Transformation Groups
Vol. 24, No. 2, 2019, pp. 301–354

SCHUR–WEYL DUALITY FOR HEISENBERG COSETS

T. CREUTZIG*

Department of
Mathematical and Statistical
Sciences
University of Alberta
Edmonton, Canada T6G2G1
creutzig@ualberta.ca

S. KANADE**

Department of
Mathematical and Statistical
Sciences
University of Alberta
Edmonton, Canada T6G2G1
shashank.kanade@du.edu

A. R. LINSHAW***

Department of
Mathematics
University of Denver
Denver, USA, 80208
andrew.linshaw@du.edu

D. RIDOUT****

School of
Mathematics and Statistics
The University of Melbourne
Parkville, Australia 3010
david.ridout@unimelb.edu.au

Abstract. Let V be a simple vertex operator algebra containing a rank n Heisenberg vertex algebra H and let $\mathsf{C} = \mathrm{Com}\,(\mathsf{H}, \mathsf{V})$ be the coset of H in V. Assuming that the module categories of interest are vertex tensor categories in the sense of Huang, Lepowsky and Zhang, a Schur–Weyl type duality for both simple and indecomposable but reducible modules is proven. Families of vertex algebra extensions of C are found and every simple C-module is shown to be contained in at least one V-module. A corollary of this is that if V is rational, C_2-cofinite and CFT-type, and $\mathrm{Com}\,(\mathsf{C}, \mathsf{V})$ is a rational lattice vertex operator algebra, then C is likewise rational. These results are illustrated with many examples and the C_1-cofiniteness of certain interesting classes of modules is established.

DOI: 10.1007/s00031-018-9497-2
*Supported by the NSERC discovery grant #RES0020460.
** *Current Address:* Department of Mathematics, University of Denver, Denver, USA, 80208.
**Supported by PIMS postdoctoral fellowship.
***Supported by the Simons Foundation Grant #318755.
****Supported by the Australian Research Council Discovery Projects DP1093910 and DP160101520.

Received April 4, 2017. Accepted March 5, 2018.

Published online October 15, 2018.
Corresponding Author: S. Kanade, e-mail: shashank.kanade@du.edu

1. Introduction

Let V be a vertex operator algebra.[1] If $\mathcal{G}$ is a subgroup of the automorphism group of V, then the invariants $\mathsf{V}^{\mathcal{G}}$ form a vertex operator subalgebra called the $\mathcal{G}$-orbifold of V. If W is any vertex operator subalgebra of V, then the W-coset of V is the commutant $\mathsf{C} = \mathrm{Com}(\mathsf{W}, \mathsf{V})$. Both the orbifold and coset constructions provide a way to construct new vertex operator algebras from known ones. Unfortunately, few general results concerning the structure of the resulting vertex operator subalgebras are known, but it is believed that many nice properties of V are inherited by its orbifolds and cosets. We remark that while most of the literature is primarily concerned with semisimple modules of vertex operator algebras, we are also interested in the logarithmic case in which the vertex operator algebra admits indecomposable but reducible modules.

We begin by recalling some important results in the invariant theory of vertex operator algebras that are connected to the questions addressed in this work.

1.1. From classical to vertex-algebraic invariant theory

It is valuable to view invariant-theoretic results about vertex operator algebras as generalisations of the classical results concerning Lie algebras and groups, à la Howe and Weyl [63], [109]. For example, a well-known result of Dong, Li and Mason [50] amounts to a type of Schur–Weyl duality for orbifolds, stating that for a simple vertex operator algebra V and a compact subgroup $\mathcal{G}$ of $\mathrm{Aut}\,\mathsf{V}$ (acting continuously and faithfully), the following decomposition holds as a $\mathcal{G} \times \mathsf{V}^{\mathcal{G}}$-module:

$$\mathsf{V} = \bigoplus_{\lambda} \lambda \otimes \mathsf{V}_{\lambda}.$$

Here, the sum runs over all the simple $\mathcal{G}$-modules λ and is multiplicity-free in the sense that $\mathsf{V}_{\lambda} \ncong \mathsf{V}_{\mu}$ if $\lambda \neq \mu$. They moreover prove that the V_{λ} are simple modules for the orbifold vertex operator algebra $\mathsf{V}^{\mathcal{G}}$. Similar results have also been obtained by Kac and Radul [68] (see Section 2.4).

Invariant theory for the classical groups [109] can be used to obtain generators and relations for orbifold vertex operator algebras $\mathsf{V}^{\mathcal{G}}$, provided that V is of free field type (meaning that the only field appearing in the singular terms of the operator product expansions of the strong generators is the identity field). Interestingly, the relations can be used to show that these vertex operator algebras are strongly finitely generated and, in many cases, explicit minimal strong generating sets can be obtained [87], [86], [88], [89], [90], [34]. Questions concerning cosets are usually more involved than their orbifold counterparts. However, the notion of a deformable family of vertex operator algebras [33] can sometimes be used to reduce the identification of a minimal strong generating set for a coset to a known orbifold problem for a free field algebra [32].

It is of course desirable to understand the representation theory of coset vertex operator algebras. An important first question to ask is if there is also a Schur–Weyl type duality, as in the orbifold case. Let V be a simple vertex operator

[1] We mention that much of this discussion generalises immediately to vertex operator superalgebras. However, we shall generally state results for vertex operator algebras for simplicity, leaving explicit mention of the super-case to exceptions and examples.

algebra that is self-contragredient and let $\mathsf{A}, \mathsf{B} \subseteq \mathsf{V}$ be vertex operator subalgebras satisfying the double commutant condition

$$\mathsf{A} = \mathrm{Com}(\mathsf{B}, \mathsf{V}) \quad \text{and} \quad \mathsf{B} = \mathrm{Com}(\mathsf{A}, \mathsf{V}).$$

Under the further assumption that A and B are both simple, self-contragredient, regular and CFT-type,[2]

$$\mathsf{V} = \bigoplus_i \mathsf{M}_i \otimes \mathsf{N}_i$$

as an $\mathsf{A} \otimes \mathsf{B}$-module, where each M_i is a simple A-module and each N_i is a simple B-module. Under further conditions, Lin finds [84] that this decomposition is multiplicity-free and the argument relies on knowing that the module categories of A and B are both semisimple modular tensor categories.

We are aiming for similar results, but generalised to include decompositions of modules that are not necessarily semisimple. Our setup is that V is a simple vertex operator algebra containing a Heisenberg vertex operator subalgebra H. We then study the commutant $\mathsf{C} = \mathrm{Com}(\mathsf{H}, \mathsf{V})$. For this, we assume that C has a module category $\mathscr{C}$ that is a vertex tensor category in the sense of Huang, Lepowsky and Zhang [67] and that the C-modules obtained upon decomposing V as an $\mathsf{H} \otimes \mathsf{C}$-module belong to $\mathscr{C}$. In Section 2.1, we summarise some known statements about vertex tensor categories that are relevant for our study. These statements make it clear that the C_1-cofiniteness of modules in $\mathscr{C}$ is a key concept. In Section 6, we establish the C_1-cofiniteness of Heisenberg coset modules for two families of examples.

1.2. Rational parafermion vertex operator algebras

Heisenberg cosets of rational affine vertex operator algebras are usually called parafermion vertex operator algebras. They first appeared in the form of the Z-algebras discovered by Lepowsky and Wilson in [76], [77], [78], [79], see also [75]. In physics, parafermions first appeared in the work of Fateev and Zamolodchikov [110] where they were given their standard appellation. The relation between parafermion vertex operator algebras and Z-algebras was subsequently clarified in [49].

Parafermions are surely among the best understood coset vertex operator algebras and there has been substantial recent progress towards establishing a complete picture of their properties. Key results include C_2-cofiniteness [19], see also [48, 54], and rationality [53], using previous results on strong generators [47]. In principle, strong generators can now also be determined as in [32], where this was detailed for the parafermions related to $\mathfrak{sl}_3$. We remark that C_2-cofiniteness also follows from a recent result of Miyamoto on orbifold vertex operator algebras [95]. These powerful results also allow one, for example, to compute fusion coefficients [55].

One of the central open conjectures in vertex operator algebra theory is if a simple rational C_2-cofinite CFT-type self-contragredient vertex operator algebra

[2] We recall that a vertex operator algebra V is said to be (of) CFT-type if its conformal weights are non-negative integers and the zeroth weight space is spanned by vacuum vector.

contains a rational vertex operator subalgebra, as, e.g., a lattice vertex operator algebra (corresponding to an even positive-definite lattice), then the corresponding coset vertex operator algebra will also be rational. This has recently been established for a series of examples in [17]. We prove this statement for cosets by lattice vertex operator algebras in general (see Theorem 4.12).

1.3. Results

This work is, at least in part, a continuation of our previous work on simple current extensions of vertex operator algebras [30]. In this vein, we start by proving some properties of simple currents (Theorem 2.8), in particular that fusing with a simple current defines an autoequivalence of any suitable category of modules. As further preparation, we also prove (Theorem 3.1) that if V is simple, $\mathcal{G}$ is an abelian group of automorphisms acting semisimply on V, and

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L} \subset \widehat{\mathcal{G}}} \mathsf{V}_\lambda, \tag{1.1}$$

then V_λ is a simple current for every λ in $\mathcal{L}$. The proof essentially amounts to adding details to a very similar result of Miyamoto [93, Sect. 6], see also [26].

Schur–Weyl duality. We next prove a Schur–Weyl duality for Heisenberg cosets $\mathsf{C} = \mathrm{Com}(\mathsf{H}, \mathsf{V})$. The setup is as follows. Let V be a simple vertex operator algebra, $\mathsf{H} \subseteq \mathsf{V}$ be a Heisenberg vertex operator subalgebra that acts semisimply on V, C be the commutant of H in V, and $\mathcal{L}$ be the lattice of Heisenberg weights of V (V being regarded here as an H-module). Then $\mathsf{W} = \mathrm{Com}(\mathsf{C}, \mathsf{V})$ is an extension of H by an abelian intertwining algebra. Of course, it might happen that this extension is trivial, that is, that $\mathsf{H} = \mathsf{W}$. In any case, Equation (1.1) translates into

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda. \tag{1.2}$$

Let $\mathcal{N}$ be the sublattice of all $\lambda \in \mathcal{L}$ for which $\mathsf{C}_\lambda \cong \mathsf{C}_0 = \mathsf{C}$. Theorem 3.5 now says that the abelian group $\mathcal{L}/\mathcal{N}$ controls the decomposition of V as a $\mathsf{W} \otimes \mathsf{C}$-module:

$$\mathsf{V} = \bigoplus_{[\lambda] \in \mathcal{L}/\mathcal{N}} \mathsf{W}_{[\lambda]} \otimes \mathsf{C}_{[\lambda]}. \tag{1.3}$$

Moreover, the $\mathsf{C}_{[\lambda]}$, $\lambda \in \mathcal{L}/\mathcal{N}$, are simple currents for C whose fusion products include

$$\mathsf{C}_{[\lambda]} \boxtimes_{\mathsf{C}} \mathsf{C}_{[\mu]} = \mathsf{C}_{[\lambda+\mu]}.$$

This decomposition is multiplicity free in the sense that $\mathsf{C}_{[\lambda]} \ncong \mathsf{C}_{[\mu]}$ if $[\lambda] \neq [\mu]$. The vertex operator algebra

$$\mathsf{W} = \bigoplus_{\lambda \in \mathcal{N}} \mathsf{F}_\lambda$$

is a simple current extension of H and the $\mathsf{W}_{[\lambda]}$, $[\lambda] \in \mathcal{L}/\mathcal{N}$, are simple currents for W with fusion products $\mathsf{W}_{[\lambda]} \boxtimes_{\mathsf{W}} \mathsf{W}_{[\mu]} = \mathsf{W}_{[\lambda+\mu]}$. We note that Li has proven [80] that $\bigoplus_{\lambda \in \mathcal{L}/\mathcal{N}} \mathsf{C}_{[\lambda]}$ is a generalised vertex algebra.

The main Schur–Weyl duality result is then a similar decomposition for vertex operator algebra modules, see Theorem 3.8. For this, let V, H, C, W, $\mathcal{L}$ and $\mathcal{N}$ be as above and let M be a V-module upon which H acts semisimply. Then, M decomposes as

$$\mathsf{M} = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}_\mu = \bigoplus_{[\mu] \in \mathcal{M}/\mathcal{N}} \mathsf{W}_{[\mu]} \otimes \mathsf{D}_{[\mu]}, \tag{1.4}$$

where $\mathcal{M}$ is a union of $\mathcal{L}$-orbits and the $\mathsf{D}_\mu = \mathsf{D}_{[\mu]}$ are C-modules satisfying $\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{D}_\mu = \mathsf{D}_{\lambda+\mu}$ for all $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$. We moreover show that each of the D_μ has the same decomposition structure as that of M. One example of this is if $0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M}'' \to 0$ is exact, with M' and M'' non-zero, then M' and M'' decompose as in (1.4):

$$\mathsf{M}' = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}'_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}'_\mu, \quad \mathsf{M}'' = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}''_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}''_\mu.$$

Moreover, $0 \to \mathsf{D}'_\mu \to \mathsf{D}_\mu \to \mathsf{D}''_\mu \to 0$ is also exact, with D'_μ and D''_μ non-zero, for all $\mu \in \mathcal{M}$.

