EXTENSIONS OF ISOMORPHISMS OF SUBVARIETIES IN FLEXIBLE VARIETIES

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Abstract. Let X be an algebraic variety isomorphic to the complement of a closed subvariety of dimension at most $n-3$ in $\mathbb{A}^n_{\mathbf{k}}$. We find some conditions under which an isomorphism of two closed subvarieties of X can be extended to an automorphism of X. We also study the similar problem for subvarieties of affine quadrics and $\mathrm{SL}(n,\mathbf{k})$.

Introduction

Let X be a smooth quasi-affine variety over an algebraically closed field $\mathbf{k}$ of characteristic zero and Y_1 and Y_2 be closed subvarieties of X. We study the following *extension problem*:

Under what restrictions on Y_i and TY_i an isomorphism $Y_1 \to Y_2$ extends to an automorphism of X?

This question makes sense when X itself possesses a large automorphism group $\mathrm{Aut}(X)$ which leads to the notion of a flexible variety [AFKKZ]. Recall that it is a quasi-affine algebraic variety of dimension at least 2 on which the group $\mathrm{SAut}(X)$ (generated by elements of all one-parameter unipotent subgroups of $\mathrm{Aut}(X)$) acts m-transitively for every $m > 0$. The simplest example of a flexible variety is $X = \mathbb{A}^n_{\mathbf{k}}$ and the extension problem was studied extensively for such an X.

The starting point (and an inspiration) of that research was, of course, the Abhyankar–Moh–Suzuki theorem [AMo], [Su] which states that given two plane curves isomorphic to a line one can be transferred to the other by an automorphism of $\mathbb{A}^2_{\mathbf{k}}$. Then, disproving an Abhyankar's conjecture, Jelonek [Je] established that if one requires that $4 \dim Y_1 + 2 \leqslant n$ then one gets a positive answer to the extension problem in $\mathbb{A}^n_{\mathbf{k}}$ for the case of smooth Y_i [1]. In the non-smooth case we have to take into consideration $\dim TY_i$ and the more general result established by the author [Ka] and Vasudevan Srinivas [Sr] states the following:

[1] Abhyankar's conjecture was also disproved slightly earlier by Craighero [Cr] but he did not consider the extension problem in the full generality.

Theorem 0.1. *Let* $\varphi : Y_1 \to Y_2$ *be an isomorphism of two closed subvarieties of* $\mathbb{A}_{\mathbf{k}}^n$ *such that* $n \geqslant \mathrm{ED}(Y_1) + 1$ *where* $\mathrm{ED}(Y_1) = \max(2 \dim Y_1 + 1, \dim TY_1)$. *Then* φ *extends to an automorphism of* $\mathbb{A}_{\mathbf{k}}^n$.

The first result in the case of a flexible variety different from $\mathbb{A}_{\mathbf{k}}^n$ is due to Van Santen (formerly Stampfli) [St] who proved that given two curves isomorphic to a line in an algebraic variety X isomorphic to the special linear group $\mathrm{SL}(n, \mathbf{k})$ over $\mathbf{k}$ one can be transferred to the other by an automorphism of X provided that $n \geqslant 3$ (in particular, any such curve is an orbit of a $\mathbb{G}_a$-action). This theorem was generalized later in the paper of Van Santen and Feller [FS] where they showed that the same is true if one considers two curves isomorphic to a line in a connected linear algebraic group modulo some exceptions. Then Van Santen jointly with J. Blanc showed (among other facts) that there are closed surfaces isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ in $\mathrm{SL}(2, \mathbf{k})$ which cannot be transferred to each other by an automorphism of $\mathrm{SL}(2, \mathbf{k})$ (as an algebraic variety).

In the present paper besides $\mathrm{SL}(n, \mathbf{k})$ we study smooth quadrics in $\mathbb{A}_{\mathbf{k}}^n$ and the case of X equal to the complement to a codimension at least 2 subvariety in $\mathbb{A}_{\mathbf{k}}^n$ (we call it the Gromov–Winkelmann case since these authors established the flexibility of such an X [Wi], [Gr]). The main results of our paper are the following.

Theorem 0.2. *Let* Z, Y_1, *and* Y_2 *be closed subvarieties of* $\mathbb{A}_{\mathbf{k}}^n$ *such that* $Y_1 \cap Z = Y_2 \cap Z = \varnothing$, $\dim Z \leqslant n-3$ *and* $\mathrm{ED}(Y_1) \leqslant n-2$. *Let* $\varphi : Y_1 \to Y_2$ *be an isomorphism and* $X = \mathbb{A}_{\mathbf{k}}^n \setminus Z$. *Suppose also that either*

   (a) $\dim Z + \dim Y_1 \leqslant n - 3$, *or*
   (b) $\dim Y_1 = 1$ *and* $\dim Z \leqslant n - 3$.

   Then there exists an automorphism $\gamma \in \mathrm{SAut}(X)$ *for which* $\gamma|_{Y_1} = \varphi$.[2]

Theorem 0.3. *Let* $m \geqslant 6$ *and* X *be a hypersurface in* $\mathbb{A}_{\mathbf{k}}^m$ *that is a nonzero fiber of a non-degenerate quadratic form. Suppose that* $\varphi : Y_1 \to Y_2$ *is an isomorphism of two closed subvarieties of* X. *Let* $\mathrm{ED}(Y_i) + \dim Y_i \leqslant m - 2$. *Then* φ *extends to an automorphism of* X *which belongs to* $\mathrm{SAut}(X)$.

Theorem 0.4. *Let* $X = \mathrm{SL}(n, \mathbb{C})$ *and* $\varphi : Y_1 \to Y_2$ *be an isomorphism of two closed subvarieties of* X *such that either*

   (i) $\mathrm{ED}(Y_i) + \dim Y_i \leqslant n - 2$, $H_i(Y_1) = 0$ *for* $i \geqslant 3$ *and* $H_2(Y_1)$ *is a free abelian group; or*
   (ii) Y_1 *is a curve and* $\mathrm{ED}(Y_i) \leqslant n - 2$, *or;*
   (iii) Y_1 *is a once-punctured curve and* $\mathrm{ED}(Y_1) \leqslant 2n - 3$.

   Then there exists a holomorphic automorphism β *of* X *such that* $\beta|_{Y_1} = \varphi$.

Theorem 0.5. *Let* $\varphi : Y_1 \to Y_2$ *be an isomorphism of two closed subvarieties of* $X \simeq \mathrm{SL}(n, \mathbf{k})$ *with* $n \geqslant 3$ *such that* Y_i *is isomorphic to* $\mathbb{A}_{\mathbf{k}}^k$. *Suppose that either* $k \leqslant n/3 - 1$ *or* $k = 1$. *Then there exists* $\alpha \in \mathrm{SAut}(X)$ *such that* $\alpha|_{Y_1} = \varphi$.

The paper is organized as follows. In the first six sections we develop some technique which is valid for a wide class of flexible varieties. More precisely, in

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[2]If $\mathrm{ED}(Z) \leqslant n - 1$ then there is no need for this theorem. Indeed, one can consider the isomorphism $\psi : Y_1 \cup Z \to Y_2 \cup Z$ such that $\psi|_{Y_1} = \varphi$ and $\psi|_Z = \mathrm{id}_Z$. Then Theorem 0.1 implies that ψ extends to an automorphism of $\mathbb{A}_{\mathbf{k}}^n$ as soon as $\mathrm{ED}(Y_1) \leqslant n - 1$.

Section 1 we consider a morphism $\kappa : X \to P$ of smooth irreducible varieties such that X is equipped with an action of a group $G \subset \mathrm{Aut}(X)$ which preserves every fiber $\kappa^{-1}(p)$, $p \in P$ and acts transitively on it. We describe some conditions under which one can find an algebraic family $\mathcal{A}$ of automorphisms of X contained in G such that for two subvarieties Y and Z of X and a general element $\alpha \in \mathcal{A}$ the varieties Y and $\alpha(Z)$ are transversal (this is a relative version of the transversality theorem from [AFKKZ] that dealt with the case when P was a singleton). In Section 2 we recall some facts about flexible varieties and in Section 3 we prove a relative version of a theorem from [AFKKZ] which yields automorphisms of a given flexible variety with prescribed jets at a finite number of points.

Sections 4–6 are crucial. Namely, the proof of Theorem 0.1 is heavily based on the fact that for a general linear projection $\theta : \mathbb{A}_{\mathbf{k}}^n \to \mathbb{A}_{\mathbf{k}}^{n-1}$ the variety $\theta(Y_i)$ is closed in $\mathbb{A}_{\mathbf{k}}^{n-1}$ and the restriction of θ yields an isomorphism $Y_i \to \theta(Y_i)$. Note that θ can be viewed as the composition of a fixed projection θ_0 and a general linear automorphism of $\mathbb{A}_{\mathbf{k}}^n$. Hence, in Section 4 we imitate this idea for a morphism $\varrho : X \to Q$ over P as above but with each fiber of κ being a G-flexible variety. We show that when the morphism ϱ and the variety Q are smooth and Z is a closed subvariety of X with $\mathrm{ED}(Z) \leqslant \dim Q$ then for a general element α of some algebraic family $\mathcal{A}$ contained in G the morphism $\varrho|_{\alpha(Z)} : \alpha(Z) \to Q$ is an injection and, furthermore, the induced morphism $T\alpha(Z) \to TQ$ of the Zariski tangent bundles is also an injection. However, a priori this map is not proper and in Section 5 we describe some conditions under which the morphism $\varrho|_{\alpha(Z)}$ is also proper and, therefore, $\varrho \circ \alpha(Z)$ is closed in Q and $\varrho|_{\alpha(Z)} : \alpha(Z) \to \varrho \circ \alpha(Z)$ is an isomorphism. These facts are already sufficient for the proof of Theorem 0.2 in Section 7. Section 6 is devoted to the independently interesting case when ϱ is a partial quotient morphism of a $\mathbb{G}_a$-action on X. In this situation we cannot claim that ϱ and Q are smooth and cannot guarantee the properness of $\varrho|_{\alpha(Z)}$. However, we establish that for any given finite subset $S \subset Z$ and a general $\alpha \in \mathcal{A}$ one can find a neighborhood V' of $\varrho(\alpha(S))$ in $\overline{\varrho(\alpha(Z))} \subset Q$ such that for $V = \varrho^{-1}(V') \cap \alpha(Z)$ the restriction of $\varrho|_{\alpha(Z)}$ yields an isomorphism $V \to V'$. Applications of this result go beyond the present paper (e.g., see [KKT]).

Section 8 contains Theorem 0.3 and in Sections 9 and 10 we prove some technical results that enable us to obtain Theorems 0.4 and 0.5 in Section 11 where we also present an example of a topological obstruction for the extension problem in the case of general flexible varieties.

1. Algebraically generated groups of automorphisms

Let X be an irreducible algebraic variety and $\mathrm{Aut}(X)$ be the group of its algebraic automorphisms. Recall the following terminology introduced by Ramanujam [Ra].

Definition 1.1. (1) Given an irreducible algebraic variety $\mathcal{A}$ and a map $\varphi : \mathcal{A} \to \mathrm{Aut}(X)$ we say that $(\mathcal{A}, \varphi)$ is an *algebraic family of automorphisms on X* if the

induced map $\mathcal{A} \times X \to X$, $(\alpha, x) \mapsto \varphi(\alpha).x$ is a morphism.

(2) If we want to emphasize additionally that $\varphi(\mathcal{A})$ is contained in a subgroup G of $\mathrm{Aut}(X)$ then we say that $\mathcal{A}$ is an algebraic G-family of automorphisms of X.

(3) In the case when $\mathcal{A}$ is a connected algebraic group and the induced map $\mathcal{A} \times X \to X$ is not only a morphism but also an action of $\mathcal{A}$ on X we call this family a connected algebraic subgroup of $\mathrm{Aut}(X)$.

Definition 1.2. Following [AFKKZ, Def. 1.1] we call a subgroup G of $\mathrm{Aut}(X)$ algebraically generated if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic subgroups of $\mathrm{Aut}(X)$.

We have the following important fact [AFKKZ, Thm. 1.15] (which is the analogue of the Kleiman transversality theorem [Kl] for algebraically generated groups).

Theorem 1.3 (Transversality Theorem). *Let a subgroup $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that G acts with an open orbit $O \subseteq X$.*

Then there exist subgroups $H_1, \ldots, H_m \in \mathcal{G}$ such that for any locally closed reduced subschemes Y and Z in O one can find a Zariski dense open subset $U = U(Y, Z) \subseteq H_1 \times \cdots \times H_m$ such that every element $(h_1, \ldots, h_m) \in U$ satisfies the following:

(a) *The translate $(h_1 \cdot \cdots \cdot h_m).Z_{\mathrm{reg}}$ meets Y_{reg} transversally.*
(b) *$\dim(Y \cap (h_1 \cdot \cdots \cdot h_m).Z) \le \dim Y + \dim Z - \dim X$.[3]*
   In particular, $Y \cap (h_1 \cdot \cdots \cdot h_m).Z = \varnothing$ if $\dim Y + \dim Z < \dim X$.

We need to generalize [AFKKZ, Thm. 1.15] further.

Theorem 1.4 (Collective Transversality Theorem). *Let X and P be smooth irreducible algebraic varieties and $\kappa : X \to P$ be a smooth morphism (in particular $X \times_P X$ is smooth and $\dim X \times_P X = 2 \dim X - \dim P$). Let a group $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that the G-action transforms every fiber $\kappa^{-1}(p)$ into itself and, furthermore, the restriction of the G-action to $\kappa^{-1}(p)$ is transitive for every $p \in P$.*

Then there exist subgroups $H_1, \ldots, H_m \in \mathcal{G}$ such that for any locally closed reduced subschemes Y and Z in X one can find a Zariski dense open subset $U = U(Y, Z) \subseteq H_1 \times \cdots \times H_m$ so that every element $(h_1, \ldots, h_m) \in U$ satisfies the following:

(i) *$\dim(Y \cap (h_1 \cdot \cdots \cdot h_m).Z) \le \dim(Y \times_P Z) + \dim P - \dim X$.*
   In particular, when $\dim Y \times_P Z \le \dim Y + \dim Z - \dim P$ one has
(ii) *$\dim(Y \cap (h_1 \cdot \cdots \cdot h_m).Z) \le \dim Y + \dim Z - \dim X$.*
   Furthermore, suppose that the inequality $\dim Y \times_P Z \le \dim Y + \dim Z - \dim P$ holds and also that Z, $Y \times_P Z$, and $Y \times_P X$ are smooth. Then
(iii) *$(h_1 \cdot \cdots \cdot h_m).Z$ meets Y transversally.*

The proof of Theorem 1.4 is an adjustment of the proof of [AFKKZ, Thm. 1.15]. Hence, following the latter we establish first three facts which in the case of a singleton P are nothing but Propositions 1.5, 1.8, and 1.16 in [AFKKZ].

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[3]We put the dimension of empty sets equal to $-\infty$.

Proposition 1.5 (Analogue of [AFKKZ, Prop. 1.5]). *Let the assumptions of Theorem* 1.4 *hold. There are (not necessarily distinct) subgroups* $H_1, \ldots, H_m \in \mathcal{G}$ *such that for every* $p \in P$ *and each* $x \in \kappa^{-1}(p)$ *one has*

$$\kappa^{-1}(p) = H_1.(H_2.\cdots.(H_m.x)). \tag{1}$$

Proof. Let us introduce the partial order on the set of sequences in $\mathcal{G}$ such that for $\mathcal{H} = (H_1, \ldots, H_m)$ and $\mathcal{H}' = (H'_1, \ldots H'_s)$ one has

$$\mathcal{H} \succcurlyeq \mathcal{H}' \iff \exists i_1 < \cdots < i_s : \quad (H'_1, \ldots, H'_s) = (H_{i_1}, \ldots, H_{i_s}).$$

Assuming first that κ has a section $\lambda : P \to X$ with $S = \lambda(P)$ we consider $\mathcal{H}.S = \bigcup_{x \in S} \mathcal{H}.x$ where $\mathcal{H}.x = H_1.(H_2.\cdots.(H_m.x))$. Then such a set $\mathcal{H}.S$ is constructible (since it is the image of the algebraic variety $H_1 \times \cdots \times H_m \times S$ under a morphism). In particular, $X_{\mathcal{H}} := X \setminus \mathcal{H}.S$ is a constructible set. Furthermore, the following property holds:

$$\text{if } \mathcal{H} \succcurlyeq \mathcal{H}', \text{ then } \mathcal{H}'.S \subset \mathcal{H}.S \text{ and, therefore, } X_{\mathcal{H}} \subset X_{\mathcal{H}'}.$$

By [AFKKZ, Prop. 1.8] for every point $p_0 \in P$ we can find a sequence $\mathcal{H}$ in $\mathcal{G}$ such that the morphism

$$H_1 \times \cdots \times H_m \to \kappa^{-1}(p_0), \quad (h_1, \ldots, h_m) \mapsto h_1 \cdot \cdots \cdot h_m.\lambda(p_0)$$

is surjective and smooth. Since smoothness is an open condition the morphism

$$H_1 \times \cdots \times H_m \times X \to X \text{ given by } ((h_1, \ldots, h_m), x) \mapsto (h_1 \cdot \cdots \cdot h_m).x$$

is smooth on $(e, \ldots, e) \times (S \cap \kappa^{-1}(P_0))$ where e is the identity element of the group G and $P_0 = P_0(\mathcal{H})$ is a Zariski neighborhood of p_0 in P. Consequently, $\mathcal{H}.S$ contains a Zariski neighborhood of $S \cap \kappa^{-1}(P_0)$ in X, since smooth morphisms are open. Suppose that $P_0(\mathcal{H})$ is the largest neighborhood of p_0 with this property and $R = R(\mathcal{H})$ is the complement of $P_0(\mathcal{H})$ in P. For a given $p' \in R$ choose again a sequence $\mathcal{H}' = (H'_1, \ldots H'_s)$ such that the morphism $H'_1 \times \cdots \times H'_s \to \kappa^{-1}(p')$ is smooth and surjective. Consider $\mathcal{H}'' = (\mathcal{H}', \mathcal{H})$ and note that $R(\mathcal{H}'')$ is a proper subvariety of $R(\mathcal{H})$ since $\mathcal{H}''.S \supset \mathcal{H}.S$ and $\mathcal{H}''.S \supset \mathcal{H}'.S$. Thus, increasing $\mathcal{H}$, we can reduce the dimension of irreducible components of $R(\mathcal{H})$. The induction by dimension implies now that we can suppose from the beginning that $\mathcal{H}.S$ contains a Zariski neighborhood of S in X.

Because of transitivity for every $y \in X$ we can find a sequence $\mathcal{H}_1$ such that $y \in \mathcal{H}_1.S$. Hence, $y \in \widetilde{\mathcal{H}}.S$ for any sequence $\widetilde{\mathcal{H}}$ of the form $\widetilde{\mathcal{H}} = (\mathcal{H}_1, \mathcal{H}_2)$ where $\mathcal{H}_2$ extends the sequence $\mathcal{H}$ as before. In particular, choosing y in any given irreducible component C of $X_{\mathcal{H}_2}$ we guarantee C is not contained in $X_{\widetilde{\mathcal{H}}}$. Furthermore, $\widetilde{\mathcal{H}}.S$ contains a Zariski neighborhood of y, since $H_2.S$ contains a Zariski neighborhood of S. That is, $\dim X_{\widetilde{\mathcal{H}}} \cap C < \dim C$.

Thus, enlarging $\mathcal{H}$ we can reduce the dimension of $X_{\mathcal{H}}$ and continuing this process we can make $X_{\mathcal{H}} = \varnothing$. In particular, for every $p \in P$ and $x = \lambda(p)$ one has $\mathcal{H}.x = \kappa^{-1}(p)$. Let $\mathcal{H}^t = (H_m, H_{m-1}, \ldots, H_1)$ and $\widehat{\mathcal{H}} = (\mathcal{H}, \mathcal{H}^t)$, i.e., $x \in \mathcal{H}^t.y$ for

every $y \in \kappa^{-1}(p)$. Then one has $\widetilde{\mathcal{H}}.y = \kappa^{-1}(p)$, i.e., we get the desired conclusion in the presence of a section λ.

In the general case consider an étale neighborhood W of a point $p \in P$ and suppose that $V \subset P$ is the image of W under the natural morphism. Since κ is smooth one can suppose that for an appropriate choice of W the natural projection $\tau : X \times_P W \to W$ has a section. Consider the induced G-action on $X \times_P W$. By the previous argument there is a sequence $\mathcal{H}$ such that for every $w \in W$ and $z \in \tau^{-1}(w)$ one has $\mathcal{H}.z = \tau^{-1}(w)$. Applying the natural projection $X \times_P W \to X$ we see that for every $p \in V$ and every $y \in \kappa^{-1}(p)$ we have $\mathcal{H}.y = \kappa^{-1}(p)$. Choosing now a finite number of étale neighborhoods that cover X, we can enlarge $\mathcal{H}$ so that it works for each of these neighborhoods. This implies the desired conclusion. $\square$

Proposition 1.6 (Analogue of [AFKKZ, Prop. 1.8]). *Let the assumptions of Theorem 1.4 hold. Assume that the generating family $\mathcal{G}$ of connected algebraic subgroups is closed under conjugation in G, i.e., $gHg^{-1} \in \mathcal{G}$ for all $g \in G$ and $H \in \mathcal{G}$. Then there is a sequence $\mathcal{H} = (H_1, \ldots, H_m)$ in $\mathcal{G}$ such that for all $p \in P$ and $x \in \kappa^{-1}(p)$ the tangent space $T_x\kappa^{-1}(p)$ is spanned by the tangent spaces*

$$T_x(H_1.x), \ldots, T_x(H_m.x)$$

to the orbits $H_1.x, \ldots, H_m.x$ at x.

Proof. Let $\mathcal{H} = (H_1, \ldots, H_m)$ be a sequence in $\mathcal{G}$ satisfying the conclusion of Proposition 1.5. Consider the map

$$\Phi_\mathcal{H} : H_1 \times \cdots \times H_m \times X \to X \times_P X \text{ given by } ((h_1, \ldots, h_m), x) \mapsto ((h_1 \cdots h_m).x, x).$$

The fiber $\tau^{-1}(x)$ of the second projection $\tau : X \times_P X \to X$ over $x \in X$ is naturally isomorphic to $\kappa^{-1}(p)$ where $p = \kappa(x)$. Hence, by Proposition 1.5 $\Phi_\mathcal{H}$ is a surjective map while the assumptions of Theorem 1.4 imply that τ is a smooth morphism. Let us consider the map of relative tangent bundles

$$d\Phi_\mathcal{H} : T(H_1 \times \cdots \times H_m \times X/X) \to \Phi_\mathcal{H}^*(T((X \times_P X)/X))$$

and its restriction to $\{(e, \ldots, e)\} \times X \cong X$ (where e is the identity in the group G),

$$d\Phi_\mathcal{H} : T_eH_1 \times \cdots \times T_eH_m \times X \to \Phi_\mathcal{H}^*(T((X \times_P X)/X)).$$

The set $U_\mathcal{H}$ of points in X where this map is surjective is, of course, open. By [AFKKZ, Prop. 1.8] for every $p \in P$ and $x \in \kappa^{-1}(p)$ the tangent space $T_x\kappa^{-1}(p)$ is spanned by the tangent spaces $T_x(H.x)$, where $H \in \mathcal{G}$. Hence, $\bigcup_\mathcal{H} U_\mathcal{H}$ coincides with X. Since an increasing union of open subsets stabilizes, we obtain that $X = U_\mathcal{H}$ for $\mathcal{H}$ sufficiently large (with respect to the partial order introduced in Proposition 1.5). This yields the desired conclusion. $\square$

Proposition 1.7 (Analogue of [AFKKZ, Prop. 1.16]). *Let the assumption of Theorem 1.4 hold. Then there is a sequence $H_1, \ldots, H_m$ in $\mathcal{G}$ so that for a suitable open dense subset $U \subseteq H_m \times \cdots \times H_1$, the map*

$$\Phi_m : H_m \times \cdots \times H_1 \times X \to X \times_P X \text{ with } (h_m, \ldots, h_1, x) \mapsto ((h_m \cdots h_1).x, x) \quad (2)$$

is smooth on $U \times X$.

Proof. By Proposition 1.5 there are subgroups $H_1, \ldots, H_m \subseteq G$ in $\mathcal{G}$ such that Φ_m is surjective. Let $U_m \subset H_m \times \cdots \times H_1 \times X$ be the set of points where Φ_m is smooth. Then U_m is non-empty by [Ha, Chapter III, Corollary 10.7] and it is open by [SGA1, Exp. II, Prop. 1.1]. Consider the complement $A_m = (H_m \times \cdots \times H_1 \times X) \backslash U_m$. Let us study the effect of increasing the number m of factors in the product $H_m \times \cdots \times H_1$.

Suppose that H is an element of $\mathcal{G}$ and Φ' plays the same role for the sequence $H_1, \ldots, H_m, H$ as Φ_m for the sequence $H_1, \ldots, H_m$. Note that for a fixed $h \in H$ the restriction of Φ' yields the morphism $\{h\} \times H_m \times \cdots \times H_1 \times X \to X \times_P X$ that is the composition of Φ_m and the automorphism φ of $X \times_P X$ given by $(x, y) \to (h.x, y)$ which implies smoothness of Φ' on $H \times U_m$. Thus, $U_{m+1} \supseteq H_{m+1} \times U_m$ and $A_{m+1} \subseteq H_{m+1} \times A_m$. Increasing the number of factors by $H_{m+1}, \ldots, H_{m+k}$ in a suitable way, we can achieve that

$$\dim A_{m+k} < \dim(H_{m+k} \times \cdots \times H_{m+1} \times A_m). \tag{3}$$

Indeed, if $(h_m, \ldots, h_1, x) \in A_m$ and $y = (h_m \cdot \ldots \cdot h_1).x$ then by Proposition 1.6 for suitable $H_{m+k}, \ldots, H_{m+1}$ the map

$$H_{m+k} \times \cdots \times H_{m+1} \times X \to X \times_P X, \ (h_{m+k}, \ldots, h_{m+1}, z) \mapsto ((h_{m+k} \cdots h_{m+1}).z, z)$$

is smooth in all points $(e, \ldots, e, y)$ where e is the identity of the group G. In particular, Φ_{m+k} is smooth in all points $(e, \ldots, e, h_m, \ldots, h_1, x)$ with $x \in X$, i.e.,

$$\{(e, \ldots, e)\} \times A_m \cap A_{m+k} = \varnothing.$$

Now (3) follows.

Thus increasing the number of factors suitably we can achieve that

$$\dim A_m < \dim(H_m \times \cdots \times H_1).$$

That is, the image of A_m under the projection

$$\pi : H_m \times \cdots \times H_1 \times X \to H_m \times \cdots \times H_1$$

is nowhere dense. Hence, there is an open dense subset $U \subseteq H_m \times \cdots \times H_1$ such that $\Phi_m : U \times X \to X \times_P X$ is smooth. $\quad\square$

Remark 1.8. Let $\Phi_m, H_1, \ldots, H_m$ and $U \subset H_m \times \cdots \times H_1$ be as in Proposition 1.7 and let H be an element $\mathcal{G}$. Suppose that Φ' (resp. Φ'') plays the same role for the sequence $H_1, \ldots, H_m, H$ (resp. $H, H_1, \ldots, H_m$) as Φ_m for the sequence $H_1, \ldots, H_m$. Then Φ' is smooth on $H \times U \times X$, since $\Phi'|_{H \times U \times X}$ is the following composition of smooth morphisms

$$H \times U \times X \xrightarrow{\mathrm{id}_H \times \Phi_m|_{U \times X}} H \times X \times_P X \xrightarrow{(h, x, y) \mapsto (h.x, y)} X \times_P X.$$

Similarly, Φ'' is smooth on $U \times H \times X$, since $\Phi''|_{U \times H \times X}$ is the composition of the following smooth morphisms:

$$U \times H \times X \xrightarrow[\sim]{(u, h, x) \mapsto (h, u, h.x)} H \times U \times X \xrightarrow{\mathrm{id}_H \times \Phi_m|_{U \times X}} H \times X \times_P X$$

$$\hspace{6cm} {\scriptstyle (h, x, y) \mapsto (x, h^{-1}.y)} \downarrow \hspace{2cm} .$$

$$\hspace{6cm} X \times_P X$$

Proof of Theorem 1.4. By Proposition 1.7 there are subgroups $H_1, \ldots, H_m$ in $\mathcal{G}$ such that $\Phi_m : U \times X \to X \times_P X$ is smooth for some open subset $U \subseteq H_m \times \cdots \times H_1$. Let $\mathcal{C} = \Phi_m^*(Y \times_P Z) \cap (U \times X)$.

Consider first the case in (i) when $Y \times_P Z$ is smooth. Then $\mathcal{C}$ is smooth. By [Ha, Chap. III, Cor. 10.7] the general fibers of the projection $\pi_{\mathcal{C}} : \mathcal{C} \to U$ are smooth as well. Suppose that $\pi_{\mathcal{C}}$ is dominant (otherwise the general fibers of $\pi_{\mathcal{C}}$ are empty). Shrinking U we may now assume that all fibers of $\pi_{\mathcal{C}}$ are smooth. Then the dimension of every fiber of Φ_m is $\dim U - \dim X + \dim P$. Thus $\dim \mathcal{C} = \dim U + \dim P - \dim X + \dim Y \times_P Z$ and the dimension of every fiber $\pi_{\mathcal{C}}^*(h)$ of $\pi_{\mathcal{C}}$ is

$$\dim Y \times_P Z + \dim P - \dim X. \tag{4}$$

Observe now that for a point $h = (h_m, \ldots, h_1) \in U$ the fiber $\pi_{\mathcal{C}}^*(h)$ maps bijectively via $\{h\} \times X \to X$, $(h, x) \mapsto (h_m \cdot \cdots \cdot h_1).x$ onto $Y \cap (h_1 \cdot \cdots \cdot h_m).Z$ which yields (i) (and therefore (ii)) in the case of smooth $Y \times_P Z$.

In the general case stratifying Z and Y we can find Zariski dense open subsets $Z_0 \subset Z$ and $Y_0 \subset Y$ such that $Y_0 \times_P Z_0$ is smooth. By formula (4) we see that for a general $h \in U$ the dimension of $Y_0 \cap (h_1 \cdots h_m).Z_0$ is at most $\dim Y \times_P Z + \dim P - \dim X$. Let $Z' = Z \setminus Z_0$ and $Y' = Y \setminus Y_0$ and consider, say, the pair (Y_0, Z'). We can suppose that $Y_0 \times_P Z'$ is smooth (otherwise stratify further). Then the same argument with formula (4) implies that the dimension of $Y_0 \cap (h_1 \cdots h_m).Z'$ is at most $\dim Y_0 \times_P Z' + \dim P - \dim X \leqslant \dim Y \times_P Z + \dim P - \dim X$. Repeating this procedure for the pairs (Y', Z_0) and (Y', Z') we get (i) and (ii) in full generality.

For (iii) consider $\mathcal{Z} = U \times Z$ and $\mathcal{Y} = \Phi_m^*(Y \times_P X) \cap (U \times X)$, i.e., $\mathcal{C}$ (as a scheme) is the intersection of $\mathcal{Z}$ and $\mathcal{Y}$. As before, shrinking U we can suppose that all fibers of the natural projections $\pi_{\mathcal{Y}} : \mathcal{Y} \to U$ and $\pi_{\mathcal{Z}} : \mathcal{Z} \to U$ are smooth. Observe also that $\pi_{\mathcal{Z}}^*(h) = \{h\} \times Z \subset \{h\} \times X$ and $\pi_{\mathcal{Y}}^*(h) = \{h\} \times (h_1^{-1} \cdot \cdots \cdot h_m^{-1}).Y \subset \{h\} \times X$.[4] It remains to note that if these two smooth subvarieties of $\{h\} \times X$ do not meet transversely and the dimension of their intersection $\pi_{\mathcal{C}}^*(h)$ is $\dim Y + \dim Z - \dim X$ then $\pi_{\mathcal{C}}^*(h)$ cannot be smooth (as a scheme). Indeed, the absence of transversality implies that at some closed point of $x \in \pi_{\mathcal{C}}^*(h)$ the dimension of the intersection of the tangent spaces $T_x \pi_{\mathcal{Y}}^*(h)$ and $T_x \pi_{\mathcal{Z}}^*(h)$ is greater than $\dim Y + \dim Z - \dim X$. Note that this intersection coincides with the intersection of kernels of the differentials of all functions from the defining ideals of $\pi_{\mathcal{Z}}^*(h)$ and $\pi_{\mathcal{Y}}^*(h)$, and, therefore, from the defining ideal of $\pi_{\mathcal{C}}^*(h)$. However, for a smooth $\pi_{\mathcal{C}}^*(h)$ this dimension must be equal to $\dim Y + \dim Z - \dim X$. Hence, the smoothness of $\pi_{\mathcal{C}}^*(h)$ established before yields (iii) which concludes the proof. $\square$

Remark 1.9.

(1) Suppose that $\kappa(Y)$ is dense in P and all fibers of $\kappa|_Y$ are of the same dimension (say, $\kappa|_Y$ is flat). That is, the dimension of each of these fibers is $\dim Y - \dim P$. Then $\dim Y \times_P Z = \dim Y + \dim Z - \dim P$ and we are under the assumption of (ii) in Theorem 1.4, i.e., the dimension of $Y \cap (h_1 \cdots h_m).Z$ is at most $\dim Y + \dim Z - \dim X$.

