 4.1 for some $v \in T_{eK}^\perp N$. Recall the decompositions given in Proposition 4.5:

$$T_{eK}N = \bigoplus_{\lambda \in \Lambda_0} (\mathfrak{m}_0 \cap S_\lambda) \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\lambda \in \Lambda_\alpha} (\mathfrak{m}_\alpha \cap S_\lambda), \tag{5.1}$$

$$T_{eK}^\perp N = \mathfrak{m}_0 \cap T_{eK}^\perp N \oplus \bigoplus_{\alpha \in \Delta^+} \left(\mathfrak{m}_\alpha \cap T_{eK}^\perp N \right). \tag{5.2}$$

Set

$$m(0,\lambda) := \dim(\mathfrak{m}_0 \cap S_\lambda), \qquad m(\alpha,\lambda) := \dim(\mathfrak{m}_\alpha \cap S_\lambda),$$
$$m(0,\perp) := \dim(\mathfrak{m}_0 \cap T_{eK}^\perp N), \quad m(\alpha,\perp) := \dim\left(\mathfrak{m}_\alpha \cap T_{eK}^\perp N\right).$$

Take bases

$$\left\{y_j^{0,\lambda}\right\}_{j=1}^{m(0,\lambda)} \text{ of } \mathfrak{m}_0 \cap S_\lambda, \qquad \left\{y_k^{\alpha,\lambda}\right\}_{k=1}^{m(\alpha,\lambda)} \text{ of } \mathfrak{m}_\alpha \cap S_\lambda,$$
$$\left\{\{y_l^{0,\perp}\right\}_{l=1}^{m(0,\perp)} \text{ of } \mathfrak{m}_0 \cap T_{eK}^\perp N, \quad \left\{y_r^{\alpha,\perp}\right\}_{r=1}^{m(\alpha,\perp)} \text{ of } \mathfrak{m}_\alpha \cap T_{eK}^\perp N.$$

Then we obtain a basis

$$\bigcup_{\lambda \in \Lambda_0} \left\{y_j^{0,\lambda}\right\}_{j=1}^{m(0,\lambda)} \cup \left\{y_l^{0,\perp}\right\}_{l=1}^{m(0,\perp)}$$

of $\mathfrak{m}_0$ and a basis

$$\bigcup_{\lambda\in\Lambda_\alpha}\left\{y_k^{\alpha,\lambda}\right\}_{k=1}^{m(\alpha,\lambda)}\cup\left\{y_r^{\alpha,\perp}\right\}_{r=1}^{m(\alpha,\perp)}$$

of $\mathfrak{m}_\alpha$. Via an isometry $\psi_\alpha:\mathfrak{m}_\alpha\to\mathfrak{k}_\alpha$ defined by (3.1) we take a basis

$$\bigcup_{\lambda\in\Lambda_\alpha}\left\{x_k^{\alpha,\lambda}\right\}_{k=1}^{m(\alpha,\lambda)}\cup\left\{x_r^{\alpha,\perp}\right\}_{r=1}^{m(\alpha,\perp)}$$

of $\mathfrak{k}_\alpha$. Finally we choose a basis $\{x_i^0\}_{i=1}^{\dim\mathfrak{k}_0}$ of $\mathfrak{k}_0$. Then the relations

$$\begin{aligned}
&\left[\xi,x_i^0\right]=0, && \left[\xi,y_j^{0,\lambda}\right]=\left[\xi,y_l^{0,\perp}\right]=0,\\
&\left[\xi,x_k^{\alpha,\lambda}\right]=-\langle\alpha,\xi\rangle\,y_k^{\alpha,\lambda}, && \left[\xi,y_k^{\alpha,\lambda}\right]=\langle\alpha,\xi\rangle\,x_k^{\alpha,\lambda},\\
&\left[\xi,x_r^{\alpha,\perp}\right]=-\langle\alpha,\xi\rangle\,y_r^{\alpha,\perp}, && \left[\xi,y_r^{\alpha,\perp}\right]=\langle\alpha,\xi\rangle\,x_r^{\alpha,\perp}.
\end{aligned}$$

hold for any $\xi\in\mathfrak{t}$.

We write $V(\mathfrak{g})$ for the Hilbert space $V_{\mathfrak{g}}=L^2([0,1],\mathfrak{g})$ and decompose

$$\begin{aligned}
V(\mathfrak{g})\;=\;&V(\mathfrak{k}_0)\oplus V(\mathfrak{m}_0\cap T_{eK}N)\oplus V(\mathfrak{m}_0\cap T_{eK}^{\perp}N)\\
&\oplus\bigoplus_{\alpha\in\Delta^+}(V(\mathfrak{k}_\alpha)\oplus V(\mathfrak{m}_\alpha\cap T_{eK}N)\oplus V(\mathfrak{m}_\alpha\cap T_{eK}^{\perp}N)).
\end{aligned}$$

We equip a suitable basis with each term above. Recall that in addition to

$$\left\{1,\sqrt{2}\cos 2n\pi t,\sqrt{2}\cos 2n\pi t\right\}_{n=1}^{\infty}$$

there are two other kinds of orthonormal bases of $L^2([0,1],\mathbb{R})$, namely

$$\left\{1,\sqrt{2}\cos 2n\pi t\right\}_{n=1}^{\infty}\qquad\text{and}\qquad\left\{\sqrt{2}\sin n\pi t\right\}_{n=1}^{\infty}.$$

We consider bases

$$\begin{aligned}
&\left\{x_i^0\sin n\pi t\right\}_{i,n} && \text{of } V(\mathfrak{k}_0),\\
&\left\{y_j^{0,\lambda}\right\}_{\lambda,j}\cup\left\{y_j^{0,\lambda}\cos n\pi t\right\}_{\lambda,j,n} && \text{of } V(\mathfrak{m}_0\cap T_{eK}N),\\
&\left\{y_l^{0,\perp}\right\}_{l}\cup\left\{y_l^{0,\perp}\cos n\pi t\right\}_{l,n} && \text{of } V(\mathfrak{m}_0\cap T_{eK}^{\perp}N),\\
&\left\{x_k^{\alpha,\lambda}\sin n\pi t\right\}_{\lambda,k,n}\cup\left\{x_r^{\alpha,\perp}\sin n\pi t\right\}_{r,n} && \text{of } V(\mathfrak{k}_\alpha),\\
&\left\{y_k^{\alpha,\lambda}\right\}_{\lambda,k}\cup\left\{y_k^{\alpha,\lambda}\cos n\pi t\right\}_{\lambda,n,k} && \text{of } V(\mathfrak{m}_\alpha\cap T_{eK}N),\\
&\left\{y_r^{\alpha,\perp}\right\}_{r}\cup\left\{y_r^{\alpha,\perp}\cos n\pi t\right\}_{n,r} && \text{of } V(\mathfrak{m}_\alpha\cap T_{eK}^{\perp}N).
\end{aligned}$$

Then all these bases form a basis of $V(\mathfrak{g}) = V_{\mathfrak{g}} \cong T_{\hat{0}} V_{\mathfrak{g}}$. Since $\Phi : V_{\mathfrak{g}} \to G$ is a Riemannian submersion with the orthogonal direct sum decomposition [31, p. 686]

$$T_{\hat{0}} V_{\mathfrak{g}} = T_{\hat{0}} \Phi^{-1}(e) \oplus \hat{\mathfrak{g}}, \qquad X = (X - \textstyle\int_0^1 X(t)dt) \oplus \int_0^1 X(t)dt,$$

we have the orthogonal direct sum decomposition

$$T_{\hat{0}} V_{\mathfrak{g}} \cong T_{\hat{0}} \Phi_K^{-1}(N) \oplus T_{eK}^{\perp} N, \qquad X = (X - (\textstyle\int_0^1 X(t)dt)^{\perp}) \oplus (\int_0^1 X(t)dt)^{\perp},$$

where $\perp$ denotes the projection from $\mathfrak{g} = \mathfrak{k} \oplus T_{eK} N \oplus T_{eK}^{\perp} N$ onto $T_{eK}^{\perp} N$. Thus, we obtain a basis

$$\begin{aligned}
&\left\{x_i^0 \sin n\pi t\right\}_{i,n} \cup \left\{y_j^{0,\lambda}\right\}_{\lambda, j} \cup \left\{y_j^{0,\lambda} \cos n\pi t\right\}_{\lambda, j, n} \cup \left\{y_r^{0,\perp} \cos n\pi t\right\}_{r,n} \\
&\cup \bigcup_{\alpha \in \Delta^+} \left(\left\{x_k^{\alpha,\lambda} \sin n\pi t\right\}_{\lambda, k, n} \cup \left\{y_k^{\alpha,\lambda}\right\}_{\lambda, k} \cup \left\{y_k^{\alpha,\lambda} \cos n\pi t\right\}_{\lambda, k, n} \right) \\
&\cup \bigcup_{\alpha \in \Delta^+} \left(\left\{x_r^{\alpha,\perp} \sin n\pi t\right\}_{r,n} \cup \left\{y_r^{\alpha,\perp} \cos n\pi t\right\}_{r,n} \right)
\end{aligned}$$

of $T_{\hat{0}} \Phi_K^{-1}(N)$.

For each $\xi \in \mathfrak{t}$, we denote by $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$ the shape operator of $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi}$. Similarly to [18, Lemma 3.1], the following lemma holds.

