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ZARISKI'S FINITENESS THEOREM AND PROPERTIES OF SOME RINGS OF INVARIANTS

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Abstract. In this paper we will give a short proof of a special case of Zariski's result about finite generation in connection with Hilbert's 14^{th} problem using a new idea. Our result is useful for invariant subrings of unipotent or connected semisimple groups. We will also prove an analogue of Miyanishi's result for the ring of invariants of a $\mathbb{G}_a$-action on $R[X, Y, Z]$ for an affine Dedekind domain R using topological methods.

1. Introduction

In this paper k will be an algebraically closed field of characteristic 0. All varieties and morphisms are defined over k, unless otherwise specified. When some topological argument is used, k will be tacitly assumed to be the field of complex numbers.

In this paper our aim is to prove the following Theorems 2, 3 and 4 by using some topological ideas.

(1) We also prove a stronger form of a result of A. Tyc [19].

Theorem 1. *Let $\mathbb{G}_a$ act regularly on the affine space $\mathbb{A}^n$ by automorphisms over k. Assume that $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ exists as an affine variety and the induced quotient morphism is a surjection. Then $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ is Gorenstein with at most rational singular points. In particular, $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ has at worst canonical singularities* [16].

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Our proof of this result shows that we need only that the singular locus of $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ is in the image of the quotient morphism.

(2) From the above theorem we get the following.

Corollary 1.1. *Let $\mathbb{G}_a$ act linearly on $\mathbb{A}^n$ such that $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ has an isolated singular point at its vertex. Then $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ has a Gorenstein canonical singularity at the vertex.*

In [19], it was proved that $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ is Gorenstein assuming that $\mathbb{G}_a$ acts linearly on $\mathbb{A}^n$, but without assuming surjectivity of the quotient morphism.

(3) O. Zariski's result [20], in connection with his generalization of Hilbert's Fourteenth Problem, usually known as *Zariski's Finiteness Theorem*, is the following.

Let T be an affine normal domain over k with quotient field L. Suppose K is a subfield of L containing k such that $\operatorname{tr.deg}_k K$ *is at most* 2. *Then $T \cap K$ is finitely generated over k.*

We prove the following special cases of Zariski's Finiteness Theorem.

Theorem 2. *Let T be an affine normal domain over k with quotient field L. Suppose $k \subset K \subset L$, where K is a field of transcendence degree* 1 *over k. Then $S := T \cap K$ is finitely generated as a k-algebra.*

Theorem 3. *Let T be an affine factorial domain over k. Let S be an inert subring of T such that* $\operatorname{tr.deg}_k S = 2$. *Then S is finitely generated over k.*

(4) The above theorems prove the following.

Corollary 3.1. *Let T be an affine factorial domain over k. Assume that G is either a unipotent group, or a connected semisimple group defined over k. Let G act on T regularly such that* $\operatorname{tr.deg}_k(T^G) \leq 2$. *Then T^G is finitely generated over k.*

Remark 1. Note that it is a classical result that for an affine domain T with a regular action of a semisimple algebraic group the ring of invariants is always finitely generated.

(5) We further prove a sufficient criterion for the quotient $\mathbb{A}^3_C /\!\!/ \mathbb{G}_a$, where C is a smooth curve, to be a Zariski locally trivial $\mathbb{A}^2$-bundle over C.

Theorem 4. *Let R be a regular affine domain of dimension* 1 *over $\mathbb{C}$. Let $\mathbb{G}_a$ act by R-automorphisms on $R[X,Y,Z]$ and let $S := R[X,Y,Z]^{\mathbb{G}_a}$. Assume that a fiber F_0 of the morphism* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is normal. Then F_0 is isomorphic to $\mathbb{A}^2$ and* $\operatorname{Spec} S$ *is a trivial $\mathbb{A}^2$-bundle over* $\operatorname{Spec} R$ *in a neighbourhood of F_0.*

(6) We get the following global analogue of M. Miyanishi's result [12, Thm. 4].

Corollary 4.1. *With the above notation, if every fiber of* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is normal then this morphism is a locally trivial $\mathbb{A}^2$-bundle.*

Note that by a result of S. M. Bhatwadekar–D. Daigle the ring S in the above result is finitely generated over R [2].

In [3], examples are given to show that in general F_0 can be non-normal. In fact, the embedding dimension of F_0 at a singular point of F_0 can be arbitrarily large.

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2. Preliminaries

(A) In some latter proofs we will use relative cohomology groups of a pair (X, Y) where X is a paracompact simplicial complex of finite dimension and Y is a subcomplex. In our applications, X will be an algebraic variety and Y a proper closed subvariety of X (but not necessarily irreducible). In this article whenever we write homologies or cohomologies without mentioning the coefficient ring it would mean that the coefficient ring is the ring of integers. In other cases the coefficient ring will always be mentioned appropriately. However in our context the coefficient group will be either the group of integers or rational numbers. We refer to [18, Chap. 6, §9] for results related to various cohomology theories with compact supports, their connection with usual singular cohomology, and the duality result stated below. Theorem 10 in the above reference is of special interest for us.

This cohomology has the following properties.

(1) For any variety X and a closed subvariety Y there is a long exact sequence

$$0 \to H_c^0(X, Y) \to H_c^0(X) \to H_c^0(Y) \to H_c^1(X, Y) \to H_c^1(X) \to H_c^1(Y) \to \cdots$$

If $X - Y$ is smooth of pure dimension d then $H_c^i(X, Y) \cong H_{2d-i}(X - Y)$. This duality is crucially used in the proof of Theorem 4.

In [18, Thm. 10, Chap. 6, §9] we will take A to be X and B to be Y. In our situation since $X - Y$ is smooth, the proof of the above theorem is valid.

(2) For an irreducible variety X of dimension $d \geq 1$, we have $H_c^0(X) = 0$. Moreover, $H_c^{2d}(X) = \mathbb{Z}$.