(2) Let us emphasize the following fact, the first part of which follows from the argument in the proof of Theorem 1.4.

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[4]In particular, the last equality implies that Y is smooth under the assumption of (ii).

Proposition 1.10. *If a sequence $H_1, \ldots, H_m \in \mathcal{G}$ satisfies the conclusions of Proposition 1.7 then it satisfies also the conclusions of Theorem 1.4. Furthermore, for any element H of $\mathcal{G}$ the sequence $H_1, \ldots, H_m, H$ (resp. $H, H_1, \ldots, H_m$) satisfies the conclusions of Theorem 1.4 as well.*

Proof. The second statement is true because by Remark 1.8 the sequence $H_1, \ldots$ $\ldots, H_m, H$ (resp. $H, H_1, \ldots, H_m$) satisfies the conclusions of Proposition 1.7. □

Remark 1.11.

(1) Let us consider an application of Proposition 1.10. Suppose that X and P are smooth irreducible algebraic varieties and $\kappa : X \to P$ is a morphism which is not in general smooth or even dominant. Let a group $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that the G-action transforms every fiber $\kappa^{-1}(p)$ into itself and, furthermore, the restriction of the G-action to $\kappa^{-1}(p)$ is transitive for every $p \in P$.

By the Generic Smoothness theorem (e.g., see [Ha, Chap. III, Cor. 10.7]) we can present $\kappa(X)$ as a disjoint union $\bigcup_{k=1}^{n} P_k$ of smooth varieties such that for $X_k = \kappa^{-1}(P_k)$ the morphism $\kappa|_{X_k} : X_k \to P_k$ is smooth. For Y and Z as in Theorem 1.4 let $Y_k = X_k \cap Y$ and $Z_k = X_k \cap Z$. Then by Theorem 1.4 there exist subgroups $H_1^k, \ldots, H_{m_k}^k \in \mathcal{G}$ such that one can find a Zariski dense open subset $U_k = U(Y_k, Z_k) \subseteq H_1^k \times \cdots \times H_{m_k}^k$ so that for every element $(h_1^k, \ldots, h_{m_k}^k) \in U_k$ we have the inequality

$$\dim(Y_k \cap (h_1^k \cdot \cdots \cdot h_{m_k}^k).Z_k) \leq \dim(Y_k \times_{P_k} Z_k) + \dim P_k - \dim X_k .$$

Consider now a general element α in

$$H_1^1 \times \cdots \times H_{m_1}^1 \times H_1^2 \times \cdots \times H_1^n \times \cdots \times H_{m_n}^n .$$

Then Proposition 1.10 implies that one has

$$\dim(Y \cap \alpha.Z) \leq \max\{\dim(Y_k \times_{P_k} Z_k) + \dim P_k - \dim X_k \mid k = 1, \ldots n\}.$$

(2) Let us consider now the case when in (1) all nonempty fibers of the morphism $\kappa|_Y : Y \to P$ are of the same dimension and all morphisms $\kappa|_{Y_k} : Y_k \to P_k$ are dominant. By Remark 1.9 (1) we have $\dim(Y_k \times_{P_k} Z_k) = \dim Y_k + \dim Z_k - \dim P_k$ for every $k = 1, \ldots, n$. Hence,

$$\dim(Y \cap \alpha.Z) \leq \max\{\dim Y_k + \dim Z_k - \dim X_k \mid k = 1, \ldots n\}.$$

Furthermore, if, additionally, all nonempty fibers of $\kappa : X \to P$ are of the same dimension then $\dim Y_k - \dim X_k = \dim Y - \dim \kappa^{-1}(\kappa(Y))$ and we have the following.

Proposition 1.12. *Let X and P be smooth irreducible algebraic varieties and $\kappa : X \to P$ be a flat morphism. Let a group $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that the G-action transforms every fiber $\kappa^{-1}(p)$ into itself and, furthermore, the restriction of the G-action to $\kappa^{-1}(p)$ is transitive for every $p \in P$. Let Y and Z*

be locally closed reduced subschemes of X such that all the assumptions of Remark 1.11(2) for Y and κ are satisfied (i.e., the nonempty fibers of the morphism $\kappa|_Y : Y \to \kappa(Y)$ are of the same dimension and every morphism $\kappa|_{Y_k} : Y_k \to P_k$ is dominant). Then there exist subgroups $H_1, \ldots, H_m \in \mathcal{G}$ such that for any Y and Z as above one can find a Zariski dense open subset $U = U(Y,Z) \subseteq H_1 \times \cdots \times H_m$ so that for every element $(h_1, \ldots, h_m) \in U_k$ we have the inequality $\dim(Y \cap (h_1 \cdot \cdots \cdot h_m).Z) \le \dim Y + \dim Z - \dim \kappa^{-1}(\kappa(Y))$.

2. Flexible varieties

Definition 2.1.

(1) A derivation σ on the ring A of regular functions on a quasi-affine algebraic variety X is called locally nilpotent if for every $0 \ne a \in A$ there exists a natural n for which $\sigma^n(a) = 0$. For the smallest n with this property one defines the degree of a with respect to σ as $\deg_\sigma a = n - 1$. This derivation can be viewed as a vector field on X which we also call locally nilpotent. The flow of this vector field is an algebraic $\mathbb{G}_a$-action on X, i.e., the action of the group $(\mathbf{k}, +)$ which can be viewed as a one-parameter unipotent group U in the group $\mathrm{Aut}(X)$ of all algebraic automorphisms of X. In fact, every $\mathbb{G}_a$-action is generated by a locally nilpotent vector field (e.g., see [Fr]).

(2) A smooth quasi-affine algebraic variety X of dimension at least 2 is called flexible if for every $x \in X$ the tangent space $T_x X$ is spanned by the tangent vectors to the orbits of one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ through x.

(3) The subgroup $\mathrm{SAut}(X)$ of $\mathrm{Aut}\, X$ generated by all one-parameter unipotent subgroups is called special.

We have the following [AFKKZ, Thm. 01] and [FKZ, Thm. 2.12].

Theorem 2.2. *For every irreducible smooth quasi-affine algebraic variety X of dimension at least 2 the following are equivalent:*

  (i) *the special subgroup $\mathrm{SAut}(X)$ acts transitively on X;*
  (ii) *the special subgroup $\mathrm{SAut}(X)$ acts infinitely transitively on X (i.e., for every natural m the action is m-transitive[5]);*
  (iii) *X is flexible.*

Definition 2.3.

(1) For every locally nilpotent vector field σ and each function $f \in \mathrm{Ker}\, \sigma$ from its kernel the field $f\sigma$ is called a replica of σ. Recall that such a replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on X. Then $G_{\mathcal{N}} \subset \mathrm{SAut}(X)$ denotes the subgroup that is generated by all flows of elements of $\mathcal{N}$. We say that $G_{\mathcal{N}}$ is the subgroup of $\mathrm{SAut}(X)$ that is generated by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called saturated if $\mathcal{N}$ is closed under conjugation by elements in $G_{\mathcal{N}}$ and for every $\sigma \in \mathcal{N}$ each replica of σ is also contained in $\mathcal{N}$.

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[5]Recall that a group G acts m-transitively on a space Y if for any two m-tuples $(y_1, \ldots, y_m)$ and $(y_1', \ldots, y_m')$ of distinct points in Y there is an element $\alpha \in G$ such that $\alpha(y_i) = y_i'$ for every $i = 1, \ldots, m$.

Definition 2.4. Let $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on X and $G = G_\mathcal{N}$ be the subgroup of $\mathrm{SAut}(X)$ that is generated by $\mathcal{N}$. Then X is called G-flexible if for any $x \in X$ the vector space $T_x X$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at x.

Remark 2.5. A priori the notion of G-flexibility depends on $\mathcal{N}$ while $\mathcal{N}$ is not determined uniquely by the group $G = G_\mathcal{N}$. However, the following generalization of Theorem 2.2 (see [FKZ, Thm. 2.12]) shows that this notion depends on G only.

Theorem 2.6. *Let X be an irreducible smooth quasi-affine algebraic variety of dimension at least 2 and G be the subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X. Then the following are equivalent:*

  (i) *G acts transitively on X;*
  (ii) *G acts infinitely transitively on X;*
  (iii) *X is G-flexible.*

The next fact is a straightforward consequence of Theorem 2.6.

Proposition 2.7. *Let X be a G-flexible variety where G is as in Definition 2.4 and Δ be the diagonal in $X \times X$. Then the natural action of G on the variety $X \times X \setminus \Delta$ is transitive.*[6]

The following result will be very useful later in this paper.

Theorem 2.8 ([AFKKZ, Thm. 4.14 and Rem. 4.16]). *Let $x_1, \ldots, x_m$ be distinct points in a G-flexible variety X of $\dim X = n$ where G is generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X. Then there exists an automorphism $\alpha \in G \subset \mathrm{SAut}(X)$ such that it fixes the points $x_1, \ldots, x_m$ and for every i the linear map $\mathrm{d}\alpha|_{T_{x_i} X}$ coincides with a prescribed element β_i of $\mathrm{SL}(n, \mathbf{k})$. Furthermore, let $k \in \mathbb{N}$ and γ_i be a k-jet of an isomorphism between two étale neighborhoods of x_i in X which preserves x_i and a local volume form at x_i. Then α can be chosen so that for every $i = 1, \ldots, m$ the k-jet of α at x_i coincides with γ_i.*

By the Rosenlicht Theorem (e.g., see [PV, Thm. 2.3]) for X, A, and U as in Definition 2.1 (1) one can find a finite set of U-invariant functions $a_1, \ldots, a_m \in A$, which separate general U-orbits in X. They generate a morphism $\varrho : X \to Q$ into an affine algebraic variety Q (in particular, $\dim Q = \dim X - 1$ because general U-orbits are one-dimensional). Note that this set of invariant functions can be chosen so that Q is normal (since X is normal).

Definition 2.9. Such a morphism $\varrho : X \to Q$ into a normal Q is called a partial quotient. In the case when $a_1, \ldots, a_m$ generate the subring A^U of U-invariant elements of A such a morphism is called the categorical quotient.[7]

Proposition 2.10. *Let $G \subset \mathrm{SAut}(X)$ be generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on a smooth quasi-affine algebraic variety X which is G-flexible. Suppose that $\mathcal{N}_0$ is a nontrivial subset of $\mathcal{N}$ which is also saturated and closed under conjugation by elements of G. Then X is G_0-flexible where the group G_0 is generated by $\mathcal{N}_0$.*

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[6]Precaution: this natural action of G on $X \times X \setminus \Delta$ is not infinitely transitive.

[7]However, in general A^U is not finitely generated by Nagata's example. That is, why, following [FKZ], we prefer to formulate some results for partial quotients.

Proof. By Theorem 2.8 for any point $x \in X$ where $\sigma \in \mathcal{N}_0$ does not vanish we can find $\alpha_1, \ldots, \alpha_n \in G$ where $n = \dim X$ such that $\alpha_i(x) = x$ and the values of the fields $\alpha_{1*}(\sigma), \ldots, \alpha_{n*}(\sigma)$ at x generate $T_x X$. That is, $T_x X$ is generated by the values of the fields from $\mathcal{N}_0$. Since G acts transitively on X we can guarantee the same for every $x \in X$ using conjugation by elements in G. Thus, X is G_0-flexible which is the desired conclusion. $\square$

Definition 2.11. Let δ_1 and δ_2 be a pair of locally nilpotent vector fields on X. Suppose that $\operatorname{Ker} \delta_1$ and $\operatorname{Ker} \delta_2$ are finitely generated algebras (in particular, δ_i admits the categorical quotient morphism $\varrho : X \to Q_i$). We say that (δ_1, δ_2) is a compatible pair if

(i) the vector space $\operatorname{Span}(\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2)$ generated by elements from $\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2$ contains a nonzero ideal in $\mathbf{k}[X]$ (so-called associated ideal of the pair) and

(ii) some element $a \in \operatorname{Ker} \delta_2$ is of degree 1 with respect to δ_1, i.e., $\delta_1(a) \in \operatorname{Ker} \delta_1 \setminus \{0\}$.

Remark 2.12.

(1) In [KK] the vector field δ_2 was also allowed to be semi-simple, but we do not consider this case here.

(2) For every locally nilpotent vector field δ on X its nonzero replica $f\delta$ has the same kernel and for each $a \in \mathbf{k}[X]$ the degree of a with respect to δ coincides with its degree with respect to $f\delta$. Hence, for a compatible pair (δ_1, δ_2) and $f_i \in \operatorname{Ker} \delta_i \setminus \{0\}$, $i = 1, 2$ the pair $(f_1\delta_1, f_2\delta_2)$ is also compatible. Furthermore, the conjugation by any element of $\operatorname{Aut}(X)$ transfers (δ_1, δ_2) into another compatible pair.

(3) The assumption that $\operatorname{Ker} \delta_1$ and $\operatorname{Ker} \delta_2$ are finitely generated was unfortunately missed in [KK].

(4) It is worth mentioning that in (ii) the field $a\delta_1$ is complete and, furthermore, (ii) holds for any commuting pair of nontrivial non-equivalent[8] locally nilpotent derivations.

Notation 2.13. For every affine algebraic variety X we denote by $\operatorname{AVF}(X)$ the space of all algebraic vector fields on X. If K is an ideal in $\mathbf{k}[X]$ then $\operatorname{AVF}_K(X)$ is the subspace of $\operatorname{AVF}(X)$ generated by all fields of the form $f\sigma$ where $f \in K$ and $\sigma \in \operatorname{AVF}(X)$. Given a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X and a closed subvariety Z in X we denote by $\operatorname{Lie}_{\operatorname{alg}}^{\mathcal{N}}(X, Z)$ the Lie algebra generated by the complete vector fields vanishing on Z that are of the form $b\delta$ where $\delta \in \mathcal{N}$ and $b \in \mathbf{k}[X]$ has degree $\deg_\delta b \leqslant 1$. If $x \in X$ then μ_x is its vanishing maximal ideal in $\mathbf{k}[X]$ and for every $\mathbf{k}[X]$-module M its localization at μ_x will be denoted by $(M)_{\mu_x}$.

Lemma 2.14. *Let X and Z be as in Notation 2.13 and $I \subset \mathbf{k}[X]$ be the vanishing ideal of Z. Let M be a $\mathbf{k}[X]$-submodule of $\operatorname{AVF}(X)$ such that for every $x \in X \setminus Z$ the localization $(M)_{\mu_x}$ coincides with the localization of $\check{M} = \operatorname{AVF}(X)$ at μ_x. Then there exists $k > 0$ such that M contains $\operatorname{AVF}_{I^k}(X)$.*

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[8]Recall that two locally nilpotent derivations δ and σ on $\mathbf{k}[X]$ are equivalent if $\operatorname{Ker} \delta = \operatorname{Ker} \sigma$.

Proof. Since X is affine the $\mathbf{k}[X]$-module $\breve{M}$ is finitely generated, i.e., we have $\breve{M} = \sum_{i=1}^{m} \breve{M}_i$ where $\breve{M}_i = \mathbf{k}[X]\sigma_i$ for some vector field $\sigma_i \in \mathrm{AVF}[X]$. Let $M_i = M \cap \breve{M}_i$. Note that $\breve{M}_i$ has a natural structure of the ring $\mathbf{k}[X]$ and M_i can be viewed as an ideal K_i in $\mathbf{k}[X]$. Since the operations of localization and intersection commute we have $(M_i)_{\mu_x} = (\breve{M}_i)_{\mu_x}$ for every $x \notin Z$. Hence, the zero locus of K_i is contained in Z. By Nullstellensatz there exists k_i for which $I^{k_i} \subset K_i$ and, therefore, $I^{k_i}\breve{M}_i \subset M_i$. Letting $k = \max\{k_i | i = 1, \ldots, m\}$ we get the desired conclusion. $\square$

Theorem 2.15. *Let X, Z and $\mathcal{N}$ be as in Notation 2.13 and $I \subset \mathbf{k}[X]$ be the vanishing ideal of Z. Let $G \subset \mathrm{SAut}(X)$ be the group generated by $\mathcal{N}$ and let $X \setminus Z$ be G-flexible. Suppose that X admits a pair of compatible locally nilpotent vector fields δ_1 and $\delta_2 \in \mathcal{N}$. Then for some $k \geqslant 1$ the algebra $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ contains the space $\mathrm{AVF}_{I^k}(X)$.*

Proof. Let $\varrho_i : X \to Q_i := \mathrm{Spec}\,\mathrm{Ker}\,\delta_i$ be the quotient morphism associated with δ_i, let Z_i be the closure of $\varrho_i(Z)$ in Q_i, and let a be as in Definition 2.11. Choose a nonzero function $h_i \in \mathrm{Ker}\,\delta_i \simeq \mathbf{k}[Q_i]$ that vanishes on Z_i and note that for every $f_i \in \mathrm{Ker}\,\delta_i$, $i = 1, 2$ the fields $f_1 h_1 \delta_1, a f_2 h_2 \delta_2, a f_1 h_1 \delta_1$, and $f_2 h_2 \delta_2$ are contained in $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$. Hence,

$$[f_1 h_1 \delta_1, a f_2 h_2 \delta_2] - [a f_1 h_1 \delta_1, f_2 h_2 \delta_2] = f_1 f_2 h_1 h_2 \delta_1(a)\delta_2 \tag{5}$$

also belongs to $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$. By condition (i) in Definition 2.11 $\mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$ contains a nonzero ideal J. Hence, $J_2 := h_1 h_2 \delta_1(a)J \subset J \cap I$ is also a nonzero ideal in $\mathbf{k}[X]$. By formula (5) one has $J_2 \delta_2 \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$.

Let $\widetilde{\mathcal{N}}$ be the set $\{\delta''\}$ of locally nilpotent vector fields $\delta'' \in \mathcal{N}$ such that for some $\delta' \in \mathcal{N}$ the pair (δ', δ'') is compatible. By Remark 2.12 (2) this set is saturated and closed under conjugation by elements of G. By Proposition 2.10 $\widetilde{\mathcal{N}}$ generates a subgroup $\widetilde{G}$ of $\mathrm{SAut}(X \setminus Z)$ such that $X \setminus Z$ is $\widetilde{G}$-flexible. Let $\sigma_1, \ldots, \sigma_m \in \widetilde{\mathcal{N}}$, $\sigma_2 = \delta_2$ and J_i plays the same role for σ_i as J_2 above for δ_2, i.e., $J_i \sigma_i \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$. Applying [AFKKZ, Prop. 1.8] (with H_i being the one-parameter unipotent group associated with σ_i) we can suppose that for every $x \in X \setminus Z$ the values of the fields $\sigma_1, \ldots, \sigma_m$ generate $T_x X$. Set $L_1 = J_1 \cdot \cdots \cdot J_m$, i.e., $L_1 \sigma_i \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ for every i and L_1 is contained in I.

Let $\breve{M} = \mathrm{AVF}(X)$ and W_1 be the zero locus of L_1. Then $M = \sum_{i=1}^{m} L_1 \sigma_i$ is a $\mathbf{k}[X]$-submodule of $\breve{M}$ such that for every point $x \in X \setminus W_1$ and every nonzero $v \in T_x X$ there exists a vector field $\sigma \in M$ whose value at x is v. Hence, we have $(\breve{M})_{\mu_x} = \mu_x(\breve{M})_{\mu_x} + (M)_{\mu_x}$ for all $x \in X \setminus W_1$ and the Nakayama lemma [AM, Cor. 2.7] implies that $(M)_{\mu_x} = (\breve{M})_{\mu_x}$. By Lemma 2.14 M contains $\mathrm{AVF}_{K_1}(X)$ where K_1 is some power of the vanishing ideal of W_1. In particular, $\mathrm{AVF}_{K_1}(X) \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$.

Using conjugations by elements of $\widetilde{G}$, we can transform K_1 into a sequence of ideals $K_1, \ldots, K_k \subset I$ such that for every $x \in X \setminus Z$ there exists i for which x is not in the zero locus of K_i. Consider the $\mathbf{k}[X]$-module $N = \sum_{i=1}^{k} \mathrm{AVF}_{K_i}(X)$. By construction $(N)_{\mu_x}$ coincides with the localization of $\mathrm{AVF}(X)$ at μ_x for every $x \in X \setminus Z$. Hence, by Lemma 2.14 N contains $\mathrm{AVF}_{I^k}(X)$ for some $k > 0$. Since $\mathrm{AVF}_{K_i}(X) \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ for every i, we see that $N \subset \mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ which yields the desired conclusion. $\square$

Remark 2.16.

(1) Let the assumptions of Theorem 2.15 hold with the following exception: we do not assume that the fields from $\mathcal{N}$ are tangent to Z and that $X \setminus Z$ are G-flexible, but we suppose that X is G-flexible. Then the conclusion of Theorem 2.15 remains valid. Indeed, consider the saturated subset $\mathcal{N}_Z$ of $\mathcal{N}$ that consists of all fields that vanish on Z and let $G_Z \subset \mathrm{SAut}(X)$ be the group generated by $\mathcal{N}_Z$. Then $X \setminus Z$ is G_Z-fleixible by [FKZ]. Hence, replacing $\mathcal{N}$ and G by $\mathcal{N}_Z$ and G_Z respectively we get the assumptions and, therefore, the conclusion of Theorem 2.15.

(2) If $\mathcal{N}$ is the set of all locally nilpotent vector fields on X and for some $k \geqslant 1$ the algebra $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ contains the space of all algebraic vector fields on X that vanish on Z with multiplicity at least k then we say that the pair (X, Z) has the algebraic density property. In this terminology Theorem 2.15 is a generalization of [KK, Thm. 4] which established the algebraic density property for pairs of the form $(\mathbb{C}^n, Z)$ where Z is a closed subvariety of $\mathbb{C}^n$ with $\dim Z \leqslant n - 2$. The algebraic density property in the complex case has some remarkable consequences. In particular, as in [KK] we get two interesting facts which will not be used in the sections below.

Theorem 2.17 (cf. [F2, Thm. 4.10.6]). *Let X be a complex affine flexible variety and Z be a closed subvariety of X whose codimension is at least 2. Suppose that X admits a pair of compatible vector fields. Let $\Phi_t : \Omega_0 \to \Omega_t = \Phi_t(\Omega_0) \subset X \setminus Z$ $(t \in [0,1])$ be a $\mathcal{C}^1$-isotopy consisting of injective holomorphic maps between Runge domains[9] with $\Phi_0 = \mathrm{Id}_{\Omega_0}$. Suppose also that each Ω_t is Stein. Then Φ_1 can be approximated uniformly on compacts of Ω_0 by holomorphic automorphisms of X identical on Z.*

Proof. Consider generators $f_1, \ldots, f_m$ of the vanishing ideal $I \subset \mathbf{k}[X]$ of Z. These functions have no common zeros on any Ω_t. By the weak Nullstellensatz for Stein spaces (e.g., see [On, Thm. 4.25]) there are holomorphic functions $g_1^t, \ldots, g_m^t$ on Ω_t for which $\sum_{i=1}^m f_i g_i^t = 1$ on Ω_t. Since Ω_t is a Runge domain on every compact $K_t \subset \Omega_t$ these functions $g_1^t, \ldots, g_m^t$ can be uniformly approximated by global holomorphic functions on X, i.e., we get a holomorphic function h on X vanishing on Z which is as close to 1 on K_t as we wish. Furthermore, replacing h by h^k for a given $k \geqslant 1$ we can suppose that h is contained in $\widetilde{I}^k$ where $\widetilde{I}$ is the vanishing ideal of Z in the algebra $\mathrm{Hol}(X)$ of holomorphic functions on X. Hence, every holomorphic vector field ν_t on Ω_t can be approximated in the compact-open topology by holomorphic fields from $\mathrm{HVF}_{\widetilde{I}^k}(X)$ where $\mathrm{HVF}(X)$ is the space of all holomorphic vector fields on X. Since X is affine every holomorphic function (resp. vector field) on X can be approximated by regular functions (resp. algebraic vector fields) on X in the compact-open topology and also $\widetilde{I}$ is generated by I over $\mathrm{Hol}(X)$ (e.g., see [Ka, Thm. 4]). Hence, ν_t can be approximated in the compact-open topology by elements of $\mathrm{AVF}_{I^k}(X)$ and, therefore, by Theorem 2.15 by elements of $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$. Let $\widetilde{\nu}_t$ be an element of $\mathrm{Lie}_{\mathrm{alg}}^{\mathcal{N}}(X, Z)$ uniformly close

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[9]Recall that an open subset Ω of a Stein manifold Y is called a Runge domain if every holomorphic function on Ω can be approximated in the compact-open topology by holomorphic functions on Y.

to ν_t on K_t and let ψ_s and $\widetilde{\psi}_s$ be the flows of ν_t and $\widetilde{\nu}_t$ respectively with s being the time parameter. Suppose that for some $s_0 > 0$ both of these flows are defined for every $x \in K_t$. Recall that all generators of $\mathrm{Lie}^{\mathcal{N}}_{\mathrm{alg}}(X, Z)$ can be chosen as complete algebraic vector fields that vanish on Z. By [F2, Cor. 4.8.4] $\widetilde{\psi}_{s_0}$ can be approximated by compositions of flows of these generators. Hence, we have the following

$$\text{the element } \psi_{s_0} \text{ of the flow of } \nu_t \text{ can be uniformly approximated} \atop \text{on } K_t \text{ by global holomorphic automorphisms of } X \text{ identical on } Z. \qquad (*)$$

To make use of $(*)$ following the proof of [F2, Thm. 4.9.2] we consider the trace $\widetilde{\Omega} = \{(t, z) \mid t \in [0, 1], z \in \Omega_t\}$ of the isotopy $\{\Phi_t\}$ in $\mathbb{R} \times X$ and we treat Φ_t as the flow of the continuous time dependent vector field

$$V(t, z) = \dot{\Phi}_t(\Phi_t^{-1}(z))$$

where dot denotes the derivative on t. The field V is continuous on $\widetilde{\Omega}$ and holomorphic on Ω_t for every fixed $t \in [0, 1]$. Divide the interval $[0, 1]$ into N subintervals $[t_k, t_{k+1}]$ of length $1/N$ for a given natural N and consider the locally constant holomorphic vector field $\widetilde{V}(t, z)$ which is equal to $V(t_k, z)$ on every interval $[t_k, t_{k+1}]$. Let φ_t be the flow of $\widetilde{V}$. Similarly to the estimates in the proof of [F2, Thm. 4.8.2] one can check that as $N \to +\infty$ the flow φ_t converges to Φ_t uniformly on compacts in Ω for all $t \in [0, 1]$. Since by $(*)$ φ_t can be approximated by global holomorphic automorphisms of X identical on Z so can Φ_t. This yields the desired conclusion. $\square$

Corollary 2.18. *Let X be a complex affine flexible variety of dimension n and Z be a closed subvariety of X whose codimension is at least 2. Suppose that X admits a pair of compatible vector fields. Then every $x \in X \setminus Z$ has a neighborhood Ω in $X \setminus Z$ that is a Fatou–Bieberbach domain (i.e., U is biholomorphic to $\mathbb{C}^n$).*

Proof. Since $X \setminus Z$ is flexible it suffices to prove this statement for some point x_0 in $X \setminus Z$ only. Choose a dominant morphism $\varphi : X \to \mathbb{C}^n$ and choose $x_0 \in X \setminus Z$ so that for a ball $B_0 \subset \mathbb{C}^n$ with center at $\varphi(x_0)$ the component B of the preimage $\varphi^{-1}(B_0)$, containing x_0, is naturally biholomorphic to B_0. Taking B small enough we can suppose that there are regular functions $g_1, \ldots, g_m \in \mathbb{C}[X]$ such that each $|g_i|$ does not exceed 1 on B while it is greater than 1 at each point of $\varphi^{-1}(B_0) \setminus B$. This implies that B is $\mathrm{Hol}(X)$-convex. Hence, it is a Runge domain in X by the Oka-Weil theorem [F2, Thm. 2.2.5]. For an analytic coordinate system $(z_1, \ldots, z_n)$ on B with the origin at x_0 consider the homothety $\Phi : (z_1, \ldots, z_n) \to (z_1/2, \ldots, z_n/2)$. By Theorem 2.17 Φ can be approximated by a global holomorphic automorphism F of X identical on Z. Since $F(B) \subset B$ by the Brouwer fixed point theorem F has a fixed point in B. Without loss of generality we can suppose that this point is x_0 and reducing the size of B we can suppose that B contains no other fixed point but x_0. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the map $\mathrm{d}F$ at x_0 must be close to those of $\mathrm{d}\Phi$ which are $1/2$. In particular, we can suppose that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$ and $|\lambda_1|^2 < |\lambda_n|$. In particular, these eigenvalues satisfy the assumptions of [RR,

Thm. 9.1] and we can copy the argument of Rosay and Rudin. Namely, consider the basin Ω of attraction of F, i.e., $x \in \Omega$ if there exists $N > 0$ such that $F^N(x) \in B$. By continuity for every y in a small neighborhood of x one has $F^N(y) \in B$. Thus Ω is open (in the standard topology). Treating B as a neighborhood of the origin in $T_{x_0}X \simeq \mathbb{C}^n$ and letting $A = dF$ we can consider an injective map $A^{-M} \circ F^M : F^{-N}(B) \to \mathbb{C}^n$ where $M \geqslant N$. Choose positive constants $\beta > |\lambda_1|$ and $\alpha < |\lambda_n|$ such that $\beta^2 < \alpha$. Since the local Jacobi matrix of the map $A^{-1} \circ F$ at x_0 is the identity matrix the computation in [RR, Thm. 9.1][10] implies that for every compact $K \subset F^{-N}(B)$ there exist some $M_0 > N$ and a positive real number b such that for every $x \in K$ one has

$$\|A^{-M} \circ F^M(x) - A^{-M-1} \circ F^{M+1}(x)\| = \|A^{-M}(F^M(x) - A^{-1} \circ F(F^M(x)))\|$$
$$\leqslant b(\beta^2/\alpha)^M$$

where $M \geqslant M_0$. Hence, we have a well-defined holomorphic map $\Phi : \Omega \to \mathbb{C}^n$ where $\Psi = \lim_{M \to \infty} A^{-M} \circ F^M$. Since at every point of $F^{-N}(B)$ the local Jacobian of $A^{-M} \circ F^M$ does not vanish we have the following alternative: either the local Jacobian of Ψ does not vanish or it is identically zero. However, the local Jacobian of Ψ at x_0 is 1 and we have the former, i.e., Ψ is an open map. Furthermore, it is injective since otherwise the maps $A^{-M} \circ F^M$ are not injective for large M. Note also that $\Psi = A^{-1} \circ \Psi \circ F$, i.e., Ψ and $A^{-1} \circ \Psi$ have the same range. Since the linear operator A^{-1} is an expansion it follows that $\Psi(\Omega) = \mathbb{C}^n$. Thus the basin Ω of attraction of F is biholomorphic to $\mathbb{C}^n$ and we have the desired conclusion. $\square$

Remark 2.19. For $X = \mathbb{C}^n$ the question about Fatou–Bieberbach domains in the complement of subvariety of codimension 2 was posed by Siu and answered by Buzzard and Hubbard [BH] (see also [F1]).

3. Relative version of Theorem 2.8

Let us prove first the following analogue of [AFKKZ, Thm. 3.1].

Theorem 3.1. *Let $\varrho : X \to Q$ be a dominant morphism of quasi-affine algebraic varieties, Q_0 be a Zariski open dense subset of Q, and $X_0 = \varrho^{-1}(Q_0)$. Let every fiber $\varrho^{-1}(q)$, $q \in Q_0$ be G-flexible where $G \subset \mathrm{SAut}(X_0)$ is a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X_0 which are tangent to the fibers of ϱ. Suppose that $q_1, \ldots, q_m$ are distinct points in $\varrho(X_0)$ and $\alpha_i \in G|_{\varrho^{-1}(q_i)}$. Then there exists an automorphism α of X over Q such that $\alpha|_{\varrho^{-1}(q_i)} = \alpha_i$ for every $i = 1, \ldots, m$ and $\alpha|_{X_0} \in G$.*

Proof. Suppose first that $Q = Q_0$. For a locally nilpotent vector field σ denote by $\exp(t\sigma)$ the element of the one-parameter group associated with σ at time $t \in \mathbf{k}$. By definition α_i is of the form

$$\exp(t_{1,i}\sigma_{1,i}) \circ \cdots \circ \exp(t_{n(i),i}\sigma_{n(i),i})|_{\varrho^{-1}(q_i)}$$

---

[10]See the formula immediately after formula (9) in [RR, Thm. 9.1]. Formally, this formula is proven for $X = \mathbb{C}^n$ but it works in our case as well without change.

where $n(i)$ is a natural number depending on i and $t_{j,i} \in \mathbf{k}$. Choose regular functions f_i on Q such that $f_i(q_i) = 1$ and $f_i(q_j) = 0$ for every $j \neq i$. Using the natural embedding $\mathbf{k}[Q] \subset \mathbf{k}[X]$ we treat f_i as a function on X. Then by the assumption $f_i \in \operatorname{Ker} \sigma$ for every $\sigma \in \mathcal{N}$, i.e., the replica $f_i \sigma \in \mathcal{N}$. It remains to put

$$\alpha = \exp(t_{1,1} f_1 \sigma_{1,1}) \circ \cdots \circ \exp(t_{n(1),1} f_1 \sigma_{n(1),1}) \circ \cdots$$
$$\circ \exp(t_{1,m} f_m \sigma_{1,m}) \circ \cdots \circ \exp(t_{n(m),m} f_m \sigma_{n(m),m})$$

and we are done in the case of $Q = Q_0$.