However, these module decompositions need not be multiplicity-free in general. For example, the parafermion coset of $\mathsf{L}_2(\mathfrak{sl}_2)$ yields an example of a coset module that appears twice in the decomposition of a simple $\mathsf{L}_2(\mathfrak{sl}_2)$-module. We give three criteria to guarantee that a given decomposition is multiplicity-free — one based on characters, one based on the signature of the lattice $\mathcal{L}$, and one based on open Hopf link invariants following [28], [27]. Most of these statements also hold if we replace V by a vertex operator superalgebra.

Extensions of vertex operator algebras. Let $\mathcal{E}$ be a sublattice of $\mathcal{L}$. We would like to know if

$$\mathsf{C}_{\mathcal{E}} = \bigoplus_{\lambda \in \mathcal{E}} \mathsf{C}_\lambda$$

carries the structure of a vertex operator algebra extending that of $\mathsf{C} = \mathsf{C}_0$. Theorem 4.1, which itself follows immediately from [81], implies that this is the case provided that

$$\mathsf{W}_{\mathcal{E}} = \bigoplus_{\lambda \in \mathcal{E}} \mathsf{W}_\lambda$$

is a vertex operator algebra. If $\mathcal{E}$ is a rank one subgroup, then this conclusion also follows from [30].

Lifting modules. Let D be a C-module. We would like to know if D lifts to a $\mathsf{C}_{\mathcal{E}}$-module and also if there exists a H-module F_β such that $\mathsf{F}_\beta \otimes \mathsf{D}$ lifts to a V-module. This question is decided by the monodromy (composition of braidings)

$$M_{\mathsf{C}_\lambda, \mathsf{D}} \colon \mathsf{C}_\lambda \boxtimes \mathsf{D} \to \mathsf{C}_\lambda \boxtimes \mathsf{D}.$$

We have the following result (Theorem 4.3): Let D be a generalised C-module that appears as a subquotient of the fusion product of some finite collection of simple

C-modules. Let $\mathcal{L}'$ be the dual lattice of $\mathcal{L}$ and let $U = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$. Then, there exists $\alpha \in U$ such that

$$M_{\mathsf{C}_\lambda,\mathsf{D}} = \mathsf{e}^{-2\pi \mathsf{i} \langle \alpha, \lambda \rangle} \, \mathrm{Id}_{\mathsf{C}_\lambda \boxtimes \mathsf{D}}$$

and $\mathsf{F}_\beta \otimes \mathsf{D}$ lifts to a V-module if and only if $\beta \in \alpha + \mathcal{L}'$. Moreover, the lifted module is $\mathsf{V} \boxtimes_{\mathsf{H} \otimes \mathsf{C}} (\mathsf{F}_\beta \otimes \mathsf{D})$. Note that the lifting problem, assuming that all involved vertex operator algebras are regular, was treated in [73].

We also show that D lifts to a $\mathsf{C}_{\mathcal{E}}$-module if and only if α is in a certain lattice associated to $\mathcal{E}$ (see Theorem 4.4) and that every $\mathsf{C}_{\mathcal{E}}$-module is a quotient of a lifted module (this follows essentially from [74]). The lifted module is then $\mathsf{C}_{\mathcal{E}} \boxtimes_{\mathsf{C}} \mathsf{D}$.

Rationality. Miyamoto [95] has proven that C is C_2-cofinite provided that W is the lattice vertex operator algebra of a positive definite even lattice and V is C_2-cofinite. Together with our lifting results and the exactness of fusion with simple currents, this implies a rationality theorem (Theorem 4.12): If V is simple, rational, C_2-cofinite and CFT-type, then every grading-restricted generalised C-module is semisimple. In particular, we thereby obtain an alternative proof of the rationality of the parafermion cosets [53], [26] as well as of the Heisenberg cosets of the rational Bershadsky–Polyakov algebras [16].

Examples. Our results rely on the applicability of the vertex tensor theory of Huang, Lepowsky and Zhang [67]. It is in general very difficult to verify this beyond C_2-cofinite vertex operator algebras. We remark that this has recently been done successfully for the category of ordinary modules of affine vertex operator algebras at admissible level [29] and it is work in progress to study Heisenberg cosets of affine vertex operator superalgebras of type $\mathfrak{sl}(2|1)$ that are extensions of affine vertex operator algebras at admissible level times certain rational vertex operator algebras. We also note that Theorem 5 of [44] and Example 4.3 of [21] give examples where the Heisenberg coset is C_2-cofinite and non-rational.

We illustrate our results with various examples, both rational and non-rational, though our main interest is applications to the vertex operator algebras of logarithmic conformal field theory, that is, to indecomposable but reducible modules. Schur–Weyl duality is exemplified in the well-known rational example of $\mathsf{L}_2(\mathfrak{sl}_2)$ (Example 1) and then, in much detail, for the case of $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ (Example 2). We explain how Schur–Weyl duality works for the (conjectured) projective covers of the simple modules. Extensions of the Heisenberg cosets of $\mathsf{L}_k(\mathfrak{g})$ for rational and non-zero k are discussed in Example 3.

Example 4 then deals with the relation via Heisenberg cosets of various archetypal logarithmic vertex operator algebras, most notably the singlet algebra $\mathsf{I}(2)$ and the affine vertex operator superalgebra $\mathsf{V}_k(\mathfrak{gl}(1|1))$. In particular, we give the decomposition of the projective indecomposable modules of the latter in terms of projective $\mathsf{H} \otimes \mathsf{I}(2)$-modules. The triplet algebra $\mathsf{W}(2)$ is then an example of an extended vertex operator algebra that is C_2-cofinite.

The lifting of modules is illustrated in Example 5 for the modules of the $N = 2$ vertex operator superalgebra. Finally, we use the opportunity to prove that $\mathsf{L}_{-1}(\mathfrak{sl}(m|n))$ appears as a Heisenberg coset of an appropriate tensor product of $\beta\gamma$- and bc-ghost vertex operator superalgebras. This generalises the case $n = 0$

of [10]. We mention that the case $m = 2$ and $n = 0$ is exceptional and is identified with a rectangular W-algebra of $\mathfrak{sl}_4$.

On C_1-cofiniteness. Our results rely on the applicability of the vertex tensor theory of Huang, Lepowsky and Zhang [67]. Our belief is that the key criterion for this applicability is the C_1-cofiniteness of the modules with finite composition length, see also [38, Sect. 6]. In Section 6, we prove a few C_1-cofiniteness results for modules of Heisenberg cosets of the affine vertex operator algebras of type $\mathfrak{sl}_2$ as well as those of the Bershadsky–Polyakov algebras.

Outlook on fusion. The main concern of this work is the relationship between the modules of the Heisenberg coset vertex operator algebra C and those of its parent algebra V. A valid question is then if there is also a clear relation between the fusion product of the C-modules and the corresponding V-modules. One can prove that the induction functor is a tensor functor under appropriate assumptions on the module category [31]. Modulo these assumptions, this rigorously establishes the connection between fusion and extended algebras that has been proposed in the physics literature [102].

1.4. Application: Towards new C_2-cofinite logarithmic vertex operator algebras

Presently, there are very few known examples of C_2-cofinite non-rational vertex operator algebras; these include the triplet algebras [7], [106], [107] and their close relatives [1]. In order to gain more experience with such logarithmic C_2-cofinite vertex operator algebras, new examples are needed. The main application we have in mind for the work reported here is the construction of new examples of this type.

The idea is a two-step process illustrated as follows:

$$\mathsf{V} \xrightarrow{\text{H-coset}} \mathsf{C} \xrightarrow{\text{extension}} \mathsf{C}_{\mathcal{E}}.$$

A series of examples that confirms this idea was explored in [44], see also Example 3. There, the $\mathsf{I}(p)$ singlet algebras of Kausch [72] were (conjecturally) obtained as Heisenberg cosets of the Feigin–Semikhatov algebras [57], see also [61]. The extension in the above process is then an infinite order simple current extension and the results [36], [100] are the best understood C_2-cofinite logarithmic vertex operator algebras, the $\mathsf{W}(p)$ triplet algebras.

New examples may be obtained by taking V to be the simple affine vertex operator algebra associated to the simple Lie algebra $\mathfrak{g}$ at admissible, but negative, level k and H to be the Heisenberg vertex operator subalgebra generated by the affine fields associated to the Cartan subalgebra of $\mathfrak{g}$. The module categories of such admissible level affine vertex operator algebras remain quite mysterious despite strong results concerning category $\mathscr{O}$ [69], [14]. Beyond category $\mathscr{O}$, detailed results are currently only known for $\mathfrak{g} = \mathfrak{sl}_2$, see [6], [59], [97], [98], [99], [39], [41], [101], and $\mathfrak{g} = \mathfrak{sl}_3$, see [18]. A first feasible task here would be to compute the characters of the coset modules that appear in the decomposition of modules in category $\mathscr{O}$. We expect the appearance of Kostant false theta functions [37] as they are the natural generalisation of the ordinary false theta functions that appear in the case of the admissible level parafermion coset of $\mathsf{L}_k(\mathfrak{sl}_2)$ [21].

In [21], we will study $\mathsf{C}_{\mathcal{E}}$ for $\mathfrak{g} = \mathfrak{sl}_2$ and k negative and admissible. Under the assumption that the tensor theory of Huang–Lepowsky–Zhang applies to C, we shall prove that there are only finitely many inequivalent simple $\mathsf{C}_{\mathcal{E}}$-modules. It is thus natural to conjecture that $\mathsf{C}_{\mathcal{E}}$ is C_2-cofinite. A consequence of C_2-cofiniteness is the modularity of characters (supplemented by pseudotrace functions) [92]. In [21], we shall also demonstrate this modularity of characters (plus pseudotraces) for all modules that are lifts of C-modules. We will prove the C_2-cofiniteness of $\mathsf{C}_{\mathcal{E}}$, for various choices of $\mathcal{E}$, in subsequent works.

A third family of examples that fit this idea concerns simple minimal W-algebras in the sense of Kac and Wakimoto [71]. These are quantum Hamiltonian reductions that are strongly generated by fields in conformal dimension 1 and 3/2, together with the Virasoro field. For certain levels, these W-algebras have a one-dimensional associated variety and they contain a rational affine vertex operator subalgebra. The Heisenberg coset of the coset of the minimal W-algebra by the rational affine vertex operator algebra thus seems to be another candidate for new C_2-cofinite algebras as infinite order simple current extensions. These cosets are explored in [15].

1.5. Organisation

We start with a background section. There, we review the vertex tensor theory of Huang, Lepowsky and Zhang and discuss it in the case of the Heisenberg vertex operator algebra. Next, we prove various properties of simple currents and then discuss vertex operator algebra orbifolds following Kac and Radul. Section 3 then details our results on Schur–Weyl duality for Heisenberg cosets. Section 4 is concerned with extended algebras, lifting of modules and, as a special application, proves our rationality theorem. In Section 5, we give a short proof that $\mathsf{L}_{-1}(\mathfrak{sl}(m|n))$ is a Heisenberg coset of an appropriate ghost vertex operator superalgebra. In Section 6, we prove the C_1-cofiniteness of the modules that appear in the Heisenberg cosets of the Bershadsky–Polyakov algebras and $\mathsf{L}_k(\mathfrak{sl}_2)$.

Acknowledgements. T. C. and S. K. would like to thank Yi-Zhi Huang and Robert McRae for helpful discussions regarding vertex tensor categories, [67]. T. C. also thanks Antun Milas for discussions on the applicability of the theory of vertex tensor categories.

2. Background

In this section, we give a brief exposition of the results of Huang, Lepowsky and Zhang regarding the vertex tensor categories that we shall use. We mention the case of Heisenberg vertex operator algebras separately in detail. Then, we present our new results regarding properties of simple currents under fusion. After that, we review a useful result of Kac and Radul on the simplicity of orbifold models.

2.1. Conditions and assumptions regarding the theory of Huang–Lepowsky–Zhang

We begin with a quick glossary of the terminology that we shall use.