These results can be proved by induction on d using (1), considering the following stratification.

$$X \supset \operatorname{Sing} X \supset \operatorname{Sing}(\operatorname{Sing} X) \supset \cdots.$$

(Here $\operatorname{Sing} V$ denotes the singular locus of an algebraic variety V.)

(B) The next three lemmas play a crucial role to show that a certain strictly increasing sequence of normal affine domains stabilizes after a finite stage.

We will implicitly use the result that for any non-empty Zariski open subset U of a normal irreducible variety V the induced homomorphism $\pi_1(U) \to \pi_1(V)$, and hence also $H_1(U) \to H_1(V)$, is a surjection. If V is smooth and $V - U$ has codimension ≥ 2 in V then these homomorphisms are isomorphisms.

Lemma 5. *Let V, W be normal irreducible algebraic varieties, and let $f : W \to V$ be a dominant morphism. Then $b_1(W) \geq b_1(V)$, where b_1 denotes the first Betti number.*

This can be proved by using the observation that there is a proper closed subvariety S of V such that $W - f^{-1}(S) \to V - S$ is a differentiable fiber bundle, and the long exact homotopy sequence of a fiber bundle.

It is well known that a Stein space X of dimension d has the property that $H_i(X) = 0$ for $i > d$ and $H_d(X)$ is torsion-free. We will use this implicitly in our proofs.

Lemma 6. *Let $R_1 \subset R_2 \subset \cdots$ be normal affine domains over an uncountable field (not necessarily algebraically closed) k of characteristic 0, all contained in an affine domain S. If each of the induced maps $\operatorname{Spec} R_{i+1} \to \operatorname{Spec} R_i$ is quasi-finite, then the above sequence stabilizes after a finite stage.*

Proof. Let $V_i = \operatorname{Spec} R_i$ for all i, and $W = \operatorname{Spec} S$. The quotient fields of R_i form an increasing chain, all contained in the quotient field of S. Hence without loss of generality we can assume that all the R_i have the same quotient field. This is because $Q(R_{i+1})$ is algebraic over $Q(R_i)$ for large enough i and $\bigcup_{i\geq 1} Q(R_i)$ is finitely generated over k since it is contained in $Q(S)$ and S is affine. We can redesignate the sequence of R_i suitably so that all have the same quotient field. Now by Zariski's Main Theorem, each V_{i+1} is a Zariski-open subset of V_i. We want to prove that V_i's are equal for large i.

If this is false, then V_i's are obtained by removing more and more divisors from V_1. Consider the image of W in V_1. It is a constructible set which contains a non-empty affine open set U. Then $V_1 - U$ can contain only finitely many divisors. But the image of W is contained in each V_i, hence disjoint from the infinite number of divisors which are removed from V_1, a contradiction. □

Lemma 7. *Let Z be a complete algebraic variety of dimension d over k. Let $D_1, D_2, \ldots$ be a sequence of distinct irreducible divisors in Z. Then $b_1(Z - (\operatorname{Sing} Z \cup D_1 \cup \cdots \cup D_n))$ tends to infinity with n.*

Proof. By taking a resolution of singularities of Z such that the inverse image of Sing Z is a union of divisors we can assume that Z is smooth. Consider the relative cohomology sequence with rational coefficients of the pair $\left(Z, \bigcup_{i=1}^n D_i\right)$. This has the terms

$$\cdots \longrightarrow H^{2d-2}(Z;\mathbb{Q}) \to H^{2d-2}\left(\bigcup_{i=1}^n D_i;\mathbb{Q}\right) \to H^{2d-1}\left(Z, \bigcup_{i=1}^n D_i;\mathbb{Q}\right) \to \cdots.$$

By duality,

$$H^{2d-1}\left(Z, \bigcup_{i=1}^n D_i;\mathbb{Q}\right) \cong H_1\left(Z - \bigcup_{i=1}^n D_i;\mathbb{Q}\right).$$

The vector space $H^{2d-2}\left(\bigcup_{i=1}^n D_i;\mathbb{Q}\right)$ has rank n over $\mathbb{Q}$. This shows that as n tends to ∞ the result follows. □

Remark 2. We will use the above lemma for an affine variety V and divisors $D_1, D_2, \ldots$ and deduce that $b_1(V - (\operatorname{Sing} V \cup D_1 \cup \cdots \cup D_n))$ tends to infinity with n by taking a suitable compactification of V.

(C) Let $\mathbb{G}_a$ act regularly on an integral domain R. Then the ring of invariants $R^{\mathbb{G}_a}$ is *inert* in R, i.e., if $r \in R^{\mathbb{G}_a}$ is written as $r_1 \cdot r_2$ with $r_i \in R$ then r_1, r_2 are in $R^{\mathbb{G}_a}$. In particular, R and $R^{\mathbb{G}_a}$ have the same group of units. Also, if R is factorial then so is $R^{\mathbb{G}_a}$. This result follows from the fact that the group in question does not have a non-trivial algebraic homomorphism to $k^* = k - \{0\}$. For some basic properties of $\mathbb{G}_a$ actions, or equivalently locally nilpotent derivations on integral domains, we refer the reader to [10, Chap. I].

Let $\mathbb{G}_a$ act regularly and linearly on a polynomial ring $R := k[X_1, \ldots, X_n]$. By a well-known result of Weitzenböck the ring of invariants $R^{\mathbb{G}_a}$ is finitely generated and positively graded over k. This defines an affine variety $V := \operatorname{Spec} R^{\mathbb{G}_a}$. The irrelevant maximal ideal of $R^{\mathbb{G}_a}$ corresponds to a (closed) point of V. We call this point the *vertex* of V.

Let $\mathbb{G}_a$ act regularly on an integral domain T. Suppose S is a multiplicative subset of $R^{\mathbb{G}_a}$ of non-zero elements. Then $(R^{\mathbb{G}_a})_S = (R_S)^{\mathbb{G}_a}$. This observation is used often to reduce the finite generation questions to suitable localizations.

We will implicitly use the following well-known result [6, Lem. 1.10].

Let Y be a factorial affine scheme with a regular action of $\mathbb{G}_a$. Assume that $Y /\!/ \mathbb{G}_a$ is an affine scheme having dimension more than 1. Then no fiber of the quotient morphism contains a divisor.