In the general case this α is only an automorphism of X_0 and its extension to X may have poles on $X \setminus X_0$. However, one can choose f_i from the above so that they vanish on $Q \setminus Q_0$ with sufficiently high multiplicity. That is, we can assume that each $f_i \sigma_{j,i}$ is a regular vector field on X that vanishes on $X \setminus X_0$. Then the extension of α becomes a regular automorphism on X whose restriction to $X \setminus X_0$ is the identity map. $\square$

Remark 3.2.

(1) Let Q be affine. Then Theorem 3.1 remains valid with the same proof when the finite set $q_1, \ldots, q_m$ is replaced by a collection of m disjoint closed suvarieties of Q which are contained in Q_0.

(2) One can consider a more general situation when every α_i is the restriction of an element of G to a k-infinitesimal neighborhood V_i of $\varrho^{-1}(q_i)$.[11] Then we can still find $\alpha \in G$ for which $\alpha|_{V_i} = \alpha_i$, $i = 1, \ldots, m$. For this it suffices to require that each f_i vanishes with multiplicity at least k at q_j (where $j \neq i$) and takes the value 1 with multiplicity at least k at q_i.

Notation 3.3. Further in this section X is a smooth algebraic variety of dimension n, $G \subset \operatorname{SAut}(X)$ is a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X, $G_z \subset G$ is the isotropy group of a point $z \in X$, $\mathfrak{m}_z$ is the maximal ideal in the local ring $\mathcal{O}_{X,z}$ at z, and $A_m(X, z) = \mathfrak{m}_z / \mathfrak{m}_z^{m+1}$ (in particular, $A_1(X, z)$ coincides with the cotangent space $T_z^* X$). We consider the set $\operatorname{Aut}(A_m(X, z))$ of $\mathbf{k}$-algebra isomorphisms $f : A_m(X, z) \to A_m(X, z)$ satisfying the following condition:

$$\text{the Jacobian } J(f) = 1 \mod \mathfrak{m}_z^{m+1}. \tag{6}$$

Let $u_1, \ldots, u_n \in \mu_z$ be such that they generate the cotangent space. Then we call the n-tuple $(u_1, \ldots, u_n)$ a local coordinate system at z (indeed, if the ground field $\mathbf{k} = \mathbb{C}$ then $u_1, \ldots, u_n$ form a local analytic coordinate system in a small neighborhood of z in the standard topology). In terms of this local coordinate system elements of $\operatorname{Aut}(A_m(X, z))$ can be described as follows. The $\mathbf{k}$-algebra A_m is contained in the quotient $A/\mathfrak{m}_A^{m+1}$ of the local ring $A = \mathbf{k}[[u_1, \ldots, u_n]]$ of formal power series with respect to the power of its maximal ideal $\mathfrak{m}_A$. Therefore, we

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[11] For every reduced subvariety Y of X with a defining ideal $I \subset \mathbf{k}[X]$ one can treat an automorphism of the k-infinitesimal neighborhood of Y as an automorphism of the ring $\mathbf{k}[X]/I^k$.

treat any map $f \in \mathrm{Aut}(A_m(X, z))$ as an n-tuple of polynomials $(F_1, \ldots, F_n) \in (A/\mathfrak{m}_A^{m+1})^n$ in n variables $u_1, \ldots, u_n$ of degree at most m such that they vanish at the origin and the determinant of the matrix $[\partial F_i/\partial x_j]_{i,j}$ is 1 modulo terms of degree higher than m. In particular, each F_i is the sum of homogeneous k-forms where k runs from 1 to m. Let F_i' be the m-form present in this sum and $\theta_{z,m}$ be the linear map from $\mathrm{Aut}(A_m(X, z))$ to the space of n-tuples of m-forms given by

$$\theta_{z,m}(f) = (F_1', \ldots, F_n').$$

Suppose also that $\lambda(f)$ is the n-tuple of linear parts of f. In particular, $\lambda(f) \in \mathrm{SL}(n, \mathbf{k})$ (because of the assumption on the Jacobian). Note that $\mathrm{SL}(n, \mathbf{k})$ admits different natural actions on the space $\theta_{z,m}(\mathrm{Aut}(A_m(X, z)))$ of n-tuples $F(\bar{u})$ of m forms in n variables (i.e., $\bar{u} = (u_1, \ldots, u_n)$) for which we use the following notations

$$\lambda._l F(\bar{u}) = \lambda(F(\bar{u})), \quad \lambda._r F(\bar{u}) = F(\lambda(\bar{u})), \quad \text{and} \quad \lambda.F(\bar{u}) = \lambda^{-1}(F(\lambda(\bar{u}))).$$

Lemma 3.4. *Let Notation* 3.3 *hold and* $\mathrm{Aut}_{m-1}(A_m(X, z))$ *be the subgroup of the group* $\mathrm{Aut}(A_m(X, z))$ *consisting of those automorphisms* f *for which* $f \equiv \mathrm{id}$ *mod* $\mathfrak{m}_z^m$ *(i.e.,* $(f - \theta_{z,m}(f))(\bar{u})$ *coincides with the n-tuple* $(u_1, \ldots, u_n)$*). Then we have the following.*

(a) *For every* $f \in \mathrm{Aut}_{m-1}(A_m(X, z))$ *and* $g \in \mathrm{Aut}(A_m(X, z))$ *one has*

$$g - \theta_{z,m}(g) = f \circ g - \theta_{z,m}(f \circ g) = g \circ f - \theta_{z,m}(g \circ f),$$
$$\theta_{z,m}(g \circ f) = \lambda(g)._l(\theta_{z,m}(f)) + \theta_{z,m}(g) \text{ and } \theta_{z,m}(f \circ g) = \lambda(g)._r(\theta_{z,m}(f)) + \theta_{z,m}(g),$$
$$g^{-1} \circ f \circ g \in \mathrm{Aut}_{m-1}(A_m(X, z)) \text{ and } \theta_{z,m}(g^{-1} \circ f \circ g) = \lambda(g).\theta_{z,m}(f).$$

In particular, if g *is also in* $\mathrm{Aut}_{m-1}(A_m(X, z))$ *then* $\theta_{z,m}(g \circ f) = \theta_{z,m}(f \circ g) = \theta_{z,m}(f) + \theta_{z,m}(g)$.

(b) *For* $m \geqslant 2$ *the set* $\mathcal{F}_{z,m} := \theta_{z,m}(\mathrm{Aut}_{m-1}(A_m(X, z)))$ *is the linear space of n-tuples* $F(\bar{u})$ *of m-forms in n variables of divergence zero and the* $\mathrm{SL}(n, \mathbf{k})$*-action on* $\mathcal{F}_{z,m}$ *given by* $\lambda.F(\bar{u})$ *is irreducible.*

(c) *There is a natural homomorphism* $J_{z,m} : G_z \to \mathrm{Aut}(A_m(X, z))$ *such that in the case when X is G-flexible one has* $J_{z,1}(G_z) = \mathrm{Aut}(A_1(X, z)) \simeq \mathrm{SL}(n, \mathbf{k}) = \mathrm{SL}(T_z^* X)$.

(d) *If* ∂ *is a locally nilpotent vector field on X with a zero of order $m \geq 2$ at z then* $\theta_{z,m}(J_{z,m}(\exp(t\partial))) = t\theta_{z,m}(J_{z,m}(\exp(\partial)))$.

Proof. Statement (a) is straightforward (see also [AFKKZ, Lem. 4.12]). The first clause in statement (b) can be found in [AFKKZ, Lem. 4.13] and the second in [Pr, IX.10.2]. Define $J_{m,z}(\alpha)$ as the operation of taking the m-jet of $\alpha \in G_z$. Since $\alpha \in \mathrm{SAut}(X)$ we see that the Jacobian $J(\alpha) \equiv 1$. Hence, $J_{z,m}(G_z) \subset \mathrm{Aut}(A_m(X, z))$. The fact that $J_{z,1}(G_z) \simeq \mathrm{SL}(T_z^* X)$ when X is G-flexible follows from [AFKKZ, Cor. 4.3] which concludes (c).

For (d) note (as in [AFKKZ, Lem. 4.12]) that $\exp(\partial) \in G_{z,m}$ and it induces the map $\mathrm{id} + \hat{\partial} \in \mathrm{Aut}_{m-1}(A_m)$ where $\hat{\partial}$ denotes the derivation on A_m induced by ∂. Hence, $\exp(t\partial)$ induces $\mathrm{id} + t\hat{\partial}$ which is (d) and we are done. $\square$

Notation 3.5. In addition to Notation 3.3 suppose that $\varrho : X \to Q$ is a smooth morphism of smooth quasi-affine algebraic varieties such that every fiber $Y = \varrho^{-1}(q)$, $q \in Q$ is of dimension at least 2 and G-flexible (i.e., we are under the assumptions of Theorem 3.1 and G is generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields such that every $\delta \in \mathcal{N}$ is tangent to the fibers of ϱ). In particular, for every $z \in X$ and $q = \varrho(z)$ a local coordinate system can be chosen in the form $(\bar{u}, \bar{v}) := (u_1, \dots, u_k, v_1, \dots, v_{n-k})$ where u_i and v_i are regular functions on X such that $v_1, \dots, v_{n-k}$ are the lifts of functions on Q that form a local coordinate system at $q \in Q$ while the restriction $(u_1, \dots, u_k)$ yields a local coordinate system at $z \in Y$.

Note that in such a coordinate system for every $\alpha \in G_z$ its image $J_{z,m}(\alpha) \in \text{Aut}(A_m(X, z))$ is of the form

$$J_{z,m}(\alpha) = (F_1(\bar{u}, \bar{v}), \dots, F_k(\bar{u}, \bar{v}), v_1, \dots, v_{n-k}) \tag{7}$$

where F_i is a polynomial of degree at most m and the determinant of the matrix $[\frac{\partial F_i}{\partial u_j}]_{i,j}$ is 1 up to terms of degree higher than m.

Lemma 3.6. *Let $\delta \in \mathcal{N}$ and δ_q be the restriction of δ to $Y = \varrho^{-1}(q)$ for $q \in Q$. Then a partial quotient morphism $\tau : X \to P$ (resp. $\tau_q : Y \to P_q$) of δ (resp. δ_q) can be chosen so that ϱ factors through τ and $\tau|_Y$ factors through τ_q (i.e., $\varrho = \theta \circ \tau$ and $\tau|_Y = \kappa_q \circ \tau_q$ for some morphisms $\theta : P \to Q$ and $\kappa_q : P_q \to P$).*

Proof. The quasi-affine variety Q is contained as an open subset in an affine variety $\check{Q}$. Under the natural embedding $\mathbf{k}[\check{Q}] \hookrightarrow \mathbf{k}[X]$ generators of $\mathbf{k}[\check{Q}]$ can be treated as elements of $\text{Ker}\,\delta$. Thus we can choose τ so that $\mathbf{k}[P]$ contains these generators which implies that ϱ factors through τ, i.e., $\varrho = \theta \circ \tau$. Similarly, we can choose τ_q so that $\mathbf{k}[P_q]$ contains generators of the ring of regular functions on an affine variety in which $\theta^{-1}(q)$ is an open subset. This implies that $\tau|_Y$ factors through τ_q. $\square$

Lemma 3.7. *Let Z be a finite subset of $Y = \varrho^{-1}(q)$ and $z \in Z$. Suppose that there exists $\delta \in \mathcal{N}$ such that $\delta|_Y \neq 0$ and partial quotient morphisms τ and τ_q from Lemma 3.6 can be chosen so that*

$$\kappa_q(P_q) \text{ is dense in } \theta^{-1}(q). \tag{8}$$

Then such a field δ and a coordinate system $(\bar{u}, \bar{v})$ as in Notation 3.5 can be chosen so that the following holds:

(i) *δ induces a trivial derivation on $A_m(X, w)$ for every $w \in Z \setminus \{z\}$;*

(ii) *u_i belongs to the kernel $\text{Ker}\,\delta$ for every $i \geqslant 2$;*

(iii) *the derivation σ on $A_m(X, z) \subset A/\mathfrak{m}_A^{m+1}$ (where $A = \mathbf{k}[[\bar{u}, \bar{v}]]$) induced by δ coincides with $\sigma := \partial/\partial u_1$.*

Proof. Treating δ as a derivation $\delta : \mathbf{k}[X] \to \mathbf{k}[X]$ consider its conjugate $\widetilde{\delta} = g^* \circ \delta \circ (g^*)^{-1} : \mathbf{k}[X] \to \mathbf{k}[X]$ for $g \in G$. Similarly, let $\widetilde{\delta}_q = g|_Y^* \circ \delta_q \circ (g|_Y^*)^{-1}$. Note that $\text{Ker}\,\widetilde{\delta} = g^*(\text{Ker}\,\delta)$ and $\text{Ker}\,\widetilde{\delta}_q = g|_Y^*(\text{Ker}\,\delta_q)$. Therefore, for $\widetilde{\tau} = \tau \circ g : X \to P$ and $\widetilde{\tau}_q = \tau_q \circ g|_Y : Y \to P_q$ one has $\widetilde{\tau}^*(\mathbf{k}[P]) \subset \text{Ker}\,\widetilde{\delta}$ and $\widetilde{\tau}_q^*(\mathbf{k}[P_q]) \subset \text{Ker}\,\widetilde{\delta}_q$. Hence, these morphisms are invariant under the $\mathbb{G}_a$-actions associated with $\widetilde{\delta}$ and

$\widetilde{\delta}_q$ respectively. Each fiber of $\widetilde{\tau}$ (resp. $\widetilde{\tau}_q$) is the image of a fiber of τ (resp. τ_q) under the action of g. This implies that the general fibers of $\widetilde{\tau}$ and $\widetilde{\tau}_q$ are isomorphic to lines and these morphisms are partial quotient morphisms for $\widetilde{\delta}$ and $\widetilde{\delta}_q$ respectively. Furthermore, one can see that $\varrho = \varrho \circ g = \theta \circ \tau \circ g = \theta \circ \widetilde{\tau}$ and, similarly, $\widetilde{\tau}|_Y = \kappa_q \circ \widetilde{\tau}_q$. In particular, the assumption (8) holds for the locally nilpotent vector field $\widetilde{\delta}$. Since $\mathcal{N}$ is saturated $\widetilde{\delta} \in \mathcal{N}$ and we can replace δ by $\widetilde{\delta}$ while trying to achieve (i)-(iii) for the finite set Z.

Note that the validity of (i)-(iii) for the pair $(\widetilde{\delta}, Z)$ is equivalent to the validity of these conditions for the pair $(\delta, g^{-1}(Z))$. Indeed, (i) is obvious and for the rest let $(u_1, \ldots, u_k, v_1, \ldots, v_{n-k})$ be a local coordinate system at z such that (i)-(iii) are true for the pair $(\widetilde{\delta}, Z)$. Let $(\widehat{u}_1, \ldots, \widehat{u}_k, v_1, \ldots, v_{n-k})$ be a local coordinate system at $g^{-1}(z)$ such that $\widehat{u}_i = u_i \circ g$, i.e., the local form of g is

$$(\widehat{u}_1, \ldots, \widehat{u}_k, v_1, \ldots, v_{n-k}) \to (\widehat{u}_1 \circ g^{-1}, \ldots, \widehat{u}_k \circ g^{-1}, v_1, \ldots, v_{n-k}).$$

Since $u_i \in \operatorname{Ker} \widetilde{\delta}_i$ for $i \geqslant 2$ and $\widetilde{\delta} = g^* \circ \delta \circ (g^*)^{-1}$ we see that $\widehat{u}_i \in \operatorname{Ker} \delta$, i.e., we have (ii) for the pair $(\delta, g^{-1}(Z))$. Condition (iii) is equivalent to the fact that modulo t^2 the exponent $\exp(t\widetilde{\delta})$, $t \in \mathbf{k}$ has the following local form at z

$$(u_1, \ldots, u_k, v_1, \ldots, v_{n-k}) \to (u_1 + t + tf(\bar{u}, \bar{v}), u_2 \ldots, u_k, v_1, \ldots, v_{n-k})$$

where f is a function vanishing at the origin with multiplicity $m+1$ or higher. Since $\exp(t\widetilde{\delta}) = g^* \circ \exp(t\delta) \circ (g^*)^{-1}$ the local form of $\exp(t\delta)$ is

$$(\widehat{u}_1, \ldots, \widehat{u}_k, v_1, \ldots, v_{n-k}) \to (\widehat{u}_1 + t + tf(\widehat{u}_1, \ldots, \widehat{u}_k, \bar{v}), \widehat{u}_2 \ldots, \widehat{u}_k, v_1, \ldots, v_{n-k})$$

and we have (iii) for the pair $(\delta, g^{-1}(Z))$.

Therefore, we replace Z by $g(Z)$ while keeping δ intact. By virtue of infinite transitivity (Theorem 2.6) we can suppose now that Z consists of general points $\{w_i\}$ of Y with $z = w_1$. Hence, $p_i' = \tau_q(w_i)$ are distinct general (and, therefore, smooth) points p_i' of P_q such that for some neighborhood $U_i \subset P_q$ of p_i' one has a natural U_i-isomorphism $\tau_q^{-1}(U_i) \simeq U_i \times \mathbb{A}_\mathbf{k}$. Furthermore, we can suppose also that δ is nontrivial on $\tau_q^{-1}(p_1')$.

Let $p_i = \kappa_q(p_i')$. By (8) we can suppose that $\{p_i\}$ are general points of $\theta^{-1}(q)$, i.e., there is a function $f \in \mathbf{k}[P]$ that vanishes at each p_i, $i \geqslant 2$ with multiplicity at least m but has $f(p_1) = 1$ also with multiplicity at least m. Then replacing δ by its replica $f\delta \in \mathcal{N}$ we get (i).

Since p_1 is a smooth point of $\theta^{-1}(q)$ a local coordinate system at $p_1 \in P$ can be chosen in the form $(u_2, \ldots, u_k, v_1, \ldots, v_{n-k})$ where each u_i, $i \geqslant 2$ is a regular function on P. Taking u_1 as an appropriate extension to X of a coordinate function on the $\mathbb{G}_a$-orbit $\tau_q^{-1}(p_1') \simeq \mathbb{A}_\mathbf{k}$ we can treat $(u_1, u_2, \ldots, u_k, v_1, \ldots, v_{n-k})$ as a local coordinate system at $z \in X$. This is the desired coordinate system with δ being a desired derivation which concludes the proof. $\square$

Remark 3.8. Condition (8) is very mild and it is automatically true when q is a general point of Q.

Lemma 3.9. *Let $z \in Z \subset Y$ and $(\bar{u}, \bar{v})$ be as in Lemma 3.7. Suppose that $G_Z = \bigcap_{w \in Z} G_w$ and $G_{Z,z}^m$ is the subgroup of G_Z such that it induces the identity map on the m-th infinitesimal neighborhood of every point $w \in Z \setminus \{z\}$. Then the image $I_m := J_{z,m}(G_{Z,z}^m)$ contains all automorphisms of $A_m(X, z)$ with the following coordinate form*

$$(\bar{u}, \bar{v}) := (u_1, \ldots, u_k, v_1, \ldots, v_{n-k}) \to (u_1 + \ell_1^m(\bar{v}), \ldots, u_k + \ell_k^m(\bar{v}), v_1, \ldots, v_{n-k}) \quad (9)$$

where every $\ell_i^m(\bar{v})$ is an m-form in variables $v_1, \ldots, v_{n-k}$. Furthermore, for every $\lambda(\bar{u}) = (\lambda_1(\bar{u}), \ldots, \lambda_k(\bar{u}))$ in $\mathrm{SL}(T_z^*Y)$ and each k-tuple $(\ell_1^1(\bar{v}), \ldots, \ell_k^1(\bar{v}))$ of 1-forms the subgroup I_1 contains the following automorphism of T_z^*X

$$(\bar{u}, \bar{v}) \to (\lambda_1(\bar{u}) + \ell_1^1(\bar{v}), \ldots, \lambda_k(\bar{u}) + \ell_k^1(\bar{v}), v_1, \ldots, v_{n-k}). \quad (10)$$

Proof. Let σ be as in Lemma 3.7. Then for every $h \in \mathrm{Ker}\, \delta$ the automorphism $\exp(h\delta) \in G_{Z,z}^m$ induces the automorphism $\exp(h'\sigma)$ on $A_m(X, z)$ (where h' is the image of h in $A_m(X, z)$) of the following coordinate form

$$(\bar{u}, \bar{v}) \to (u_1 + h', u_2, \ldots, u_k, v_1, \ldots, v_{n-k}). \quad (11)$$

In particular, choosing h so that h' is equal to an m-form $l_1^m(\bar{v})$ we see that the automorphism

$$(\bar{u}, \bar{v}) \to (u_1 + \ell_1^m(\bar{v}), u_2, \ldots, u_k, v_1, \ldots, v_{n-k}) \quad (12)$$

is contained in I_m. Note the variety $(Y \setminus Z) \cup \{z\}$ is $G_{Z,z}^m$-flexible by [FKZ]. Therefore, by Theorem 2.8 for every $\lambda(\bar{u}) \in \mathrm{SL}(T_z^*Y)$ the subgroup I_1 contains an automorphism as in formula (10) for certain 1-forms $\ell_i^1(\bar{v})$.

Using notations of Lemma 3.4 we suppose that f and $g \in G_{Z,z}$ are such that $J_{z,1}(g)$ is given by formula (10) and $J_{z,m}(f)$ is given by formula (12). In particular, $J_{z,m}(f) \in \mathrm{Aut}_{m-1}(A_m(X, z))$ and $\theta_{z,m}(J_{z,m}(f)) = (\ell_1^m(\bar{v}), 0, \ldots, 0)$ while

$$\theta_{z,1}(J_{z,m}(g)) = (\lambda_1(\bar{u}) + \ell_1^1(\bar{v}), \ldots, \lambda_k(\bar{u}) + \ell_k^1(\bar{v}), v_1, \ldots, v_{n-k}).$$

By Lemma 3.4(a) $J_{z,m}(g^{-1} \circ f \circ g) \in \mathrm{Aut}_{m-1}(A_m(X, z))$ and $\theta_{z,m}(J_{z,m}(g^{-1} \circ f \circ g))$ is obtained from $\theta_{z,m}(J_{z,m}(f))$ via conjugation by $\theta_{z,1}(J_{z,m}(g))$. Using such conjugations we see that automorphisms of the form

$$(\bar{u}, \bar{v}) \to (u_1, \ldots, u_{i-1}, u_i + \ell_i^m(\bar{v}), u_{i+1}, \ldots, u_k, v_1, \ldots, v_{n-k}) \quad (13)$$

are also contained in I_m. Taking the product of automorphism in formula (13) with i running from 1 to k we obtain every automorphism from formula (9) as an element of I_m. This implies in turn that any automorphism from formula (10) is contained in I_1 regardless of the choice of $\ell_i^1(\bar{v})$ which concludes the proof. $\square$

Now we can formulate the main result of this section.

Theorem 3.10. *Let Notation 3.5 hold and $V(A_m(X, z))$ be the subset of the group* $\mathrm{Aut}(A_m(X, z))$ *which consists of automorphisms as in formula* (7) *satisfying the assumption on the determinant of* $[\partial F_i / \partial u_j]_{i,j}$ *in formula* (6). *Let Z be a finite subset of X such that for every $q \in \varrho(Z)$ there exists $\delta \in \mathcal{N}$ (where δ may depend on q) for which condition* (8) *in Lemma 3.7 holds. Suppose that $G_Z = \bigcap_{z \in Z} G_z$ and $J_{Z,m} : G_Z \to \prod_{z \in Z} V(A_m(X, z))$ is the natural homomorphism. Then $J_{Z,m}$ is surjective* (*in brief, one can choose an element $\alpha \in G$ so that for every $z \in Z$ the m-jet of α at z coincides with a prescribed jet from formula* (7) *satisfying the assumption on the determinant*).

Proof. By Theorem 3.1 and Remark 3.2(2) it suffices to consider the case when $\varrho(Z)$ is a singleton $q \in Q$. Furthermore, assume that for every $z \in Z$ we can find an element α_z in the subgroup $G^m_{Z,z} \subset G_Z$ from Lemma 3.9 such that at z this automorphism has a prescribed m-jet from formula (7) satisfying the assumption on the determinant. Recall that for every $w \in Z \setminus \{z\}$ the m-jet of α_z at w is the identity map in the m-th infinitesimal neighborhood of w. Hence, the composition of such automorphisms α_z with z running over Z (in any order) yields an automorphism $\alpha \in G_Z$ such that at every point of Z it has a prescribed m-jet from formula (7) satisfying the assumption on the determinant. Therefore, it is enough to consider the case when Z is a singleton z, which we do below.

We shall use induction on m with the case of $m = 1$ provided by Lemma 3.9. Assume now that the statement is true for $m - 1$. Let $h \in V(A_m(X, z))$. Then $h - \theta_{z,m}(h) = \widetilde{h} \in V(A_{m-1}(X, z))$. By the assumption there exists $\alpha \in G_Z$ for which $J_{z,m-1}(\alpha) = \widetilde{h}$. Let $J_{Z,m}(\alpha) = g$ and let $\lambda(g)$ be as in Notation 3.3. Consider the n-tuple $\widetilde{f} = \theta_{z,m}(h) - \theta_{z,m}(g)$ of m-forms and let $f \in \mathrm{Aut}_{m-1}(A_m(X, z))$ be such that $\theta_{z,m}(f) = \lambda(g)^{-1} \cdot_r \widetilde{f}$. Then by Lemma 3.4 (a) we have $f \circ g = h$. That is, it suffices to show that f belongs to $J_{z,m}(G_Z)$, or, equivalently $\theta_{z,m}(f)$ is contained in

$$I := \theta_{z,m}(J_{z,m}(G_Z) \cap \mathrm{Aut}_{m-1}(A_m(X, z))).$$

Note that since h and g are in $V(A_m(X, z))$ the last $n - k$ coordinates of the n-tuple $\widetilde{f}$ (and, therefore, $\theta_{z,m}(f)$) are equal to zero for $m > 1$. Note also that any element of I (and, thus, $\theta_{z,m}(f)$) is of the form $\sum_\mu \mu p_\mu(\bar{u})$ where μ is a monomial in coordinates $\bar{v}$ (with $(\bar{u}, \bar{v})$ from Notation 3.5) and $p_\mu(\bar{u})$ is an n-tuple of homogeneous polynomials in $\bar{u}$ of degree $l = m - \deg \mu$ such that the last $n - k$ coordinates of this n-tuple are equal to zero and its divergence is also zero (the latter fact follows from the assumption of the Jacobian, see [AFKKZ, Lem. 4.13]). Thus applying Lemma 3.4(a) again we see that it suffices to show that $\mu p_\mu(\bar{u})$ belongs to I. Furthermore, we can suppose that $l > 0$ since the case of $l = 0$ is taken care of by Lemma 3.9.

Recall that δ can be chosen so that conditions (i)-(iii) from Lemma 3.7 are satisfied. In particular, u_2 belongs to the kernel $\mathrm{Ker}\,\delta$ as well as every v_j and, therefore, μ. Since $\mathcal{N}$ is saturated the replica $\mu u_2^{l-1} \delta$ belongs to $\mathcal{N}$. Thus, we have $\exp(\mu u_2^{l-1} \delta) \in G_Z$, and

$$e := J_{z,m}(\exp(\mu u_2^{l-1} \delta))$$

belongs to $J_{z,m}(G_Z) \cap \mathrm{Aut}_{m-1}(A_m(X, z))$. Consider a finte set $\{\gamma_i\}$ in G_Z and

$g_i = J_{z,m}(\gamma_i)$. Then

$$e_i := J_{z,m}(\gamma_i^{-1} \circ \exp(\mu u_2^{l-1}\delta) \circ \gamma_i) \in J_{z,m}(G_Z) \cap \mathrm{Aut}_{m-1}(A_m(X,z))$$

and $\theta_{z,m}(e_i) = \lambda(g_i).\theta_{z,m}(e)$ is the result of the natural action of $\lambda(g_i) \in \mathrm{SL}(T_z X)$ from Lemma 3.4(b). Consider the projection κ of the space of n-tuples of m-forms in $\bar{u}$ and $\bar{v}$ to the space of k-tuples forgetting the last $n-k$ coordinates. Note that $\kappa(\theta_{z,m}(e))$ belongs to $\mu\theta_{z,l}(\mathrm{Aut}_{l-1}(A_l(Y,z)))$. Recall also that by Lemma 3.9 we can suppose that $\lambda(g_i)$ is given by

$$(\bar{u}, \bar{v}) \to (\lambda_1(\bar{u}), \ldots, \lambda_k(\bar{u}), v_1, \ldots, v_{n-k})$$

where $(\lambda_1(\bar{u}), \ldots, \lambda_k(\bar{u}))$ is a prescribed element of $\mathrm{SL}(T_z Y) \simeq \mathrm{SL}(k, \mathbf{k})$. Therefore, $\kappa(\theta_{z,m}(e_i)) \in \mu\theta_{z,l}(\mathrm{Aut}_{l-1}(A_l(Y,z)))$ and by Lemma 3.4(b) we can suppose that these elements generate the vector space $\mu\mathcal{F}_{z,l}$ where $\mathcal{F}_{z,l} = \theta_{z,l}(\mathrm{Aut}_{l-1}(A_l(Y,z)))$ is the space of k-tuples of l-forms in $\bar{u}$ whose divergence is zero. Lemma 3.4(d) implies that $t_i\lambda(g_i).\theta_{z,m}(e)$ is also contained in I for every $t_i \in \mathbf{k}$. Applying Lemma 3.4(a) again we see that every linear combination of the elements $\lambda(g_i).\theta_{z,m}(e)$ belongs to I and, therefore, $\kappa(I)$ contains the space $\mu\mathcal{F}_{z,l}$. Hence, $\mu p(\bar{u}) \in I$ which concludes the proof. $\square$

Corollary 3.11. *Let Notation* 3.5 *hold and condition* (8) *in Lemma* 3.7 *be valid for every* $q \in Q$. *Suppose that* Z *is a finite subset of* X, *and* m *is a natural number. Let* S'_z *and* S''_z *be étale sections of* $\varrho : X \to Q$ *through* $z \in Z$. *Then there exists an automorphism* $\alpha \in G$ *such that for every* $z \in Z$ *one has* $\alpha(z) = z$ *and* $\alpha(S'_z)$ *is tangent to* S''_z *at* z *with multiplicity at least* m.

4. General projections for flexible varieties. I

Notation 4.1.

(1) If $\kappa : X \to P$ is a morphism of algebraic varieties then we denote by $\mathrm{Aut}(X/P)$ (resp. $\mathrm{SAut}(X/P)$) the subgroup of $\mathrm{Aut}(X)$ (resp. $\mathrm{SAut}(X)$) that preserves each fiber of κ.

(2) If $\kappa : X \to P$ is a smooth morphism of smooth varieties we denote by $T(X/P)$ the relative tangent bundle (which is the kernel of the induced map $TX \to \kappa^*(TY)$). Furthermore, if Z is a subvariety of X then we still denote by $T(Z/P)$ the intersection of the Zariski tangent bundle TZ with $T(X/P)$. Similarly, for every $z \in Z$ we let $T_z(Z/P) = T_z Z \cap T(Z/P)$.

The aim of this section is to describe analogues of general linear projections of $\mathbb{A}_{\mathbf{k}}^n$ for flexible varieties. More precisely, we shall prove the following fact.

Theorem 4.2. *Let* X *and* P *be smooth algebraic varieties and* Q *be a normal algebraic variety. Let* $\varrho : X \to Q$ *and* $\tau : Q \to P$ *be dominant morphisms such that* $\kappa : X \to P$ *is smooth for* $\kappa = \tau \circ \varrho$. *Suppose that* Q_0 *is a non-empty Zariski open smooth subset of* Q *so that for* $X_0 = \varrho^{-1}(Q_0)$ *the morphism* $\varrho|_{X_0} : X_0 \to Q_0$ *is smooth. Let* $G \subset \mathrm{Aut}(X/P)$ *be an algebraically generated group acting* 2-*transitively on each fiber of* $\kappa : X \to P$ *and* Z *be a locally closed reduced subvariety in* X.

