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GENERALISED GELFAND–GRAEV REPRESENTATIONS IN BAD CHARACTERISTIC ?

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Abstract. Let G be a connected reductive algebraic group defined over a finite field with q elements. In the 1980's, Kawanaka introduced generalised Gelfand–Graev representations of the finite group $G(\mathbb{F}_q)$, assuming that q is a power of a good prime for G. These representations have turned out to be extremely useful in various contexts. Here we investigate to what extent Kawanaka's construction can be carried out when we drop the assumptions on q. As a curious by-product, we obtain a new, conjectural characterisation of Lusztig's concept of special unipotent classes of G in terms of weighted Dynkin diagrams.

1. Introduction

Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let G be a connected reductive algebraic group over k and assume that G is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where q is a power of p. Let $F\colon G \to G$ be the corresponding Frobenius map. We are interested in studying the representations (over an algebraically closed field of characteristic 0) of the finite group of fixed points $G^F = \{g \in G \mid F(g) = g\}$.

Assuming that p is a "good" prime for G, Kawanaka [16], [17], [18] described a procedure by which one can associate with any unipotent element $u \in G^F$ a representation Γ_u of G^F, obtained by induction of a certain one-dimensional representation from a unipotent subgroup of G^F. If u is the identity element, then Γ_1 is the regular representation of G^F; if u is a regular unipotent element, then Γ_u is a Gelfand–Graev representation as defined, for example, in [2, §8.1] or [35, §14]. For arbitrary u, the representation Γ_u is called a *generalised Gelfand–Graev representation* (GGGR for short); it only depends on the G^F-conjugacy class of u.

A fundamental step in understanding the GGGRs is achieved by Lusztig [23] where the characters of GGGRs are expressed in terms of characteristic functions of intersection cohomology complexes on G. In [23] it is assumed that p is sufficiently large; in [36] it is shown that one can reduce these assumptions so that everything

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works as in Kawanaka's original approach. These results have several consequences. By [12] the characters of the various Γ_u span the $\mathbb{Z}$-module of all unipotently supported virtual characters of G^F. In addition to the original applications in [16], [17], [18], GGGRs have turned out to be very useful in various questions concerning ℓ-modular representations of G^F where ℓ is a prime not equal to p; see, e.g., [13], [6]. Thus, it seems desirable to explore the possibilities for a definition without any restriction on p, q. These notes arose from an attempt to give such a definition. Recall that p is "good" for G if p is good for each simple factor involved in G; the conditions for the various simple types are as follows.

$$\begin{aligned} \mathsf{A}_n &: \quad \text{no condition,} \\ \mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n &: \quad p \neq 2, \\ \mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7 &: \quad p \neq 2, 3, \\ \mathsf{E}_8 &: \quad p \neq 2, 3, 5. \end{aligned}$$

Easy examples indicate that one can not expect a good definition of GGGRs for all unipotent elements of G^F. Instead, it seems reasonable to restrict oneself to those unipotent classes which "come from characteristic 0", where the classical Dynkin–Kostant theory is available; see Section 2. This is also consistent with the picture presented by Lusztig [25], [26], [28], [29] for dealing with unipotent classes in small characteristic. Based on this framework, we formulate in Definition 3.4 some precise conditions under which it should be possible to define GGGRs for a given unipotent class. Of course, these conditions will be satisfied if p is a good prime for G, and lead to Kawanaka's original GGGRs; so then the question is how far we can go beyond this. Our answer to this question is as follows.

An essential feature of GGGRs in good characteristic is that they are very closely related to the "unipotent supports" of the irreducible representations of G^F, in the sense of Lusztig [23]. (In a somewhat different way, and without complete proofs, this concept appeared under the name of "wave front set" in Kawanaka [18].) For example, using this concept, one can show that every irreducible representation of G^F occurs with "small" multiplicity in some GGGR; see [23, Thm. 11.2]. One would hope that a useful theory of GGGRs in bad characteristic preserves some of these features. As a first step in this direction, let $\mathfrak{C}^\bullet$ be the set of unipotent classes of G which arise as the unipotent support of some irreducible representation of G^F, or of G^{F^n} for some $n \geqslant 1$. (We shall see that $\mathfrak{C}^\bullet$ only depends on G but not on the particular Frobenius map F; also note that "unipotent supports" exist in any characteristic by [14].) If p is a good prime for G, then it is known that $\mathfrak{C}^\bullet$ is the set of all unipotent classes of G. In general, all classes in $\mathfrak{C}^\bullet$ indeed "come from characteristic 0". Based on the methods in [27], an explicit description of the sets $\mathfrak{C}^\bullet$, for G simple and p bad, is given in Proposition 4.3. This result complements the general results on "unipotent support" in [14], [23] and may be of independent interest.

Now, extensive experimentation (using the computer algebra systems GAP [10] and SINGULAR [15]) lead us to the expectation, formulated as Conjecture 4.4, that our conditions in Definition 3.4 will work for all the classes in $\mathfrak{C}^\bullet$, without any restriction on p, q. Furthermore, in Conjecture 4.9, we propose an intrinsic characterisation of the set $\mathfrak{C}^\bullet$ (just in terms of Dynkin–Kostant theory). Finally, our experiments suggest a new characterisation of Lusztig's special unipotent classes in terms of an integrality condition, formulated as Conjecture 4.10. In Section 5, we work out in detail the example where G is of type F_4. In fact, using similar methods, we will be able to verify Conjectures 4.9 and 4.10 for all G of exceptional type; see Corollary 5.11. (The analogous conjectures for G of classical type have recently been established in [4].)

These notes merely contain examples and conjectures; nevertheless we hope that they show that the story about GGGRs is by no means complete and that there is some evidence for a further theory in bad characteristic.

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2. Weighted Dynkin diagrams

We use Carter [2] as a general reference for results on algebraic groups and unipotent classes. Results on unipotent classes or nilpotent orbits that are stated for large characteristic in [2] typically remain valid whenever the characteristic is a good prime; see [31]. Let $G, k, p, \ldots$ be as in Section 1. Also recall that G is defined over the finite field $\mathbb{F}_q \subseteq k$, with corresponding Frobenius map $F\colon G \to G$. We fix an F-stable maximal torus $T \subseteq G$ and an F-stable Borel subgroup $B \subseteq G$ containing T. We have $B = U \rtimes T$ where U is the unipotent radical of B. Let Φ be the set of roots of G with respect to T and $\Pi \subseteq \Phi$ be the set of simple roots determined by B. Let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. Then G acts on $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$.

2.1. For each $\alpha \in \Phi$, we have a corresponding homomorphism of algebraic groups $x_\alpha\colon k^+ \to G$, $u \mapsto x_\alpha(u)$, which is an isomorphism onto its image; furthermore, $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$ for all $t \in T$ and $u \in k$. Setting $U_\alpha := \{x_\alpha(u) \mid u \in k\}$, we have $G = \langle T, U_\alpha \ (\alpha \in \Phi)\rangle$. Note that $U_\alpha \subseteq G_{\mathrm{der}}$ for all $\alpha \in \Phi$, where G_{der} denotes the derived subgroup of G. On the level of $\mathfrak{g}$, we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ is the Lie algebra of T and $\mathfrak{g}_\alpha$ is the image of the differential $d_0x_\alpha\colon k \to \mathfrak{g}$; furthermore, $\mathrm{Ad}(t)(y) = \alpha(t)y$ for $t \in T$ and $y \in \mathfrak{g}_\alpha$. We set

$$e_\alpha := d_0x_\alpha(1) \in \mathfrak{g}_\alpha.$$

Then $e_\alpha \neq 0$ and $\mathfrak{g}_\alpha = ke_\alpha$. (For all this see, e.g., [34, §8.1].)

2.2. For $\alpha \in \Phi$, we can write uniquely $\alpha = \sum_{\beta\in\Pi} n_\beta \beta$ where $n_\beta \in \mathbb{Z}$ for all $\beta \in \Pi$. Then $\mathrm{ht}(\alpha) := \sum_{\beta\in\Pi} n_\beta$ is called the height of α. We fix once and for all a total ordering $\preceq$ of Φ^+ which is compatible with the height, that is, if $\alpha, \beta \in \Phi^+$ are such that $\alpha \preceq \beta$, then $\mathrm{ht}(\alpha) \leqslant \mathrm{ht}(\beta)$. Then every $u \in U$ has a unique expression $u = \prod_{\alpha\in\Phi^+} x_\alpha(u_\alpha)$ where $u_\alpha \in k$ (and the product is taken in the given order $\preceq$ of Φ^+). Let $\alpha, \beta \in \Phi^+$, $\alpha \neq \beta$. Let $u, v \in k$. Then we have Chevalley's commutator relations

$$x_\alpha(u)x_\beta(v)x_\alpha(u)^{-1}x_\beta(v)^{-1} = \prod_{i,j>0;\; i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(C_{\alpha,\beta,i,j}u^i v^j)$$

where the constants $C_{\alpha,\beta,i,j} \in k$ only depend on α, β, i, j but not on u, v (and, again, the product on the right-hand side is taken in the given order $\preceq$ of Φ^+). Furthermore, if $\alpha + \beta \in \Phi$, then

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \quad \text{where} \quad N_{\alpha,\beta} := C_{\alpha,\beta,1,1} \in k.$$

(Since all U_α, $\alpha \in \Phi$, are contained in the semisimple algebraic group G_{der}, this follows from [35, Lem. 15 (p. 22) and Rem. (p. 64)].)