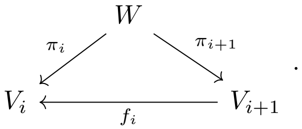
3. On Zariski's Finiteness Theorem

In this section first we prove a special case of Zariski's Finiteness Theorem. The proof of this result contains the germ of an idea which is used several times in later proofs.

For simplicity we are assuming that the rings involved are always normal.

Theorem 8. *Let T be an affine normal domain over k with quotient field L. Suppose $k \subset K \subset L$, where K is a field of transcendence degree 1 over k. Then $S := T \cap K$ is finitely generated as a k-algebra.*

Proof. Since T is normal, S is also normal. This follows from the definition of S. Since S is a countable dimensional k-vector space we can find affine normal domains $R_1 \subset R_2 \subset \ldots$, each with quotient field $Q(S)$ whose union is S. We will prove that $S = R_i$ for $i \gg 0$. Let $V_i = \operatorname{Spec} R_i$, for all i and $W = \operatorname{Spec} T$. There are dominant morphisms $\pi_i : W \to V_i$ induced from $R_i \subset T$ and $f_i : V_{i+1} \to V_i$ induced from $R_i \subset R_{i+1}$ for all i with the following obvious commutativity of diagrams:



Since V_i's are curves, all the maps $f_i : V_{i+1} \to V_i$ are quasi-finite. Since each R_i is contained in T and T is normal, by Lemma 6, the above sequence of R_i stabilizes after a finite number of steps. This completes the proof. $\square$

Before going to our next result let us fix a notation which will be used in later proofs.

Notation. We use the notation $\Gamma(X)$ to denote the ring of global sections of the structure sheaf of a variety X.

Theorem 9. *Let T be an affine factorial domain over k. Let S be an inert subring of T such that* $\operatorname{tr.deg}_k S = 2$*. Then S is finitely generated over k.*

Proof. Let $W := \operatorname{Spec} T$. Since T is factorial, T is a normal domain and S, being an inert subring of T, is a normal domain too. Let $K = Q(S)$. Thus $\operatorname{tr.deg}_k K = 2$.

Since T has countable dimension over k, S is also countable dimensional over k. Find $R_{11} \subset S$, a normal affine k-algebra with $Q(R_{11}) = K$. Thus $\dim R_{11} = 2$. We have a dominant map $\pi_{11} : W \to \operatorname{Spec} R_{11}$ induced from $R_{11} \subset T$. Then there exist at most finitely many closed points, say, $p_1, p_2, \ldots, p_r$ in $\operatorname{Spec} R_{11}$ each of whose pre-image in W contains a divisor. Let these finitely many prime divisors be Δ_1, Δ_2, ..., Δ_n. Each prime divisor in W corresponds to a height 1 prime ideal in T and since T is factorial, every height 1 prime ideal is principal. So, each Δ_i in W is defined by a prime element, say, t_i in T, for all $i = 1, 2, \ldots, n$.

Without loss of generality assume that $\Delta_1 \subset \pi_{11}^{-1}(p_1)$. Let M_1 be the maximal ideal of R_{11} corresponding to p_1. Thus $t_1 T \cap R_{11} = M_1$ and hence M_1 is contained in the height 1 prime ideal $t_1 T$ of T. Since S is inert in T, all the prime factors of each generator of M_1 in T lie in S. Adjoin all these prime factors to R_{11} and call this new affine domain R_{12}, which is clearly contained in S. We get the induced dominant morphisms $f_{11} : \operatorname{Spec} R_{12} \to \operatorname{Spec} R_{11}$ and $\pi_{12} : W \to \operatorname{Spec} R_{12}$ such that $f_{11} \circ \pi_{12} = \pi_{11}$.

Observe that if there is a point, say, q in $\operatorname{Spec} R_{12}$ such that a prime divisor Δ lies in the fiber $\pi_{12}^{-1}(q)$, then $\Delta \in \{\Delta_1, \Delta_2, \ldots, \Delta_n\}$. Suppose that $\Delta_1 \subset \pi_{12}^{-1}(q)$. Then $f_{11}(q) = p_1$. Repeat the above process with the maximal ideal in R_{12} corresponding to $q \in \operatorname{Spec} R_{12}$ to construct another affine domain, say, R_{13} contained in S, and so on.

Claim. *The above process stops after finitely many steps.*

Proof of Claim. Let $M_1 = (a_1, a_2, \ldots, a_n)$. Each a_i is divisible by t_1 in T. For each i, let $a_i = t_1 \cdot a_i'$ for some $a_i' \in T$. Since S is inert in T every $a_i' \in S$. So $R_{12} = R_{11}[a_1', a_2' \ldots, a_n']$. Let M_2 be a maximal ideal of R_{12} lying over M_1 such that $t_1 T \cap R_{12} = M_2$. Then for each i, $a_i' - \lambda_i = t_1 \cdot a_i''$ for some $a_i'' \in S$, by inertness of S in T. After substituting, and repeating this process we get that each $a_i \in M_1$ is a power series in t_1 with k-coefficients. This is a contradiction since then ${R_{11}}_{M_1}$ embeds in the power series ring $k[[t_1]]$. But any two elements in a system of parameters of ${R_{11}}_{M_1}$ are analytically independent. Hence we can get an affine domain $R_{1\ell}$ contained in S such that no fiber of the induced morphism $\pi_{1\ell} : W \to \operatorname{Spec} R_{1\ell}$ contains the prime divisor Δ_1, for some $\ell \in \mathbb{N}$. Repeat the same process for all other Δ_j's to finally get an affine domain, say, R_1 contained in S such that no fiber of the induced morphism $\pi_1 : W \to \operatorname{Spec} R_1$ contains any prime divisor. This completes the proof of the claim. □

Now by taking the normalisation of R_1 in K we can assume that R_1 is normal, by abuse of notation. Let $V_1 := \operatorname{Spec} R_1$. Clearly $Q(R_1) = K = Q(S)$ and thus $\dim R_1 = 2$.