- By a *generalised* module of a vertex operator algebra, we shall mean a module that is graded by generalised eigenvalues of L_0. A generalised module need not satisfy any of the other restrictions mentioned below regarding grading. For $n \in \mathbb{C}$ and a generalised module W, we let $\mathsf{W}_{[n]}$ denote the generalised L_0-eigenspace of eigenvalue n.
- A generalised module W is called *lower truncated* if $\mathsf{W}_{[n]} = 0$ whenever the real part of n is sufficiently negative.
- A generalised module W is called *grading-restricted* if it is lower truncated and if, moreover, for all n, $\dim(\mathsf{W}_{[n]}) < \infty$.
- A generalised module W is called *strongly graded* if $\dim(\mathsf{W}_{[n]}) < \infty$ and, for each $n \in \mathbb{C}$, $\mathsf{W}_{[n+k]} = 0$ for all sufficiently negative integers k. This notion is slightly more general than that of being grading-restricted.
- In the definitions above, we shall replace the qualifier "generalised" with "ordinary" if the module is graded by eigenvalues of L_0 as opposed to generalised eigenvalues.
- Henceforth, by "module", without qualifiers, we shall mean a grading-restricted generalised module. For convenience in the applications to follow, we shall also assume that every vertex operator algebra module is of at most countable dimension. This implies, of course, that the dimension of all vertex operator algebras will also be at most countable.
- We will sometimes need broader analogues of the concepts above, wherein the restrictions pertain to doubly-homogeneous spaces with respect to Heisenberg zero modes and L_0. The actual statements in [67] pertain to such broader situations. However, the theorems in [65], that guarantee that [67] may be applied in specific scenarios, assume the definitions that we have recalled above. We expect that the theorems and concepts in [65] may be generalised to the broader setting that we require.

Recall the notion [67, Def. 3.10] of a (*logarithmic*) *intertwining operator* among a triple of modules. When the formal variable in a logarithmic intertwining operator is carefully specialised to a fixed $z \in \mathbb{C}^\times$, one gets the notion of a $P(z)$-*intertwining map*, [67, Def. 4.2]. These maps form the backbone of the logarithmic tensor category theory developed in [67]. There, tensor products (fusion products) of modules are defined via certain universal $P(z)$-intertwining maps $\boxtimes_{P(z)}$ and the monoidal structure on the module category is obtained by fixing $z \in \mathbb{C}^\times$, typically chosen to be $z = 1$ for convenience.[3] We remark that the products $\boxtimes_{P(z)}$, for different values of z, together form a structure richer than that of a braided monoidal category, called a *vertex tensor category*. This richer structure is exploited in the proofs of many important theorems, see [66] for some examples.

For convenience, and especially with a view towards the proof of Theorem 3.3 below, we give a definition of the fusion product of two modules, equivalent to that of [67], using intertwining operators instead of intertwining maps.

[3]We mention that the same notation is generally used to denote both the fusion product operation and the universal $P(z)$-intertwining map corresponding to said fusion product.

Definition 2.1. Given modules W_1 and W_2, the *fusion product* $\mathsf{W}_1 \boxtimes \mathsf{W}_2$ is the pair $(\mathsf{W}_1 \boxtimes \mathsf{W}_2, \mathcal{Y}^{\boxtimes})$, where $\mathsf{W}_1 \boxtimes \mathsf{W}_2$ is a module and $\mathcal{Y}^{\boxtimes}$ is an intertwining operator of type $\binom{\mathsf{W}_1 \boxtimes \mathsf{W}_2}{\mathsf{W}_1 \ \mathsf{W}_2}$, that satisfies the following universal property: Given any other "test module" W and an intertwining operator $\mathcal{Y}$ of type $\binom{\mathsf{W}}{\mathsf{W}_1 \ \mathsf{W}_2}$, there exists a *unique* morphism $\eta \colon \mathsf{W}_1 \boxtimes \mathsf{W}_2 \to \mathsf{W}$ such that $\mathcal{Y} = \eta \circ \mathcal{Y}^{\boxtimes}$.

Note that the universal intertwining operator $\mathcal{Y}^{\boxtimes}$ will often be clear from the context and hence we shall often refer to the fusion product by its underlying module.

Below, we shall need the following property of the universal intertwining operators:

Lemma 2.2 ([67, Prop. 4.23])**.** *The universal intertwining operator $\mathcal{Y}^{\boxtimes}$ is surjective, in the sense that the linear span of its expansion coefficients equals $\mathsf{W}_1 \boxtimes \mathsf{W}_2$.*

Now, let V be a vertex operator algebra and let $\mathscr{C}$ be a category of generalised V-modules that satisfies the following properties:

(1) $\mathscr{C}$ is a full abelian subcategory of the category of all strongly graded generalised V-modules.
(2) $\mathscr{C}$ is closed under taking contragredient duals and the $P(z)$-tensor product $\boxtimes_{P(z)}$ (recall [67, Def. 4.15]).
(3) V is itself an object of $\mathscr{C}$.
(4) For each object W of $\mathscr{C}$, the (generalised) L_0-eigenvalues are real and the size of the Jordan blocks of L_0 is bounded above (the bound may depend on W).
(5) Assumption 12.2 of [67] holds.

A precise formulation of (5) may be found in [67]. In essence, this assumption guarantees the convergence of products and iterates of intertwining operators in a specific class of multivalued analytic functions. It, moreover, guarantees that products of intertwining operators can be written as iterates and vice versa.

Theorem 2.3 ([67, Thm. 12.15, Cor. 12.16])**.** *Under these conditions, the category $\mathscr{C}$ equipped with the tensor product bifunctor $\boxtimes = \boxtimes_{P(1)}$ is naturally a braided monoidal category.*

We shall not need an explicit description — an easily accessible account of which may be found in [67, Sect. 12], [66], [31, Sect. 3.3] — of the associativity, unit and braiding isomorphisms required to specify the braided monoidal category structure of Theorem 2.3.

We shall require the following fundamental property of $\boxtimes$:

Lemma 2.4 ([67, Prop. 4.26])**.** *For any $\mathsf{W} \in \mathscr{C}$, the functors $\mathsf{W} \boxtimes -$ and $- \boxtimes \mathsf{W}$ are right-exact.*

The condition 5 is quite technical; the following theorem provides situations in which it holds.

Theorem 2.5 ([65])**.** *Let V be a vertex operator algebra satisfying the following conditions:*

- V *is* $C_1^{\text{alg.}}$*-cofinite, meaning that the space spanned by*

$$\left\{\operatorname{Res}_z z^{-1}Y(u,z)v \,\middle|\, u,v \in \mathsf{V}_{[n]} \text{ with } n>0\right\} \cup L_{-1}\mathsf{V}$$

  has finite codimension in V*.*
- *There exists a positive integer* N *that bounds the differences between the real parts of the lowest conformal weights of the simple* V*-modules and is such that the* N*-th Zhu algebra* $A_N(\mathsf{V})$ *(see* [52]*) is finite-dimensional.*
- *Every simple* V*-module is* $\mathbb{R}$*-graded and* C_1*-cofinite.*

Then, the category of grading-restricted generalised modules of V *satisfies the conditions 1–5 given above, hence is a vertex tensor category.*

If V *is* C_2*-cofinite, has no states of negative conformal weight, and the space of conformal weight* 0 *states is spanned by vacuum, then these conditions are satisfied and so the theory of vertex tensor categories may be applied to the grading-restricted generalised* V*-modules.*

As is amply clear from Theorem 2.5, [94] and [67, Rem. 12.3], C_1-cofiniteness already takes us a long way towards establishing that a given category of V-modules is a vertex tensor category. Our hope is that, in the future, C_1-cofiniteness will be, along with other minor conditions (such as conditions on the eigenvalues and Jordan blocks of L_0), essentially enough to invoke the theory developed by Huang, Lepowsky and Zhang. With this hope in mind, we shall prove several useful C_1-cofiniteness results in Section 6.

We would also like to remark that there are still many examples of vertex operator algebras, some quite fundamental, which do not meet the known conditions that guarantee the applicability of the vertex tensor theory of [67]. It is an important problem to analyse the module categories of these examples and bring them "into the fold", as it were. Not only will this make the theory more wide-reaching, but we expect that accommodating these new examples will lead to further crucial insights into the true nature of vertex operator algebra module categories.

2.2. Vertex tensor categories for the Heisenberg algebra

For Heisenberg vertex operator algebras, there exist simple modules with non-real conformal weights and, therefore, one can not invoke Theorem 2.5. In this section, we shall deal with general Heisenberg vertex operator algebras, bypassing Theorem 2.5 and instead relying (mostly) on the results in [49]. For related discussions, including self-extensions of simple modules (which are not relevant for our purposes), see [91], [38], [103].

We shall verify that a certain semisimple category $\mathscr{C}_{\mathbb{R}}$ of modules with real conformal weights (see 2.2 below) is closed under fusion and satisfies the associativity requirements for intertwining operators, by invoking results in [49]. Once this is done, it is straightforward to verify that $\mathscr{C}_{\mathbb{R}}$ satisfies the assumptions for being vertex tensor category as in [67, Sect. 12].

Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra over $\mathbb{C}$, equipped with a symmetric non-degenerate bilinear form $\langle \cdot\,,\cdot\rangle$. We shall identify $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via this form. As in [82, Chap. 6], let $\widehat{\mathfrak{h}}$ denote the Heisenberg Lie algebra and H

the corresponding Heisenberg vertex operator algebra (of level 1, for convenience). Given $\alpha \in \mathfrak{h}$, we denote the (simple) Fock module of H, with highest weight $\lambda \in \mathfrak{h}$, by F_λ. It is known (see [77]), as an algebraic analogue of the Stone-von Neumann theorem, that these simple Fock modules exhaust the isomorphism classes of the simple H-modules. Let $\mathscr{C}$ be the semisimple abelian category of H modules generated by these simple H-modules and let $\mathscr{C}_{\mathbb{R}}$ be the full subcategory generated by the Fock modules with real highest weights.

Theorem 2.6. *The subcategory $\mathscr{C}_{\mathbb{R}}$ can be given the structure of a vertex tensor category.*

Proof. The proof splits into the following steps. Let $\lambda, \mu, \nu \in \mathfrak{h} = \mathfrak{h}^*$.

(1) Using [49, Eq. (12.10)], the fusion coefficient $\binom{\mathsf{W}}{\mathsf{F}_\mu \ \mathsf{F}_\nu}$ is zero if W does not have $\mathsf{F}_{\mu+\nu}$ as a direct summand.

(2) Proceeding exactly as in [49, Lem. 12.6–Prop. 12.8], we see that the fusion coefficient $\binom{\mathsf{F}_{\mu+\nu}}{\mathsf{F}_\mu \ \mathsf{F}_\nu}$ is either 0 or 1.

(3) Let $\mathcal{L}$ be the lattice spanned by μ and ν. One can check that the (generalised) lattice vertex operator algebra $\mathsf{V}_{\mathcal{L}}$ satisfies the Jacobi identity given in [49, Thm. 5.1], even though $\mathcal{L}$ is not necessarily rational. This implies that the vertex map Y of $\mathsf{V}_{\mathcal{L}}$ furnishes explicit (non-zero) intertwining operators of type $\binom{\mathsf{F}_{\mu+\nu}}{\mathsf{F}_\mu \ \mathsf{F}_\nu}$, thereby implying that the fusion coefficient $\binom{\mathsf{F}_{\mu+\nu}}{\mathsf{F}_\mu \ \mathsf{F}_\nu}$ is always 1.

(4) We conclude that $\mathscr{C}$ is closed under $\boxtimes_{P(z)}$ (recall [67, Def. 4.15]). In general, if $\mathcal{M}$ is a subgroup of $\mathfrak{h}$, regarded as an additive abelian group, and if $\mathscr{C}'$ is the semisimple category generated by the Fock modules with highest weights in $\mathcal{M}$, then $\mathscr{C}'$ is closed under $\boxtimes_{P(z)}$. In particular, the subcategory $\mathscr{C}_{\mathbb{R}}$ is closed under $\boxtimes_{P(z)}$.

(5) Given $\mu_1, \dots, \mu_j \in \mathfrak{h}_{\mathbb{R}}$, let $\mathcal{L}$ be the lattice that they span. Then, $\mathsf{V}_{\mathcal{L}}$ again satisfies the Jacobi identity [49, Thm. 5.1] and the duality results of [49, Chap. 7] also go through. As a consequence, the expected convergence and associativity properties of intertwining operators among Fock modules in $\mathscr{C}_{\mathbb{R}}$ hold.

(6) Since the conformal weights of all modules in $\mathscr{C}_{\mathbb{R}}$ are real, the associativity of the intertwining operators yields a natural associativity isomorphism for $\mathscr{C}_{\mathbb{R}}$ as detailed in [67, Sect. 12.2].

(7) Finally, one can proceed as in [67, Sect. 12.4] to verify the remaining properties satisfied by the braiding and associativity isomorphisms. Thus, $\mathscr{C}_{\mathbb{R}}$ forms a vertex tensor category in the sense of Huang–Lepowsky and, in particular, is a braided tensor category. □

2.3. Simple currents

An important concept in the theory of vertex operator algebras is the simple current extension, wherein a given algebra V is embedded in a larger one W that is constructed from certain V-modules called simple currents. The utility of this construction is that, unlike general embeddings, the representation theories of V and W are very closely related.

