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COTANGENT BUNDLES OF PARTIAL FLAG VARIETIES AND CONORMAL VARIETIES OF THEIR SCHUBERT DIVISORS

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Abstract. Let P be a parabolic subgroup in $G = \mathrm{SL}_n(\mathbf{k})$, for $\mathbf{k}$ an algebraically closed field. We show that there is a G-stable closed subvariety of an affine Schubert variety in an affine partial flag variety which is a natural compactification of the cotangent bundle T^*G/P. Restricting this identification to the conormal variety $N^*X(w)$ of a Schubert divisor $X(w)$ in G/P, we show that there is a compactification of $N^*X(w)$ as an affine Schubert variety. It follows that $N^*X(w)$ is normal, Cohen–Macaulay, and Frobenius split.

1. Introduction

Let the base field $\mathbf{k}$ be algebraically closed. Consider a cyclic quiver with h vertices and dimension vector $\underline{d} = (d_1, \dots, d_h)$. Let

$$\mathrm{Rep}(\underline{d}, \widehat{\mathsf{A}}_h) = \mathrm{Hom}(V_1, V_2) \times \cdots \times \mathrm{Hom}(V_h, V_1), \quad \mathrm{GL}_{\underline{d}} = \prod_{1 \le i \le h} \mathrm{GL}(V_i).$$

We have a natural action of $\mathrm{GL}_{\underline{d}}$ on $\mathrm{Rep}(\underline{d}, \widehat{\mathsf{A}}_h)$: for $g = (g_1, \dots, g_h) \in \mathrm{GL}_{\underline{d}}$ and $f = (f_1, \dots, f_h) \in \mathrm{Rep}(\underline{d}, \widehat{\mathsf{A}}_h)$,

$$g \cdot f = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \dots, g_1 f_h g_h^{-1}).$$

Let

$$\mathfrak{Z} = \left\{ (f_1, \dots, f_h) \in \mathrm{Rep}(\underline{d}, \widehat{\mathsf{A}}_h) \,\middle|\, f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \to V_1 \text{ is nilpotent} \right\}.$$

Clearly $\mathfrak{Z}$ is $\mathrm{GL}_{\underline{d}}$-stable. Set $n = \sum d_i$, and let $\widehat{\mathrm{SL}}_n$ be the affine type Kac–Moody group with Dynkin diagram $\widehat{\mathsf{A}}_{n-1}$. Lusztig (cf. [12]) has shown that an orbit closure

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in $\mathfrak{Z}$ is canonically isomorphic to an open subset of a Schubert variety in $\widehat{\mathrm{SL}}_n/Q$, where Q is the parabolic subgroup of $\widehat{\mathrm{SL}}_n$ corresponding to omitting the simple roots $\alpha_0, \alpha_{d_1}, \alpha_{d_1+d_2}, \ldots, \alpha_{d_1+\cdots+d_{h-1}}$.

Let now $h = 2$ and consider the subvariety $\mathfrak{Z}_0$ of $\mathfrak{Z}$ given by

$$\mathfrak{Z}_0 = \left\{(f_1, f_2) \in \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1) \,\middle|\, f_2 \circ f_1 = 0, f_1 \circ f_2 = 0\right\}$$

with the dimension vector (d_1, d_2). Strickland (cf. [17]) has shown that the irreducible components of $\mathfrak{Z}_0$ give the conormal varieties of the determinantal varieties in $\mathrm{Hom}(V_1, V_2) = \mathrm{Mat}_{d_2,d_1}(\mathbf{k})$, the set of $d_2 \times d_1$ matrices with entries in $\mathbf{k}$. A determinantal variety in $\mathrm{Hom}(V_1, V_2)$ being canonically isomorphic to an open subset in a certain Schubert variety in $\mathrm{Gr}_{d_2,d_1+d_2}$ (the Grassmannian variety of d_2-dimensional subspaces of $\mathbf{k}^{d_1+d_2}$) (cf. [8]), the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in the (finite-dimensional) flag variety and affine Schubert varieties.

Let $G = \mathrm{SL}_n(\mathbf{k})$. We view the loop group $LG = \mathrm{SL}_n\left(\mathbf{k}[t, t^{-1}]\right)$ as a Kac–Moody group of type $\widehat{\mathsf{A}}_{n-1}$. Let P be the parabolic subgroup of G corresponding to omitting the simple root α_d for some $1 \le d \le n-1$, and $\mathcal{P}$ the parabolic subgroup of LG corresponding to omitting the simple roots α_0, α_d. Lakshmibai ([7]) has shown that the cotangent bundle T^*G/P is an open subset of a Schubert variety in $LG/\mathcal{P}$. Let w_0 be the longest element in the Weyl group W of G. In [10], we have shown that the conormal variety of the Schubert variety $X_P(w)$ is an open subset of a Schubert variety in $LG/\mathcal{P}$ if and only if the Schubert variety $X_P(w_0 w)$ is smooth.

The approach adopted in [7], [10] seems to be quite successful in relating cotangent bundles and conormal varieties of classical Schubert varieties to affine Schubert varieties. In this paper, for any parabolic subgroup P of G, we first construct an embedding of the cotangent bundle T^*G/P inside an affine Schubert variety $X(\kappa)$ (Theorem 4.11). We then show that the conormal variety of a Schubert divisor in G/P is an open subset of some affine Schubert subvariety of $X(\kappa)$ (Proposition 5.4). In particular, we obtain that the conormal variety of a Schubert divisor is normal, Cohen–Macaulay, and Frobenius split (Corollary 5.5).

Let $L^+G = G(\mathbf{k}[t])$ and let $\mathcal{B}$ be a Borel subgroup of LG contained in L^+G. The action of of G on $V = \mathbf{k}^n$ defines an action of LG on $V[t, t^{-1}]$. The affine Grassmannian LG/L^+G is an ind-variety whose points are the lattices with virtual dimension 0 (see Section 3.8 for the definition of a lattice and its virtual dimension). The affine flag variety $LG/\mathcal{B}$ is an ind-variety whose points are affine flags $L_0 \subset L_1 \subset \cdots \subset L_{n-1}$ satisfying the incidence relation $tL_{n-1} \subset L_0$.

Let $\mathcal{N}$ be the variety of $n \times n$ nilpotent matrices on which G acts by conjugation. In [11], Lusztig constructs a G-equivariant embedding $\psi : \mathcal{N} \hookrightarrow LG/L^+G$ which takes G-orbit closures in $\mathcal{N}$ to G-stable Schubert varieties in LG/L^+G. Let $\boldsymbol{\theta}$ be the Springer resolution $T^*G/B \to \mathcal{N}$ given by $(g, X) \mapsto gXg^{-1}$. In [9] Lakshmibai et al. construct a lift of ψ, namely a map η embedding T^*G/B into the affine flag variety $LG/\mathcal{B}$, leading to the following commutative diagram:

$$\begin{array}{ccc} T^*G/B & \xhookrightarrow{\eta} & LG/\mathcal{B} \\ \Big\downarrow \boldsymbol{\theta} & & \Big\downarrow \mathrm{pr} \\ \mathcal{N} & \xhookrightarrow{\psi} & LG/L^+G, \end{array}$$

where pr is the projection map. Under the identifications of the previous paragraph, it takes the affine flag $L_0 \subset \cdots \subset L_{n-1}$ to the lattice L_0. In this paper, for any parabolic subgroup P of G, we define a generalization ϕ_P, as described below, of the map η, and then study the conormal variety of a Schubert divisor by identifying its image under ϕ_P.

Let P be a parabolic subgroup of G corresponding to some subset $S_P \subset S_0$. We denote by $\mathcal{P}$ the parabolic subgroup of LG corresponding to $S_P \subset S = S_0 \sqcup \{\alpha_0\}$. Let $\mathfrak{u}$ be the Lie algebra of the unipotent radical of P. Using the identification $T^*G/P = G \times^P \mathfrak{u}$ (see Section 3.6), we define in Section 3.11 the map $\phi_P : T^*G/P \hookrightarrow LG/\mathcal{P}$ by $\phi_P(g, X) = g\left(1 - t^{-1}X\right) (\operatorname{mod} \mathcal{P})$, which sits in the following commutative diagram (see Section 3.13):

$$\begin{array}{ccc} T^*G/P & \xhookrightarrow{\phi_P} & LG/\mathcal{P} \\ \Big\downarrow \theta_P & & \Big\downarrow \mathrm{pr} \\ \mathcal{N}_{\boldsymbol{\nu}} & \xhookrightarrow{\psi} & LG/L^+G. \end{array}$$

The variety $\mathcal{N}_{\boldsymbol{\nu}}$ is the closure of a G-orbit in $\mathcal{N}$. The map $\theta_P : T^*G/P \to \mathcal{N}_{\boldsymbol{\nu}}$, given by $(g, X) \mapsto gXg^{-1}$ is a resolution of $\mathcal{N}_{\boldsymbol{\nu}}$ (cf. [2]), and pr is the quotient map. The closure of $\operatorname{Im}(\phi_P)$ in $LG/\mathcal{P}$ is a G-stable compactification of T^*G/P. We also identify the $\kappa \in \widehat{W}^{\mathcal{P}}$ (see Section 4.5 and Theorem 4.11) for which $\operatorname{Im} \phi_P \subset X_{\mathcal{P}}(\kappa)$, and further, κ is minimal for this property. Finally, given a Schubert divisor $X_P(w) \subset G/P$, we identify a Schubert variety $X_{\mathcal{P}}(v)$ which is a compactification of the conormal variety $N^*X_P(w)$ (see Proposition 5.4).

When P is maximal, ϕ_P is the same as the map in [7], and gives a dense embedding of T^*G/P, i.e., $X_{\mathcal{P}}(\kappa)$ is a compactification of T^*G/P. Mirković and Vybornov [14] have constructed another lift of Lusztig's embedding along the Springer resolution, which we briefly discuss in Section 3.14. Their lift is in general different, but agrees with ϕ_P exactly when P is maximal. The reader is cautioned that Mirković and Vybornov work with a different choice of Lusztig's map given by $X \mapsto (1 - t^{-1}X)^{-1} (\operatorname{mod} L^+G)$. This difference is simply a matter of convention.