2.3. It will also be convenient to fix some notation concerning the action of the Frobenius map $F\colon G \to G$. By the results in [35, §10], the maps $x_\alpha\colon k^+ \to G$ can be chosen such that the following holds. There exist a permutation $\tau\colon \Phi \to \Phi$ and signs $\epsilon_\alpha = \pm 1$ ($\alpha \in \Phi$) such that

$$F(x_\alpha(u)) = x_{\tau(\alpha)}(\epsilon_\alpha u^q) \quad \text{for all } u \in k.$$

Here, we can assume that $\tau(\Pi) = \Pi$ and $\epsilon_{\pm\beta} = 1$ for all $\beta \in \Pi$. Now F also induces a Frobenius map on the Lie algebra $\mathfrak{g}$ which we denote by the same symbol. We have $F(uy) = u^q F(y)$ for all $u \in k$ and $y \in \mathfrak{g}$; furthermore,

$$F(e_\alpha) = \epsilon_\alpha e_{\tau(\alpha)} \quad \text{for all } \alpha \in \Phi.$$

Finally, for $g \in G$ and $y \in \mathfrak{g}$, we have $\mathrm{Ad}(F(g))(F(y)) = F(\mathrm{Ad}(g)(y))$.

2.4. Let G_0 be a connected reductive algebraic group over $\mathbb{C}$ of the same type as G; let $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ be its Lie algebra. Then, by the classical Dynkin–Kostant theory (see, e.g., [2, §5.6]), the nilpotent $\mathrm{Ad}(G_0)$-orbits in $\mathfrak{g}_0$ are parametrized by a certain set Δ of so-called *weighted Dynkin diagrams*, i.e., maps $d\colon \Phi \to \mathbb{Z}$ such that

(a) $d(-\alpha) = -d(\alpha)$ for all $\alpha \in \Phi$ and $d(\alpha+\beta) = d(\alpha) + d(\beta)$ for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$;

(b) $d(\beta) \in \{0, 1, 2\}$ for every simple root $\beta \in \Pi$.

Furthermore, these nilpotent orbits in $\mathfrak{g}_0$ are naturally in bijection with the unipotent classes of G_0 (see [2, §1.15]). If G_0 is a simple algebraic group, then the corresponding set Δ of weighted Dynkin diagrams is explicitly known in all cases; see [2, §13.1] and the references there. (Several examples will be given below.) For each $d \in \Delta$, we denote by $\mathscr{O}_d$ the corresponding nilpotent orbit in $\mathfrak{g}_0$ and set

$$\mathbf{b}_d := \tfrac{1}{2}(\dim G_0 - \mathrm{rank}(G_0) - \dim \mathscr{O}_d).$$

This number is not really relevant for the further discussion, but it is simply a very useful invariant for distinguishing nilpotent orbits. (The number $\mathbf{b}_d$ is also the dimension of the variety of Borel subgroups of G_0 containing an element in the unipotent class corresponding to $\mathscr{O}_d$; see [2, §1.15, §5.10].).

2.5. Let us fix $d \in \Delta$. For $i \in \mathbb{Z}$, we set $\Phi_i := \{\alpha \in \Phi \mid d(\alpha) = i\}$ and define

$$\mathfrak{g}(i) := \begin{cases} \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha & \text{if } i \neq 0, \\ \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha & \text{if } i = 0. \end{cases}$$

Thus, as in [17, §2.1], we obtain a grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$; note that we do have $[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$ for all $i, j \in \mathbb{Z}$. For any $i \geqslant 0$, we also set $\mathfrak{g}(\geqslant i) := \bigoplus_{j \geqslant i} \mathfrak{g}(j)$. Furthermore, we define subgroups of G as follows:

$$\begin{aligned} P &:= \langle T, U_\alpha \mid \alpha \in \Phi_i \text{ for all } i \geqslant 0\rangle, \\ U_1 &:= \langle U_\alpha \mid \alpha \in \Phi_i \text{ for all } \geqslant 1\rangle, \\ L &:= \langle T, U_\alpha \mid \alpha \in \Phi_0\rangle. \end{aligned}$$

Then P is a parabolic subgroup of G with unipotent radical U_1 and Levi decomposition $P = U_1 \rtimes L$. The Lie algebra of P is given by $\mathfrak{p} := \mathrm{Lie}(P) = \mathfrak{g}(\geqslant 0) \subseteq \mathfrak{g}$. More generally, for any integer $i \geqslant 1$, we set

$$U_i := \langle U_\alpha \mid \alpha \in \Phi_j \text{ for all } j \geqslant i\rangle \subseteq G.$$

Thus, we obtain a chain of subgroups $P \supseteq U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$; using Chevalley's commutator relations, one immediately sees that each U_i is a normal subgroup of P and that U_i/U_{i+1} is abelian.

2.6. Let us fix a weighted Dynkin diagram $d \in \Delta$ as above. For any integer $i \geqslant 1$, we have a corresponding subgroup $U_i \subseteq G$ and a corresponding subspace $\mathfrak{g}(i) \subseteq \mathfrak{g}$. Following Kawanaka [17, (3.1.1)], we define a map

$$f\colon U_1 \to \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

as follows. Let $u \in U_1$. As in 2.2, we have a unique expression

$$u = \prod_{\alpha \in \Phi_i \text{ for } i \geqslant 1} x_\alpha(u_\alpha) \quad (u_\alpha \in k)$$

where the product is taken in the given order $\preceq$ on Φ^+. Then we set

$$f(u) = f\Big(\prod_{\alpha \in \Phi_i \text{ for } i \geqslant 1} x_\alpha(u_\alpha)\Big) := \sum_{\alpha \in \Phi_1 \cup \Phi_2} u_\alpha e_\alpha.$$

Lemma 2.7 (cf. Kawanaka [17, §3.1])**.** *Let $u, v \in U_1$. Then the following hold.*

(a) *If* $u \in U_2$ *or* $v \in U_2$, *then* $f(uv) = f(u) + f(v)$.
(b) $f(uvu^{-1}v^{-1}) \equiv [f(u), f(v)] \bmod \mathfrak{g}(\geqslant 3)$.
(c) $f(F(u)) = F(f(u))$.

Proof. This is a rather straightforward application of Chevalley's commutator relations. For (b), we use the fact that $[e_\alpha, e_\beta] = C_{\alpha,\beta,1,1}e_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$; see 2.2. For (c), we use the formulae in 2.3. We omit further details. □

2.8. The general idea for defining GGGRs corresponding to a fixed $d \in \Delta$ is as follows. (In the following discussion we avoid any reference to the characteristic of k.) First of all, we assume that d is invariant under the permutation $\tau\colon \Phi \to \Phi$ induced by F. Consequently, all the subgroups P, U_i ($i \geqslant 1$) of G are F-stable and all the subspaces $\mathfrak{g}(i)$ ($i \geqslant 0$) are F-stable. Let us fix a non-trivial character $\psi\colon \mathbb{F}_q^+ \to \mathbb{C}^\times$.

Let us also consider a linear map $\lambda\colon \mathfrak{g}(2) \to k$ defined over $\mathbb{F}_q$, that is, we have $\lambda(F(y)) = \lambda(y)^q$ for all $y \in \mathfrak{g}(2)$. Then Lemma 2.7(a) shows that $U_2 \to k^+$, $u \mapsto \lambda(f(u))$, is a group homomorphism and so, by Lemma 2.7(c), we also obtain a group homomorphism

$$\chi_\lambda\colon U_2^F \to \mathbb{C}^\times, \quad u \mapsto \psi\big(\lambda(f(u))\big).$$

We shall require that λ is in "sufficiently general position" (where this term will have to be further specified; see Definition 3.4 below). Let us assume that this is the case. If $\mathfrak{g}(1) = \{0\}$, then the GGGR corresponding to d, λ will simply be given by the induced representation

$$\Gamma_{d,\lambda} := \mathrm{Ind}_{U_2^F}^{G^F}\big(\chi_\lambda\big).$$

Now consider the case where $\mathfrak{g}(1) \neq \{0\}$. Since $[\mathfrak{g}(1), \mathfrak{g}(1)] \subseteq \mathfrak{g}(2)$, we obtain a well-defined alternating bilinear form

$$\sigma_\lambda\colon \mathfrak{g}(1) \times \mathfrak{g}(1) \to k, \quad (y, z) \mapsto \lambda\big([y, z]\big).$$

Assume also that the radical of this bilinear form is zero. Then we choose an F-stable Lagrangian subspace in $\mathfrak{g}(1)$ and pull back this subspace to an F-stable subgroup $U_{1.5} \subseteq U_1$ via the map f. Using Lemma 2.7(b) we see that $\ker(\chi_\lambda)$ is normal in $U_{1.5}^F$ and $U_{1.5}^F/\ker(\chi_\lambda)$ is an abelian p-group. (See also the proof of [17, Lem. 3.1.9].) So we can extend χ_λ to a character $\tilde{\chi}_\lambda\colon U_{1.5}^F \to \mathbb{C}^\times$. In this case, the GGGR corresponding to d, λ will be given by the induced representation

$$\Gamma_{d,\lambda} := \mathrm{Ind}_{U_{1.5}^F}^{G^F}\big(\tilde{\chi}_\lambda\big).$$

Note that $[U_1^F : U_{1.5}^F] = [U_{1.5}^F : U_2^F]$. Furthermore, it turns out that

$$\mathrm{Ind}_{U_2^F}^{G^F}\big(\chi_\lambda\big) = [U_1^F : U_{1.5}^F] \cdot \Gamma_{d,\lambda},$$

which shows that the definition of $\Gamma_{d,\lambda}$ does not depend on the choice of the Lagrangian subspace or the extension $\tilde{\chi}_\lambda$ of χ_λ (cf. [16, 1.3.6], [17, 3.1.12]).

In Kawanaka's set-up [16, §1.2], [17, §3.1], the above assumption on the radical of σ_λ is always satisfied. (See also Remark 3.5 below.) Our plan for the definition of GGGRs in bad characteristic is to follow the above general procedure, but we have to find out in which situations this still makes sense at all. The following two examples show that there is a serious issue concerning the radical of σ_λ when $\mathfrak{g}(1) \neq \{0\}$.