If $R_1 \neq S$, find a normal affine k-algebra R_2 such that $R_1 \subsetneq R_2 \subset S$ and let $V_2 = \operatorname{Spec} R_2$. Clearly, $Q(R_2) = K$. This way we will have a chain $R_1 \subsetneq R_2 \subsetneq R_3 \subsetneq \cdots \subset S$ of normal affine domains contained in S, all having the same quotient field K and $S = \bigcup_{i \geq 1} R_i$. Note that $\dim R_i = 2$ for all $i \geq 1$.

Let $V_i = \operatorname{Spec} R_i$ for all $i \geq 1$. We get the dominant morphisms $f_i : V_{i+1} \to V_i$ and $\pi_i : W \to V_i$ induced from $R_i \subset R_{i+1}$ and $R_i \subset T$ respectively for all $i \geq 1$ with the following commutative diagram:

$$\begin{array}{ccc} & W & \\ {\scriptstyle \pi_i} \swarrow & & \searrow {\scriptstyle \pi_{i+1}} \\ V_i & \xleftarrow{\quad f_i \quad} & V_{i+1} \end{array} .$$

Suppose that there is a curve in V_2 which maps to a point in V_1 under f_1. This can happen for at most finitely many curves, say, $C_1, C_2, \ldots, C_r$ in V_2 mapping to finitely many points in V_1. Let $\operatorname{Exc} f_1 := \bigcup_{i=1}^{r} C_i$

Suppose that there exists a divisor Δ in W which maps to C_i for some i. Then Δ maps to a point in V_1, a contradiction. Hence there exists no divisor in W mapping to any C_i. Then $\pi_2^{-1}(C_i)$ has co-dimension ≥ 2. Let $\Sigma := \bigcup_{i=1}^{r} \pi_2^{-1}(C_i) = \pi_2^{-1}(\operatorname{Exc} f_1)$. Thus $\operatorname{codim}_W(\Sigma) \geq 2$. Therefore, by a version of Hartog's Theorem,[2] we have

$$\Gamma(V_2 - \operatorname{Exc} f_1) \subset \Gamma\big(W - \Sigma\big) = \Gamma(W) = T.$$

Thus $\Gamma(V_2 - \operatorname{Exc} f_1) \subset K \cap T$. Since S is inert in T having quotient field K, we have $K \cap T = S$. Therefore $\Gamma(V_2 - \operatorname{Exc} f_1) \subset S$.

Suppose infinitely many f_i are not quasi-finite, i.e., taking a subsequence we can assume that not a single map f_i is a quasi-finite morphism. Similarly as before, for all $i \geq 1$, denote by $\operatorname{Exc} f_i$ the union of all those finitely many curves in V_{i+1} each of which contracts to a point in V_i under f_i and let $P_i = f_i(\operatorname{Exc} f_i)$ be a finite set of points in V_i.

Define,

$$\begin{aligned} V_1' &:= V_1 - P_1, \\ V_2' &:= V_2 - \operatorname{Exc} f_1 - P_2, \\ V_{i+1}' &:= V_{i+1} - \bigcup_{1 \leq j < i} (f_{j+1} \cdots f_i)^{-1}(\operatorname{Exc} f_j) - \operatorname{Exc} f_i - P_{i+1} \quad \text{for all } i \geq 2. \end{aligned}$$

Therefore we have,

$$V_1' \xleftarrow{f_1} V_2' \xleftarrow{f_2} V_3' \xleftarrow{f_3} \cdots ,$$

such that each dominant map $f_i|_{V_{i+1}'} : V_{i+1}' \to V_i'$ is now quasi-finite for all $i \geq 1$. So by Zariski's Main Theorem,

$$V_1' \supset V_2' \supset V_3' \supset \cdots ,$$

all inclusions are open immersions. Also note that each $\Gamma(V_i')$ is contained in S by what we observed above and Hartog's Theorem, since $\operatorname{codim}_{V_i} P_i = 2$. Now

$$\Gamma(V_1') \subset \Gamma(V_2') \subset \Gamma(V_3') \subset \cdots \subset S \subset T.$$

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[2] **Hartog's Theorem.** Let V be a normal complex algebraic variety with W a subvariety of codimension at least 2. Then every holomorphic function on $V - W$, extends across V.

Since $\dim V_i = 2$, the rings $\Gamma(V_i')$ are again finitely generated k-algebras, by Nagata's result [15, Thm. 5, Chap. 5]. Since $\bigcup_{i\geq 1} R_i = S$ and $R_i \subset \Gamma(V_i') \subset S$ for all i, we have $\bigcup_{i\geq 1} \Gamma(V_i') = S$.

By taking a subsequence we can assume that all the inclusions $\Gamma(V_i') \subset \Gamma(V_{i+1}')$ are strict (otherwise finite generation of S follows). This will lead to a contradiction using Lemma 7, since $b_1(V_i') \leq b_1(W)$ for all i.

This proves that there exists $n \in \mathbb{N}$ such that the morphism $f_i : V_{i+1} \to V_i$ is quasi-finite for all $i \geq n$.

So by Lemma 6, the chain of R_i's must stabilize after a finite stage and thus S is a finitely generated k-algebra. □

The above theorem has the following immediate consequence.

Corollary 9.1. *Let T be an affine factorial domain over k. Assume that G is either a unipotent group, or a connected semisimple group defined over k. Let G act on T regularly such that* $\operatorname{tr.deg}_k(T^G) \leq 2$. *Then T^G is finitely generated over k.*

Proof. Observe that for the group G as in the hypothesis, T^G is an inert subring of T. The rest follows from Theorem 9. □

4. Rationality of singularities of $\mathbb{A}^n /\!\!/ \mathbb{G}_a$

In this section we will prove the following stronger form of A. Tyc's result [19].