The paper is organized as follows. In §2, we discuss some generalities on loop groups, and the associated root system and Weyl group. In §3, we define the embedding $\phi_P : T^*G/P \hookrightarrow \mathcal{G}/\mathcal{P}$, relate it to Lusztig's embedding $\psi : \mathcal{N} \to \mathcal{G}/L^+G$ via the Springer resolution, and discuss the relationship between ϕ_P and the map $\widetilde{\psi}$ of [14]. In §4, we identify the minimal $\kappa \in \widehat{W}$ for which $\phi_P(T^*G/P) \subset X_{\mathcal{P}}(\kappa)$. In particular, ϕ_P identifies a G-stable subvariety of $X_{\mathcal{P}}(\kappa)$ as a compactification of T^*G/P. We also compute the dimension of $X_{\mathcal{P}}(\kappa)$. This allows us to recover the result of [7] and further show that $X_{\mathcal{P}}(\kappa)$ is a compactification of T^*G/P if and only if P is a maximal parabolic subgroup. Finally, in §6, we apply ϕ_P to the conormal variety $N^*X_P(w)$ of a Schubert divisor $X_P(w)$ in G/P to show that $N^*X_P(w)$ can be embedded as an open subset of an affine Schubert variety.

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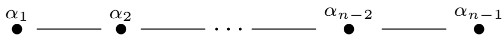
2. The loop group

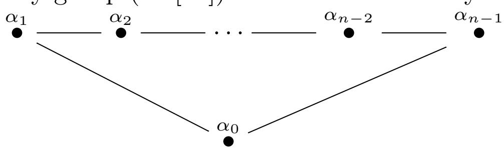
Let $\mathbf{k}$ be an algebraically closed field. In this section, we discuss some standard results on the root system, Weyl groups, and the Bruhat decomposition of the loop group $LG = \mathrm{SL}_n\left(\mathbf{k}[t, t^{-1}]\right)$. For further details, the reader may refer to [5], [6], [15].

2.1. The loop group and its Dynkin diagram

Let G be the special linear group $\mathrm{SL}_n(\mathbf{k})$. The subgroup B (resp. B^-) of upper (resp. lower) triangular matrices in G is a Borel subgroup of G, and the subgroup T of diagonal matrices is a maximal torus in G. The root system of G with respect to (B, T) has Dynkin diagram A_{n-1}. We use the standard labeling of simple roots S_0



Let Δ_0 be the set of roots and Δ_0^+ the set of positive roots. Let $L^{\pm}G = \mathrm{SL}_n\left(k\left[t^{\pm 1}\right]\right)$ and $\pi_{\pm} : L^{\pm}G \to G$ the surjective map given by $t^{\pm 1} \mapsto 0$. Then $\mathcal{B} := \pi_+^{-1}(B)$ and $\mathcal{B}^- := \pi_-^{-1}(B^-)$ are *opposite* Borel subgroups of LG. The loop group LG is a (minimal) Kac–Moody group (cf. [15]) with torus T and Dynkin diagram $\widehat{\mathsf{A}}_{n-1}$.



We denote the Borel subgroup $\mathcal{B}^+$ as just $\mathcal{B}$. Parabolic subgroups containing B (resp. $\mathcal{B}$) in G (resp. LG) correspond to subsets of S_0 (resp. S). For P the parabolic subgroup in G corresponding to $S_P \subset S_0$, the parabolic subgroup $\mathcal{P} \subset LG$ corresponding to $S_P \subset S$ is given by $\mathcal{P} = \pi^{-1}(P)$. In particular, the subgroup L^+G is the parabolic subgroup corresponding to $S_0 \subset S$.

2.2. The root system of A_{n-1}

Consider the vector space of $n \times n$ diagonal matrices, with basis $\{E_{i,i} \mid 1 \leq i \leq n\}$. Writing $\{\epsilon_i \mid 1 \leq i \leq n\}$ for the basis dual to $\{E_{i,i} \mid 1 \leq i \leq n\}$, we can identify the set of roots Δ_0 of G as follows:

$$\Delta_0 = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}.$$

The positive root $\alpha_i + \cdots + \alpha_{j-1}$, where $i < j$, corresponds to $\epsilon_i - \epsilon_j$, and the negative root $-(\alpha_i + \cdots + \alpha_{j-1})$ corresponds to $\epsilon_j - \epsilon_i$. For convenience, we shall denote the root $\epsilon_i - \epsilon_j$ by (i, j), where $1 \leq i \neq j \leq n$. The action of $W (\cong S_n)$ on Δ_0 is given by $w(i, j) = (w(i), w(j))$.

2.3. The root system of $\widehat{\mathsf{A}}_{n-1}$

Let Δ be the root system associated to the Dynkin diagram $\widehat{\mathsf{A}}_{n-1}$. We have

$$\Delta = \{k\delta + \alpha \,|\, k \in \mathbb{Z}, \alpha \in \Delta_0\} \sqcup \{k\delta \,|\, k \in \mathbb{Z}, k \neq 0\}\,.$$

where $\delta = \alpha_0 + \theta$ is the basic imaginary root in Δ, and $\theta = \alpha_1 + \cdots + \alpha_{n-1}$ is the highest root in Δ_0^+. The set of positive roots Δ^+ has the following description:

$$\Delta^+ = \{k\delta + \alpha \,|\, k > 0, \alpha \in \Delta_0 \sqcup \Delta_0^+ \sqcup \{k\delta \,|\, k > 0\}. \tag{2.4}$$

Let $\sim$ be the equivalence relation on $\mathbb{Z} \times \mathbb{Z}$ given by

$$(i, j) \sim (i + kn, j + kn) \quad \forall\, k \in \mathbb{Z}.$$

We identify Δ with $\mathbb{Z} \times \mathbb{Z}/\sim$ as follows:

$$\begin{aligned} k\delta &\mapsto (0, kn), \\ (i, j) + k\delta &\mapsto (i, j + kn) \quad \text{for } (i, j) \in \Delta_0. \end{aligned} \tag{2.5}$$

Under this identification, $(i, j) \in \Delta^+$ if and only if $i < j$.

2.6. The Weyl group

Let N be the normalizer of T in G. We identify the Weyl group $\widehat{W}$ associated to $\widehat{\mathsf{A}}_{n-1}$ with $N(\mathbf{k}[t, t^{-1}])/T$. Let $E_{i,j}$ be the $n \times n$ matrix with 1 in the (i, j) position, and 0 elsewhere. The elements of $N\big(\mathbf{k}[t, t^{-1}]\big)$ are matrices of the form $\sum\limits_{1 \le i \le n} t_i E_{\sigma(i),i}$, where

- (t_i) is a collection of non-zero monomials in t and t^{-1}.
- σ is a permutation of $\{1 \ldots n\}$.
- $\det\big(\sum_{i=1}^n t_i E_{\sigma(i),i}\big) = 1$.

Consider the homomorphism $N\big(\mathbf{k}[t, t^{-1}]\big) \to \mathrm{GL}_n\big(\mathbf{k}[t, t^{-1}]\big)$ given by

$$\sum_{i=1}^n t_i E_{\sigma(i),i} \mapsto \sum_{i=1}^n t^{\mathrm{ord}(t_i)} E_{\sigma(i),i}. \tag{2.7}$$

The kernel of this map is T. Hence we can identify $\widehat{W}$ with the group of $n \times n$ permutation matrices M, with each non-zero entry a power of t, and $\mathrm{ord}(\det M) = 0$. For $w \in \widehat{W}$, we call the matrix corresponding to w the *affine permutation matrix* of w.

2.8. Generators for $\widehat{W}$

We shall work with the set of generators for $\widehat{W}$ given by $\{s_0, s_1, \ldots, s_{n-1}\}$, where $s_i, 0 \le i \le n-1$ are the reflections with respect to $\alpha_i, 0 \le i \le n-1$. Note that $\{\alpha_i \,|\, 1 \le i \le n-1\}$ being the set of simple roots of G, the Weyl group W of G is the subgroup of $\widehat{W}$ generated by $s_1, \dots, s_{n-1}$. The affine permutation matrix of $w \in W$ is $\sum E_{w(i),i}$. The affine permutation matrix of s_0 is given by

$$\begin{pmatrix} 0 & 0 & \cdots & t^{-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ t & 0 & 0 & 0 \end{pmatrix}.$$

For $1 \leq a < b \leq n$, the reflection with respect to the positive root (a,b) is given by the affine permutation matrix

$$s_{(a,b)} = E_{a,b} + E_{b,a} + \sum_{\substack{1 \leq i \leq n \\ i \neq a,b}} E_{i,i}.$$

2.9. Decomposition of $\widehat{W}$ as a semi-direct product

Consider the element $s_\theta \in W$, reflection with respect to the highest root θ. There exists (cf. [6, §13.1.6]) a group isomorphism $\widehat{W} \to W \ltimes Q$ given by

$$\begin{aligned} s_i &\mapsto (s_i, 0) \quad \text{for } 1 \leq i \leq n-1, \\ s_0 &\mapsto (s_\theta, -\theta^\vee), \end{aligned}$$

where Q is the coroot lattice and $\theta^\vee = \alpha_1^\vee + \cdots + \alpha_{n-1}^\vee$. The simple coroot $\alpha_i^\vee \in \mathfrak{h}$, $1 \leq i \leq n-1$, is given by the matrix $E_{i,i} - E_{i+1,i+1}$. As shown in [9, Sect. 2.4], a lift to $N(\mathbf{k}[t,t^{-1}])$ of $(\mathrm{id}, \alpha_i^\vee) \in \widehat{W}$, for $1 \leq i \leq n-1$ is given by

$$\tau_{\alpha_i^\vee} = \sum_{k \neq i,i+1} E_{k,k} + t^{-1} E_{i,i} + t E_{i+1,i+1}.$$