Example 2.9. Let $G = \mathrm{Sp}_4(k)$. We have $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$ where $\Pi = \{\alpha, \beta\}$; here, α is a short simple root and β is a long simple root. By [2, p. 394], there are 4 weighted Dynkin diagrams $d \in \Delta$, where:

$$(d(\alpha), d(\beta)) \in \{(0,0), (1,0), (0,2), (2,2)\}.$$

Let $d_0 \in \Delta$ be such that $d_0(\alpha) = 1$ and $d_0(\beta) = 0$. Then $\mathbf{b}_{d_0} = 2$ and

$$\mathfrak{g}(1) = \langle e_\alpha, e_{\alpha+\beta} \rangle_k, \quad \mathfrak{g}(2) = \langle e_{2\alpha+\beta} \rangle_k.$$

We have $[e_\alpha, e_{\alpha+\beta}] = \pm 2 e_{2\alpha+\beta}$. Let $\lambda\colon \mathfrak{g}(2) \to k$ be a linear map. If $p \neq 2$, then the radical of the alternating form σ_λ is zero whenever $\lambda(e_{2\alpha+\beta}) \neq 0$. Now assume that $p = 2$. Then $[e_\alpha, e_{\alpha+\beta}] = 0$ and so σ_λ is identically zero for any λ. The commutator relations in 2.2 also show that the subgroup $U_1 = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle$ associated with d_0 is abelian. In this case, it is not clear to us at all how one should proceed in order to define a GGGR associated with d_0.

2.10. To simplify the notation for matrices, we define $\mathrm{antidiag}(x_1, \ldots, x_n)$ to be the $n \times n$-matrix with entry x_i at position $(i, n+1-i)$ for $1 \leqslant i \leqslant n$, and entry 0 otherwise. Thus, for example,

$$\mathrm{antidiag}(x_1, x_2, x_3) = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & x_2 & 0 \\ x_3 & 0 & 0 \end{pmatrix}.$$

Example 2.11. Let $G = \mathsf{G}_2(k)$. We have

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$$

where $\Pi = \{\alpha, \beta\}$; here, α is a short simple root and β is a long simple root. By [2, p. 401], there are 5 weighted Dynkin diagrams $d \in \Delta$, where:

$$(d(\alpha), d(\beta)) \in \{(0,0), (0,1), (1,0), (0,2), (2,2)\}.$$

(a) Let $d \in \Delta$ be such that $d(\alpha) = 1$ and $d(\beta) = 0$. Then $\mathbf{b}_d = 2$ and

$$\mathfrak{g}(1) = \langle e_\alpha, e_{\alpha+\beta} \rangle_k, \quad \mathfrak{g}(2) = \langle e_{2\alpha+\beta} \rangle_k.$$

We have $[e_\alpha, e_{\alpha+\beta}] = \pm 2 e_{2\alpha+\beta}$. Let $\lambda\colon \mathfrak{g}(2) \to k$ be a linear map. If $p = 2$, then σ_λ is identically zero for any λ. If $p \neq 2$, then the radical of σ_λ is zero whenever $\lambda(e_{2\alpha+\beta}) \neq 0$.

(b) Let $d \in \Delta$ be such that $d(\alpha) = 0$ and $d(\beta) = 1$. Then $\mathbf{b}_d = 3$ and

$$\mathfrak{g}(1) = \langle e_\beta, e_{\alpha+\beta}, e_{2\alpha+\beta}, e_{3\alpha+\beta} \rangle_k, \quad \mathfrak{g}(2) = \langle e_{3\alpha+2\beta} \rangle_k.$$

Here, the only non-zero Lie brackets are $[e_\beta, e_{3\alpha+\beta}] = \pm e_{3\alpha+2\beta}$, $[e_{\alpha+\beta}, e_{2\alpha+\beta}] = \pm 3 e_{3\alpha+2\beta}$. Hence, if $\lambda\colon \mathfrak{g}(2) \to k$ is any linear map, then the Gram matrix of σ_λ with respect to the above basis of $\mathfrak{g}(1)$ is given by

$$\pm x_1 \cdot \mathrm{antidiag}(1, 3, -3, -1) \quad \text{where} \quad x_1 := \lambda(e_{3\alpha+2\beta}).$$

The determinant of this matrix is $9x_1^4$. Hence, if $p = 3$, then there is no λ such that the radical of σ_λ is zero. On the other hand, if $p \neq 3$, then the radical of σ_λ is zero whenever $x_1 = \lambda(e_{3\alpha+2\beta}) \neq 0$.

3. Nilpotent and unipotent pieces

We keep the set-up of the previous section. Given a weighted Dynkin diagram $d \in \Delta$, our main task is to find suitable conditions under which a linear map $\lambda\colon \mathfrak{g}(2) \to k$ may be considered to be in "sufficiently general position" (cf. 2.8). For this purpose, we use Lusztig's framework [25]–[29] for dealing with unipotent elements in G and nilpotent elements in $\mathfrak{g}$ when p (the characteristic of k) is small.

3.1. Let $\mathscr{U}$ be the variety of unipotent elements of G. In [25, §1], Lusztig introduced a natural partition

$$\mathscr{U} = \coprod_{d \in \Delta} H_d$$

where each H_d is an irreducible locally closed subset stable under conjugation by G. A general, case-free proof for the existence of this partition was given by Clarke–Premet [3, Thm. 1.4]. The sets $\{H_d \mid d \in \Delta\}$ are called the *unipotent pieces* of G. In each such piece H_d, there is a unique unipotent class C_d of G such that C_d is open dense in H_d. If p is a good prime for G, then $H_d = C_d$. In general, H_d is the union of C_d and a finite number of unipotent classes of dimension strictly smaller than $\dim C_d$. We will say that the unipotent classes $\{C_d \mid d \in \Delta\}$ "come from characteristic 0". (Alternatively, the latter notion can be defined using the Springer correspondence, see [25, 1.3, 1.4], or the results of Spaltenstein [33]; see also [14, §2]. All these definitions agree as can be checked using the explicit knowledge of the unipotent classes and the Springer correspondence in all cases.)

3.2. We recall some further notation and some results from [29, §2]. There is a coadjoint action of G on the dual vector space $\mathfrak{g}^*$ which we denote by $g.\xi$ for $g \in G$ and $\xi \in \mathfrak{g}^*$; thus, $(g.\xi)(y) = \xi(\mathrm{Ad}(g^{-1})(y))$ for all $y \in \mathfrak{g}$. We denote by G_ξ the stabilizer of $\xi \in \mathfrak{g}^*$ under this action. As in [29], an element $\xi \in \mathfrak{g}^*$ is called nilpotent if there exists some $g \in G$ such that the Lie algebra of the Borel subgroup $B \subseteq G$ is contained in $\mathrm{Ann}(g.\xi)$. Let

$$\mathscr{N}_{\mathfrak{g}^*} := \{\xi \in \mathfrak{g}^* \mid \xi \text{ is nilpotent}\}.$$

For any $Y \subseteq \mathfrak{g}$, we denote $\mathrm{Ann}(Y) := \{\xi \in \mathfrak{g}^* \mid \xi(y) = 0 \text{ for all } y \in Y\}$. Let us fix a weighted Dynkin diagram $d \in \Delta$. As in 2.5, we have a corresponding grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$. In order to indicate the dependence on d, we shall now write $\mathfrak{g}_d(i) = \mathfrak{g}(i)$ for all $i \in \mathbb{Z}$; similarly, we write $P_d = P$ for the corresponding parabolic subgroup of G. Now, we also have a grading $\mathfrak{g}^* = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_d(j)^*$ where we set

$$\mathfrak{g}_d(j)^* := \mathrm{Ann}\Big(\bigoplus_{i \in \mathbb{Z}\colon\, i \neq -j} \mathfrak{g}_d(i)\Big) \quad \text{for any } j \in \mathbb{Z}.$$

We note that the subspace $\mathfrak{g}_d(\geqslant j)^* := \bigoplus_{j' \in \mathbb{Z}\colon\, j' \geqslant j} \mathfrak{g}_d(j')^*$ is stable under the coadjoint action of P_d. Let

$$\mathfrak{g}_d(2)^{*!} := \{\xi \in \mathfrak{g}_d(2)^* \mid G_\xi \subseteq P_d\}$$

and $\sigma_d^* := \mathfrak{g}_d(2)^{*!} + \mathfrak{g}_d(\geqslant 3)^* \subseteq \mathfrak{g}_d(\geqslant 2)^*$. Then σ_d^* is stable under the coadjoint action of P_d on $\mathfrak{g}_d(\geqslant 2)^*$. Finally, let $\hat{\sigma}_d^* \subseteq \mathfrak{g}^*$ be the union of the orbits of the elements in σ_d* under the coadjoint action of G. Then $\xi \mapsto \xi$ is a map

$$\Psi_{\mathfrak{g}^*}\colon \coprod_{d\in\Delta} \hat{\sigma}_d^* \to \mathscr{N}_{\mathfrak{g}^*}.$$

By [29, Thm. 2.2], the map $\Psi_{\mathfrak{g}^*}$ is a bijection if the adjoint group of G is a direct product of simple groups of types A, C and D. By the main result of [37], this also holds if there is a direct factor of type B. In the remarks just following [29, Thm. 2.2], Lusztig expresses the expectation that $\Psi_{\mathfrak{g}^*}$ is a bijection without any restriction on G.

3.3. In order to apply the above results to the situation in Section 2, we need a mechanism by which we can pass back and forth between the vector spaces $\mathfrak{g}_d(i)$ and $\mathfrak{g}_d(i)^*$. If there exists a G-equivariant vector space isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, then there is a canonical way of doing this, as explained in [29, 2.3]. However, such an isomorphism will not always exist. To remedy this situation, we follow Kawanaka [16, §1.2], [17, §3.1] and fix an $\mathbb{F}_q$-opposition automorphism $\mathfrak{g} \to \mathfrak{g}$, $y \mapsto y^\dagger$. This is a linear isomorphism, defined over $\mathbb{F}_q$, such that $\mathfrak{t}^\dagger = \mathfrak{t}$ and $e_\alpha^\dagger = \pm e_{-\alpha}$ for all $\alpha \in \Phi$. (See also [36, Lem. 5.2].) If $\xi \in \mathfrak{g}^*$, then we define $\xi^\dagger \in \mathfrak{g}^*$ by $\xi^\dagger(y) := \xi(y^\dagger)$ for $y \in \mathfrak{g}$.

