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ON SOME VERTEX ALGEBRAS RELATED TO $V_{-1}(\mathfrak{sl}(n))$ AND THEIR CHARACTERS

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Dedicated to Mirko Primc on the occasion of his 70th birthday

Abstract. We consider several vertex operator algebras and superalgebras closely related to $V_{-1}(\mathfrak{sl}(n))$, $n \geq 3$: (a) the parafermionic subalgebra $K(\mathfrak{sl}(n), -1)$ for which we completely describe its inner structure, (b) the vacuum algebra $\Omega(V_{-1}(\mathfrak{sl}(n)))$, and (c) an infinite extension $\mathcal{U}$ of $V_{-1}(\mathfrak{sl}(n))$ obtained from certain irreducible ordinary modules with integral conformal weights. It turns out that $\mathcal{U}$ is isomorphic to the coset vertex algebra $\mathfrak{psl}(n|n)_1/\mathfrak{sl}(n)_1$, $n \geq 3$. We show that $V_{-1}(\mathfrak{sl}(n))$ admits precisely n ordinary irreducible modules, up to isomorphism. This leads to the conjecture that $\mathcal{U}$ is *quasi-lisse*. We present evidence in support of this conjecture: we prove that the (super)character of $\mathcal{U}$ is quasi-modular of weight one by virtue of being the constant term of a meromorphic Jacobi form of index zero. Explicit formulas and MLDE for characters and supercharacters are given for $\mathfrak{g} = \mathfrak{sl}(3)$ and outlined for general n. We present a conjectural family of 2nd order MLDEs for characters of vertex algebras $\mathfrak{psl}(n|n)_1$, $n \geq 2$. We finish with a theorem pertaining to characters of $\mathfrak{psl}(n|n)_1$ and $\mathcal{U}$-modules.

1. introduction

Orbifolding, coset constructions and simple current extensions are standard methods for producing new examples of vertex algebras. For irrational vertex algebras it is also important to consider *infinite* simple current extension. For instance, lattice vertex algebras are infinite simple current extensions of the Heisenberg vertex algebras. Infinite simple current extensions are also important in

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logarithmic conformal field theory. As demonstrated by the authors, the triplet vertex algebra $\mathcal{W}(p)$ (which is C_2-cofinite) is indeed an infinite simple current extension of the non C_2-cofinite singlet vertex algebra [4].

This paper aims to provide a study of the simple affine vertex operator algebra of level -1 for $\mathfrak{sl}(n)$, denoted by $V_{-1}(\mathfrak{sl}(n))$, and some of its subalgebras and infinite extensions. This vertex algebra is known to be irrational and non C_2-cofinite, and has been studied from several points of view. Early work [25] was focused primarily on various properties of characters of representations. The first author and Perše obtained a complete classification of ordinary (atypical) irreducible $V_{-1}(\mathfrak{sl}(n))$-modules and fusion rules of ordinary modules [12] (see also [11]). They also showed that $V_{-1}(\mathfrak{sl}(n))$ admits a generic (or typical) series of irreducible representations. Kac and Wakimoto recently obtained a Weyl–Kac type character formula [26] involving higher rank partial theta series (cf. also [15]). Asymptotic and modular-type properties of characters of $V_{-1}(\mathfrak{sl}(n))$-modules were studied recently in the work of Bringmann, Mahlburg and the second author [17]. Although these characters are mixed quantum modular forms [15], [17], it seems difficult to formulate and prove a continuous version of the Verlinde formula of characters even for the ordinary modules.

Instead of studying the vertex algebra $V_{-1}(\mathfrak{sl}(n))$, here we focus on two somewhat better behaved objects: (i) the parafermionic algebra(s) and (ii) a certain infinite (simple current) extension which we denote by $\mathcal{U}$. Both vertex algebras have interesting properties from algebraic and number theoretic standpoints; we explore both aspects in great depth.

Let us outline the content and the main results. Throughout we assume that $n \geq 3$. We first review the construction of the simple affine vertex algebra $V_{-1}(\mathfrak{sl}(n))$. Here we utilize the rank n symplectic fermion vertex algebra $\mathcal{A}(n)$ [1], a certain lattice vertex algebra V_L, and the beta-gamma system (or the Weyl vertex algebra) $W_{(n)}$. Then $V_{-1}(\mathfrak{gl}(n))$ (and then of course $V_{-1}(\mathfrak{sl}(n))$) is embedded inside the zero "charge" subalgebra $W_{(n)}^{(0)} \subset \mathcal{A}(n) \otimes V_L$. Similarly, we obtain explicit realizations of irreducible ordinary $V_{-1}(\mathfrak{sl}(n))$-modules denoted by V_s, $s \in \mathbb{Z}$ (see Proposition 2.3).

Then we move on to study parafermionic and vacuum subalgebras. Recall that the parafermionic subalgebra $K(\mathfrak{sl}(n), -1)$ is defined as

$$K(\mathfrak{sl}(n), -1) := \{v \in V_{-1}(\mathfrak{sl}(n)) : a(m)v = 0,\ a \in M(1), m \geq 0\},$$

where $M(1)$ is the Heisenberg subalgebra, and the vacuum algebra is similarly defined as

$$\Omega_n := \{v \in V_{-1}(\mathfrak{sl}(n)) : a(m)v = 0,\ a \in M(1), m > 0\}.$$

Our next result pertains to the structure of these vertex algebras.

Theorem 1.1. *We have*

(1) $K(\mathfrak{sl}(n), -1) \cong \overline{M(1)}^{\otimes n}$, *where* $\overline{M(1)}$ *is the singlet vertex algebra of central charge* -2 *(cf.* [1, 4, 33]*), and*

(2) $\Omega_n \cong \mathcal{A}(n)^{(0)}$, *the charge zero subalgebra of the symplectic fermion vertex algebra.*

Then we consider an infinite extension of $V_{-1}(\mathfrak{sl}(n))$. We first prove that for every $n \geq 3$,

$$\mathcal{U}^{(n)} := \bigoplus_{s \in \mathbb{Z}} V_{s \cdot n}$$

has a simple vertex algebra structure for n even, and $\mathbb{Z}$-graded vertex superalgebra if n is odd. Then we can prove

Theorem 1.2.

(1) *The vertex (super)algebra $\mathcal{U} := \mathcal{U}^{(n)}$ has precisely n ordinary irreducible modules $\mathcal{U}_i$, $0 \leq i \leq n-1$, up to equivalence, such that*

$$\mathcal{U}_i \cong \bigoplus_{s \equiv i \mod n} V_s.$$

(2) *For $n \geq 3$, we have*

$$\mathcal{U} \cong \frac{\mathfrak{psl}(n|n)_1}{\mathfrak{sl}(n)_1}.$$

Since our newly introduced vertex algebra has finitely many ordinary modules, it is natural to ask whether it is quasi-lisse in the sense of [14]. As we are currently unable to prove this property, instead, we investigate the (super)characters of $\mathcal{U}$ and of its modules. If a vertex algebra is quasi-lisse, then necessarily characters and supercharacters must be solutions of modular linear differential equation (MLDE) [14]. In particular, solutions of such equations are known to be either modular (as in the case of ordinary admissible representations) or quasimodular (as in the case of Deligne's series at non-admissible levels). We prove

Theorem 1.3. *The characters and supercharacters of $\mathcal{U}$ are quasi-modular forms. More precisely, for n even (resp. odd) the character $\mathrm{ch}[\mathcal{U}](\tau)$ (resp. the supercharacter $\mathrm{sch}[\mathcal{U}](\tau)$) is a quasi-modular form (with a multiplier) of weight 1 and depth 1 on $\Gamma_0(n)$.*

Motivated again by [14] we conjecture that the (super)character of $\mathcal{U}$ is a component of a vector-valued logarithmic modular form coming from modular linear differential equations (MLDEs). Compared to Deligne's series where this differential equation is of order two, here the situation is more complicated because the order of the equation grows with n. We hope to return to vector-valuedness and properties of MLDEs in our future publications. Here we only analyze an MLDE corresponding to $\mathfrak{g} = \mathfrak{sl}(3)$ (see Proposition 6.3).

The vertex algebras associated to $\mathfrak{psl}(n|n)$ and $\mathfrak{gl}(n|n)$ have recently attracted much attention in the literature (cf. [3], [8], [9], [20], [19]). In the present paper we identify the coset $\mathfrak{psl}(n|n)_1/\mathfrak{sl}(n)_1$ as a vertex algebra $\mathcal{U}$, for $n \geq 3$. We prove in Theorem 7.2 that for every $n \geq 3$, the supercharacter of the simple vertex algebra $V_1(\mathfrak{psl}(n|n))$ equals the supercharacter of the symplectic fermion vertex algebra, which is known to be $\eta(\tau)^2$. In the case $n = 3$, we present a different proof by using the (super)character of $\mathcal{U}$ from the previous section together with branching rules for conformal embeddings.

We have the following conjecture, based on the analysis in the case $\mathfrak{psl}(n|n)_1$ and results from the paper [8] and [14].

Conjecture 1.4. *For every even $n \geq 0$, we have*

$$\mathrm{sch}[V_{-2}(\mathfrak{osp}(n+8|n))](\tau) = \mathrm{ch}[V_{-2}(\mathfrak{so}(8))](\tau).$$

We should also mention that the vertex algebra $V_{-2}(\mathfrak{osp}(n+8|n))$ has recently appeared in the work of K. Costello and D. Gaiotto [19, Sect. 5] in the context of $SU(2)$-gauge theory with $N \geq 4$ flavors.

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2. The affine vertex algebra $V_{-1}(\mathfrak{sl}(n))$

In Section 2 we recall the basic properties of the affine vertex algebra $V_{-1}(\mathfrak{sl}(n))$. Here we use the standard notation: $V^k(\mathfrak{g})$ denotes the universal vertex algebra of level k and $V_k(\mathfrak{g})$ is the corresponding simple vertex algebra. All affine vertex algebras are equipped with the usual conformal structure (via Sugawara's construction).