For $q \in Q$, we will write τ_q for the image of $(\mathrm{id}, q) \in W \ltimes Q$ in $\widehat{W}$. Observe that for $\alpha \in \Delta_0$, $q \in Q$ such that $\alpha = (a,b)$ and $\tau_q = \sum t_i E_{ii}$, we have

$$\alpha(q) = \mathrm{ord}(t_b) - \mathrm{ord}(t_a). \tag{2.10}$$

The action of τ_q on Δ is given by $\tau_q(\delta) = \delta$, and

$$\tau_q(\alpha) = \alpha - \alpha(q)\delta \quad \text{for } \alpha \in \Delta_0. \tag{2.11}$$

Hence, for $\alpha \in \Delta_0^+$, it follows from Equation (2.4) that

$$\tau_q(\alpha) > 0 \iff q(\alpha) \leq 0. \tag{2.12}$$

Corollary 2.13. *For $\alpha \in \Delta_0^+$, $\tau_q s_\alpha > \tau_q$ if and only if $\alpha(q) \leq 0$ and $s_\alpha \tau_q > \tau_q$ if and only if $\alpha(q) \geq 0$.*

Proof. Follows from the equivalences (cf. [6]) $ws_\alpha > w \iff w(\alpha) > 0$ and $s_\alpha w > w \iff w^{-1}(\alpha) > 0$ applied to $w = \tau_q$. $\square$

2.14. The Coxeter length

Let $w \in \widehat{W}$ be given by the affine permutation matrix $w = \sum t^{c_i} E_{\sigma(i),i}$, i.e., $w(i) = \sigma(i) - c_i n$. We have the formula:

$$l(w) = \sum_{1 \le i < j \le n} |c_i - c_j - f_\sigma(i,j)|, \tag{2.15}$$

where

$$f_\sigma(i,j) = \begin{cases} 0 & \text{if } \sigma(i) < \sigma(j), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We use Equation (2.5) to write $\Delta^+ = \{(i,j) \,|\, 1 \le i \le n,\, i < j\}$. We know from [6] that

$$\begin{aligned} l(w) &= \#\left\{\alpha \in \Delta^+ \,\middle|\, w(\alpha) < 0\right\} \\ &= \#\left\{(i,j) \,|\, 1 \le i \le n,\, i < j,\, w(i) > w(j)\right\} \\ &= \sum_{1 \le i \le n} \sum_{1 \le j \le n} \#\left\{(i,j') \,|\, j' = j \bmod n,\, i < j',\, w(i) > w(j')\right\} \\ &= \sum_{1 \le i < j \le n} \#\left\{(i, j+kn) \,|\, k \ge 0,\, w(i) > w(j+kn)\right\} \\ &\qquad + \#\left\{(j, i+kn) \,|\, k \ge 1,\, w(j) > w(i+kn)\right\} \\ &= \sum_{1 \le i < j \le n} \#\left\{k \,|\, 0 \le k < (\sigma(i) - \sigma(j))/n + c_j - c_i\right\} \\ &\qquad + \#\left\{k \,|\, 1 \le k < (\sigma(j) - \sigma(i))/n - c_j + c_i\right\} \\ &= \sum_{1 \le i < j \le n} |c_i - c_j - f_\sigma(i,j)|. \qquad \square \end{aligned}$$

2.16. The Bruhat decompostion

The Bruhat decomposition of LG is given by $LG = \bigsqcup_{w \in \widehat{W}} \mathcal{B}w\mathcal{B}$. For $w \in \widehat{W}$, let $X_\mathcal{B}(w) \subset LG/\mathcal{B}$ be the *affine Schubert variety*:

$$X_\mathcal{B}(w) = \overline{\mathcal{B}w\mathcal{B}}(\operatorname{mod} \mathcal{B}) = \bigsqcup_{v \le w} \mathcal{B}v\mathcal{B}(\operatorname{mod} \mathcal{B}).$$

The Bruhat order $\le$ on $\widehat{W}$ reflects inclusion of Schubert varieties, i.e., $v \le w$ if and only if $X_\mathcal{B}(v) \subseteq X_\mathcal{B}(w)$.

Let $\widehat{W}^\mathcal{P} \subset \widehat{W}$ be the set of minimal (in the Bruhat order) representatives of $\widehat{W}/\widehat{W}_\mathcal{P}$. The Bruhat decomposition with respect to $\mathcal{P}$ is given by

$$LG = \bigsqcup_{w \in \widehat{W}^\mathcal{P}} \mathcal{B}w\mathcal{P}.$$

For $w \in \widehat{W}^{\mathcal{P}}$, the *affine Schubert variety* $X_{\mathcal{P}}(w) \subset LG/\mathcal{P}$ is given by

$$X_{\mathcal{P}}(w) = \overline{\mathcal{B}w\mathcal{P}}(\operatorname{mod}\mathcal{P}) = \bigsqcup_{\substack{v \leq w \\ v,w \in \widehat{W}^{\mathcal{P}}}} \mathcal{B}w\mathcal{P}(\operatorname{mod}\mathcal{P}).$$

For $w \in \widehat{W}^{\mathcal{P}}$, $X_{\mathcal{P}}(w)$ is a normal (cf. [3]), projective variety of dimension $l(w)$, the length of w.

Proposition 2.17. *Suppose $w \in \widehat{W}$ is given by the affine permutation matrix*

$$\sum t_i E_{\sigma(i),i} = \sum E_{\sigma(i),i} \sum t_i E_{i,i} = \sigma\tau_q,$$

with $\sigma \in S_n$ and $\tau_q = \sum t_i E_{i,i} \in Q$, where Q is the coroot lattice as in Section 2.9. *Let $1 \leq a < b \leq n$ and set $s_r = s_{(a,b)}$, $s_l = \sigma s_r \sigma^{-1} = s_{(\sigma(a),\sigma(b))}$. The unique minimal element in the set $\{w, s_l w, w s_r, s_l w s_r\}$ has the description as given by the following cases:*

Case 1. *Suppose* $\operatorname{ord}(t_a) = \operatorname{ord}(t_b)$. *Then $s_l w = w s_r$, and the set $\{w, s_l w\}$ has a unique minimal element u, given by*

$$u = \begin{cases} w & \text{if } \sigma(a) < \sigma(b), \\ s_l w & \text{if } \sigma(a) > \sigma(b). \end{cases}$$

Case 2. *Suppose* $\operatorname{ord}(t_a) \neq \operatorname{ord}(t_b)$ *and $\sigma(a) < \sigma(b)$. Then $s_l w \neq w s_r$, and the set $\{w, s_l w, w s_r, s_l w s_r\}$ has a unique minimal element u, given by*

$$u = \begin{cases} w s_r & \text{if } \operatorname{ord}(t_a) < \operatorname{ord}(t_b), \\ s_l w & \text{if } \operatorname{ord}(t_a) > \operatorname{ord}(t_b). \end{cases}$$

Further, we have $u < s_l u < s_l u s_r$ and $u < u s_r < s_l u s_r$.

Case 3. *Suppose* $\operatorname{ord}(t_a) \neq \operatorname{ord}(t_b)$ *and $\sigma(a) > \sigma(b)$. Then $s_l w \neq w s_r$, and the set $\{w, s_l w, w s_r, s_l w s_r\}$ has a unique minimal element u, given by*

$$u = \begin{cases} s_l w s_r & \text{if } \operatorname{ord}(t_a) < \operatorname{ord}(t_b), \\ w & \text{if } \operatorname{ord}(t_a) > \operatorname{ord}(t_b). \end{cases}$$

Further, we have $u < s_l u < s_l u s_r$ and $u < u s_r < s_l u s_r$.

Proof. Recall from Equation (2.10) that $q(a,b) = \operatorname{ord}(t_b) - \operatorname{ord}(t_a)$.

Case 1. Suppose first that $\operatorname{ord}(t_a) = \operatorname{ord}(t_b)$, which from Equation (2.11) is equivalent to $\tau_q(a,b) = (a,b)$. It follows that

$$w s_r = \sigma\tau_q s_{(a,b)} = \sigma s_{(a,b)} \tau_q = s_{(\sigma(a),\sigma(b))} \sigma\tau_q = s_l w.$$

Recall that for $\alpha \in \Delta_0^+$, we have $w s_\alpha > w$ if and only if $w(\alpha) > 0$. Now, $w(a,b) = \sigma\tau_q(a,b) = \sigma(a,b) = (\sigma(a),\sigma(b))$ being positive if and only if $\sigma(a) < \sigma(b)$, it follows that $w < ws_r$ if and only if $\sigma(a) < \sigma(b)$. This is precisely the description of the unique minimal element in Case 1.

Case 2. Suppose $\operatorname{ord}(t_a) \neq \operatorname{ord}(t_b)$ and $\sigma(a) < \sigma(b)$. In particular, $(\sigma(a), \sigma(b))$ is a positive root. Set $k = -q(a,b) = \operatorname{ord}(t_a) - \operatorname{ord}(t_b)$. By Equation (2.11), we have that $\tau_q(a,b) = (a,b) + k\delta$. Then

$$w(a,b) = \sigma\tau_q(a,b) = \sigma((a,b)+k\delta) = (\sigma(a),\sigma(b)) + k\delta,$$
$$w^{-1}(\sigma(a),\sigma(b)) = \tau_q^{-1}\sigma^{-1}(\sigma(a),\sigma(b)) = \tau_{-q}(a,b) = (a,b) - k\delta.$$

It follows that $w(a,b)$ is positive if and only if $k > 0$, while $w^{-1}(\sigma(a),\sigma(b))$ is positive if and only if $k < 0$. Hence, we have

$$\begin{aligned} s_l w < w < ws_r \quad &\text{if } \operatorname{ord}(t_a) > \operatorname{ord}(t_b), \\ ws_r < w < s_l w \quad &\text{if } \operatorname{ord}(t_a) < \operatorname{ord}(t_b). \end{aligned} \tag{2.18}$$

In particular, we have $s_l w \neq ws_r$. Next, observe that

$$s_l w s_r = \sum_{i \neq a,b} t_i E_{\sigma(i),i} + t_b E_{\sigma(a),a} + t_a E_{\sigma(b),b}.$$

Further, the affine permutation matrix of $s_l w s_r$ is obtained by interchanging t_a and t_b in the affine permutation matrix of w. Applying Equation (2.18) to $v = s_l w s_r$, we have

$$\begin{aligned} s_l w = vs_r < v < s_l v = ws_r \quad &\text{if } \operatorname{ord}(t_a) > \operatorname{ord}(t_b), \\ ws_r = s_l v < v < vs_r = s_l w \quad &\text{if } \operatorname{ord}(t_a) < \operatorname{ord}(t_b). \end{aligned}$$

From Equations (2.18) and (2.19), we deduce the description of the minimal element in the set $\{w, s_l w, ws_r, s_l w s_r\}$, as asserted in Case 2 of the Proposition.