Definition 3.4. Let $d \in \Delta$ be a weighted Dynkin diagram and consider the corresponding grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_d(i)$. Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be a linear map. We regard λ as an element of $\mathfrak{g}^*$ by setting λ equal to zero on $\mathfrak{g}_d(i)$ for all $i \neq 2$. We say that λ is in "sufficiently general position" if the following conditions hold.

(K1) We require that $\lambda^\dagger \in \mathfrak{g}_d(2)^{*!}$, that is, $G_{\lambda^\dagger} \subseteq P_d$; see 3.2, 3.3.
(K2) If $\mathfrak{g}_d(1) \neq \{0\}$, then we also require that the radical of the corresponding alternating form $\sigma_\lambda\colon \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \to k$ in 2.8 is zero.

Note that (K1), (K2) only refer to the algebraic group G, but not to the Frobenius map F. If (K1), (K2) hold and if λ is defined over $\mathbb{F}_q$, then we can follow the procedure in 2.8 and define the corresponding GGGR $\Gamma_{d,\lambda}$ of the finite group G^F.

Remark 3.5. Kawanaka's work [16], [17] fits into this setting as follows. Assume that p is a good prime for G and that there exists a non-degenerate, symmetric and G-invariant bilinear form $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to k$. We also need to make a certain technical assumption on the isogeny type of the derived subgroup of G. (For details see [36, 3.22], [31].) Let $d \in \Delta$. Then there is a dense open orbit under the adjoint action of the standard Levi factor of P_d on $\mathfrak{g}_d(2)$. Let e be an element of this orbit and define a linear map $\lambda_e\colon \mathfrak{g}_d(2) \to k$ as follows.

$$\lambda_e(y) := \kappa(e^\dagger, y) \quad \text{for } y \in \mathfrak{g}_d(2).$$

Then (K1), (K2) are satisfied for $\lambda = \lambda_e$; see [16, §1.2], [17, §3.1]. Furthermore, if d is invariant under the permutation of Φ induced by F, then e can be chosen such that (K1), (K2) hold and λ_e is defined over $\mathbb{F}_q$. For example, all this holds for $G = \mathrm{SL}_n(k)$ with no restriction on p; see [16, 1.2].

Remark 3.6. Let $d \in \Delta$. Then, by [3, Thm. 7.3 and Remark 1 (p. 665)], the subset $\mathfrak{g}_d(2)^{*!} \subseteq \mathfrak{g}_d(2)^*$ contains a dense open subset of $\mathfrak{g}_d(2)^*$ (denoted by $X^\triangle(\mathfrak{g}^*)$ in [3, 7.1]) and so $\mathfrak{g}_d(2)^{*!}$ itself is a dense subset of $\mathfrak{g}_d(2)^*$. Thus, there always exists a dense set of linear maps $\lambda\colon \mathfrak{g}_d(2) \to k$ such that condition (K1) in Definition 3.4 is satisfied.

As illustrated by the examples at the end of Section 2, the condition (K2) requires more attention.

Remark 3.7. Let $d \in \Delta$ and assume that $\mathfrak{g}_d(1) \neq \{0\}$. Let $\Phi_1 = \{\beta_1, \ldots, \beta_n\}$ and $\Phi_2 = \{\gamma_1, \ldots, \gamma_m\}$. Given a linear map $\lambda\colon \mathfrak{g}_d(2) \to k$, we denote by $\mathscr{G}_\lambda \in M_n(k)$ the Gram matrix of σ_λ with respect to the basis $\{e_{\beta_1}, \ldots, e_{\beta_n}\}$ of $\mathfrak{g}_d(1)$. The entries of $\mathscr{G}_\lambda$ are given as follows. We set $x_l := \lambda(e_{\gamma_l})$ for $1 \leqslant l \leqslant m$. For $1 \leqslant i, j \leqslant n$, we define an element $\nu_{ij} \in k$ as follows. If $\beta_i + \beta_j \notin \Phi$, then $\nu_{ij} := 0$. Otherwise, there is a unique $l(i,j) \in \{1, \ldots, m\}$ and some $\nu_{ij} \in k$ such that $[e_{\beta_i}, e_{\beta_j}] = \nu_{ij} e_{\gamma_{l(i,j)}}$. Then we have

$$(\mathscr{G}_\lambda)_{ij} = \sigma_\lambda(e_{\beta_i}, e_{\beta_j}) = \begin{cases} x_{l(i,j)}\nu_{ij} & \text{if } \beta_i + \beta_j \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

In order to work this out explicity, we may assume without loss of generality that G is semisimple (since $U_\alpha \subseteq G_{\text{der}}$ for all $\alpha \in \Phi$; see 2.1). But then, by [35, Rem. (p. 64)], the structure constants of the Lie algebra $\mathfrak{g}$ are obtained from those of a Chevalley basis of $\mathfrak{g}_0$ by reduction modulo p. Thus, we can explicitly determine the elements ν_{ij}, via a computation inside $\mathfrak{g}_0$. Using one of the two canonical Chevalley bases in [11, §5] (the two bases only differ by a global sign), one can even avoid the issue of choosing certain signs. Hence, there is a purely combinatorial algorithm for computing $\mathscr{G}_\lambda$, and this can be easily implemented in GAP [10]. In particular, we have:

(∗) Up to a global sign, the Gram matrix $\mathscr{G}_\lambda$ only depends on the root system Φ and the values $\lambda(e_\alpha)$ $(\alpha \in \Phi_2)$.

The radical of σ_λ is zero if and only if $\det(\mathscr{G}_\lambda) \neq 0$. Now we notice that this determinant is given by evaluating a certain m-variable polynomial at $x_1, \ldots, x_m$. In particular, we see that condition (K2) is an "open" condition: either there is no λ at all for which (K2) holds, or (K2) holds for a non-empty open set of linear maps $\lambda\colon \mathfrak{g}_d(2) \to k$. Combining this with the discussion concerning (K1) in Remark 3.6, we immediately obtain the following conclusion.

Corollary 3.8. *Let $d \in \Delta$ and assume that $\mathfrak{g}_d(1) \neq \{0\}$. Then either there is a non-empty open set of linear maps $\lambda\colon \mathfrak{g}_d(2) \to k$ in "sufficiently general position", or there is no such linear map at all.*

With these preparations, we now obtain our first example where bad primes exist but (K2) holds without any restriction on the field k.

Example 3.9. Let G be of type D_4. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha_1, \alpha_2, \alpha_4$ are all connected to α_3. By [2, pp. 396–397], there are 12 weighted Dynkin diagrams $d \in \Delta$. There are two of them with $\mathfrak{g}_d(1) \neq \{0\}$.

(a) Let $d(\alpha_1) = d(\alpha_2) = d(\alpha_4) = 0$ and $d(\alpha_3) = 1$. We have $\mathbf{b}_d = 7$ and

$$\begin{aligned} \mathfrak{g}_d(1) &= \langle e_{\alpha_3}, e_{\alpha_1+\alpha_3}, e_{\alpha_2+\alpha_3}, e_{\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3}, \\ &\qquad e_{\alpha_1+\alpha_3+\alpha_4}, e_{\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\rangle_k, \\ \mathfrak{g}_d(2) &= \langle e_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}\rangle_k. \end{aligned}$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. As explained in Remark 3.7, we can work out the Gram matrix $\mathscr{G}_\lambda$ of the alternating form σ_λ. It is given by

$$\mathscr{G}_\lambda = \pm\mathrm{antidiag}(-x_1, x_1, x_1, x_1, -x_1, -x_1, -x_1, x_1).$$

where we set $x_1 := \lambda(e_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4})$. If $x_1 \neq 0$, then $\det(\mathscr{G}_\lambda) \neq 0$ and so the radical of σ_λ is zero. Hence, condition (K2) is satisfied for such choices of λ, and this works for any field k.

(b) Let $d(\alpha_1) = d(\alpha_2) = d(\alpha_4) = 1$ and $d(\alpha_3) = 0$. We have $\mathbf{b}_d = 4$ and

$$\mathfrak{g}_d(1) = \langle e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_4}, e_{\alpha_1+\alpha_3}, e_{\alpha_2+\alpha_3}, e_{\alpha_3+\alpha_4}\rangle_k,$$
$$\mathfrak{g}_d(2) = \langle e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_3+\alpha_4}, e_{\alpha_2+\alpha_3+\alpha_4}\rangle_k.$$

Again, let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. As above, we now obtain

$$\mathscr{G}_\lambda = \pm\begin{pmatrix} 0 & 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & x_1 & 0 & x_3 \\ 0 & 0 & 0 & x_2 & x_3 & 0 \\ 0 & -x_1 & -x_2 & 0 & 0 & 0 \\ -x_1 & 0 & -x_3 & 0 & 0 & 0 \\ -x_2 & -x_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where $x_1 := \lambda(e_{\alpha_1+\alpha_2+\alpha_3})$, $x_2 := \lambda(e_{\alpha_1+\alpha_3+\alpha_4})$, $x_3 := \lambda(e_{\alpha_2+\alpha_3+\alpha_4})$. We compute $\det(\mathscr{G}_\lambda) = 4x_1^2x_2^2x_3^2$. Hence, if $p = 2$, then the radical of σ_λ will never be zero and so condition (K2) will never be satisfied. On the other hand, if $p \neq 2$, then the radical of σ_λ will be zero whenever $x_1x_2x_3 \neq 0$.

In the above examples, it was easy to compute the determinant of the Gram matrix $\mathscr{G}_\lambda$. However, we will encounter examples below where the computation of $\det(\mathscr{G}_\lambda)$ becomes a very serious issue.

4. Unipotent support

We are now looking for a unifying principle behind the various examples that we have seen so far. In Conjecture 4.4 below, we propose such a unification. First we need some preparations.