(i) *Let* $\dim Z \times_P Z \leqslant 2 \dim Z - \dim P$ *and* $\dim Q \geqslant \dim Z + m$ *where* $m \geqslant 1$. *Then there exists an algebraic G-family* $\mathcal{A}$ *of automorphisms of* X *(see Definition 1.1) such that for a general element* $\alpha \in \mathcal{A}$ *one can find a constructible subset* R *of* $\alpha(Z) \cap X_0$ *of dimension* $\dim R \leqslant \dim Z - m$ *for which* $\varrho(R) \cap \varrho(\alpha(Z) \setminus R) = \varnothing$ *and the restriction* $\varrho|_{(\alpha(Z) \cap X_0) \setminus R} : (\alpha(Z) \cap X_0) \setminus R \to Q_0$ *of* ϱ *is injective. In particular, if* $\dim Q \geqslant 2 \dim Z + 1$ *and* $\bar{Z}'_\alpha$ *is the closure of* $Z'_\alpha = \varrho \circ \alpha(Z)$ *in* Q *then for a general element* $\alpha \in \mathcal{A}$ *the map* $\varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0$ *is a bijection, while in the case of a pure-dimensional* Z *and* $\dim Q \geqslant \dim Z + 1$ *the morphism* $\varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to \bar{Z}'_\alpha \cap Q_0$ *is birational.*

(ii) *Let* $\dim Z \times_P Z \leqslant 2 \dim Z - \dim P$, $\dim Q \geqslant \dim Z + 1$ *and* F *be a closed subvariety of* Z *such* $\dim F \times_P Z < \dim Q - \dim P$ *(which is the case when* F *is a finite set and* P *is a singleton). Then for a general element* α *in the algebraic family* $\mathcal{A}$ *from* (i) *and every* $z \in \alpha(F) \cap X_0$ *one has* $\varrho^{-1}(\varrho(z)) \cap \alpha(Z) = z$.

(iii) *Suppose that* G *is generated by a saturated set* $\mathcal{N}$ *of locally nilpotent vector fields on* X *(in particular, every fiber of* κ *is G-flexible). Let* $P_0 = \tau(Q_0)$, $Z_0 = Z \cap \kappa^{-1}(P_0)$ *and* $\dim T(Z_0/P_0) \leqslant \dim Q - \dim P$ [12]. *Then there exists an algebraic family* $\mathcal{A}$ *of G-automorphisms of* X *such that for a general element* $\alpha \in \mathcal{A}$, *every* $z \in \alpha(Z_0) \cap X_0$ *the induced map* $\varrho_* : T_z \alpha(Z_0) \to T_{\varrho(z)} Q$ *of the tangent spaces is injective.*

(iv) *Suppose that* G *is again generated by a saturated set* $\mathcal{N}$ *of locally nilpotent vector fields on* X. *Let* $\dim Z \times_P Z \leqslant 2 \dim Z - \dim P$, $2 \dim Z + 1 \leqslant \dim Q$ *and* $\dim T(Z_0/P_0) \leqslant \dim Q - \dim P$. *Then the algebraic family* $\mathcal{A}$ *from* (i) *can be chosen so that for a general element* $\alpha \in \mathcal{A}$ *the morphism* $\varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0$ *is bijective and it induces an injective map of the Zariski tangent bundle of* $\alpha(Z) \cap X_0$ *into the Zariski tangent bundle of* Q_0.

Proof. For every variety $\mathcal{X}$ over P denote by $S_{\mathcal{X}}$ the variety $S_{\mathcal{X}} = (\mathcal{X} \times_P \mathcal{X}) \setminus \Delta_{\mathcal{X}}$ where $\Delta_{\mathcal{X}}$ is the diagonal in $\mathcal{X} \times \mathcal{X}$. Then every automorphism in $\mathrm{Aut}(X/P)$ can be lifted to an automorphism of S_X. In particular, we have a G-action on S_X and by the assumption this action is transitive on every fiber of the projection $S_X \to P$. Consider the subvariety $Y \subset S_X$ that is the intersection of $X_0 \times_{Q_0} X_0$ and S_X in $X \times_P X$. The codimension of Y in S_X is $\dim Q - \dim P$ and, because of smoothness, all non-empty fibers of the natural morphism $Y \to P$ are of the same dimension. Hence, by Remark 1.9(1) $\dim Y \times_P S_Z = \dim Y + \dim S_Z - \dim P$. By Theorem 1.4(ii) we can choose algebraic subgroups $H_1, \ldots, H_m$ of G such that for a general element $(h_1, \ldots, h_m) \in H_1 \times \cdots \times H_m$ one has

$$\dim W \leqslant \dim Y + \dim S_Z - \dim S_X = \dim S_Z - \dim Q + \dim P \leqslant 2 \dim Z - \dim Q$$

where $W = Y \cap \alpha(S_Z)$ for $\alpha = h_1 \cdot \cdots \cdot h_m$. Hence, in case (i) the dimension of W is at most $\dim Z - m$. Let R be the image of W under one of the two natural projections $X \times_Q X \to X$. In particular, R is a constructible set by Chevalley's theorem, $R \subset \alpha(Z) \cap X_0$ and $\dim R \leqslant \dim Z - m$. Note that for $z \in \alpha(Z) \cap X_0$ one has $\varrho^{-1}(\varrho(z)) \cap \alpha(Z) = z$ iff $z \notin R$. Hence, the restriction of ϱ to $(\alpha(Z) \cap X_0) \setminus R$ is injective. Therefore, letting $\mathcal{A} = H_1 \times \cdots \times H_m$, we get (i).

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[12]In particular, when P is a singleton the latter inequality can be written as $\dim TZ \leqslant \dim Q$.

In (ii) we let $S_{F,Z} = (F \times_P Z) \cap S_X$. By the assumption of (ii) we have $\dim S_{F,Z} < \dim Q - \dim P$. By Theorem 1.4(i) for a general element $(h_1, \ldots, h_m) \in H_1 \times \cdots \times H_m$ the dimension of the intersection of $\alpha(S_{F,Z})$ (where $\alpha = h_1 \cdot \ldots \cdot h_m$) with Y is at most $\dim S_{F,Z} + \dim Y - \dim S_X = \dim S_{F,Z} + \dim P - \dim Q < 0$, i.e., this intersection is empty. It remains to note that the fact that $\alpha(S_{F,Z}) \cap Y = \varnothing$ is exactly the statement that for every $z \in \alpha(F) \cap X_0$ one has $\varrho^{-1}(\varrho(z)) \cap \alpha(Z) = z$.

In (iii) for every variety $\mathcal{X}$ and a subvariety $\mathcal{Y}$ of the tangent bundle $T\mathcal{X}$ let $\mathcal{Y}^* = \mathcal{Y} \setminus \mathcal{S}$ where $\mathcal{S}$ is the zero section of the natural morphism $T\mathcal{X} \to \mathcal{X}$. Every automorphism $\alpha \in \operatorname{Aut}(X/P)$ generates an automorphism of $T(X/P)$. In particular, G acts on $T(X/P)^*$ and by Theorem 2.8 this action is transitive on every fiber of $T(X/P)^* \to X \xrightarrow{\kappa} P$. Note that the codimension of $Y^* = T(X_0/Q_0)^*$ in $T(X/P)^*$ is equal to $\dim Q - \dim P$ and all fibers of the natural projection $Y^* \to P_0$ are of the same dimension. Hence, by Remark 1.9

$$\dim Y^* \times_P T(Z_0/P_0)^* = \dim Y^* + \dim T(Z_0/P_0)^* - \dim P.$$

By Theorem 1.4 and the inequality $\dim T(Z_0/P_0) \leqslant \dim Q - \dim P$ we can choose one-parameter unipotent algebraic subgroups $\widetilde{H}_1, \ldots, \widetilde{H}_{\widetilde{m}}$ of G such that for a general element $(\widetilde{h}_1, \ldots, \widetilde{h}_{\widetilde{m}}) \in \widetilde{H}_1 \times \cdots \times \widetilde{H}_{\widetilde{m}}$ and $Z'' = (\widetilde{h}_1 \cdot \ldots \cdot \widetilde{h}_{\widetilde{m}})(Z_0)$ one has $\dim Y^* \cap T(Z''/P_0)^* \leqslant 0$. Note that if $Y^* \cap T(Z''/P_0)^*$ contains a point then $\dim Y^* \cap T(Z''/P_0)^*$ must be at least 1 (since this point is a vector in TZ'' and then $Y^* \cap T(Z''/P_0)^*$ contains all nonzero vectors proportional to that one). That is, $Y^* \cap T(Z''/P_0)^* = \varnothing$. This implies that for every $z \in Z'' \cap X_0$ the restriction of ϱ_* to $T_z(Z''/P_0)$ is injective. Consequently, the restriction of ϱ_* to $T_z Z''$ is injective i.e., we have (iii).

In the last statement we note that (i) and (iii) in combination with Proposition 1.10 imply that for a general element α of $H_1 \times \cdots \times H_m \times \widetilde{H}_1 \times \cdots \times \widetilde{H}_{\widetilde{m}}$ the restriction of ϱ to $\alpha(Z) \cap X_0$ is bijective and it induces an injective map of the Zariski tangent bundle of $\alpha(Z) \cap X_0$ into the Zariski tangent bundle of Q_0. Thus we have (iv). $\square$

Remark 4.3.

(1) Since the algebraic family $\mathcal{A}$ of automorphisms of X in Theorem 4.2 is the product of connected groups we see that $\mathcal{A}$ is irreducible and contains the identity map.

(2) Let $\widetilde{P}$ be a smooth subvariety of P and let $\widetilde{X} = \kappa^{-1}(\widetilde{P})$ be also smooth. Let V be either $\widetilde{X}$, or $S_{\widetilde{X}}$ or $T(\widetilde{X}/\widetilde{P})^*$. That is, in any case the natural action of G on V is transitive on every fiber of the natural morphism $V \to \widetilde{P}$. Then Proposition 1.7 yields one-parameter unipotent subgroups $H_1, \ldots, H_m$ of G and an open dense subset $U \subseteq H_1 \times \cdots \times H_m$ such that the natural map

$$\Psi : H_1 \times \cdots \times H_m \times V \to V \times_{\widetilde{P}} V \tag{14}$$

is smooth on $U \times V$. Note that the proof of Theorem 4.2 implies that the statements (i)–(ii) are true for general elements of an algebraic G-family $\mathcal{A}$ of automorphisms of X of the form $\mathcal{A} = H_1 \times \cdots \times H_m$ as soon as the map Ψ is smooth on $U \times V$ in the case of $V = S_X$. Similarly, the conclusions of Theorem 4.2(iii) for general elements

of a similar family $\mathcal{A} = H_1 \times \cdots \times H_m$ are true as soon as the map Ψ is surjective and smooth on $U \times V$ for $V = (TX/P)^*$. Suppose that V_i, $i = 1, \ldots, n$ are smooth varieties over smooth subvarieties $P_i \subset P$ satisfying the same description as V. Let $\mathcal{P}_i$ be any statement about the G-action on V_i which is true for general elements of any algebraic G-family of the same form $\mathcal{A} = H_1 \times \cdots \times H_m$ as soon as for a suitable open dense subset $U \subseteq H_1 \times \cdots \times H_m$, the natural map

$$\Psi_i : H_1 \times \cdots \times H_m \times V_i \to V_i \times_{P_i} V_i \qquad (15)$$

is smooth on $U \times V_i$. Then Remark 1.8 and Proposition 1.12 imply that such family $\mathcal{A}$ can be chosen so that the statements $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are true for a general element of $\mathcal{A}$ simultaneously.

(3) Let P be a singleton and X be G-flexible. Under the assumption of (iv) we can find $\alpha \in \mathcal{A}$ such that for a given $z_0 \in Z$ one has $z := \alpha(z_0) \in X_0$. In particular, for a general α the morphism $\varrho|_{\alpha(Z)} : \alpha(Z) \to \bar{Z}'_\alpha$ induces the injective map $\varrho_* : T_z\alpha(Z) \to T_{\varrho(z)}Q$. Note also that under the assumption of (iv) we can suppose that for some neighborhood U of z in $\alpha(Z)$ the morphism $\varrho|_U : U \to Q$ induces an injective map of the Zariski tangent bundles. However, even if $\varrho|_{\alpha(Z)} : \alpha(Z) \to Q$ induces an injective map of the Zariski tangent bundles it may not be proper (and, in particular, Z'_α may not be closed in Q). As a counterexample one can consider a bijective morphism of $\mathbb{C}^*$ onto a polynomial curve which has only one singular point and this singularity is a node.

(4) Note that the natural morphism $T(X/P) \to P$ (resp. $Y^* \to P_0$) in the proof of Theorem 4.2(iii) has equidimensional fibers. Hence, by Proposition 1.12 we conclude that the assumption that ϱ and τ are dominant and κ is smooth can be omitted in the formulation of Theorem 4.2(iii).

(5) Similarly, if κ is not smooth or even dominant we can suppose as in Remark 1.11 that $\kappa(X)$ as a disjoint union $\bigcup_{k=1}^n P_k$ of smooth varieties such that for $X_k = \kappa^{-1}(P_k)$ the morphism $\kappa|_{X_k} : X_k \to P_k$ is smooth. Letting $Y_k = X_k \cap Y$ and $Z_k = X_k \cap Z$ and using further stratification we can assume that for every k the morphism $\kappa|_{Z_k} : Z_k \to P_k$ has all fibers of the same dimension (and, in particular $\dim Z_k \times_{P_k} Z_k = 2\dim Z_k - \dim P_k$ [13]). Suppose that for every k and $Q_k = \tau^{-1}(P_k)$ one has

$$\dim Q_k \geqslant \dim Z_k + m. \qquad (16)$$

By Theorem 4.2 (i) (with X, Q, P and Z replaced by X_k, Q_k, P_k and Z_k) there exists an algebraic family $\mathcal{A}_k$ of G-automorphisms of X_k such that for a general element $\alpha \in \mathcal{A}_k$ one can find a subvariety R_k of $\alpha(Z_k) \cap X_0$ of dimension $\dim R_k \leqslant \dim Z_k - m$ for which $\varrho(R_k) \cap \varrho(\alpha(Z_k) \setminus R_k) = \varnothing$ and the morphism $\varrho|_{(\alpha(Z_k) \cap X_0) \setminus R_k} : (\alpha(Z_k) \cap X_0) \setminus R_k \to \varrho \circ \alpha(Z_k) \cap Q_0$ is injective. More precisely, by (2) this is true for an algebraic G-family $\mathcal{A}_k = H_1 \times \cdots \times H_m$ of automorphisms satisfying the following assumption: for a suitable open dense subset $U \subseteq H_1 \times \cdots \times H_m$, the natural map $H_1 \times \cdots \times H_m \times S_{X_k} \to S_{X_k} \times_{P_k} S_{X_k}$ is smooth on $U \times S_{X_k}$. Furthermore, as we mentioned, one can choose $\mathcal{A}$ such that the latter assumption is true for all $k = 1, \ldots, n$ simultaneously. Thus we see that the conclusion of Theorem

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[13]Of course, this does not imply that $\dim Z \times_P Z \leqslant 2\dim Z - \dim P$.

4.2 (i) (and, similarly, (iv)) remains valid even in the case when κ is not dominant provided that one replaces the assumption that $\dim Z \times_P Z \leqslant 2 \dim Z - \dim P$ and $\dim Q \geqslant \dim Z + m$ by the inequalities in formula (16).

Corollary 4.4. *Let the assumptions of Theorem 4.2(iv) hold with $Q = Q_0$ being quasi-affine. Suppose that Z is a once-punctured curve[14]. Then there exists an algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general $\alpha \in \mathcal{A}$ the set $\varrho(\alpha(Z))$ is a closed curve in Q and the restriction of ϱ yields an isomorphism between $\alpha(Z)$ and $\varrho(\alpha(Z))$.*

Proof. By Theorem 4.2(iv) there is an algebraic family $\mathcal{A}$ such that for a general $\alpha \in \mathcal{A}$ the restriction of ϱ yields an injective map of the Zariski tangent bundle of $\alpha(Z)$ into the Zariski tangent bundle of $\overline{\varrho(\alpha(Z))}$. Note also that, being an injective image of Z, $\varrho(\alpha(Z))$ is closed in Q since any once-punctured curve in a quasi-affine algebraic variety is automatically closed. Therefore, the bijective morphism $\varrho|_{\alpha(Z)} : \alpha(Z) \to \varrho(\alpha(Z))$ is proper. This yields the finiteness of $\varrho|_{\alpha(Z)}$. Furthermore, since the map $T_z \alpha(Z) \to T_q Q$ is injective, [Ka, Prop. 7]) implies that $\varrho|_{\alpha(Z)} : \alpha(Z) \to \varrho(\alpha(Z))$ is an isomorphism which is the desired conclusion. $\square$

We shall need later the following fact.

Proposition 4.5. *Let X be a G-flexible variety for a group $G \subset \mathrm{SAut}(X)$ generated by a saturated set of locally nilpotent vector fields, $\varrho : X \to Q$ be a dominant morphism into another algebraic variety Q and Z be a locally closed reduced subvariety in X. Suppose that F is a finite subset of Z such that $\dim T_{z_0} Z \leqslant \dim Q$ for every $z_0 \in F$. Then one can find an automorphism $\alpha_0 \in G$ with the following property: for any irreducible algebraic G-family $\mathcal{A}$ of automorphisms of X containing α_0 and any general element $\alpha \in \mathcal{A}$ there is a neighborhood V_α of $\alpha(F)$ in $\alpha(Z)$ such that for every $z \in V_\alpha$ and $q = \varrho(z)$ the induced map $\varrho_* : T_z \alpha(Z) \to T_q Q$ of the tangent spaces is injective.*

Proof. By flexibility and Theorem 2.8 there is an automorphism α_0 of X such that for every $z_0 \in F$ and $z_0' = \alpha_0(z_0)$ the variety Q (resp. the morphism ϱ) is smooth at (resp. over) $q_0' = \varrho(z_0')$ and the induced map $T_{z_0'} \alpha(Z) \to T_{q_0'} Q$ of the tangent spaces is injective. In the case when α_0 is contained in an irreducible algebraic family $\mathcal{A}$ of automorphisms put $z_1 = \alpha(z_0)$, $Z_1 = \alpha(Z)$ and $q_1 = \varrho(z_1)$ for a general $\alpha \in \mathcal{A}$. Then by continuity the variety Q (resp. the morphism ϱ) is smooth at (resp. over) q_1 and the induced map $\varrho_*|_{T_{z_1} Z_1} : T_{z_1} Z_1 \to T_{q_1} Q$ of the tangent spaces is also injective. That is, the kernel of $\varrho_*|_{T_{z_1} Z_1}$ is zero. Hence, for every z in some neighborhood of z_1 in Z_1 the kernel of the induced map $T_z Z_1 \to T_{\varrho(z)} Q$ is also zero. This yields the desired conclusion. $\square$

We shall also need a parametric version of Theorem 4.2(iii).

Proposition 4.6. *Let the assumptions and notations of Theorem 4.2(iii) hold with $Q_0 = Q$ and let U be a smooth algebraic variety. Let $\mathcal{Z} = U \times Z$, $\mathcal{P} = U \times P$, $\mathcal{X} = U \times X$, $\mathcal{Q} = U \times Q$, $\check{\kappa} = (\mathrm{id}, \kappa) : \mathcal{X} \to \mathcal{P}$ and $\check{\tau} = (\mathrm{id}, \tau) : \mathcal{Q} \to \mathcal{P}$. Suppose that $\check{\varrho} : \mathcal{X} \to \mathcal{Q}$ is a smooth morphism such that $\check{\kappa} = \check{\tau} \circ \check{\varrho}$. Consider the G-action on $\mathcal{X}$ such that the natural projection $\mathcal{X} \to X$ is G-equivariant.*

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[14]That is, Z is the complement to a smooth point in a complete curve.

(1) *Then there exists an algebraic G-family $\mathcal{A}$ of automorphisms of $\mathcal{X}$ such that for a general element $\alpha \in \mathcal{A}$, a general $u \in U$ and every $z \in \alpha(u \times Z)$ the induced map $\check{\varrho}_* : T_z\alpha(u \times Z) \to T_{\check{\varrho}(z)}(u \times Q)$ is injective.*

(2) *Suppose that the assumptions of Theorem 4.2(iv) are satisfied as well. Then the morphism $\check{\varrho}|_{\alpha(u \times Z)} : \alpha(u \times Z) \to u \times Q$ in (1) is also injective for general $\alpha \in \mathcal{A}$ and $u \in U$.*

(3) *Furthermore, suppose that W is a subvariety of X such that $\dim W \times_P Z + \dim P < \dim X$ (resp. $\dim W \times_P Z + \dim P \leqslant \dim X$). Let $\mathcal{W} = U \times W$. Then one can suppose additionally that $\mathcal{W} \cap \mathcal{X}_u$ does not meet $\alpha(\mathcal{Z}_u)$ (resp. $\mathcal{W} \cap \mathcal{X}_u \cap \alpha(\mathcal{Z}_u)$ is finite) for a general $u \in U$.*

Proof. Let $T(X/P)^*$, $T(Z/P)^*$ and Y^* be as in the proof of Theorem 4.2(iii). Let $\mathcal{T}^* = U \times T(X/P)^*$, $\mathcal{Y}^* = U \times Y^*$ and $\mathcal{E} = U \times T(Z/P)^*$. Then G acts on $\mathcal{T}^*$ and by Theorem 2.8 this action is transitive on every fiber of $\mathcal{T}^* \to \mathcal{X} \overset{\check{\kappa}}{\to} \mathcal{P}$. Note that $\dim \mathcal{E}^* = \dim T(Z/P) + \dim U \leqslant \dim Q - \dim P + \dim U$ (because $\dim T(Z/P) \leqslant \dim Q - \dim P$ by the assumption). Since all fibers of the natural projection $\mathcal{Y}^* \to \mathcal{P}$ are of the same dimension, by Remark 1.9 (1) one has $\dim \mathcal{Y}^* \times_{\mathcal{P}} \mathcal{E}^* = \dim \mathcal{Y}^* + \dim \mathcal{E}^* - \dim \mathcal{P}$. Since the codimension of $\mathcal{Y}^*$ in $\mathcal{T}^*$ is equal to $\dim Q - \dim P$, by Theorem 1.4 we can choose one-parameter unipotent algebraic subgroups $H_1, \ldots, H_m$ of G such that for a general element $\alpha = (\widetilde{h}_1, \ldots, \widetilde{h}_{\widetilde{m}}) \in H_1 \times \cdots \times H_m$ and $\mathcal{E}'' = \alpha_*(\mathcal{E}^*)$ one has $\dim \mathcal{Y}^* \cap \mathcal{E}'' \leqslant \dim U$. Hence, for a general $u \in U$ the fiber E_u of $\mathcal{E}'' \cap \mathcal{Y}^*$ over u is at most finite. Note that up to the zero vector the set $(T\alpha(u \times Z)) \cap \mathcal{Y}^*$ is the kernel of the natural projection $T\alpha(u \times Z) \to T(u \times Q)$. This implies that E_u is empty and, therefore, the map $(T\alpha(u \times Z)) \cap \mathcal{E}^* \to T(u \times Q)$ is injective. Consequently, this yields (1).

As in Remark 4.3(2) we observe that the subgroups $H_1, \ldots, H_m$ fit this description if the following assumption is satisfied: for a suitable open dense subset $U \subseteq H_1 \times \cdots \times H_m$, the natural map

$$H_1 \times \cdots \times H_m \times T(\mathcal{X}/\mathcal{P})^* \to T(\mathcal{X}/\mathcal{P})^* \times_{\mathcal{P}} T(\mathcal{X}/\mathcal{P})^* \tag{17}$$

is smooth on $U \times T(\mathcal{X}/\mathcal{P})^*$.

The similar argument works in the case of Theorem 4.2(i) with $\dim Q \geqslant 2 \dim Z + 1$. That is, choose again an algebraic G-family $\mathcal{A}$ of automorphisms of $\mathcal{X}$ of the form $\mathcal{A} = H_1 \times \cdots \times H_m$ such that for a suitable open dense subset $U \subseteq H_1 \times \cdots \times H_m$, the natural map

$$H_1 \times \cdots \times H_m \times S_{\mathcal{X}} \to S_{\mathcal{X}} \times_{\mathcal{P}} S_{\mathcal{X}}$$

is smooth on $U \times S_X$. Then by Remark 4.3 (2) and Theorem 4.2(i) for a general $\alpha \in \mathcal{A}$ the morphism $\check{\varrho}|_{\alpha(Z)}$ is injective outside a constructible subset of dimension less than $\dim U$. Hence, $\check{\varrho}|_{\alpha(u \times Z)} : \alpha(u \times Z) \to u \times Q$ is injective for a general $u \in U$. By Proposition 1.10 we can suppose that such $\mathcal{A}$ satisfies also the assumption for formula (17), i.e., $\check{\varrho}|_{\alpha(u \times Z)}$ is an injective immersion for a general $u \in U$ which yields (2).

For (3) note that $W \times_P Z$ is a fiber of the natural morphism $\mathcal{W} \times_{\mathcal{P}} \mathcal{Z} \to U$ and, hence, by Chevalley's semi-continuity the general fibers of this morphism are

at most of dimension $\dim W \times_P Z$. In particular, replacing U by its Zariski dense open subset, we can suppose that $\dim \mathcal{W} \times_{\mathcal{P}} \mathcal{Z} \leqslant \dim W \times_P Z + \dim U$. Theorem 1.4 implies that for a general α in some family $\mathcal{A}$ (of the same form as before) $\alpha(\mathcal{Z})$ meets $\mathcal{W}$ along a subvariety of dimension

$$d := \dim \mathcal{W} \times_{\mathcal{P}} \mathcal{Z} + \dim \mathcal{P} - \dim \mathcal{X} \leqslant \dim W \times_P Z + \dim U + \dim P - \dim X.$$

By the assumption $d < \dim U$ (resp. $d \leqslant \dim U$). Hence, for a general $u \in U$ the set $\mathcal{W} \cap \mathcal{X}_u$ does not meet $\alpha(\mathcal{Z}_u)$ (resp. $\mathcal{W} \cap \mathcal{X}_u \cap \alpha(\mathcal{Z}_u)$ is finite). By Remark 4.3 (2) we can suppose, additionally, that $\breve{\varrho}|_{\alpha(u \times Z)}$ is still an injective immersion. This yields the desired conclusion. $\square$

Remark 4.7. By Remark 4.3 (4) one can drop the assumption that κ is dominant in the first statement of Propostion 4.6. Similarly, this assumption can be dropped in the second statement if one replaces the inequality $\dim Z \times_P Z \leqslant 2 \dim Z - \dim P$ and $\dim Q \geqslant \dim Z + m$ by the ones in formula (16).

5. General projections for flexible varieties. II

Notation 5.1. In this section X and P are smooth algebraic varieties, P is a closed affine subvariety of $\mathbb{A}_{\mathbf{k}}^m$, $\kappa : X \to P$ is a surjective morphism, and Z is a closed subvariety of X. We suppose that $\widehat{P}$ is a completion of P, $\widehat{X}$ is a completion of X, $\widehat{D} = \widehat{X} \setminus X$, and $\widehat{\kappa} : \widehat{X} \dashrightarrow \widehat{P}$ is a rational extension of κ. Let $\widehat{f}$ be the rational extension of a regular function $f \in \mathbf{k}[X]$ to $\widehat{X}$. Denote by $\mathcal{R}(\widehat{f})$ the subvariety of $\widehat{D}$ that consists of points which are either indeterminacy points of $\widehat{f}$ or points at which $\widehat{f}$ is regular and takes finite values. Given any morphism of the form $\lambda = (f_1, \ldots, f_N) : X \to \mathbb{A}_{\mathbf{k}}^N$ we let $\mathcal{R}(\widehat{\lambda}) = \bigcap_{i=1}^N \mathcal{R}(\widehat{f_i})$ where $\widehat{\lambda}$ is the extension of λ to $\widehat{X}$. In particular, since κ is given by m coordinate functions we can define $\mathcal{R}(\widehat{\kappa})$.

The aim of this section is to describe some conditions under which the morphism $\kappa|_Z : Z \to P$ is proper and, in particular, $\kappa(Z)$ is closed in P.

Proposition 5.2. *Let $\breve{Z}$ be the intersection of $\widehat{D}$ with the closure of Z in $\widehat{X}$. If $\breve{Z} \cap \mathcal{R}(\widehat{\kappa}) = \varnothing$ then $\kappa|_Z : Z \to P$ is proper.*

Proof. Using a resolution $\pi : \bar{X} \to \widehat{X}$ of the indeterminacy points of $\widehat{\kappa}$ one gets a morphism $\bar{\kappa} = \widehat{\kappa} \circ \pi : \bar{X} \to \widehat{P}$. Treat X as a subvariety of $\bar{X}$ and denote by $\bar{Z}$ the closure of Z in $\bar{X}$. Note that $\kappa|_Z : Z \to P$ is proper if and only if $(\bar{\kappa}^{-1}(P) \setminus X) \cap \bar{Z} = \varnothing$. Note also that $\pi(\bar{\kappa}^{-1}(P) \setminus X)$ consists of all point in $\widehat{D}$ where either $\widehat{\kappa}$ has indeterminacy or where $\widehat{\kappa}$ is regular and takes values in P. Since P is closed in $\mathbb{A}_{\mathbf{k}}^m$ the latter means that all coordinate functions of $\widehat{\kappa}$ take finite values and, therefore, $\pi(\bar{\kappa}^{-1}(P) \setminus X)$ is contained in $\mathcal{R}(\widehat{\kappa})$. Furthermore, if $\widehat{Z}$ is the closure of Z in $\widehat{X}$ the properness of π implies that $(\bar{\kappa}^{-1}(P) \setminus X) \cap \bar{Z} = \varnothing$ if $\pi(\bar{\kappa}^{-1}(P) \setminus X) \cap \widehat{Z} = \varnothing$. Since $\pi(\bar{\kappa}^{-1}(P) \setminus X) \cap \widehat{Z} = \breve{Z} \cap \mathcal{R}(\widehat{\kappa})$ we have the desired conclusion. $\square$

Remark 5.3. One may have properness of $\kappa|_Z : Z \to P$ even if $\breve{Z} \cap \mathcal{R}(\widehat{\kappa}) \neq \varnothing$. Indeed, consider $X = \mathbb{A}_{\mathbf{k}}^2 \subset \mathbb{P}_{\mathbf{k}}^2 = \widehat{X}$, $P = \mathbb{A}_{\mathbf{k}}^1$ and the map $X \to P$ given by

$(x, y) \to x$ for a coordinates system (x, y) on $\mathbb{A}^2_{\mathbf{k}}$. Let $Z \subset X$ be the parabola given by $y = x^2$. Then $\kappa|_Z : Z \to P$ is proper but $\bar{Z}$ meets $\widehat{D}$ at an indeterminacy point of $\widehat{x}$.

Corollary 5.4. *Let $\widehat{D}$ be irreducible and $I(\widehat{f_i})$ be the set of indeterminacy points of $\widehat{f_i}$ for $\kappa = (f_1, \dots, f_m)$. Suppose that every f_i is non-constant on Z and $\check{Z} \cap \bigcap_{i=1}^m I(\widehat{f_i}) = \varnothing$. Then the morphism $\kappa|_Z : Z \to P$ is proper and $\kappa(Z)$ is closed in P.*

Proof. Let $x \in \check{Z}$. By the assumption at least for one index i the rational function $\widehat{f_i}$ yields a regular morphism into $\mathbb{P}^1$ in a neighborhood of x. Note the value of $\widehat{f_i}$ at x is ∞ (indeed, $\widehat{f_i}^{-1}(\infty) \subset \widehat{D}$ and, therefore, $\widehat{f_i}^{-1}(\infty)$ must be equal to $\widehat{D}$ since the latter is irreducible). Thus $\check{Z} \cap \mathcal{R}(\widehat{\kappa})$ is empty and Proposition 5.2 yields the desired conclusion. $\square$

Theorem 5.5. *Let $\widehat{\kappa} : \widehat{X} \to \widehat{P}$ be regular and $\widetilde{\kappa} : \widetilde{X} \to P$ be its restriction to $\widetilde{X} = \widehat{\kappa}^{-1}(P)$. Suppose that H is an algebraically generated group acting on $\widetilde{X}$ so that it transforms every fiber of κ into itself. Let $\widetilde{D} = \widehat{D} \cap \widetilde{X}$ be a finite disjoint union of irreducible smooth subvarieties $\{T_i\}$ such that every set $\widetilde{\kappa}(T_i)$ is a smooth variety and $\widetilde{\kappa}|_{T_i} : T_i \to \widetilde{\kappa}(T_i)$ is a smooth morphism with each fiber being an orbit of H. Let $\theta : X \to P \times \mathbb{A}^N_{\mathbf{k}}$ be a morphism of the form $\theta = (\kappa, \lambda)$ and let $S_i = \mathcal{R}(\widehat{\lambda}) \cap T_i$ and $Z_i = \check{Z} \cap T_i$ where $\check{Z}$ from Proposition 5.2. Suppose that for every i either S_i is empty or $\dim Z_i \times_P S_i + \dim \widetilde{\kappa}(T_i) < \dim T_i$. Then there exists an algebraic H-family $\mathcal{A}$ of automorphisms of X such that for a general element $h \in \mathcal{A}$ the morphism $\theta|_{h(Z)} : h(Z) \to P \times \mathbb{A}^N_{\mathbf{k}}$ is proper.*

Proof. Since P is closed in $\mathbb{A}^n_{\mathbf{k}}$ the description of $\widetilde{X}$ implies that for every point $x \in \widehat{D} \setminus \widetilde{D}$ there is a coordinate function f of κ for which $\widehat{f}$ has a regular value ∞ at x. Thus $(\widehat{D} \setminus \widetilde{D}) \cap \mathcal{R}(\widehat{\kappa}) = \varnothing$ and $\check{Z} \cap \mathcal{R}(\widehat{\theta}) \subset \check{Z} \cap \mathcal{R}(\widehat{\lambda}) \cap \widetilde{D}$. Applying Theorem 1.4 to the morphism $\widetilde{\kappa}|_{T_i} : T_i \to \widetilde{\kappa}(T_i)$ in the case of a non-empty S_i we see that for a general element h of some algebraic family $\mathcal{A} \subset H$ the variety $h(Z_i) \cap S_i$ is empty when $\dim Z_i \times_P S_i + \dim \widetilde{\kappa}(T_i) < \dim T_i$. Hence, by Proposition 1.10 $h(\check{Z}) \cap \mathcal{R}(\widehat{\lambda}) \cap \widetilde{D} = \varnothing$ and we are done by Proposition 5.2. $\square$

Corollary 5.6. *Let $\widetilde{\kappa} : \widetilde{X} \to P$, H, T_i, $\theta = (\kappa, \lambda)$ and S_i be as in Theorem 5.5. Suppose that for every i either S_i is empty or $\widetilde{\kappa}|_{S_i} : S_i \to \widetilde{\kappa}(T_i)$ is a flat morphism with fibers of dimension l_i. Let $k = \dim Z$ and m_i be the dimension of the fibers of $\widetilde{\kappa}|_{T_i} : T_i \to \widetilde{\kappa}(T_i)$. Suppose that for every i one has $l_i + k - 1 < m_i$. Then for a general element $h \in H$ the morphism $\theta|_{h(Z)} : h(Z) \to P \times \mathbb{A}^N_{\mathbf{k}}$ is proper.*

Proof. Let Z_i be as in Theorem 5.5. By the same theorem we can assume that S_i is not empty. Since $\widetilde{\kappa}(S_i) = \widetilde{\kappa}(T_i)$ Remark 1.9 (1) implies that $Z_i \times_P S_i = \dim S_i + \dim Z_i - \dim \widetilde{\kappa}(T_i)$, i.e., the desired inequality from Theorem 5.5 can be rewritten as $\dim Z_i + \dim S_i < \dim T_i$. By the assumption we have $l_i + \dim \widetilde{\kappa}(T_i) + k - 1 < m_i + \dim \widetilde{\kappa}(T_i)$. Note that $\dim S_i = l_i + \dim \widetilde{\kappa}(T_i)$, $\dim T_i = m_i + \dim \widetilde{\kappa}(T_i)$, and $\dim \check{Z} = k - 1$. Hence, we have this inequality $\dim S_i + \dim \check{Z} < \dim T_i$ which concludes the proof. $\square$

Notation 5.7. From now on by $\mathrm{ED}(X)$ we denote $\mathrm{ED}(X) = \max(2\dim X + 1, \dim TX)$.