2.1. Symplectic fermions and the $c = -2$ singlet vertex algebra

The symplectic fermion vertex algebra $\mathcal{A}(n)$ (see [1] for more details) is the universal vertex superalgebra generated by odd fields/vectors ξ_i and η_i $(i = 1, \ldots, n)$ with the following non-trivial λ-bracket

$$[(\xi_i)_\lambda \eta_j] = \delta_{i,j}\lambda.$$

$\mathcal{A}(n)$ can be realized on the irreducible level one module for the Lie superalgebra with generators

$$\{K, \xi_i(n), \eta_i(n), n \in \mathbb{Z}\}$$

and relations

$$\{\xi_i(n), \xi_j(m)\} = \{\eta_i(n), \eta_j(m)\} = 0, \quad \{\xi_i(n), \eta_j(m)\} = n\delta_{i,j}\delta_{n+m,0}K.$$

Here K is central and other super-commutators are trivial. As a vector space,

$$\mathcal{A}(n) = \bigwedge \mathrm{span}\left\{\xi_i(-m), \eta_i(-m),\ m \in \mathbb{Z}_{>0}, i = 1, \ldots, n\right\}.$$

The fields ξ_i, η_j can be identified as formal Laurent series acting on $\mathcal{A}(n)$,

$$\xi_i(x) = \sum_{n\in\mathbb{Z}} \xi_i(n)x^{-n-1}, \quad \eta_i(x) = \sum_{n\in\mathbb{Z}} \eta_i(n)x^{-n-1}.$$

The vertex algebra $\mathcal{A}(n)$ has the following Virasoro element of central charge $c = -2n$:

$$\omega_{\mathcal{A}(n)} = \sum_{i=1}^{n} :\xi_i \eta_i: .$$

There is a charge operator $J \in \operatorname{End}(\mathcal{A}(n))$ such that

$$[J, \xi_i(n)] = \xi_i(n), \quad [J, \eta_i(n)] = -\eta_i(n),$$

which defines on $\mathcal{A}(n)$ the $\mathbb{Z}$-gradation:

$$\mathcal{A}(n) = \sum_{\ell \in \mathbb{Z}} \mathcal{A}(n)^{(\ell)}, \quad \mathcal{A}(n)^{(\ell)} = \{v \in \mathcal{A}(n) \mid Jv = \ell v\}.$$

The character of $\mathcal{A}(n)$ is given by

$$\begin{aligned} \operatorname{ch}[\mathcal{A}(n)](\tau, \zeta) &= \operatorname{tr}|\mathcal{A}(n)\ q^{L(0)-c/24}\zeta^J \\ &= q^{n/12} \prod_{i=1}^{\infty} (1 + q^i \zeta)^n (1 + q^i \zeta^{-1})^n \qquad (1) \\ &= \sum_{\ell \in \mathbb{Z}} \operatorname{ch}[\mathcal{A}(n)^{(\ell)}](\tau) \zeta^\ell. \qquad (2) \end{aligned}$$

Recall that the automorphism group of $\mathcal{A}(n)$ is $\operatorname{Aut}(\mathcal{A}(n)) = \operatorname{Sp}(2n, \mathbb{C})$ (cf. [1]). Let $g_m = \exp[(2\pi i/m)J]$. Then g_m generates the subgroup of $\operatorname{Aut}(\mathcal{A}(n))$ isomorphic to $\mathbb{Z}_m$. One can show that

$$\mathcal{A}(n)^{\mathbb{Z}_m} = \sum_{\ell \in \mathbb{Z}} \mathcal{A}(n)^{(m\ell)}. \qquad (3)$$

The vertex algebra $\mathcal{A}(1)^{(0)}$ is isomorphic to the singlet vertex algebra $\overline{M(1)}$ of central charge $c = -2$ (cf. [33], [2]). For every $i \in \{0, \dots, n\}$ we set $\mathcal{G}_i^0 = 1$, and for $m \in \mathbb{Z}_{\geq 1}$ we define

$$\mathcal{G}_i^m = \xi_i(-m) \cdots \xi_i(-1), \quad \mathcal{G}_i^{-m} = \eta_i(-m) \cdots \eta_i(-1).$$

Each $u_r = \mathcal{G}_i^r.\mathbf{1}$, $r \in \mathbb{Z}$, is a singular vector for the singlet vertex algebra, which generates an irreducible module π_r (note that we drop the index i).

It was proven in [5] that these are simple current $\mathcal{A}(1)^{(0)}$-modules with the following fusion rules:

$$\pi_r \times \pi_s = \pi_{r+s}.$$

2.2. The Clifford vertex algebra

The Weyl vertex algebra $F_{(n)}$ is the universal vertex algebra generated by the odd fields $\Psi_i^{\pm}$ and the following non-trivial λ-bracket:

$$[(\Psi_i^+)_\lambda \Psi_j^-] = \delta_{i,j}, \quad (i, j = 1, \dots, n).$$

The vertex algebra $F_{(n)}$ has the structure of the irreducible level one module for Clifford algebra with generators $\{K, \Psi^{\pm}(n+1/2) \mid n \in \mathbb{Z}\}$ and super-commutation relations:

$$\{\Psi_i^+(r), \Psi_j^-(s)\} = \delta_{i,j}\delta_{r+s,0}K, \quad \{\Psi_i^{\pm}(r), \Psi_j^{\pm}(s)\} = 0 \quad (r, s \in \tfrac{1}{2} + \mathbb{Z},\ i, j = 1, \dots, n),$$

where K is the central element. The fields $\Psi_i^{\pm}$ act on $F_{(n)}$ as the following Laurent series

$$\Psi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \Psi^{\pm}\big(n + \tfrac{1}{2}\big) z^{-n-1}.$$

2.3. The Weyl vertex algebra and its bosonization.

The Weyl vertex algebra $W_{(n)}$ is the universal vertex algebra generated by the even fields $a_i^{\pm}$ and the following non-trivial λ-bracket:

$$[(a_i^+)_\lambda a_j^-] = \delta_{i,j} \quad (i, j = 1, \dots, n).$$

The vertex algebra $W_{(n)}$ has the structure of the irreducible level one module for the Lie algebra with generators $\{K, a^{\pm}(n + 1/2) \mid n \in \mathbb{Z}\}$ and commutation relations:

$$[a_i^+(r), a_j^-(s)] = \delta_{i,j}\delta_{r+s,0}K, \quad [a_i^{\pm}(r), a_j^{\pm}(s)] = 0 \quad (r, s \in \tfrac{1}{2} + \mathbb{Z},\ i, j = 1, \dots, n),$$

where K is central element. The fields $a_i^{\pm}$ acts on $W_{(n)}$ as the following Laurent series

$$a^{\pm}(z) = \sum_{n \in \mathbb{Z}} a^{\pm}\big(n + \tfrac{1}{2}\big) z^{-n-1}.$$

For $i = \{1, \dots, n\}$ and $r \in \mathbb{Z}$, we define

$$X_i^r := a_i^+(-1/2)^r, \quad r \geq 0 \;\text{ and }\; X_i^r := a_i^-(-1/2)^{-r}\mathbf{1},\; r < 0.$$

Let $V_L = M_n(1) \otimes \mathbb{C}[L]$ be the lattice vertex superalgebra associated to the lattice

$$L = \mathbb{Z}\varphi_1 \oplus \cdots \oplus \mathbb{Z}\varphi_n$$

with products:

$$\langle \varphi_i, \varphi_j \rangle = -\delta_{i,j} \quad (i, j = 1, \dots, n).$$

Here $M_n(1)$ denotes the level one module for the Heisenberg vertex algebra associated to the Heisenberg Lie algebra $\widehat{\mathfrak{h}}_n = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_n \oplus \mathbb{C}K$, where $\mathfrak{h}_n = \mathbb{C} \otimes_{\mathbb{Z}} L$.

We have the embedding

$$W_{(n)} \to \mathcal{A}(n) \otimes V_L$$

such that

$$a_i^+ =: \xi_i e^{\varphi_i} : \quad , \quad a_i^- = - : \eta_i e^{-\varphi_i} :$$

2.4. Realization of $V_{-1}(\mathfrak{sl}(n))$ and its ordinary modules

Define $c := -(\varphi_1 + \cdots + \varphi_n)$. Then $c(0)$ defines on $W_{(n)}$ the natural $\mathbb{Z}$-gradation:

$$W_{(n)} = \bigoplus_{\ell \in \mathbb{Z}} W_{(n)}^{(\ell)}.$$

Let $M_c(1)$ be the Heisenberg vertex algebra of level 1 generated by c. Let $M_c(1, r)$ be the irreducible $M_c(1)$-module on which $c(0)$ acts as $r\mathrm{Id}$.

The vertex subalgebra of $W_{(n)}^{(0)}$ generated by the vectors

$$\{e_{i,j} = - : a_i^+ a_j^- : \mid i, j = 1, \ldots, n\}$$

is isomorphic to the simple affine vertex algebra $V_{-1}(\mathfrak{gl}(n))$ at level -1 (cf. [12]).

We also have for $i \neq j$:

$$e_{i,j} :=: \xi_i \eta_j e^{\varphi_i - \varphi_j} : . \tag{4}$$

Then we have:

Theorem 2.1. *$W_{(n)}^{(0)}$ is a simple vertex algebra and the following holds:*

- ([27]) *For $n = 1$, $W_{(n)}^{(0)}$ is a $\mathcal{W}_{1+\infty}$-algebra at central charge $c = -1$.*
- ([21, Thm. 5.2]) *For $n = 2$, $W_{(n)}^{(0)} \cong \mathcal{W} \otimes M_c(1)$, where $\mathcal{W}$ is a certain W-algebra of type $W(1, 1, 1, 2, 2, 2)$ at central charge $c = -3$ (conjecturally isomorphic to $W_{-5/2}(\mathfrak{sl}(4), f_{sh})$ (cf. [21], [6]) where f_{sh} is a short nilpotent element of $\mathfrak{sl}(4)$).*
- ([12]) *For $n \geq 3$: $W_{(n)}^{(0)} \cong V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1)$.*

We need the following result on fusion rules.

Proposition 2.2. *[12] Assume that $n \geq 3$. For $s \in \mathbb{Z}_{\geq 0}$, let*

$$V_s := L_{\mathfrak{sl}(n)}(-(1+s)\Lambda_0 + s\Lambda_1), \quad V_{-s} := L_{\mathfrak{sl}(n)}(-(1+s)\Lambda_0 + s\Lambda_{n-1}).$$

- *The set $\{V_s \mid s \in \mathbb{Z}\}$ provides a complete list of irreducible $V_{-1}(\mathfrak{sl}(n))$-modules in the category KL_{-1} (= the category of ordinary modules).*
- *The following fusion rules hold in the category KL_{-1}.*

$$V_{s_1} \times V_{s_2} = V_{s_1+s_2} \qquad (s_1, s_2 \in \mathbb{Z}). \tag{5}$$

Let us now present a realization of irreducible $V_{-1}(\mathfrak{sl}(n))$-modules. Let

$$Q_n = \{z_1\varphi_1 + \cdots + z_n\varphi_n \mid z_i \in \mathbb{Z}, z_1 + \cdots + z_n = 0\}$$

be the root lattice of $\mathfrak{sl}(n)$ with negative-definite signature. Since $V_{-1}(\mathfrak{sl}(n)) \subset \mathcal{A}(n) \otimes V_{Q_n}$, we have that for every $\lambda \in Q_n^0$ (= the dual lattice of Q_n), $\mathcal{A}(n) \otimes V_{\lambda + Q_n}$ is a $V_{-1}(\mathfrak{sl}(n))$-module. Let

$$\omega_1 = \frac{1}{n}((n-1)\varphi_1 - \varphi_2 - \cdots - \varphi_n), \quad \omega_{n-1} = \frac{1}{n}(\varphi_1 + \cdots + \varphi_{n-1} - (n-1)\varphi_n).$$

We set $v^{(0)} = \mathbf{1}$. For $j \in \mathbb{Z}_{>0}$ we define

$$v^{(j)} = b_1(-j) \cdots b_1(-1)\mathbf{1} \otimes e^{j\omega_1},$$
$$v^{(-j)} = c_n(-j) \cdots c_n(-1)\mathbf{1} \otimes e^{j\omega_{n-1}}.$$

Proposition 2.3. *For $s \in \mathbb{Z}$, we have:*

$$V_s \cong V_{-1}(\mathfrak{sl}(n)).v^{(s)}.$$

Proof. First we notice that $v^{(s)}$ is a singular vector for $\widehat{\mathfrak{sl}}(n)$. Then the space $\widetilde{U}_s = V_{-1}(\mathfrak{sl}(n)).v^{(s)}$ is a highest weight $V_{-1}(\mathfrak{sl}(n))$-module, having the same highest weight as V_s. By using the bosonization of the Weyl vertex algebra, we show that as a $V_{-1}(\mathfrak{gl}(n)) = V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1)$-module, we have $W_{(n)}^{(s)} \cong \widetilde{U}_s \otimes M_c(1,s)$. Since $W_{(n)}^{(s)}$ is an irreducible $V_{-1}(\mathfrak{gl}(n))$-module, we conclude that $\widetilde{U}_s$ is an irreducible $V_{-1}(\mathfrak{sl}(n))$-module, and thus $\widetilde{U}_s \cong V_s$. □

3. Parafermionic algebra and the vacuum of $V_{-1}(\mathfrak{sl}(n))$

Recall the definition of the parafermion vertex algebra of level k:

$$K(\mathfrak{g}, k) := \{v \in V_k(\mathfrak{g}) \mid (\mathfrak{h} \otimes t^m).v = 0,\ m \in \mathbb{Z}_{\geq 0}\}.$$

Theorem 3.1. *Assume that $n \geq 3$. Then*

$$K(\mathfrak{sl}(n), -1) \cong (\mathcal{A}(1)^{(0)})^{\otimes n}.$$

Proof. Let $M_{n-1}(1)$ (resp. $M_n(1)$) be the Heisenberg vertex algebra generated by the Cartan Lie subalgebra of $\mathfrak{sl}(n)$ (resp. $\mathfrak{gl}(n)$). Let $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{C}c$. As usual, we identify $x = x_{(-1)}\mathbf{1}$ for $x \in \mathfrak{sl}(n)$. Then $M_n(1) = M_{n-1}(1) \otimes M_c(1)$, where $M_c(1)$ is the Heisenberg vertex algebra generated by c.