4.1. Let $\mathrm{Irr}(G^F)$ be the set of complex irreducible representations of G^F (up to isomorphism). Then there is a canonical map

$$\mathrm{Irr}(G^F) \to \{F\text{-stable unipotent classes of } G\}, \quad \rho \mapsto C_\rho,$$

defined in terms of the notion of “unipotent support”. To explain this, we need to introduce some notation. Let C be an F-stable unipotent conjugacy class of G. Then C^F is a union of conjugacy classes of G^F. Let $u_1, \ldots, u_r \in C^F$ be representatives of the classes of G^F contained in C^F. For $1 \leqslant i \leqslant r$ we set $A(u_i) := C_G(u_i)/C_G^\circ(u_i)$. Since $F(u_i) = u_i$, the Frobenius map F induces an automorphism of $A(u_i)$ which we denote by the same symbol. Let $A(u_i)^F$ be the group of fixed points under F. Then we set

$$\mathrm{AV}(\rho, C) := \sum_{1\leqslant i\leqslant r} |A(u_i) : A(u_i)^F|\,\mathrm{trace}\big(\rho(u_i)\big)$$

for any $\rho \in \mathrm{Irr}(G^F)$. Note that this does not depend on the choice of the representatives u_i. Now the desired map is obtained as follows. Let $\rho \in \mathrm{Irr}(G^F)$ and set $a_\rho := \max\{\dim C \mid \mathrm{AV}(\rho, C) \neq 0\}$ (where the maximum is taken over all F-stable unipotent classes C of G). By the main results of [14], [23], there is a unique C such that $\dim C = a_\rho$ and $\mathrm{AV}(\rho, C) \neq 0$. This C will be denoted by C_ρ and called the *unipotent support* of ρ.

By [14, Rem. 3.9], it is known that C_ρ comes from characteristic 0 (see 3.1) and, hence, equals C_{d_ρ} for a well-defined weighted Dynkin diagram $d_\rho \in \Delta$. We set

$$\Delta_{k,F}^\bullet := \{d_\rho \in \Delta \mid \rho \in \mathrm{Irr}(G^F)\}.$$

Thus, $\Delta_{k,F}^\bullet$ consists precisely of those weighted Dynkin diagrams for which the corresponding unipotent class of G occurs as the unipotent support of some irreducible representation of G^F. We also set

$$\Delta_k^\bullet := \bigcup_{n\geqslant 1} \Delta_{k,F^n}^\bullet.$$

Thus, $\mathfrak{C}^\bullet = \{C_d \mid d \in \Delta_k^\bullet\}$ is precisely the set of unipotent classes mentioned at the end of Section 1. We shall see below that $\Delta_k^\bullet$ and, hence, also $\mathfrak{C}^\bullet$ only depend on G but not on the choice of the particular Frobenius map F.

TABLE 1. The sets $\Delta_k^\bullet \setminus \Delta_{\mathrm{spec}}$ for G of exceptional type

G_2	$\mathbf{b}_d$	condition
A_1	3	$p \neq 3$
$\tilde{\mathsf{A}}_1$	2	$p \neq 2$

F_4	$\mathbf{b}_d$	condition
A_1	16	$p \neq 2$
$\mathsf{A}_2+\tilde{\mathsf{A}}_1$	7	$p \neq 2$
B_2	6	$p \neq 2$
$\tilde{\mathsf{A}}_2+\mathsf{A}_1$	6	$p \neq 3$
$\mathsf{C}_3(a_1)$	5	$p \neq 2$

E_6	$\mathbf{b}_d$	condition
$3\mathsf{A}_1$	16	$p \neq 2$
$2\mathsf{A}_2+\mathsf{A}_1$	9	$p \neq 3$
$\mathsf{A}_3+\mathsf{A}_1$	8	$p \neq 2$
A_5	4	$p \neq 2$

E_7	$\mathbf{b}_d$	condition
$(3\mathsf{A}_1)'$	31	$p \neq 2$
$4\mathsf{A}_1$	28	$p \neq 2$
$2\mathsf{A}_2+\mathsf{A}_1$	18	$p \neq 3$
$(\mathsf{A}_3+\mathsf{A}_1)'$	17	$p \neq 2$
$\mathsf{A}_3+2\mathsf{A}_1$	16	$p \neq 2$
$\mathsf{D}_4+\mathsf{A}_1$	12	$p \neq 2$
A_5'	9	$p \neq 2$
$\mathsf{A}_5+\mathsf{A}_1$	9	$p \neq 3$
$\mathsf{D}_6(a_2)$	8	$p \neq 2$
D_6	4	$p \neq 2$

E_8	$\mathbf{b}_d$	condition
$3\mathsf{A}_1$	64	$p \neq 2$
$4\mathsf{A}_1$	56	$p \neq 2$
$\mathsf{A}_2+3\mathsf{A}_1$	43	$p \neq 2$
$2\mathsf{A}_2+\mathsf{A}_1$	39	$p \neq 3$
$\mathsf{A}_3+\mathsf{A}_1$	38	$p \neq 2$
$2\mathsf{A}_2+2\mathsf{A}_1$	36	$p \neq 3$
$\mathsf{A}_3+2\mathsf{A}_1$	34	$p \neq 2$
$\mathsf{A}_3+\mathsf{A}_2+\mathsf{A}_1$	29	$p \neq 2$
$\mathsf{D}_4+\mathsf{A}_1$	28	$p \neq 2$
$2\mathsf{A}_3$	26	$p \neq 2$
A_5	22	$p \neq 2$
$\mathsf{A}_4+\mathsf{A}_3$	20	$p \neq 5$
$\mathsf{A}_5+\mathsf{A}_1$	19	$p \neq 2, 3$
$\mathsf{D}_5(a_1)+\mathsf{A}_2$	19	$p \neq 2$
$\mathsf{D}_6(a_2)$	18	$p \neq 2$
$\mathsf{E}_6(a_3)+\mathsf{A}_1$	18	$p \neq 3$
$\mathsf{E}_7(a_5)$	17	$p \neq 2$
$\mathsf{D}_5+\mathsf{A}_1$	16	$p \neq 2$
D_6	12	$p \neq 2$
A_7	11	$p \neq 2$
$\mathsf{E}_6+\mathsf{A}_1$	9	$p \neq 3$
$\mathsf{E}_7(a_2)$	8	$p \neq 2$
D_7	7	$p \neq 2$
E_7	4	$p \neq 2$

(Notation from [2, §13.1])

Remark 4.2. Recall from Lusztig [21], [24] the notion of *special unipotent classes*. The precise definition of these classes is not elementary; it involves the Springer correspondence and the notion of *special representations* of the Weyl group of G. By [14, Prop. 4.2], an F-stable unipotent class is special if and only if it is the unipotent support of some unipotent representation of G^F. Thus, we conclude that special unipotent classes come from characteristic 0 and we have:

(a) $\Delta_{\text{spec}} \subseteq \Delta_k^\bullet$, where Δ_{spec} denotes the set of all $d \in \Delta$ such that the corresponding unipotent class C_d of G is special.

Explicit descriptions of the sets Δ_{spec} are contained in the tables in [2, §13.4]. Using the above concepts, one can give an alternative description of the map $\rho \mapsto C_\rho$; see [22, 13.4], [23, 10.9], [27, §1], [14, §3.C]. This alternative description immediately yields that $\Delta_{k,F}^\bullet \subseteq \Delta_{k,F^n}^\bullet$ for any integer $n \geqslant 1$. Consequently, we have:

(b) The set $\Delta_k^\bullet$ only depends on G but not on the Frobenius map F.

(Note also that, if $F_1 \colon G \to G$ is another Frobenius map, then there always exist integers $n, n_1 \geqslant 1$ such that $F_1^{n_1} = F^n$.) Furthermore, that alternative description allows one to compute $\Delta_{k,F}^\bullet$ explicitly, without knowing any character values of G^F. In particular, this yields the following statement:

(c) If p is a good prime for G, then $\Delta_k^\bullet = \Delta$.

This was first stated (in terms of the alternative description of $\rho \mapsto C_\rho$ and for p large) as a conjecture in [21, §9]; see also [22, 13.4]. A full proof eventually appeared in [27, Thm. 1.5].

Proposition 4.3. *Assume that G is simple and p is a bad prime for G. If G is of classical type $\mathsf{B}_n, \mathsf{C}_n$ or D_n, then $\Delta_k^\bullet = \Delta_{\text{spec}}$. If G is of exceptional type G_2, F_4, E_6, E_7 or E_8, then the sets $\Delta_k^\bullet \setminus \Delta_{\text{spec}}$ are specified in Table* 1.

(In Table 1, we list all $d \in \Delta \setminus \Delta_{\text{spec}}$; for each such d, the last column gives the condition on p such that $d \in \Delta_k^\bullet$.)

Proof. This follows by analogous methods as in [27]. The special feature of the case where G is of classical type and $p = 2$ is the fact that then the centraliser of a semisimple element is a Levi subgroup of some parabolic subgroup (see, e.g., [1, §4]). If G is of exeptional type, then one uses explicit computations completely analogous to those in [27, §7]; here, it is convenient to use Lübeck's tables [20] concerning possible centralisers of semisimple elements in these cases. We omit further details. □

Conjecture 4.4. *Let $d \in \Delta_k^\bullet$ be invariant under the permutation $\tau \colon \Phi \to \Phi$ induced by F. Then there exist linear maps $\lambda \colon \mathfrak{g}_d(2) \to k$ which are defined over $\mathbb{F}_q$ and are in "sufficiently general position"* (*see Definition* 3.4)*. Hence, following the general procedure described in 2.8, we can define the corresponding* GGGR*s $\Gamma_{d,\lambda}$ of G^F.*

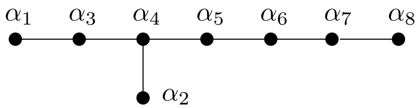
By Remarks 3.5, 3.7(∗) and 4.2(c), the conjecture holds for all $d \in \Delta_k^\bullet = \Delta$ if p is a good prime for G. In particular, it holds when G is of type A_n. We will now discuss a number of examples supporting the conjecture in cases where p is a bad prime. As already explained in the previous section, the main issue is the validity of condition (K2) in Definition 3.4.