Definition 2.7. A *simple current* J of a vertex operator algebra V is a V-module that possesses a fusion inverse: $\mathsf{J} \boxtimes \mathsf{J}^{-1} \cong \mathsf{V} \cong \mathsf{J}^{-1} \boxtimes \mathsf{J}$.

Simple currents and simple current extensions were introduced by Schellekens and Yankielowicz in [105]. We note that more general definitions of a simple current exist, see [51] for example, but that the one adopted above will suffice for the vertex operator algebras treated below. Pertinent examples of simple currents are the Heisenberg Fock modules F_λ discussed in Section 2.2: the fusion inverse of F_λ is $\mathsf{F}_{-\lambda}$.

The great advantage of requiring invertibility is that each simple current J gives rise to a functor $\mathsf{J} \boxtimes -$ which is an autoequivalence of any V-module category that is closed under $\boxtimes$. The following theorem gives some consequences of this; we provide proofs in order to prepare for the similar, but more subtle arguments of the next section.

Proposition 2.8. *Let J be a simple current of a vertex operator algebra V.*

(1) *If M is a non-zero V-module, then $\mathsf{J} \boxtimes \mathsf{M}$ is non-zero.*
(2) *If M is an indecomposable V-module, then $\mathsf{J} \boxtimes \mathsf{M}$ is indecomposable.*
(3) *If M is a simple V-module, then $\mathsf{J} \boxtimes \mathsf{M}$ is simple. In particular, J is simple if V is.*
(4) *The covariant functor $\mathsf{J} \boxtimes -$ is exact (hence, so is $- \boxtimes \mathsf{J}$).*
(5) *If M has a composition series with composition factors S_i, $1 \le i \le n$, then $\mathsf{J} \boxtimes \mathsf{M}$ has a composition series with composition factors $\mathsf{J} \boxtimes \mathsf{S}_i$, $1 \le i \le n$.*
(6) *If M has a radical or socle, then so does $\mathsf{J} \boxtimes \mathsf{M}$. Moreover, the latter's radical or socle is then given by $\mathsf{J} \boxtimes \operatorname{rad} \mathsf{M} \cong \operatorname{rad}(\mathsf{J} \boxtimes \mathsf{M})$ or $\mathsf{J} \boxtimes \operatorname{soc} \mathsf{M} \cong \operatorname{soc}(\mathsf{J} \boxtimes \mathsf{M})$.*
(7) *If M has a radical or socle series, then so does $\mathsf{J} \boxtimes \mathsf{M}$. In particular, the corresponding Loewy diagrams of $\mathsf{J} \boxtimes \mathsf{M}$ are obtained by replacing each composition factor S_i of M by $\mathsf{J} \boxtimes \mathsf{S}_i$.*

Proof. If $\mathsf{J} \boxtimes \mathsf{M} = 0$, then $0 = \mathsf{J}^{-1} \boxtimes \mathsf{J} \boxtimes \mathsf{M} \cong \mathsf{V} \boxtimes \mathsf{M} \cong \mathsf{M}$. Thus, (1) follows:

$$\mathsf{M} \neq 0 \quad \Rightarrow \quad \mathsf{J} \boxtimes \mathsf{M} \neq 0. \tag{2.1}$$

Similarly, if $\mathsf{J} \boxtimes \mathsf{M} \cong \mathsf{M}' \oplus \mathsf{M}''$, then $\mathsf{M} \cong \mathsf{J}^{-1} \boxtimes \mathsf{J} \boxtimes \mathsf{M} \cong (\mathsf{J}^{-1} \boxtimes \mathsf{M}') \oplus (\mathsf{J}^{-1} \boxtimes \mathsf{M}'')$. In other words, M indecomposable implies that $\mathsf{J} \boxtimes \mathsf{M}$ is indecomposable, which is (2).

Suppose now that M is simple, but that $\mathsf{J} \boxtimes \mathsf{M}$ has a proper submodule M'. Then,

$$0 \to \mathsf{M}' \to \mathsf{J} \boxtimes \mathsf{M} \to \mathsf{M}'' \to 0$$

is exact, for $\mathsf{M}'' \cong (\mathsf{J} \boxtimes \mathsf{M})/\mathsf{M}' \neq 0$. But, fusion is right-exact as recalled in Theorem 2.4, so

$$\mathsf{J}^{-1} \boxtimes \mathsf{M}' \to \mathsf{M} \to \mathsf{J}^{-1} \boxtimes \mathsf{M}'' \to 0$$

is exact. However, $\mathsf{M}'' \neq 0$ implies that $\mathsf{J}^{-1} \boxtimes \mathsf{M}''$ is a non-zero quotient of M, by 1, so we must have $\mathsf{J}^{-1} \boxtimes \mathsf{M}'' \cong \mathsf{M}$, as M is simple. Fusing with J now gives $\mathsf{J} \boxtimes \mathsf{M} \cong \mathsf{M}''$, so we conclude that $\mathsf{M}' = 0$ and that $\mathsf{J} \boxtimes \mathsf{M}$ is simple. The simplicity of $\mathsf{J} \cong \mathsf{J} \boxtimes \mathsf{V}$ now follows from that of V, completing the proof of (3).

To prove (4), note that applying right-exactness to the short exact sequence $0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M}'' \to 0$ results in

$$\mathsf{J} \boxtimes \frac{\mathsf{M}}{\mathsf{M}'} \cong \frac{\mathsf{J} \boxtimes \mathsf{M}}{(\mathsf{J} \boxtimes \mathsf{M}')/\ker f}, \tag{2.2}$$

where f is the induced map from $\mathsf{J} \boxtimes \mathsf{M}'$ to $\mathsf{J} \boxtimes \mathsf{M}$ that might not be an inclusion. Fusing with J^{-1} and applying (2.2), we arrive at

$$\frac{\mathsf{M}}{\mathsf{M}'} \cong \mathsf{J}^{-1} \boxtimes \frac{\mathsf{J} \boxtimes \mathsf{M}}{(\mathsf{J} \boxtimes \mathsf{M}')/\ker f} \cong \frac{\mathsf{M}}{\Big(\mathsf{J}^{-1} \boxtimes \dfrac{\mathsf{J} \boxtimes \mathsf{M}'}{\ker f}\Big)/\ker g},$$

where $g \colon \mathsf{J}^{-1} \boxtimes \big((\mathsf{J} \boxtimes \mathsf{M}')/\ker f\big) \to \mathsf{M}$ might not be an inclusion. Thus,

$$\mathsf{M}' \cong \frac{\mathsf{J}^{-1} \boxtimes \dfrac{\mathsf{J} \boxtimes \mathsf{M}'}{\ker f}}{\ker g} \cong \frac{\dfrac{\mathsf{M}'}{(\mathsf{J}^{-1} \boxtimes \ker f)/\ker h}}{\ker g},$$

where $h \colon \mathsf{J}^{-1} \boxtimes \ker f \to \mathsf{M}'$ might not be an inclusion. We conclude that $\ker g = 0$ and $\ker h = \mathsf{J}^{-1} \boxtimes \ker f$. But, both require that

$$\mathsf{M}' \cong \mathsf{J}^{-1} \boxtimes \frac{\mathsf{J} \boxtimes \mathsf{M}'}{\ker f} \quad \Rightarrow \quad \mathsf{J} \boxtimes \mathsf{M}' \cong \frac{\mathsf{J} \boxtimes \mathsf{M}'}{\ker f} \quad \Rightarrow \quad \ker f = 0.$$

$f \colon \mathsf{J} \boxtimes \mathsf{M}' \to \mathsf{J} \boxtimes \mathsf{M}$ is therefore an inclusion, hence $\mathsf{J} \boxtimes -$ is exact.

Suppose now that $0 = \mathsf{M}_0 \subset \mathsf{M}_1 \subset \cdots \subset \mathsf{M}_{n-1} \subset \mathsf{M}_n = \mathsf{M}$ is a composition series for M, so that each $\mathsf{S}_i = \mathsf{M}_i/\mathsf{M}_{i-1}$ is simple. By (4), applying $\mathsf{J} \boxtimes -$ to each exact sequence $0 \to \mathsf{M}_{i-1} \to \mathsf{M}_i \to \mathsf{S}_i \to 0$ gives another exact sequence $0 \to \mathsf{J} \boxtimes \mathsf{M}_{i-1} \to \mathsf{J} \boxtimes \mathsf{M}_i \to \mathsf{J} \boxtimes \mathsf{S}_i \to 0$. Moreover, $\mathsf{J} \boxtimes \mathsf{S}_i$ is simple, by (3). Assembling all of these exact sequences gives (5).

For (6), first recall that $\operatorname{rad} \mathsf{M}$ is the intersection of the maximal proper submodules of M and that $\mathsf{M}_i \subset \mathsf{M}$ is maximal proper if and only if M/M_i is simple. In this case, (3) and (4) now imply that $\mathsf{J} \boxtimes (\mathsf{M}/\mathsf{M}_i)$ is simple and isomorphic to $(\mathsf{J}\boxtimes\mathsf{M})/(\mathsf{J}\boxtimes\mathsf{M}_i)$, whence $\mathsf{J}\boxtimes\mathsf{M}_i$ is maximal proper in $\mathsf{J}\boxtimes\mathsf{M}$. Applying $\mathsf{J}^{-1}\boxtimes -$ gives the converse. Second, given a collection $\mathsf{M}_i \subseteq \mathsf{M}$, (4) also implies that $\mathsf{J} \boxtimes (\cap_i \mathsf{M}_i)$ is a submodule of each $\mathsf{J} \boxtimes \mathsf{M}_i$, hence of $\cap_i(\mathsf{J} \boxtimes \mathsf{M}_i)$. But now, $\cap_i(\mathsf{J} \boxtimes \mathsf{M}_i) \cong \mathsf{J} \boxtimes \mathsf{J}^{-1} \boxtimes \big(\cap_i(\mathsf{J} \boxtimes \mathsf{M}_i)\big) \subseteq \mathsf{J} \boxtimes (\cap_i \mathsf{M}_i)$, hence we have $\mathsf{J} \boxtimes (\cap_i \mathsf{M}_i) \cong \cap_i(\mathsf{J} \boxtimes \mathsf{M}_i)$. These two conclusions together give $\mathsf{J}\boxtimes \operatorname{rad}\mathsf{M} \cong \operatorname{rad}(\mathsf{J}\boxtimes\mathsf{M})$. A similar, but easier, argument establishes $\mathsf{J} \boxtimes \operatorname{soc}\mathsf{M} \cong \operatorname{soc}(\mathsf{J} \boxtimes \mathsf{M})$.

Finally, (7) follows by combining (6) with slight generalisations of the arguments used to prove (5). □

This proposition has a simple summary: fusing with a simple current preserves module structure. We remark, obviously, that a simple current J need not be simple if the vertex operator algebra V is not simple.

2.4. Orbifold modules

Here, we review a result of Kac and Radul [68] on the simplicity of orbifold modules. For a very similar result, see [50].

Let A be an associative algebra, for example the mode algebra of a vertex operator algebra, and let $\mathcal{G}$ be a subgroup of $\operatorname{Aut}\mathsf{A}$ acting semisimply on A. We consider A-modules M which admit a semisimple $\mathcal{G}$-action that is compatible with the $\mathcal{G}$-action on A and which decompose as a countable direct sum of finite-dimensional simple $\mathcal{G}$-modules. This compatibility means that

$$g(am) = (ga)(gm) \quad \text{for all } g \in \mathcal{G},\ a \in \mathsf{V} \text{ and } m \in \mathsf{M}. \tag{2.3}$$

If we now define A_0 to be the space of $\mathcal{G}$-invariants $a \in \mathsf{A}$, so $ga = a$ for all $g \in \mathcal{G}$, then the actions of each $g \in \mathcal{G}$ and each $a \in \mathsf{A}_0$ commute on every such module M.

Choose an M satisfying (2.3) and let N be a simple $\mathcal{G}$-module. Then, we may define the $\mathcal{G}$-module

$$\mathsf{M}_{\mathsf{N}} = \sum \{\mathsf{N}_i \subseteq \mathsf{M} \,:\, \mathsf{N}_i \cong \mathsf{N}\}.$$

As the action of A_0 commutes with that of $\mathcal{G}$, every $a \in \mathsf{A}_0$ maps a given N_i to some N_j or 0, by Schur's lemma. Thus, M_{N} is an A_0-module.