By [12], we have for $n \geq 3$:

$$V_{-1}(\mathfrak{gl}(n)) = V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1) \cong W_{(n)}^{(0)} = \mathrm{Ker}_{W_{(n)}} c(0).$$

A. Linshaw proved in [29, Thm. 7.2] that

$$\mathrm{Com}(M_n(1), W_{(n)}) \cong (\mathcal{A}(1)^{(0)})^{\otimes n}.$$

(This corresponds to the case $m = n$ in [29]).

Since $c \in M_n(1)$, we have that $\mathrm{Com}(M_n(1), W_{(n)}) \subset W_{(n)}^{(0)}$, which implies that for $n \geq 3$:

$$\mathrm{Com}(M_n(1), W_{(n)}) = \mathrm{Com}(M_n(1), V_{-1}(\mathfrak{gl}(n))) \cong \mathrm{Com}(M_{n-1}(1), V_{-1}(\mathfrak{sl}(n)).$$

Therefore (for $n \geq 3$):

$$\mathrm{Com}(M_n(1), W_{(n)}) \cong \mathrm{Com}(M_{n-1}(1), V_{-1}(\mathfrak{sl}(n))) \cong K(\mathfrak{sl}(n), -1).$$

The proof follows. □

If $\mathcal{M}$ is any $V_{-1}(\mathfrak{sl}(\mathfrak{n}))$-module, the (q, ζ)-character and q-character are defined as

$$\mathrm{ch}[\mathcal{M}](\tau, \zeta) = \mathrm{tr}|_{\mathcal{M}}\ q^{L(0)-c/24} \zeta^{h(0)},$$
$$\mathrm{ch}[\mathcal{M}](\tau) = \mathrm{tr}|_{\mathcal{M}}\ q^{L(0)-c/24}.$$

3.1. The vacuum space

The vacuum space is defined as

$$\Omega(V_k(\mathfrak{g})) = \{v \in V_k(\mathfrak{g}) \mid h(j)v = 0 \quad j \geq 1,\ h \in \mathfrak{h}\},$$

and it has the structure of a generalized vertex algebra [30], [23].

Theorem 3.2.

(1) *Assume that $n \geq 2$. The vacuum algebra*

$$\Omega_n := \Omega(V_{-1}(\mathfrak{sl}(n))) = \{v \in V_{-1}(\mathfrak{sl}(n)) \mid h(j)v = 0 \quad j \geq 1,\ h \in \mathfrak{h}\}$$

is isomorphic to a vertex subalgebra of $\mathcal{A}(n)$ generated by

$$\{Z_{i,j} =: \xi_i \eta_j : \ \mid 1 \leq i \neq j \leq n\}.$$

(2) *Assume that $n \geq 3$. Then $\Omega_n \cong \mathcal{A}(n)^{(0)}$.*

(3) *The q-character of Ω_n is given by*

$$\mathrm{ch}[\mathcal{A}(n)^{(0)}](\tau) = q^{n/12}\mathrm{CT}_\zeta \prod_{i=1}^{\infty}(1+q^i\zeta)^n(1+q^i\zeta^{-1})^n,$$

where CT_ζ denotes the constant term.

Proof. The proof uses the explicit realization, the bosonization and the formula for Z-operators.

By using [30, Thm. 6.4], we see that $\Omega(V_{-1}(\mathfrak{sl}(n))$ is generated by the following (generalized) vertex operators

$$Z_{i,j} = Y_\Omega(e_{i,j}, z) := E^-(-h_{i,j}, z)e_{i,j}(z)(z)E^+(-h_{i,j}, z)z^{h_{i,j}(0)},$$

where $h_{i,j} = \varphi_i - \varphi_j$ and

$$E^{\pm}(\alpha, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{\pm n} z^{\mp n}\right).$$

Using (4) we see that on $\Omega_n = \Omega(V_{-1}(\mathfrak{sl}(n))$ we have that

$$Z_{i,j} =: \xi_i \eta_j : .$$

Therefore Ω_n is generated by $\{Z_{i,j} =: \xi_i \eta_j : \ \mid 1 \leq i \neq j \leq n\}$. This proves (1).

(2) Since Ω_n is generated by $Z_{i,j}$, $i \neq j$ we have:

$$u_{i,j} = (Z_{i,j})_1 Z_{j,i} = (\xi_i(-1)\eta_j(-1)\mathbf{1})_1 \xi_j(-1)\eta_i(-1)\mathbf{1} =: \xi_i\eta_i : + : xi_j\eta_j :\in \Omega_n.$$

This implies

$$(*) : \xi_i \eta_i :\in \Omega_n$$

Since $\Omega_n \subset \mathcal{A}(n)^{(0)}$, (2) will follow from the following claim:

(2′) For $n \geq 3$, $\mathcal{A}(n)^{(0)}$ is generated by the set $\{: \xi_i \eta_j :, \quad i, j = 1, \dots, n\}$.

The claim (2′) can be proved using completely analogous methods to those from [21, Sect. 5]. We omit details.

The proof of assertion (3) follows from (2) and character formulas for $\mathcal{A}(n)$ from (1)–(2). □

Remark 1. In [21], the authors denoted the maximal vertex operator subalgebra of the generalized vertex operator algebra $\Omega(V_k(\mathfrak{g}))$ by $E_{k,\mathfrak{g}}$ (see [21, Example 3]). In our case, $\Omega(V_{-1}(\mathfrak{sl}(n)))$ is a vertex algebra, so we have

$$E_{-1,\mathfrak{sl}(n)} \cong \mathcal{A}(n)^{(0)}.$$

One can consider the $V_{-1}(\mathfrak{sl}(n))$-module

$$\mathcal{U}^{\mathrm{large}} = \bigoplus_{s\in\mathbb{Z}} V_s,$$

and show that it is a generalized vertex algebra.

On the other hand, one can prove the following theorem.

Theorem 3.3.

(1) *The $V_{-1}(\mathfrak{sl}(n))$-module*

$$\mathcal{U}^{(n)} = \bigoplus_{s\in\mathbb{Z}} V_{ns},$$

carries the structure of a vertex operator algebra if n is even and a $\mathbb{Z}$-graded vertex operator superalgebra if n is odd.

(2) *In the category of ordinary modules, $\mathcal{U}^{(n)}$ has n non-equivalent irreducible ordinary modules:*

$$\mathcal{U}_i := \bigoplus_{s\in\mathbb{Z}} V_{ns+i} \quad (i = 0, \dots, n-1)$$

with the following fusion rules

$$\mathcal{U}_i \times \mathcal{U}_j = \mathcal{U}_{i+j \bmod n}.$$

Proof. The proof of assertion (1) is based on the explicit realization discussed in Section 2. Note that $mn\omega_1, mn\omega_{n-1} \in L$ and that $e^{mn\omega_1}$ and $e^{mn\omega_{n-1}}$ are even vectors in V_L.

Consider the vertex subalgebra $\widetilde{\mathcal{U}}$ of $\mathcal{A}(n) \otimes V_L$ generated by U_0 and highest weight vectors

$$v^{(n)} = \xi_1(-n)\cdots\xi_1(-2)\xi_1(-1) \otimes e^{n\omega_1},$$
$$v^{(-n)} = \eta_n(-n)\cdots\eta_n(-2)\eta_n(-1) \otimes e^{n\omega_{n-1}}.$$

Note that the vector $v^{(\pm n)}$ has conformal weight n. Moreover, vectors $v^{(\pm n)}$ are even (resp. odd) if n is even (resp. odd). Therefore, $\widetilde{\mathcal{U}}$ is a vertex operator algebra if n is even, and a $\mathbb{Z}$-graded vertex operator superalgebra if n is odd.

By Proposition 2.3 we have that $m \in \mathbb{Z}_{>0}$ modules $V_{\pm mn}$ are realized as $V_{\pm mn} = V_0.v^{(\pm mn)}$. Since $v^{(\pm mn)} \in \widetilde{\mathcal{U}}$, we get that $\widetilde{\mathcal{U}} \supset \mathcal{U}^{(n)}$. By using fusion rules (5), we see that $\mathcal{U}^{(n)}$ is a vertex subalgebra of $\widetilde{\mathcal{U}}$. Since both vertex algebras are generated by V_0 and $v^{(\pm n)}$, we conclude that $\mathcal{U}^{(n)} = \widetilde{\mathcal{U}}$. This proves the assertion (1).

Let us now discuss the construction and classification of irreducible $\mathcal{U}^{(n)}$-modules. Clearly $\mathcal{L}_i = \mathcal{U}^{(n)} v^{(i)} = \bigoplus_{s\in\mathbb{Z}} V_{ns+i}$ is an irreducible $\mathcal{U}^{(n)}$-module for $i = 0,\dots,n-1$.

Assume that M is an irreducible ordinary module for $\mathcal{U}^{(n)}$. Then M is in the category KL_{-1} as a $V_{-1}(\mathfrak{sl}(n))$-module. Since the top component $\Omega(M)$ is a finite-dimensional module for $U(\mathfrak{sl}(n))$, we conclude that $\Omega(M)$ contains a singular vector for $\widehat{sl}(n)$. Thus, M contains a $V_{-1}(\mathfrak{sl}(n))$-submodule isomorphic to V_i for certain $i \in \mathbb{Z}$. By using the fusion rules (5) again, we conclude that $M \cong \bigoplus_{s\in\mathbb{Z}} V_{ns+i} = \mathcal{U}_i$. The proof follows. □

Conjecture 3.4. *The vertex algebra $\mathcal{U}^{(n)}$ is quasi-lisse in the sense of* [14].

Remark 2. In our paper we present some evidence for Conjecture 3.4.