Example 4.5. (a) Let $G = \mathrm{Sp}_4(k)$ and assume that k has characteristic 2. By Proposition 4.3, or by inspection of the known character table of G^F (see [7]), we note that the unipotent class corresponding to the weighted Dynkin diagram d_0 in Example 2.9 is not the unipotent support of any irreducible character of G^F. Thus, $d_0 \notin \Delta_k^\bullet$ and so this critical case does not enter in the range of validity of Conjecture 4.4. In fact, $\Delta_k^\bullet = \Delta \setminus \{d_0\}$ if $p = 2$.

(b) The situation is similar for G of type G_2. Consider the two weighted Dynkin diagrams with $\mathfrak{g}_d(1) \neq \{0\}$ in Example 2.11. By Table 1, or by inspection of the known character table of G^F (see [9] for $p = 2$ and [8] for $p = 3$), we see that $\Delta_k^\bullet = \Delta \setminus \{d\}$ where d is as in Example 2.11(a) if $p = 2$, and d is as in Example 2.11(b) if $p = 3$.

Example 4.6. Let again G be of type D_4 and return to the discussion in Example 3.9. The special unipotent classes are explicitly described in [2, p. 439]; there is only one class which is not special (it corresponds to elements with Jordan blocks of sizes $3, 2, 2, 1$), and this is precisely the one considered in Example 3.9(b). But, by Proposition 4.3, the corresponding weighted Dynkin diagram does not belong to $\Delta_k^\bullet$ if $p = 2$.

Example 4.7. Let G be of type E_8 with diagram



Let $d_0 \in \Delta$ correspond to the class denoted $\mathsf{A}_4{+}\mathsf{A}_3$ in Table 1. We have $d_0(\alpha_4) = d_0(\alpha_7) = 1$ and $d_0(\alpha_i) = 0$ for $i \neq 4, 7$; see [2, p. 406]. By Table 1, we have $\Delta_k^\bullet = \Delta \setminus \{d_0\}$ if $p = 5$; furthermore, $d_0 \in \Delta_k^\bullet$ if $p \neq 5$. Let $\lambda\colon \mathfrak{g}_{d_0}(2) \to k$ be a linear map and consider the Gram matrix $\mathscr{G}_\lambda$ of the alternating form σ_λ. We claim:

(a) If $p \neq 5$, then $\det(\mathscr{G}_\lambda) \neq 0$ for some $\lambda\colon \mathfrak{g}_{d_0}(2) \to k$.
(b) If $p = 5$, then $\det(\mathscr{G}_\lambda) = 0$ for all $\lambda\colon \mathfrak{g}_{d_0}(2) \to k$.

First, we find that $\dim \mathfrak{g}_{d_0}(1) = 24$ and $\dim \mathfrak{g}_{d_0}(2) = 21$. As explained in Remark 3.7, we then explicitly work out $\mathscr{G}_\lambda$. We have

$$\mathscr{G}_\lambda = \big(f_{ij}(x_1, \ldots, x_{21})\big)_{1 \leqslant i,j \leqslant 24}$$

where f_{ij} are certain polynomials with integer coefficients in 21 indeterminates. In order to verify (a), we argue as follows. If $p > 5$, then (a) holds by Remark 3.5. If $p = 2, 3$, then we simply run through all vectors of values $(x_1, \ldots, x_{21}) \in \{0, 1\}^{21}$ (starting with the vector $1, 1, \ldots, 1$ and then increasing step by step the number of zeroes) until we find one such that $\det(\mathscr{G}_\lambda) \neq 0$. It turns out that this search is successful just after four steps. The verification of (b) is much harder. Let $p = 5$ and denote by $\bar{f}_{ij}$ the reduction of f_{ij} modulo p. Then we need to check that $\det(\bar{f}_{ij}) = 0$. It seems to be practically impossible to compute such a determinant directly, or even just the rank. (Using special values of the x_i as above, one quickly sees that the rank of $(\bar{f}_{ij})$ is at least 22.) Now, since $\mathscr{G}_\lambda$ is anti-symmetric, we can use the fact that the desired determinant is given by $\mathrm{Pf}(\bar{f}_{ij})^2$, where $\mathrm{Pf}(\bar{f}_{ij})$ denotes the Pfaffian of the matrix $(\bar{f}_{ij})$; see, for example, [5], [19]. (I am indebted to Ulrich Thiel for pointing this out to me.) A simple recursive algorithm (via row expansion, as in [5, 1.5]) is sufficient to compute $\mathrm{Pf}(\bar{f}_{ij}) = 0$ in this case and yields (b). (Over $\mathbb{Z}$, the Pfaffian of (f_{ij}) is a non-zero polynomial which is a linear combination of 1386 monomials in 21 indeterminates, where all coefficients are divisible by 5.)

The class $\mathsf{A}_4{+}\mathsf{A}_3$ in type E_8 also plays a special role in [32, §4.2]. (I thank Alexander Premet for pointing this out to me.)

Example 4.8. Let again G be of type E_8, with diagram as above. Let $d_1 \in \Delta$ correspond to the class denoted $\mathsf{A}_5{+}\mathsf{A}_1$ in Table 1. We have $d_1(\alpha_1) = d_1(\alpha_4) = d_1(\alpha_8) = 1$ and $d_1(\alpha_i) = 0$ for $i \neq 1, 4, 8$; see [2, p. 406]. By Table 1, we have $d_1 \in \Delta_k^\bullet$ if $p \neq 2, 3$, and $d_1 \notin \Delta_k^\bullet$ otherwise. Let $\lambda\colon \mathfrak{g}_{d_1}(2) \to k$ be a linear map and consider the Gram matrix $\mathscr{G}_\lambda$ of the alternating form σ_λ. Here, we find that $\dim \mathfrak{g}_{d_1}(1) = 22$ and $\dim \mathfrak{g}_{d_1}(2) = 18$. Using computations as in the previous example, we obtain:

(a) If $p \neq 2, 3$, then $\det(\mathscr{G}_\lambda) \neq 0$ for some $\lambda\colon \mathfrak{g}_{d_1}(2) \to k$.
(b) If $p \in \{2, 3\}$, then $\det(\mathscr{G}_\lambda) = 0$ for all $\lambda\colon \mathfrak{g}_{d_1}(2) \to k$.

(In (b), there are λ such that $\mathscr{G}_\lambda$ has rank 20.)

The examples suggest the following characterisation of the set $\Delta_k^\bullet$.

Conjecture 4.9. *Let $d \in \Delta$. Then $d \in \Delta_k^\bullet$ if and only if either $\mathfrak{g}_d(1) = \{0\}$, or there exists a linear map $\lambda\colon \mathfrak{g}_d(2) \to k$ such that the radical of σ_λ is zero.*

Finally, we state the following conjecture concerning special unipotent classes. As in 2.4, let G_0 be a connected reductive algebraic group over $\mathbb{C}$ of the same type as G; let $\mathfrak{g}_0$ be its Lie algebra. For $d \in \Delta$ and $i = 1, 2$, we set

$$\mathfrak{g}_{\mathbb{Z},d}(i) := \langle e_\alpha \mid d(\alpha) = i \rangle_{\mathbb{Z}} \subseteq \mathfrak{g}_0.$$

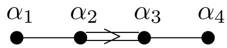
As in 2.8, given a homomorphism $\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(2) \to \mathbb{Z}$, we obtain an alternating form $\sigma_\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \to \mathbb{Z}$ and we may consider its Gram matrix with respect to the $\mathbb{Z}$-basis $\{e_\alpha \mid \alpha \in \Phi_1\}$ of $\mathfrak{g}_{\mathbb{Z},d}(1)$. If this Gram matrix has determinant ± 1, then we say that σ_λ is non-degenerate over $\mathbb{Z}$.

Conjecture 4.10. *With the above notation, let $d \in \Delta$. Then we have $d \in \Delta_{\mathrm{spec}}$ if and only if either $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$, or there exists a homomorphism $\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(2) \to \mathbb{Z}$ such that σ_λ is non-degenerate over $\mathbb{Z}$.*

Note that, if $\mathfrak{g}_d(1) = \{0\}$, then we certainly have $d \in \Delta_{\mathrm{spec}}$. (This easily follows from [30, Prop. 1.9(b)].) Hence, in order to verify the above conjectures for a given example, it is sufficient to consider the cases where $\mathfrak{g}_d(1) \neq \{0\}$. This will be further discussed in the following section.

5. A worked example: type F_4

In this section, we work out in detail the example where G is of type F_4. We believe that the results of our computations are strong evidence for the truth of Conjectures 4.9 and 4.10; the discussion of the various cases will also provide a good illustration of the computational issues involved. So, from now until 5.8, assume that G is of type F_4. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a set of simple roots such that the Dynkin diagram of G looks as follows:



By [2, p. 401], there are 16 weighted Dynkin diagrams in Δ; together with some additional information, these are printed in Table 2. (The entries in the last column are determined by Remark 4.2(a) and Table 1.)