If we choose a one-dimensional subspace $\mathbb{C} \subseteq \mathsf{N}$, then Schur's lemma picks out a one-dimensional subspace $\mathbb{C}_i \subseteq \mathsf{N}_i$, for each i. Then, each $a \in \mathsf{A}_0$ maps each $\mathbb{C}_i$ to some $\mathbb{C}_j$ or to 0, hence

$$\mathsf{M}^{\mathsf{N}} = \sum_{\mathsf{N}_i \cong \mathsf{N}} \mathbb{C}_i$$

is an A_0-module. But, because $\mathsf{N}_i \cong \mathsf{N} \cong \mathsf{N} \otimes \mathbb{C}_i$, we may write

$$\mathsf{M}_{\mathsf{N}} \cong \sum_{\mathsf{N}_i \cong \mathsf{N}} \mathsf{N} \otimes \mathbb{C}_i = \mathsf{N} \otimes \mathsf{M}^{\mathsf{N}}$$

as a $\mathbb{C}\mathcal{G} \otimes \mathsf{A}_0$-module. The semisimplicity of M, as a $\mathcal{G}$-module, now gives us the decomposition

$$\mathsf{M} \cong \bigoplus_{[\mathsf{N}]} \mathsf{M}_{\mathsf{N}} \cong \bigoplus_{[\mathsf{N}]} \mathsf{N} \otimes \mathsf{M}^{\mathsf{N}}, \tag{2.4}$$

again as a $\mathbb{C}\mathcal{G} \otimes \mathsf{A}_0$-module. Here, $[\mathsf{N}]$ denotes the isomorphism class of the simple $\mathcal{G}$-module N.

The result of Kac and Radul gives conditions under which the A_0-modules M^{N}, appearing in (2.4), are guaranteed to be simple.

Theorem 2.9 ([68, Thm. 1.1 and Rem. 1.1])**.** *With the above setup, the (non-zero)* M^{N} *appearing in* (2.4) *will be simple* A_0*-modules provided that* M *is a simple* A*-module.*

3. Schur–Weyl duality

In this section, we state and prove results concerning the decomposition of a vertex operator algebra and its modules into modules over a Heisenberg vertex operator subalgebra and its commutant. We regard this decomposition as a vertex-algebraic analogue of the well-known Schur–Weyl duality familiar for symmetric groups and general linear Lie algebras. These results are enhanced by deducing sufficient conditions for the decompositions, and their close relations, to be multiplicity-free. Finally, we illustrate our results with several carefully chosen examples.

3.1. Heisenberg cosets

Let $\mathcal{G}$ be a finitely generated abelian subgroup of the automorphism group of a simple vertex operator algebra V. We assume that $\mathcal{G}$ grades V, meaning that the actions of these automorphisms may be simultaneously diagonalised, hence that V decomposes into a direct sum of $\mathcal{G}$-modules:

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathsf{V}_\lambda. \tag{3.1}$$

Here, the λ are elements of the (abelian) dual group $\hat{\mathcal{G}}$ of inequivalent (complex, not necessarily unitary) one-dimensional modules of $\mathcal{G}$ (recall that addition is tensor product and negation is contragredient dual), V_λ denotes the simultaneous eigenspace upon which each $g \in \mathcal{G}$ acts as multiplication by $\lambda(g) \in \mathbb{C}$, and $\mathcal{L}$ is the subset of $\lambda \in \hat{\mathcal{G}}$ for which $\mathsf{V}_\lambda \neq 0$. Note that the cardinality of $\mathcal{L}$ is at most countable.

The action of V on itself restricts to an action of each V_λ on each V_μ. For $\lambda = \mu = 0$, where 0 denotes the trivial $\mathcal{G}$-module, this implies that V_0 is a vertex operator subalgebra of V; for $\lambda = 0$, this implies that each V_μ is a V_0-module. From the simplicity of V, it now easily follows that $\mathcal{L}$ is a subgroup of $\hat{\mathcal{G}}$: closure under addition follows from annihilating ideals being trivial [82, Cor. 4.5.15] and closure under negation follows similarly, see [83, Prop. 3.6].

Applying Theorem 2.9, with $\mathsf{M} = \mathsf{V}$ and A being the mode algebra of V, we can now improve upon (3.1). Indeed, in this setting, (2.4) becomes

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathbb{C}_\lambda \otimes \mathsf{V}_\lambda,$$

where $\mathbb{C}_\lambda$ denotes the one-dimensional module upon which $g \in \mathcal{G}$ acts as multiplication by $\lambda(g)$, and we learn that the V_λ are simple as V_0-modules. In particular, V_0 is a simple vertex operator algebra.

If we assume that V_0 satisfies the conditions required to invoke the tensor category theory of Huang, Lepowsky and Zhang (Section 2.1), then more is true. As Miyamoto has shown, the V_λ are then simple currents for V_0, see [93, 26]. It should be noted that the proof in [93], [26] assumes that the group of automorphisms under consideration is finite. However, their proof works more generally under the assumption that tensor category theory for the fixed-point algebra can be invoked. For completeness, we include a detailed exposition of their proof in our slightly more general setting in Appendix A.

Theorem 3.1 **(**[93, Sect. 6]**).** *Assume the above setup and that* $\mathsf{V}_0 = \mathsf{V}^{\mathcal{G}}$ *satisfies conditions sufficient to invoke Huang, Lepowsky and Zhang's tensor category theory, for example those of Theorem* 2.5*. Then, the* V_λ *are simple currents for* V_0 *with* $\mathsf{V}_\lambda \boxtimes_{\mathsf{V}_0} \mathsf{V}_\mu \cong \mathsf{V}_{\lambda+\mu}$*, for all* $\lambda, \mu \in \mathcal{L}$*.*

Let us now restrict to vertex operator algebras V that contain a Heisenberg vertex operator subalgebra H, generated by r fields $h^i(z)$, $i = 1, \dots, r$, of conformal weight 1. We will assume throughout that the action of H on V is semisimple[4] and that the eigenvalues of the zero modes h_0^i, $i = 1, \dots, r$, are all real. Let C denote the commutant vertex operator algebra of H in V and let $\mathcal{G} \cong \mathbb{Z}^r$ be the lattice generated by the h_0^i. Each V_λ of the $\mathcal{G}$-decomposition (3.1) is a module for H since the fields of H commute with the zero modes of $\mathcal{G}$. As $\mathcal{G}$ acts semisimply on V_λ and the only simple H-module with h_0^i-eigenvalues $\lambda = (\lambda^1, \dots, \lambda^r)$ is the Fock module F_λ, we must have the following $\mathsf{H} \otimes \mathsf{C}$-module decomposition:

$$\mathsf{V}_\lambda = \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda, \quad \text{for all } \lambda \in \mathcal{L}.$$

In this setting, we may take $\mathcal{L}$ to be the lattice of all $\lambda \in \mathbb{R}^r$ for which $\mathsf{V}_\lambda \neq 0$. Moreover, the C-module C_λ is simple because V_λ and F_λ are. In particular, the commutant $\mathsf{C} = \mathsf{C}_0$ is a simple vertex operator algebra. We summarise this as follows.

Proposition 3.2. *Let V be a simple vertex operator algebra with a Heisenberg vertex operator subalgebra H that acts semisimply on V. Then, the coset vertex operator algebra $\mathsf{C} = \mathrm{Com}(\mathsf{H}, \mathsf{V})$ is likewise simple.*

From here on, we make the following natural assumption:

> We assume that we are working with categories of (generalised) V_0- and C-modules for which the tensor category theory of Huang, Lepowsky and Zhang [67] may be invoked.

Of course, we have confirmed in Section 2.2 that this theory may be invoked for semisimple H-modules with real weights. In general, we would like to apply our results to vertex operator algebras for which we are not currently able to verify this assumption. Such illustrations should therefore be regarded as conjectural. However, we view the results in these cases as strong evidence that the conditions required to invoke Huang–Lepowsky–Zhang are, in fact, significantly weaker than those that were given in Section 2.1.

Given now the fusion rules $\mathsf{F}_\lambda \boxtimes_{\mathsf{H}} \mathsf{F}_\mu \cong \mathsf{F}_{\lambda+\mu}$ and $\mathsf{V}_\lambda \boxtimes_{\mathsf{V}_0} \mathsf{V}_\mu \cong \mathsf{V}_{\lambda+\mu}$, which imply that

$$\left(\mathsf{F}_\lambda \otimes \mathsf{C}_\lambda\right) \boxtimes_{\mathsf{V}_0} \left(\mathsf{F}_\mu \otimes \mathsf{C}_\mu\right) \cong \mathsf{F}_{\lambda+\mu} \otimes \mathsf{C}_{\lambda+\mu}, \tag{3.2}$$

one is naturally led to suppose that $\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{C}_\mu \cong \mathsf{C}_{\lambda+\mu}$. Proving this, however, is a little subtle because we are not assuming that the corresponding module categories are semisimple. We therefore present a technical result that we shall use to confirm this supposition and other similar assertions. We remark that this result can be greatly strengthened when one of the vertex operator algebras involved is of Heisenberg or lattice type, or when the vertex operator algebras involved are rational (see [84]).

Proposition 3.3. *Let A and B be vertex operator algebras and let A_i and B_i, for $i = 1, 2, 3$, be A-modules and B-modules, respectively. Suppose that*

$$\left((\mathsf{A}_1 \otimes \mathsf{B}_1) \boxtimes_{\mathsf{A}\otimes\mathsf{B}} (\mathsf{A}_2 \otimes \mathsf{B}_2), \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}}\right) = (\mathsf{A}_3 \otimes \mathsf{B}_3, \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}}).$$

[4]Examples on which a Heisenberg vertex operator subalgebra does not act semisimply are provided by the Takiff vertex operator algebras of [23], [22].

Also assume that either of the fusion coefficients $\binom{\mathsf{A}_3}{\mathsf{A}_1\ \mathsf{A}_2}$ *or* $\binom{\mathsf{B}_3}{\mathsf{B}_1\ \mathsf{B}_2}$ *is finite. Then,* $(\mathsf{A}_3 \otimes \mathsf{B}_3, \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}})$ *may be taken to be* $\big((\mathsf{A}_1 \boxtimes_{\mathsf{A}} \mathsf{A}_2) \otimes (\mathsf{B}_1 \boxtimes_{\mathsf{B}} \mathsf{B}_2), \mathcal{Y}^{\boxtimes}_{\mathsf{A}} \otimes \mathcal{Y}^{\boxtimes}_{\mathsf{B}}\big)$*. In particular,*

$$\mathsf{A}_1 \boxtimes_{\mathsf{A}} \mathsf{A}_2 \cong \mathsf{A}_3 \quad \textit{and} \quad \mathsf{B}_1 \boxtimes_{\mathsf{B}} \mathsf{B}_2 \cong \mathsf{B}_3.$$

Proof. The key here is [2, Thm. 2.10] which, as stated, applies to non-logarithmic intertwining operators but in fact also holds when logarithmic intertwiners are present. Using this, we may write

$$\mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}} = \sum_{j=1}^{N} \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{A}} \otimes \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{B}},$$

for some N, where each $\widetilde{\mathcal{Y}}^{(j)}_{\mathsf{A}}$ is an intertwiner for A of type $\binom{\mathsf{A}_3}{\mathsf{A}_1\ \mathsf{A}_2}$ and each $\widetilde{\mathcal{Y}}^{(j)}_{\mathsf{B}}$ is of type $\binom{\mathsf{B}_3}{\mathsf{B}_1\ \mathsf{B}_2}$ for B. The universality of the fusion product now guarantees the existence of (unique) A-module morphisms $\mu^{(j)}_{\mathsf{A}} \colon \mathsf{A}_1 \boxtimes_{\mathsf{A}} \mathsf{A}_2 \to \mathsf{A}_3$, such that $\mu^{(j)}_{\mathsf{A}} \circ \mathcal{Y}^{\boxtimes}_{\mathsf{A}} = \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{A}}$, and B-module morphisms $\mu^{(j)}_{\mathsf{B}} \colon \mathsf{B}_1 \boxtimes_{\mathsf{B}} \mathsf{B}_2 \to \mathsf{B}_3$, such that $\mu^{(j)}_{\mathsf{B}} \circ \mathcal{Y}^{\boxtimes}_{\mathsf{B}} = \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{B}}$. Setting $\mu = \sum_{j=1}^{N} \mu^{(j)}_{\mathsf{A}} \otimes \mu^{(j)}_{\mathsf{B}}$, we obtain

$$\mu \circ \big(\mathcal{Y}^{\boxtimes}_{\mathsf{A}} \otimes \mathcal{Y}^{\boxtimes}_{\mathsf{B}}\big) = \sum_{j=1}^{N} \big(\mu^{(j)}_{\mathsf{A}} \otimes \mu^{(j)}_{\mathsf{B}}\big) \circ \big(\mathcal{Y}^{\boxtimes}_{\mathsf{A}} \otimes \mathcal{Y}^{\boxtimes}_{\mathsf{B}}\big) = \sum_{j=1}^{N} \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{A}} \otimes \widetilde{\mathcal{Y}}^{(j)}_{\mathsf{B}} = \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}}. \tag{3.3}$$