Case 3. Suppose $\operatorname{ord}(t_a) \neq \operatorname{ord}(t_b)$ and $\sigma(a) > \sigma(b)$. Then Case 2 applies to the element $v = s_l w$. Hence, we deduce the answers in this case by replacing w with $s_l w$ everywhere in Case 2.

3. Lusztig's embedding and the Springer resolution

In this section, we define the embedding $\phi_P : T^*G/P \hookrightarrow LG/\mathcal{P}$ and relate it to Lusztig's embedding $\psi : \mathcal{N} \to LG/L^+G$ via the Springer resolution. We also discuss Mirković and Vybornov's ([14]) compactification of T^*G/P and show how it relates to the map ϕ_P.

Let V be the vector space $\mathbf{k}^n$ with standard basis $\{e_i \mid 1 \leq i \leq n\}$, and let the group $G = \mathrm{SL}_n(\mathbf{k})$ act on V in the usual way. Fix a sequence $0 = d_0 < d_1 < \ldots < d_{r-1} < d_r = n$, and let $P \supset B$ be the parabolic subgroup corresponding to the set of roots $S_P = S_0 \setminus \{\alpha_{d_i} \mid 1 \leq i < r\}$ in S_0. Let $\lambda_i = d_i - d_{i-1}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$. We denote by $\mathcal{P}$ the parabolic subgroup $\mathcal{B} \subset \mathcal{P} \subset LG$ corresponding to $S_P \subset S$.

3.1. Partitions

A partition $\boldsymbol{\mu}$ of n is a non-increasing sequence $(\mu_1, \mu_2, \ldots, \mu_r)$ of positive integers such that $\sum \mu_i = n$. Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$, we sometimes view $\boldsymbol{\mu}$ as an infinite non-negative sequence by defining $\mu_i = 0$ for $i > r$. Let $\mathcal{P}ar$ denote the set of partitions of n. The *dominance order* $\preceq$ on $\mathcal{P}ar$ is given by

$$\boldsymbol{\mu} \preceq \boldsymbol{\nu} \iff \sum_{j \leq i} \mu_j \leq \sum_{j \leq i} \nu_j \quad \forall i \in \mathbb{Z}.$$

3.2. The Nilpotent cone

Let $\mathcal{N}$ be the variety of nilpotent $n\times n$ matrices, and let G act on $\mathcal{N}$ by conjugation. The variety $\mathcal{N}$ has the G-orbit decomposition $\mathcal{N} = \bigsqcup_{\boldsymbol{\nu}\in\mathcal{P}ar} \mathcal{N}^\circ_{\boldsymbol{\nu}}$, where $\mathcal{N}^\circ_{\boldsymbol{\nu}}$ is the set of nilpotent matrices of Jordan type $\boldsymbol{\nu}$. We write $\mathcal{N}_{\boldsymbol{\nu}}$ for the closure (in $\mathcal{N}$) of $\mathcal{N}^\circ_{\boldsymbol{\nu}}$. Then $\mathcal{N}_{\boldsymbol{\mu}} \subset \mathcal{N}_{\boldsymbol{\nu}}$ if and only if $\boldsymbol{\mu} \preceq \boldsymbol{\nu}$.

3.3. Lusztig's embedding

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s)$ be the partition of n which is conjugate to the partition of n obtained from $\boldsymbol{\lambda}$ by rearranging the λ_i in non-increasing order. In particular, $r = \nu_1$ and $s = \max\{\lambda_i \,|\, 1 \le i \le r\}$. Consider the element $\tau_q \in \widehat{W}$ given by the affine permutation matrix

$$\tau_q = \sum_{i=1}^{s} t^{\nu_i - 1} E_{i,i} + \sum_{i=s+1}^{n} t^{-1} E_{i,i}. \tag{3.4}$$

There exists a G-equivariant injective map $\psi : \mathcal{N}_{\boldsymbol{\nu}} \hookrightarrow X_{L^+G}(\tau_q)$ given by

$$\psi(X) = \left(1 - t^{-1}X\right) (\mathrm{mod}\, L^+G) \quad \text{for } X \in \mathcal{N}_{\boldsymbol{\nu}}. \tag{3.5}$$

The map ψ is an open immersion onto the *opposite Bruhat cell*

$$Y_{L^+G}(\tau_q) := \mathcal{B}^-/L^+G \cap X_{L^+G}(\tau_q),$$

where $\mathcal{B}^-/L^+G$ denotes the image of $\mathcal{B}^-$ under the map $LG \to LG/L^+G$. This statement is well known. A proof of can be found in §4.1 of [1]. A variant of the map ψ was first introduced in [11].

3.6. The cotangent bundle

We identify the points of the variety G/P with the partial flags in $V(= \mathbf{k}^n)$ of shape $\boldsymbol{\lambda}$:

$$G/P \cong \{(F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r) \,|\, \dim F_i/F_{i-1} = \lambda_i\}.$$

Using the Killing form on $\mathfrak{sl}_n$, we can identify the cotangent space at identity T^*_eG/P with the Lie algebra $\mathfrak{u}$ of the unipotent radical U_P of P. Then T^*G/P is the fiber bundle over G/P associated to the principal P-bundle $G \to G/P$, for the adjoint action of P on $\mathfrak{u}$:

$$T^*G/P = G \times^P \mathfrak{u} = G \times \mathfrak{u}/\!\sim.$$

The equivalence relation $\sim$ is given by $(g, Y) \sim (gp, p^{-1}Yp)$, where $g \in G$, $Y \in \mathfrak{u}$, $p \in P$. We also have the identification

$$T^*G/P = \{(X, F_0 \subset \cdots \subset F_k) \,|\, X \in \mathcal{N},\, \dim F_i/F_{i-1} = \lambda_i,\, X(F_i) \subset F_{i-1}\}. \tag{3.7}$$

The two identifications are related via the isomorphism

$$(g, X) \mapsto (gXg^{-1}, gV_0 \subset \cdots \subset gV_k),$$

where $(V_0 \subset \cdots \subset V_r)$ is the flag of shape $\boldsymbol{\lambda}$ fixed by P.

3.8. The affine Grassmannian

A *lattice* in $L \subset V[t,t^{-1}]$ is a $\mathbf{k}[t]$ module satisfying $\mathbf{k}[t,t^{-1}] \underset{\mathbf{k}[t]}{\otimes} L = V[t,t^{-1}]$. The *virtual dimension* of L is defined as

$$\mathrm{vdim}(L) := \dim_{\mathbf{k}}(L/L \cap E) - \dim_{\mathbf{k}}(E/L \cap E).$$

where E is the standard lattice, namely the $\mathbf{k}[t]$ span of V. The quotient LG/L^+G is an ind-variety whose points are identified with the lattices of virtual dimension 0 (cf. [3]).

$$LG/L^+G = \{L \text{ a lattice} \mid \mathrm{vdim}(L) = 0\}.$$

3.9. Affine flag varieties

A *partial affine flag* of shape $\boldsymbol{\lambda}$ is a sequence of lattices $L_0 \subset L_1 \subset \cdots \subset L_r$ satisfying $tL_r = L_0$ and $\dim L_i/L_{i-1} = \lambda_i$ for $1 \le i \le r$. For P and $\mathcal{P}$ as above, we can identify the points of the ind-variety $LG/\mathcal{P}$ with partial affine flags of shape $\boldsymbol{\lambda}$ satisfying the additional condition $\mathrm{vdim}(L_0) = 0$ (cf. [3]).

$$LG/\mathcal{P} \cong \{(L_0 \subset \cdots \subset L_r) \mid \dim L_i/L_{i-1} = \lambda_i,\, tL_r = L_0,\, \mathrm{vdim}(L_0) = 0\}. \quad (3.10)$$

The ind-variety $LG/\mathcal{P}$ is called the *affine flag variety* associated to $\mathcal{P}$.

3.11. The map ϕ_P

Let $\phi_P : G \times^P \mathfrak{u} \to LG/\mathcal{P}$ be defined by

$$\phi_P(g, X) = g(1 - t^{-1}X)(\operatorname{mod} \mathcal{P}) \quad g \in G,\ X \in \mathfrak{u}.$$

For $g \in G$, $p \in P$ and $X \in \mathfrak{u}$, we have

$$\begin{aligned}\phi_P\left(gp, p^{-1}Xp\right) &= gp\left(1 - t^{-1}p^{-1}Xp\right)(\operatorname{mod} \mathcal{P}) \\ &= g\left(p - t^{-1}Xp\right)(\operatorname{mod} \mathcal{P}) \\ &= g\left(1 - t^{-1}X\right)(\operatorname{mod} \mathcal{P}) \\ &= g\,\phi_P(1, X).\end{aligned}$$

It follows that ϕ_P is well-defined and G-equivariant. Under the identifications of Equations (3.7) and (3.10), we have

$$\phi_P(X, F_0 \subset F_1 \subset \cdots \subset F_r) = L_0 \subset L_1 \subset \cdots \subset L_r.$$

where $L_i = \left(1 - t^{-1}X\right)\left(V[t] \oplus t^{-1}F_i\right)$.