Theorem 10. *Let $\mathbb{G}_a$ act regularly on the affine space $\mathbb{A}^n$ such that the quotient $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ exists as an affine variety. Assume that the image of the quotient morphism $\mathbb{A}^n \to \mathbb{A}^n /\!\!/ \mathbb{G}_a$ contains* $\operatorname{Sing}\left(\mathbb{A}^n /\!\!/ \mathbb{G}_a\right)$. *Then $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ is Gorenstein with rational singularities.*

In particular, $\mathbb{A}^n /\!\!/ \mathbb{G}_a$ has canonical singularities.

Proof. Since $\mathbb{G}_a$ has no non-trivial characters, $V := \mathbb{A}^n /\!\!/ \mathbb{G}_a$ is an affine factorial variety of dimension d. If $d = 1$ then we see easily that $V \cong \mathbb{A}^1$. So we assume that $d > 1$. It follows that the canonical divisor of V is trivial (which by definition means V is quasi-Gorenstein). Moreover, it is well known that the inverse image of any codimension > 1 subvariety of V in $\mathbb{A}^n$ contains no divisor. This follows from the proof of [6, Lem. 1.10]. Then by the proof of [8, Lem. 2], applied to the morphism $\widetilde{V} \to V$, any regular d-form on $V - \operatorname{Sing} V$ extends as a regular d-form on $\widetilde{V}$, where $\widetilde{V}$ is a resolution of singularities of V. Now by [4, 1.3 Satz] it follows that V has rational singularities. Thus, V is Cohen Macaulay with a trivial canonical divisor. So V is Gorenstein. Finally, a Gorenstein rational singular point is a canonical singularity by [16]. □

Remark 3. Alternatively, we can argue as follows. By Tyc's result V is Cohen Macaulay. By the above argument every regular d-form on the smooth locus of V extends to a regular d form on a resolution of singularities of V. These two properties are equivalent to rationality of singularities on V. Finally, since V is factorial the canonical divisor of V is trivial. Hence V is Gorenstein with rational singularities. Note that in the previous proof Cohen-Macaulayness of V was proved indirectly without using Tyc's result.

Corollary 10.1. *Let $\mathbb{G}_a$ act regularly and linearly on $\mathbb{A}^n$. Assume that $\mathbb{A}^n/\!\!/\mathbb{G}_a$ has an isolated singular point at its vertex. Then $\mathbb{A}^n/\!\!/\mathbb{G}_a$ has a Gorenstein canonical singular point at its vertex.*

Proof. By Weitzenböck's theorem, $V := \mathbb{A}^n/\!\!/\mathbb{G}_a$ exists as an affine variety. Since the action of $\mathbb{G}_a$ is linear the ring of invariants is positively graded. Clearly, the vertex of V is the image of the origin in $\mathbb{A}^n$. Now the result follows from the above theorem. This completes the proof of the corollary. □

Remark 4.

(1) There is an example of $\mathbb{G}_a$-action on $\mathbb{A}^4$ such that the quotient $\mathbb{A}^4/\!\!/\mathbb{G}_a$ exists as an affine variety but the morphism $\mathbb{A}^4 \to \mathbb{A}^4/\!\!/\mathbb{G}_a$ is not a surjection [1]. In this example $\mathbb{A}^4/\!\!/\mathbb{G}_a \cong \mathbb{A}^3$.

(2) Is Theorem 10 true if $\mathbb{A}^n/\!\!/\mathbb{G}_a$ exists as an affine variety but the image of the quotient morphism does not contain $\operatorname{Sing}(\mathbb{A}^n/\!\!/\mathbb{G}_a)$?

5. A sufficient criterion for $\mathbb{A}^3_C/\!\!/\mathbb{G}_a$ to be a trivial $\mathbb{A}^2$-bundle over a smooth curve C

Theorem 11. *Let R be a regular affine domain of dimension 1 over $\mathbb{C}$. Let t be a uniformizing parameter on R at a point p_0 in $C := \operatorname{Spec} R$. Let $\mathbb{G}_a$ act by R-automorphisms on $R[X,Y,Z]$ and $S := R[X,Y,Z]^{\mathbb{G}_a}$. Assume that $F_0 := \operatorname{Spec}(S/tS)$ is normal. Then $F_0 \cong \mathbb{A}^2$ and $\operatorname{Spec} S$ is a trivial $\mathbb{A}^2$-bundle over $\operatorname{Spec} R$ in a neighbourhood of p_0.*

Proof. We know that S is finitely generated over R by [2]. Also, S is inert in $R[X,Y,Z]$. By shrinking C we can assume that t is a prime element in R and thus t is a prime element in S due to the inertness of S in $R[X,Y,Z]$. Let $\mathfrak{m}$ be the maximal ideal in R generated by t. Since $R_{\mathfrak{m}}[X,Y,Z]$ is factorial, so is $S_{\mathfrak{m}} = R_{\mathfrak{m}}[X,Y,Z]^{\mathbb{G}_a}$. Thus $K_{\operatorname{Spec} S_{\mathfrak{m}}}$ is trivial.

Let $F_0 := \operatorname{Spec}(S/tS)$. Since t is a prime element in S, the ring S/tS is an integral domain. By inertness of S, we get $S/tS \hookrightarrow (R/tR)[X,Y,Z]$, with $R/tR \cong \mathbb{C}$. Hence there is an induced dominant morphism $\pi : \mathbb{A}^3 \to F_0$. Note that $\operatorname{Spec} S$ is a normal affine variety of dimension 3 over $\mathbb{C}$.

If $K = Q(R)$, then by combining Miyanishi's result [12, Thm. 4] with Kambayashi's result [7],

$$K[X,Y,Z]^{\mathbb{G}_a} = K[U_1, U_2],$$

for suitable algebraically independent polynomials U_1 and U_2 in $K[X,Y,Z]$. Since the rest of the proof is mainly topological, we will think of C as a small Euclidean open disc Δ_{p_0} in $\mathbb{C}$ around $p_0 \in C$. Thus $\operatorname{Spec} S$ should be replaced by $V := \operatorname{Spec} S \times_C \Delta_{p_0}$ and hence

$$V - F_0 \cong \Delta^*_{p_0} \times \mathbb{C}^2,$$

with $\Delta^*_{p_0}$ the punctured disc $\Delta_{p_0} - \{p_0\}$ in $\mathbb{C}$.