TABLE 2. Unipotent classes in type F_4

Name	$d \in \Delta$	$\mathbf{b}_d$	special?	condition $d \in \Delta_k^\bullet$
1	$0 - 0 \Rightarrow 0 - 0$	24	yes	–
A_1	$1 - 0 \Rightarrow 0 - 0$	16	no	$p \neq 2$
$\tilde{\mathsf{A}}_1$	$0 - 0 \Rightarrow 0 - 1$	13	yes	–
$\mathsf{A}_1{+}\tilde{\mathsf{A}}_1$	$0 - 1 \Rightarrow 0 - 0$	10	yes	–
A_2	$2 - 0 \Rightarrow 0 - 0$	9	yes	–
$\tilde{\mathsf{A}}_2$	$0 - 0 \Rightarrow 0 - 2$	9	yes	–
$\mathsf{A}_2{+}\tilde{\mathsf{A}}_1$	$0 - 0 \Rightarrow 1 - 0$	7	no	$p \neq 2$
B_2	$2 - 0 \Rightarrow 0 - 1$	6	no	$p \neq 2$
$\tilde{\mathsf{A}}_2{+}\mathsf{A}_1$	$0 - 1 \Rightarrow 0 - 1$	6	no	$p \neq 3$
$\mathsf{C}_3(a_1)$	$1 - 0 \Rightarrow 1 - 0$	5	no	$p \neq 2$
$\mathsf{F}_4(a_3)$	$0 - 2 \Rightarrow 0 - 0$	4	yes	–
B_3	$2 - 2 \Rightarrow 0 - 0$	3	yes	–
C_3	$1 - 0 \Rightarrow 1 - 2$	3	yes	–
$\mathsf{F}_4(a_2)$	$0 - 2 \Rightarrow 0 - 2$	2	yes	–
$\mathsf{F}_4(a_1)$	$2 - 2 \Rightarrow 0 - 2$	1	yes	–
F_4	$2 - 2 \Rightarrow 2 - 2$	0	yes	–

(Notation from [2, p. 401])

There are eight weighted Dynkin diagrams which satisfy $\mathfrak{g}_d(1) \neq \{0\}$. We now consider these eight cases in detail, where we just focus on the validity of condition (K2) in Definition 3.4. (In particular, the Frobenius map $F\colon G \to G$ will not play a role in this section.)

5.1. Let $d(\alpha_1) = 1$, $d(\alpha_2) = d(\alpha_3) = d(\alpha_4) = 0$. We have $\mathbf{b}_d = 16$ and

$$\begin{aligned} \mathfrak{g}_d(1) = \langle & e_{1000}, e_{1100}, e_{1110}, e_{1120}, e_{1111}, e_{1220}, e_{1121}, \\ & e_{1221}, e_{1122}, e_{1231}, e_{1222}, e_{1232}, e_{1242}, e_{1342}\rangle_k, \\ \mathfrak{g}_d(2) = \langle & e_{2342}\rangle_k, \end{aligned}$$

where, for example, 1342 stands for the root $\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. As in Example 3.9, we work out the corresponding Gram matrix $\mathscr{G}_\lambda$, where we set $x_1 := \lambda(e_{2342})$. It is given by

$$\mathscr{G}_\lambda = \pm x_1 \cdot \mathrm{antidiag}(1, -1, 2, -1, -2, 1, 2, -2, -1, 2, 1, -2, 1, -1).$$

We have $\det(\mathscr{G}_\lambda) = 64x_1^{14}$. Hence, if $p = 2$, then the radical of σ_λ is not zero. If $p \neq 2$, then the radical is zero whenever $x_1 \neq 0$.

5.2. Let $d(\alpha_1) = d(\alpha_2) = d(\alpha_3) = 0$, $d(\alpha_4) = 1$. We have $\mathbf{b}_d = 13$ and

$$\begin{aligned} \mathfrak{g}_d(1) &= \langle e_{0001}, e_{0011}, e_{0111}, e_{1111}, e_{0121}, e_{1121}, e_{1221}, e_{1231} \rangle_k, \\ \mathfrak{g}_d(2) &= \langle e_{0122}, e_{1122}, e_{1222}, e_{1232}, e_{1242}, e_{1342}, e_{2342} \rangle_k, \end{aligned}$$

where we use the same notational conventions as above. Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let $x_1, \dots, x_8$ be the values of λ on the 8 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 0 & 0 & 0 & -2x_1 & -2x_2 & -2x_3 & -x_4 \\ 0 & 0 & 2x_1 & 2x_2 & 0 & 0 & -x_4 & -2x_5 \\ 0 & -2x_1 & 0 & 2x_3 & 0 & x_4 & 0 & -2x_6 \\ 0 & -2x_2 & -2x_3 & 0 & -x_4 & 0 & 0 & -2x_7 \\ 2x_1 & 0 & 0 & x_4 & 0 & 2x_5 & 2x_6 & 0 \\ 2x_2 & 0 & -x_4 & 0 & -2x_5 & 0 & 2x_7 & 0 \\ 2x_3 & x_4 & 0 & 0 & -2x_6 & -2x_7 & 0 & 0 \\ x_4 & 2x_5 & 2x_6 & 2x_7 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In principle, we could work out $\det(\mathscr{G}_\lambda)$ and then try to find out for which values of $x_1, \dots, x_8$ it is non-zero. However, this determinant is already quite complicated; it is a linear combination of 34 monomials in $x_1, \dots, x_8$. But we can just notice that, if we set $x_4 := 1$ and $x_i := 0$ for all $i \neq 4$, then $\det(\mathscr{G}_\lambda) = 1$. So the radical of σ_λ will be zero for this choice of λ, and this works for any field k.

5.3. Let $d(\alpha_1) = d(\alpha_3) = d(\alpha_4) = 0$, $d(\alpha_2) = 1$. We have $\mathbf{b}_d = 10$ and

$$\begin{aligned} \mathfrak{g}_d(1) = \langle & e_{0100}, e_{1100}, e_{0110}, e_{1110}, e_{0120}, e_{0111}, \\ & e_{1120}, e_{1111}, e_{0121}, e_{1121}, e_{0122}, e_{1122} \rangle_k, \\ \mathfrak{g}_d(2) = \langle & e_{1220}, e_{1221}, e_{1231}, e_{1222}, e_{1232}, e_{1242} \rangle_k. \end{aligned}$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map, and denote by $x_1, \dots, x_6$ the values of λ on the 6 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_2 & 0 & -x_4 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_2 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 2x_1 & 0 & 0 & 0 & x_2 & 0 & -x_3 & 0 & -x_5 \\ 0 & 0 & -2x_1 & 0 & 0 & -x_2 & 0 & 0 & x_3 & 0 & x_5 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 & -x_6 \\ 0 & 0 & 0 & x_2 & 0 & 0 & x_3 & 2x_4 & 0 & x_5 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 & x_6 & 0 \\ 0 & 0 & -x_2 & 0 & -x_3 & -2x_4 & 0 & 0 & -x_5 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -x_3 & 0 & 0 & 0 & x_5 & 0 & 2x_6 & 0 & 0 \\ x_2 & 0 & x_3 & 0 & 0 & -x_5 & 0 & 0 & -2x_6 & 0 & 0 & 0 \\ 0 & -x_4 & 0 & -x_5 & 0 & 0 & -x_6 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & x_5 & 0 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here we notice that, if we set $x_3 := 1$, $x_4 := 1$ and $x_i := 0$ for $i \neq 3, 4$, then $\det(\mathscr{G}_\lambda) = 1$. Hence, the radical of σ_λ is zero for this choice of λ, and this works for any field k.

5.4. Let $d(\alpha_1) = d(\alpha_2) = d(\alpha_4) = 0$, $d(\alpha_3) = 1$. We have $\mathbf{b}_d = 7$ and

$$\mathfrak{g}_d(1) = \langle e_{0010}, e_{0110}, e_{0011}, e_{1110}, e_{0111}, e_{1111}\rangle_k,$$
$$\mathfrak{g}_d(1) = \langle e_{0120}, e_{1120}, e_{0121}, e_{1220}, e_{1121}, e_{0122}, e_{1221}, e_{1122}, e_{1222}\rangle_k.$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let $x_1, \ldots, x_9$ be the values of λ on the 9 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 2x_1 & 0 & 2x_2 & x_3 & x_5 \\ -2x_1 & 0 & -x_3 & 2x_4 & 0 & x_7 \\ 0 & x_3 & 0 & x_5 & 2x_6 & 2x_8 \\ -2x_2 & -2x_4 & -x_5 & 0 & -x_7 & 0 \\ -x_3 & 0 & -2x_6 & x_7 & 0 & 2x_9 \\ -x_5 & -x_7 & -2x_8 & 0 & -2x_9 & 0 \end{pmatrix}.$$

We have $\det(\mathscr{G}_\lambda) = 16(x_1x_5x_9 - x_1x_7x_8 - x_2x_3x_9 + x_2x_6x_7 + x_3x_4x_8 - x_4x_5x_6)^2$. So, if $p = 2$, then $\det(\mathscr{G}_\lambda) = 0$ (for any λ). If $p \neq 2$, then we notice that $\det(\mathscr{G}_\lambda) = 16$ for $x_4 := 1$, $x_5 := 1$, $x_6 := 1$ and $x_i := 0$ for $i \neq 4, 5, 6$. Hence, the radical of σ_λ is zero for this choice of λ.

5.5. Let $d(\alpha_1) = 2$, $d(\alpha_2) = d(\alpha_3) = 0$, $d(\alpha_4) = 1$. We have $\mathbf{b}_d = 6$ and

$$\mathfrak{g}_d(1) = \langle e_{0001}, e_{0011}, e_{0111}, e_{0121}\rangle_k,$$
$$\mathfrak{g}_d(2) = \langle e_{1000}, e_{1100}, e_{1110}, e_{1120}, e_{1220}, e_{0122}\rangle_k.$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let $x_1, \ldots, x_6$ be the values of λ on the 6 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm 2x_6 \cdot \operatorname{antidiag}(-1, 1, -1, 1).$$

We have $\det(\mathscr{G}_\lambda) = 16x_6^4$. Hence, if $p = 2$, then the radical of σ_λ is not zero. If $p \neq 2$, then the radical is zero whenever $x_6 \neq 0$.