Lemma 3.12. *The map ϕ_P is injective.*

Proof. Suppose $\phi_P(g, Y) = \phi_P(g_1, Y_1)$, i.e.,

$$\begin{aligned}&g\left(1 - t^{-1}Y\right) = g_1\left(1 - t^{-1}Y_1\right)(\operatorname{mod} \mathcal{P}) \\ \Longrightarrow\ &g\left(1 - t^{-1}Y\right) = g_1\left(1 - t^{-1}Y_1\right)x \quad \text{for some } x \in \mathcal{P}.\end{aligned}$$

Denoting $h = g_1^{-1} g$ and $Y' = hYh^{-1}$, we have

$$\begin{aligned} h\left(1 - t^{-1}Y\right) &= \left(1 - t^{-1}Y_1\right) x \\ \implies x &= \left(1 - t^{-1}Y_1\right)^{-1} h \left(1 - t^{-1}Y\right) \\ &= \left(1 - t^{-1}Y_1\right)^{-1} \left(1 - t^{-1}Y'\right) h \\ \implies xh^{-1} &= \left(1 + t^{-1}Y_1 + t^{-2}Y_1^2 + \cdots\right) \left(1 - t^{-1}Y'\right). \end{aligned}$$

Now since $x \in \mathcal{P}$, $h \in G$, the left-hand side is integral, i.e., does not involve negative powers of t. Hence both sides must equal identity. It follows that $x = h \in P = \mathcal{P} \bigcap G$ and $Y_1 = Y' = hYh^{-1}$. In particular, $(g_1, Y_1) = \left(gh^{-1}, hYh^{-1}\right) \sim (g, Y)$ as required. □

3.13. The Springer resolution

Let $\boldsymbol{\nu}$ be the partition of n which is conjugate to the partition of n obtained from $\boldsymbol{\lambda}$ by rearranging the λ_i in non-increasing order. The Springer map $\theta_P : T^*G/P \to \mathcal{N}$, given by $\theta_P(g, X) = gXg^{-1}$ where $g \in G$, $X \in \mathfrak{n}$, is a resolution of singularities for the G-orbit $\mathcal{N}_{\boldsymbol{\nu}} \subset \mathcal{N}$. The maps ϕ_P and ψ from Section 3.11 and equation (3.5) sit in the following commutative diagram:

$$\begin{array}{ccc} T^*G/P & \overset{\phi_P}{\hookrightarrow} & LG/\mathcal{P} \\ \downarrow \theta_P & & \downarrow \mathrm{pr} \\ \mathcal{N}_{\boldsymbol{\nu}} & \overset{\psi}{\hookrightarrow} & LG/L^+G, \end{array}$$

where $\mathrm{pr} : LG/\mathcal{P} \to LG/L^+G$ is the natural projection. We present a more precise version of this statement in Section 4.5.

3.14. The Mirković–Vybornov compactification

Consider the *convolution Grassmannian* $\widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}}$ whose points are identified with certain lattice flags:

$$\widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}} = \left\{ L_0 \subset L_1 \subset \cdots \subset L_r \,\middle|\, L_r = t^{-1}V[t],\, \dim L_i/L_{i-1} = \lambda_i,\, tL_i \subset L_{i-1} \right\}.$$

Mirković and Vybornov [14] have constructed an embedding $\widetilde{\psi} : T^*G/P \hookrightarrow \widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}}$ given by

$$\widetilde{\psi}(X, F_0 \subset \cdots \subset F_r) = L_0 \subset \cdots \subset L_r,$$

where $L_i = (1 - t^{-1}X)V[t] \oplus t^{-1}F_i$. Once again, we have a commutative diagram

$$\begin{array}{ccc} T^*G/P & \overset{\widetilde{\psi}}{\hookrightarrow} & \widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}} \\ \downarrow \theta_P & & \downarrow \mathrm{pr} \\ \mathcal{N}_{\boldsymbol{\nu}} & \overset{\psi}{\hookrightarrow} & LG/L^+G, \end{array}$$

where the map $\mathrm{pr} : \widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}} \to LG/L^+G$ is given by $(L_0, L_1, \ldots, L_r) \mapsto L_0$. As we can see, the incidence relations $tL_i \subset L_{i-1}$ are in general different from the incidence relations $tL_r = L_0$ of the partial affine flag variety; accordingly, $\widetilde{\mathcal{G}r}^{\boldsymbol{\lambda}}$ is different from $LG/\mathcal{P}$. However, when P is maximal, i.e., $\boldsymbol{\lambda} = (d, n-d)$ for some d, we have an isomorphism $\beta : \widetilde{\mathcal{G}r}^{\lambda} \xrightarrow{\sim} LG/\mathcal{P}$ given by

$$(L_0 \subset L_1 \subset t^{-1}V[t]) \mapsto (L_0 \subset L_1 \subset L_2),$$

where $L_2 = t^{-1}L_0$. In this case, one can verify that $\beta \circ \widetilde{\psi} = \phi_P$.

4. The element $\boldsymbol{\kappa}$

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$, P, and $\mathcal{P}$ be as in the previous section. Further, let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_s)$ be the partition of n which is conjugate to the partition of n obtained from $\boldsymbol{\lambda}$ by rearranging the λ_i in non-increasing order.

In this section, we describe the element $\kappa \in \widehat{W}^{\mathcal{P}}$ for which $\phi_P(T^*G/P) \subset X_{\mathcal{P}}(\kappa)$; further, κ is minimal for this property. We compute $\dim X_{\mathcal{P}}(\kappa) = l(\kappa)$ and show that when P is maximal, $X_{\mathcal{P}}(\kappa)$ is a compactification of T^*G/P.

4.1. Tableaux

We draw a left-aligned tableau with r rows, with the i^{th} row from top having λ_i boxes. Fill the boxes of the tableau as follows: the entries of the i^{th} row are the integers k satisfying $d_i < k \le d_{i+1}$, written in increasing order. We denote by $\mathcal{R}ow(i)$ the set of entries in the i^{th} row of the tableau. Observe that the number of boxes in the i^{th} column from the left is ν_i. The Weyl group W_P is the set of elements in S_n that preserve the partition $\{1, \dots, n\} = \bigsqcup_i \mathcal{R}ow(i)$.

We define a co-ordinate system $\chi(\bullet, \bullet)$ on $\{1, \dots, n\}$ as follows: For $1 \le i \le r$, $1 \le j \le \nu_i$, let $\chi(j, i)$ denote the j^{th} entry (from the top) of the i^{th} column. Note that $\chi(j, i)$ need not be in $\mathcal{R}ow(j)$.

Finally, let $F^i_{j,k}$ denote the elementary matrix $E_{\chi(j,i),\chi(k,i)}$. Observe that $\chi(b, i) = \chi(c, j)$ if and only if $i = j$ and $b = c$. In particular,

$$F^i_{a,b} F^j_{c,d} = \delta_{ij}\delta_{bc} F^i_{a,d}. \tag{4.2}$$

4.3. $\mathcal{R}ed$ and $\mathcal{B}lue$

We split the set $\{1, \dots, n\}$ into disjoint subsets $\mathcal{R}ed$ and $\mathcal{B}lue$, depending on their positions in the tableau. Let $\mathcal{S}_1 = \{\chi(1, i) \mid 1 \le i \le s\}$ be the set of entries which are topmost in their column. We write $\mathcal{S}_1(i) = \mathcal{S}_1 \bigcap \mathcal{R}ow(i)$. For convenience, we also define $\mathcal{S}_2(i) = \mathcal{R}ow(i) \backslash \mathcal{S}_1(i)$ and $\mathcal{S}_2 = \bigcup_i \mathcal{S}_2(i)$.

The set $\mathcal{R}ed(i)$ is the collection of the $\#\mathcal{S}_1(i)$ smallest entries in $\mathcal{R}ow(i)$:

$$\mathcal{R}ed(i) := \{j \mid d_{i-1} < j \le d_i - \max\{\lambda_k \mid k < i\}\}.$$

We set $\mathcal{B}lue(i) = \mathcal{R}ow(i) \backslash \mathcal{R}ed(i)$, $\mathcal{R}ed = \bigcup_i \mathcal{R}ed(i)$, and $\mathcal{B}lue = \bigcup_i \mathcal{B}lue(i)$. The elements of $\mathcal{R}ed(i)$ are smaller than the elements of $\mathcal{B}lue(i)$.

The elements of $\mathcal{R}ed$, arranged in increasing order are written $l(1), \dots, l(s)$. We enumerate the elements of $\mathcal{B}lue$, written row by row from bottom to top, each row written left to right, as $m(1), \dots, m(n-s)$.

Example 4.4. Let $n = 17$ and the sequence (d_i) be $(1, 5, 9, 11, 17)$. The corresponding tableau is

1					
2	3	4	5		
6	7	8	9		
10	11				
12	13	14	15	16	17

- $r = 5$, $s = 6$.
- The sequence (λ_i) is $(1, 4, 4, 2, 6)$.
- The sequence (ν_i) is $(5, 4, 3, 3, 1, 1)$.
- $\mathcal{R}ow(3) = \{6, 7, 8, 9\}$.
- $\chi(4, 1) = 10$, $\chi(3, 4) = 15$, $\chi(1, 6) = 17$, etc.
- $F^1_{2,4} = E_{2,10}$, $F^3_{3,3} = E_{14,14}$, etc.
- $\mathcal{S}_1 = \{1, 3, 4, 5, 16, 17\}$.
- The sequence $l(i)$ is $(1, 2, 3, 4, 12, 13)$.
- The sequence $m(i)$ is $(14, 15, 16, 17, 10, 11, 6, 7, 8, 9, 5)$.