Here, we have used A. Sathaye's result in [17] that if every fiber of an affine morphism $Y \to D$ from a smooth affine 3-fold to a smooth affine curve is isomorphic to $\mathbb{A}^2$ then this map is a locally trivial $\mathbb{A}^2$-bundle.

Goal: Our aim is to prove that $F_0 \cong \mathbb{A}^2$.

Since $\operatorname{Spec} S - F_0 \cong (C - \{p_0\}) \times \mathbb{A}^2$, it is easy to see that S has a locally nilpotent derivation δ which restricts to a locally nilpotent derivation $\delta|_{F_0}$ on F_0, but $\delta|_{F_0}$ can be identically zero. By considering $t^a \cdot \delta$ for a suitable integer a, we can assume that, by changing the notation, $\delta|_{F_0} \neq 0$.

Since F_0 is assumed to be normal, F_0 is a normal affine surface with a $\mathbb{G}_a$-action. Hence there is an $\mathbb{A}^1$-fibration $F_0 \to D$ for some smooth affine curve D. By Miyanishi's result F_0 has at worst cyclic quotient singularities [11, Chap. I, §6].

Recall that the canonical divisor $K_{\operatorname{Spec} S_{\mathfrak{m}}}$ is trivial. Since F_0 is defined by a single function, F_0 is a principal divisor, i.e., F_0 corresponds to a trivial line bundle. Hence

$$K_{F_0} = (K_{\operatorname{Spec} S_{\mathfrak{m}}} + F_0)|_{F_0}$$

is also a trivial line bundle. This implies that the singular points of F_0 are just cyclic rational double points.

Recall that V is smooth outside F_0. Since F_0 is reduced and defined by $\{t = 0\}$, we see that every smooth point of F_0 is a smooth point of V and hence V has only isolated singular points which lie on F_0. Let $P := \operatorname{Sing} V = \{q_1, q_2, \ldots, q_m\}$.

Since q_i is a hypersurface singularity of V, there exists a contractible neighbourhood N_i of q_i such that

$$\pi_1(N_i - \{q_i\}) = (1) \text{ for all } i = 1, 2, \ldots, m.$$

For this we refer to [9]. Let $N := \sqcup_{i=1}^m N_i$. Thus $N - P = \sqcup_{i=1}^m (N_i - \{q_i\})$.

Now we will prove that $F_0 \cong \mathbb{A}^2$ through the following sequence of claims.

Claim 1.

(i) $H_i(F_0) = 0$ *for all* $i \geq 3$ *and* $H_2(F_0)$ *is torsion-free.*
(ii) $H_i(V) = 0$ *for all* $i \geq 4$ *and* $H_3(V)$ *is torsion-free.*

Proof. The claim follows since F_0 and V are Stein spaces of dimensions 2 and 3, respectively. □

Claim 2.

(i) $H_1(F_0; \mathbb{Q}) = 0$.
(ii) $H_1(V - P) = 0$. *In fact,* $\pi_1(V - P) = (1)$.

Proof. Recall that we have a dominant morphism $\pi : \mathbb{A}^3 \to F_0$. Therefore we have $H_1(F_0; \mathbb{Q}) = 0$.

The quotient morphism, say, $\varphi : C \times \mathbb{A}^3 \to \operatorname{Spec} S = (C \times \mathbb{A}^3) /\!\!/ \mathbb{G}_a$, restricts to a holomorphic map $\psi : (\Delta_{p_0} \times \mathbb{C}^3) - \varphi^{-1}(P) \to V - P$. Also note that the general fiber of ψ is connected. Since outside of F_0, fibers of φ are $\mathbb{A}^1$, we can find a non-empty Zariski open neighbourhood $U \subset V - P$ such that $\psi|_{\psi^{-1}(U)} : \psi^{-1}(U) \to U$ is a locally trivial fiber bundle with connected general fiber. Therefore the induced map

$$(\psi|_{\psi^{-1}(U)})_* : \pi_1(\psi^{-1}(U)) \to \pi_1(U)$$

is surjective. We have the following commutative diagram

$$\begin{array}{ccc} \pi_1(\psi^{-1}(U)) & \xrightarrow{(\psi|_{\psi^{-1}(U)})_*} & \pi_1(U) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ \pi_1\big((\Delta_{p_0} \times \mathbb{C}^3) - \varphi^{-1}(P)\big) & \xrightarrow[\psi_*]{} & \pi_1(V - P) \end{array}$$

with both the vertical maps are onto induced from the inclusions $i : \psi^{-1}(U) \hookrightarrow (\Delta_{p_0} \times \mathbb{C}^3) - \varphi^{-1}(P)$ and $j : U \hookrightarrow V - P$ respectively. Therefore from the above commutative diagram we see that ψ_* is onto.

Let $\mathfrak{m}$ be the maximal ideal in R corresponding to the point p_0 in C. Clearly t generates $\mathfrak{m}$. Let $W := \operatorname{Spec}(R_{\mathfrak{m}}[X, Y, Z])$. Since R is regular, $R_{\mathfrak{m}}$ is a factorial domain and so is $R_{\mathfrak{m}}[X, Y, Z]$. Consider the quotient morphism $\eta : W \to W/\!\!/\mathbb{G}_a$. By [6, Lem. 1.10], $\eta^{-1}\operatorname{Sing}(W/\!\!/\mathbb{G}_a)$ does not contain any divisor in W.

Therefore $\operatorname{codim}\varphi^{-1}(P) \geq 2$. Thus $\pi_1(V - P) = (1)$ and hence $H_1(V - P) = 0$. $\square$

Claim 3.

(i) $H_c^i(V, P) \cong H_c^i(V)$ for all $i \geq 2$.
(ii) $H_c^i(F_0, P) \cong H_c^i(F_0)$ for all $i \geq 2$.