- There are finitely many (ordinary) irreducible $\mathcal{U}^{(n)}$-modules.
- Characters and super-characters of (ordinary) $\mathcal{U}^{(n)}$-modules are quasi-modular forms. For $\mathfrak{g} = \mathfrak{sl}(3)$, the supercharacters are solutions of an MLDE (see Proposition 6.3).
- The vacuum space is a C_2-cofinite vertex operator algebra.
  For simplicity, let us discuss the case $n = 3$. Then we will see that the vacuum space $\Omega(\mathcal{U}^{(3)})$ is a $\mathbb{Z}_3$-orbifold of the symplectic fermion vertex algebra $\mathcal{A}(3)$. Since every cyclic orbifold of a C_2-cofinite vertex algebra is C_2-cofinite (cf. [32]), then the vacuum $\Omega(\mathcal{U}^{(3)})$ is C_2-cofinite.

Remark 3. Assume that V is a vertex operator (super) algebra containing a Heisenberg vertex subalgebra $M(1)$. We believe that if the vacuum space $\Omega(V)$ is C_2-cofinite, then V is quasi-lisse.

4. A decomposition of $V_{-1}(\mathfrak{sl}(n))$ as a $K(\mathfrak{sl}(n),-1) \otimes M_{n-1}(1)$-module

4.1. Decomposition of $V_{-1}(\mathfrak{sl}(3))$ as a $K(\mathfrak{sl}(3),-1) \otimes M_2(1)$-module: from the realization

Let Q be the root lattice of $\mathfrak{sl}(3)$. For $(r,s) \in \mathbb{Z}^2$, we set $\gamma_{r,s} = r\varphi_1 + s\varphi_2 - (r+s)\varphi_3$, We have:

$$\begin{aligned} V_{-1}(\mathfrak{sl}(3)) &= \bigoplus_{(r,s)\in\mathbb{Z}^2} \left(K(\mathfrak{sl}(3),-1) \otimes M_2(1)\right).P_{r,s} \\ &= \bigoplus_{(r,s)\in\mathbb{Z}^2} \left(K(\mathfrak{sl}(3),-1) \otimes M_2(1)\right).(v_{r,s} \otimes e^{\gamma_{r,s}}) \\ &= \bigoplus_{(r,s)\in\mathbb{Z}^2} K_{r,s} \otimes M_2(1).e^{\gamma_{r,s}}, \end{aligned}$$

where

$$P_{r,s} = X_1^r X_2^s X_3^{-r-s}\mathbf{1} = v_{r,s} \otimes e^{\gamma_{r,s}},$$

and

$$K_{r,s} = \{v \in V_{-1}(\mathfrak{sl}(3)) \mid h(n)v = \delta_{n,0}\langle h, \gamma_{r,s}\rangle v \quad \forall h \in \mathfrak{h},\ n \in \mathbb{Z}_{\geq 0}\}$$

is an irreducible $K(\mathfrak{sl}(3),-1)$-module generated by the lowest weight vector

$$v_{r,s} = \mathcal{G}_1^r \mathcal{G}_2^s \mathcal{G}_r^{-r-s}\mathbf{1}.$$

We conclude that $K_{r,s} = \pi_r \otimes \pi_s \otimes \pi_{-r-s}$. In this way we have proved the following theorem:

Theorem 4.1. *The vertex algebra $V_{-1}(\mathfrak{sl}(3))$ is a simple current extension of $(\mathcal{A}(1)^{(0)})^{\otimes 3} \otimes M_2(1)$, and*

$$V_{-1}(\mathfrak{sl}(3)) = \bigoplus_{(r,s)\in\mathbb{Z}^2} \pi_r \otimes \pi_s \otimes \pi_{-r-s} \otimes M_2(1)e^{\gamma_{r,s}}$$

4.2. Decomposition of $V_{-1}(\mathfrak{sl}(3))$ as a $K(\mathfrak{sl}(3),-1) \otimes M_2(1)$-module II: from character formulas

It is possible to prove Theorem 4.1 directly from the character formula (7). Recall a well-known identity [13]

$$\frac{1}{\prod_{n\geq 1}(1-zq^{n-1})(1-z^{-1}q^n)} = \frac{\sum_{m\in\mathbb{Z}} F_m(q)z^m}{\prod_{n\geq 1}(1-q^n)^2},$$

where

$$F_m(q) = \sum_{r\geq 0} q^{(2r+1)r+2mr} - \sum_{r\geq 0} q^{(2r-1)r+m(2r-1)}, \quad m \geq 0$$

and

$$F_m(q) = q^m F_{-m}(q), m < 0$$

We use this formula to expand the character (three times). Then we extract the coefficients of z_1 and z_2, which computes characters of modules for the vacuum algebra. Finally, we use a well-known formula for $\mathrm{ch}[K_{r,s}]$ [18], where $K_{r,s}$ is an irreducible module for the tensor product of three copies of the singlet algebra. This gives

$$\mathrm{ch}[V_{-1}(\mathfrak{sl}(3))](\tau) = \sum_{(r,s)\in\mathbb{Z}^2} \mathrm{ch}[K_{r,s}](\tau)\mathrm{ch}[M_2(1).e^{\gamma_{r,s}}](\tau).$$

4.3. q-hypergometric formula for $\mathrm{ch}[V_{-1}(\mathfrak{sl}(3))](\tau)$

Now we use the discussion from the last section to prove:

Proposition 4.2.

$$(q;q)_\infty^2 q^{-1/6}\mathrm{ch}[V_{-1}(\mathfrak{sl}(3))](\tau)$$
$$= \sum_{k_1,k_2\in\mathbb{Z}^2} \sum_{n_1,n_2,n_3\geq 0} \frac{q^{n_1^2+n_2^2+n_3^2+(|k_1|+1)n_1+(|k_2|+1)n_2+(|-k_1-k_2|+1)n_3+\frac{|k_1|+|k_2|+|-k_1-k_2|}{2}}}{(q;q)_{n_1}(q;q)_{n_1+|k_1|}(q;q)_{n_2}(q;q)_{n_2+|k_2|}(q;q)_{n_3}(q;q)_{n_3+|-k_1-k_2|}},$$

where $(q;q)_\infty = \prod_{i\geq 1}(1-q^i)$, $(q;q)_n = \prod_{i=1}^n(1-q^i)$.

Proof. Follows directly from the q-hypergeometric representations for the $p=2$ false theta functions, which are essentially characters of modules for the $p=2$ singlet algebra (here $k \in \mathbb{Z}$):

$$\frac{q^{k/2}\sum_{n\geq 0}\left(q^{2n^2+n(2k+1)}-q^{2n^2+n(2k+3)+k+1}\right)}{(q;q)_\infty}=\sum_{n\geq 0}\frac{q^{n^2+n(|k|+1)+|k|/2}}{(q;q)_n(q;q)_{n+|k|}}.$$

These relations are well-known and implicitly proven in [18, 34]. They generalize a well-known Ramanujan's identity corresponding to $k=0$. □

4.4. The vacuum space $\Omega(\mathcal{U}^{(3)})$

Let us again consider the case $n=3$, so that $\mathcal{U}=\mathcal{U}^{(3)}$.

Moreover, for $m \in \mathbb{Z}_{>0}$, the modules $V_{\pm 3m}$ are realized as $V_{\pm 3m}=V_0.v^{(\pm 3m)}$, where

$$v^{(3m)}=\xi_1(-3m)\cdots\xi_1(-1)\otimes e^{3m\omega_1},$$
$$v^{(-3m)}=\eta_3(-3m)\cdots\eta_3(-1)\otimes e^{3m\omega_2}.$$

Recall from Subsection 2.1 that the action of the cyclic group $\mathbb{Z}_3$ on $\mathcal{A}(3)$ is generated by $g_3=\exp[\pi i J/3]$.

Theorem 4.3. *We have:*

$$\Omega(\mathcal{U})\cong\mathcal{A}(3)^{\mathbb{Z}_3}.$$

Proof. Applying formula (3) for $m=3$ we get

$$\begin{aligned}\mathcal{A}(3)^{\mathbb{Z}_3}&\cong\bigoplus_{m_1+m_2+m_3\in 3\mathbb{Z}}\pi_{m_1}\otimes\pi_{m_2}\otimes\pi_{m_3}\\&=\bigoplus_{m_1+m_2+m_3\in 3\mathbb{Z}}\left(\mathcal{A}(1)^{(0)}\right)^{\otimes 3}.\mathcal{G}_1^{m_1}\mathcal{G}_2^{m_2}\mathcal{G}_3^{m_3}\mathbf{1}.\end{aligned}\tag{6}$$

Using realization we see that

- $\Omega(\mathcal{U})\subset\mathcal{A}(3)^{\mathbb{Z}_3}$;
- $\mathcal{G}_1^{m_1}\mathcal{G}_2^{m_2}\mathcal{G}_3^{m_3}\mathbf{1}\in\Omega(\mathcal{U})$ for all $(m_1,m_2,m_3)\in\mathbb{Z}^3$, $m_1+m_2+m_3\in 3\mathbb{Z}$.

Now the claim follows by (6). □

4.5. Decomposition for the general case $n\geq 3$

For $s\in\mathbb{Z}$ we define

$$Q_n^{(s)}=\{z=(z_1,\dots,z_n)\in\mathbb{Z}^n \mid z_1+\cdots+z_n=s\}.$$

For $z\in Q_n^{(s)}$ we define

$$\gamma_z=z_1\varphi_1+\cdots+z_n\varphi_n,$$
$$P_z=X_1^{z_1}\cdots X_n^{z_n}\mathbf{1}\in W_{(n)}^{(0)}\cong V_{-1}(\mathfrak{sl}(n)),$$
$$v_z=\mathcal{G}_1^{z_1}\cdots\mathcal{G}_n^{z_n}\mathbf{1}\in\mathcal{A}(n)^{(s)}.$$

We have

$$P_z=\nu v_z\otimes e^{\gamma_z}\quad(\nu=\pm 1).$$

Set $Q_n:=Q_n^{(0)}$.

Theorem 4.4. *The vertex algebra $V_{-1}(\mathfrak{sl}(n))$ is a simple current extension of $(\mathcal{A}(1)^{(0)})^{\otimes n} \otimes M_{n-1}(1)$ and*

$$V_{-1}(\mathfrak{sl}(n)) = \bigoplus_{z \in Q_n} \pi_{z_1} \otimes \pi_{z_2} \cdots \otimes \pi_{z_n} \otimes M_{n-1}(1) e^{\gamma_z}.$$

For $s \in \mathbb{Z}$ we have:

$$V_s = \bigoplus_{z \in Q_n^{(s)}} \pi_{z_1} \otimes \pi_{z_2} \cdots \otimes \pi_{z_n} \otimes M_{n-1}(1) e^{\gamma_z}.$$

4.6. Decomposition in the case $n = 2$

Let us consider also the case $n = 2$.

Theorem 4.5. *The vertex algebra $\mathcal{W} = \mathrm{Com}(M_c(1), W_{(2)})$ (which is isomorphic to the affine W-algebra $W_{-5/2}(\mathfrak{sl}(4), f_{sh})$) is a simple current extension of the algebra $(\mathcal{A}(1)^{(0)})^{\otimes 2} \otimes M_1(1)$, and*

$$\mathcal{W} = \bigoplus_{m \in \mathbb{Z}} \pi_m \otimes \pi_{-m} \otimes M_1(1) e^{m(\varphi_1 - \varphi_2)}.$$

Remark 4. A detailed study of the representation theory of the vertex algebra $\mathcal{W}$ will be discussed in our forthcoming papers. In particular, we will analyze the fusion algebra for $\mathcal{W}$ by using methods developed in [5], [7].