Lemma 5.1

(i) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(x_i^0 \sin n\pi t\right) = 0\,, \; A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(y_j^{0,\lambda}\right) = \langle \lambda, \xi \rangle y_j^{0,\lambda}$,

(ii) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(y_j^{0,\lambda} \cos n\pi t\right) = A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(y_l^{0,\perp} \cos n\pi t\right) = 0$,

(iii) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} (x_r^{\alpha,\perp} \sin n\pi t) = -\frac{\langle \alpha, \xi \rangle}{n\pi} y_r^{\alpha,\perp} \cos n\pi t$,

$A_{\hat{\xi}}^{\Phi_K^{-1}(N)} (y_r^{\alpha,\perp} \cos n\pi t) = -\frac{\langle \alpha, \xi \rangle}{n\pi} x_r^{\alpha,\perp} \sin n\pi t$,

(iv) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(y_k^{\alpha,\lambda}\right) = \langle \lambda, \xi \rangle y_k^{\alpha,\lambda} + \frac{2\langle \alpha, \xi \rangle}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(x_k^{\alpha,\lambda} \sin n\pi t\right)$,

(v) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(x_k^{\alpha,\lambda} \sin n\pi t\right) = -\frac{\langle \alpha, \xi \rangle}{n\pi} y_k^{\alpha,\lambda} (-1 + \cos n\pi t)$,

(vi) $A_{\hat{\xi}}^{\Phi_K^{-1}(N)} \left(y_k^{\alpha,\lambda} \cos n\pi t\right) = -\frac{\langle \alpha, \xi \rangle}{n\pi} x_k^{\alpha,\lambda} \sin n\pi t$.

The following theorem describes the principal curvatures of the PF submanifold $\Phi_K^{-1}(N)$ of $V_{\mathfrak{g}}$. This theorem refines [18, Theorem 3.2] (see also [13, Theorem 3.3]). In fact, if $c = 1$ then it is equivalent to the original one. It can be proven by the similar arguments using Lemma 5.1.

Theorem 5.2 *Let $M = G/K$ be a symmetric space of compact type, $\Phi_K : V_{\mathfrak{g}} \to M$ the parallel transport map, N a c-curvature-adapted submanifold of M through eK, and $\mathfrak{t}$ an arbitrary c-dimensional abelian subspace in $\mathfrak{m}$ satisfying the condition* (ii) *of Definition* 4.1. *Then for each $\xi \in \mathfrak{t}$ the principal curvatures of the PF submanifold*

$\Phi_K^{-1}(N)$ *in the direction of* $\hat{\xi}$ *are given by*

$$\{0\} \cup \{\langle \lambda, \xi \rangle \mid \lambda \in \Lambda_0 \cup \textstyle\bigcup_{\beta \in \Delta_\xi^+} \Lambda_\beta\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda, \xi \rangle} + m\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta_\xi^+, \ \lambda \in \Lambda_\alpha, \ m \in \mathbb{Z} \right\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta_\xi^+, \ \mathfrak{m}_\alpha \cap T_{eK}^\perp N \neq \{0\}, \ n \in \mathbb{Z} \backslash \{0\} \right\},$$

where we set $\Delta_\xi^+ := \{\beta \in \Delta^+ \mid \langle \beta, \xi \rangle = 0\}$ *and* $\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda, \xi \rangle} := \frac{\pi}{2}$ *if* $\langle \lambda, \xi \rangle = 0$. *The eigenfunctions and the multiplicities are given in* Table 1.

6 Principal Curvatures of $P(G, H \times K)$-Orbits

In this section, from Theorem 5.2, we derive an explicit formula for the principal curvatures of orbits of $P(G, H \times K)$-actions induced by Hermann actions.

Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Choose and fix a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. Then $\pi(\exp \mathfrak{t})$ is a section of the Hermann action $H \curvearrowright M$ and $\hat{\mathfrak{t}} = \{\hat{x} \mid x \in \mathfrak{t}\}$ is a section of the hyperpolar $P(G, H \times K)$-action on $V_{\mathfrak{g}}$. We take arbitrary $w, \xi \in \mathfrak{t}$ and consider the principal curvatures of $P(G, H \times K) * \hat{w}$ in the direction of $\hat{\xi}$.

Recall that the tangent space and the normal space of the orbit $N = H \cdot aK$ where $a := \exp w$ are decomposed as follows (cf. Section 3 and Example 4.6):

$$T_{aK} N = \bigoplus_{\epsilon \in U(1)_0^\top} dL_a(\mathfrak{m}_{0,\epsilon}) \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^\top} dL_a(\mathfrak{m}_{\alpha,\epsilon}),$$
$$T_{aK}^\perp N = dL_a(\mathfrak{t}) \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^\perp} dL_a(\mathfrak{m}_{\alpha,\epsilon}),$$

Table 1 Eigenfunctions and multiplicities

Eigenvalue	Basis of the eigenspace	Multiplicity
0	$\left\{x_i^0 \sin n\pi t, \ y_j^{0,\lambda} \cos n\pi t, \ y_l^{0,\perp} \cos n\pi t\right\}_{\lambda \in \Lambda_0, n \in \mathbb{Z}_{\geq 1}, i, j, l}$ $\cup \left\{x_k^{\beta,\lambda} \sin n\pi t, \ y_k^{\beta,\lambda} \cos n\pi t\right\}_{\beta \in \Delta_\xi, \lambda \in \Lambda_\beta, n \in \mathbb{Z}_{\geq 1}, k}$ $\cup \left\{x_r^{\beta,\perp} \sin n\pi t, \ y_r^{\beta,\perp} \cos n\pi t\right\}_{\beta \in \Delta_\xi, n \in \mathbb{Z}_{\geq 1}, r}$	∞
$\langle \lambda, \xi \rangle$	$\left\{y_j^{0,\lambda}\}_j \cup \{y_k^{\beta,\lambda}\right\}_{\beta \in \Delta_\xi, k}$	$m(0,\lambda) + \sum_\beta m(\beta, \lambda)$
$\frac{\langle \alpha, \xi \rangle}{\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda, \xi \rangle} + m\pi}$	$\left\{\sum_{n \in \mathbb{Z}} \frac{\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda, \xi \rangle} + m\pi}{\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda, \xi \rangle} + (m+n)\pi} \left(x_k^{\alpha,\lambda} \sin n\pi t + y_k^{\alpha,\lambda} \cos n\pi t\right)\right\}_k$	$m(\alpha, \lambda)$
$\frac{\langle \alpha, \xi \rangle}{n\pi}$	$\left\{x_r^{\alpha,\perp} \sin n\pi t - y_r^{\alpha,\perp} \cos n\pi t\right\}_r$	$m(\alpha, \perp)$

where we set $U(1)_\alpha := \{\epsilon \in U(1) \mid \mathfrak{m}_{\alpha,\epsilon} \neq \{0\}\}$ and

$$U(1)_\alpha^\top := \left\{\epsilon \in U(1)_\alpha \mid \langle\alpha, w\rangle + \frac{1}{2}\arg\epsilon \notin \pi\mathbb{Z}\right\},$$
$$U(1)_\alpha^\perp := \left\{\epsilon \in U(1)_\alpha \mid \langle\alpha, w\rangle + \frac{1}{2}\arg\epsilon \in \pi\mathbb{Z}\right\}.$$

Here, $dL_a(\mathfrak{m}_{0,\epsilon})$ and $dL_a(\mathfrak{m}_{\alpha,\epsilon})$ are the eigenspaces of the shape operator $A^N_{dL_a(\xi)}$ associated with the eigenvalues 0 and $-\langle\alpha, \xi\rangle \cot\left(\langle\alpha, w\rangle + \frac{1}{2}\arg\epsilon\right)$ respectively.

Using the above information, we can describe the principal curvatures of orbits of $P(G, H \times K)$-actions induced by Hermann actions:

Theorem 6.1 *Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ and $w \in \mathfrak{t}$. Then for each $\xi \in \mathfrak{t}$ the principal curvatures of $P(G, H \times K) * \hat{w}$ in the direction of $\hat{\xi}$ are given by*

$$\{0\} \cup \left\{ \frac{\langle\alpha, \xi\rangle}{-\langle\alpha, w\rangle - \frac{1}{2}\arg\epsilon + m\pi} \;\middle|\; \alpha \in \Delta^+\backslash\Delta^+_\xi,\ \epsilon \in U(1)^\top_\alpha,\ m \in \mathbb{Z} \right\}$$
$$\cup \left\{ \frac{\langle\alpha, \xi\rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+\backslash\Delta^+_\xi \text{ satisfying } U(1)^\perp_\alpha \neq \emptyset,\ n \in \mathbb{Z}\backslash\{0\} \right\}.$$

Taking bases $\left\{x_i^0\right\}_i$ of $\mathfrak{k}_0$, $\left\{x_k^{\alpha,\epsilon}\right\}_k$ of $\mathfrak{k}_{\alpha,\epsilon}$, $\left\{y_j^{0,\epsilon}\right\}_j$ of $\mathfrak{m}_{0,\epsilon}$, $\{\eta_l\}_l$ of $\mathfrak{t}$ and $\left\{y_k^{\alpha,\epsilon}\right\}_k$ of $\mathfrak{m}_{\alpha,\epsilon}$ with the relation (3.3) we can describe the eigenfunctions and the multiplicities as in Table 2. *Here, we are identifying $T_{\hat{w}}V_\mathfrak{g}$ with $T_{\hat{0}}V_\mathfrak{g}$ via the gauge transformation $g* : V_\mathfrak{g} \to V_\mathfrak{g}$ for a unique $g \in P(G, G \times \{e\})$ satisfying $g * \hat{0} = \hat{w}$.*

In particular, if $w \in \mathfrak{t}$ is a regular point then the term $\frac{\langle\alpha,\xi\rangle}{n\pi}$ vanishes.