Proof. Consider the relative cohomology sequences with compact support for the pairs (V, P) and (F_0, P), respectively. We have the following exact sequences for all $n \in \mathbb{N}$.

$$H_c^n(P) \to H_c^{n+1}(V, P) \to H_c^{n+1}(V) \to H_c^{n+1}(P),$$

and

$$H_c^n(P) \to H_c^{n+1}(F_0, P) \to H_c^{n+1}(F_0) \to H_c^{n+1}(P).$$

Since $H_c^i(P) = 0$ for all $i \geq 1$, the claim follows. $\square$

Claim 4.

(i) $H_c^i(V) \cong H_c^i(F_0)$ *for* $0 \leq i \leq 3$.
(ii) $H_c^4(V) = 0$.

Proof. Note that $V - F_0$ is smooth. So by duality,

$$H_c^i(V, F_0) \cong H_{6-i}(V - F_0), \text{ for all } 0 \leq i \leq 6.$$

Since $V - F_0 \cong \Delta_{p_0}^* \times \mathbb{C}^2$, thus $V - F_0$ is homotopy equivalent to the unit circle $\mathbb{S}^1$. So $H_0(V - F_0) = H_1(V - F_0) = \mathbb{Z}$ and $H_i(V - F_0) = 0$ for all $i \geq 2$. Hence

$$H_c^i(V, F_0) = \begin{cases} 0 & \text{for } 0 \leq i \leq 4 \\ \mathbb{Z} & \text{for } i = 5, 6. \end{cases} \tag{1}$$

Consider the relative cohomology sequences with compact support for the pairs (V, F_0). We have the following exact sequences for all $n \in \mathbb{N}$:

$$H_c^n(V, F_0) \to H_c^n(V) \to H_c^n(F_0) \to H_c^{n+1}(V, F_0).$$

By (1), we get $H_c^i(V) \cong H_c^i(F_0)$ for $i = 0, 1, 2, 3$.

Since by (1), $H_c^4(V, F_0) = 0$ we have the following exact sequence:

$$0 \to H_c^4(V) \to H_c^4(F_0) \to H_c^5(V, F_0) \to H_c^5(V).$$

Using Claim 3 and duality we get $H_c^5(V) \cong H_c^5(V, P) \cong H_1(V - P)$ and thus $H_c^5(V) = 0$ by Claim 2. Moreover, since F_0 is an irreducible surface over $\mathbb{C}$, $H_c^4(F_0) \cong \mathbb{Z}$. By (1), $H_c^5(V, F_0) \cong \mathbb{Z}$. Thus we conclude from the above exact sequence that $H_c^4(V) = 0$. □

Claim 5.

(i) $H_i(N - P) = 0$ *for* $1 \le i \le 4$.
(ii) $H_3(V) = 0$.

Proof. Note that for each i, $N_i - \{q_i\}$ deformation retracts to the boundary ∂N_i of N_i. Since $\pi_1(N_i - \{q_i\}) = (1)$, we have $H_1(N_i - \{q_i\}) \cong H_1(\partial N_i) = 0$ for all i and thus $H_1(N - P) = 0$. Observe that ∂N_i is a compact, connected manifold of real dimension 5. Therefore by duality we have

$$H_4(N_i - \{q_i\}) \cong H_4(\partial N_i) \cong H^1(\partial N_i)$$

and by the universal coefficient theorem,

$$H^1(\partial N_i) \cong \operatorname{Ext}^1\big(H_0(\partial N_i), \mathbb{Z}\big) \oplus \operatorname{Hom}\big(H_1(\partial N_i), \mathbb{Z}\big) = 0,$$

since $\operatorname{Ext}^1\big(H_0(\partial N_i), \mathbb{Z}\big) = \operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0$. Thus $H_4(N - P) = 0$.

Recall that $H_4(V) = 0$, by Claim 1. Also since $V - P$ is smooth, we get

$$H_2(V - P) \cong H_c^4(V, P) \cong H_c^4(V) = 0,$$

by duality, Claim 3 and Claim 4 respectively. Thus we consider the following part of the Mayer–Vietoris sequence for $V = N \cup (V - P)$.

$$0 \to H_3(N - P) \to H_3(V - P) \to H_3(V) \to H_2(N - P) \to 0.$$

Note that in the above sequence we have used $H_4(V) = H_2(V - P) = 0$.

Duality, Claim 3 and Claim 4 together yield

$$H_3(V - P) \cong H_c^3(V, P) \cong H_c^3(V) \cong H_c^3(F_0).$$

Also $H_c^3(F_0; \mathbb{Q}) \cong H_1(F_0; \mathbb{Q}) = 0$, by duality and Claim 2. Hence $H_3(V - P)$ is a torsion $\mathbb{Z}$-module.

Thus from the above Mayer–Vietoris sequence, we can see that $H_3(N - P)$ is also a torsion $\mathbb{Z}$-module. Again $H_3(N - P) \cong \bigoplus_i H_3(N_i - \{q_i\}) \cong \bigoplus_i H_3(\partial N_i)$. Since ∂N_i is compact, duality and the universal coefficient theorem together imply that

$$H_3(\partial N_i) \cong H^2(\partial N_i) \cong \operatorname{Ext}^1(H_1(\partial N_i), \mathbb{Z}) \oplus \operatorname{Hom}(H_2(\partial N_i), \mathbb{Z}).$$

Since $H_1(\partial N_i) = 0$ as we have observed earlier, we conclude that $H_3(\partial N_i)$ is torsion-free and hence $H_3(N-P)$ is torsion-free. Also we observed earlier that $H_3(N-P)$ is a torsion $\mathbb{Z}$-module. Hence $H_3(N-P)=0$.

Therefore we get the following short exact sequence from the above Mayer–Vietoris sequence.

$$0 \to H_3(V-P) \to H_3(V) \to H_2(N-P) \to 0.$$

Since $H_3(V-P)$ is a torsion $\mathbb{Z}$-module and $H_3(V)$ is torsion-free by Claim 1, the above short exact sequence yields $H_3(V-P)=0$. Hence $H_3(V) \cong H_2(N-P)$. Again $H_2(N-P) \cong \bigoplus_i H_2(N_i - \{q_i\}) \cong \bigoplus_i H_2(\partial N_i)$. So $H_2(N-P)$ and hence each $H_2(\partial N_i)$ is torsion-free.