4.5. The element $\boldsymbol{\kappa}$

Let $\kappa \in \widehat{W}$ be given by the affine permutation matrix

$$\sum_{i=1}^{s} t^{\nu_i - 1} E_{i,l(i)} + \sum_{i=1}^{n-s} t^{-1} E_{i+s,m(i)}.$$

Recall the element $\tau_q \in \widehat{W}$ from Equation (3.4). The commutative diagram of Section 3.13 can be refined to the following:

$$\begin{array}{ccc} T^*G/P & \xhookrightarrow{\phi_P} & X_{\mathcal{P}}(\kappa) \\ \Big\downarrow \theta_P & & \Big\downarrow \mathrm{pr} \\ \mathcal{N}_{\boldsymbol{\nu}} & \xhookrightarrow{\psi} & X_{L^+G}(\tau_q). \end{array}$$

The only additional part is the claim $\phi_P(T^*G/P) \subset X_{\mathcal{P}}(\kappa)$. This is the content of Theorem 4.11.

4.6. The matrix $\boldsymbol{Z}$

Let V be the n-dimensional vector space with basis $e_1, \ldots, e_n$, and let Z be the linear endomorphism of V given by

$$Ze_{\chi(j,i)} = \begin{cases} e_{\chi(j-1,i)} & \text{if } \ j > 1, \\ 0 & \text{if } \ j = 1. \end{cases}$$

Writing Z as a matrix with respect to the basis $e_1, \ldots, e_n$, we have

$$Z = \sum_{i=1}^{s} \sum_{j=1}^{\nu_i - 1} F^i_{j,j+1}.$$

Observe that $\{e_{\chi(\nu_i,i)} \mid 1 \le i \le s\}$ is a minimal generating set for V as a module over $\mathbf{k}[Z]$ (the $\mathbf{k}$-algebra generated by Z). Consequently, the Jordan type of Z is $\boldsymbol{\nu}$, i.e., $Z \in \mathcal{N}_{\boldsymbol{\nu}}^{\circ}$. Further, observe that $Z(V_i) \subset V_{i-1}$ for all i, hence $Z \in \mathfrak{u}$, where $\mathfrak{u}$ is as in Section 3.6.

An element in $x \in \mathfrak{u}$ is called a *Richardson element* of $\mathfrak{u}$ if the P-orbit of x is dense in $\mathfrak{u}$, or equivalently, the G-orbit of $(1, Z)$ is dense in $T^*G/P = G \times^P \mathfrak{u}$. A comprehensive study of Richardson elements in $\mathfrak{u}$ can be found in [4]. In particular, we will need the following:

Lemma 4.7. *The matrix Z is a Richardson element in $\mathfrak{u}$.*

Proof. This follows from Theorem 3.3 of [4], along with the above observation that the Jordan type of Z is $\boldsymbol{\nu}$, i.e., $Z \in \mathcal{N}_{\boldsymbol{\nu}}^{\circ}$. $\square$

Proposition 4.8. *Let $\mathcal{B}\varpi\mathcal{B}$ be the Bruhat cell containing $1 - t^{-1}Z$. A lift of ϖ to $N\left(\mathbf{k}[t, t^{-1}]\right)$ is given by*

$$\tilde{\varpi} = \sum_{i=1}^{s} \Big(t^{\nu_i - 1} F^i_{\nu_i,1} - \sum_{j=2}^{\nu_i} t^{-1} F^i_{j-1,j} \Big).$$

Proof. For $1 \le i \le s$, let

$$b_i := \sum_{j=1}^{\nu_i} \sum_{k=j}^{\nu_i} t^{k-j} F^i_{k,j} = \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i - j} t^k F^i_{j+k,j},$$

$$c_i := \sum_{j=1}^{\nu_i} F^i_{j,j} + \sum_{j=2}^{\nu_i} t^{j-1} F^i_{j,1},$$

$$Z_i := \sum_{j=1}^{\nu_i} F^i_{j,j} - t^{-1} \sum_{j=1}^{\nu_i - 1} F^i_{j,j+1}.$$

We compute

$$\begin{aligned}
b_i Z_i c_i &= \Big(\sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i - j} t^k F^i_{j+k,j} \Big) \Big(\sum_{j=1}^{\nu_i} F^i_{j,j} - t^{-1} \sum_{j=1}^{\nu_i - 1} F^i_{j,j+1} \Big) c_i \\
&= \Big(\sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i - j} t^k F^i_{j+k,j} - \sum_{j=1}^{\nu_i - 1} \sum_{k=0}^{\nu_i - j} t^{k-1} F^i_{j+k,j+1} \Big) c_i \\
&= \Big(\sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i - j} t^k F^i_{j+k,j} - \sum_{j=2}^{\nu_i} \sum_{k=-1}^{\nu_i - j} t^k F^i_{j+k,j} \Big) c_i \\
&= \Big(\sum_{k=0}^{\nu_i - 1} t^k F^i_{1+k,1} - \sum_{j=2}^{\nu_i} t^{-1} F^i_{j-1,j} \Big) \Big(\sum_{j=1}^{\nu_i} F^i_{j,j} + \sum_{j=2}^{\nu_i} t^{j-1} F^i_{j,1} \Big) \\
&= \sum_{k=0}^{\nu_i - 1} t^k F^i_{k+1,1} - \sum_{j=2}^{\nu_i} t^{-1} F^i_{j-1,j} - \sum_{j=2}^{\nu_i} t^{j-2} F^i_{j-1,1} \\
&= t^{\nu_i - 1} F^i_{\nu_i,1} - \sum_{j=2}^{\nu_i} t^{-1} F^i_{j-1,j}.
\end{aligned}$$

Observe that $1-t^{-1}Z = \sum_{1\le i\le s} Z_i$. It follows from Equation (4.2) that for $i \neq j$, $b_iZ_j = 0$ and $Z_jc_i = 0$. Writing $b = \sum_i b_i$ and $c = \sum_i c_i$, we see

$$b(1-t^{-1}Z)c = \sum_i b_iZ_ic_i = \varpi.$$

The result now follows from the observation $b, c \in \mathcal{B}$. $\square$

Lemma 4.9. *There exist $w_g \in \widehat{W}_{L^+G}(=W)$, $w_p \in \widehat{W}_{\mathcal{P}}(=W_P)$ such that $\varpi = w_g\kappa w_p$. In particular, $X_{\mathcal{P}}(\varpi) \subset \overline{L^+G\kappa\mathcal{P}}(\mathrm{mod}\,\mathcal{P})$.*

Proof. Recall the disjoint subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\{1,\dots,n\}$ from Section 4.3. Consider the bijection $\iota : \mathcal{S}_2 \to \{\chi(j,i) \,|\, 1 \le j \le \nu_i - 1\}$ given by $\iota(\chi(j,i)) = \chi(j-1,i)$. We reformulate Proposition 4.8 as

$$\tilde{\varpi} = \sum_{i=1}^{s} t^{\nu_i-1}E_{\chi(\nu_i,i),\chi(1,i)} - \sum_{i\in\mathcal{S}_2} E_{\iota(i),i}.$$

It follows from Equation (2.7) that the affine permutation matrix of ϖ is given by

$$\varpi = \sum_{i=1}^{s} t^{\nu_i-1}E_{\chi(\nu_i,i),\chi(1,i)} + \sum_{i\in\mathcal{S}_2} E_{\iota(i),i}.$$

Observe that $\#\mathcal{R}ed(k) = \#\mathcal{S}_1(k)$ and $\#\mathcal{B}lue(k) = \#\mathcal{S}_2(k)$ for all $1 \le k \le r$. Since both $(l(i))_{1\le i\le s}$ and $(\chi(1,i))_{1\le i\le s}$ are increasing sequences, $\chi(1,i)$ and $l(i)$ are in the same row for each i. Furthermore, there exists an enumeration $t(1),\dots,t(n-s)$ of $\mathcal{S}_2$ such that $t(i)$ is in the same row as $m(i)$ for all i. We define $w_g \in W$ and $w_p \in W_P$ via their affine permutation matrices:

$$w_g = \sum_{i=1}^{s} E_{i,\chi(\nu_i,i)} + \sum_{i=1}^{n-s} E_{i+s,\iota(t(i))},$$
$$w_p = \sum_{i=1}^{s} E_{\chi(1,i),l(i)} + \sum_{i=1}^{n-s} E_{t(i),m(i)}.$$

A simple calculation shows $\varpi = w_g\kappa w_p$. $\square$

Proposition 4.10. *The Schubert variety $X_{\mathcal{P}}(\kappa)$ is stable under left multiplication by L^+G, i.e., $X_{\mathcal{P}}(\kappa) = \overline{L^+G\kappa\mathcal{P}/\mathcal{P}}$.*

Proof. Consider the affine permutation matrix of κ. We will show, for $1 \le i < n$, either $s_i\kappa = \kappa(\mathrm{mod}\,\widehat{W}_{\mathcal{P}})$ or $s_i\kappa < \kappa$. We split the proof into several cases, and use Proposition 2.17 with $s_l = s_i$:

(1) $i < s$ and $\nu_i = \nu_{i+1}$: We deduce from $\nu_i = \nu_{i+1}$ that the entries $l(i)$ and $l(i+1)$ appear in the same row of the tableau. In particular, $l(i) \in S_P$. The non-zero entries of the i^{th} and $(i+1)^{\text{th}}$ row of κ are $t^{\nu_i-1}E_{i,l(i)}$ and $t^{\nu_{i+1}-1}E_{i+1,l(i+1)}$ respectively. We see $s_i\kappa = \kappa s_{l(i)} = \kappa(\mathrm{mod}\,\widehat{W}_{\mathcal{P}})$.

(2) $i < s$ and $\nu_i > \nu_{i+1}$: The non-zero entries of the i^{th} and $(i+1)^{\text{th}}$ row of κ are $t^{\nu_i - 1}E_{i,l(i)}$ and $t^{\nu_{i+1}-1}E_{i+1,l(i+1)}$ respectively. Case 2 of Proposition 2.17 applies with $a = l(i)$, $b = l(i+1)$, and we have $s_i\kappa < \kappa$.