Corollary 5.8. *Let the assumptions of Theorem 5.5 or Corollary 5.6 hold with θ being a dominant morphism. Suppose also that X is a G-flexible variety where $G \subset \mathrm{SAut}(X)$ is generated by a saturated set of locally nilpotent derivations and that $\mathrm{ED}(Z) \leqslant \dim P + N$. Then there exists an algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general $\alpha \in \mathcal{A}$ the morphism $\theta|_{\alpha(Z)} : \alpha(Z) \to P \times \mathbb{A}_{\mathbf{k}}^N$ is a closed embedding.*

Proof. Consider Theorem 4.2(iv) with P in *its formulation* being a singleton unlike P in the formulation above. Applying this special case one can see that there exists an algebraic G-family $\mathcal{A}_0$ of automorphisms of X such that for a general $\beta \in \mathcal{A}_0$ the morphism $\theta|_{\beta(Z)} : \beta(Z) \to P \times \mathbb{A}_{\mathbf{k}}^N$ is an embedding. By Proposition 1.10 this property remains valid if one replaces $\mathcal{A}_0$ by $\mathcal{A} = H \times \mathcal{A}$. Hence, by Theorem 5.5 for a general $h \in H$ and $\alpha = h \circ \beta \in \mathcal{A}$ the morphism $\theta|_{\alpha(Z)} : \alpha(Z) \to P \times \mathbb{A}_{\mathbf{k}}^N$ is also proper. Repeating now the argument from Corollary 4.4 we conclude that it is a closed embedding which is the desired conclusion. $\square$

Remark 5.9. Suppose that unlike in Notation 5.1 P is not an affine variety but only a quasi-affine one and let R be an affine variety containing P as an open subset. Assume additionally that $\kappa(Z)$ is closed in R. Then the conclusion about the properness of $\theta|_{h(Z)} : h(Z) \to P \times \mathbb{A}_{\mathbf{k}}^N$ in Theorem 5.5 remains valid. For a more general setting we need the following.

Notation 5.10. Let $\varrho : \mathcal{X} \to \mathcal{P}$ be an affine morphism of smooth quasi-affine varieties and $\mathcal{P} = \bigcup_{j=1}^m P_j$, where each P_j is an affine Zariski dense open subset of $\mathcal{P}$. Let $X_j = \varrho_i^{-1}(P_j)$, $\varrho_j = \varrho|_{X_j} : X_j \to P_j$ and $\widetilde{\varrho}_j : \widetilde{X}_j \to P_j$ be an extension of ϱ_j to a proper morphism, where $\widetilde{X}_j$ is a smooth variety and $\widetilde{D}_j = \widetilde{X}_j \setminus X_j$. Suppose that for every $j = 1, \dots, m$ there is a set $\mathcal{N}_j$ of locally nilpotent vector fields on X_j tangent to the fibers of ϱ_j and extendable to complete vector fields on $\widetilde{X}_j$. Let H_j be the group of automorphisms of X_j (and $\widetilde{X}_j$) generated by the elements of the flows of the vector fields from $\mathcal{N}_j$. Let $\lambda : \mathcal{X} \to \mathbb{A}_{\mathbf{k}}^N$ be a morphism, $\Theta = (\varrho, \lambda) : \mathcal{X} \to \mathcal{P} \times \mathbb{A}_{\mathbf{k}}^N$, $\mathcal{Z}$ be a closed subvariety of $\mathcal{X}$ and $\mathcal{Z}_j = \mathcal{Z} \cap X_j$. Suppose also that for every $j = 1, \dots, m$ the assumptions (and, therefore, the conclusion) of Theorem 5.5 are true if $\kappa : X \to P$, $\widetilde{\kappa} : \widetilde{X} \to P$, Z, $\widetilde{D}$, H and θ in the formulation of that theorem are replaced with $\varrho_j : X_j \to P_j$, $\widetilde{\varrho}_j : \widetilde{X}_j \to P_j$, $\mathcal{Z}_j$ $\widetilde{D}_j$, H_j and $\theta_j = \Theta|_{X_j}$.

Theorem 5.11. *Let Notation 5.10 hold. Then there exists an algebraic family $\mathcal{A}$ of automorphisms of $\mathcal{X}$ over $\mathcal{P}$ such that for a general element $h \in \mathcal{A}$ the morphism $\Theta|_{h(\mathcal{Z})} : h(\mathcal{Z}) \to P \times \mathbb{A}_{\mathbf{k}}^N$ is proper.*

Proof. Let $I_j \subset \mathbf{k}[\mathcal{P}]$ be the defining ideal of the variety $\mathcal{P} \setminus P_j$. For every $\delta \in \mathcal{N}_j$ one can find $f \in I_j$ such that $f\delta$ is a locally nilpotent vector field on $\mathcal{X}$ tangent to the fibers of ϱ. Furthermore, for a given point $p \in P$ such f can be chosen so that $f(p) = 1$. Consider the set $\mathcal{N}_j'$ of all locally nilpotent vector fields of this form $f\delta$. The elements of their flows generate a group $H_j' \subset \mathrm{SAut}(\mathcal{X}/\mathcal{P})$. Without loss of generality we can suppose that for every $\sigma \in \mathcal{N}_j'$ and each $r \in \mathbf{k}$ one has

$r\sigma \in \mathcal{N}_j$. Then for every point $p \in P_j$ the restrictions of $\mathcal{N}_j$ and $\mathcal{N}_j'$ to $\varrho_j^{-1}(p)$ (resp. $\widetilde{\varrho}_j^{-1}(p)$) coincide. Hence, the restriction of the H_j'-action to $\varrho_j^{-1}(p)$ (resp. $\widetilde{\varrho}_j^{-1}(p)$) is the same as the H_j-action. Thus, we can replace H_j in Notation 5.10 by H_j'. In particular, from the beginning we can assume that $H_j \subset \mathrm{SAut}(\mathcal{X}/\mathcal{P})$. By Theorem 5.5 there exists an algebraic H_j-family $\mathcal{A}_j$ of automorphisms of X such that for a general element $h \in \mathcal{A}_j$ the morphism $\Theta|_{h(\mathcal{Z}_j)} : h(\mathcal{Z}_j) \to P_j \times \mathbb{A}_{\mathbf{k}}^N$ is proper. Arguing as in Remark 4.3(2) we can find an algebraic $\mathrm{SAut}(\mathcal{X}/\mathcal{P})$-family $\mathcal{A}$ of automorphisms of X such that for a general element $\alpha \in \mathcal{A}$ the morphism $\Theta|_{\alpha(\mathcal{Z}_j)} : \alpha(\mathcal{Z}_j) \to P_j \times \mathbb{A}_{\mathbf{k}}^N$ is proper for every j which yields the desired conclusion. $\square$

6. General projections for partial quotient morphisms of flexible varieties

In the case of partial quotients of $\mathbb{G}_a$-actions one can get local properness under much milder assumptions than in Theorem 5.5. It is reflected in the following fact which plays an important role in [KKT].

Theorem 6.1. *Let X be a smooth quasi-affine algebraic variety, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on X, and $G \subset \mathrm{SAut}(X)$ be the group generated by $\mathcal{N}$. Suppose also that X is G-flexible. Let $\varrho_0 : X \to Q$ be a partial quotient morphism associated with a nontrivial $\delta_0 \in \mathcal{N}$, Z be a locally closed reduced subvariety of X of codimension at least 2 and F be a finite subset of Z such that $\dim T_{z_0} Z \leqslant \dim Q$ for every $z_0 \in F$. Then there exists a connected algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general element $\alpha \in \mathcal{A}$ and the closure $\bar{Z}_\alpha'$ of $Z_\alpha' = \varrho_0 \circ \alpha(Z)$ in Q one can find a neighborhood V_0' of $\varrho_0(\alpha(F))$ in $\bar{Z}_\alpha'$ such that for $V_0 = \varrho_0^{-1}(V_0') \cap \alpha(Z)$ the morphism $\varrho_0|_{V_0} : V_0 \to V_0'$ is an isomorphism.*

The proof consists mostly of recalling some results from [FKZ].

Proposition 6.2 ([FKZ, Prop. 2.15]). *Let X, G, and $\mathcal{N}$ be as in Theorem 6.1. Then for any locally nilpotent derivation $\delta_0 \in \mathcal{N}$ one can find another one $\delta_1 \in \mathcal{N}$ such that the subgroup $H \subset G$ generated by δ_0, δ_1 and all their replicas acts with an open orbit on X.*

Remark 6.3. In fact we have more. It follows from the proof of [FKZ, Prop. 2.15]) that δ_1 can be chosen so that the open orbit of H contains a given finite subset of X.

Notation 6.4.

(a) Let δ_0 and δ_1 be as in Proposition 6.2 and U^i be the one-parameter unipotent subgroup of $\mathrm{SAut}(X)$ associated with δ_i. Any function $f \in \ker \delta_0 \setminus \ker \delta_1$ yields the one-parameter group U_f^0 associated with the replica $f\delta_0$, and similarly $g \in \ker \delta_1 \setminus \ker \delta_0$ yields the one-parameter group U_g^1 associated with the replica $g\delta_1$.

(b) To any sequence of invariant functions

$$\mathcal{F} = \{f_1, \ldots, f_s, g_1, \ldots, g_s\}, \quad \text{where } f_i \in \ker \delta_1 \setminus \ker \delta_0 \text{ and } g_i \in \ker \delta_0 \setminus \ker \delta_1, \quad (18)$$

we associate an algebraic family of automorphisms defined by the product

$$U^{\mathcal{F}} = U^1_{f_s} \cdot U^0_{g_s} \cdot \cdots \cdot U^1_{f_1} \cdot U^0_{g_1} \subseteq H. \tag{19}$$

More generally, given a tuple $\kappa = (k_i, l_i)_{i=1,\ldots,s} \in \mathbb{N}^{2s}$ the product

$$U_\kappa = U^{\mathcal{F}}_\kappa = U^1_{f_s^{k_s}} \cdot U^0_{g_s^{l_s}} \cdot \cdots \cdot U^1_{f_1^{k_1}} \cdot U^0_{g_1^{l_1}} \subseteq H \tag{20}$$

is as well an algebraic family of automorphisms.

Proposition 6.5 ([FKZ, Cor. 5.4]). *There is a finite collection of invariant functions $\mathcal{F}$ as in (18) such that for any sequence $\kappa = (k_i, l_i)_{i=1,\ldots,s} \in \mathbb{N}^{2s}$ the algebraic family of automorphisms U_κ as in (20) has an open orbit $O(U_\kappa)$ that coincides with $O(H)$ and so does not depend on the choice of $\kappa \in \mathbb{N}^{2s}$.*

Notation 6.6. We keep the notation and assumptions from 6.4(a).

(a) Let $\varrho_0 : X \to Q_0$ and $\varrho_1 : X \to Q_1$ be partial quotient morphisms with respect to the unipotent subgroups U^0 and U^1, respectively. It is proven in [FKZ, Lem. 3.3] that there are open embeddings $X \hookrightarrow \bar{X}$, $Q_0 \hookrightarrow \bar{Q}_0$, and $Q_1 \hookrightarrow \bar{Q}_1$ into normal projective varieties such that the following conditions are satisfied.

  (i) ϱ_0 and ϱ_1 extend to morphisms $\bar{\varrho}_0 : \bar{X} \to \bar{Q}_0$ and $\bar{\varrho}_1 : \bar{X} \to \bar{Q}_1$.
  (ii) The unique "horizontal" divisors $D_0 \subset \bar{X} \setminus X$ and $D_1 \subset \bar{X} \setminus X$, that are mapped birationally (by $\bar{\varrho}_0$ and $\bar{\varrho}_1$) onto $\bar{Q}_0$ and $\bar{Q}_1$ respectively, are smooth.
  (iii) The completion $\bar{X}$ satisfies some other conditions which we assume to be true but omit because they are not needed for the formulation of Proposition 6.7 below.

(b) Given a closed subscheme $Y \subseteq X$ of codimension at least 2 we call

$$\partial_0 Y = \bar{Y} \cap D_0 \quad \text{and} \quad \partial_1 Y = \bar{Y} \cap D_1$$

the *partial boundaries.*

(c) For a one-parameter group U we let $U^* = U \setminus \{\mathrm{id}\}$ and for $U_\kappa = U^1_{f_s^{k_s}} \cdot U^0_{g_s^{l_s}} \cdot \cdots \cdot U^1_{f_1^{k_1}} \cdot U^0_{g_1^{l_1}}$ as in (20) we let

$$U^*_\kappa = U^{1*}_{f_s^{k_s}} \cdot U^{0*}_{g_s^{l_s}} \cdot \cdots \cdot U^{1*}_{f_1^{k_1}} \cdot U^{0*}_{g_1^{l_1}}.$$

Proposition 6.7 ([FKZ, Prop. 5.11]). *Let $(Y_\alpha)_{\alpha \in A}$ be a flat family of proper closed subsets of X. Assume that the partial boundaries $\partial_i Y_\alpha$ are contained in $E_{\alpha,i}$, where the $(E_{\alpha,i})_{\alpha \in A}$, $i = 0, 1$, form flat families of proper closed subsets of D_i. Then one can find an open dense subset A° of A, flat families of proper, closed subsets $(E^\circ_{\alpha,i})_{\alpha \in A^\circ}$ of D_i $(i = 0, 1)$, and a sequence $\kappa = (k_1, l_1, \ldots, k_s, l_s) \in \mathbb{N}^{2s}$ such that for any element $h \in U^*_\kappa$ we have*

$$\partial_i(h.Y_\alpha) \subseteq E^\circ_{\alpha,i}, \qquad i = 0, 1, \ \forall \alpha \in A^\circ.$$

Proof of Theorem 6.1. Let Q_0 be a smooth Zariski open dense subset of Q such that for $X_0 = \varrho_0^{-1}(Q_0)$ the morphism $\varrho_0|_{X_0} : X_0 \to Q_0$ is smooth. Since the G-action on X is m-transitive for every m replacing Z by $g(Z)$ for some $g \in G$ we can suppose that $F \subset X_0 \cap O(H)$ where $O(H)$ is the open orbit of H from Proposition 6.2. Let $\bar{Z}$ be the closure of Z in X. By Theorem 4.2(i)–(ii) there exists an irreducible algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general $\alpha \in \mathcal{A}$ one has

(a) $\varrho_0|_{\alpha(\bar{Z})} : \alpha(\bar{Z}) \cap X_0 \to \varrho_0 \circ \alpha(\bar{Z})$ is birational and $\varrho_0^{-1}(\varrho_0(\alpha(F))) \cap \alpha(\bar{Z}) = \alpha(F)$
(here we use the fact that $\alpha(F) \subset X_0 \cap O(H)$ since $F \subset X_0 \cap O(H)$).

Recall that by Remark 4.3 (1) we can suppose that $\mathcal{A}$ is irreducible and it contains the identity map e. By the definition of $\mathrm{SAut}(X)$ any automorphism $\alpha_0 \in \mathrm{SAut}(X)$ is an element of an algebraic family $H_1 \times \cdots \times H_m$ of automorphisms on X as in Theorem 1.3 where every H_i is a unipotent one-parameter group. Using the way to enlarge $\mathcal{A}$ as in Remark 4.3 (2) we can replace $\mathcal{A}$ by $H_1 \times \cdots \times H_m \times \mathcal{A}$ where the last family is still irreducible and contains e but it contains also α_0 now. Hence, since $\dim T_{z_0} Z \leqslant \dim TQ$ for every $z_0 \in F$, choosing α_0 as in Proposition 4.5 we also have

(b) a neigborhood $\widetilde{V}_\alpha$ of $\alpha(F)$ in $\alpha(Z)$ such that for every $z \in \widetilde{V}_\alpha$ the induced map $T_z \alpha(Z) \to T_{\varrho_0(z)} Q$ of the tangent spaces is injective.

Let U_κ be from Proposition 6.7 and let $\beta = h \circ \alpha$ be a general element of the family $U_\kappa \cdot \mathcal{A}$. By Proposition 1.10 and Remark 4.3 (1) we still have conditions (a) and (b) for this bigger family. By Proposition 6.7 the partial boundary $\partial_0(\beta(\bar{Z})) = \partial_0(h.\alpha(\bar{Z})) \subseteq E^o_{\alpha,0}$ where $E^o_{\alpha,0}$ is a proper subvariety of D_0 from Notation 6.6. This implies that the morphism $\varrho_0|_{\beta(\bar{Z})} : \beta(\bar{Z}) \to Q$ is proper over $Q \setminus \bar{\varrho}_0(E^o_{\alpha,0})$ where $\bar{\varrho}_0$ is again from Notation 6.6. On the other hand, by Proposition 6.5 $\beta(z_0) = h.\alpha(z_0)$ runs over the open set $O(H)$ when h runs over U_κ. Hence, $\varrho_0 \circ \beta(F)$ does not meet $\bar{\varrho}_0(E^o_{\alpha,0})$ for a general β. Therefore, the morphism $\varrho_0|_{\beta(\bar{Z})} : \beta(\bar{Z}) \to Q$ is proper over an neighborhood of $F' = \varrho_0 \circ \beta(F)$ in Q.

Assume that there is an irreducible subvariety $Y' \subset \varrho_0 \circ \beta(Z) \cap Q_0$ such that $\dim Y' < \dim Y$ where Y is an irreducible component of $\varrho_0^{-1}(Y') \cap \beta(Z)$ with some $z_0' = \varrho_0(\beta(z_0))$ from F' contained in the closure of $\varrho_0(Y)$. Then by Chevalley's theorem [EGA, 13.1.3] $\dim \varrho_0^{-1}(z_0') \cap Y \geqslant 1$ (this fiber cannot be empty because of properness) contrary to the fact that by (a) $\varrho_0^{-1}(z_0') \cap \beta(Z) = \beta(z_0)$. Hence, we can suppose that $\varrho_0|_{\beta(Z)}$ is not only birational but also quasi-finite over a neighborhood $V_0' \subset \varrho_0 \circ \beta(Z)$ of F' which in combination with properness implies finiteness by the Grothendieck theorem [EGA, Thm. 8.11.1].

If for a sufficiently small V_0' and $V_0 = \varrho_0^{-1}(V_0') \cap \beta(Z)$ the morphism $\varrho_0|_{V_0}$ is not injective then for some curve $C' \in V_0$ through a point $z_0' \in F'$ and $C = \varrho_0^{-1}(C') \cap \beta(Z)$ the finite morphism $\varrho_0|_C : C \to C'$ must be ramified at $\beta(z_0)$. However, this is contrary to the fact that the induced map $T_{\beta(z_0)} T(\beta(Z)) \to T_{z_0'} Q$ of the tangent spaces is injective by (b). That is, we can suppose that $\varrho_0|_{V_0} : V_0 \to V_0'$ is injective and proper. Since it is also finite, condition (b) implies that this map is an embedding (e.g., see [Ka, Prop. 7]) which yields the desired conclusion. $\square$

Remark 6.8.
(1) Note that by Remark 4.3 (1) and construction the family $\mathcal{A}$ from Theorem 6.1 is a Zariski open subset in a larger family of automorphisms which contains

the identity map.

(2) Similar to Remark 4.3 (2) we also note that the family $\mathcal{A}$ from Theorem 6.1 does not depend on the choice of the subvariety Z.

(3) Let X, Z, F, $\mathcal{N}$ and G be as in Theorem 6.1, $S = \{\delta_1, \ldots, \delta_s\} \subset \mathcal{N}$ and $\varrho_i : X \to Q_i$ be a partial quotient morphism associated with δ_i for $i = 1, \ldots s$. Then an easy adjustment of the proof yields the following generalization of Theorem 6.1.

There exists a connected algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general element $\alpha \in \mathcal{A}$, every $i = 1, \ldots, s$, and the closure $Z_\alpha^i := \overline{\varrho_i \circ \alpha(Z)}$ of $\varrho_i \circ \alpha(Z)$ in Q_i the following holds: one can find a neighborhood V_i' of $\varrho_i(\alpha(F))$ in Z_α^i such that for $V_i = \varrho_i^{-1}(V_i') \cap \alpha(Z)$ the morphism $\varrho_i|_{V_i} : V_i \to V_i'$ is an isomorphism.

7. The case of Gromov–Winkelmann flexible varieties

The aim of this section is to prove the following statement.

Theorem 7.1. *Let Z, Y_1, and Y_2 be closed subvarieties of $\mathbb{A}_{\mathbf{k}}^n$ such that $Y_1 \cap Z = Y_2 \cap Z = \varnothing$, $\dim Z \leqslant n - 3$ and $\mathrm{ED}(Y_1) \leqslant n - 2$ (where $\mathrm{ED}(Y_1)$ is as in Notation 5.7). Let $\varphi : Y_1 \to Y_2$ be an isomorphism and $X = \mathbb{A}_{\mathbf{k}}^n \setminus Z$. Suppose also that either*

(a) $\dim Z + \dim Y_1 \leqslant n - 3$, *or*
(b) $\dim Y_1 = 1$ *and* $\dim Z = n - 3$.

Then there exists an automorphism $\gamma \in \mathrm{SAut}(X)$ for which $\gamma|_{Y_1} = \varphi$.

The proof requires some preparations.

Notation 7.2. Further in this section we write $\mathbb{A}^n$ instead of $\mathbb{A}_{\mathbf{k}}^n$ and the symbol $\mathbb{A}_{u_1, \ldots, u_n}^n$ means that $\mathbb{A}^n$ is equipped with a fixed coordinate system $\bar{u} = (u_1, \ldots, u_n)$. In particular, this system induces an embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ into a projective space. We also suppose that Z, Y_1 and Y_2 are closed subvarieties of $\mathbb{A}_{u_1, \ldots, u_n}^n$ such that $\dim Z \leqslant n - 3$, $\mathrm{ED}(Y_1) \leqslant n - 2$, Y_1 and Y_2 are disjoint from Z, and there exists an isomorphism $\varphi : Y_1 \to Y_2$.

The following result will be important in this section: in the case when Z is of codimension at least 2 Gromov [Gr] observed that $\mathbb{A}^n \setminus Z$ is a flexible variety and Winkelmann [Wi] showed that $\mathrm{SAut}(\mathbb{A}^n \setminus Z)$ acts transitively on $\mathbb{A}^n \setminus Z$ which is equivalent by virtue of Theorem 2.2. (In particular, for a finite Y_1 Theorem 7.1 is valid even when $\dim Z = n - 2$.)

The next fact is well known (e.g., see [Ka]).

Proposition 7.3. *Let $\mathrm{ED}(Z) \leqslant k \leqslant n$ and $\mathrm{Lin}(\mathbb{A}^n, \mathbb{A}^k)$ be the affine variety of surjective linear maps $\mathbb{A}^n \to \mathbb{A}^k$. Then for a general element $\varrho \in \mathrm{Lin}(\mathbb{A}^n, \mathbb{A}^k)$ the restriction $\varrho|_Z : Z \to \mathbb{A}^k$ is a closed embedding.*

Proposition 7.4. *Let $\varrho : \mathbb{A}^n \to \mathbb{A}^k$ be a general element of $\mathrm{Lin}(\mathbb{A}^n, \mathbb{A}^k)$.*

(1) *If $k \leqslant \dim Z$ then $\varrho|_Z : Z \to \mathbb{A}^k$ is surjective and for every $w \in \mathbb{A}^k$ the fiber $F = \varrho^{-1}(w) \cap Z$ is of dimension $\dim Z - k$.*
(2) *If $k = \dim Z$ then $\varrho|_Z : Z \to \mathbb{A}^k$ is finite.*
(3) *Let $T_z(Z/\mathbb{A}^k)$ be as in Notation 4.1 and E be a finite subset of Z. Then for every $z \in E$ and every $k \leqslant \dim T_z Z$ one has $\dim T_z(Z/\mathbb{A}^k) \leqslant \dim T_z Z - k$.*

Proof. Let $\varrho = (f_1, \ldots, f_k)$ be the coordinate form of ϱ and $\bar{\varrho} = (\bar{f}_1, \ldots, \bar{f}_k)$ be the rational extension of ϱ to $\mathbb{P}^n$. Let $\mathcal{R}(\bar{\varrho}) = \bigcap_{i=1}^k \mathcal{R}(\bar{f}_i)$ be the same as in Notation 5.1. Since ϱ is surjective (i.e., $f_1, \ldots, f_k$ are linearly independent) $\mathcal{R}(\bar{\varrho})$ is of codimension k in $D = \mathbb{P}^n \setminus \mathbb{A}^n$. The natural action of $\mathrm{SL}(n, \mathbf{k})$ on $\mathbb{A}^n$ extends to an action to $\mathbb{P}^n$ whose restriction to D is transitive. In particular, replacing ϱ by $\varrho \circ h$ where h is a general element of $\mathrm{SL}(n, \mathbf{k})$, by virtue of Theorem 1.3 we can assure that the intersection $\bar{Z} \cap \mathcal{R}(\bar{\varrho})$ is of dimension $\dim Z - 1 - k$ where $\bar{Z}$ is the closure $\bar{Z}$ of Z in $\mathbb{P}^n$ (and, therefore, $\bar{Z} \cap \mathcal{R}(\bar{\varrho}) = \varnothing$ when $\dim Z = k$).

Assume that there exists a fiber F of $\varrho|_Z$ with $\dim F > \dim Z - k$. Then the closure $\bar{F}$ of F meets D along a subvariety of dimension at least $\dim Z - k$. However, $\bar{F} \cap D \subset \bar{Z} \cap \mathcal{R}(\bar{\varrho})$ which yields a contradiction. Thus $\dim F = \dim Z - k$ (since it cannot be less than $\dim Z - k$ [Sh, Chap. 1, Sec. 6, Thm. 7]). In particular, $\varrho|_Z : Z \to \mathbb{A}^k$ is quasi-finite when $\dim Z = k$. Since $\bar{Z} \cap \mathcal{R}(\bar{\varrho}) = \varnothing$ in the latter case the morphism $\varrho|_Z : Z \to \mathbb{A}^k$ is proper by Proposition 5.2 and, therefore, it is finite by [EGA, Thm. 8.11.1]. Hence, we have (2) and, in particular, this map is surjective. This implies that when $k < \dim Z$ we have also surjectivity for a general ϱ which is (1).

In (3) we treat $T_z Z$ as a vector subspace of $\mathbb{A}^n$ and denote by V_z (resp. V_z') the closure of $T_z Z$ (resp. $T_z(Z/\mathbb{A}^k)$) in $\mathbb{P}^n$. Then for every $z \in E$, a general ϱ and $\mathcal{R}(\bar{\varrho})$ as above the dimension of $V_z \cap \mathcal{R}(\bar{\varrho})$ is equal to $\dim T_z Z - k - 1$. It remains to note that $V_z' \cap D$ is contained in $V \cap \mathcal{R}(\bar{\varrho})$ (indeed, if $v \in \mathrm{Ker}\, \varrho_*$ then $f_1(v) = \ldots = f_k(v) = 0$ and the closure of every line $\{tv | t \in \mathbf{k}\}$ in $\mathbb{P}^n$ meets D at a point of $\mathcal{R}(\bar{\varrho})$). That is, $\dim V_z' \cap \mathcal{R}(\bar{\varrho}) \leqslant \dim T_z Z - k - 1$ and, thus, $\dim V_z' \leqslant \dim T_z Z - k$ which concludes the proof. $\square$

Notation 7.5. For every group G acting on an algebraic variety X and every subscheme W of X we denote by G_W the subgroup of G such that the action of every element of G_W yields the identity map on W.

Lemma 7.6. *Let $X = P \times \mathbb{A}^{N+1}$ where P is a smooth affine algebraic variety, $\varrho : X \to P$ be the natural projection, and $m > 0$. Suppose that $G = \mathrm{SAut}(X/P)$ and $X_p := \varrho^{-1}(p)$, $p \in P$. Let $\mathrm{Lin}(\mathbb{A}^{N+1}, \mathbb{A}^N)$ be the variety of surjective linear maps $\mathbb{A}^{N+1} \to \mathbb{A}^N$ and Z be a closed subvariety of X such that for every $p \in P$ one has $\dim Z \cap X_p \leqslant N - 1$ and for a general $\lambda \in \mathrm{Lin}(\mathbb{A}^{N+1}, \mathbb{A}^N)$ and $\theta = (\varrho, \lambda)$ the morphism $\theta|_Z : Z \to P \times \mathbb{A}^N$ is proper. Then each variety $X_p \setminus Z$ is G_{Z_m}-flexible where Z_m is the m-th infinitesimal neighborhood of Z in X and G_{Z_m} has the same meaning as in Notation 7.5.*

Proof. Let $\psi : P \times \mathbb{A}^N \to P$ be the natural projection, i.e., $\theta(Z \cap X_p) = \theta(Z) \cap \psi^{-1}(p)$. Let x be any point in $X_p \setminus Z$. Since $\dim Z \cap X_p \leqslant N - 1$ we see that $\theta(x) \notin \theta(Z \cap X_p)$ for a general λ and hence $\theta(x) \notin \theta(Z)$. Note also that θ can be treated as a quotient morphism of a locally nilpotent vector δ field on X whose value at x is a general vector ν in $T_x X_p$ (since λ is general). Note also that the kernel of δ coincides with $\mathbf{k}[P \times \mathbb{A}^N]$ where we treat $\mathbf{k}[P \times \mathbb{A}^N]$ as a subring of $\mathbf{k}[X]$ under the natural embedding. Hence, for every regular function f on $P \times \mathbb{A}^N$ that vanishes on $\theta(Z)$ but not at $\theta(x)$ (which exists since $\theta(Z)$ is closed in $P \times \mathbb{A}^N$) the vector field $f^m \delta$ is locally nilpotent and its value at x coincides with this general vector ν up to a nonzero factor. Furthermore, the flow of $f^m \delta$ is a one-parameter

group in G_{Z_m}. Hence, $X_p \setminus Z$ is G_{Z_m} flexible by the definition of flexibility and we are done. $\square$

Notation 7.7. Further in this section by X we denote $X = \mathbb{A}^n_{u_1,\ldots,u_n} \setminus Z$.

Lemma 7.8. *Let* $\kappa : \mathbb{A}^n \to \mathbb{A}^s$ *be the natural projection* $(u_1,\ldots,u_n) \to (u_1,\ldots,u_s)$ *where* $s \leqslant \mathrm{ED}(Y_1)$. *Let* $\underline{\kappa} = \kappa \circ \varphi$, $\dim Z \leqslant n - s - 2$ *and* $F \subset Y_1$ *be a finite set for which* $\varphi(y) = y$ *for every* $y \in F$. *Then for every* $m > 0$ *there exists an automorphism* β *of* $X \subset \mathbb{A}^n$ *over* $\mathbb{A}^s$ *such that for each* $y \in F$ *the* m-jet of φ at y *coincides with the restriction of the* m-jet of β.