Table 2 Eigenfunctions and multiplicities

Eigenvalue	Basis of the eigenspace	Multiplicity
0	$\left\{x_i^0 \sin n\pi t,\ y_j^{0,\epsilon} \sin n\pi t,\ \eta_l \cos n\pi t\right\}_{\epsilon\in U(1)_0^\top,\, n\in\mathbb{Z}_{\geq 1}, i, j, l}$ $\cup \left\{x_k^{\beta,\epsilon} \sin n\pi t,\ y_k^{\beta,\epsilon} \sin n\pi t\right\}_{\beta\in\Delta^+_\xi,\, \epsilon\in U(1)_\beta^\top,\, n\in\mathbb{Z}_{\geq 1}, k}$ $\cup \left\{x_r^{\beta,\epsilon} \sin n\pi t,\ y_r^{\beta,\epsilon} \cos n\pi t\right\}_{\beta\in\Delta^+_\xi,\, \epsilon\in U(1)_\beta^\perp,\, n\in\mathbb{Z}_{\geq 1}, r}$	∞
$\frac{\langle\alpha,\xi\rangle}{-\langle\alpha,w\rangle-\frac{1}{2}\arg\epsilon+m\pi}$	$\left\{\sum_{n\in\mathbb{Z}} \frac{\langle\alpha,w\rangle+\frac{1}{2}\arg\epsilon+m\pi}{\langle\alpha,w\rangle+\frac{1}{2}\arg\epsilon+(m+n)\pi} \left(x_k^{\alpha,\epsilon} \sin n\pi t + y_k^{\alpha,\epsilon} \cos n\pi t\right)\right\}_k$	$m(\alpha, \epsilon)$
$\frac{\langle\alpha,\xi\rangle}{n\pi}$	$\left\{x_r^{\alpha,\epsilon} \sin n\pi t - y_r^{\alpha,\epsilon} \cos n\pi t\right\}_{\epsilon\in U(1)_\alpha^\perp,\, r}$	$\sum_\epsilon m(\alpha, \epsilon)$

Proof Take a unique $g \in P(G, G \times \{e\})$ satisfying $g * \hat{0} = \hat{w}$. By (2.2) we have $g(0) = \exp w = a$. From (2.3) the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ \Phi_K \downarrow & & \Phi_K \downarrow \\ M & \xrightarrow{L_a} & M \end{array}$$

commutes. Thus setting $\bar{N} := L_a^{-1}(N)$ we have $g*\Phi_K^{-1}(\bar{N}) = \Phi_K^{-1}(N) = P(G, H \times K) * \hat{w}$ by (2.4). Moreover since $w \in \mathfrak{t}$ it follows from $g * \hat{0} = \hat{w}$ that $g(t) \in \exp \mathfrak{t}$ for all $t \in [0, 1]$. Thus we have $d(g*)\hat{\xi} = g\hat{\xi}g^{-1} = \hat{\xi}$. Hence it suffices to compute the principal curvatures of $\Phi_K^{-1}(\bar{N})$ in the direction of $\hat{\xi}$. Since $\mathfrak{t}$ is a c-dimensional abelian subspace in $\mathfrak{m}$ satisfying the condition (ii) of Definition 4.1 we can apply Theorem 5.2 to $\bar{N}$. From (4.3) and (4.4), the tangent space and the normal space of $\bar{N}$ are

$$\begin{aligned} T_{eK}\bar{N} &= \bigoplus_{\epsilon \in U(1)_0^{\top}} \mathfrak{m}_{0,\epsilon} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^{\top}} \mathfrak{m}_{\alpha,\epsilon}, \\ T_{eK}^{\perp}\bar{N} &= \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\epsilon \in U(1)_\alpha^{\perp}} \mathfrak{m}_{\alpha,\epsilon}. \end{aligned}$$

From (4.5) and (4.6), the above decompositions are rewritten as

$$\begin{aligned} T_{eK}\bar{N} &= \mathfrak{m}_0 \cap \bar{S}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \bigoplus_{\lambda \in \Lambda_\alpha} (\mathfrak{m}_\alpha \cap \bar{S}_\lambda), \\ T_{eK}^{\perp}\bar{N} &= \mathfrak{m}_0 \cap T_{eK}^{\perp}\bar{N} \oplus \bigoplus_{\alpha \in \Delta^+} \left(\mathfrak{m}_\alpha \cap T_{eK}^{\perp}\bar{N}\right), \end{aligned}$$

where $\bar{S}_0 := dL_a^{-1}(S_0)$ and $\bar{S}_\lambda := dL_a^{-1}(S_\lambda)$. Since $\langle \beta, \xi \rangle = 0$ implies $\langle \lambda(\beta, \epsilon), \xi \rangle = 0$ the eigenvalue $\langle \lambda, \xi \rangle$ in the theorem is equal to 0. Moreover taking a unique $m' \in \mathbb{Z}$ satisfying $-\pi/2 < \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m'\pi \leq \pi/2$ we have

$$\arctan \frac{\langle \alpha, \xi \rangle}{\langle \lambda(\alpha, \epsilon), \xi \rangle} = -\langle \alpha, w \rangle - \frac{1}{2} \arg \epsilon - m'\pi.$$

Since $m \in \mathbb{Z}$ in the theorem is arbitrary, the assertion follows. □

Applying Theorem 6.1 to (3.6) and (3.7), we obtain the following corollary.

Corollary 6.2 *Suppose that* $\sigma \circ \tau = \tau \circ \sigma$*. Then the principal curvatures of the orbit* $P(G, H \times K) * \hat{w}$ *in the direction of* $\hat{\xi}$ *are given by*

$$\begin{aligned}
&\{0\} \\
&\cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + m\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \mathfrak{m}_\alpha \cap \mathfrak{p} \neq \{0\},\ \langle \alpha, w \rangle \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\} \\
&\cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle - \frac{\pi}{2} + m\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \mathfrak{m}_\alpha \cap \mathfrak{h} \neq \{0\},\ \langle \alpha, w \rangle + \frac{\pi}{2} \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\} \\
&\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \mathfrak{m}_\alpha \cap \mathfrak{p} \neq \{0\},\ \langle \alpha, w \rangle \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right. \\
&\qquad\qquad \left. \text{or } \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \mathfrak{m}_\alpha \cap \mathfrak{h} \neq \{0\},\ \langle \alpha, w \rangle + \frac{\pi}{2} \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right\}.
\end{aligned}$$

The multiplicities are respectively given by

$$\infty, \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}) + \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}).$$

In particular, if $w \in \mathfrak{t}$ *is a regular point then the term* $\frac{\langle \alpha, \xi \rangle}{n\pi}$ *vanishes.*

Applying Theorem 6.1 to (4.5) and (4.6), we obtain the following corollary.

Corollary 6.3 *Suppose that* $\sigma = \tau$*. Then the principal curvatures of the orbit* $P(G, H \times K) * \hat{w}$ *in the direction of* $\hat{\xi}$ *are given by*

$$\begin{aligned}
\{0\} &\cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + m\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \langle \alpha, w \rangle \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\} \\
&\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+ \backslash \Delta^+_\xi,\ \langle \alpha, w \rangle \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right\}.
\end{aligned}$$

The multiplicities are respectively given by

$$\infty, \qquad \dim \mathfrak{m}_\alpha, \qquad \dim \mathfrak{m}_\alpha.$$

In particular, if $w \in \mathfrak{t}$ *is a regular point then the term* $\frac{\langle \alpha, \xi \rangle}{n\pi}$ *vanishes.*

Remark 6.4 Terng [29] showed that any principal orbit of the $P(G, \Delta G)$-action, where ΔG is the diagonal of $G \times G$, is an isoparametric PF submanifold of $V_\mathfrak{g}$ and computed its principal curvatures. This result was extended by Pinkall and Thorbergsson [25] to the case of $P(G, K \times K)$-action, where K is a symmetric subgroup of G. (Note that in the equation (28) of [25] the term $\alpha(Y)$ should be $-\alpha(Y)$.) More generally, Koike [14] computed the principal curvatures of principal orbits of the $P(G, H \times K)$-action induced by a Hermann action with the assumption that the involutions σ and τ commute [14, p. 114]. Theorem 6.1 above does not require such assumptions at all.

Remark 6.5 For each $\alpha \in \Delta^+$ and $\epsilon \in U(1)^\top_\alpha$ it is clear that

$$\left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle - \frac{1}{2}\arg\epsilon + m\pi} \;\middle|\; m \in \mathbb{Z} \right\} = \left\{ -\frac{\langle \alpha, \xi \rangle}{\langle \alpha, w \rangle + \frac{1}{2}\arg\epsilon + m\pi} \;\middle|\; m \in \mathbb{Z} \right\}.$$

We will alternatively use the latter expression to describe the principal curvatures.

7 The Austere Property: Reduced Case

In this section, we study the relation between the austere properties of H- and $P(G, H \times K)$-orbits under the assumption that the root system Δ is reduced; the non-reduced case will be dealt with in the next section. Notice that this assumption is independent of the choice of a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. The main result of this section is the following theorem (Theorem I in Introduction):

Theorem 7.1 *Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Suppose that the root system Δ of a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ is reduced. Then for $w \in \mathfrak{g}$ the following conditions are equivalent:*

(i) *The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M,*
(ii) *The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

To prove this theorem, we need the following lemma. The statement (i) was essentially shown by Ohno [22, Proposition 13]. Note that this lemma is still valid in the non-reduced case.