(3) $i = s$: The non-zero entries of the i^{th} and $(i+1)^{\text{th}}$ row of κ are $t^{\nu_s - 1}E_{s,l(s)}$ and $t^{-1}E_{s+1,m(1)}$ respectively. Since $m(1) \in \mathcal{B}lue(r)$, it follows from Section 4.3 that $l(s) < m(1)$. Case 2 of Proposition 2.17 applied with $a = l(s)$, $b = m(1)$ tells us $s_i\kappa < \kappa$.

(4) $i > s$ and $m(i-s) \in S_P$: The non-zero entries of the i and $(i+1)^{\text{th}}$ row of κ are $t^{-1}E_{i,m(i-s)}$ and $t^{-1}E_{i+1,m(i+1-s)}$ respectively. It follows that $s_i\kappa = \kappa s_{m(i-s)} = \kappa (\text{mod}\, \widehat{W}_{\mathcal{P}})$.

(5) $i > s$ and $m(i-s) \notin S_P$: It follows from $m(i-s) \notin S_P$ that if $m(i-s) \in \mathcal{R}ow(j)$ then $m(i+1-s) \in \mathcal{R}ow(j-1)$. In particular, $m(i+1-s) < m(i-s)$. The non-zero entries of the i^{th} and $(i+1)^{\text{th}}$ row of κ are $t^{-1}E_{i,m(i-s)}$ and $t^{-1}E_{i+1,m(i+1-s)}$ respectively. Case 1 of Proposition 2.17 applied with $a = m(i+1-s)$, $b = m(i+s)$ tells us $s_i\kappa < \kappa$. □

Theorem 4.11. *Let ϕ_P be as in Section* 3.11. *Then* $\text{Im}(\phi_P) \subset X_{\mathcal{P}}(\kappa)$. *In particular, the map ϕ_P gives a compactification of T^*G/P. Further $\kappa \in \widehat{W}^{\mathcal{P}}$ is minimal for the property* $\text{Im}(\phi_P) \subset X_{\mathcal{P}}(\kappa)$.

Proof. Let $\mathcal{O}$ denote the G-orbit of $(1, Z) \in G \times^P \mathfrak{u}$. It follows from Lemma 4.7 that $T^*G/P = \overline{\mathcal{O}}$, and from Proposition 4.8 that $\phi_P(1, Z) \in \mathcal{B}\varpi\mathcal{P}/\mathcal{P}$. Since ϕ_P is G-equivariant, it follows that $\phi_P(\mathcal{O}) \subset G\mathcal{B}\varpi\mathcal{P}/\mathcal{P} \subset L^+G\varpi\mathcal{P}/\mathcal{P}$, and so

$$\phi_P\left(T^*G/P\right) = \phi_P\left(\overline{\mathcal{O}}\right) \subset \overline{\phi_P\left(\mathcal{O}\right)} \subset \overline{L^+G\varpi\mathcal{P}/\mathcal{P}} = X_{\mathcal{P}}(\kappa),$$

where the last equality follows from Lemma 4.9 and Proposition 4.10. Further, $\overline{\phi_P(T^*G/P)}$, being a closed subvariety of $X_{\mathcal{P}}(\kappa)$, is compact.

To prove the minimality of κ, we show that there exists an element $a \in G$ such that $\phi_P(a, Z) \in \mathcal{B}\kappa\mathcal{P}/\mathcal{P}$. Proposition 4.10 implies that κ is maximal in the right coset $W\varpi$. In particular, there exists $w \in W$ such that $\kappa = w\varpi$ and $l(\kappa) = l(w) + l(\varpi)$. It follows that $\mathcal{B}w\mathcal{B}\varpi\mathcal{P} = \mathcal{B}\kappa\mathcal{P}$, and $\phi_P(a, Z) \in \mathcal{B}\kappa\mathcal{P}$ for any $a \in BwB$. □

Proposition 4.12. *The dimension of $X_{\mathcal{P}}(\kappa)$ is $l(\kappa)$, the length of κ.*

Proof. We need to show that $\kappa \in \widehat{W}^{\mathcal{P}}$. For $\alpha_i \in S_P$, we show $\kappa s_i > \kappa$. Recall the partitioning of $\{1, \ldots, n\}$ from Section 4.3 into $\mathcal{R}ed$ and $\mathcal{B}lue$. Note that $\alpha_i \in S_P$, implies i and $i+1$ appear in the same row of the tableau. In particular, if $i \in S_P \bigcap \mathcal{B}lue(j)$ then $i+1 \in \mathcal{B}lue(j)$.

(1) Suppose $i \in \mathcal{B}lue$. Then $i = m(k)$ and $i+1 = m(k+1)$ for some k. The non-zero entries in the i^{th} and $(i+1)^{\text{th}}$ columns of κ are $t^{-1}E_{k+s,i}$ and $t^{-1}E_{k+s+1,i+1}$. We apply Case 1 of Proposition 2.17 with $a = i$, $b = i+1$.

(2) Suppose $i \in \mathcal{R}ed$ and $i+1 \in \mathcal{B}lue$. The non-zero entries in the i^{th} and $(i+1)^{\text{th}}$ columns of κ are $t^{\nu_k - 1}E_{k,i}$ and $t^{-1}E_{j,i+1}$. Since $k \le s < j$, we can apply Case 2 of Proposition 2.17 with $a = i$, $b = i+1$, $\nu_k - 1 = \text{ord}(t_a) > \text{ord}(t_b) = -1$ and $k = \sigma(a) < \sigma(b) = j$ to get $\kappa < \kappa s_i$.

(3) Suppose $i, i+1 \in \mathcal{R}ed$. Since i and $i+1$ are in the same row of the tableau, we have $i = l(k)$ and $i+1 = l(k+1)$ for some k. The non-zero entries in the i^{th}

and $(i+1)^{\text{th}}$ columns of κ are $t^{\nu_k - 1}E_{k,i}$ and $t^{\nu_{k+1}-1}E_{k+1,i+1}$. If $\nu_k = \nu_{k+1}$, Case 1 of Proposition 2.17 applies with $a = i$, $b = i+1$ and $k = \sigma(a) < \sigma(b) = k+1$ to give $\kappa < \kappa s_i$. If $\nu_k > \nu_{k+1}$, Case 2 of Proposition 2.17 applies with $a = i$, $b = i+1$, $k = \sigma(a) < \sigma(b) = k+1$ and $\nu_k - 1 = \mathrm{ord}(t_a) > \mathrm{ord}(t_b) = \nu_{k+1} - 1$ to give $\kappa < \kappa s_i$. □

Lemma 4.13. *The length of κ is given by the formula*

$$l(\kappa) = 2\dim G/P + \sum_{k'<k} \#\mathcal{R}ow(k)\#\mathcal{B}lue(k').$$

Proof. Note that $\kappa = \tau_q\sigma$, where τ_q is given by Equation (3.4), and $\sigma \in W$ is given by the permutation matrix

$$\sigma = \sum_{i=1}^{s} E_{i,l(i)} + \sum_{i=1}^{n-s} E_{i+s,m(i)}. \tag{4.14}$$

Viewing σ as an element of S_n, we have

$$\sigma^{-1}(i) = \begin{cases} l(i) & i \leq s, \\ m(i-s) & i > s. \end{cases}$$

In particular,

$$\begin{aligned} l(\sigma) =& \# \left\{ (i,j) \,\middle|\, 1 \leq i < j \leq n,\, \sigma^{-1}(i) > \sigma^{-1}(j) \right\} \\ =& \# \left\{ (i,j) \,|\, i < j \leq s, l(i) > l(j) \right\} \\ &+ \# \left\{ (i,j) \,|\, i \leq s < j,\, l(i) > m(j-s) \right\} \\ &+ \# \left\{ (i,j) \,|\, s < i < j,\, m(i-s) > m(j-s) \right\}. \end{aligned}$$

Recall that $l(i)$ is an increasing sequence, i.e., $i < j < s \implies l(i) < l(j)$, and so,

$$\begin{aligned} l(\sigma) =& \# \left\{ (i,j) \,|\, i \leq s,\, j \leq n-s,\, l(i) > m(j) \right\} \\ &+ \# \left\{ (i,j) \,|\, i < j \leq n-s,\, m(i) > m(j) \right\} \\ =& \# \left\{ (i,j) \,|\, i \in \mathcal{R}ed,\, j \in \mathcal{B}lue,\, i > j \right\} \\ &+ \# \left\{ (i,j) \,|\, i < j \leq n-s,\, m(i) > m(j) \right\} \\ =& \sum_{k'<k} \# \left\{ (i,j) \,|\, i \in \mathcal{R}ed(k),\, j \in \mathcal{B}lue(k') \right\} \\ &+ \sum_{k'<k} \# \left\{ (i,j) \,|\, i \in \mathcal{B}lue(k),\, j \in \mathcal{B}lue(k') \right\} \\ =& \sum_{k'<k} \# \left\{ (i,j) \,|\, i \in \mathcal{R}ow(k),\, j \in \mathcal{B}lue(k') \right\} \\ =& \sum_{k'<k} \#\mathcal{R}ow(k)\#\mathcal{B}lue(k'). \end{aligned}$$

Now, it follows from Equations (2.10) and (2.12) that $\tau_q \in \widehat{W}^{L^+G}$. In particular,

$$l(\tau_q) = \dim X_{L^+G}(\tau_q) = \dim \mathcal{N}_{\boldsymbol{\nu}} = 2\dim G/P.$$

Further, $l(\kappa) = l(\tau_q) + l(\sigma) = 2\dim G/P + \sum_{k'<k} \#\mathcal{R}ow(k)\#\mathcal{B}lue(k')$ as claimed. □

Corollary 4.15. *The Schubert variety $X_{\mathcal{P}}(\kappa)$ is a compactification of T^*G/P if and only if P is a maximal parabolic subgroup.*

Proof. The parabolic subgroup P is maximal if and only if $S_P = S_0 \setminus \{\alpha_d\}$ for some d, equivalently, the corresponding tableau has exactly 2 rows. In this case $\mathcal{B}lue \subset \mathcal{R}ow(2)$, (recall from Section 4.3 that $\mathcal{B}lue = \bigsqcup_i \mathcal{B}lue(i)$). It follows that the second term in Lemma 4.13 is an empty sum, which implies $l(\kappa) = 2\dim G/P$. Suppose now that the tableau has $r \geq 3$ rows. In this case, both $\mathcal{B}lue(2)$ and $\mathcal{R}ow(r)$ are non-empty, and so the second term in Lemma 4.13 is strictly greater than 0. □

5. Conormal variety of the Schubert divisor

Let $\boldsymbol{\lambda}$, P, and $\mathcal{P}$ be as in the previous section. In this section, we show that for $X_P(w)$ a Schubert divisor in G/P, the conormal variety $N^*X_P(w)$ is an open subset of a Schubert variety in $LG/\mathcal{P}$. In particular, $N^*X_P(w)$ is normal, Cohen–Macaulay, and Frobenius split.