5. The character $\mathrm{ch}[\mathcal{U}]$ for $\mathfrak{g} = \mathfrak{sl}(3)$

We now discuss graded dimensions (or characters) of ordinary $V_{-1}(\mathfrak{sl}(3))$-modules. These characters were thoroughly analyzed in [17] using different methods.

The next formula is a consequence of the explicit construction of V_s (here $s \geq 0$):

$$\mathrm{ch}[V_s](\tau) = q^{h_s + 1/6} \mathrm{Coeff}_{\zeta^s} \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - \zeta q^{n-1})^3 (1 - \zeta^{-1} q^n)^3}, \tag{7}$$

where h_s is the lowest conformal weight of V_s and we also used that $c = -4$. We also have the full character formula of Kac and Wakimoto [25]

$$\mathrm{ch}[V_s](\tau; z_1, z_2) = q^{h_s + 1/6}$$
$$\cdot \mathrm{Coeff}_{\zeta^s} \frac{(q;q)_\infty}{(\zeta;q)_\infty (\zeta z_2;q)_\infty (\zeta z_1 z_2;q) \infty (\zeta^{-1} z_1^{-1} z_2^{-1} q;q)_\infty (\zeta^{-1} z_2^{-1} q;q)_\infty (\zeta^{-1} q;q)_\infty},$$

where $(a;q)_\infty = \prod_{i \geq 0} (1 - a q^i)$.

Very recently, Kac and Wakimoto [26] gave another Weyl–Kac type character formula for $\mathrm{ch}[V_s]$, expressed as a rank two Jacobi false theta function (see also [17] for a different formula). After a specialization $(z_1, z_2) \to (1,1)$ in their formula, we easily get

$$\begin{aligned} F_0(q) &:= (q;q)_\infty^8 q^{-1/6} \mathrm{ch}[V_0](\tau) \\ &= 4 \sum_{\substack{n_1 \in \mathbb{N}_0, \\ n_2 \in \mathbb{Z}}} \left(2n_1 - n_2 + \tfrac{1}{2}\right) \left(2n_2 - n_1 + \tfrac{1}{2}\right) (n_1 + n_2 + 1) q^{2n_1^2 + 2n_2^2 - 2n_1 n_2 + n_1 + n_2}. \end{aligned}$$

We also have

$$\begin{aligned} F_s(q) :&= (q;q)_\infty^8 q^{-1/6-h_s}\mathrm{ch}[V_s](\tau) \\ &= 4 \sum_{\substack{n_1\in\mathbb{N}_0,\\ n_2\in\mathbb{Z}}} \left(2n_1 - n_2 + \frac{s}{2} + \frac{1}{2}\right)\left(2n_2 - n_1 + \frac{1}{2}\right)\left(n_1 + n_2 + \frac{s}{2} + 1\right) \\ &\quad \cdot q^{2n_1^2+2n_2^2-2n_1n_2+(s+1)n_1+n_2}. \end{aligned}$$

Observe that the summation over n_1 is only over the set of non-negative integers. On the other hand, it is easy to see that the sum over the integers vanishes (by changing $n_j \mapsto -n_j - 1$)

$$\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{Z}}} \left(2n_1 - n_2 + \tfrac{1}{2}\right)\left(2n_2 - n_1 + \tfrac{1}{2}\right)(n_1 + n_2 + 1)q^{2n_1^2+2n_2^2-2n_1n_2+n_1+n_2} = 0.$$

Let

$$G(\tau) := 4 \sum_{\substack{n_1\in-\mathbb{N}_0\\ n_2\in\mathbb{Z}}} \left(2n_1 - n_2 + \tfrac{1}{2}\right)\left(2n_2 - n_1 + \tfrac{1}{2}\right)(n_1 + n_2 + 1)q^{2n_1^2+2n_2^2-2n_1n_2+n_1+n_2}.$$

Proposition 5.1. *We have*

$$G(\tau) = q^3 F_3(\tau).$$

Proof. Straightforward computation with q-series. □

Corollary 5.2. *We have*

$$F_{n_2=0}(q) = F_0(q) + q^3 F_3(q),$$

where

$$F_{n_2=0}(q) := 4 \sum_{n_1\in\mathbb{Z}} \left(2n_1 + \tfrac{1}{2}\right)\left(-n_1 + \tfrac{1}{2}\right)(n_1 + 1)q^{2n_1^2+n_1}.$$

Moreover, this series is quasi-modular.

Proof. Follows directly from the previous proposition and vanishing of the sum over the full lattice. Quasi-modularity is clear as this series is obtained by differentiating a unary theta function. □

Next we combine the characters of modules appearing in the decomposition of $\mathcal{U}$ in pairs:

$$q^{-1/6}(q;q)_\infty^8 \mathrm{ch}[\mathcal{U}](\tau) = \sum_{m\geq 0} (q^{3m^2/2+3m/2}F_{3m}(q) + q^{3(m+1)^2/2+3(m+1)/2}F_{3m+3}(q)),$$

where $(q;q)_\infty = \prod_{i=1}^\infty (1-q^i)$. For each pair in the summation we have a similar q-series identity (the proof is almost identical)

Lemma 5.3. *For every $m \geq 0$,*

$$q^{3m^2/2+3m/2}F_{3m}(q) + q^{3(m+1)^2/2+3(m+1)/2}F_{3m+3}(q)$$
$$= 4\sum_{n_1\in\mathbb{Z}}\Big(2n_1+\frac{1}{2}\Big)\Big(-n_1+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n_1+\frac{3m}{2}\Big)q^{2n_1^2+n_1+3m^2/2+3m/2}.$$

Theorem 5.4. *We have*

(i) *The following equality holds:*

$$\mathrm{ch}[\mathcal{U}](\tau) = \frac{4}{2\eta(\tau)^8}$$
$$\cdot\sum_{n,m\in\mathbb{Z}}\Big(2n+\frac{1}{2}\Big)\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)q^{2(n+1/4)^2+3(m+1/2)^2/2}.$$

(ii) *Let $\mathcal{U}_{\pm 1} = \bigoplus_{n\in\mathbb{Z}} V_{3n\pm 1}$. Then $\mathrm{ch}[\mathcal{U}_1](\tau) = \mathrm{ch}[\mathcal{U}_{-1}](\tau)$ and*

$$\mathrm{ch}[\mathcal{U}_{\pm 1}](q) = \frac{4}{2\eta(\tau)^8}$$
$$\cdot\sum_{n,m\in\mathbb{Z}}\Big(2n+\frac{1}{2}\Big)\Big(-n+\frac{3m}{2}+\frac{1}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+\frac{1}{2}+1\Big)q^{2(n+1/4)^2+3(m+5/6)^2/2}.$$

(iii) *For the supercharacter we have*

$$\mathrm{sch}[\mathcal{U}](q) = \mathrm{tr}_{\mathcal{U}}\sigma q^{L(0)-c/24} = \frac{4}{\eta(\tau)^8}$$
$$\cdot\sum_{n,m\in\mathbb{Z}}\Big(2n+\frac{1}{2}\Big)^2\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)q^{2(n+1/4)^2+3(m+1/2)^2/2}.$$

(iv) *Both $\mathrm{ch}[\mathcal{U}]$ and $\mathrm{sch}[\mathcal{U}]$ are quasi-modular.*

Proof. We only prove (i) here — formula (ii) can be proven using similar ideas. Proof of (iii) is slightly different and is postponed for Section 8 (see Remark 7).

We first apply Lemma 5.3 to write

$$\mathrm{ch}[\mathcal{U}](\tau) = \mathrm{tr}|_{\mathcal{U}}q^{L(0)+1/6} = \sum_{m\geq 0}\frac{q^{1/3}}{(q;q)_\infty^8}F(m),$$

where $F(m) := q^{3m^2/2+3m/2}F_{3m}(q) + q^{3(m+1)^2/2+3(m+1)/2}F_{3m+3}(q)$, in the form

$$\mathrm{ch}[\mathcal{U}](\tau) = \frac{4}{\eta(\tau)^8}\sum_{n\in\mathbb{Z},m\geq 0}\Big(2n+\frac{1}{2}\Big)\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)q^{2(n+1/4)^2+3(m+1/2)^2/2}.$$

Now observe that

$$\mathrm{ch}[\mathcal{U}](\tau) = \frac{4}{\eta(\tau)^8}\sum_{n\in\mathbb{Z},m<0}\Big(2n+\frac{1}{2}\Big)\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)q^{2(n+1/4)^2+3(m+1/2)^2/2},$$

hence taking the summation over $m \in \mathbb{Z}$ and dividing by 2 yields the formula (i).

In part (iv) we prove quasi-modularity only for $\mathrm{ch}[\mathcal{U}]$. In other cases proof is very similar. Using

$$\Big(2n+\frac{1}{2}\Big)\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)=-2(n+1/4)^3+9/2(m+1/2)^2(n+1/4),$$

we get

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}} &(-2(n+1/4)^3+9/2(m+1/2)^2(n+1/4))q^{2(n+1/4)^2+3(m+1/2)^2/2}\\ &= -\sum_{m\in\mathbb{Z}} q^{3(m+1/2)^2/2}\Theta_q\Big(\sum_{n\in\mathbb{Z}}(n+1/4)q^{2(n+1/4)^2}\Big)\\ &+3\Theta_q\Big(\sum_{m\in\mathbb{Z}} q^{3(m+1/2)^2/2}\Big)\sum_{n\in\mathbb{Z}}(n+1/4)q^{2(n+1/4)^2},\end{aligned}$$

where $\Theta_q := q\, d/dq$. As it is known, the Θ_q-derivative of weight $1/2$ and $3/2$ theta functions gives quasi-modular forms. □

Remark 5. Observe that irreducible $\mathcal{U}$-modules are also $\mathbb{Z}_2$-graded. Their super-characters are given by

$$\mathrm{sch}[\mathcal{U}_1] = -\mathrm{ch}[V_1] + \sum_{i\geq 1}(-1)^{i-1}(\mathrm{ch}[V_{3i+1}]+\mathrm{ch}[V_{-3i+1}]),$$
$$\mathrm{sch}[\mathcal{U}_2] = \mathrm{ch}[V_2] + \sum_{i\geq 1}(-1)^{i-1}(\mathrm{ch}[V_{3i+2}]+\mathrm{ch}[V_{-3i+2}]).$$

Since $\mathrm{ch}[V_i]=\mathrm{ch}[V_{-i}]$, this easily implies that $\mathrm{sch}[\mathcal{U}_1]=\mathrm{sch}[\mathcal{U}_2]$. One can also show that

$$\begin{aligned}\mathrm{sch}[\mathcal{U}_1](\tau) = \mathrm{sch}[\mathcal{U}_2](\tau) &= \frac{4}{\eta(\tau)^8}\\ \cdot\sum_{n,m\in\mathbb{Z}}\Big(2n+\frac{1}{2}\Big)^2&\Big(-n+\frac{3m}{2}+\frac{1}{2}\Big)\Big(n+\frac{3m}{2}+1\Big)q^{2(n+1/4)^2+3(m+5/6)^2/2}.\end{aligned}$$

Remark 6. The above approach to modularity is difficult to generalize to $\mathfrak{sl}(n)$ because it requires explicit formulae as in Theorem 5.4. But these formulae are non-trivial to extract from [26]. In the remainder of the paper we show how to solve the (quasi)-modularity problem via meromorphic Jacobi forms and explicit construction.