Lemma 7.2 *Let $\mathfrak{t}$ be a maximal abelian subspace in $\mathfrak{m} \cap \mathfrak{p}$ and $w \in \mathfrak{t}$. Set*

$$U(1)^*_\alpha := \{\epsilon \in U(1)_\alpha \mid \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \notin \frac{\pi}{2}\mathbb{Z}\},$$

which is a subset of $U(1)^\top_\alpha$. Then

(i) (Ohno [22]) *The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M if and only if the set*

$$\left\{ \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) \alpha \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1)^*_\alpha \right\}$$

with multiplicities is invariant under the multiplication by (-1), where the multiplicity of $\cot(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon)\alpha$ is defined to be $m(\alpha, \epsilon)$,

(ii) *The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$ if and only if the set*

$$\left\{ \frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \alpha \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1)^*_\alpha,\ m \in \mathbb{Z} \right\}$$

with multiplicities is invariant under the multiplication by (-1), where the multiplicity of $\frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \alpha$ is defined to be $m(\alpha, \epsilon)$.

In connection with the proof of (ii), we reprove (i) here.

Proof (i) Set $a := \exp w$ and $N := H \cdot aK$. From the straightforward computations [22, pp. 15–16], the normal space of N is expressed as

$$T^{\perp}_{aK} N = dL_a(\mathfrak{m} \cap \mathrm{Ad}(a)^{-1}\mathfrak{p}) = dL_a\Big(\bigcup_{b \in K \cap a^{-1}Ha} \mathrm{Ad}(b)\mathfrak{t}\Big). \tag{7.1}$$

Thus for each $v \in T^{\perp}_{aK}N$ there exist $\xi \in \mathfrak{t}$ and $b \in K \cap a^{-1}Ha$ such that $v = dL_a(\mathrm{Ad}(b)\xi)$. Since b belongs to $a^{-1}Ha$ the isometry L_b leaves the submanifold $\bar{N} := L_a^{-1}N$ invariant. Moreover since b belongs to K the differential dL_b of the isometry L_b at eK is identified with $\mathrm{Ad}(b)$. Thus, the shape operators satisfy $A^{\bar{N}}_{\mathrm{Ad}(b)\xi} = dL_b \circ A^{\bar{N}}_{\xi} \circ dL_b^{-1}$. From this, we obtain

$$A^N_{dL_a(\mathrm{Ad}(b)\xi)} = dL_a \circ dL_b \circ dL_a^{-1} \circ A^N_{dL_a(\xi)} \circ dL_a \circ dL_b^{-1} \circ dL_a^{-1}.$$

This shows that the eigenvalues with multiplicities of the shape operators A^N_v and $A^N_{dL_a(\xi)}$ coincide. Thus, to consider the austere property, it suffices to consider normal vectors $\{dL_a(\xi)\}_{\xi \in \mathfrak{t}}$ of N. Thus, it follows from the common eigenspace decomposition (3.4) that the orbit $H \cdot (\exp w)K$ is an austere submanifold of M if and only if the set

$$\left\{ \langle \alpha, \xi \rangle \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1)^{\top}_{\alpha} \right\}$$

with multiplicities is invariant under the multiplication by (-1) for each $\xi \in \mathfrak{t}$. Notice that this condition is equivalent to the condition that the set

$$\left\{ \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) \alpha \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1)^{\top}_{\alpha} \right\}$$

with multiplicities is invariant under the multiplication by (-1) cf. [11, p. 459]. Hence, the assertion follows from the fact $\cot\left(\frac{\pi}{2} + \pi\mathbb{Z}\right) = \{0\}$.

(ii) Choose a unique $g \in P(G, G \times \{e\})$ satisfying $g * \hat{0} = \hat{w}$. Then we have $a = g(0)$ and the commutative diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ {\scriptstyle \Phi_K}\downarrow & & {\scriptstyle \Phi_K}\downarrow \\ M & \xrightarrow{L_a} & M. \end{array} \tag{7.2}$$

Since Φ_K is a Riemannian submersion it follows from (7.1) and (7.2) that each normal vector of $\Phi_K^{-1}(N)$ is expressed as $(dg*)\,\mathrm{Ad}(b)\hat{\xi}$ for some $\xi \in \mathfrak{t}$ and $b \in K \cap a^{-1}Ha$. Denote by $\hat{b} \in \mathcal{G}$ the constant path with value b. Then by (2.3) we have the commutative diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\hat{b}*} & V_{\mathfrak{g}} \\ {\scriptstyle \Phi_K}\downarrow & & {\scriptstyle \Phi_K}\downarrow \\ M & \xrightarrow{L_b} & M, \end{array}$$

where $\hat{b}*$ is identified with $\mathrm{Ad}(b)$ acting on $V_{\mathfrak{g}}$ by pointwise operation. Since L_b leaves $\bar{N}$ invariant, it follows that $\hat{b}*$ leaves $\Phi_K^{-1}(\bar{N})$ invariant. Thus, we have

$$A^{\Phi_K^{-1}(\bar{N})}_{\mathrm{Ad}(b)(\hat{\xi})} = (d\hat{b}*) \circ A^{\Phi_K^{-1}(\bar{N})}_{\hat{\xi}} \circ (d\hat{b}*)^{-1}.$$

This together with the equality $g * \Phi_K^{-1}(\bar{N}) = \Phi_K^{-1}(N)$ implies

$$A^{\Phi_K^{-1}(N)}_{(dg*)(\mathrm{Ad}(b)\hat{\xi})} = (dg*) \circ (d\hat{b}) * \circ (dg*)^{-1} \circ A^{\Phi_K^{-1}(N)}_{(dg*)(\hat{\xi})} \circ (dg*) \circ (d\hat{b}*)^{-1} \circ (dg*)^{-1}.$$

Thus, similarly, it suffices to consider normal vectors $\{d(g*)\hat{\xi}\}_{\xi \in \mathfrak{t}}$ of $\Phi_K^{-1}(N)$. Note that $g * \hat{0} = \hat{w}$ implies $d(g*)\hat{\xi} = \hat{\xi}$ as mentioned in the proof of Theorem 6.1. Thus, from Theorem 6.1 and Remark 6.5, it follows that the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$ if and only if the set

$$\left\{ \frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \alpha \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1)^{\top}_{\alpha},\ m \in \mathbb{Z} \right\}$$

with multiplicities is invariant under the multiplication by (-1). Hence, the assertion follows from the fact that the set $\left\{ \frac{1}{\pi/2 + m\pi} \alpha \right\}_{m \in \mathbb{Z}}$ with multiplicities is invariant under the multiplication by (-1) due to the equality $\frac{1}{\pi/2+m\pi}\alpha = (-1) \times \frac{1}{\pi/2+(-m-1)\pi}\alpha$. □

We are now in position to prove Theorem 7.1.

Proof of Theorem 7.1 Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. Since $\pi(\exp \mathfrak{t})$ is a section of the H-action, we can assume $w \in \mathfrak{t}$ without loss of generality.

"(i) ⇒ (ii)" : Let $\alpha \in \Delta^+$ and $\epsilon \in U(1)^*_{\alpha}$. Since the orbit $H \cdot (\exp w)K$ is austere, it follows from Lemma 7.2 (i) that there exist $\alpha' \in \Delta^+$ and $\epsilon' \in U(1)^*_{\alpha'}$ such that

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) \alpha = (-1) \times \cot\left(\langle \alpha', w \rangle + \frac{1}{2} \arg \epsilon' \right) \alpha'. \tag{7.3}$$

Since $\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right) \neq 0$ and $\cot\left(\langle \alpha', w \rangle + \frac{1}{2} \arg \epsilon'\right) \neq 0$, it follows from the reduced property of Δ that $\alpha' = \alpha$. Moreover since the map $\epsilon \mapsto \cot(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon)$ is injective we have $m(\alpha, \epsilon) = m(\alpha, \epsilon')$. Then we have

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) = (-1) \times \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon' \right).$$

Since $\cot x$ is strictly decreasing on $\mathbb{R}/\pi\mathbb{Z}$, there exists a unique $n \in \mathbb{Z}$ such that

$$\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon = (-1) \times \left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon' \right) + n\pi.$$

For each $m \in \mathbb{Z}$ we set $m' := -n - m$. Then we obtain

$$\frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \alpha = (-1) \times \frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon' + m'\pi} \alpha.$$

Thus, by Lemma 7.2 (ii) the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

"(ii) $\Rightarrow$ (i)": Since the orbit $P(G, H \times K) * \hat{w}$ is austere, it follows from Lemma 7.2 (ii) that for each $\alpha \in \Delta^+$, $\epsilon \in U(1)^*_\alpha$ and $m \in \mathbb{Z}$ there exist $\alpha' \in \Delta^+$, $\epsilon' \in U(1)^*_{\alpha'}$ and $m' \in \mathbb{Z}$ such that

$$\frac{1}{\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon + m\pi}\alpha = (-1) \times \frac{1}{\langle \alpha', w\rangle + \frac{1}{2}\arg\epsilon' + m'\pi}\alpha'.$$

Since Δ is reduced, we have $\alpha = \alpha'$. Moreover since the map $(\epsilon, m) \mapsto \frac{1}{\langle \alpha, w\rangle + (\arg\epsilon)/2 + m\pi}$ is injective we have $m(\alpha, \epsilon) = m(\alpha, \epsilon')$. Then we have

$$\frac{1}{\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon + m\pi} = (-1) \times \frac{1}{\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon' + m'\pi},$$

that is,

$$\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon + m\pi = (-1) \times \left(\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon' + m'\pi\right).$$

Hence, we have

$$\cot\left(\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon\right)\alpha = (-1) \times \cot\left(\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon'\right)\alpha.$$

Thus by Lemma 7.2 (i) the orbit $H \cdot (\exp w)K$ is an austere submanifold of M. $\square$

Remark 7.3 In the above proof, we essentially showed that the following conditions are equivalent when Δ is reduced:

(i) The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M,

(ii) For each $\alpha \in \Delta^+$ the set

$$\left\{\cot\left(\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon\right)\alpha \;\middle|\; \epsilon \in U(1)^*_\alpha\right\}$$

with multiplicities is invariant under the multiplication by (-1),

(iii) For each $\alpha \in \Delta^+$ the set

$$\left\{\frac{1}{\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon + m\pi}\alpha \;\middle|\; \epsilon \in U(1)^*_\alpha,\ m \in \mathbb{Z}\right\}$$

with multiplicities is invariant under the multiplication by (-1).