Now, let X be a "test" $\mathsf{A} \otimes \mathsf{B}$-module and let $\mathcal{Y}$ be an intertwining operator of type $\binom{\mathsf{X}}{\mathsf{A}_1\otimes\mathsf{B}_1\ \mathsf{A}_2\otimes\mathsf{B}_2}$. By the universal property satisfied by $(\mathsf{A}_3 \otimes \mathsf{B}_3, \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}})$, there exists a (unique) $\eta \colon \mathsf{A}_3 \otimes \mathsf{B}_3 \to \mathsf{X}$ such that $\eta \circ \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}} = \mathcal{Y}$. It follows that

$$(\eta \circ \mu) \circ \big(\mathcal{Y}^{\boxtimes}_{\mathsf{A}} \otimes \mathcal{Y}^{\boxtimes}_{\mathsf{B}}\big) = \eta \circ \mathcal{Y}^{\boxtimes}_{\mathsf{A}\otimes\mathsf{B}} = \mathcal{Y}. \tag{3.4}$$

It remains to prove that $\eta \circ \mu \colon (\mathsf{A}_1 \boxtimes_{\mathsf{A}} \mathsf{A}_2) \otimes (\mathsf{B}_1 \boxtimes_{\mathsf{B}} \mathsf{B}_2) \to \mathsf{X}$ is the unique $\mathsf{A} \otimes \mathsf{B}$-module morphism satisfying (3.4). However, as recalled in Theorem 2.2, $\mathcal{Y}^{\boxtimes}_{\mathsf{A}}$ and $\mathcal{Y}^{\boxtimes}_{\mathsf{B}}$ are surjective intertwining operators — this surjectivity goes hand-in-hand with the "uniqueness" requirement in the universal property, see [67, Prop. 4.23] — and so, therefore, is $\mathcal{Y}^{\boxtimes}_{\mathsf{A}} \otimes \mathcal{Y}^{\boxtimes}_{\mathsf{B}}$. This means that (3.4) uniquely specifies the morphism $\eta \circ \mu$, completing the proof. □

From this theorem, we immediately obtain the following theorem.

Corollary 3.4. *If* A *and* B *are simple vertex operator algebras and* $\mathsf{M} \otimes \mathsf{N}$ *is a simple current for* $\mathsf{A} \otimes \mathsf{B}$*, then* M *and* N *are simple currents for* A *and* B*, respectively. Moreover, the inverse of* $\mathsf{M} \otimes \mathsf{N}$ *is* $\mathsf{M}^{-1} \otimes \mathsf{N}^{-1}$*.*

Proof. Because $\mathsf{A}\otimes\mathsf{B}$ is assumed to be simple, $\mathsf{M}\otimes\mathsf{N}$ and its inverse are simple $\mathsf{A}\otimes\mathsf{B}$-modules, by Theorem 2.83. Moreover, this simplicity hypothesis also guarantees that the inverse has the form $\widetilde{\mathsf{M}} \otimes \widetilde{\mathsf{N}}$ [58, Thm. 4.7.4]. Applying Theorem 3.3 to $(\widetilde{\mathsf{M}} \otimes \widetilde{\mathsf{N}}) \boxtimes_{\mathsf{A}\otimes\mathsf{B}} (\mathsf{M} \otimes \mathsf{N}) \cong \mathsf{A} \otimes \mathsf{B}$, we obtain $\widetilde{\mathsf{M}} \boxtimes_{\mathsf{A}} \mathsf{M} \cong \mathsf{A}$ and $\widetilde{\mathsf{N}} \boxtimes_{\mathsf{B}} \mathsf{N} \cong \mathsf{B}$, hence $\widetilde{\mathsf{M}} \cong \mathsf{M}^{-1}$ and $\widetilde{\mathsf{N}} \cong \mathsf{N}^{-1}$. □

In any case, (3.2) and Theorem 3.3 give the desired conclusion:

$$\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{C}_\mu \cong \mathsf{C}_{\lambda+\mu}. \tag{3.5}$$

In particular, the C_λ are simple currents for all $\lambda \in \mathcal{L}$. We have therefore arrived at the following decomposition of V into simple currents of H and C:

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda. \tag{3.6}$$

However, this may be further refined if $\lambda \neq \mu$ in $\mathcal{L}$ does not imply that $\mathsf{C}_\lambda \neq \mathsf{C}_\mu$ (this implication is obviously true for Fock modules). Suppose that $\mathsf{C}_\lambda = \mathsf{C}_{\lambda+\mu}$ for some $\lambda, \mu \in \mathcal{L}$. Then, we must have $\mathsf{C}_\mu = \mathsf{C}$ and hence $\mathsf{C}_{n\mu} = \mathsf{C}$ for all $n \in \mathbb{Z}$. More generally, let $\mathcal{N}$ denote the sublattice of $\mu \in \mathcal{L}$ for which $\mathsf{C}_\mu = \mathsf{C}$. Then, we may define

$$\mathsf{W}_{[\lambda]} = \bigoplus_{\mu \in \mathcal{N}} \mathsf{F}_{\lambda+\mu}$$

and note that $\mathsf{W} = \mathsf{W}_{[0]}$ will be a lattice vertex operator algebra if the conformal weights of the fields of each F_μ, with $\mu \in \mathcal{N}$, are all integers.[5] The decomposition (3.6) then becomes a decomposition as a $\mathsf{W} \otimes \mathsf{C}$-module:

$$\mathsf{V} = \bigoplus_{[\lambda] \in \mathcal{L}/\mathcal{N}} \mathsf{W}_{[\lambda]} \otimes \mathsf{C}_{[\lambda]}. \tag{3.7}$$

Now the $\mathsf{C}_{[\lambda]} \equiv \mathsf{C}_\lambda$, with $[\lambda] \in \mathcal{L}/\mathcal{N}$, are mutually inequivalent: $[\lambda] \neq [\mu]$ implies that $\mathsf{C}_{[\lambda]} \ncong \mathsf{C}_{[\mu]}$. We remark that $\mathcal{L}/\mathcal{N}$ may still be infinite because the rank of $\mathcal{N}$ may be smaller than that of $\mathcal{L}$.

We summarise these results as follows.

Theorem 3.5. *Let:*

- V *be a simple vertex operator algebra.*
- $\mathsf{H} \subseteq \mathsf{V}$ *be a Heisenberg vertex operator subalgebra that acts semisimply on* V.
- $\mathsf{C} = \mathsf{C}_0$ *be the commutant of* H *in* V.
- $\mathcal{L}$ *be the lattice of Heisenberg weights of* V (V *being regarded as an* H*-module*).

Then the decompositions (3.6) *and* (3.7) *hold, where:*

- *The* C_λ, $\lambda \in \mathcal{L}$, *are simple currents for* C *whose fusion products include* $\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{C}_\mu = \mathsf{C}_{\lambda+\mu}$.
- $\mathsf{W} = \bigoplus_{\lambda \in \mathcal{N}} \mathsf{F}_\lambda$ *is a simple current extension of* H ($\mathcal{N}$ *is the sublattice of* $\lambda \in \mathcal{L}$ *for which* $\mathsf{C}_\lambda \cong \mathsf{C}$).
- *The* $\mathsf{W}_{[\lambda]}$, $[\lambda] \in \mathcal{L}/\mathcal{N}$, *are simple currents for* W *with fusion products* $\mathsf{W}_{[\lambda]} \boxtimes_{\mathsf{W}} \mathsf{W}_{[\mu]} = \mathsf{W}_{[\lambda+\mu]}$.

In particular, the $\mathsf{C}_{[\lambda]}$, $[\lambda] \in \mathcal{L}/\mathcal{N}$, *of* (3.7) *are mutually non-isomorphic.*

[5]If the conformal weights are not all integers, then W is a vertex operator superalgebra, or another type of generalised vertex operator algebra. This does not significantly affect the following analysis.

Remark 3.6. Note that we may instead choose $\mathcal{N}$ to be any subgroup of $\mathcal{L}$ in which every $\lambda \in \mathcal{N}$ satisfies $\mathsf{C}_\lambda \cong \mathsf{C}$. In particular, we may take $\mathcal{N} = 0$, in which case the decomposition (3.7) reduces to that of (3.6). Obviously, the conclusion that the $\mathsf{C}_{[\lambda]}$ are mutually non-isomorphic will only hold if $\mathcal{N}$ is taken to be maximal.

The corresponding decomposition for V-modules proceeds similarly. Let M be a non-zero V-module upon which H acts semisimply. The H-weight space decomposition of M then gives $\mathsf{M} = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}_\mu$, where $\mathcal{M} = \{\mu \in \mathbb{R}^r : \mathsf{M}_\mu \neq 0\}$ is countable. Using the triviality of annihilating ideals [82, Cor. 4.5.15] as before, we see that $\mathcal{M}$ is closed under the additive action of $\mathcal{L}$, meaning that $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$ imply that $\lambda + \mu \in \mathcal{M}$. It follows that each M_μ is a V_0-module. Decomposing as an $\mathsf{H} \otimes \mathsf{C}$-module, we get $\mathsf{M}_\mu = \mathsf{F}_\mu \otimes \mathsf{D}_\mu$, for some C-module D_μ. The key step towards proving a decomposition theorem for modules is now to establish certain fusion products involving the M_μ and D_μ.

Proposition 3.7. *Let* V, H, C, W *and* $\mathcal{L}$ *be as in Theorem* 3.5 *and let* M, $\mathcal{M}$ *and* $\mathsf{M}_\mu = \mathsf{F}_\mu \otimes \mathsf{D}_\mu$ *be as in the previous paragraph. Then, the following fusion rules hold for all* $\lambda \in \mathcal{L}$ *and* $\mu \in \mathcal{M}$*:*

$$\mathsf{V}_\lambda \boxtimes_{\mathsf{V}_0} \mathsf{M}_\mu \cong \mathsf{M}_{\lambda+\mu}, \tag{3.8a}$$

$$\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{D}_\mu \cong \mathsf{D}_{\lambda+\mu}. \tag{3.8b}$$

We mention that when $\mathsf{M} = \mathsf{V}$, the fusion rule (3.8a) is precisely the result of Miyamoto reported in Theorem 3.1. However, we cannot use Miyamoto's proof in this more general setting because it would amount to assuming the simplicity of the M_μ as V_0-modules.

Proof. We will detail the proof of the fusion rule (3.8a), noting that (3.8b) will then follow immediately by applying Theorem 3.3.

To prove (3.8a), let $\widetilde{\mathsf{M}}$ denote the V-submodule of M generated by M_μ. Then, $(\mathsf{M}/\widetilde{\mathsf{M}})_\mu = 0$. If $v \in \mathsf{V}_{-\lambda}$ is non-zero, for some $\lambda \in \mathcal{L}$, and $w \in (\mathsf{M}/\widetilde{\mathsf{M}})_{\lambda+\mu}$, then it follows that v must annihilate w, hence that $w = 0$ by the triviality of annihilating ideals [82, Cor. 4.5.15]. We conclude that $(\mathsf{M}/\widetilde{\mathsf{M}})_{\lambda+\mu} = 0$, that is $\widetilde{\mathsf{M}}_{\lambda+\mu} = \mathsf{M}_{\lambda+\mu}$, for all $\lambda \in \mathcal{L}$.

The action of V on M now restricts to an action of V_λ on M_μ. The space generated by the latter action is therefore precisely $\mathsf{M}_{\lambda+\mu}$ [82, Prop. 4.5.6]. It now follows from the universal property of fusion products that there exists a surjection

$$\mathsf{V}_\lambda \boxtimes_{\mathsf{V}_0} \mathsf{M}_\mu \twoheadrightarrow \mathsf{M}_{\lambda+\mu}, \tag{3.9}$$

for each $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$. Fusing with the simple current $\mathsf{V}_{-\lambda}$ therefore gives

$$\mathsf{M}_\mu \cong \mathsf{V}_{-\lambda} \boxtimes_{\mathsf{V}_0} (\mathsf{V}_\lambda \boxtimes_{\mathsf{V}_0} \mathsf{M}_\mu) \twoheadrightarrow \mathsf{V}_{-\lambda} \boxtimes_{\mathsf{V}_0} \mathsf{M}_{\lambda+\mu} \twoheadrightarrow \mathsf{M}_\mu,$$

the first surjection being the right-exactness of fusion and the second surjection being (3.9) with (λ, μ) replaced by $(-\lambda, \lambda + \mu)$. Since these surjections preserve conformal weights and the dimensions of the generalised eigenspaces of L_0 are finite, by hypothesis, it follows that $\mathsf{V}_{-\lambda} \boxtimes_{\mathsf{V}_0} \mathsf{M}_{\lambda+\mu} = \mathsf{M}_\mu$, for all $\lambda \in \mathcal{L}$, proving (3.8a). ☐

If $\lambda \in \mathcal{N}$, then the fusion rules (3.8b) imply that $\mathsf{D}_{\lambda+\mu} = \mathsf{D}_\mu$, hence that the $\mathsf{D}_{[\mu]} \equiv \mathsf{D}_\mu$ are well defined. The decomposition of M as a $\mathsf{W} \otimes \mathsf{C}$-module now follows as before. Before stating this formally, it is convenient to observe that if $\mathcal{M} = \mathcal{M}^1 \cup \cdots \cup \mathcal{M}^n$ is a disjoint union of orbits under the action of $\mathcal{L}$, then $\mathsf{M} = \mathsf{M}^1 \oplus \cdots \oplus \mathsf{M}^n$ as a V-module, where $\mathsf{M}^i = \bigoplus_{\mu \in \mathcal{M}^i} \mathsf{M}^i_\mu$. While the M_i need not be indecomposable as V-modules, several of the arguments to come will be simplified if we assume that $\mathcal{M}$ consists of a single $\mathcal{L}$-orbit. Conclusions about more general M then follow immediately from the properties of direct sums.