6. Quasi-modularity of (s)ch$[\mathcal{U}](\tau)$

In this part we prove the quasi-modularity of $(\mathrm{s})\mathrm{ch}[\mathcal{U}](\tau)$, generalizing our explicit computation for $\mathfrak{g}=\mathfrak{sl}(3)$ in Section 6. Let

$$(a)_\infty := \prod_{i\geq 1}(1-aq^{i-1}).$$

We will make use of a Jacobi theta function

$$\vartheta(z;\tau) := (-i)q^{1/8}\zeta^{-1/2}(q)_\infty(\zeta)_\infty(q\zeta^{-1})_\infty,$$

where $\zeta = e^{2\pi iz}$. Recall the elliptic and modular transformation formulae (here $\lambda, \mu \in \mathbb{Z}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$):

$$\vartheta(z+\lambda\tau+\mu;\tau) = (-1)^{\lambda+\mu}q^{-\lambda^2/2}\zeta^{-\lambda}\vartheta(z;\tau),$$
$$\vartheta\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(c\tau+d)^{1/2}e^{(\pi icz^2)/(c\tau+d)}\vartheta(z;\tau),$$

where χ is a certain multiplier. In particular,

$$\vartheta\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\pi iz^2/\tau}\vartheta(z;\tau).$$

As in the $\mathfrak{sl}(3)$ case, from the explicit construction [25] we see that

$$\tilde{\mathrm{ch}}(V_s) = \mathrm{Coeff}_{\zeta^s}\frac{(q)_\infty}{(\zeta)_\infty^n(q\zeta^{-1})_\infty^n}, \tag{8}$$

where $\tilde{\mathrm{ch}}(V_s)$ is the character of V_s up to a multiplicative q-shift. More precisely, for $s \geq 0$

$$\tilde{\mathrm{ch}}(V_s) = \dim(L(s\omega_1)) + O(q)$$

and for $s < 0$

$$\tilde{\mathrm{ch}}(V_s) = q^{-s}(\dim(L(s\omega_{n-1})) + O(q)),$$

where $L(m\omega_i)$ denotes an irreducible $\mathfrak{sl}(3)$-module of highest weight $m\omega_i$. Thus in order to compute the genuine character we must multiply with

$$q^{h_{V_s}-c_n/24},$$

for $s \geq 0$, and in addition shift with q^s for $s < 0$. It is easy to see that for $s \geq 0$,

$$h_{V_s} = h_{V_{-s}} = \frac{s^2}{2n} + \frac{s}{2}$$

and the central charge is

$$c_n = -(n+1).$$

Combined, this yields

$$h_{V_{ns}} - \frac{c_n}{24} = \frac{s^2n}{2} + \frac{sn}{2} + \frac{n+1}{24}.$$

Putting this together with (8), and taking into account the q-multiplicative shift for $s < 0$, we get

$$\mathrm{ch}[\mathcal{U}](\tau) = \mathrm{CT}_\zeta \frac{\sum_{s\in\mathbb{Z}} q^{s^2n/2+sn/2+(n+1)/24}\zeta^{-sn}\prod_{i=1}^\infty(1-q^i)}{(\zeta)_\infty^n(q\zeta^{-1})_\infty^n}.$$

Next we multiply the numerator and the denominator with $\zeta^{n/2}q^{n/8}(q;q)_\infty^n$ so that in the denominator we have a power of $\theta(z,\tau)$, a weight 1/2 Jacobi form of index 1/2. So we obtain

$$\mathrm{ch}[\mathcal{U}](\tau) = \mathrm{CT}_\zeta \frac{\sum_{s\in\mathbb{Z}} q^{s^2n/2+sn/2+(n+1)/24}\zeta^{-sn-n/2}q^{n/8}(q;q)_\infty^n(q;q)_\infty}{\zeta^{-n/2}(\zeta)_\infty^n(q\zeta^{-1})_\infty^n(q;q)_\infty^n q^{n/8}}.$$

Continuing with the numerator, we have

$$\sum_{s\in\mathbb{Z}} q^{s^2n^2/2+sn/2+(n+1)/24}\zeta^{-sn-n/2} = \sum_{s\in\mathbb{Z}} q^{n(s+1/2)^2/2-n/8+(n+1)/24}\zeta^{-(s+1/2)n}.$$

Notice that $q^{n/8}$ cancels out, and an extra $q^{(n+1)/24}$ term nicely combines with $(q;q)_\infty^{n+1}$ giving $\eta(\tau)^{n+1}$. We conclude

$$\begin{aligned}\mathrm{ch}[\mathcal{U}](\tau) &= i^n\eta(\tau)^{n+1}\mathrm{CT}_\zeta \frac{\sum_{s\in\mathbb{Z}} q^{n(s+1/2)^2/2}\zeta^{-(s+1/2)n}}{\vartheta(z,\tau)^n}\\ &= i^n\eta(\tau)^{n+1}\mathrm{CT}_\zeta \frac{\sum_{s\in\mathbb{Z}} q^{n(s+1/2)^2/2}\zeta^{(s+1/2)n}}{\vartheta(z,\tau)^n},\end{aligned}$$

where we also used the Jacob triple product identity in the denominator

$$\begin{aligned}\vartheta(z,\tau) &= (-i)q^{1/8}\zeta^{-1/2}(\zeta)_\infty(q\zeta^{-1})_\infty\\ &= \sum_{s\in\mathbb{Z}+1/2} q^{s^2/2}e^{2\pi i s(z+1/2)} = i\sum_{s\in\mathbb{Z}}(-1)^s q^{(s+1/2)^2/2}e^{2\pi i(s+1/2)z}.\end{aligned}$$

6.1. n is odd

In this case we have

$$\mathrm{ch}[\mathcal{U}](\tau) = \epsilon_n\eta(\tau)^{n+1}\mathrm{CT}_\zeta\left(\frac{\vartheta(nz+1/2,n\tau)}{\vartheta(z,\tau)^n}\right),$$

where $\epsilon_n = -i^n$, for the character.

6.2. n is even

For n even we also have

$$\mathrm{ch}[\mathcal{U}](\tau) = \epsilon_n\eta(\tau)^{n+1}\mathrm{CT}_\zeta\left(\frac{\vartheta(zn+1/2,n\tau)}{\vartheta(z,\tau)^n}\right)$$

6.3. Supercharacter

For n odd we can also compute the supercharacter

$$\mathrm{sch}[\mathcal{U}](\tau) = i\epsilon_n\eta(\tau)^{n+1}\mathrm{CT}_\zeta\left(\frac{\vartheta(zn,n\tau)}{\vartheta(z,\tau)^n}\right)$$

6.4. Characters of modules

Straightforward computation gives for $0 \le k \le n-1$,

$$\mathrm{ch}[\mathcal{U}_k](\tau) = \eta(\tau)^{n+1}\mathrm{CT}_\zeta \frac{\sum_{s\in\mathbb{Z}} q^{n(s+(k+n)/2n)^2/2}\zeta^{-(s+(n+k)/2n)n}}{\vartheta(z,\tau)^n}.$$

For n even we can write this as

$$\mathrm{ch}[\mathcal{U}_k](\tau) = \epsilon_{n,k}\eta(\tau)^{n+1}\mathrm{CT}_\zeta \left(\frac{\vartheta((z+k/2n)n, n\tau)}{\vartheta(z,\tau)^n}\right),$$

where $\epsilon_{n,k}$ is a normalization constant as above. Similarly we compute $\mathrm{sch}[\mathcal{U}_k](\tau)$.

6.5. Quasimodularity

Here we prove a general theorem on quasimodularity of the (super)character of $\mathcal{U}$, which extends our previous calculations for $\mathfrak{sl}(3)$.

Theorem 6.1. *The supercharacter of $\mathcal{U}$ (for n odd) and the character of $\mathcal{U}$ (for n even) are quasi-modular forms (with multipliers) of weight one and depth one on $\Gamma_0(n)$.*

Proof. Case 1: n is odd.

First we observe that

$$G(\tau, z) := \eta(\tau)^{n+1}\left(\frac{\vartheta(zn, n\tau)}{\vartheta(z,\tau)^n}\right)$$

is a meromorphic Jacobi form of weight $(n+1)/2+1/2-n/2 = 1$. After we multiply with $1/\eta(\tau)^2$, $H(\tau;z) := G(\tau;z)/\eta(\tau)^2$ we get a meromorphic Jacobi function of weight zero.

Claim: $G(\tau, z)$ is a Jacobi form of index zero for the congruence group $\Gamma_0(n)$ (transforming with a character).

We consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(n)$ (thus $ad-cb=1$ and $n|c$). Then

$$\begin{aligned}\vartheta\left(\frac{nz}{c\tau+d}; n\frac{a\tau+b}{c\tau+d}\right) &= \vartheta\left(\frac{(nz)}{(c/n)(n\tau)+d}; \frac{a(n\tau)+nb}{(c/n)(\tau n)+d}\right)\\ &= \chi'\begin{pmatrix} a & b \\ c & d\end{pmatrix}(c\tau+d)^{1/2}e^{(\pi icnz^2)/(c\tau+d)}\vartheta(nz;n\tau)\end{aligned}$$

where we used that $\begin{bmatrix} a & bn \\ c/n & d \end{bmatrix} \in \Gamma(1)$. For the denominator,

$$\vartheta(z;\tau)^n|_{(\tau\to(a\tau+b)/(c\tau+d); z\to z/(c\tau+d))} = \chi^n(c\tau+d)^{n/2}e^{(\pi icnz^2)/(c\tau+d)}\vartheta(z;\tau)^n.$$

For translations, for $\lambda, \nu \in \mathbb{Z}$, we have

$$\vartheta(nz+n\lambda\tau+n\mu, n\tau) = (-1)^{n(\lambda+\mu)}q^{-n\lambda^2/2}e^{-2\pi i\lambda nz}\vartheta(nz, n\tau),$$

$$\vartheta(n+\lambda\tau+\mu,\tau)^n=(-1)^{n(\lambda+\mu)}q^{-n\lambda^2/2}e^{-2\pi i\lambda nz}\vartheta(z,\tau)^n.$$

After taking the quotient this implies the claim.

Notice that $H(\tau;z)$ is even with respect to z and has a pole of order n at $z=0$, so we can write the Laurent expansion [16] (see also [22])

$$H(\tau;z)=\frac{H_n(\tau)}{(2\pi iz)^n}+\frac{H_{n-2}(\tau)}{(2\pi iz)^{n-2}}+\cdots+\frac{H_2(\tau)}{(2\pi iz)^2}+H_0(\tau), \tag{9}$$

where $H_{2j}(\tau)$ is a modular form of weight $-2j$ with respect to $\Gamma_0(n)$ (transforming with the same character as the Jacobi form).

Then, by using [16], [22] we can write the "finite" part as

$$H^F(\tau):=H_0(\tau)+\sum_{j=1}^{n/2}\frac{B_{2j}}{(2j)!}H_{2j}(\tau)E_{2j}(\tau), \tag{10}$$

which is quasi-modular of weight zero. Here $E_{2j}(\tau)$ denotes the Eisenstein series and B_{2j} are the Bernoulli numbers. Finally, the constant term is given by

$$\mathrm{sch}[\mathcal{U}](\tau)=\eta(\tau)^2H^F(\tau).$$

It is a modular form of weight one. The depth is one due to the appearance of $E_2(\tau)$.