(iv) The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$,

8 The Austere Property: General Case

In this section, without supposing that the root system Δ is reduced, we study the relation between the austere properties of H- and $P(G, H \times K)$-orbits.

As a preliminary, we prove the following lemma, which generalizes Lemma 4.32 in [9]. In fact, if $\sigma = \tau$ then it is just the original one.

Lemma 8.1 *Let $M = G/K$ be a symmetric space of compact type, H a symmetric subgroup of G and $\mathfrak{t}$ a maximal abelian subspace in $\mathfrak{m} \cap \mathfrak{p}$. Suppose that there exists $\alpha \in \Delta$ satisfying $2\alpha \in \Delta$. Then the multiplicities satisfy $m(\alpha) > m(2\alpha)$.*

Proof We extend the inner product of $\mathfrak{g}$ to the complex symmetric bi-linear form on $\mathfrak{g}^{\mathbb{C}}$ which is still denoted by $\langle \cdot, \cdot \rangle$. Choose $\epsilon \in U(1)$ satisfying $\mathfrak{g}(\alpha, \epsilon) \neq \{0\}$. Since $\overline{\mathfrak{g}(\alpha, \epsilon)} = \sigma(\mathfrak{g}(\alpha, \epsilon)) = \mathfrak{g}(-\alpha, \epsilon^{-1})$ the involution $z \mapsto \sigma(\bar{z})$ leaves $\mathfrak{g}(\alpha, \epsilon)$ invariant. Thus we have the (± 1)-eigenspace decomposition

$$\mathfrak{g}(\alpha, \epsilon) = \mathfrak{g}(\alpha, \epsilon)^{+} \oplus \mathfrak{g}(\alpha, \epsilon)^{-}.$$

Take $z_0 \in \mathfrak{g}(\alpha, \epsilon)^{+} \backslash \{0\}$. Then by definition we have

$$\sigma(z_0) = \bar{z}_0, \quad \sigma(\bar{z}_0) = z_0, \quad \tau(z_0) = \epsilon \bar{z}_0, \quad \tau(\bar{z}_0) = \epsilon^{-1} z_0. \tag{8.1}$$

Since $[\mathfrak{g}(\alpha), \mathfrak{g}(\alpha)] \subset \mathfrak{g}(2\alpha)$ we have the linear map $\mathrm{ad}(z_0) : \mathfrak{g}(\alpha) \to \mathfrak{g}(2\alpha)$. We restrict this map to the subspace

$$(\mathbb{C} z_0)^{\perp} := \{z \in \mathfrak{g}(\alpha) \mid \langle z, \bar{z}_0 \rangle = 0\}.$$

It suffices to show that the restriction $\mathrm{ad}(z_0) : (\mathbb{C} z_0)^{\perp} \to \mathfrak{g}(2\alpha)$ is surjective. Take arbitrary $y \in \mathfrak{g}(2\alpha)$. We define $x \in (\mathbb{C} z_0)^{\perp}$ by

$$x := \frac{-1}{2\|\alpha\|^2 \|z_0\|^2} [\bar{z_0}, y], \quad \text{where } \|z_0\|^2 := \langle z_0, \bar{z}_0 \rangle.$$

Then, by the Jacobi identity, we have

$$\mathrm{ad}(z_0)[\bar{z}_0, y] = -[\bar{z}_0, [y, z_0]] - [y, [z_0, \bar{z}_0]] = [[z_0, \bar{z}_0], y], \tag{8.2}$$

where the last equality follows from $[y, z_0] \in [\mathfrak{g}(2\alpha), \mathfrak{g}(\alpha)] \subset \mathfrak{g}(3\alpha) = \{0\}$. Notice that $[z_0, \bar{z}_0] \in [\mathfrak{g}(\alpha), \mathfrak{g}(-\alpha)] \subset \mathfrak{g}(0)$. Moreover, from (8.1), we have $[z_0, \bar{z}_0] \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{p}^{\mathbb{C}}$. Hence, we have $[z_0, \bar{z}_0] \in \mathfrak{t}^{\mathbb{C}}$ by maximality. Since

$$\langle [z_0, \bar{z_0}], \eta \rangle = \langle \bar{z}_0, [\eta, z_0] \rangle = \langle \bar{z}_0, \sqrt{-1} \langle \alpha, \eta \rangle z_0 \rangle = \sqrt{-1} \|z_0\|^2 \langle \alpha, \eta \rangle$$

for all $\eta \in \mathfrak{t}$ we get $[z_0, \bar{z}_0] = \sqrt{-1} \|z_0\|^2 \alpha$. Applying (8.2) to this, we obtain

$$\mathrm{ad}(z_0)[\bar{z}_0, y] = \sqrt{-1} \|z_0\|^2 [\alpha, y] = -2 \|z_0\|^2 \|\alpha\|^2 y.$$

Therefore, we have $\mathrm{ad}(z_0)(x) = y$. This proves the lemma. □

Using this lemma, we study the relation between the austere properties of H- and $P(G, H \times K)$-orbits in the rest of this section. First we consider the case $\sigma = \tau$ (Theorem II (i) in "Introduction"):

Theorem 8.2 *Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Suppose that $\sigma = \tau$. Then for $w \in \mathfrak{g}$ the following conditions are equivalent:*

(i) *The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M,*
(ii) *The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Remark 8.3 The above conditions (i) and (ii) are also equivalent to the following conditions (see [9, Proposition 4.27, Theorem 4.31] and [19, Theorem 8]. See also [20, Theorem 1] for the irreducible case):

(iii) The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is a totally geodesic submanifold of M,
(iv) The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is a reflective submanifold of M,
(v) The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Note that the orbit $P(G, H \times K) * \hat{w}$ is not totally geodesic [19, Corollary 2] and thus not reflective.

Proof of Theorem 8.2 Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} = \mathfrak{p}$. Since $\pi(\exp \mathfrak{t})$ is a section of the H-action we can assume $w \in \mathfrak{t}$ without loss of generality.

"(i) $\Rightarrow$ (ii)": Let $\alpha \in \Delta^+$ satisfy $\langle \alpha, w \rangle \notin \pi\mathbb{Z}$. Suppose that $\langle \alpha, w \rangle \notin \frac{\pi}{2}\mathbb{Z}$. Then from Lemma 7.2 (i) there exists $\alpha' \in \Delta^+$ satisfying $\langle \alpha', w \rangle \notin \frac{\pi}{2}\mathbb{Z}$ such that

$$\alpha \cot\langle \alpha, w \rangle = (-1) \times \alpha' \cot\langle \alpha', w \rangle. \tag{8.3}$$

Since $\langle \alpha, w \rangle \notin \frac{\pi}{2}\mathbb{Z}$ we have $\alpha' \neq \alpha$. Then by the property of root systems α' is either 2α or $\frac{1}{2}\alpha$. Suppose that $\alpha' = 2\alpha$. Then the multiplicities of left and right terms of (8.3) are $m(\alpha)$ and $m(2\alpha)$ respectively. However $m(\alpha) > m(2\alpha)$ holds by Lemma 8.1. Thus $\alpha' \neq 2\alpha$. Similarly $\alpha' \neq \frac{1}{2}\alpha$. This is a contradiction. Thus, $\langle \alpha, w \rangle \in \frac{\pi}{2}\mathbb{Z}$ holds for all $\alpha \in \Delta^+$ satisfying $\langle \alpha, w \rangle \notin \pi\mathbb{Z}$. (Thus, N is totally geodesic.) Hence from Lemma 7.2 (ii) the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

"(ii) $\Rightarrow$ (i)": Let $\alpha \in \Delta^+$ satisfy $\langle \alpha, w \rangle \notin \pi\mathbb{Z}$. Suppose that $\langle \alpha, w \rangle \notin \frac{\pi}{2}\mathbb{Z}$. Take $m \in \mathbb{Z}$. Then it follows from Lemma 7.2 (ii) that there exist $\alpha' \in \Delta^+$ satisfying $\langle \alpha', w \rangle \notin \frac{\pi}{2}\mathbb{Z}$ and $m' \in \mathbb{Z}$ such that

$$\frac{1}{\langle \alpha, w \rangle + m\pi}\alpha = (-1) \times \frac{1}{\langle \alpha', w \rangle + m'\pi}\alpha'. \tag{8.4}$$