Theorem 3.8. *Let* V, H, C, W, $\mathcal{L}$ *and* $\mathcal{N}$ *be as in Theorem* 3.5 *and let* M *be a* V*-module upon which* H *acts semisimply. Then,* M *decomposes as*

$$\mathsf{M} = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}_\mu = \bigoplus_{[\mu] \in \mathcal{M}/\mathcal{N}} \mathsf{W}_{[\mu]} \otimes \mathsf{D}_{[\mu]}, \tag{3.10}$$

where $\mathcal{M}$ *is a union of* $\mathcal{L}$*-orbits and the* $\mathsf{D}_\mu = \mathsf{D}_{[\mu]}$ *are* C*-modules satisfying* $\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{D}_\mu = \mathsf{D}_{\lambda+\mu}$*, for all* $\lambda \in \mathcal{L}$ *and* $\mu \in \mathcal{M}$*. Moreover, if we assume (for convenience) that* $\mathcal{M}$ *is a single* $\mathcal{L}$*-orbit, then the following hold:*

(1) *If* M *is a non-zero* V*-module, then all of the* D_μ *are non-zero.*
(2) *If* M *is a simple* V*-module, then all of the* D_μ *are simple.*
(3) *If* M *is an indecomposable* V*-module, then all of the* D_μ *are indecomposable.*

(4) *If* $0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M}'' \to 0$ *is exact, with* M' *and* M'' *non-zero, then* M' *and* M'' *decompose as in* (3.10)*:*

$$\mathsf{M}' = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}'_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}'_\mu, \quad \mathsf{M}'' = \bigoplus_{\mu \in \mathcal{M}} \mathsf{M}''_\mu = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}''_\mu. \tag{3.11}$$

Moreover, $0 \to \mathsf{D}'_\mu \to \mathsf{D}_\mu \to \mathsf{D}''_\mu \to 0$ *is also exact, for all* $\mu \in \mathcal{M}$*.*
(5) *If* M *has a composition series with composition factors* S^i, $1 \le i \le n$*, then each* S^i *decomposes into an* $\mathsf{H} \otimes \mathsf{C}$*-module as* $\mathsf{S}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{T}^i_\mu$*, where the* T^i_μ, $1 \le i \le n$*, are the composition factors of* D_μ*, for each* $\mu \in \mathcal{M}$*. In particular, each* D_μ *has the same composition length as* M*.*
(6) *If* M *has a socle, then so do the* D_μ *and* $\operatorname{soc} \mathsf{M} = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \operatorname{soc} \mathsf{D}_\mu$*.*
If M *has a radical, then so do the* D_μ*. If, in addition,* M *has no subquotient isomorphic to the direct sum of two isomorphic simple* V*-modules, then* $\operatorname{rad} \mathsf{M} = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \operatorname{rad} \mathsf{D}_\mu$*.*
(7) *If* M *has a socle series, then so do the* D_μ *and the corresponding Loewy diagram is obtained by replacing each composition factor* S^i *by* T^i_μ*, where* $\mathsf{S}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{T}^i_\mu$*.*
If M *has a radical series, then so do the* D_μ*. If, in addition,* M *has no subquotient isomorphic to the direct sum of two isomorphic simple* V*-modules, then the corresponding Loewy diagram is obtained by replacing each composition factor* S^i *by* T^i_μ*, where* $\mathsf{S}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{T}^i_\mu$*.*

Proof. We have already proven the non-numbered statements. For (1), suppose that $\mathsf{D}_\mu = 0$, for some $\mu \in \mathcal{M}$. Then, $\mathsf{M}_\mu = \mathsf{F}_\mu \otimes \mathsf{D}_\mu$ would be 0, contradicting the definition of $\mathcal{M}$. The argument for (2) is likewise short: M simple implies that each M_μ, with $\mu \in \mathcal{M}$, is simple, by Theorem 2.9, which forces each of the D_μ to be simple. To prove (3), note that if some D_ν, $\nu \in \mathcal{M}$, were decomposable, then every D_μ, $\mu \in \mathcal{M}$, would be decomposable because $\mu - \nu \in \mathcal{L}$, hence $\mathsf{D}_\mu \cong \mathsf{C}_{\mu-\nu} \boxtimes_{\mathsf{C}} \mathsf{D}_\nu$. But then, every M_μ would be decomposable, hence so would M, a contradiction.

Given the exact sequence in (4), it is clear that H acts semisimply on both M' and M'', hence that we have the decompositions (3.11) except that some of the M'_μ or M''_μ might be zero, for some $\mu \in \mathcal{M}$. However, $\mathcal{M}$ is assumed to consist of a single $\mathcal{L}$-orbit, so either all the M'_μ are zero or none of them are (and the same for the M''_μ). But, either being zero would imply that the corresponding module is zero, which is ruled out by hypothesis. Thus, the M'_μ and M''_μ are non-zero, for all $\mu \in \mathcal{M}$.

Since restricting to a V_0-module and projecting onto the (simultaneous) eigenspaces of the h^i_0 (which commute with $\mathsf{V}_0 = \mathsf{H} \otimes \mathsf{C}$) are exact functors, the sequence $0 \to \mathsf{F}_\mu \otimes \mathsf{D}'_\mu \to \mathsf{F}_\mu \otimes \mathsf{D}_\mu \to \mathsf{F}_\mu \otimes \mathsf{D}''_\mu \to 0$ is exact, for all $\mu \in \mathcal{M}$. However, $\mathrm{End}_{\mathsf{H}} \mathsf{F}_\mu \cong \mathbb{C}$ implies that each non-trivial map in this exact sequence has the form $\mathrm{id}_{\mathsf{F}_\mu} \otimes d_\mu$, where d_μ is a C-module homomorphism. The required exactness of the sequence of C-modules thus follows, proving (4).

For (5), let $0 = \mathsf{M}^0 \subset \mathsf{M}^1 \subset \cdots \subset \mathsf{M}^{n-1} \subset \mathsf{M}^n = \mathsf{M}$ be a composition series, so that $\mathsf{S}^i = \mathsf{M}^i / \mathsf{M}^{i-1}$ is simple, for all $1 \le i \le n$. Then, $0 \to \mathsf{M}^{i-1} \to \mathsf{M}^i \to \mathsf{S}^i \to 0$ is exact, hence so is $0 \to \mathsf{D}^{i-1}_\mu \to \mathsf{D}^i_\mu \to \mathsf{T}^i_\mu \to 0$, for all $1 \le i \le n$ and $\mu \in \mathcal{M}$, by (4). Here, we have decomposed each M^i as $\mathsf{M}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}^i_\mu$, so that $\mathsf{D}^0_\mu = 0$ and $\mathsf{D}^n_\mu = \mathsf{D}_\mu$, and each S^i as $\mathsf{S}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{T}^i_\mu$. Since the T^i_μ are non-zero and simple, by (1) and (2), they are the composition factors of D_μ.

We turn to (6). Let $\{\mathsf{M}^i\}_{i \in I}$ be the set of all simple submodules of M so that $\mathrm{soc}\, \mathsf{M} = \sum_{i \in I} \mathsf{M}^i$. Then, each M^i decomposes as $\mathsf{M}^i = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}^i_\mu$, where D^i_μ is a simple submodule of D_μ, for each $i \in I$ and $\mu \in \mathcal{M}$, by (2) and (4). As sums distribute over tensor products, we have

$$\mathrm{soc}\, \mathsf{M} = \sum_{i \in I} \Big[\bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}^i_\mu \Big] = \bigoplus_{\mu \in \mathcal{M}} \mathsf{F}_\mu \otimes \Big(\sum_{i \in I} \mathsf{D}^i_\mu \Big).$$

It remains to show that for each $\mu \in \mathcal{M}$, every simple submodule of D_μ is one of the D^i_μ.

Consider therefore a simple submodule $\mathsf{E}_\mu \subseteq \mathsf{D}_\mu$, for some given $\mu \in \mathcal{M}$. Form $\mathsf{E}_\nu = \mathsf{C}_{\nu-\mu} \boxtimes_{\mathsf{C}} \mathsf{E}_\mu$, for all $\nu \in \mathsf{M}$ (so that $\nu - \mu \in \mathcal{L}$), and note that each E_ν is a simple submodule of D_ν, by parts (3) and (4) of Theorem 2.8. Tensoring over $\mathbb{C}$ is exact, so $\bigoplus_{\nu \in \mathcal{M}} \mathsf{F}_\nu \otimes \mathsf{E}_\nu$ is a submodule of $\bigoplus_{\nu \in \mathcal{M}} \mathsf{F}_\nu \otimes \mathsf{D}_\nu = \mathsf{M}$. Moreover, it is a simple submodule because it has the same number of composition factors as E_μ, by (5). It is therefore one of the M^i, hence E_μ is one of the D^i_μ. It follows that $\sum_{i \in I} \mathsf{D}^i_\mu = \mathrm{soc}\, \mathsf{D}_\mu$, as required.

The same argument works for the radical, which we recall is the intersection of the maximal proper submodules, except that intersections need not distribute over sums. The additional condition on M guarantees this [24]. The proof of 6 is thus complete and the proof of (7) now follows similarly to that of (5). $\square$

Remark 3.9. It is not clear if the condition imposed on M in the radical parts 6 and 7 is required. However, if $\operatorname{rad}\mathsf{M}$ decomposes as $\operatorname{rad}\mathsf{M} = \bigoplus_{\mu\in\mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{R}_\mu$, then without this condition, the argument used in the proof only establishes that $\mathsf{R}_\mu \subseteq \operatorname{rad}\mathsf{D}_\mu$, for each $\mu \in \mathcal{M}$.

Unlike the $\mathsf{C}_{[\mu]}$ in (3.7), the coset modules $\mathsf{D}_{[\mu]}$, $[\mu] \in \mathcal{M}/\mathcal{N}$, appearing in (3.10) need not be mutually non-isomorphic. We shall illustrate this with a simple example in Section 3.3. In the following section, we first give three useful criteria which guarantee that the $\mathsf{D}_{[\mu]}$ are all non-isomorphic.

3.2. Criteria for being multiplicity-free

In this section, we discuss whether the decomposition (3.10) is multiplicity-free or not. In other words, we investigate when one can assert that the D_μ or the $\mathsf{D}_{[\mu]}$ are mutually non-isomorphic, in the notation of Theorem 3.8.

Criterion based on conformal weights. It may so happen that the conformal weights of the highest-weight vectors of the Heisenberg subalgebra H immediately rule out multiplicities. For example, consider the case of an affine vertex operator algebra V of *negative* level k and a V-module M whose conformal weights are bounded below. We shall assume, as in Theorem 3.8, that the corresponding set $\mathcal{M}$ is a single orbit of $\mathcal{L}$. Suppose that the decomposition of M is not multiplicity-free, so that $\mathsf{D}_{\mu+\lambda} = \mathsf{D}_\mu$, for some $\lambda \in \mathcal{L}$. Then, $\mathsf{C}_\lambda \boxtimes_{\mathsf{C}} \mathsf{D}_\mu = \mathsf{D}_\mu$ and so $\mathsf{D}_{\mu+n\lambda} = \mathsf{D}_\mu$, for all $n \in \mathbb{Z}$. However, the conformal weight of the highest-weight vector of $\mathsf{F}_{\mu+n\lambda}$ is $\frac{1}{2k}\left\|\mu+n\lambda\right\|^2$, which becomes arbitrarily negative for $|n|$ large, because $k<0$. It follows that the conformal weights of $\mathsf{F}_{\mu+n\lambda} \otimes \mathsf{D}_{\mu+n\lambda} = \mathsf{F}_{\mu+n\lambda} \otimes \mathsf{D}_\mu$ would become arbitrarily negative, for all $\mu \in \mathcal{M}$. This contradicts the hypothesis that the conformal weights of $\mathsf{M} = \bigoplus_{\mu\in\mathcal{M}} \mathsf{F}_\mu \otimes \mathsf{D}_\mu$ are bounded below, hence the D_μ, with $\mu \in \mathcal{M}$, must all be mutually non-isomorphic.