Proof. By Lemma 7.6 every fiber of $\kappa|_X$ is G_Z-flexible. By Theorem 3.10 we can suppose now that the tangent spaces of Y_1 and Y_2 at every $y \in F$ coincide and the isomorphism of these tangent spaces induced by φ is the identity map. Without loss of generality we can also suppose that $T_y Y_1$ is contained in $\{u_{k+1} = \ldots = u_n = 0\}$ where $k \geqslant s$ and y is the origin $\bar{0}$ of $\mathbb{A}^n$. Then the m-jet φ_m of φ at y can be chosen in the following form $(u_1,\ldots,u_n) \to (u_1 + h_1,\ldots,u_n + h_n)$ where $h_1,\ldots,h_n$ are polynomials in $u_1,\ldots,u_k$ without free and linear terms. Let $\widetilde{h}_n = \partial h_1/\partial u_1 + \cdots + \partial h_k/\partial u_k$. Then the m-jet $(u_1,\ldots,u_{n-1},u_n) \to (u_1 + h_1,\ldots,u_{n-1} + h_{n-1}, u_n - u_n\widetilde{h}_n)$ is of divergence 0 and, hence, it satisfies the assumption on the Jacobian in Notation 3.5 (e.g., see the proof of [AFKKZ, Lem. 4.13]). By Theorem 3.10 we can find an automorphism $\beta_1 \in G_Z$ of X over $\mathbb{A}^s$ with such precise jets at every point of F. Hence, φ_m is a composition of β_1 with an m-jet ψ_m of the form $(u_1,\ldots,u_{n-1},u_n) \to (u_1,\ldots,u_{n-1},u_n + h)$ where a priori h depends on $u_1,\ldots,u_n$. However, restricting ψ_m to the étale germ of Y_1 at y we can replace h with a function of $u_1,\ldots,u_k$, i.e., we can view ψ_m as the restriction of the m-jet of an automorphism $\beta_2 \in G_Z$. Letting $\beta = \beta_2 \circ \beta_1$ we get the desired conclusion. $\square$

Lemma 7.9. *Let* $\varrho : \mathbb{A}^n \to \mathbb{A}^k$ *be the natural projection* $(u_1,\ldots,u_n) \to (u_1,\ldots,u_k)$ *with* $k \geqslant \max(\dim Z + 1, \mathrm{ED}(Y_1))$. *Assume that for the closure* Z' *of* $\varrho(Z)$ *in* $\mathbb{A}^k$ *the set* $Y_1 \cap \varrho^{-1}(Z')$ *is finite. Let* $\varrho|_{Y_1} : Y_1 \to \mathbb{A}^k$ *be a closed embedding and* $\varphi(y) = y$ *for every* $y \in Y_1 \cap \varrho^{-1}(Z')$. *Then there exists* $m > 0$ *such that whenever the* m-th jets of φ are equal to the jets of the identity map for all $y \in Y_1 \cap \varrho^{-1}(Z')$, one has an automorphism α of $\mathbb{A}^n$ over $\mathbb{A}^k$ for which the following holds:

(a) $\alpha(z) = z$ *for every* $z \in Z$ *(in particular,* α *is an automorphism of* $X = \mathbb{A}^n \setminus Z$*) and*

(b) *after replacing* Y_1 *by* $\alpha(Y_1)$ *and* φ *by* $\varphi \circ \alpha^{-1}$ *one has* $u_i|_{Y_1} = u_i \circ \varphi$ *for every* u_i *with* $i \geqslant k + 1$.

Proof. Let $I' \subset \mathbf{k}[\mathbb{A}^k]$ be the vanishing ideal of Z', $Y_1' = \varrho(Y_1)$, $I \subset \mathbf{k}[Y_1']$ be the image of I' under the natural morphism $\mathbf{k}[\mathbb{A}^k] \to \mathbf{k}[Y_1']$ and $J \subset \mathbf{k}[Y_1']$ be the vanishing ideal of $Y_1' \cap Z'$. By Nullstellensatz one has $J^m \subset I$ for some $m > 0$. Let $g_i = u_i \circ \varphi - u_i|_{Y_1}$ for $i \geqslant k + 1$. Since and Y_1 and Y_1' are isomorphic we can consider g_i as a function on Y_1'. Note that it always belongs to J and, furthermore, under the assumption it belongs to J^m. Thus g_i is the restriction of a polynomial $\widetilde{g}_i$ on $\mathbb{A}^k$ which vanishes on Z'. Consider the automorphism ψ_i of X given by $u_i \to u_i + \widetilde{g}_i(u_1,\ldots,u_k)$ and $u_j \to u_j$ for $j \neq i$. Then the composition $\psi_{k+1} \circ \ldots \circ \psi_n$ yields the desired automorphism α. $\square$

Remark 7.10. Since m in the proof above can be chosen as large as we wish one can suppose that the extension $\widetilde{g}_i$ vanishes not only on Z' but also on the m-th infinitesimal neigborhood of Z'. This implies that the automorphisms $\{\psi_i\}$ in the proof are elements of the flows of locally nilpotent vector fields that vanish on Z with multiplicity m. Since α from Lemma 7.9 is a composition of such automorphisms we can suppose that the restriction of α to the m-th infinitesimal neigborhood Z_m of Z is the identity map. The same is true for β from Lemma 7.8 since by Lemma 7.6 $X_p \setminus Z$ is not only G_Z-flexible but also G_{Z_m}-flexible.

Proof of Theorem 7.1(a). Let $H = \mathrm{SL}(n, \mathbf{k})$ act naturally on $\mathbb{A}^n$, $\varrho_k : \mathbb{A}^n \to \mathbb{A}^k$ be the natural projection given by $(u_1, \dots, u_n) \to (u_1, \dots, u_k)$ and $\varrho_{k,h}$ be the composition $\varrho_k \circ h$ where $h \in H$. By Propositions 7.3 and 7.4 we have the following for a general $h \in H$.

(1) For $k = \dim Z$ the morphism $\varrho_{k,h}|_Z : Z \to \mathbb{A}^k$ is finite.

(2) For every $k < \dim Y_2$ the morphism $\varrho_{k,h}|_{Y_2} : Y_2 \to \mathbb{A}^k$ is surjective with the dimension of each fiber equal to $\dim Y_2 - k$; for $k = \dim Y_2$ the morphism $\varrho_{k,h}|_{Y_2} : Y_2 \to \mathbb{A}^m$ is finite.

Let us show inductively that for every $k \leqslant \dim Y_1$ and a general h in H one can also assume the following.

(3) $\varrho_{k,h} \circ \varphi = \varrho_{k,h}|_{Y_1}$ and, in particular, by (2) if $k < \dim Y_1$ then $\varrho_{\sigma,k}|_{Y_1} : Y_1 \to \mathbb{A}^k$ is surjective with the dimension of each fiber equal to $\dim Y_1 - k$, if $k = \dim Y_1$ then $\varrho_{k,h}|_{Y_1} : Y_1 \to \mathbb{A}^m$ is finite.

(4) $\varrho_{n-2,h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-2}$ is a closed embedding.

(5) $\varrho_{n-2,h}(Y_1)$ does not meet $\varrho_{n-2,h}(Z)$.

For $k = 0$ the assumption (3) is automatically true while (4) follows from Proposition 7.3. Note that (5) is equivalent to the fact that Y_1 does not meet $\varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z))$ where the latter variety is of dimension $\dim Z + 2$ which is less than $n - \dim Y_1$ by the assumption of Theorem 7.1. Hence, replacing, if necessary, Y_1 (resp. Y_2) by $\widetilde{h}(Y_1)$ (resp. $\widetilde{h}(Y_2)$) for a general $\widetilde{h} \in H$ we see that Theorem 1.3 implies that neither Y_1 nor Y_2 meets $\varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z))$. This yields (5) for $k = 0$ without violating (1)–(4).

Let us assume that (3)–(5) are valid for $k - 1$ and let us establish that this implies their validity for k. Since (4) and (5) hold for $k - 1$, by Lemma 7.9 we obtain an automorphism $\beta \in \mathrm{SAut}(X)$ over $\mathbb{A}^{n-2}$ such that after replacing Y_1 by $\beta(Y_1)$ one has $u_i|_{Y_1} = u_i \circ \varphi$ for $i = n - 1$ and $i = n$. Consider the element σ of H such that $\sigma_*(u_k) = u_n, \sigma_*(u_n) = -u_k$ while the remaining coordinates remain intact. Note that since h is general we can suppose that $\sigma \circ \varrho_{k,h}$ is still of the form ϱ_{k,h_1} where h_1 is a general element of H. Hence, replacing $\varrho_{k,h}$ with $\sigma \circ \varrho_{k,h}$ we get (3) while keeping (1) and (2).

Treat now $\varrho_{k,h} : X \to \mathbb{A}^k$ (resp. $\varrho_{n-2,h} : X \to \mathbb{A}^{n-2}$) as $\kappa : X \to P$ (resp. $\varrho : X \to Q$)) in Theorem 4.2. Note the assumptions of Theorem 4.2 are valid for the group $G = \mathrm{SAut}(X/P)$ and the subvariety Y_1 of X (indeed, the fibers of κ is G-flexible by Lemma 7.6 while $\dim Y_1 \times_P Y_1 \leqslant 2 \dim Y_1 - \dim P$ by Remark 1.9 and the fact that the fibers of $\kappa|_{y_1} : Y_1 \to P$ are equidimensional because of (3)). By Theorem 4.2(i) and (iii) we can find an algebraic G-family $\mathcal{A}_1$ of automorphisms of X (a priori depending on h) such that for a general element $\alpha_1 \in \mathcal{A}_1$ the morphism $\varrho|_{\alpha_1(Y_1)} : \alpha_1(Y_1) \to Q$ is injective and generates an injective map of the

Zariski tangent bundles. Similarly, since $\varrho^{-1}(\varrho(Z))$ is of dimension $\dim Z + 2 <$ $\dim X - \dim Y_1$ by Theorem 1.4 we can find an algebraic family $\mathcal{A}_2 \subset G$ (also depending on h) such that for a general element $\alpha_2 \in \mathcal{A}_2$ the variety $\alpha_2(Y_1)$ does not meet $\varrho^{-1}(\varrho(Z))$ which is equivalent to (5). By Remark 4.3 (2) we can suppose that $\mathcal{A}_1 = \mathcal{A}_2$.

Hence, in order to prove (4) and (5) one needs to show that the morphism $\varrho|_{\alpha(Y_1)} : \alpha(Y_1) \to Q$ can be made proper.

Let H' be the subgroup of H consisting of all elements whose action on the first summand of $\mathbb{A}^n = \mathbb{A}^k_{u_1,\ldots,u_k} \oplus \mathbb{A}^{n-k-2}_{u_{k+1},\ldots,u_{n-2}} \oplus \mathbb{A}^2_{u_{n-1},u_n}$ is the identity map. Consider Proposition 4.6 with $U = H'$ and $\breve{\varrho} : H' \times X = \mathcal{X} \to \mathcal{Q} = H \times Q$ given by $(h', x) \to (h', \varrho_{n-2,h'h}(x))$. Then Proposition 4.6 implies that we can find a family $\mathcal{A}_1$ as above which provides the injectivity condition for general h' in H' simultaneously. Note that the composition of the H'-action with the projection $\mathbb{A}^n \to \mathbb{A}_{u_{k+1},\ldots,u_{n-2}}$ can be viewed as the variety $\mathrm{Lin}(\mathbb{A}^n, \mathbb{A}^{n-k-2})$ of surjective linear maps $\varrho'_{n-k-2,h'h} : \mathbb{A}^n \to \mathbb{A}^{n-k-2}$ and, hence, Proposition 7.4 implies that $\varrho'_{n-k-2,h'h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-k-2}$ is proper for a general h' because $\dim Y_1 < n - k - 2$ (the latter inequality follows from the fact that $k \leqslant \dim Y$ and by the assumption of Theorem 7.1 $2 \dim Y_1 + 1 \leqslant \mathrm{ED}(Y_1) \leqslant n - 2$). Since for a general h' the element $h'h$ is still general in H, replacing h by $h'h$ we get (4) without ruining (1)-(3) while the third statement of Proposition 4.6 provides us with (5).

Now we shall establish by induction conditions (3)–(5) for a general h and $k = k_1 + 1, \ldots, n$ where $k_1 = \dim Y_1$. Since $\varrho_{h,k_1}|_{Y_1} : Y_1 \to \mathbb{A}^{k_1}$ is finite by (3) we can suppose now every $\varrho_{h,k}|_{Y_1} : Y_1 \to \mathbb{A}^k$ is proper. Assume that (4) and (5) hold for $k - 1$. Applying Lemma 7.9 as above we can achieve (3) for k without ruining this properness (but may be violating (4) and (5)). Using notation $\kappa : X \to P$ and $\varrho : X \to Q$ in the same meaning as before we observe that though the assumption $\dim Y_1 \times_P Y_1 \leqslant 2 \dim Y_1 - \dim P$ does not hold we can still apply Theorem 4.2(i) and (iv) by virtue of Remark 4.3 (5) and finiteness of κ. Hence, the same argument as above yields an algebraic G-family $\mathcal{A}$ of automorphisms of X over $\mathbb{A}^k$ such that for a general $\alpha \in \mathcal{A}$ the morphism $\varrho_{n-2,h}|_{\alpha(Y_1)} : \alpha(Y_1) \to Q$ is injective and generates an injective map of the Zariski tangent bundles, and, furthermore, $\alpha(Y_1)$ does not meet $\varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z))$. The latter fact is (5) and in combination with the properness the former one yields (4).

Note that this process of replacing Y_1 by its automorphic image in X in the case of $k = n$ yields the equality $Y_1 = Y_2$ and, hence, the desired conclusion. $\square$

Lemma 7.11. *Let $m > 0$ and $\mathcal{L}$ be a finite collection of 3^m distinct lines in $\mathbb{P}^2$ such that the intersection of any three of them is empty. Then for every finite subset $F \subset \mathbb{P}^2$ of cardinality m there is a line $L \in \mathcal{L}$ such that $L \cap F = \varnothing$.*

Proof. Let us use induction by m. For $m = 1$ the claim follows from the definition of $\mathcal{L}$. Suppose that the statement is true for $m - 1$ and $\mathcal{L}$ is a collection of 3^m lines with the desired property. Present $\mathcal{L}$ as a disjoint union $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ where each collection $\mathcal{L}_i$ consists of 3^{m-1} lines. Let $F = F_0 \cup \{x\} \subset \mathbb{P}^2$ be of cardinality m, i.e., F_0 is of cardinality $m - 1$. By the assumption we can find lines $L_1 \in \mathcal{L}_1$, $L_2 \in \mathcal{L}_2$ and $L_3 \in \mathcal{L}_3$ such that $L_i \cap F_0 = \varnothing$ for every i. Since $L_1 \cap L_2 \cap L_3 = \varnothing$ one of these lines does not contain x and we are done. $\square$

Proof of Theorem 7.1(b). Let $\bar{u} = (u_1, \ldots, u_n)$ and $\varrho_{k,h} : \mathbb{A}^n \to \mathbb{A}^k$ have the same meaning as in the proof of Theorem 7.1(a) where $h \in H = \mathrm{SL}(n, \mathbf{k})$. We can suppose again that (1) as in the proof of Theorem 7.1(a) holds.

Let us show inductively that for $k \leqslant n - 3$ and some general h (depending on k) we can also assume

(2) $\varrho_{k,h} \circ \varphi = \varrho_{k,h}|_{Y_1}$.

(3) $\varrho_{n-2,h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-2}$ is a closed embedding.

(4) $\varrho_{n-2,h}(Y_1)$ meets $\varrho_{n-2,h}(Z)$ at a finite number of points and, furthermore, these points are the images of smooth points of Y_1.

(5) For every singular point y of Y_1 the image of $T_y Y_1$ in $T\mathbb{A}^k$ under the map induced by $\varrho_{k,h}$ is of dimension $\min(k, \dim T_y Y_1)$ (which implies the inequality $\dim T(Y_1/\mathbb{A}^k) \leqslant \dim TY_1 - k$) and the cardinalities of $E := \mathrm{Sing}\, Y_1$ and $Y_1 \cap \varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(E))$ are the same.

Step 1 (condition (5) and the case of $k = 0$). Let $k = 0$ and h be a general element of H. Then the assumption (2) is automatic and (3) follows from Proposition 7.3. Since X is $\mathrm{SAut}(X)$-flexible, replacing Y_2 by its automorphic image in X we can suppose by Theorem 2.6 that E is also the set of singular points of Y_2 and $\varphi(y) = y$ for every $y \in E$. Replacing, if necessary, Y_1 and Y_2 by $\tilde{h}(Y_1)$ and $\tilde{h}(Y_2)$ respectively, where $\tilde{h}$ is a general element of H, we can suppose that E does not meet $\varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z))$ and, in particular, the sets $\varrho_{n-2,h}(E)$ and $\varrho_{n-2,h}(Z)$ are disjoint. Furthermore, for a general $h \in H$ one has always the second clause of (5), i.e., $Y_1 \cap \varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(E)) = E$ (indeed, otherwise, since h is general almost every line through some point of E meets another point of Y_1 contrary to the fact that Y_1 is a curve). By Theorem 2.8 we can also suppose that for every $y \in E$ and every $k \geqslant 0$ the image of $T_y Y_2$ in $T\mathbb{A}^k$ under the map induced by $\varrho_{k,h}$ is of dimension $\min(k, \dim T_y Y_2)$. Applying now an automorphism of X to Y_1 we can suppose also that $\varphi_* : T_y Y_1 \to T_y Y_2$ is the identity map. Let G be the subgroup of $\mathrm{SAut}(X)$ whose restriction to the second infinitesimal neighborhood E_2 of E is the identity map. From now on we are going to use replacements of Y_1 and Y_2 by their images under the actions of some elements of G. Note that this quarantees (5) for all $k \geqslant 0$. Returning to the case of $k = 0$ we observe that $\dim Y_1 + \dim \varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z)) \leqslant \dim X$. Hence, since $X' = X \setminus E$ is G-flexible, applying to Y_1 an element of G we get (4) by Theorem 1.4 which concludes Step 1.

Step 2. Now let us assume that (1)–(4) are valid for $k - 1$ and proceed with the case of $k \leqslant n - 3$.

Let $F_1 = Y_1 \cap \varrho_{n-2,h}^{-1}(\varrho_{n-2,h}(Z))$ and $F_2 = \varphi(F_1)$. Since $\varrho_{k-1,h} \circ \varphi = \varrho_{k-1,h}|_{Y_1}$ we see that the sets $\varrho_{k-1,h}^{-1}(w) \cap F_1$ and $\varrho_{k-1,h}^{-1}(w) \cap F_2$ are of the same cardinality for every point $w \in \mathbb{A}^{k-1}$. Since by Lemma 7.6 every fiber of $\varrho_{k-1,h}|_{X'}$ is G-flexible, by Theorems 2.8 and 3.1 we can find an automorphism $\alpha_1 \in G$ such that replacing Y_2 by $\alpha_1(Y_2)$ one has $F_1 = F_2$ and $\varphi|_{F_1} = \mathrm{id}|_{F_1}$. Suppose that m is as in Lemma 7.9 (with ϱ replaced by $\varrho_{n-2,h}$). Choose β as in Lemma 7.8 and replace Y_2 by $\beta(Y_2)$, i.e., the m-jet of φ at any point of F_1 is now the m-jet of the identity map[15]. By Lemma 7.9 we can find $\alpha_2 \in G$ such that after replacement of Y_1 by $\alpha_2(Y_1)$ one

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[15]By flexibility such β can be chosen as an element of G since the finite sets E and F_1 are disjoint.

has $u_n|_{Y_1} = u_n \circ \varphi$. Consider $\sigma \in H$ such that $\sigma_*(u_k) = u_n, \sigma_*(u_n) = -u_k$ while the remaining coordinates remain intact. Since we started with a general h one has $\sigma \circ \varrho_{k,h} = \varrho_{k,h_1}$ where h_1 is a general element of H. Thus, replacement of h by h_1 leaves (1) valid while providing us with (2).

If we want to apply Theorem 4.2 further we need to guarantee the condition (16) in Remark 4.3 (5) with $\varrho_{k,h}$ as κ. For $k = 1$ we can get this condition by choosing a couple of points y_1 and y_2 in each irreducible component of Y_2 and requiring that for α_1 as above one has additionally $\varrho_{1,h}(\alpha_1(y_1)) \neq \varrho_{1,h}(\alpha_1(y_2))$. Then $\varrho_{1,h}|_{Y_2} : Y_2 \to \mathbb{A}^1$ is a quasi-finite morphism (after the replacement of Y_2 by $\alpha_1(Y_2)$) and we have (16) for Y_2. By (2) we have it also for Y_1. Furthermore, this condition will survive for $k > 1$ provided that in further replacements of h by another general $\widetilde{h} \in H$ one has $\varrho_{2,h} = \varrho_{2,\widetilde{h}}$.

As in the previous proof we treat $\varrho_{k,h} : X' \to \mathbb{A}^k$ (resp. $\varrho_{n-2,h} : X' \to \mathbb{A}^{n-2}$) as $\kappa : X \to P$ (resp. $\varrho : X \to Q$) in Theorem 4.2. By Theorem 4.2(i) and (iii) we can find an algebraic G-family $\mathcal{A}_3$ of automorphisms of X such that for a general element $\alpha_3 \in \mathcal{A}_3$ the morphism $\varrho|_{\alpha_3(Y_1 \setminus E)} : \alpha_3(Y_1 \setminus E) \to Q$ is injective and generates an injective map of the Zariski tangent bundles. By (5) this is also true for the morphism $\varrho|_{\alpha_3(Y_1)} : \alpha_3(Y_1) \to Q$. Similarly, since $\varrho^{-1}(\varrho(Z))$ is of dimension $\dim Z + 2 \leqslant \dim X - \dim Y_1$ by Theorem 1.4 we can find an algebraic G-family $\mathcal{A}_4$ of automorphisms of X such that for a general element $\alpha_4 \in \mathcal{A}_4$ the variety $\alpha_4(Y_1)$ meet $\varrho^{-1}(\varrho(Z))$ at a finite number of points which is equivalent to (4). By Remark 4.3 (2) we can suppose that $\mathcal{A}_3 = \mathcal{A}_4$. We need to show that the morphism $\varrho|_{\alpha(Y_1)} : \alpha(Y_1) \to Q$ can be made proper for which we need to change our general element $h \in H$.

Let H' be the subgroup of H as in the proof of Theorem 7.1(a). Applying the same argument as in that proof (i.e., using Proposition 4.6(1)) we see that for a general $h' \in H'$ the replacement of h by $\widetilde{h} = h'h$ provides us with properness of the morphism $\varrho'_{n-k-2,h'h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-k-2}$ as soon as $n - k - 2 \geqslant \dim Y_1 = 1$ (i.e., $k \leqslant n - 3$) while preserving the injectivity of these morphism and injectivity of the induced morphism of the Zariski tangent bundles (as well as (1), (2), and (5)). This yields (3) and, similarly, by Proposition 4.6(3) we get (4). Thus we have $\varrho_{n-3,h} \circ \varphi = \varrho_{n-3,h}|_{Y_1}$ which concludes Step 2.

Step 3. Let us show that replacing Y_1 and Y_2 with their automorphic images in X and replacing h by another general element of H we can suppose that $\varrho_{n-2,h} \circ \varphi = \varrho_{n-2,h}|_{Y_1}$ while keeping the morphism $\varrho_{n-2,h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-2}$ finite.

Consider the embedding $\mathbb{A}^n = P \times \mathbb{A}^3_{u_{n-2},u_{n-1},u_n} \subset P \times \mathbb{P}^3 =: W$ where $P = \mathbb{A}^{n-3}_{u_1,\ldots,u_{n-3}}$. Let $\widetilde{D} = W \setminus \mathbb{A}^n = P \times \mathbb{P}^2$ and for every closed subvariety Y of $\mathbb{A}^n$ denote by $\check{Y}$ the intersection of $\widetilde{D}$ with the closure of Y in W. Note that if Y is a curve then $\check{Y}$ consists of no more than l points where l is the number of punctures of Y (that are the points in the complement to a normalization of Y in a smooth completion of this normalization). Furthermore, let $V \subset \mathbb{A}^n$ be given by $u_n = 0$. Then $\check{V} = P \times L$ where L is a line in $\mathbb{P}^2$ and the natural morphism $Y \to \mathbb{A}^1_{u_n}$ is finite if and only if $\check{Y}$ does not meet $P \times L$. Consider the natural action of $H'' = \mathrm{SL}(3, \mathbf{k})$ on $\mathbb{A}^3_{u_{n-2},u_{n-1},u_n}$ and extend this action to the action on $\mathbb{A}^n = \mathbb{A}^{n-3} \oplus \mathbb{A}^3$ such that it is trivial on the first summand (this enables us to

treat this H'' as a subgroup of $H = SL(n, \mathbf{k})$ acting naturally on $\mathbb{A}^n$). Consider general elements $h_1'', \ldots, h_{3l}''$ of H'' where l is the same as above but for $Y = Y_1$. Let V_i be the zero locus of $u_n \circ (h_i'' h)$. Then $\breve{V}_i = P \times L_i$ where $L_1, \ldots, L_{3l}$ are general lines in $\mathbb{P}^2$ and, hence, the intersection of any three of them is empty.

Consider $F_1^i = Y_1 \cap \varrho_{n-2,h_i'' h}^{-1}(\varrho_{n-2,h_i'' h}(Z))$ and $F_2^i = \varphi(F_1^i)$. As before replacing Y_2 by its automorphic image one has $F_1^i = F_2^i$ and $\varphi|_{F_1^i} = \mathrm{id}|_{F_1^i}$ for every i. Furthermore, for every point in $\bigcup F_1^i$ we can suppose that the m-jet of φ coincides with the m-jet of the identity map where m is as in Lemma 7.9. By Lemma 7.11 for one of the lines, say L_1, the intersection $\breve{Y}_2$ with $P \times L_1$ is empty and thus for $u_n'' = u_n \circ (h_1'' h)$ the morphism $Y_2 \to \mathbb{A}_{u_n''}^1$ is finite. Applying Lemma 7.9 we can replace Y_1 by its automorphic image such that $u_n''|_{Y_1} = u_n'' \circ \varphi$. Exchanging the role of u_{n-2} and u_n we can suppose that $\varrho_{n-2,h'' h} \circ \varphi = \varrho_{n-2,h_1'' h}|_{Y_1}$. Since $h_1'' h$ is a general element of H, after the replacement of h by $h_1'' h$ we still have (1) and (5). The finiteness of the morphism $Y_2 \to \mathbb{A}_{u_{n-2}''}^1$ implies also that the morphism $\varrho_{n-2,h_1'' h}|_{Y_1} : Y_1 \to \mathbb{A}^{n-2}$ is finite which concludes Step 3.

Step 4. To finish the proof let us treat $\varrho_{n-2,h} : X' \to \mathbb{A}^{n-2}$ (resp. $\varrho_{n-1,h} : X' \to \mathbb{A}^{n-1}$) as $\kappa : X \to P$ (resp. $\varrho : X \to Q$)) in Theorem 4.2. By Theorem 4.2(i) and (iii) (in combination with (5)) we can find an algebraic G-family $\mathcal{A}_5$ of automorphisms of X such that for a general element $\alpha_5 \in \mathcal{A}_5$ the morphism $\varrho|_{\alpha_5(Y_1)} : \alpha_5(Y_1) \to Q$ is injective and generates an injective map of the Zariski tangent bundles. Being also finite (since $\kappa|_{Y_1} : Y_1 \to P$ is finite and α_5 is in $\mathrm{Aut}(X/P)$) it is a closed embedding. Furthermore, by Theorem 1.4 we can suppose that $\alpha_5(Y_1)$ does not meet $\varrho^{-1}(\varrho(Z))$ since the sum of the dimensions of these varieties is less than $\dim X$. Consider the natural action of $H''' = \mathrm{SL}(2, \mathbf{k})$ on $\mathbb{A}_{u_{n-1}, u_n}^2$ and extend this action to the action on $\mathbb{A}^n = \mathbb{A}^{n-2} \oplus \mathbb{A}^2$ such that it is trivial on the first summand (i.e., H''' is again a subgroup of H acting naturally on $\mathbb{A}^n$). By Proposition 4.6 for a general $h''' \in H'''$ the morphism $\varrho_{n-1,h''' h}|_{\alpha_5(Y_1)} : \alpha_5(Y_1) \to Q$ is still a closed embedding with $\alpha_5(Y_1)$ not meeting $\varrho_{n-1,h''' h}^{-1}(\varrho_{n-1,h''' h}(Z))$. Since h''' is general the same is true for the replacement of h by $\sigma''' h''' h$ where $\sigma''' \in H'''$ is up to a sign the transposition of coordinates u_{n-1} and u_n. By Lemma 7.9 (with $k = n - 1$) replacing Y_1 by its automorphic image we can suppose that for $u_n''' = u_n \circ (h''' h)$ one has $u_n'''|_{Y_1} = u_n''' \circ \varphi$. Exchanging the role of u_{n-1} and u_n we can suppose that $\varrho_{n-1,\sigma''' h''' h} \circ \varphi = \varrho_{n-1,\sigma''' h''' h}|_{Y_1}$. The assumptions of Lemma 7.9 are true with ϱ in that lemma equal to $\varrho_{n-1,\sigma''' h''' h}$. Thus, after additional replacement of Y_1 by its automorphic image we can make $Y_1 = Y_2$ which is the desired conclusion. $\square$

Remark 7.12.

(1) The assumption that Y_1 and Y_2 are closed in $\mathbb{A}^n$ (and not in $\mathbb{A}^n \setminus Z$) cannot be dropped. Indeed, consider Z that does not admit nontrivial automorphisms, and let $L_1 \simeq \mathbb{A}$ and $L_2 \simeq \mathbb{A}$ be disjoint curves in $\mathbb{A}^n$ each of which meets Z at one point only. Then $Y_1 = L_1 \setminus Z$ and $Y_2 = L_2 \setminus Z$ are isomorphic but there is no way to extend this isomorphism to an automorphism of $\mathbb{A}^n \setminus Z$.

(2) We constructed an automorphism γ of X such that $\gamma|_{Y_1} = \varphi$ as a composition of elements of G_Z. However, Remark 7.10 implies that we can choose γ as a

composition of elements of G_{Z_m}. More precisely, the proof of Theorem 7.1 yields the following fact.

Proposition 7.13. *For every $m > 0$ the automorphism γ from Theorem 7.1 can be chosen as a composition of elements of the flows of locally nilpotent vector fields that vanish on Z with multiplicity m. In particular, the restriction of γ to the m-th infinitesimal neighborhood Z_m of Z is the identity map.*

It is interesting to understand how sharp is Theorem 7.1. Therefore, we would like to pose the following.

Question. *Let Z be a closed subvariety in $\mathbb{A}^n$ of codimension 2, and let Y_1 and Y_2 be two smooth closed isomorphic curves in $\mathbb{A}^n$ disjoint from Z. Can one always find an automorphism of $\mathbb{A}^n \setminus Z$ transforming Y_1 onto Y_2?*

8. The case of quadrics

Notation 8.1. In this section $m \geqslant 6$ and X is a hypersurface in $\mathbb{A}^m_{\mathbf{k}}$ that is a nonzero fiber of a non-degenerate quadratic form. That is, when $m = 2n$ (resp. $m = 2n+1$) we can suppose that X is given by $u_1 v_1 + \cdots + u_n v_n = 1$ (resp. $u_0^2 + u_1 v_1 + \cdots + u_n v_n = 1$) in a suitable coordinate system $(\bar{u}, \bar{v}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$(resp. $(u_0, \bar{u}, \bar{v})$). Recall that such an X is a homogeneous space of a special orthogonal group $G := \mathrm{SO}(m, \mathbf{k})$ acting linearly on $\mathbb{A}^m_{\mathbf{k}}$. In particular, it is flexible since homogeneous spaces of any semi-simple group are flexible [AFKKZ]. Furthermore, the coordinate system determines an embedding $\mathbb{A}^m_{\mathbf{k}} \hookrightarrow \mathbb{P}^m_{\mathbf{k}}$ such that the action of G extends to $\mathbb{P}^m_{\mathbf{k}}$. The closure $\bar{X}$ of X in $\mathbb{P}^m_{\mathbf{k}}$ yields a completion of X for which $\bar{X} \cap H$ is a quadric in $H = \mathbb{P}^m_{\mathbf{k}} \setminus \mathbb{A}^m_{\mathbf{k}} \simeq \mathbb{P}^{m-1}_{\mathbf{k}}$.

Lemma 8.2. *Let Notation 8.1 hold. Then G acts transitively on $\bar{X} \cap H$.*

Proof. Making a linear coordinate change we can suppose that the coordinate system $(w_1, \ldots, w_m)$ on $\mathbb{A}^m_{\mathbf{k}}$ is such that X is given by $w_1^2 + \cdots + w_m^2 = 1$. Then $\mathbb{P}^m_{\mathbf{k}}$ is equipped with the coordinate system $(\widetilde{w}_0 : \widetilde{w}_1 : \ldots : \widetilde{w}_m)$ so that $w_i = \widetilde{w}_i / \widetilde{w}_0$ for $i = 1, \ldots, m$ and H is given by $\widetilde{w}_0 = 0$. Hence, the equation of $\bar{X} \cap H$ in H is $\widetilde{w}_1^2 + \cdots + \widetilde{w}_m^2 = 0$.