Since ∂N_i is compact, duality and the universal coefficient theorem together imply that

$$H_2(\partial N_i) \cong H^3(\partial N_i) \cong \operatorname{Ext}^1(H_2(\partial N_i),\mathbb{Z}) \oplus \operatorname{Hom}(H_3(\partial N_i),\mathbb{Z}).$$

Since $H_2(\partial N_i)$ is a torsion-free finitely generated $\mathbb{Z}$-module, $H_2(\partial N_i)$ is a free abelian group for all $i=1,2,\dots,m$. Thus $\operatorname{Ext}^1(H_2(\partial N_i),\mathbb{Z})=0$. Also we have $\operatorname{Hom}(H_3(\partial N_i),\mathbb{Z})=0$, since $H_3(\partial N_i)=0$ for all $i=1,2,\dots,m$. So $H_2(\partial N_i)=0$ for all $i=1,2,\dots,m$ and hence $H_2(N-P)=0$. Therefore $H_3(V) \cong H_2(N-P) = 0$. □

Claim 6. $H_2(F_0)=0$.

Proof. By duality, $H_2(F_0;\mathbb{Q}) \cong H_c^2(F_0;\mathbb{Q})$ and using Claim 4 we conclude that $H_2(F_0;\mathbb{Q}) \cong H_c^2(V;\mathbb{Q})$. Again by Claim 3 and duality, $H_2(F_0;\mathbb{Q}) \cong H_c^2(V,P;\mathbb{Q}) \cong H_4(V-P;\mathbb{Q})$. Consider the following part of the Mayer–Vietoris sequence for $V = N \cup (V-P)$:

$$H_5(V) \to H_4(N-P) \to H_4(V-P) \to H_4(V).$$

Since $H_4(V)=0=H_5(V)$ by Claim 1, we have $H_4(V-P) \cong H_4(N-P)$. Since the latter group is trivial by Claim 5, we conclude that $H_2(F_0;\mathbb{Q})=0$. This implies that $H_2(F_0)$ is a torsion $\mathbb{Z}$-module. Claim 1 implies that $H_2(F_0)=0$.

Recall that F_0 is a normal affine surface with $b_1(F_0)=0=b_2(F_0)$ by Claim 2 and Claim 6. We already observed that the singularities of F_0 are cyclic rational double points. These together imply that F_0 is a logarithmic $\mathbb{Q}$-homology plane.

This implies that the $\mathbb{A}^1$-fibration on F_0 has $\mathbb{A}^1$ as a base and all the fibers are irreducible, isomorphic to $\mathbb{A}^1$ if taken with reduced structure. We refer to [5, 4.15] for this. □

Claim 7. $H_1(F_0-P)=0$.

Proof. Note that F_0-P is smooth. By duality $H_1(F_0-P) \cong H_c^3(F_0,P)$. Again by Claim 3 and Claim 4 we get

$$H_1(F_0-P) \cong H_c^3(F_0) \cong H_c^3(V).$$

Consider the following part of the Mayer–Vietoris sequence for $V = (V - P) \cup N$:

$$H_3(N - P) \to H_3(V - P) \to H_3(V) \to H_2(N - P).$$

Thus $H_3(V - P) \cong H_3(V)$, using Claim 5. Now use the duality on the pair (V, P) and Claim 3 respectively to conclude that $H_c^3(V) \cong H_3(V)$.

Thus, $H_1(F_0 - P) \cong H_c^3(V) \cong H_3(V)$. The latter group is trivial by Claim 5. □

Claim 8.

(i) $H_i(F_0) = 0$ *for all* $i > 0$.
(ii) F_0 *is smooth.*

Proof. Let M be a disjoint union of contractible neighbourhoods M_j in F_0 around the singular points $q_j \in F_0$. Therefore $H_i(M - P) \cong \bigoplus_j H_i(M_j - \{q_j\})$ for all i. Consider the following part of the Mayer–Vietoris sequence for the covering $F_0 = (F_0 - P) \cup M$:

$$H_2(F_0) \to H_1(M - P) \to H_1(F_0 - P) \to H_1(F_0).$$

Note that the map $H_1(F_0 - P) \to H_1(F_0)$ is always surjective, since F_0 is normal (see comment following (B) in the Preliminaries). Using Claim 6 and Claim 7, we conclude that $H_1(M - P) = 0$ and $H_1(F_0) = 0$.

Therefore combining with Claim 1 we conclude, $H_i(F_0) = 0$ for all $i > 0$. Also $H_1(M_i - \{q_i\}) = 0$ for every singular point q_i of F_0. Since these are cyclic quotient singular points of F_0 by Mumford's result [14], F_0 is smooth. □

By the usual ramified covering trick [13, III, 3.2.1], we conclude that all the fibers are reduced and isomorphic to $\mathbb{A}^1$. Hence F_0 is isomorphic to $\mathbb{A}^2$.

Finally, by Sathaye's result [17] V is an $\mathbb{A}^2$-bundle over C, and hence a trivial $\mathbb{A}^2$-bundle if C is shrunk to a suitable Zariski open neighbourhood of p_0.

This completes the proof of the theorem. □

Example 1. Daigle and Freudenburg found examples of $\mathbb{G}_a$-actions on the polynomial ring $\mathbb{C}[X_1, X_2, X_3, X_4]$ keeping X_1 invariant such that $\mathbb{C}^4/\!\!/\mathbb{G}_a$ need arbitrarily large number of generators. In fact the embedding dimension at some singular point of $\mathbb{C}^4/\!\!/\mathbb{G}_a$ can be arbitrarily large. We refer to [3] for this.

By our result, we conclude that $\{X_1 = \alpha\}$ for some $\alpha \in \mathbb{C}$, treated as a subscheme of the quotient, is non-normal for these examples.

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