5.6. Let $d(\alpha_1) = 0$, $d(\alpha_2) = 1$, $d(\alpha_3) = 0$, $d(\alpha_4) = 1$. We have $\mathbf{b}_d = 6$ and

$$\mathfrak{g}_d(1) = \langle e_{0100}, e_{0001}, e_{1100}, e_{0110}, e_{0011}, e_{1110}, e_{0120}, e_{1120}\rangle_k,$$
$$\mathfrak{g}_d(2) = \langle e_{0111}, e_{1111}, e_{0121}, e_{1220}, e_{1121}\rangle_k.$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let $x_1, \ldots, x_5$ be the values of λ on the 5 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & -x_1 & 0 & -x_2 & -x_3 & -x_5 \\ 0 & 0 & 0 & 0 & -x_2 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & 0 & -x_3 & 2x_4 & 0 & 0 \\ x_1 & 0 & x_2 & x_3 & 0 & x_5 & 0 & 0 \\ 0 & x_2 & 0 & -2x_4 & -x_5 & 0 & 0 & 0 \\ 0 & x_3 & -x_4 & 0 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\det(\mathscr{G}_\lambda) = 9(x_1x_4^2x_5 - x_2x_3x_4^2)^2$. Hence, if $p = 3$, then the radical of σ_λ is not zero. Now assume that $p \neq 3$. Then we notice that $\det(\mathscr{G}_\lambda) = 9$ for $x_1 := 1$, $x_4 := 1$, $x_5 := 1$ and $x_i := 0$ for $i = 2, 3$. So the radical is zero for this choice of λ.

5.7. Let $d(\alpha_1) = 1$, $d(\alpha_2) = 0$, $d(\alpha_3) = 1$, $d(\alpha_4) = 0$. We have $\mathbf{b}_d = 5$ and

$$\mathfrak{g}_d(1) = \langle e_{1000}, e_{0010}, e_{1100}, e_{0110}, e_{0011}, e_{0111}\rangle_k,$$
$$\mathfrak{g}_d(2) = \langle e_{1110}, e_{0120}, e_{1111}, e_{0121}, e_{0122}\rangle_k.$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let $x_1, \ldots, x_5$ be the values of λ on the 5 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 & x_3 \\ 0 & 0 & x_1 & 2x_2 & 0 & x_4 \\ 0 & -x_1 & 0 & 0 & -x_3 & 0 \\ -x_1 & -2x_2 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & x_3 & x_4 & 0 & 2x_5 \\ -x_3 & -x_4 & 0 & 0 & -2x_5 & 0 \end{pmatrix}.$$

We have $\det(\mathscr{G}_\lambda) = 4(x_1^2x_5 - x_1x_3x_4 + x_2x_3^2)^2$. Hence, if $p = 2$, then the radical of σ_λ is not zero. Now assume that $p \neq 2$. Then we notice that $\det(\mathscr{G}_\lambda) = 4$ for $x_1 := 1$, $x_5 := 1$ and $x_i := 0$ for $i = 2, 3, 4$. So the radical of σ_λ is zero for this choice of λ.

5.8. Let $d(\alpha_1) = 1$, $d(\alpha_2) = 0$, $d(\alpha_3) = 1$, $d(\alpha_4) = 2$. We have $\mathbf{b}_d = 3$ and

$$\mathfrak{g}_d(1) = \langle e_{1000}, e_{0010}, e_{1100}, e_{0110}\rangle_k,$$
$$\mathfrak{g}_d(2) = \langle e_{0001}, e_{1110}, e_{0120}\rangle_k.$$

Let $\lambda\colon \mathfrak{g}_d(2) \to k$ be any linear map. Let x_1, x_2, x_3 be the values of λ on the 3 basis vectors of $\mathfrak{g}_d(2)$ (ordered as above). Then we obtain

$$\mathscr{G}_\lambda = \pm \begin{pmatrix} 0 & 0 & 0 & x_2 \\ 0 & 0 & x_2 & 2x_3 \\ 0 & -x_2 & 0 & 0 \\ -x_2 & -2x_3 & 0 & 0 \end{pmatrix}.$$

Hence, we see that the radical of σ_λ is zero for any linear map $\lambda\colon \mathfrak{g}_d(2) \to k$ such that $\lambda(e_{1110}) = x_2 \neq 0$, and this works for any field k.

Similar computations can, of course, be performed for other types of groups. The results are summarized as follows.

5.9. Let G be simple of exceptional type G_2, F_4, E_6, E_7 or E_8. Assume that the characteristic p of k is a bad prime for G. Let $d \in \Delta$ be such that $\mathfrak{g}_d(1) \neq \{0\}$. Consider the following two statements.

(a) If $d \in \Delta_k^\bullet$, then there exist linear maps $\lambda\colon \mathfrak{g}_d(2) \to k$ such that $\det(\mathscr{G}_\lambda) \neq 0$ and $\lambda(e_\alpha) \in \{0, 1\}$ for all $\alpha \in \Phi_2$.

(b) If $d \notin \Delta_k^\bullet$, then $\det(\mathscr{G}_\lambda) = 0$ for all linear maps $\lambda\colon \mathfrak{g}_d(2) \to k$.

Then (a) can be verified by exactly the same kind of computations as in Example 4.7(a), by systematically running through all possibilities where $\lambda(e_\alpha) \in \{0, 1\}$ for $\alpha \in \Phi_2$, until we find one such that $\det(\mathscr{G}_\lambda) \neq 0$. In each case, we find a desired λ with at most four values equal to 0. The verification of (b) is much harder. As in the verification of Example 4.7(b), we need to show that the determinant of a certain matrix with entries in a polynomial ring over $\mathbb{F}_p$ is 0. Except for some cases in type E_8, it is sufficient to use the Pfaffian of that matrix, as in Example 4.7(b). For types G_2, F_4, we can see all this immediately from the results of the computations in Example 2.11 and in 5.1–5.8, by comparing with the entries in the last column of Table 2. However, there are critical cases in type E_8 (for example, $\mathsf{A}_2{+}3\mathsf{A}_1$, $2\mathsf{A}_2{+}\mathsf{A}_1$, $2\mathsf{A}_2{+}2\mathsf{A}_1$, $\mathsf{A}_3{+}2\mathsf{A}_1$, $\mathsf{A}_3{+}\mathsf{A}_2{+}\mathsf{A}_1$) where the computation of the Pfaffian appears to be practically impossible. In these cases, some more sophisticated computational methods are required.

Proposition 5.10 (Steel–Thiel)**.** *Let G be simple of type E_8 and assume that p is a bad prime for G. Then the statement in* 5.9(b) *holds.*

Proof. For d as in 5.9(b), let $m = \dim \mathfrak{g}_d(2)$ and $n = \dim \mathfrak{g}_d(1)$. Let R be the polynomial ring over $\mathbb{F}_p$ in m indeterminates. As discussed above, we must show that $\det(F) = 0$, for a certain matrix $F \in M_n(R)$ (which specialises to the various possible matrices $\mathscr{G}_\lambda$). For this purpose, we consider the R-module homomorphism $\varphi\colon R^n \to R^n$ defined by F. The SINGULAR function syz (which relies on Groebner bases techniques) computes a system of generators for $\ker(\varphi)$, as explained in [15, §2.8.7]. In all cases under consideration here, syz finds non-zero elements in $\ker(\varphi)$. (This just takes a few seconds.) Consequently, we do have $\det(F) = 0$, as required. □

Corollary 5.11. *If G is simple of exceptional type G_2, F_4, E_6, E_7 or E_8, then Conjectures* 4.9 *and* 4.10 *hold for G.*

Proof. First consider Conjecture 4.9. Let $d \in \Delta_k^\bullet$. If $\mathfrak{g}_d(1) \neq \{0\}$, then we must show that there exists some $\lambda\colon \mathfrak{g}_d(2) \to k$ such that $\det(\mathscr{G}_\lambda) \neq 0$. If p (the characteristic of k) is a good prime for G, then this holds by Remarks 3.5 and 3.7(∗). If p is a bad prime, then this holds since 5.9(a) is known to hold. Conversely, assume that either $\mathfrak{g}_d(1) = \{0\}$, or there exists a linear map $\lambda\colon \mathfrak{g}_d(2) \to k$ such that $\det(\mathscr{G}_\lambda) \neq 0$. If $\mathfrak{g}_d(1) = \{0\}$, then $d \in \Delta_{\text{spec}} \subseteq \Delta_k^\bullet$, as already remarked at the end of Section 4. If $\mathfrak{g}_d(1) \neq \{0\}$, then we have $d \in \Delta_k^\bullet$ since 5.9(b) is known to hold (by Proposition 5.10 for type E_8).

Now consider Conjecture 4.10. Let $d \in \Delta_{\text{spec}}$. If $\mathfrak{g}_{\mathbb{Z},d}(1) \neq \{0\}$, then we must show that there exists a homomorphism $\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(2) \to \mathbb{Z}$ such that $\sigma_\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \to \mathbb{Z}$ is non-degenerate over $\mathbb{Z}$. The verification is similar to that in 5.9(a), but now we work over $\mathbb{Z}$. Again, we systematically run through all possibilities where $\lambda(e_\alpha) \in \{0, 1\}$ for $\alpha \in \Phi_2$, until we find one such that the Gram matrix of σ_λ has determinant equal to 1. In each case we can find a desired λ with at most four values equal to 0. Conversely, assume that either $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$, or there exists a homomorphism $\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(2) \to \mathbb{Z}$ such that $\sigma_\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \to \mathbb{Z}$ is non-degenerate over $\mathbb{Z}$. If $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$, then $d \in \Delta_{\text{spec}}$ (see again the remark at the end of Section 4). Now assume that $\mathfrak{g}_{\mathbb{Z},d}(1) \neq \{0\}$. By reduction modulo p, we obtain a linear map $\lambda_k\colon \mathfrak{g}_d(2) \to k$ and a corresponding alternating form $\sigma_{\lambda_k}\colon \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \to k$. Since σ_λ is non-degenerate over $\mathbb{Z}$, the radical of σ_{λ_k} will be zero and so $d \in \Delta_k^\bullet$, since we already know that Conjecture 4.9 is true for G. Note that this holds for all choices of k. So Table 1 shows that $d \in \Delta_{\text{spec}}$. □

In order to verify Conjecture 4.4 in full, one would also need a description of the sets $\mathfrak{g}_d(2)^{*!}$; this will be discussed elsewhere. (For G of type A_n, B_n, C_n, D_n, such a description is available from [29], [37].) Furthermore, one would need to check whether or not there exists some λ which is defined over $\mathbb{F}_q$ and is in sufficiently general position. It would be highly desirable to find a more conceptual explanation for all this.

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