Let $Z_i = \bar{X} \cap H \cap \{\widetilde{w}_i = 0\}$ and $Z_i' = (\bar{X} \cap H) \setminus Z_i$. Note that Z_i' is isomorphic to a nonzero fiber of a non-degenerate quadratic form on $\mathbb{A}^{m-1}_{\mathbf{k}}$ and, therefore, the subgroup $G_i \simeq \mathrm{SO}(m-1, \mathbf{k})$ of G that preserves the coordinate w_i acts transitively on Z_i'. On the other hand, for a point $z \in Z_m$ at least one of the coordinates $\widetilde{w}_1 : \ldots : \widetilde{w}_{m-1}$ is nonzero (say $\widetilde{w}_1$). Furthermore, applying an element of G_m we can make $\widetilde{w}_2$ also different from zero. Hence, the subgroup G_2 can transform z into a point of Z_m' since the action of G_2 on Z_2' can switch coordinates $\widetilde{w}_1$ and $\widetilde{w}_m$ (and change the sign of one of them). This yields the desired statement about transitivity because $\bar{X} \cap H = Z_m \cup Z_m'$. $\square$

The aim of this section is to prove the following theorem.

Theorem 8.3. *Let Notation 8.1 hold and $\varphi : Y_1 \to Y_2$ be an isomorphism of two closed subvarieties of X. Let $\mathrm{ED}(Y_i) + \dim Y_i \leqslant m - 2$. Then φ extends to an automorphism of X which belongs to $\mathrm{SAut}(X)$.*

Proof. Consider the case of $m = 2n$, i.e., $X \subset \mathbb{A}_{\mathbf{k}}^m$ is given by $u_1 v_1 + \cdots + u_n v_n = 1$. Note that for $m \geqslant 6$ the assumption that $\mathrm{ED}(Y_i) + \dim Y_i \leqslant m - 2$ implies that $k := \dim Y_i + 2 \leqslant n$. Let $(u_1, \ldots, u_n, v_{k+1}, \ldots, v_n)$ be a coordinate system on $\mathbb{A}_{\mathbf{k}}^{m-k}$ and $\varrho : \mathbb{A}_{\mathbf{k}}^m \to \mathbb{A}_{\mathbf{k}}^{m-k}$ be the natural projection. Suppose that $\bar{\varrho} : \bar{X} \dashrightarrow \mathbb{A}_{\mathbf{k}}^{m-k}$ is the rational extension of $\varrho|_X$. Then $\mathcal{R}(\bar{\varrho})$ as in Notation 5.1 has dimension $k - 1$. Note also that if $\bar{Y}_j$ is the closure of Y_j in $\bar{X}$ then $\dim \bar{Y}_i \cap H < m - k - 2$ since $\mathrm{ED}(Y_i) \leqslant m - k$. Hence, $\dim \bar{Y}_j \cap H + \dim \mathcal{R}(\bar{\varrho}) < \dim \bar{X} \cap H$. Lemma 8.2 implies that Corollary 5.8 is applicable and for a general element α of some algebraic family $\mathcal{A}$ of automorphisms of X the morphism $\varrho|_{\alpha(Y_j)} : \alpha(Y_j) \to \mathbb{A}_{\mathbf{k}}^{m-k}$ is a closed embedding. Furthermore, by Theorem 1.3 we can suppose that $\alpha(Y_j)$ does not meet the subvariety F of X given by $u_2 = \ldots = u_k = 0$ since $\dim Y_j < k - 1 = \mathrm{codim}_X F$. Hence, replacing each Y_j by its automorphic image we suppose that $\varrho|_{Y_j} : Y_j \to \mathbb{A}_{\mathbf{k}}^{m-k}$ is a closed embedding and $Y_j \cap F = \varnothing$. This implies that $Y_j' := \varrho(Y_j)$ is isomorphic to Y_j and it does not meet the subspace of $\mathbb{A}_{\mathbf{k}}^{m-k}$ given by the same system of equations $u_2 = \ldots = u_k = 0$. In particular, we can treat $v_1|_{Y_j}$ as the lift of a function f_j on Y_j' and by the Nullstellensatz one has $1 - f_j = \sum_{i=2}^{k} u_i g_{j,i}$ for some regular functions $g_{j,2}, \ldots, g_{j,k}$ on Y_j'.

Claim. For every Y_j as in the statement of the Theorem there exists an automorphism of X that sends Y_j onto a subvariety of the hypersurface S in X given by $v_1 = 1$ (in particular, S is isomorphic to $\mathbb{A}_{\mathbf{k}}^{m-2}$).

Indeed, let δ_i, $i = 2, \ldots, k$ be the locally nilpotent vector fields on X given by $\delta_i = u_i \, \partial/\partial v_1 - u_1 \, \partial/\partial v_i$. Let $\psi_{j,i}$ be the flow of $g_{j,i} \delta_i$ at time $t = 1$. Note that the automorphism $\psi_{j,2} \circ \ldots \circ \psi_{j,k}$ transforms Y_j into a subvariety of X on which the restriction of v_1 is identically 1. This concludes the proof of the Claim.

Thus we can suppose from the beginning that Y_1 and Y_2 are contained in S. By Theorem 0.1 there is an automorphism β of S that transforms Y_1 into Y_2. Since $X \setminus \{v_1 = 0\} \simeq \mathbb{A}_{\mathbf{k}}^* \times S$ we can suppose that β is the restriction of an automorphism of $X \setminus \{v_1 = 0\}$. Thus by Theorem 3.1 β can be extended to an automorphism of X which concludes the case of $m = 2n$.

If $m = 2n + 1$ then we treat $(u_0, u_1, \ldots, u_n, v_{k+1}, \ldots, v_n)$ as a coordinate system on $\mathbb{A}_{\mathbf{k}}^{m-k}$ and consider the natural projection $\varrho : \mathbb{A}_{\mathbf{k}}^m \to \mathbb{A}_{\mathbf{k}}^{m-k}$. The rest of the proof works without change and we are done for the case of $m = 2n + 1$ as well. $\square$

The assumption on $\mathrm{ED}(Y_i)$ implies that in the smooth case (i.e., in the case when $\mathrm{ED}(Y_i) = 2 \dim Y_i + 1$) we have the following.

Corollary 8.4. *Let X be a nonzero fiber of a non-degenerate quadratic form in $\mathbb{A}_{\mathbf{k}}^m$. Suppose that $\varphi : Y_1 \to Y_2$ is an isomorphism of two closed smooth subvarieties of X such that $\dim Y_i$ does not exceed $m/3 - 1$. Then φ extends to an automorphism of X.*

9. Comparable morphisms

Definition 9.1.

(1) Let X (resp. X') be a smooth algebraic variety and $\varrho : X \to X'$ be a

morphism. Consider a family $\mathcal{F}$ of closed subvarieties of X. Suppose that, given Y_1 and $Y_2 \in \mathcal{F}$ isomorphic over X' and such that $\varrho|_{Y_i} : Y_i \to X'$ is a closed embedding, there is an automorphism $\alpha \in \mathrm{Aut}(X/X')$ of X over X' that transforms Y_1 onto Y_2. In this case we say that ϱ is comparable on $\mathcal{F}$.

When the ground field k is $\mathbb{C}$ we also say that ϱ is holomorphically comparable on $\mathcal{F}$ if one can find a holomorphic automorphism of X over X' that transforms Y_1 onto Y_2.

(2) Let X (resp. X') be a smooth algebraic variety with a group $G \subset \mathrm{Aut}(X)$ (resp. $G' \subset \mathrm{Aut}(X')$) acting on it. We say that a morphism $\varrho : X \to X'$ is (G, G')-comparable if for every $g' \in G'$ there exists $g \in G$ such that $\varrho \circ g = g' \circ \varrho$.

The next fact follows from Definition 9.1.

Proposition 9.2. *Let $\varrho : X \to X'$ be (G, G')-comparable. Let Y_1 and Y_2 be closed subvarieties of X and $\varphi : Y_1 \to Y_2$ be an isomorphism. Suppose that each morphism $\varrho|_{Y_i} : Y_i \to X'$ is a closed embedding, $Y_i' = \varrho(Y_i)$, and $\varphi' : Y_1' \to Y_2'$ is the isomorphism for which $\varphi' \circ \varrho|_{Y_1} = \varrho|_{Y_2} \circ \varphi$. Let φ' extend to an automorphism g' of X' which is an element of G'. Then there exists an element $g \in G$ such that $g(Y_1)$ is naturally isomorphic to Y_2 over X'. Furthermore, if ϱ is also comparable (resp. holomorphically comparable) on a family $\mathcal{F}$ containing Y_1 and Y_2 then there exists an algebraic (resp. holomorphic) automorphism α of X for which $\alpha|_{Y_1} = \varphi$.*

Proposition 9.3. *Let $\varrho : X \to X'$ be a principal bundle for some algebraic group H and let $h.x$ be the action of $h \in H$ on $x \in X$. Let $\varphi : Y_1 \to Y_2$ be an isomorphism of closed subvarieties of X over X' such that $\varrho|_{Y_k} : Y_k \to X'$ is a closed embedding. Then there exists a morphism $\theta : Y' \to H$ from $Y' = \varrho(Y_k)$ such that*

(a) *$\varphi(y_1) = \theta(\varrho(y_1)).y_1$ for every $y_1 \in Y_1$ and*

(b) *φ extends to an automorphism of X over X' if θ extends to a morphism $X' \to H$.*

Furthermore, if the ground field $\mathbf{k}$ is $\mathbb{C}$ then

(c) *φ extends to a holomorphic automorphism of X over X' if θ extends to a holomorphic map $X' \to H$.*

Proof. Let $y_2 = \varphi(y_1)$, and $y' = \varrho(y_k)$. Since y_1 and y_2 are in $\varrho^{-1}(y') \simeq H$ we have a unique element $h \in H$ for which $y_2 = h.y_1$. Letting $\theta(y') = h$ we get (a). If θ extends to $\Theta : X' \to H$ then the desired extension of φ in (b) is given by $\Theta(\varrho(x)).x$ for $x \in X$. $\square$

Corollary 9.4. *Let $\varrho : X \to X'$ be a principal bundle for some algebraic group H and X' be quasi-affine.*

(i) *Suppose that $\mathcal{F}_{\mathrm{aff}}$ is the family of closed subvarieties of X isomorphic to $\mathbb{A}_{\mathbf{k}}^k$ with $k = 1, 2, \ldots$. Then ϱ is comparable on $\mathcal{F}_{\mathrm{aff}}$.*

(ii) *Suppose that H is isomorphic as an algebraic variety to $\mathbb{A}_{\mathbf{k}}^n$. Consider the family $\mathcal{F}$ of closed subvarieties Z of X such that $\varrho(Z)$ is closed in an affine variety X'' containing X' as an open subvariety. Then ϱ is comparable on $\mathcal{F}$.*

Proof. Let Y_1 and $Y_2 \in \mathcal{F}_{\mathrm{aff}}$ be as in Definition 9.1(1), i.e., $Y' = \varrho(Y_i) \simeq \mathbb{A}_{\mathbf{k}}^k$ is closed in X'. Consider a morphism $g : X' \to \mathbb{A}_{\mathbf{k}}^k$ whose restriction to Y' is the identity map (such a g exists since X' is quasi-affine). Let θ be as in Proposition 9.3. Then $\theta \circ g$ yields an extension of θ to X' which by Proposition 9.3 concludes (i).

In (ii) note that since $Z' := \varrho(Z)$, $Z \in \mathcal{F}$ is closed in the affine variety X'' this morphism θ extends to a morphism $\Theta : X'' \to \mathbb{A}_{\mathbf{k}}^n \simeq H$ and again Proposition 9.3 yields (ii). □

Remark 9.5. Let $\varrho : X \to X'$ be an affine morphism of smooth varieties and $G \subset \mathrm{Aut}(X)$ (resp. $G' \subset \mathrm{Aut}(X')$).

(1) Note that if ϱ is (G, G')-comparable and G' acts transitively on X' then all fibers of ϱ are isomorphic and, furthermore, $\varrho : X \to X'$ is a locally trivial fiber bundle in étale topology (it follows from the fact that any affine morphism $W \to V$ of algebraic varieties with pairwise isomorphic general fibers is locally trivial over an étale neighborhood of a general point of V [KR]).

(2) Let G (resp. G') be generated by a set $\mathcal{N}$ (resp. $\mathcal{N}'$) of complete algebraic vector fields on X (resp. X'). In order for ϱ to be (G, G')-comparable it suffices to require that for every $\delta' \in \mathcal{N}'$ there exists $\delta \in \mathcal{N}$ such that for every $x \in X$ and $x' = \varrho(x)$ one has

$$\varrho_*(\delta_x) = \delta'_{x'} \tag{21}$$

where δ_x (resp. $\delta'_{x'}$) is the value of the field δ at x (resp. $\delta'_{x'}$ at x').

Definition 9.6. We call a pair δ, δ' of vector fields (on X and X' respectively) comparable if they satisfy formula (21). Similarly, we call the pair $(\mathcal{N}, \mathcal{N}')$ from Remark 9.5 (2) comparable if for every $\delta' \in \mathcal{N}'$ there exists $\delta \in \mathcal{N}$ so that the pair δ, δ' is comparable.

Example 9.7. Let $X = \mathrm{SL}(n, \mathbf{k})$, thus $\dim X = n^2 - 1$. Let $A = [a_{i,j}]_{i,j=1}^n$ be a matrix from $\mathrm{SL}(n, \mathbf{k})$ and let $k \leqslant n - 1$ and $m \leqslant n$. Consider the $(m \times k)$-matrix A' obtained from A by removing all rows starting with $(m+1)$-st and all columns starting with $(k+1)$-st. Then one has the natural morphism of $\varrho : X \to X'$, $A \to A'$ into the space $X' \simeq \mathbb{A}_{\mathbf{k}}^{km}$ of $(k \times m)$-matrices. Let $1 \leqslant i \neq j \leqslant n$ and

$$\delta_{ij} = \sum_{l=1}^n a_{l,i} \frac{\partial}{a_{l,j}},$$

i.e., δ_{ij} is a locally nilpotent vector field on X whose flow is the addition of multiples of the i-th column in A to the j-th column. Note that

$$\delta'_{ij} = \sum_{l=1}^m a_{l,i} \frac{\partial}{a_{l,j}}, \ i, j \leqslant k$$

is a locally nilpotent vector field on X' for which formula (21) is valid. Similarly, the locally nilpotent vector fields $\sigma_{ij} = \sum_{l=1}^n a_{i,l} \, \partial/a_{j,l}$ and $\sigma'_{ij} = \sum_{l=1}^k a_{i,l} \, \partial/a_{j,l}$ on X and X' respectively also satisfy formula (21).

Notation 9.8. Suppose that $\varrho : X \to X'$ is a smooth morphism of smooth algebraic varieties. Let $\mathcal{N}$ (resp. $\mathcal{N}'$) be a set of locally nilpotent vector fields on X (resp. X') such that the pair $(\mathcal{N}, \mathcal{N}')$ is comparable. Furthermore, we suppose that X is a closed subvariety of $X' \times \mathbb{A}_{\mathbf{k}}^m$ with ϱ being the restriction of the natural projection $\widehat{\varrho} : X' \times \mathbb{A}_{\mathbf{k}}^m \to X'$. That is, for every δ and δ' satisfying formula (21) we can write $\delta = \delta' + \delta''$ where δ'' is tangent to each fiber $\widehat{\varrho}^{-1}(x') \simeq \mathbb{A}_{\mathbf{k}}^m$ of $\widehat{\varrho}$ (where by abuse of notation we identify δ' with its natural lift to $X' \times \mathbb{A}_{\mathbf{k}}^m$).

Recall that a vector field on $\mathbb{A}_{\mathbf{k}}^m$ (with a fixed coordinate system $\bar{u} = (u_1, \ldots, u_m)$) is a linear vector field if it is of the form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are vectors in $\mathbb{A}_{\mathbf{k}}^m$ and A is a square $(m \times m)$-matrix (i.e., it is a non-homogeneous system of linear differential equations).

Proposition 9.9. *Let Notation 9.8 hold. Suppose that for every comparable pair $(\delta, \delta') \in \mathcal{N} \times \mathcal{N}'$ the following condition is true:*

(A) the restriction of the field $\delta'' = \delta - \delta'$ to every fiber $\widehat{\varrho}^{-1}(x') \simeq \mathbb{A}_{\mathbf{k}}^m$ is a linear vector field (depending on $x' \in X'$).

Let $\widetilde{\mathcal{N}}'$ be the smallest saturated set of locally nilpotent vector fields on X' such that it contains $\mathcal{N}'$. Suppose that $\mathcal{L}'$ is the Lie algebra generated by the fields of the form $a\delta'$ where $\delta' \in \widetilde{\mathcal{N}}'$ and $a \in \mathbf{k}[X']$. Then there exists a Lie algebra $\mathcal{L}$ of vector fields on X such that for every complete (resp. locally nilpotent) vector field $\sigma' \in \mathcal{L}'$ there exists a complete (resp. locally nilpotent) vector field $\sigma \in \mathcal{L}$ such that the pair (σ, σ') is comparable.

Proof. Note that for every $a \in \mathbf{k}[X']$ and every comparable pair $(\delta, \delta') \in \mathcal{N} \times \mathcal{N}'$ the pair $(a\delta, a\delta')$ is comparable and satisfies condition (A). Furthermore, let (δ_i, δ_i'), $i = 1, 2, \ldots, s$ be a collection of comparable pairs satisfying condition (A) and let ℓ be a linear form in s variables. Let $\kappa' = \ell(\delta_1', \ldots, \delta_s')$ (resp. $\kappa = \ell(\delta_1, \ldots, \delta_s)$). Then the pair (κ, κ') is also comparable and satisfies condition (A).

For the Lie brackets $\delta_0 = [\delta_1, \delta_2]$ and $\delta_0' = [\delta_1', \delta_2']$ the pair (δ_0, δ_0') is, of course, comparable. Let us check condition (A) for this pair. Note that

$$[\delta_1, \delta_2] = [\delta_1', \delta_2'] + [\delta_1'', \delta_2''] + [\delta_1', \delta_2''] + [\delta_1'', \delta_2']$$

where the first term belongs to $\mathcal{L}'$ and the restriction of the second one to every fiber of $\widehat{\varrho}$ is a linear vector field. Consider, say, the third term $[\delta_1', \delta_2'']$. Since the restriction of δ_2'' to $\widehat{\varrho}^{-1}(x') \simeq \mathbb{A}_{\mathbf{k}}^m$ is of the form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ we see that $[\delta_1', \delta_2'']$ is of the form $\delta_1'(A)\mathbf{x} + \delta_1'(\mathbf{b})$ and, in particular, it is again a vector field whose restriction to $\widehat{\varrho}^{-1}(x')$ is linear. Thus the pair (δ_0, δ_0') satisfies condition (A).

Now, a comparable pair (δ, δ') of complete vector fields induces flows $\varphi_t : X \to X$ and $\varphi_t' : X' \to X'$ (where $t \in \mathbf{k}$) so that $\varrho \circ \varphi_t = \varphi_t' \circ \varrho$. Hence, φ_t tranforms a fiber $\varrho^{-1}(x_0') \simeq \mathbb{A}_{\mathbf{k}}^m$ to the fiber $\varrho^{-1}(\varphi_t'(x_0')) \simeq \mathbb{A}_{\mathbf{k}}^m$. When condition (A) is satisfied this map of fibers is an element of the flow of a linear non-autonomous vector field which is automatically an affine map. Hence, for every comparable pair (σ, σ') of complete vector fields the conjugation by φ_t and φ_t' yields a comparable pair $(\widetilde{\sigma}, \widetilde{\sigma}')$ of complete vector fields such that condition (A) is satisfied.

Applying these operations of taking Lie brackets, conjugations, and linear combinations we construct the desired saturated set $\widetilde{\mathcal{N}}'$ so that the pair $(\widetilde{\mathcal{N}}, \widetilde{\mathcal{N}}')$ is

comparable for some set $\widetilde{\mathcal{N}}$ of locally nilpotent vector fields on X. Let $\mathcal{L}'$ (resp. $\mathcal{L}$) be the Lie algebra generated by the fields of the form $a\delta'$ where $\delta' \in \widetilde{\mathcal{N}}'$ and $a \in \mathbf{k}[X']$ (resp. $a\delta$ where $\delta \in \widetilde{\mathcal{N}}$ and $a \in \mathbf{k}[X'] \subset \mathbf{k}[X]$). By construction for every element $\sigma' \in \mathcal{L}'$ there is an element of $\sigma \in \mathcal{L}$ so that the pair (σ, σ') is also comparable and satisfies condition (A).

Suppose now that σ' is complete and O' is an integral curve of this field (in particular, O' is isomorphic to either $\mathbb{A}_{\mathbf{k}}$ or $\mathbb{A}_{\mathbf{k}}^*$). To show that σ is complete it suffices to prove that O' admits a lift to an integral curve $O \subset \widehat{\varrho}^{-1}(O') \simeq O' \times \mathbb{A}_{\mathbf{k}}^m$ of the field $\sigma = \sigma' + \sigma''$. The restriction of σ'' to $O' \times \mathbb{A}_{\mathbf{k}}^m$ is of the form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ where the matrix A and the vector $\mathbf{b}$ depend on the parameter $t \in O'$. That is, we are dealing with a non-autonomous system of linear equations. Such a system has a solution for all values of t which yields the desired lift of O' to an integral curve of σ.

If σ' is locally nilpotent then in order to show that $\sigma = \sigma' + \sigma''$ is locally nilpotent one needs to prove that for every $b \in \mathbf{k}[X]$ there exists n for which $\sigma^n(b) = 0$. This is true when $b \in \mathbf{k}[X'] \subset \mathbf{k}[X]$ since in this case $\sigma(b) = \sigma'(b)$. Since $\mathbf{k}[X]$ is generated over $\mathbf{k}[X']$ by the coordinate functions on $\mathbb{A}_{\mathbf{k}}^m$ it suffices to prove that if b is a coordinate on $\mathbb{A}_{\mathbf{k}}^m$ then $\sigma^n(b) = 0$ for $n >> 0$. Note that in this case $\sigma(b) = \sigma''(b)$ and by condition (A) one has $\sigma''(b) \in \mathbf{k}[X']$. This yields the desired conclusion. $\square$

10. Holomorphic extension of θ

In this section the ground field $\mathbf{k}$ is $\mathbb{C}$ and its aim is to describe conditions under which $\theta : Y' \to H$ from Proposition 9.3 admits a holomorphic extension $\Theta : X' \to H$.

Proposition 10.1. *Let the assumption of Proposition 9.3 hold and X' be Stein. Suppose that the map θ is homotopy equivalent to a constant map from Y' to H. Then θ admits a holomorphic extension $\Theta : X' \to H$.*

Proof. By the Oka–Grauert principle [F2, Thm. 5.4.4] it suffices to construct a continuous extension of θ. Let $\widetilde{\theta}(t) : Y' \to H$, $t \in [0, 1]$ be a homotopy of θ to a constant map, i.e., $\widetilde{\theta}(*, 0) = \theta$ and $\theta(*, 1)$ sends Y' to a point $h_0 \in H$.

The argument is rather simple in the case of a smooth Y' since one can consider its closed tubular neighborhood U. Note that U admits a continuous map $g : U \to [0, 1]$ such that $g^{-1}(0) = Y'$ and $g^{-1}(1)$ is the boundary ∂U of U. Define a continuous extension $\theta' : U \to H$ of θ via the formula $\theta'(u) = \widetilde{\theta}(p(u), g(u))$ where $u \in U$ and $p : U \to Y'$ is the natural projection. Then we can extend θ' further to $\theta'' : X' \to H$ by putting $\Theta(x') = h_0$ for $x' \in X' \setminus U$.

In the general case we recall that complex algebraic varieties admit triangulation [Lo]. That is, one can view Y' as a subcomplex in a complex X'. Then the second baricentric derived neighborhood U of Y' is regular (e.g., see [Hi]). This means, that Y' is a deformation retract of U and the map $U \to Y'$ (which is identity on Y') can be constructed via a sequence of collapses $U_{i-1} \searrow U_i$, $i = 1, \ldots, k$ where each U_i is a subcomplex of the second baricentric subdivision, $U_0 = U$, U_k coincides with the second baricentric subdivision of Y', and $U_{i-1} \searrow U_i = \{\sigma_i, \tau_i\}$

where σ_i is an l-dimensional simplex and τ_i is the only $(l-1)$-dimensional face of σ_i that is not contained in U_i (viewed as a complex). In particular, the union $\bigcup_{i=1}^{k} \tau_i$ contains the boundary of U_0. Let us establish the following.

Claim. A continuous map $\theta_i : U_i \to H$, admitting a homotopy $\widetilde{\theta}_i$ to the constant map h_0 and such that $\widetilde{\theta}_i(\partial \tau_i) = h_0$, can be extended to a similar continuous map $\theta_{i-1} : U_{i-1} \to H$ with a homotopy $\widetilde{\theta}_{i-1}$ to the constant map h_0 such that $\widetilde{\theta}_{i-1}(\tau_i) = h_0$.

Let $\partial \sigma_i$ (resp. $\partial \tau_i$) be the boundary of σ_i (resp. τ_i). Then there is a homotopy $g : \sigma_i \times [0,1] \to \sigma_i$ such that $g(*, 1) = \mathrm{id}|_{\sigma_i}$, $\mathrm{Im}\, g(*, 0) = \partial \sigma_i \cap U_{i-1}$, and for every $t \in [0,1]$ the restriction $g(*, t)|_{\partial \sigma_i \cap U_{i-1}}$ is the identity map on $\partial \sigma_i \cap U_{i-1}$. In particular, each collapse $U_{i-1} \searrow U_i$ induces a strong deformation retract $f_i : U_{i-1} \to U_i$. Furthermore, g can be chosen so that it yields a natural homeomorphism between $\sigma_i \setminus \partial \tau_i$ and $\dot{\tau}_i \times [0,1]$ where $\dot{\tau}_i = \tau_i \setminus \partial \tau_i$ is the interior of τ_i. [16] Hence, we can now define the extension θ_{i-1} of θ_i so that every $u \in \dot{\tau}_i$ and $t \in [0,1]$ one has $\theta_{i-1}(g(u,t)) = \widetilde{\theta}_i(f_i(u), t)$. By construction $\theta_{i-1}(\tau_i) = h_0$ and the extension is homotopic to a constant map (indeed, define the homotopy $\widetilde{\theta}_{i-1}(g(u,s))$ on $U_{i-1} \setminus \partial \tau_i$ via $\widetilde{\theta}_i(f_i(u), t + s(1-t))$). This yields the the Claim.

It remains to show that for such extensions one has $\widetilde{\theta}_i|_{\partial U} = h_0$ as soon as $U_i \cap \partial U \neq \varnothing$. First note that if σ_i is one-dimensional and meets Y' then τ_i is a singleton (i.e., $\partial \tau_i = \varnothing$) and, therefore, $\widetilde{\theta}_i(\tau_i) = h_0$. Let S be the collection of $\sigma_i \notin U_k$ such that each of them has a face contained in U_k. For every $\sigma_i \in S$ there is the only one vertex not in U_k. Suppose that T is the set of such vertices. Since each of $\sigma_i \in S$ collapses before the one-dimensional simplexes mentioned above we can suppose that $\widetilde{\theta}_i|_T = h_0$. Let S_m be the collection of m-dimensional simplexes $\sigma_j \notin S$ that have faces in $\bigcup_{\sigma_i \in S} \sigma_i$. For every $\sigma_j \in S_2$ we have a one-dimensional τ_j such that $\partial \tau_j \in T$. Thus by the Claim we can suppose that the restriction of $\widetilde{\theta}_j$ to $\bigcup_{\sigma_j \in S_2} \tau_j$ is the constant map to h_0. Similarly, for every $\sigma_k \in S_3$ we have $\partial \tau_k$ in $\bigcup_{\sigma_j \in S_2} \tau_j$. Hence, proceeding by induction we see that the restriction of every $\widetilde{\theta}_i$ to $\bigcup_{j \geqslant 2} \bigcup_{\sigma_k \in S_j} \tau_k$ is h_0. Consequently, for every $\sigma_i \notin S \cup \bigcup_{j \geqslant 2} S_j$ we have $\widetilde{\theta}_i(\tau_i) = h_0$. Since we deal with the second baricentric subdivision we have $\bigcup_{\sigma_i \in S} \tau_i \cap \partial U = \varnothing$ which yields the desired conclusion. $\square$

Lemma 10.2. *Let $m \geqslant 2$ and Z be a complex affine algebraic variety such that $H_i(Z) = 0$ for $i \geqslant m + 1$. Then Z can be embedded into a contractible topological space $\widehat{Z}$ such that $\widehat{Z} \setminus Z$ is a union of components each of which is homeomorphic to a ball B whose real dimension is between 2 and $m + 2$.*

Proof. Since any algebraic variety is a finite CW-complex, Z has a finitely generated fundamental group. Gluing Z with discs along generators of this group we obtain a topological space $Z_1 \supset Z$ such that Z_1 is simply connected. Furthermore, the Mayer–Vietoris sequence implies that the embedding $Z \hookrightarrow Z_1$ induces an isomorphism $H_i(Z) \simeq H_i(Z_1)$ for $i \geqslant 3$. By the Hurewicz theorem $\pi_2(Z_1)$ is

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[16]In order to see this, treat σ_i as a closed l-dimensional ball, τ_i as the upper $(l-1)$-dimensional semi-sphere in its boundary, $\partial \sigma_i \cap U_{i-1}$ as the lower semi-sphere and $\partial \tau_i$ as its equator.

naturally isomorphic to $H_2(Z_1)$. Thus, we can choose a nontrivial element ω of $H_2(Z_1)$ presented by a two-dimensional cell in the CW-complex which is homeomorphic to a two-sphere. Glue Z_1 with a three-dimensional ball along this sphere and let $V_1 = Z_1 \cup B_1$ be the resulting simply connected topological space. Consider the Mayer–Vietoris sequence

$$\cdots \to H_{k+1}(V_1) \xrightarrow{d_{k+1}} H_k(Z_1 \cap B_1) \to H_k(Z_1) \oplus H_k(B_1)$$
$$\to H_k(V_1) \xrightarrow{d_k} H_{k-1}(Z_1 \cap B_1) \to \cdots .$$

Note that if ω generates a free group then the map $H_2(Z_1 \cap B_1) \to H_2(Z_1)$ is a monomorpism and, therefore, d_3 sends $H_3(V_1)$ to zero. That is, after such a gluing $H_i(V_1) = H_i(Z_1)$ for $i \geqslant 3$ while the rank of $H_2(V_1)$ is less than the rank of $H_2(Z_1)$ since ω induces the zero element in $H_2(V_1)$. Continuing this procedure we can embed Z_1 into a simply connected V_2 such that $H_i(V_2) = H_i(Z_1)$ for $i \geqslant 3$ while $H_2(V_2)$ is finite. Let V_3 be the result of gluing a three-ball to V_2 along ω as before and $V_3 = V_2 \cup B_2$ be the resulting simply connected topological space. In this case ω is of finite order l and the kernel of $H_2(V_2 \cap B_2) \to H_2(V_2)$ in the associated Mayer–Vietoris sequence is isomorphic to $l\mathbb{Z} \subset \mathbb{Z} \simeq H_2(V_2 \cap B_2)$. In particular, the image of the map $H_{k+1}(V_3) \to H_k(V_2 \cap B_2) \simeq \mathbb{Z}$ is $l\mathbb{Z}$. Since the latter is a free $\mathbb{Z}$-module the map $H_{k+1}(V_3) \to l\mathbb{Z}$ has a right inverse. Thus it follows from the Mayer–Vietoris sequence that $H_3(V_3)$ is naturally isomorphic to $H_3(V_2) \oplus \mathbb{Z}$ while $H_i(V_3) = H_i(V_2)$ for $i \geqslant 4$. Since the number of elements in $H_2(V_3)$ is less than the number of elements in $H_2(V_2)$, continuing this procedure we embed Z_1 into a simply connected Z_2 such that $H_2(Z_2) = 0$, $H_i(Z_2) = H_i(Z_1) = H_i(Z)$ for $i \geqslant 4$, and $H_3(Z_2)$ is naturally isomorphic to the direct sum of $H_3(Z)$ and a finitely generated free $\mathbb{Z}$-module.

In the same manner we can embed Z into a simply connected Z_m such that $H_i(Z_m) = 0$ for $i \leqslant m$, $H_i(Z_m) = H_i(Z) = 0$ for $i \geqslant m + 2$, and $H_{m+1}(Z_m)$ is naturally isomorphic to the direct sum of $H_{m+1}(Z)$ and a finitely generated free $\mathbb{Z}$-module. That is, $H_{m+1}(Z_m)$ is a finitely generated free $\mathbb{Z}$-module since $H_{m+1}(Z) = 0$. Gluing Z_m with balls of real dimension $m + 2$ we can obtain a simply connected $\widehat{Z}$ with $H_i(\widehat{Z}) = 0$ for $i \leqslant m + 1$. Furthermore, as we saw before, since $H_{m+1}(Z_m)$ is free, this procedure does not affect the i-homology groups with $i \geqslant m + 2$. Hence, $H_i(\widehat{Z}) = 0$ for $i \geqslant 1$ and by the Hurewicz theorem $\pi_i(\widehat{Z}) = 0$ for $i \geqslant 1$. By the Whitehead theorem $\widehat{Z}$ is contractible which yields the desired conclusion. $\square$

Remark 10.3. It follows from the proof that if in Lemma 10.2 the group $H_m(Z)$ is free, then in order to construct $\widehat{Z}$ one needs only balls of real dimension at most $m + 1$.