Case 2: n is even.

For n is even we have to study

$$H(z;\tau):=\eta(\tau)^{n-1}\frac{\vartheta(zn+1/2,n\tau)}{\vartheta(z,\tau)^n}.$$

Since

$$\vartheta\Big((-z)n+\frac{1}{2};\tau\Big)=\vartheta\Big(-zn+\frac{1}{2};\tau\Big)=-\vartheta\Big(zn-\frac{1}{2};\tau\Big)=\vartheta\Big(zn+\frac{1}{2};\tau\Big)$$

and n is even, $H(z;\tau)$ is an even function. It is easy to see that

$$\vartheta\Big(z+\frac{1}{2}+\lambda\tau+\mu\Big)=(-1)^{\mu}q^{-\lambda^2/2}e^{-2\pi i\lambda z}\vartheta\Big(z+\frac{1}{2};\tau\Big)$$

which implies

$$\vartheta\Big(nz+n\lambda\tau+n\mu+\frac{1}{2},n\tau\Big)=q^{-n\lambda^2/2}e^{-2\pi i\lambda nz}\vartheta(nz,n\tau).$$

Thus $H\left(z+\lambda\tau+\mu;\tau\right)=H(z;\tau)$. We also get

$$H\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=\chi''\begin{pmatrix}a&b\\c&d\end{pmatrix}H(z;\tau),$$

where χ'' is a character. The rest follows as in the odd case. $\square$

6.6. Explicit example: case $n = 3$

Here we compute the constant term of

$$G(\tau, z) := \eta(\tau)^4 \left(\frac{\vartheta(3z; 3\tau)}{\vartheta(z,\tau)^3} \right).$$

The same method can be used to compute $G(\tau; z)$ for every n. We have to compute modular forms appearing inside the series (9). Here we use a standard method of Laurent expansion following [16]. We write

$$\vartheta^*(z;\tau) := \frac{1}{z}\vartheta(z;\tau),$$

where we suppress τ from the formula for brevity. Then we have

$$\vartheta^*(z;\tau) = \vartheta^*(0;\tau) + \vartheta^{*\prime\prime}(0;\tau)\frac{z^2}{2!} + O(z^4)$$

and

$$\vartheta^*(3z;3\tau) = \vartheta^*(0;3\tau) + \vartheta^{*\prime\prime}(0)\frac{9z^2}{2!} + O(z^4),$$

hence

$$G(z) = \frac{3z\left(\vartheta^*(0;3\tau) + \vartheta^{*\prime\prime}(0;3\tau)9z^2/2! + O(z^4)\right)}{z^3(z\left(\vartheta^*(0) + \vartheta^{*\prime\prime}(0) + z^2/2! + O(z^4)\right)}. \tag{11}$$

It is clear that

$$\vartheta^*(0;\tau) = -2\pi\eta^3(\tau) = -2\pi\sum_{n\in\mathbb{Z}}(-1)^n(n+1/2)q^{(n+1/2)^2/2}$$

from the infinite expansion of $\vartheta(z;\tau)$ and Euler's theorem and

$$\vartheta^{*\prime\prime}(0;\tau) = \frac{1}{12}(2\pi i)^2 E_2(\tau)\eta^3(\tau).$$

Expanding (11) gives only even powers of z and in particular

$$H_0(\tau) + \frac{H_{-2}(\tau)}{(2\pi i z)^2} + O(1).$$

Finally

$$\begin{aligned}
&\mathrm{CT}_z\left\{\eta(\tau)^4\left(\frac{\vartheta(3z;3\tau)}{\vartheta(z,\tau)^3}\right)\right\} \\
&= H_0(\tau) + \frac{B_2}{2}H_2(\tau)E_2(\tau) \\
&= -\frac{9}{8}E_2(\tau)\frac{\eta(\tau)^3\eta(3\tau)^3}{\eta(\tau)^8} + \frac{9}{8}\frac{\eta(\tau)^3\sum_{n\geq 0}(-1)^n(2n+1)^3q^{3n(n+1)/2}}{\eta^8(\tau)} \\
&\quad + \frac{\eta^3(3\tau)\sum_{n\geq 0}(-1)^n(2n+1)^3q^{n(n+1)/2}}{\eta^8(\tau)} \\
&= -\frac{1}{8}E_2(\tau)\frac{\eta(\tau)^3\eta(3\tau)^3}{\eta(\tau)^8} + \frac{9}{8}\frac{\eta(\tau)^3\sum_{n\geq 0}(-1)^n(2n+1)^3q^{3n(n+1)/2}}{\eta^8(\tau)} \\
&= -\frac{1}{8}\left(E_2(\tau) - 9E_2(3\tau)\right)\frac{\eta(3\tau)^3}{\eta(\tau)^5}.
\end{aligned}$$

Remark 7. Observe that the above formulas, with $\eta(\tau)^3$ and $\eta(3\tau)^3$ expanded as sums, immediately imply the relation (iii) in Theorem 5.4.

It is clear that $E_2(\tau)$ is a quasi-modular form of weight 2 and depth 1 on $\Gamma(1)$. It is easy to show that $E_{2,3}(\tau) := E_2(3\tau)$ is a quasi-modular form of weight 2 and depth 1 on $\Gamma_0(3)$, i.e.

$$E_{2,3}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_{2,3}(\tau) + \frac{6c(c\tau+d)}{3i\pi}.$$

As the index of $\Gamma_0(3)$ in $\Gamma(1)$ is 4, $E_{2,3}(\tau)$ combines into a vector-valued quasi-modular form under the full modular group.

Lemma 6.2. *$\eta(\tau/3)^3$ and $\eta(3\tau)^3$ form a 2-dimensional vector-valued modular form of weight $3/2$ under the full modular group.*

Proof. Straightforward computation with Shimura's theta series of weight 3/2 together with Jacobi's identity for $\eta(\tau)^3$. □

Using this lemma and the previous discussion one can explicitly write down a vector space of quasi-modular forms closed under the modular group, which also contains the supercharacter. However, this space is difficult to analyze and does not give much evidence for the quasi-lisseness of $\mathcal{U}$ conjectured earlier. As demonstrated in [14], characters of quasi-lisse vertex algebras must satisfy a particular type of linear modular differential equation whose coefficients are *holomorphic* Eisenstein series (usually abbreviated as MLDE). For quasi-lisse $\mathbb{Z}_{\geq 0}$-graded vertex superalgebras we expect the same property to hold for supercharacters. By analyzing the leading behavior of the above function (which is quasi-modular) combined with computer computations we can conclude:

Proposition 6.3. *The supercharacter $\mathrm{sch}[\mathcal{U}](\tau)$ satisfies a 5-th order MLDE*

$$\theta^5(y(q)) - \frac{7}{36}E_4(\tau)\theta^3(y(q)) + \frac{19}{216}E_6(\tau)\theta^2(y(q)) - \frac{5}{324}E_4(\tau)^2\theta(y(q)) + \frac{5}{1944}E_4(\tau)E_6(\tau)y(q) = 0,$$

where Ramanujan–Serre's n-th derivative is defined by

$$\theta^n := \vartheta_{2n}\circ\cdots\circ\theta_0;\quad \theta_k := \left(q\frac{d}{dq} - \frac{kE_2(\tau)}{12}\right).$$

As usual, the Eisenstein series appearing in the equation are given by

$$E_2(\tau) = 1 - 24\sum_{n\geq 1}\frac{nq^n}{1-q^n}$$

$$E_4(\tau) = 1 + 240\sum_{n\geq 1}\frac{n^3q^n}{1-q^n}$$

$$E_6(\tau) = 1 - 504\sum_{n\geq 1}\frac{n^5q^n}{1-q^n}.$$

Two supercharacters (that are equal) of ordinary $\mathcal{U}$-modules also satisfy this modular equation. Two additional solutions are expected to come from σ-twisted $\mathcal{U}$-modules, not analyzed in this paper. These four solutions, together with a logarithmic solution, form a fundamental system of solutions for this MLDE.

6.7. Explicit example: case $n = 2$

Here we essentially repeat the same procedure with a notable difference that

$$\vartheta\left(2z+\frac{1}{2};2\tau\right)$$

admits Taylor expansion in even powers of the z variable:

$$\vartheta\left(2z+\frac{1}{2};2\tau\right)=\vartheta\left(\frac{1}{2},2\tau\right)+O(z^2)$$

so that

$$\frac{\vartheta(2z+1/2;2\tau)}{\vartheta(z;\tau)^2}=\frac{1}{z^2}a_0(\tau)+a_1(\tau)+O(z^2)$$

Repeating the same procedure as in the odd supercharacter case we get

$$\mathrm{ch}[\mathcal{U}](\tau)=\frac{1}{3}\frac{\eta(4\tau)^2}{\eta(\tau)^3\eta(2\tau)}\left(4E_2(2\tau)-E_2(4\tau)\right).$$

7. Vertex superalgebra $V_1(\mathfrak{psl}(n|n))$ and $V_{-1}(\mathfrak{sl}(n))$

Let $\mathfrak{g}=\mathfrak{psl}(n|n)$. We consider the simple vertex algebra $V_1(\mathfrak{g})$. We have the following result which identifies our vertex algebra $\mathcal{U}^{(n)}=\mathcal{U}_0$ as a coset subalgebra in $V_1(\mathfrak{g})$.

Proposition 7.1. *Assume that $n\geq 3$. Then we have:*

(1) *The vertex algebras $\mathcal{U}^{(n)}$ and $L_{\mathfrak{sl}(n)}(\Lambda_0)$ form a Howe dual pair inside $V_1(\mathfrak{g})$. In particular,*

$$\frac{\mathfrak{psl}(n|n)_1}{\mathfrak{sl}(n)_1}:=Com_{V_1(\mathfrak{g})}(L_{\mathfrak{sl}(n)}(\Lambda_0))\cong\mathcal{U}^{(n)}.$$

(2) $K(\mathfrak{g},1)\cong K(\mathfrak{sl}(n),-1)\cong(\mathcal{A}(1)^{(0)})^{\otimes n}$.

Proof. By using the decomposition of the conformal embedding $\mathfrak{sl}(n)\times\mathfrak{sl}(n)\hookrightarrow\mathfrak{g}$ (cf. [10]) we get

$$V_1(\mathfrak{g})=\bigoplus_{i=0}^{n-1}\mathcal{U}_i\otimes L(\Lambda_i),\tag{12}$$

where for brevity we omit the superscript $\mathfrak{sl}(n)$. Alternatively, relation (12) can be directly proved by using the fusion rules resulting from Theorem 3.3 and the well-known fact that all $V_1(\mathfrak{sl}(n))$-modules are simple currents.

The first assertion follows directly from (12). The second assertion follows again from (12) and from

$$K(\mathfrak{sl}(n),1)\cong\mathbb{C},\quad \mathrm{Com}_{\mathcal{U}^{(n)}}(M_{n-1}(1))=K(\mathfrak{sl}(n),-1).\qquad\square$$

The case $n = 1$ corresponds exactly to the symplectic fermion vertex algebra $\mathcal{A}(1)$ of central charge $c = -2$. We will see that for $n \geq 2$, the supertraces $\mathrm{sch}[V_1(\mathfrak{g})](\tau)$ are the same, and therefore they satisfy the same MLDE

$$\theta^2(y(\tau)) + \frac{1}{144}E_4(\tau)y(\tau) = 0. \tag{13}$$

Theorem 7.2. *We have:*

$$\mathrm{sch}[V_1(\mathfrak{g})](\tau) = \eta(\tau)^2.$$

7.1. Proof of Theorem 7.2

The proof of Theorem 7.2 uses the explicit realization of $V_1(\mathfrak{gl}(n|n))$-modules and the relation between supercharacters $V_k(\mathfrak{psl}(n|n))$ and $V_k(\mathfrak{gl}(n|n))$.