Since $\langle \alpha, w \rangle \notin \frac{\pi}{2}\mathbb{Z}$ we have $\alpha' \neq \alpha$. Then α' is either 2α or $\frac{1}{2}\alpha$. Suppose that $\alpha' = 2\alpha$. Then the multiplicity of the left term is $m(\alpha) + m(2\alpha)$ due to the equality $\frac{1}{\langle \alpha, w \rangle + m\pi}\alpha = \frac{1}{\langle 2\alpha, w \rangle + 2m\pi}2\alpha$. However, that of the right term is $m(2\alpha)$ since $\alpha' \neq \alpha$. Thus, we have $\alpha' \neq 2\alpha$. Similarly, we have $\alpha' \neq \frac{1}{2}\alpha$. This is a contradiction. Thus $\langle \alpha, w \rangle \in \frac{\pi}{2}\mathbb{Z}$ holds for all $\alpha \in \Delta^+$ satisfying $\langle \alpha, w \rangle \notin \pi\mathbb{Z}$. This shows that the orbit $H \cdot (\exp w)K$ is totally geodesic and therefore austere. □

To generalize Theorem 8.2, we recall an equivalence relation for involutions: For two involutive automorphisms τ and τ' of G, we write $\tau \sim \tau'$ if there exists $c \in G$ such that $\tau' = \mathrm{Ad}(c) \circ \tau \circ \mathrm{Ad}(c)^{-1}$. If $\tau \sim \tau'$ and H a symmetric subgroup of G with respect to τ then $H' := \mathrm{Ad}(c)H$ is a symmetric subgroup of G with respect to τ'. Moreover, the actions of H and H' on M are conjugate, that is, there exists an isomorphism $\phi : H \to H'$ and an isometry $\psi : M \to M$ such that $\psi(b \cdot p) = \phi(b) \cdot \psi(p)$ for $b \in H$ and $p \in M$. In fact $\phi := \mathrm{Ad}(c)$ and $\psi := L_c$ satisfy the property. Thus, we can identify H'-orbits with H-orbits via ψ and the theorem is generalized as follows:

Corollary 8.4 *Let M, H be as in Theorem 8.2. Suppose that $\sigma \sim \tau$. Then for $w \in \mathfrak{g}$ the following conditions are equivalent:*

(i) *The orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M,*
(ii) *The orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Proof Let $c \in G$ satisfy $\sigma = \mathrm{Ad}(c) \circ \tau \circ \mathrm{Ad}(c)^{-1}$. Take $g \in P(G, G \times \{e\})$ satisfying $g(0) = c$. Then from (2.3) the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ \Phi_K \downarrow & & \Phi_K \downarrow \\ M & \xrightarrow{L_c} & M \end{array}$$

commutes. Since each $P(G, H \times K)$-orbit is the inverse image of an H-orbit under Φ_K the assertion follows from Theorem 8.2. □

Next we consider the case $\sigma \circ \tau = \tau \circ \sigma$ (Theorem II (ii) in "Introduction"):

Theorem 8.5 *Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Suppose that the involutions σ and τ commute. Then if the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ where $w \in \mathfrak{g}$ is an austere submanifold of M, the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Remark 8.6 The converse of Theorem 8.5 does not hold in general. In the next section, we will show a counterexample of a minimal H-orbit which is *not* austere but the corresponding minimal $P(G, H \times K)$-orbit is austere.

Proof of Theorem 8.5 Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. We can assume $w \in \mathfrak{t}$ without loss of generality. Let $\alpha \in \Delta^+$. If the set $\mathbb{R}\alpha \cap \Delta^+$ consists of only α then it follows by the same argument as in the proof of Theorem 7.1 that the set

$$\left\{ \frac{1}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \alpha \;\middle|\; \epsilon \in U(1)^*_\alpha,\ m \in \mathbb{Z} \right\}$$

with multiplicities is invariant under the multiplication by (-1). Let us consider the other cases $\mathbb{R}\alpha \cap \Delta^+ = \{\alpha, 2\alpha\}$ or $\{\alpha, \frac{1}{2}\alpha\}$. It suffices to consider the former case. By Lemma 7.2 the union $X \cup Y$ of two sets

$$X := \left\{ \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \right) \alpha \;\middle|\; \epsilon \in U(1)^*_\alpha \right\} \quad \text{and}$$
$$Y := \left\{ \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2} \arg \delta \right) 2\alpha \;\middle|\; \delta \in U(1)^*_{2\alpha} \right\}$$

with multiplicities is invariant under the multiplication by (-1), and it suffices to show that the union $Z \cup W$ of two sets

$$Z := \left\{ \frac{1}{\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon + m\pi}\alpha \;\middle|\; \epsilon \in U(1)^*_\alpha,\ m \in \mathbb{Z} \right\} \quad \text{and}$$
$$W := \left\{ \frac{1}{\langle 2\alpha, w \rangle + \frac{1}{2}\arg \delta + m\pi}2\alpha \;\middle|\; \delta \in U(1)^*_{2\alpha},\ m \in \mathbb{Z} \right\}$$

with multiplicities is invariant under the multiplication by (-1).

Since σ and τ commute, we have $\epsilon, \delta \in \{\pm 1\}$. Thus if $\langle \alpha, w \rangle \in \frac{\pi}{2}\mathbb{Z}$ then the sets X, Y, Z, and W are empty. Suppose that $\langle \alpha, w \rangle \notin \frac{\pi}{2}\mathbb{Z}$. Then $\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon \notin \frac{\pi}{2}\mathbb{Z}$ for all $\epsilon \in U(1)_\alpha$. Thus, $U(1)^*_\alpha = U(1)_\alpha$. Hence $m(\alpha) = \sum_{\epsilon \in U(1)^*_\alpha} m(\alpha, \epsilon)$. This implies that there exists $\epsilon, \epsilon' \in U(1)^*_\alpha$ such that

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon\right)\alpha = (-1) \times \cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon'\right)\alpha. \tag{8.5}$$

In fact, if this does not hold then for each $\epsilon \in U(1)^*_\alpha$ there exists a unique $\delta(\epsilon) \in U(1)^*_{2\alpha}$ satisfying

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon\right)\alpha = (-1) \times \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2}\arg \delta(\epsilon)\right)2\alpha.$$

The multiplicity of the left term is $m(\alpha, \epsilon)$, or $m(\alpha, \epsilon) + m(2\alpha, \delta')$ if there exists $\delta' \in U(1)^*_{2\alpha}$ satisfying $\cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon\right)\alpha = \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2}\arg \delta'\right)2\alpha$. That of the right term is $m(2\alpha, \delta(\epsilon))$; due to negation of (8.5), we have $\cot\left(\langle 2\alpha, w \rangle + \frac{1}{2}\arg \delta(\epsilon)\right) \neq \cot\left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon'\right)$ for any $\epsilon' \in U(1)^*_\alpha$ and thus it is not $m(2\alpha, \delta(\epsilon)) + m(\alpha, \epsilon')$ but $m(2\alpha, \delta(\epsilon))$. Thus, we get $m(\alpha, \epsilon) \leq m(2\alpha, \delta(\epsilon))$. Hence, we obtain

$$m(\alpha) = \sum_{\epsilon \in U(1)^*_\alpha} m(\alpha, \epsilon) \leq \sum_{\epsilon \in U(1)^*_\alpha} m(2\alpha, \delta(\epsilon)) \leq m(2\alpha),$$

where the last inequality is due to the injective property of the map $\epsilon \mapsto \delta(\epsilon)$. This contradicts the fact $m(\alpha) > m(2\alpha)$ of Lemma 8.1. Thus, from (8.5), we have

$$\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon = (-1) \times \left(\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon'\right), \quad \mod \pi\mathbb{Z}.$$

Thus, $\langle \alpha, w \rangle = -\frac{1}{4}\arg \epsilon - \frac{1}{4}\arg \epsilon' \mod \pi\mathbb{Z}$. Since $\epsilon, \epsilon' \in \{\pm 1\}$, we obtain $\langle \alpha, w \rangle \in \frac{\pi}{4}\mathbb{Z}$ and $\langle 2\alpha, w \rangle \in \frac{\pi}{2}\mathbb{Z}$. Thus

$$\langle \alpha, w \rangle + \frac{1}{2}\arg \epsilon \in \frac{\pi}{4}\mathbb{Z}, \qquad \langle 2\alpha, w \rangle + \frac{1}{2}\arg \delta \in \frac{\pi}{2}\mathbb{Z}.$$

for any $\epsilon \in U(1)_\alpha$ and $\delta \in U(1)_{2\alpha}$. Thus, the sets Y and W are empty. Hence, the set X with multiplicities is invariant under the multiplication by (-1). Therefore, by the same argument as in the proof of Theorem 7.1, the set Z with multiplicities is invariant under the multiplication by (-1). This proves the theorem. □

To generalize Theorem 8.5, we recall an equivalence relation for pairs of involutions introduced by Matsuki [17]. Let (σ, τ) and (σ', τ') be two pairs of involutive automorphisms of G. We write $(\sigma, \tau) \sim (\sigma', \tau')$ if there exist an automorphism ρ of G and an element $c \in G$ such that

$$\sigma' = \rho \circ \sigma \circ \rho^{-1}, \qquad \tau' = \mathrm{Ad}(c) \circ \rho \circ \tau \circ \rho^{-1} \circ \mathrm{Ad}(c)^{-1}.$$

If K and H are symmetric subgroups of G then $K' := \rho(K)$ and $H' := \mathrm{Ad}(c)\circ\rho(H)$ are symmetric subgroups of G. Moreover, the H-action on G/K and the H'-action on G/K' are conjugate. In fact, these actions are conjugate under the isomorphism $\phi := \mathrm{Ad}(c) \circ \rho : H \to H'$ and the isometry $\psi : G/K \to G/K'$ defined by $\psi(aK) := c\rho(a)K'$. Then the theorem is generalized as follows:

Corollary 8.7 *Let M, H be as in Theorem 8.5. Suppose that there exists a pair of commuting involutions (σ', τ') of G satisfying $(\sigma, \tau) \sim (\sigma', \tau')$. Then if the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ where $w \in \mathfrak{g}$ is an austere submanifold of M, the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Proof Let $\rho \in \mathrm{Aut}(G)$ and $c \in G$ be as above. Since we equipped an $\mathrm{Aut}(G)$-invariant inner product with $\mathfrak{g}$ the automorphism ρ is an isometry of G. Thus, it induces an isometry from G/K to G/K', which is still denoted by ρ. The differential $d\rho : \mathfrak{g} \to \mathfrak{g}$ induces a linear orthogonal transformation of $V_{\mathfrak{g}}$ by pointwise operation, which is still denoted by $d\rho$. Note that $d\rho(g * \hat{0}) = (\rho \circ g) * \hat{0}$ holds for all $g \in \mathcal{G}$. Take $h \in P(G, G \times \{e\})$ satisfying $h(0) = c$. Then from (2.3) the diagram

$$\begin{array}{ccccc} V_{\mathfrak{g}} & \xrightarrow{d\rho} & V_{\mathfrak{g}} & \xrightarrow{h*} & V_{\mathfrak{g}} \\ \Phi_K \downarrow & & \Phi_{K'} \downarrow & & \Phi_{K'} \downarrow \\ G/K & \xrightarrow{\rho} & G/K' & \xrightarrow{L_c} & G/K' \end{array}$$

commutes. Since each $P(G, H \times K)$-orbit is the inverse image of an H-orbit under Φ_K the assertion follows from Theorem 8.5. □

Finally, as far as possible, we consider the general case that σ and τ do not necessarily commute. In view of Corollary 8.7, it suffices to consider non-commutative pairs of involutions which are not equivalent to commutative ones. According to the classification result [17] if G is simple then there are three kinds of such non-commutative pairs, and if G is not simple then there are many such non-commutative pairs. For a technical reason, here we focus on the case that G is simple. In this case if (σ, τ) is one of those three pairs then the order of the composition $\sigma \circ \tau$ is 3 or 4 see also [22, Section 5]. We will use this fact to prove the following theorem (Theorem II (iii) in "Introduction"):

Theorem 8.8 *Let $M = G/K$ be a symmetric space of compact type and H a symmetric subgroup of G. Suppose that G is simple. Then if the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ where $w \in \mathfrak{g}$ is an austere submanifold of M, the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Proof From the above discussion, it suffices to consider a pair of involutions (σ, τ) where the order l of $\sigma \circ \tau$ is 3 or 4. Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$. We can assume $w \in \mathfrak{t}$ without loss of generality. Let $\alpha \in \Delta^+$. By the same argument as in the proof of Theorem 8.5, it suffices to consider the case $\mathbb{R}\alpha \cap \Delta^+ = \{\alpha, 2\alpha\}$ and to show that the austere property of $X \cup Y$ implies that of $Z \cup W$.

First we show that $\langle \alpha, w \rangle \in \frac{\pi}{2l}\mathbb{Z}$. If $U(1)^*_\alpha \subsetneq U(1)_\alpha$ then there exists $\epsilon \in U(1)_\alpha$ satisfying $\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \in \frac{\pi}{2}\mathbb{Z}$. This shows $\langle \alpha, w \rangle \in \frac{\pi}{2l}\mathbb{Z}$. If $U(1)^*_\alpha = U(1)_\alpha$ then $m(\alpha) = \sum_{\epsilon \in U(1)^*_\alpha} m(\alpha, \epsilon)$. Thus, by the same argument as in the proof of Theorem 8.5, there exists $\epsilon, \epsilon' \in U(1)^*_\alpha$ such that

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right)\alpha = (-1) \times \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon'\right)\alpha.$$

From this, we have

$$\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon = (-1) \times \left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon'\right), \quad \mod \pi\mathbb{Z}.$$

Hence, $\langle \alpha, w \rangle = -\frac{1}{4} \arg \epsilon - \frac{1}{4} \arg \epsilon' \mod \pi\mathbb{Z}$. Therefore, $\langle \alpha, w \rangle \in \frac{\pi}{2l}\mathbb{Z}$ as claimed.

Since $\langle \alpha, w \rangle \in \frac{\pi}{2l}\mathbb{Z}$ we have

$$\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \in \frac{\pi}{2l}\mathbb{Z}, \qquad \langle 2\alpha, w \rangle + \frac{1}{2} \arg \delta \in \frac{\pi}{l}\mathbb{Z}$$

for any $\epsilon \in U(1)_\alpha$ and $\delta \in U(1)_{2\alpha}$. Thus if $l = 3$ then

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right) = \pm\sqrt{3},\ \pm\frac{1}{\sqrt{3}}, \qquad \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2} \arg \delta\right) = \pm\frac{1}{\sqrt{3}}$$

and if $l = 4$ then

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right) = \pm 1,\ \pm(\sqrt{2} \pm 1), \qquad \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2} \arg \delta\right) = \pm 1$$

for $\epsilon \in U(1)^*_\alpha$ and $\delta \in U(1)^*_{2\alpha}$. Therefore

$$\cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right)\alpha \neq (-1) \times \cot\left(\langle 2\alpha, w \rangle + \frac{1}{2} \arg \delta\right) 2\alpha.$$

for any $\epsilon \in U(1)^*_\alpha$ and $\delta \in U(1)^*_{2\alpha}$. This shows that the sets X and Y with multiplicities are respectively invariant under the multiplication by (-1). Thus, by the similar arguments as in the proof of Theorem 7.1, the sets Z and W with multiplicities are respectively invariant under the multiplication by (-1). This proves the theorem. □

Remark 8.9 By the same arguments, we can generalize Theorem 8.8 to the case that G is not simple but the order of $\sigma \circ \tau$ is 3 or 4.

Remark 8.10 In the proofs of Theorems 8.5 and 8.8, we essentially showed that the orbit $H \cdot (\exp w)$ is an austere submanifold of M if and only if the set

$$\left\{ \cot\left(\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon\right)\alpha \;\middle|\; \epsilon \in U(1)^*_\alpha \right\}$$

with multiplicities is invariant under the multiplication by (-1) for each $\alpha \in \Delta^+$.

Example 8.11 Ikawa [9] classified austere orbits of Hermann actions under the assumptions that G is simple and that σ and τ commute. This result was extended by Ohno [22] to the non-commutative case. Thus, applying Theorems 8.5 and 8.8 to those results, we can obtain many examples of homogeneous austere PF submanifolds in Hilbert spaces. Note that so obtained austere PF submanifolds are not totally geodesic due to Corollary 2 in [19].

9 A Counterexample to the Converse

In this section, we show a counterexample to the converse of Theorems 8.5 and 8.8; we show an example of a minimal H-orbit which is *not* austere but the corresponding minimal $P(G, H \times K)$-orbit is austere. Note that from Theorem 7.1 the root system Δ must be non-reduced. We give such an example by the triple

$$(G, K, H) = (SU(p+q), S(U(p) \times U(q)), SO(p+q)).$$

We shall suppose that $p > q$.

The involutions σ and τ of G corresponding to K and H respectively are

$$\sigma = \mathrm{Ad}(I_{pq}) \ \text{ where } \ I_{pq} = \begin{bmatrix} -E_p & 0 \\ 0 & E_q \end{bmatrix} \ \text{ and } \ \tau : \text{complex conjugation},$$

where E_p denote the unit matrix of order p. Clearly σ and τ commute. The canonical decomposition of $\mathfrak{g} = \mathfrak{su}(p+q)$ with respect to σ is given by $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q))$ and

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & 0 & Z \\ 0 & 0 & W \\ -{}^t\bar{Z} & -{}^t\bar{W} & 0 \end{bmatrix} \middle| \ Z \in \mathfrak{gl}(q, \mathbb{C}), \ W \in \mathfrak{gl}(p-q, q, \mathbb{C}) \right\}.$$

The canonical decomposition of $\mathfrak{g}$ with respect to τ is given by $\mathfrak{h} = \mathfrak{so}(p+q)$ and

$$\mathfrak{p} = \{\sqrt{-1}X \mid X \in \mathrm{Sym}(p+q, \mathbb{R}), \ \mathrm{tr}\, X = 0\}.$$

Thus, we can write

$$\mathfrak{m} \cap \mathfrak{p} = \left\{ \sqrt{-1} \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ {}^tX & {}^tY & 0 \end{bmatrix} \middle| \ X \in \mathfrak{gl}(q, \mathbb{R}), \ Y \in \mathfrak{gl}(p-q, q, \mathbb{R}) \right\}.$$

We define a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ by

$$\mathfrak{t} = \left\{ \sqrt{-1} \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ X & 0 & 0 \end{bmatrix} \middle| \ X = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_q \end{bmatrix}, \ x_1, \cdots, x_q \in \mathbb{R} \right\}.$$