Proposition 10.4. *Let $\varrho : X \to X'$ be a principal bundle for some algebraic group H and Y' be a closed subvariety of X'. Let X' be Stein and H be simply connected with $H_i(H) = 0$ for $i \leqslant m$ where $m \geqslant 2$. Suppose also that $H_i(Y') = 0$ for $i \geqslant m + 1$ and $H_m(Y')$ is free. Then any morphism $\theta : Y' \to H$ admits a holomorphic extension $\Theta : X' \to H$.*

Proof. By Lemma 10.2 we can embed Y' into a contractible topological space Y. Furthermore, by Remark 10.3 we can suppose that each component of $Y \setminus Y'$ is a ball B whose real dimension k is at most $m + 1$ and whose boundary is an image of a $(k-1)$-dimensional sphere in Y'. Composition with θ yields a continuous map of this sphere to H. Since $\pi_{k-1}(H) = 0$ we can extend any continuous map of a $(k-1)$-dimensional sphere in H to a map from a closed k-ball. Hence, we can extend the morphism $\theta : Y' \to H$ to a continuous map $\widehat{\theta} : Y \to H$. Let $\psi : Y \times [0,1] \to Y$ be a contraction of Y, i.e., $\psi(*, 0) = \mathrm{id}|_Y$ and $\psi(*, 1) = y_o \in Y$. Then $\widetilde{\theta} = \widehat{\theta} \circ \psi$ is a homotopy of θ to a constant map. Hence, we are done by Proposition 10.1. $\square$

11. The case of $\mathrm{SL}(n, \mathbf{k})$

Notation 11.1. Let us fix notation for the rest of the paper. From now on X will be always an affine algebraic variety isomorphic to $\mathrm{SL}(n, \mathbf{k})$ with $n \geqslant 3$, i.e., $\dim X = n^2 - 1$. We treat points in X as matrices $A = [a_{i,j}]_{i,j=1}^n$ from $\mathrm{SL}(n, \mathbf{k})$ and denote by $\{A_{ij}\}$ the cofactors of this matrix. For $m \leqslant n$ let P_m be the space of $(m \times n)$-matrices, P_m^0 be its subvariety consisting of matrices of rank m and $\varrho_m : X \to P_m$ be the natural projection that sends A to the matrix consisting of the first m rows of A. Consider also the natural projection $\varrho : X \to Q$ where Q is the quadric given by the equation

$$a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} = 1 \tag{22}$$

in the affine space $\mathbb{A}_{\mathbf{k}}^{2n}$ equipped with coordinates $(a_{1,1}, \ldots, a_{1,n}, A_{1,1}, \ldots, A_{1,n})$. Suppose that the group $H = \mathrm{SL}(n, \mathbf{k}) \times \mathrm{SL}(n, \mathbf{k})$ acts on X so that $(B, C) \in \mathrm{SL}(n, \mathbf{k}) \times \mathrm{SL}(n, \mathbf{k})$ sends $A \in X$ to BAC. Note that H contains the subgroup $H_m = \mathrm{SL}(m, \mathbf{k}) \times \mathrm{SL}(n, \mathbf{k})$ that acts naturally on P_m, while H_1 acts naturally on Q. Under these actions the morphism ϱ_m (resp. ϱ) is H_m-equivariant (resp. H_1-equivariant).

Lemma 11.2. *Let Notation* 11.1 *hold. Then the H_m-action (resp. H_1-action) on P_m^0 (resp. Q) is transitive.*

Proof. Since any finite sequence of column and row operations on a matrix $D \in P_m$ coincides with the action of some element of H_m on this matrix we see that for $D \in P_m^0$ there is an element $h \in H_m$ for which $h.D = [d_{i,j}]$ where $d_{i,i} = 1$ for every $i = 1, \ldots, m$, while $d_{i,j} = 0$ for $i \neq j$. This yields the transitivity of the H_m-action on P_m^0.

Similarly, an elementary column operation induced by an element of H_1 yields an automorphism of Q such that

$$a_{1,j} \to a_{1,j} + t a_{1,i} \quad \text{and} \quad A_{1,i} \to A_{1,i} - t A_{1,j} \tag{23}$$

(where $t \in \mathbf{k}$ is a multiple), while it keeps the rest of coordinates the same. Since $a_{1,1}, \ldots, a_{1,n}$ cannot vanish simultaneously we can send an arbitrary point q in Q via such automorphism to a point q_1 with

$$a_{1,1} = 1 \quad \text{and} \quad a_{1,2} = \cdots = a_{1,n} = 0. \tag{24}$$

In this case $A_{1,1} = 1$ because of formula (22). Let us use now the similar automorphisms which send $a_{1,1} \to a_{1,1} + ta_{1,j}$ and $A_{1,j} \to A_{1,j} - tA_{1,1}$, while preserving the rest of coordinates on Q. Then we can send q_1 to a point q_0 with $A_{1,2} = \ldots = A_{1,n} = 0$, while keeping formula (24) valid. That is, we can send an arbitrary point q to this given point q_0 which yields the transitivity of the H_1-action on Q and the desired conclusion. $\square$

Notation 11.3. Let I_k be the identity $(k \times k)$-matrix for $k \geqslant 1$. Consider the following subgroups of block matrices in $\mathrm{SL}(n, \mathbf{k})$.

$$F'_{n-m} = \begin{bmatrix} I_m & \bar{0} \\ M & \mathrm{SL}(n-m, \mathbf{k}) \end{bmatrix}, \text{ resp. } F'' = \begin{bmatrix} I_1 & \bar{0} \\ \bar{0} & \mathrm{SL}(n-1, \mathbf{k}) \end{bmatrix}$$

where M is the set of $(n-m) \times m$ matrices. Then we have the subgroup $H'_{n-m} = F'_{n-m} \times I_n$ of H acting on X and the subgroup $H'' = F'' \times I_n$ acting on X.

Proposition 11.4. *Let Notations 11.1 and 11.3 hold and $n - m \geqslant 2$. Then*

(i) $\varrho_m : X \to P_m^0$ *is a principal H'_{n-m}-bundle over P_m^0;*

(ii) $\varrho : X \to Q$ *is a principal H''-bundle over Q.*

Proof. Note that the action of H'_{n-m} on X is free and it preserves every fiber of the morphism ϱ_m. Therefore, the fibers of ϱ_m are of dimension at least $\dim H'_{n-m} = m(n-m)+(n-m)^2-1 = n(n-m)-1$. Furthermore, the action of H_m transforms each orbit of H'_{n-m} into another orbit. Hence, the fibers of ϱ_m are of the same dimension because ϱ_m is H_m-equivariant and the H_m-action on P_m^0 is transitive. Since $\dim P_m^0 = nm$ and $\dim X = n^2 - 1$, observing the equality and $n^2 - 1 = nm+n(n-m)-1$ we see that the fibers of ϱ_m are nothing but the orbits of H'_{n-m} which shows that $\varrho_m : X \to P_m^0$ is a principal H'_{n-m}-bundle. Thus we have (i).

Similarly, the H''-action on Q is free, it preserves every fiber of the morphism ϱ and it commutes with the H_1-action from Notation 11.1. Hence, the fibers of ϱ are of dimension at least $\dim \mathrm{SL}(n-1, \mathbf{k}) = (n-1)^2 - 1$. All these fibers are again of the same dimension because ϱ is H_1-equivariant and the H_1-action on Q is transitive by Lemma 11.2. On the other hand, the dimension of Q is $2n - 1$. Observing the equality $\dim \mathrm{SL}(n, \mathbf{k}) = n^2 - 1 = (n-1)^2 - 1 + (2n-1)$ we conclude that the dimension of each fiber is $(n-1)^2 - 1$. That is, the fibers of ϱ are nothing but the orbits of the H''-action which yields (ii) and the desired conclusion. $\square$

Notation 11.5.

(1) Let $\mathcal{N} = \{\delta_{i,j} \mid 1 \leqslant i \neq j \leqslant n\} \bigcup \{\sigma_{i,j} \mid 1 \leqslant i \neq j \leqslant n\}$ be the set of locally nilpotent vector fields on X given by

$$\delta_{i,j} = \sum_{l=1}^{n} a_{l,i} \frac{\partial}{\partial a_{l,j}} \quad \text{and} \quad \sigma_{i,j} = \sum_{l=1}^{n} a_{i,l} \frac{\partial}{\partial a_{j,l}},$$

$\mathcal{N}'_m = \{\delta'_{i,j} \mid 1 \leqslant i \neq j \leqslant n\} \bigcup \{\sigma'_{i,j} \mid 1 \leqslant i \neq j \leqslant m\}$ be the set of locally nilpotent vector fields on P_m given by

$$\delta'_{i,j} = \sum_{l=1}^{m} a_{l,i} \frac{\partial}{\partial a_{l,j}} \quad \text{and} \quad \sigma'_{i,j} = \sum_{l=1}^{n} a_{il} \frac{\partial}{\partial a_{j,l}},$$

and $\mathcal{N}'' = \{\delta''_{i,j}|1 \leqslant i \neq j \leqslant n\}$ be the set of locally nilpotent vector fields on Q given by

$$\delta''_{i,j} = a_{1,i}\frac{\partial}{a_{1,j}} - A_{1,j}\frac{\partial}{A_{1,i}}.$$

(2) Let $\widetilde{\mathcal{N}}'_m$ (resp. $\widetilde{\mathcal{N}}''$) be the smallest saturated set of locally nilpotent vector fields on P_m containing $\mathcal{N}'_m$ (resp. on Q containing $\mathcal{N}''$). In particular, $\widetilde{\mathcal{N}}'_m$ generates a group G'_m of automorphisms of P_m and $\widetilde{\mathcal{N}}''$ generates a group G'' of automorphisms of Q.

Lemma 11.6. *Let Notation 11.5 hold, $n \geqslant 3$ and $1 \leqslant m \leqslant n-2$.*

    (1) *The pair $(\mathcal{N}, \mathcal{N}'_m)$ (resp. $(\mathcal{N}, \mathcal{N}'')$) is comparable for the morphism ϱ_m (resp. ϱ).*

    (2) *Every element of $(\mathcal{N}, \mathcal{N}'_m)$ (resp. $(\mathcal{N}, \mathcal{N}'')$) satisfies Condition (A) in Proposition 9.9.*

    (3) *The variety P^0_m is G'_m-flexible and the variety Q is G''-flexible.*

    (4) *Let $\mathcal{L}'_m$ be the Lie algebra of vector fields on P_m generated by all fields of the form $a'\delta'$ where $\delta' \in \widetilde{\mathcal{N}}'_m$ and $\deg_{\delta'} a' \leqslant 1$. Then $\mathcal{L}'_m$ contains the space of all algebraic vector fields on P_m that vanish at $Z = P_m \setminus P^0_m$ with some multiplicity $s > 0$.*

    (5) *Let $\mathcal{L}''$ be the Lie algebra of vector fields on Q generated by all fields of the form $a''\delta''$ where $\delta'' \in \widetilde{\mathcal{N}}''$ and $\deg_{\delta''} a'' \leqslant 1$. Then $\mathcal{L}''$ coincides with the space of all algebraic vector fields on Q.*

Proof. The first and second statements follow from Notation 11.5 and Definition 9.6. For (3) note that all elementary row and column operations on P_m can be viewed as elements of flows of fields from $\mathcal{N}'_m$. Since such operations generate the action of the group H_m we see that G'_m contains H_m. By the similar reason G'' contains H''. Since by Lemma 11.2 H_m acts transitively on P^0_m (resp. H_1 acts transitively on Q) we have (3) by Theorem 2.6.

The vector fields $\delta'_{1,2}$ and $\delta'_{3,2}$ commute, i.e., we have a compatible pair of locally nilpotent vector fields in $\widetilde{\mathcal{N}}'_m$. Since $P^0_m = P_m \setminus Z$ is G'_m-flexible Theorem 2.15 implies that the Lie algebra $\mathcal{L}'_m$ contains all algebraic vector fields $\mathrm{AVF}(P_m)$ on P_m vanishing on Z with some multiplicity $s > 0$, i.e., we have (4).

The same argument and the commutativity of $\delta''_{1,2}$ and $\delta''_{3,2}$ imply (5). $\square$

Proposition 11.7. *Let $\widetilde{G} = \mathrm{SAut}(X)$, $\widetilde{G}'' = \mathrm{SAut}(Q)$ and let $\widetilde{G}'_m = \mathrm{SAut}_{Z_s}(P_m)$ be the subgroup of $\mathrm{SAut}(P_m)$ generated by the elements of the flows of locally nilpotent vector fields on P_m whose restriction to the s-infinitesimal neighborhood Z_s is zero, where s is as in Lemma 11.6. Then $\varrho_m : X \to P^0_m$ is a $(\widetilde{G}, \widetilde{G}'_m)$-comparable morphism and $\varrho : X \to Q$ is a $(\widetilde{G}, \widetilde{G}'')$-comparable morphism.*

Proof. By Lemma 11.6(4) the group $\widetilde{G}'_m$ is generated by elements of the flows of locally nilpotent vector fields from $\mathcal{L}'_m$. Since Condition (A) is satisfied by Lemma 11.6(2) we see that ϱ_m is $(\widetilde{G}, \widetilde{G}'_m)$-comparable by Proposition 9.9. Similarly, by Lemma 11.6 the group $\widetilde{G}''$ is generated by elements of the flows of locally nilpotent vector fields from $\mathcal{L}''$ and the same reasoning yields the second statement. $\square$

Lemma 11.8. *Let Notation* 11.1 *hold. Then the codimension of the subvariety* $Z = P_m \setminus P_m^0$ *in* P_m *is* $n - m + 1$.

Proof. A matrix $A \in P_m$ does not belong to P_m^0 if and only if its rank k is at most $m - 1$. Applying some element of H_m to $A = [a_{i,j}]$ with $k \leqslant m - 1$ we can suppose that $A = [a_{ij}]$ has $a_{i,j} = 1$ where $i = j$ and $i \leqslant k$, while $a_{i,j} = 0$ in all other cases. Note that the isotropy subgroup F of A in H_m consists of all elements (B, C) such that B is a block $(m \times m)$-matrix of the form

$$\begin{bmatrix} A' & B' \\ \bar{0} & D' \end{bmatrix}$$

and C is a block $(n \times n)$-matrix of the form

$$\begin{bmatrix} (A')^{-1} & \bar{0} \\ C' & E' \end{bmatrix}$$

where $A' \in \mathrm{GL}(k, \mathbf{k})$, $D' \in \mathrm{GL}(m - k, \mathbf{k})$, $E' \in \mathrm{GL}(n - k, \mathbf{k})$ with $\det D' = \frac{1}{\det A'}$ and $\det E' = \det A'$, while B' (resp. C') is an aribrary matrix of size $k \times (m - k)$ (resp. $(n - k) \times k$). Hence, the dimension of F is $k(m - k) + (m - k)^2 + (n - k)^2 + k(n - k) + k^2 - 1$. Since the subvariety of matrices of rank k in P_m is isomorphic to H_m / F we see that its dimension is $\dim H_m - \dim F = m^2 - 1 + n^2 - 1 - (k(m - k) + (m - k)^2 + (n - k)^2 + k(n - k) + k^2 - 1) = m(m - k) + k(n - k) - k^2 - 1$. For k running from 1 to $m - 1$ the maximum of latter expression is achieved for $k = m - 1$ and it is equal to $m + (m - 1)(n - m + 1) - (m - 1)^2 - 1 = (n + 1)(m - 1) = nm - n + m - 1$. Since $\dim P_m = nm$ we see that the codimension of Z in P_m is $n - m + 1$ which is the desired conclusion. $\square$

Lemma 11.9. *Let Notation* 11.1 *hold,* $0 < m < n$ *and* Y *be a closed subvariety of* X *of dimension at most* m. *Suppose that* $\check{H} \simeq \mathrm{SL}(n, \mathbf{k})$ *acts on* X *via left multiplications. Then for a general element* $h \in \check{H}$ *the morphism* $\varrho_m|_{h(Y)} : h(Y) \to P_m$ *is proper.*

Proof. Suppose that the variety $\mathcal{X} = \mathbb{A}_{\mathbf{k}}^{n^2}$ is equipped with the coordinate system $(a_{1,1}, \ldots, a_{n,n})$ and, thus, $\mathcal{X}$ contains X as a closed subvariety. Present $\mathcal{X}$ as $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_{i0}$ where $\mathcal{X}_{i0} = \mathbb{A}_{\mathbf{k}}^n$ has the coordinate system $(a_{1,i}, \ldots, a_{n,i})$. Let $\bar{\mathcal{X}}_i \simeq \mathbb{P}^n$ be a completion of $\mathcal{X}_{i0}$, i.e., $\bar{\mathcal{X}}_i = \mathcal{X}_{i0} \sqcup \mathcal{X}_{i1}$ (where $\mathcal{X}_{i1} \simeq \mathbb{P}^{n-1}$) and $\bar{\mathcal{X}} = \prod_{i=1}^{n} \bar{\mathcal{X}}_i$ is a completion of $\mathcal{X}$. Note that the $\check{H}$-action on X extends naturally to an $\check{H}$-action on $\mathcal{X}$ (resp. $\bar{\mathcal{X}}$) and we have also the transitive $\check{H}$-action on each $\mathcal{X}_{i0}$ with an extension to an action on $\bar{\mathcal{X}}_i$ for which $\mathcal{X}_{i1}$ is an $\check{H}$-orbit and the natural projection $\mathcal{X} \to \mathcal{X}_i$ (resp. $\psi_i : \bar{\mathcal{X}} \to \bar{\mathcal{X}}_i$) is $\check{H}$-equivariant. Let J be the set $\{0, 1\}^n$ without the element $(0, \ldots, 0)$. Note that the boundary $\bar{\mathcal{X}} = \bar{\mathcal{X}} \setminus \mathcal{X}$ can be presented as $\bigcup_{\bar{j} \in J} C_{\bar{j}}$ where $C_{\bar{j}} = \prod_{i=1}^{n} \mathcal{X}_{ij_i}$ and $\bar{j} = (j_1, \ldots, j_n) \in J$. In particular, for the closure $\bar{Y}$ of Y in $\bar{\mathcal{X}}$ one has $\bar{Y} \setminus Y = \bigcup \check{Y}_{\bar{j}}$ where $\check{Y}_{\bar{j}} = (\bar{Y} \setminus Y) \cap C_{\bar{j}}$. Let, say, $j_1' = 1$ for some $\bar{j}' \in J$ and, hence, $\psi_1(C_{\bar{j}'}) \subset \mathcal{X}_{11}$. Since $\bar{Y} \setminus Y$ is of dimension $m - 1$ we see that $\psi_1(\check{Y}_{\bar{j}'})$ is of dimension at most $m - 1$. Note that the set $\mathcal{R}(a_{1,1}, \ldots, a_{m,1})$ of common indeterminacy points of the functions $a_{1,1}, \ldots, a_{m,1}$ on $\bar{\mathcal{X}}_1$ is a subset of $\mathcal{X}_{11}$ of codimension m. By Theorem 1.3 for a general $h \in \check{H}$ the intersection of $h(\psi_1(\check{Y}_{\bar{j}'}))$ with $\mathcal{R}(a_{1,1}, \ldots, a_{m,1})$ is

empty. Since ψ_1 is $\breve{H}$-equivariant we have $h(\breve{Y}_{\bar{j}'}) \cap \psi_1^{-1}(\mathcal{R}(a_{11}, \ldots, a_{m,1})) = \varnothing$ where $\psi_1^{-1}(\mathcal{R}(a_{1,1}, \ldots, a_{m,1}))$ is the intersection of $C_{\bar{j}'}$ with the set of common indeterminacy points of $a_{11}, \ldots, a_{m,1}$ in $\bar{\mathcal{X}}$. Note that the similar claims are true for all $\bar{j} \in J$. Hence, for a general $h \in \breve{H}$ the variety $h(\bar{Y})$ does not meet the set of common indeterminacy points of the functions $a_{1,1}, \ldots, a_{m1}, a_{1,2}, \ldots, a_{m,n}$. Furthermore, the boundary $\bar{\mathcal{X}} \setminus \mathcal{X}$ does not contain points where these functions are regular and take finite values. Therefore, the desired conclusion follows now from Proposition 5.2. $\quad\square$

Proposition 11.10. *Let the assumptions of Lemma 11.9 hold and* $\mathrm{ED}(Y) \leqslant m$ *where* $m \leqslant n - 2$. *Then for some element* $\beta \in \mathrm{SAut}(X)$ *the morphism* $\varrho_m|_{\beta(Y)} : \beta(Y) \to P_m$ *is a closed embedding.*

Proof. By Proposition 11.4 we have $X/H'_{n-m} \simeq P_m^0$. Note that the H'_{n-m}-action on X is generated by elements of the flows of some set $\mathcal{S}$ of locally nilpotent vector fields on X. Let $\widetilde{\mathcal{S}}$ be the smallest saturated set of locally nilpotent vector fields on X containing $\mathcal{S}$ and tangent to the fibers of ϱ_m. Then $\widetilde{\mathcal{S}}$ generates a group F of automorphisms of X over P_m such that it contains H'_{n-m} and every fiber of ϱ_m is F-flexible. By Theorem 4.2(iv) there exists an algebraic F-family $\mathcal{A}$ of automorphisms of X such that for a general element $\alpha \in \mathcal{A}$ the morphism $\varrho_m \circ \alpha : Y \to P_m$ is injective and it induces an injective map of tangent bundles. By Proposition 4.6 the same is true for a morphism $(\varrho_m \circ h) \circ \alpha : Y \to P_m$ where h is a general element of $\breve{H}$. By Lemma 11.9 the latter morphism is proper. Thus, letting $\beta = h \circ \alpha$ we get the desired conclusion. $\quad\square$

Theorem 11.11. *Let* $X = \mathrm{SL}(n, \mathbb{C})$ *and* $\varphi : Y_1 \to Y_2$ *be an isomorphism of two closed subvarieties of* X *such that either*

- (i) $\mathrm{ED}(Y_i) + \dim Y_i \leqslant n - 2$, $H_i(Y_1) = 0$ *for* $i \geqslant 3$ *and* $H_2(Y_1)$ *is a free abelian group; or*
- (ii) $\dim Y_1$ *is a curve and* $\mathrm{ED}(Y_i) \leqslant n - 2$; *or*
- (iii) Y_1 *is a once-punctured curve and* $\mathrm{ED}(Y_1) \leqslant 2n - 3$.

Then there exists a holomorphic automorphism β *of* X *such that* $\beta|_{Y_1} = \varphi$.

Proof. Let $m = \mathrm{ED}(Y_i)$. By Proposition 11.10 we can suppose that $\varrho_m|_{Y_i} : Y_i \to P_m$ is a closed embedding. Let $\varphi' : Y_1' \to Y_2'$ be the isomorphism induced by φ, where $Y_i' = \varrho_m(Y_i)$. By Lemma 11.8 one has $\dim Z = nm - (n - m + 1)$, where $Z = P_m \setminus P_m^0$. Therefore, the assumption $m + \dim Y_i \leqslant n - 2$ implies that $\dim Y_i' + \dim Z \leqslant \dim P_m - 3$ in case (i). Hence, by Theorem 7.1(a) and Proposition 7.13 φ' can be extended to an automorphism $\alpha' \in \mathrm{SAut}_{Z_s}(P_m)$ for any $s > 1$. By Proposition 11.7 s can be chosen so that ϱ_m is $(\mathrm{SAut}(X), \mathrm{SAut}_{Z_s}(P_m))$-comparable and, hence, by Proposition 9.2 we can suppose that $Y_1' = Y_2'$ and φ' is the identity map.

By Proposition 9.2 it suffices to establish now that ϱ_m is holomorphically comparable on a family of algebraic varieties containing Y_i. By Proposition 11.4 ϱ_m is a principal H'_{n-m}-bundle. Consider the morphism $\theta : Y_1' \to H'_{n-m}$ as in Proposition 9.3 (with H and Y' replaced by H'_{n-m} and Y_1'). In order to prove holomorphic comparability it suffices to show that θ extends to a holomorphic map $\Theta : P_m \to H'_{n-m}$. Note that as an affine variety H'_{n-m} is isomorphic to the

direct product of $\mathbb{C}^{m(n-m)}$ and $\mathrm{SL}(n-m,\mathbb{C})$. Hence, $\pi_1(H'_{n-m}) = \pi_2(H'_{n-m}) = 0$ and the existence of an extension Θ is provided by Proposition 10.4 which yields the desired conclusion in (i).

For (ii) exactly the same argument works with Theorem 7.1(a) replaced by Theorem 7.1(b).

For (iii) we recall that $\varrho : X \to Q$ is a principal H''-bundle by Proposition 11.4. Arguing as in the proof of Proposition 11.10 we can find a subgroup $F \subset \mathrm{SAut}(X/Q)$ containing H'' such that every fiber of ϱ is F-flexible. Hence, by Corollary 4.4 we can suppose that each $\varrho|_{Y_i} : Y_i \to Q$ is a closed embedding. Let $Y''_i = \varrho(Y_i)$ and $\varphi'' : Y''_1 \to Y''_2$ be the isomorphism induced by φ. Since $\mathrm{ED}(Y_1) \leqslant 2n-3$ we see that φ'' extends to an automorphism $\alpha'' \in \mathrm{SAut}(Q)$ by Theorem 8.3. By Proposition 11.7 ϱ is $(\mathrm{SAut}(X), \mathrm{SAut}(Q))$-comparable. Hence, by Proposition 9.2 we can suppose that $Y''_1 = Y''_2$ and φ'' is the identity map. As before we have a morphism $\theta'' : Y''_1 \to H''$ as in Proposition 9.3 which extends to a holomorphic map $\Theta'' : Q \to H''$. Thus we have holomorphic comparability of ϱ which yields (iii) and concludes the proof. $\square$

Corollary 11.12. *Let $X = \mathrm{SL}(n,\mathbb{C})$ and $\varphi : Y_1 \to Y_2$ be an isomorphism of two smooth closed subvarieties of X such that $\dim Y_i \leqslant n/3 - 1$, $H_i(Y_1) = 0$ for $i \geqslant 3$ and $H_2(Y_1)$ is a free abelian group. Then there exists a holomorphic automorphism β of X such that $\beta|_{Y_1} = \varphi$.*

Proof. Since $\dim Y_i \leqslant n/3 - 1$ the smoothness assumption implies that $\mathrm{ED}(Y_i) + \dim Y_i \leqslant n-2$. Hence, Theorem 11.11 implies the desired conclusion. $\square$

Theorem 11.13. *Let $\varphi : Y_1 \to Y_2$ be an isomorphism of two closed subvarieties of $X \simeq \mathrm{SL}(n,\mathbf{k})$ with $n \geqslant 3$ such that Y_i is isomorphic to $\mathbb{A}^k_{\mathbf{k}}$. Suppose that either $k \leqslant n/3 - 1$ or $k = 1$. Then there exists $\alpha \in \mathrm{SAut}(X)$ such that $\alpha|_{Y_1} = \varphi$.[17]*

Proof. Let $k \leqslant n/3 - 1$, $m = \mathrm{ED}(Y_1)$ and let $Y'_i = \varrho_m(Y_i)$, $i = 1, 2$. Repeating the argument in the proof of Theorem 11.11 we can suppose that $\varrho_m|_{Y_i} : Y_i \to Y'_i$ is a closed embedding, $Y'_1 = Y'_2$ and the induced isomorphism $\varphi' : Y'_1 \to Y'_2$ is the identity map. Since ϱ_m is a principal H'_{n-m}-bundle, ϱ_m is comparable on the family $\mathcal{F}_{\mathrm{aff}}$ as in Corollary 9.4. The desired conclusion follows now from Proposition 9.2.

Similarly, if $k = 1$ then as in Theorem 11.11(iii) we can suppose that $\varrho|_{Y_i} : Y_i \to Q$ is a closed embedding and for $Y''_i = \varrho(Y_i)$ the induced isomorphism $\varphi'' : Y''_1 \to Y''_2$ is the identity map. Since ϱ is a principal H''-bundle, ϱ is comparable on the family $\mathcal{F}_{\mathrm{aff}}$. The desired conclusion follows again from Proposition 9.2. $\square$

The following necessary condition for the positive solution of the extension problem is straightforward.

Proposition 11.14. *Let a group $G \subset \mathrm{Aut}(W)$ act on an algebraic variety W and $\varphi : Y_1 \to Y_2$ be an isomorphism of closed subvarieties of W. Suppose that $\iota_k : Y_k \hookrightarrow W$ is the natural embedding. Suppose also that for every $\alpha \in G$ there exists $k \geqslant 1$ such that the homomorphism $\pi_k(Y_1) \to \pi_k(W)$ induced by $\alpha \circ \iota_1$ is different from the similar homomorphism induced by ι_2. Then φ cannot be extended to an automorphism from G.*

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[17]The case of $k = 1$ is, of course, the theorem of Van Santen [St] which we prove by other means.

Remark 11.15. In the complex case note that if $\iota_1 : Y_1 \hookrightarrow W$ induces a trivial homomorphism $\pi_k(Y_1) \to \pi_k(X)$ for some k while the similar homomorphism induced by $\iota_2 : Y_2 \hookrightarrow W$ is nontrivial then φ cannot be extended even to a homeomorphism of W. Furthermore, if G is contained in the connected component of identity in $\mathrm{Aut}(X)$ then the extension problem does not have a positive solution for the group G if there is no homotopy of φ to the identity map via closed embeddings of Y_1 into W.

In the rest of this section we present a concrete (and more or less obvious) example illustrating Proposition 11.14 in the case of $X \simeq \mathrm{SL}(n, \mathbb{C})$.

Lemma 11.16. *There is a closed embedding of $Y \simeq \mathrm{SL}(2, \mathbb{C})$ into $X \simeq \mathrm{SL}(n, \mathbb{C})$ such that it generates an isomorphism $\pi_3(Y) \simeq \pi_3(X)$ of the homotopy groups.*

Proof. Let $\varrho : X \to Q$ be as in Notation 11.1, i.e., by Proposition 11.4 it is a principal H''-bundle with fiber $F \simeq \mathrm{SL}(n-1, \mathbb{C})$. Since Q is a complexification of a real $(2n-1)$-dimensional sphere it has a homotopy type of this sphere.[18] Hence, $\pi_k(Q) = 0$ for $1 \leqslant k \leqslant 2n-2$ and $\pi_{2n-1}(Q) = \mathbb{Z}$. Now the exact homotopy sequence for the fiber bundle $\varrho : X \to Q$ implies that the natural embedding of $\mathrm{SL}(n-1, \mathbb{C}) \simeq F \hookrightarrow X \simeq \mathrm{SL}(n, \mathbb{C})$ induces an isomorphism $\pi_k(X) \simeq \pi_k(F)$ for $k < 2n - 2$. Choosing a natural sequence $\mathrm{SL}(2, \mathbb{C}) \subset \mathrm{SL}(3, \mathbb{C}) \subset \ldots \subset \mathrm{SL}(n-1, \mathbb{C}) \subset \mathrm{SL}(n, \mathbb{C})$ of closed embeddings we get the desired conclusion. $\square$

Theorem 11.17. *Let X be an affine algebraic variety isomorphic to $\mathrm{SL}(n, \mathbb{C})$ with $n \geqslant 3$. There are two closed subvarieties Y_1 and Y_2 in X isomorphic to $\mathrm{SL}(2, \mathbb{C})$ and such that there is no automorphism α of X for which $\alpha(Y_1) = Y_2$.*

Proof. By Lemma 11.16 we can suppose that the natural embedding $Y_1 \hookrightarrow X$ generates an isomorphism $\pi_3(Y_1) \simeq \pi_3(X)$. By Proposition 11.14 and Remark 11.15 in order to prove Theorem 11.17 it suffices to present an embedding $Y_2 \hookrightarrow X$ such that the induced homomorphism sends $\pi_3(Y_2)$ into the zero element of $\pi_3(X)$. Treat a point in X as a matrix $A = [a_{i,j}]$. Let A' be the $(n-1) \times (n-1)$ matrix obtained from A by removing the first row and the first column. Consider the subvariety X_0 of X that consists of matrices A such that A' is the identity matrix. Note that X_0 is naturally isomorphic to $\mathbb{C}^{2n-1}$ with coordinates $a_{1,2}, \ldots a_{1n}, a_{21}, \ldots, a_{n1}$ since $a_{1,1}$ can be expressed as function of these coordinates because of the equation $\det A = 1$. Choose in X_0 the quadric $Y_2 \simeq \mathrm{SL}(2, \mathbb{C})$ given by $a_{1,2}a_{1,3} - a_{2,1}a_{3,1} = 1$ and $a_{1,k} = a_{k,1} = 0$ for all $k \geqslant 4$. Then the embedding $Y_2 \hookrightarrow X$ induces the zero map $\pi_3(Y_2) \to \pi_3(X)$ since it factors through $Y_2 \hookrightarrow X_0$ and X_0 is contractible. This yields the desired conclusion. $\square$

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[18]Actually, it is well known that Q is diffeomorphic to the tangent bundle of the sphere.

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