Lemma 7.3. *For every k we have:*

$$\mathrm{sch}[V_k(\mathfrak{gl}(n|n))](\tau) = \frac{\mathrm{sch}[V_k(\mathfrak{sl}(n|n))](\tau)]}{\eta(\tau)} = \frac{\mathrm{sch}[V_k(\mathfrak{psl}(n|n))](\tau)]}{\eta(\tau)^2}.$$

Proof. Follows from the definition of $\mathfrak{psl}(n|n) = \mathfrak{gl}(n|n)/I$, where I is a two-dimensional abelian ideal. □

Lemma 7.4. *For $k = 1$ we have*

$$\mathrm{sch}[V_k(\mathfrak{gl}(n|n)](\tau) = 1.$$

Proof. Recall that $V_1(\mathfrak{gl}(n|n))$ is realized as a charge-zero component of the vertex algebra $W_{(n)} \otimes F_{(n)}$, where $W_{(n)}$ is the Weyl vertex algebra, $F_{(n)}$ is the Clifford vertex algebra. and the charge operator is $J(0)$ where

$$J = \sum_{i=1}^{n} \left(: a_i^+ a_i^- : + : \Psi_i^+ \Psi_i^- : \right).$$

Since

$$\begin{aligned}\mathrm{sch}[W_{(n)} \otimes F_{(n)}](\tau) &= \mathrm{sch}[F_{(n)}](\tau) \cdot \mathrm{ch}[W_{(n)}](\tau) \\ &= \frac{\left(\prod_{m=1}^{\infty}(1 - q^{m-1/2}z)((1 - q^{m-1/2}z^{-1})\right)^n}{\left(\prod_{m=1}^{\infty}(1 - q^{m-1/2}z)((1 - q^{m-1/2}z^{-1})\right)^n} \\ &= 1,\end{aligned}$$

we conclude that $\mathrm{sch}[V_1(\mathfrak{gl}(n|n))](\tau) = 1$. □

Now Theorem 7.2 follows from the previous two lemmas.

Remark 8. Theorem 7.2 is also in agreement with the recent results on the Duflo–Serganova functor [24].

7.2. Second proof of Theorem 7.2 for $n = 3$.

In the case $n = 3$, we have a different proof which uses the branching rules for conformal embeddings.

We have

$$V_1(\mathfrak{g}) = \mathcal{U}_0 \otimes L(\Lambda_0) \bigoplus \mathcal{U}_1 \otimes L(\Lambda_1) \bigoplus \mathcal{U}_2 \otimes L(\Lambda_2).$$

This gives

$$\mathrm{sch}[V_1(\mathfrak{g})](\tau) = \sum_{i=0}^{n-1} \mathrm{sch}[\mathcal{U}_i](\tau)\mathrm{ch}[L(\Lambda_i)](\tau).$$

Since both left and right hand side are (quasi)modular, in theory it would be sufficient to compute the first few coefficients in the q-expansion. Here we present a more conceptual proof.

Lemma 7.5.

$$\mathrm{ch}[\Lambda_0](\tau) = \frac{\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn}}{\eta(\tau)^2} = \frac{1}{\eta(\tau)^3}\left(3\eta(3\tau)^3 + \eta(\tau/3)^3\right),$$

$$\mathrm{ch}[\Lambda_1](\tau) = \mathrm{ch}[\Lambda_2](\tau) = \frac{\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2+n+1/3-mn}}{\eta(\tau)^2} = \frac{3\eta(3\tau)^3}{\eta(\tau)^3},$$

Proof. The second identity is essentially the Macdonald denominator identity for A_2. By Lemma 6.2 we have that

$$\mathcal{V} := \mathrm{Span}\left\{\frac{3\eta(3\tau)^3}{\eta(\tau)^3}, \frac{3\eta(\tau/3)^3}{\eta(\tau)^3}\right\}$$

is modularly invariant. On the other hand,

$$W := \mathrm{Span}\{\mathrm{ch}[\Lambda_0](\tau), \mathrm{ch}[\Lambda_1](\tau)\}$$

is also a two-dimensional modular invariant subspace. Since $\mathrm{ch}[\Lambda_1](\tau) \in \mathcal{V}$, we must have $\mathrm{ch}[\Lambda_0](\tau) \in \mathcal{V}$. This quickly gives the first formula by comparing the leading coefficients in the q-expansion. $\square$

Proposition 7.6. *Theorem* 7.2 *holds for* $n = 3$.

Proof. The above lemma gives

$$\mathrm{ch}[\Lambda_0](\tau) = \frac{1}{\eta(\tau)^3}\left(3\eta(3\tau)^3 + \eta(\tau/3)^3\right),$$

$$\mathrm{ch}[\Lambda_i](\tau) = \frac{3\eta(3\tau)^3}{\eta(\tau)^3}.$$

As in the previous section, for $1 \le i \le 2$ we get

$$\begin{aligned}\mathrm{sch}[\mathcal{U}_i](\tau) &= -\sum_{n\geq 0; n\equiv \pm 1 \bmod 6} \mathrm{ch}[V_n] + \sum_{n\geq 0; n\equiv \pm 2 \bmod 6} \mathrm{ch}[V_n] \\ &= \frac{1}{6}\frac{\eta(\tau)^5}{\eta(3\tau)^3} + \frac{(E_2(\tau) - 9E_2(3\tau))(\eta(\tau/3)^3 + 3\eta(3\tau)^3)}{48\eta(\tau)^5}.\end{aligned}$$

We previously derived the formula

$$\mathrm{sch}[\mathcal{U}_0](\tau) = -\frac{1}{8}\left(E_2(\tau) - 9E_2(3\tau)\right)\frac{\eta(3\tau)^3}{\eta(\tau)^5}.$$

Plugging in these q-series, we get

$$\sum_{i=0}^{2} \mathrm{sch}[\mathcal{U}_i](\tau)\mathrm{ch}[L(\Lambda_i)](\tau) = \eta(\tau)^2,$$

as desired. □

7.3. MLDE for the character of $\mathfrak{psl}(n|n)_1$

We expect the following should be true.

Conjecture 7.7. *For every $n \geq 2$, the character of $\mathfrak{psl}(n|n)_1$ satisfies the following second order MLDE (of weight zero):*

$$\left(q\frac{d}{dq}\right)^2 y(\tau) - \frac{1}{6}E_2(\tau)\left(q\frac{d}{dq}\right)y(\tau) + \left(-\frac{6n^2-5}{720}E_4(\tau) + \frac{n^2}{120}E_{4,2}(\tau)\right)y(\tau) = 0, \quad (14)$$

where

$$E_{4,2}(\tau) = 1 - 240\sum_{m\geq 1}\frac{m^3q^m}{1+q^m}$$

is an Eisenstein series on $\Gamma_0(2)$.

We are able to prove this for a few low rank cases.

Proposition 7.8. *The conjecture is true for $2 \leq n \leq 4$.*

Proof. For n even, we only comment on $n = 2$, as $n = 4$ is very similar. In the former case the character is [16]

$$y(\tau) := \mathrm{ch}[\mathfrak{psl}(2|2)_1](\tau) = \frac{\eta(2\tau)^4}{\eta(\tau)^6}\left(\frac{1}{3}E_2(\tau) - \frac{4}{3}E_2(2\tau)\right).$$

As the logarithmic derivative of the η-quotient contributes only with a linear combination of $E_2(\tau)$ and $E_2(2\tau)$, plugging in $y(\tau)$ into the left-hand side of the MLDE leaves us with the same η-quotient multiplied with a quasi-modular form of weight 6. As we know this ring is generated by $E_{2,2}(\tau)$, $E_4(\tau)$ and $E_2(\tau)$, so in order to prove that $y(\tau)$ satisfies (14) we only have to compute the first three coefficients in the q-expansion and show that they are zero (as there cannot be such a form of weight 6 with the order of vanishing at $i\infty$ greater than 3). This can be easily checked with a computer.

For $n = 3$, the character is modular [16] and computing as before gives

$$y(\tau) := \mathrm{ch}[\mathfrak{psl}(3|3)_1](\tau) = \frac{\eta(2\tau)^6}{\eta(\tau)^{10}}E_{2,2}(\tau),$$

where $E_{2,2}(\tau) = 1 + 24\sum_{n\geq 1} nq^n/(1+q^n)$ is a modular form of weight 2 on $\Gamma_0(2)$ with a character. Plugging this into (14) and applying the same argument as before gives the claim. □

The conjecture is also true for $n = 1$ (the case of symplectic fermions), with $y(\tau) = q^{1/12}\prod_{n\geq 1}(1+q^n)^2$. The degenerate case, $n = 0$, gives MLDE for $\eta(\tau)^2$ discussed earlier; see equation (13).

We note that the "constant" coefficient in our MLDE can be rewritten as

$$-4n^2 F_4(\tau) + \frac{1}{144}E_4(\tau)$$

where $F_4(\tau) = q + 8q^2 + 28q^3 + 64q^4 + \ldots$ is the unique cusp form of weight 4 on $\Gamma_0(2)$. An interesting feature of this family of MLDEs is that for *every* n there is a unique vacuum solution of the form $q^a(1 + O(q))$, where a must be $1/12$. The other (linearly independent) solution is logarithmic, though it can be expressed in an integral form. Closely related families of MLDEs appeared in studies of rational vertex operator superalgebras, e.g., $N = 1$ Ramond minimal models [31].

The method used in Proposition 7.8 cannot be used for all n. Instead, we propose to attack this conjecture by emulating the approach in [28], which is based on recursions among solutions of MLDEs.

8. The supercharacter of $V_{-2}(\mathfrak{osp}(n+8|n))$

In [14], T. Arakawa and K. Kawasetsu proved character formulae for the vertex operator algebras associated with the Deligne exceptional series at level $k = -h^\vee/6-1$. In [8] and [9], the authors discovered a family of Lie superalgebras such that the associated vertex algebras also have level $k = -h^\vee/6-1$ and share similar properties with vertex algebras in the Deligne exceptional series. In particular, vertex superalgebras $V_1(\mathfrak{psl}(n|n)) = V_{-1}(\mathfrak{psl}(n|n))$ belong to this series. Since we have demonstrated in the previous section that the supercharacters of $V_1(\mathfrak{psl}(n|n)))$ should not depend on the parameter n, one can ask if a similar situation can happen in other cases. A natural example to investigate is $V_{-2}(\mathfrak{osp}(n+8|n)$, which is a super-generalization of the affine vertex algebra $V_{-2}(\mathfrak{so}(8))$. We have the following conjecture (which is also in agreement with [24]):

Conjecture 8.1. *For every even $n \geq 0$, we have*

$$\mathrm{sch}[V_{-2}(\mathfrak{osp}(n+8|n))](\tau) = \mathrm{ch}[V_{-2}(\mathfrak{so}(8))](\tau) = \frac{(q\,d/dq)E_4(\tau)}{240\eta(\tau)^{10}}.$$

We plan to discuss its proof in our forthcoming paper.

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