Note that $\mathfrak{t}$ is maximal also in $\mathfrak{m}$. For each $i = 1, \cdots, q$ we set

$$e_i = \sqrt{-1} \begin{bmatrix} 0 & 0 & E_{ii} \\ 0 & 0 & 0 \\ E_{ii} & 0 & 0 \end{bmatrix}.$$

where E_{ij} denote the square matrix of order q having 1 in the i-th row and j-th column and zeros elsewhere. We set

$$\mathfrak{m}_{2e_i} = \left\{ \begin{bmatrix} 0 & 0 & X^{(i)} \\ 0 & 0 & 0 \\ -{}^t X^{(i)} & 0 & 0 \end{bmatrix} \,\middle|\, X^{(i)} = xE_{ii},\ x \in \mathbb{R} \right\},$$

$$\mathfrak{m}_{e_i} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & W^{(i)} \\ 0 & -{}^t\bar{W}^{(i)} & 0 \end{bmatrix} \,\middle|\, W^{(i)} = \begin{bmatrix} 0 & w_{1,i} & 0 \\ 0 & \vdots & 0 \\ 0 & w_{p-q,i} & 0 \end{bmatrix},\ w_{1,i}, \cdots, w_{p-q,i} \in \mathbb{C} \right\},$$

$$\mathfrak{m}_{e_i \pm e_j} = \left\{ \begin{bmatrix} 0 & 0 & Z^{(i,j)} \\ 0 & 0 & 0 \\ -{}^t\bar{Z}^{(i,j)} & 0 & 0 \end{bmatrix} \,\middle|\, Z^{(i,j)} = zE_{ij} \mp \bar{z}E_{ji},\ z \in \mathbb{C} \right\},$$

where

$$\dim \mathfrak{m}_{2e_i} = 1, \qquad \dim \mathfrak{m}_{e_i} = 2(p-q), \qquad \dim \mathfrak{m}_{e_k+e_l} = \dim \mathfrak{m}_{e_k-e_l} = 2.$$

Then, we obtain the root space decomposition

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{i=1}^{q} \mathfrak{m}_{2e_i} \oplus \bigoplus_{i=1}^{q} \mathfrak{m}_{e_i} \oplus \bigoplus_{1\le i<j\le q} \mathfrak{m}_{e_i+e_j} \oplus \bigoplus_{1\le i<j\le q} \mathfrak{m}_{e_i-e_j}.$$

By commutativity of involutions, this decomposition is refined as follows:

$$\mathfrak{m} \cap \mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{i=1}^{q} (\mathfrak{m}_{e_i} \cap \mathfrak{p}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i+e_j} \cap \mathfrak{p}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i-e_j} \cap \mathfrak{p}).$$

$$\mathfrak{m} \cap \mathfrak{h} = \bigoplus_{i=1}^{q} \mathfrak{m}_{2e_i} \oplus \bigoplus_{i=1}^{q} (\mathfrak{m}_{e_i} \cap \mathfrak{h}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i+e_j} \cap \mathfrak{h}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i-e_j} \cap \mathfrak{h}),$$

We now consider the orbit $N := H \cdot (\exp w)K$, where $w \in \mathfrak{t}$ is defined by

$$w := \frac{\pi}{8} \sum_{i=1}^{q} e_i.$$

Set $a := \exp w$. Then, from (3.6) and (3.7), the tangent space and the normal space of N are

$$\begin{aligned} T_{aK}N &= dL_a\Big(\bigoplus_{i=1}^{q} (\mathfrak{m}_{e_i} \cap \mathfrak{p}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i+e_j} \cap \mathfrak{p}) \Big) \oplus dL_a\Big(\bigoplus_{i=1}^{q} \mathfrak{m}_{2e_i} \\ &\quad + \bigoplus_{i=1}^{q} (\mathfrak{m}_{e_i} \cap \mathfrak{h}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i+e_j} \cap \mathfrak{h}) \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i-e_j} \cap \mathfrak{h}) \Big), \\ T^{\perp}_{aK}N &= dL_a\Big(\mathfrak{t} \oplus \bigoplus_{1\le i<j\le q} (\mathfrak{m}_{e_i-e_j} \cap \mathfrak{p}) \Big), \end{aligned}$$

and for each $\xi \in \mathfrak{t}$ the principal curvatures of N in the direction of $dL_a(\xi)$ are expressed as the inner product of ξ with vectors

$$-(\sqrt{2}+1)e_i, \quad -(e_i+e_j), \quad 2e_i, \quad (\sqrt{2}-1)e_i, \quad e_i+e_j, \quad 0,$$

whose multiplicities are respectively

$$p-q, \quad 1, \quad 1, \quad p-q, \quad 1, \quad 1.$$

Since the set $\{-(\sqrt{2}+1)e_i, 2e_i, (\sqrt{2}-1)e_i\}$ can not be invariant under the multiplication by (-1), the orbit N is not an austere submanifold of M. Note that if $p-q=1$ then it is a minimal submanifold of M but still not austere.

On the other hand, from Corollary 6.2 (see also Remark 6.5), the principal curvatures of the orbit $P(G, H \times K) * \hat{w}$ in the direction of $\hat{\xi}$ are expressed as the inner product of ξ with vectors

$$\{0\}, \quad \left\{-\frac{1}{\frac{\pi}{8}+m\pi}e_i\right\}_{m\in\mathbb{Z}}, \quad \left\{-\frac{1}{\frac{\pi}{4}+m\pi}(e_i+e_j)\right\}_{m\in\mathbb{Z}},$$
$$\left\{-\frac{1}{\frac{3}{4}\pi+m\pi}2e_i\right\}_{m\in\mathbb{Z}}, \quad \left\{-\frac{1}{\frac{5}{8}\pi+m\pi}e_i\right\}_{m\in\mathbb{Z}}, \quad \left\{-\frac{1}{\frac{3}{4}\pi+m\pi}(e_i+e_j)\right\}_{m\in\mathbb{Z}},$$
$$\left\{-\frac{1}{\frac{\pi}{2}+m\pi}(e_i-e_j)\right\}_{m\in\mathbb{Z}}, \quad \left\{\frac{1}{n\pi}(e_i-e_j)\right\}_{n\in\mathbb{Z}\setminus\{0\}},$$

whose multiplicities are respectively

$$\infty, \quad p-q, \quad 1, \quad 1, \quad p-q, \quad 1, \quad 1, \quad 1.$$

Note that

$$\left\{-\frac{1}{\frac{3}{4}\pi+m\pi}2e_i\right\}_{m\in\mathbb{Z}} = \left\{-\frac{1}{\frac{3}{8}\pi+m\pi}e_i\right\}_{m\in\mathbb{Z}} \cup \left\{-\frac{1}{\frac{7}{8}\pi+m\pi}e_i\right\}_{m\in\mathbb{Z}}.$$

Note also that the sets $\{-\frac{1}{\pi/2+m\pi}(e_i-e_j)\}_{m\in\mathbb{Z}}$ and $\{\frac{1}{n\pi}(e_i-e_j)\}_{n\in\mathbb{Z}\setminus\{0\}}$ with multiplicities are respectively invariant under the multiplication by (-1). Thus, from the equalities

$$\frac{1}{\frac{\pi}{8}+m\pi}e_i = (-1)\times\frac{1}{\frac{7}{8}\pi+(-m-1)\pi}e_i,$$
$$\frac{1}{\frac{5}{8}\pi+m\pi}e_i = (-1)\times\frac{1}{\frac{3}{8}\pi+(-m-1)\pi}e_i,$$
$$\frac{1}{\frac{\pi}{4}+m\pi}(e_i+e_j) = (-1)\times\frac{1}{\frac{3}{4}\pi+(-m-1)\pi}(e_i+e_j)$$

the orbit $P(G, H \times K) * \hat{w}$ is austere if and only if $p-q=1$. Therefore, we have shown that if $p-q=1$ then the orbit $H \cdot (\exp w)K$ is not austere but the orbit $P(G, H \times K) * \hat{w}$ is austere. This is the desired counterexample. In this case, the orbit $H \cdot (\exp w)K$ is a minimal submanifold of M as mentioned above; and thus, the orbit $P(G, H \times K) * \hat{w}$ is minimal PF submanifold of $V_{\mathfrak{g}}$ [5, 12].

Finally, we mention further remarks on the converse. As we have seen above, if the root system Δ is non-reduced then there exists a counterexample to the converse of Theorems 8.5 and 8.8. However, even if Δ is non-reduced, the converse holds in some cases. In fact, Theorem 8.2 and Corollary 8.4 are valid in the non-reduced case. Moreover, consider the case $\sigma \circ \tau = \tau \circ \sigma$ and set

$$\Delta_1^+ := \{\alpha \in \Delta^+ \mid \mathfrak{m}_\alpha \cap \mathfrak{p} \neq \{0\}\} \quad \text{and} \quad \Delta_{-1}^+ := \{\alpha \in \Delta^+ \mid \mathfrak{m}_\alpha \cap \mathfrak{h} \neq \{0\}\}.$$

Suppose that Δ is of type BC and write $\Delta^+ = \{e_i, 2e_i\}_i \cup \{e_i \pm e_j\}_{i<j}$. Suppose also that $\dim \mathfrak{t} \geq 2$ and $\Delta_1^+ \cap \Delta_{-1}^+ = \{e_i\}_i$. Then, it follows by straightforward calculations that the converse holds (cf. [21]). Note that the counterexample shown in this section satisfies $\Delta_1^+ \cap \Delta_{-1}^+ = \{e_i\}_i \cup \{e_i \pm e_j\}_{i<j}$ if $\dim \mathfrak{t} \geq 2$, and $\Delta_1^+ \cap \Delta_{-1}^+ = \{e_1\}$ if $\dim \mathfrak{t} = 1$. For the investigation of the triple $(\Delta, \Delta_1, \Delta_{-1})$ and the corresponding commutative Hermann actions, see Ikawa's papers [9] and [10].

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Declarations

Conflict of Interest The author declares no competing interests.

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