We write $\mathfrak{h}$, $\mathfrak{b}$, and $\mathfrak{g}$ for the Lie algebras of T, B, and G respectively. For $\alpha \in \Delta_0$, we denote by $\mathfrak{g}^\alpha$ the root space corresponding to α.

5.1. The conormal variety

Let X be a closed subvariety in G/P, and write X_{sm} for the smooth locus of X. For $x \in X_{\text{sm}}$, the *conormal fibre* N^*_x is the annihilator of T_xX in T^*_xG/P. The *conormal variety* N^*X of $X \hookrightarrow G/P$ is then defined to be the closure in T^*G/P of the conormal bundle N^*X_{sm}.

Proposition 5.2. *Let Δ_P be the subset of Δ_0 generated by S_P. The conormal variety $N^*X_P(w)$ is the closure in T^*G/P of*

$$\Big\{(bw, X) \in G \times^P \mathfrak{u} \,\Big|\, b \in B,\, X \in \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha\Big\},$$

where $R = \big\{\alpha \in \Delta_0^+ \,\big|\, \alpha \notin \Delta_P,\, w(\alpha) > 0\big\}$.

Proof. The tangent space of G/P at identity is $\mathfrak{g}/\mathfrak{p}$. Consider the action of P on $\mathfrak{g}/\mathfrak{p}$ induced from the adjoint action of P on $\mathfrak{g}$. The tangent bundle $T\,G/P$ is the fiber bundle over G/P associated to the principal P-bundle $G \to G/P$, for the aforementioned action of P on $\mathfrak{g}/\mathfrak{p}$, i.e., $T\,G/P = G \times^P \mathfrak{g}/\mathfrak{p}$.

Let $R' = \big\{\alpha \in \Delta_0^- \,\big|\, w(\alpha) > 0\big\}$, so that $\Delta_0^+ = \Delta_P^+ \sqcup R \sqcup -R'$. Further, let $U_w = \langle U_\alpha \,|\, \alpha \in R'\rangle$. For any point $b \in B$, we have (see, for example [16]):

$$\begin{aligned} BwP(\operatorname{mod} P) &= bBwP(\operatorname{mod} P) \\ &= b(wU_ww^{-1})wP(\operatorname{mod} P) = bwU_wP(\operatorname{mod} P). \end{aligned}$$

It follows that the tangent subspace at bw of the big cell $BwP(\operatorname{mod} P)$ is given by

$$T_wBwP(\operatorname{mod} P) = \Big\{(bw, X) \in G \times^P \mathfrak{g}/\mathfrak{p} \,\Big|\, X \in \bigoplus_{\alpha \in R'} \mathfrak{g}^\alpha/\mathfrak{p}\Big\},$$

where $\mathfrak{g}^\alpha/\mathfrak{p}$ denotes the image of a root space $\mathfrak{g}^\alpha$ under the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$. Recall that the Killing form identifies the dual of a root space $\mathfrak{g}^\alpha$ with the root space $\mathfrak{g}^{-\alpha}$. Consequently, a root space $\mathfrak{g}^\alpha \subset \mathfrak{u}$ annihilates $T_{bw}BwP(\operatorname{mod} P)$ if and only if $\alpha \in \Delta_0^+ \backslash \Delta_P^+$ and $-\alpha \notin R'$, or equivalently, $\alpha \in R$. The result now follows from the observation that $BwP(\operatorname{mod} P)$ is a dense open subset of $X_P(w)$, and is contained in the smooth locus of $X_P(w)$. $\square$

5.3. Schubert divisors

A Schubert divisor in G/P is a Schubert variety of codimension 1. Let w_0^P be the longest element in W^P. The affine permutation matrix for w_0^P is given by

$$w_0^P = \begin{pmatrix} 0 & 0 & I(\lambda_r) \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ I(\lambda_1) & 0 & 0 \end{pmatrix},$$

where, $I(k)$ denotes the $k \times k$ identity matrix. Codimension one Schubert varieties $X_P(w)$ in G/P correspond to $w = s_k w_0^P$, where $k = n - d_i$ for some $1 \leq i < r$. For $1 \leq k < n$, we define $v_k = \tau_{\alpha_k^\vee} w_0$. The affine permutation matrix of v_k is

$$v_k = \sum_{i=1}^{n} a_i E_{i,n+1-i}, \quad a_i = \begin{cases} t^{-1} & \text{if } i = k, \\ t & \text{if } i = k+1, \\ 1 & \text{otherwise.} \end{cases}$$

We denote by $v_k^{\mathcal{P}}$ the minimal representative of v_k with respect to $\widehat{W}_{\mathcal{P}}$.

Proposition 5.4. *Let $w = s_k w_0$, where $k = n - d_j$ for some $1 \leq j < r$. Then $X_{\mathcal{P}}(v_k^{\mathcal{P}})$ is a compactification of $N^* X_P(w)$ via ϕ_P.*

Proof. We first show that v_k is maximal with respect to W_P. For any $1 \leq i < n$ different from $d_1, \ldots, d_{r-1}$, we have $k \neq n - i$, hence $\alpha_{n-i}(\alpha_k^\vee) \leq 0$. For any such i, it therefore follows from Equation (2.12) that

$$v_k(\alpha_i) = \tau_{\alpha_k^\vee} w_0(\alpha_i) = \tau_{\alpha_k^\vee}(-\alpha_{n-i}) = -\tau_{\alpha_k^\vee}(\alpha_{n-i}) < 0.$$

Consequently, we have $v_k s_i < v_k$ for all $i \in \{1, \ldots, n\} \backslash \{d_1, \ldots, d_{r-1}\}$. We deduce that $v_k^{\mathcal{P}} = v_k w_P$, where w_P denotes the maximal element of W_P.

Next, as an application of Equation (2.15), we have $l(v_k) = \dim G/B$. We compute

$$l(v_k^{\mathcal{P}}) = l(v_k) - l(w_P) = \dim G/B - \dim P = \dim G/P = \dim N^* X_P(w).$$

Since $X_{\mathcal{P}}(\kappa)$ is irreducible, and has the same dimension as $N^* X_P(w)$, it suffices to show that $\phi_P\left(N^* X_P(w)\right) \subset X_{\mathcal{P}}(\kappa)$.

Let L be the bottom left square sub-matrix of w_0^P of size d_{k-1} and L' the top right square sub-matrix of w_0^P of size $(n - d_{k+1})$. We fix a lift $\mathring{w}$ of w to the normalizer of T:

$$\overset{\circ}{w} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & L' \\ 0 & 0 & 0 & I(\lambda_{k+1}-1) & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I(\lambda_k - 1) & 0 & 0 & 0 \\ L & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $I(k)$ denotes the $k \times k$ identity matrix, and $e = \pm 1$ is determined by the equation $\det \overset{\circ}{w} = 1$. Observe that the e is in the $(n - d_k, d_{k-1} + 1)$ position and the 1 in the $(n - d_k + 1, d_{k+1})$ position. Consider the root

$$\gamma := \sum_{d_{i-1} < j < d_{i+1}} \alpha_j .$$

Under the identification of Section 2.2, we have $\gamma = (d_{i-1} + 1, d_{i+1})$. We check that

$$\left\{ \alpha \in \Delta_0^+ \,\middle|\, \alpha \notin \Delta_P,\, w(\alpha) > 0 \right\} = \{\gamma\}\,.$$

In particular, we can write a generic point of $N^*X_P(w)$ as $(b\overset{\circ}{w}, aE_\gamma)$. We may further assume $a \neq 0$. It is now sufficient to show that

$$\phi_P(b\overset{\circ}{w}, aE_\gamma) = b\overset{\circ}{w}\left(1 - at^{-1}E_\gamma\right) \in X_{\mathcal{P}}(v_k^{\mathcal{P}}).$$

Consider $b_1, b_2, b_3 \in \mathcal{B}$ given by

$$b_2 = I(n) + \frac{et}{a} E_{k+1,k}, \quad b_3 = I(n) + \frac{t}{a} E_{d_{i+1}, d_{i-1}+1},$$
$$b_1 = \sum_{1 \le i \le n} c_i E_{ii}, \qquad c_i = \begin{cases} e/a & \text{for } i = k, \\ ea & \text{for } i = k+1, \\ 1 & \text{otherwise.} \end{cases}$$

It is easily verified that $(b_1 b_2 b^{-1}) b\overset{\circ}{w}(1 - t^{-1}aE_\gamma) b_3$ is an affine permutation matrix corresponding to $v_k^{\mathcal{P}} \in \widehat{W}$. $\square$

Corollary 5.5. *Let $X_P(w)$ be a Schubert divisor in G/P. The conormal variety $N^*X_P(w)$ is normal, Cohen–Macaulay, and Frobenius split.*

Proof. Schubert varieties in $LG/\mathcal{P}$ are normal, Cohen–Macaulay, and Frobenius split (cf. [3], [13]). Therefore, the same is true for $N^*X_P(w)$, since it is an open subset of $X_{\mathcal{P}}(v_k^{\mathcal{P}})$. $\square$

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