Criterion based on symmetries of characters. We can also derive a simple test to rule out multiplicities using the characters

$$\operatorname{ch}\left[\mathsf{F}_\mu\right](z;q) = \operatorname{tr}_{\mathsf{F}_\mu} z^{h_0} q^{L_0^{\mathsf{H}} - c/24} = \frac{z^\mu q^{\|\mu\|^2/2}}{\eta(q)}$$

of the Fock modules. This relies on the fact that the characters of the D_μ appearing in (3.10) will not depend on z. We remark that the factors z^{h_0} and z^μ should be interpreted here as $z_1^{h_0^1} \cdots z_r^{h_0^r}$ and $z_1^{\mu_1} \cdots z_r^{\mu_r}$, respectively, where r is the rank of the Heisenberg vertex operator algebra H.

Suppose, for simplicity, that $\mathcal{M}$ consists of a single $\mathcal{L}$-orbit, as in Theorem 3.8. Define $\mathcal{N}'$ to be the sublattice of Heisenberg weights λ such that $\mathsf{D}_\mu = \mathsf{D}_{\lambda+\mu}$, for every $\mu \in \mathcal{M}$, so that $\mathcal{N} \le \mathcal{N}' \le \mathcal{L}$. It follows that for every $\lambda \in \mathcal{N}'$, the character of the decomposition (3.10) must satisfy

$$\begin{aligned}
\mathrm{ch}\big[\mathsf{M}\big](z;q;\dots) &= \sum_{\mu\in\mathcal{M}} \frac{z^{\mu} q^{\|\mu\|^2/2}}{\eta(q)} \mathrm{ch}\big[\mathsf{D}_{\mu}\big](q;\dots) \\
&= \sum_{\mu\in\mathcal{M}} \frac{z^{\lambda+\mu} q^{\|\lambda+\mu\|^2/2}}{\eta(q)} \mathrm{ch}\big[\mathsf{D}_{\mu}\big](q;\dots) \\
&= z^{\lambda} q^{\|\lambda\|^2/2} \sum_{\mu\in\mathcal{M}} \frac{z^{\mu} q^{\langle\lambda,\mu\rangle} q^{\|\mu\|^2/2}}{\eta(q)} \mathrm{ch}\big[\mathsf{D}_{\mu}\big](q;\dots) \\
&= z^{\lambda} q^{\|\lambda\|^2/2} \mathrm{ch}\big[\mathsf{M}\big](zq^{\lambda};q;\dots),
\end{aligned}$$

where q^{λ} acts on a Heisenberg weight μ to give $q^{\langle\lambda,\mu\rangle}$. If the character of M only satisfies this equation when $\lambda \in \mathcal{N}$, then we may conclude that the $\mathsf{D}_{[\mu]}$, with $[\mu] \in \mathcal{M}/\mathcal{N}$, are mutually non-isomorphic. In the case that $\mathcal{N} = 0$, this conclusion gives the mutual inequivalence of the D_{μ}, for all $\mu \in \mathcal{M}$.

Criterion based on open Hopf links. In the case of rational vertex operator algebras, the closed Hopf links are, up to normalisation, the same as the entries of the modular S-matrix [64]. There is also a close connection between Hopf links and properties of characters for non-rational vertex operator algebras [28], [27], [38]. We will now explain how Hopf links give a criterion for the existence of fixed points under the action of fusing with a simple current. For this subsection, we assume that we are working in a ribbon category $\mathscr{C}$ of vertex operator algebra modules [56]; such categories allow us to take (partial) traces of morphisms.

Let $\mathsf{J} \in \mathscr{C}$ be a simple current and fix a module $\mathsf{X} \in \mathscr{C}$. Assume that there exists a positive integer s such that $\mathsf{J}^s \boxtimes \mathsf{X} \cong \mathsf{X}$, so that X is a fixed point of J^s. Recall that the monodromy of two modules A and B is defined by $M_{\mathsf{A},\mathsf{B}} = R_{\mathsf{B},\mathsf{A}} \circ R_{\mathsf{A},\mathsf{B}}$, where R denotes their braiding. Recall the notion [56, Def. 8.10.1] of categorical twist θ, which is a system of natural isomorphisms. The monodromy satisfies the following balancing property for any two modules A and B:

$$\theta_{\mathsf{A}\boxtimes\mathsf{B}} = M_{\mathsf{A},\mathsf{B}} \circ (\theta_{\mathsf{A}} \boxtimes \theta_{\mathsf{B}}).$$

In the formalism of vertex tensor categories, θ is given by $\mathsf{e}^{2\mathsf{i}\pi L_0}$. We will also need the open Hopf link operators from [28, 27]. These are defined as the partial traces $\Phi_{\mathsf{A},\mathsf{B}} = \mathrm{ptr}^{\mathrm{Left}}(M_{\mathsf{A},\mathsf{B}}) \in \mathrm{End}(\mathsf{B})$ and have the important property that they define a representation of the fusion ring on $\mathrm{End}(\mathsf{B})$. In particular, it follows that $\Phi_{\mathsf{J}\boxtimes\mathsf{X},\mathsf{P}} = \Phi_{\mathsf{J},\mathsf{P}} \circ \Phi_{\mathsf{X},\mathsf{P}}$, for any module $\mathsf{P} \in \mathscr{C}$, and hence that

$$\Phi_{\mathsf{X},\mathsf{P}} = \Phi_{\mathsf{J}^s\boxtimes\mathsf{X},\mathsf{P}} = \Phi_{\mathsf{J}^s,\mathsf{P}} \circ \Phi_{\mathsf{X},\mathsf{P}} = \Phi_{\mathsf{J},\mathsf{P}}^s \circ \Phi_{\mathsf{X},\mathsf{P}}. \tag{3.12}$$

We shall assume now that P is indecomposable with a finite number of composition factors, so that every endomorphism of P has a single eigenvalue, and that $M_{\mathsf{J},\mathsf{P}}$ and $\Phi_{\mathsf{J},\mathsf{P}}$ are semisimple endomorphisms of $\mathsf{J} \boxtimes \mathsf{P}$ and P, respectively. The latter assumption will be automatically satisfied if J is a simple current of finite order and both $\mathrm{End}(\mathsf{P})$ and $\mathrm{End}(\mathsf{J} \boxtimes \mathsf{P})$ are finite-dimensional [30, Lem. 2.13].

It will also be satisfied if P may be identified with a subquotient of an iterated fusion product of simple modules [30, Lem. 3.19]. With these assumptions on P, Equation (3.12) shows that the image of $\Phi_{\mathsf{X},\mathsf{P}}$ is contained in the eigenspace of $\Phi^s_{\mathsf{J},\mathsf{P}}$ with eigenvalue 1 and that this eigenspace is either 0 or P itself. We therefore have two possible conclusions: $\Phi_{\mathsf{X},\mathsf{P}} = 0$ or $\Phi^s_{\mathsf{J},\mathsf{P}} = \mathrm{Id}_{\mathsf{P}}$.

Following [28], we say that a full subcategory $\mathscr{P}$ of $\mathscr{C}$ is a left ideal if for all $\mathsf{Q} \in \mathscr{P}$, we have both $\mathsf{D} \boxtimes \mathsf{Q} \in \mathscr{P}$, for all $\mathsf{D} \in \mathscr{C}$, and $\mathsf{D} \in \mathscr{P}$ whenever there exists a composition $\mathsf{D} \to \mathsf{Q} \to \mathsf{D}$ evaluating to the identity on D. We shall assume that $\mathscr{P}$ is equipped with a modified trace $t_\bullet$ [28, 60] (for $\mathscr{P} = \mathscr{C}$, the modified trace is just the ordinary trace $t = \mathrm{tr}$) and a modified dimension $d(\bullet) = t_\bullet(\mathrm{Id}_\bullet)$. We also let $\dim(\bullet) = \mathrm{tr}(\mathrm{Id}_\bullet)$ denote the ordinary trace of the identity morphism.

We now assume that P, as introduced above, belongs to a left ideal $\mathscr{P}$ of $\mathscr{C}$. For any object D of $\mathscr{C}$, the properties of the modified trace imply that

$$\begin{aligned} t_{\mathsf{D}\boxtimes\mathsf{P}}(\mathrm{Id}_{\mathsf{D}\boxtimes\mathsf{P}}) &= t_{\mathsf{D}\boxtimes\mathsf{P}}(\mathrm{Id}_{\mathsf{D}} \boxtimes \mathrm{Id}_{\mathsf{P}}) \\ &= t_{\mathsf{P}}(\mathrm{ptr}^{\mathrm{Left}}(\mathrm{Id}_{\mathsf{D}} \boxtimes \mathrm{Id}_{\mathsf{P}})) = t_{\mathsf{P}}(\mathrm{tr}(\mathrm{Id}_{\mathsf{D}}) \boxtimes \mathrm{Id}_{\mathsf{P}}) \\ &= \dim(\mathsf{D}) t_{\mathsf{P}}(\mathrm{Id}_{\mathsf{P}}) = \dim(\mathsf{D}) d(\mathsf{P}) \end{aligned}$$

and hence that

$$\begin{aligned} t_{\mathsf{P}}(\Phi_{\mathsf{J}^s,\mathsf{P}}) &= t_{\mathsf{P}}(\mathrm{ptr}^{\mathrm{Left}}(M_{\mathsf{J}^s,\mathsf{P}})) \\ &= t_{\mathsf{J}^s\boxtimes\mathsf{P}}(M_{\mathsf{J}^s,\mathsf{P}}) = t_{\mathsf{J}^s\boxtimes\mathsf{P}}(\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})) \\ &= \dim(\mathsf{J}^s) d(\mathsf{P}) (\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})). \end{aligned}$$

Here, we have used the balancing property of monodromy and have identified $\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})$ with the scalar by which it acts. In the case that $\Phi_{\mathsf{J}^s,\mathsf{P}} = \mathrm{Id}_{\mathsf{P}}$, so $t_{\mathsf{P}}(\Phi_{\mathsf{J}^s,\mathsf{P}}) = t_{\mathsf{P}}(\mathrm{Id}_{\mathsf{P}}) = d(\mathsf{P})$, it follows that $\dim(\mathsf{J})^s(\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})) = 1$, whenever $d(\mathsf{P}) \neq 0$. We summarise this as follows.

Proposition 3.10. *Let $\mathscr{C}$ be a ribbon category, $\mathsf{J} \in \mathscr{C}$ be a simple current and $\mathsf{X} \in \mathscr{C}$ be a fixed point of J^s so that $\mathsf{J}^s \boxtimes \mathsf{X} \cong \mathsf{X}$, for some $s \in \mathbb{Z}_{>0}$. Let $\mathscr{P}$ be a left ideal of $\mathscr{C}$, equipped with a modified trace $t_\bullet$ and modified dimension $d(\bullet)$. Let $\mathsf{P} \in \mathscr{P}$ be indecomposable such that $d(\mathsf{P}) \neq 0$ and let $M_{\mathsf{J},\mathsf{P}}, \Phi_{\mathsf{J},\mathsf{P}} \in \mathrm{End}(\mathsf{P})$ be semisimple endomorphisms. Then, one of the following must hold:*

(1) $\Phi_{\mathsf{X},\mathsf{P}} = 0$, *which in turn implies that $t_{\mathsf{P}}(\Phi_{\mathsf{X},\mathsf{P}}) = 0$. If $\mathscr{C}$ is a modular tensor category, then this implies that the corresponding modular S-matrix entry is zero.*
(2) $\dim(\mathsf{J})^s(\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})) = 1$, *where we have identified $\theta_{\mathsf{J}^s\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}^s}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1})$ with the scalar by which it acts.*

As these quantities are computable, in principle, we can rule out fixed points for $\mathsf{J} = \mathsf{C}_\lambda$ or $\mathsf{W}_{[\lambda]}$ and thereby deduce a multiplicity-free decomposition. We shall illustrate this proposition below in a rational example.

3.3. Examples

Here, we discuss two simple examples involving the parafermion cosets [110, 62] to illustrate the theory developed in this section. Let $\mathsf{L}_k(\mathfrak{g})$ denote the simple vertex operator algebra of level k associated with the affine Kac-Moody (super)algebra $\widehat{\mathfrak{g}}$. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let $\mathsf{H} \subset \mathsf{L}_k(\mathfrak{g})$ be the corresponding Heisenberg vertex operator subalgebra. The commutant $\mathsf{C} = \mathrm{Com}(\mathsf{H}, \mathsf{L}_k(\mathfrak{g}))$ is called the level k parafermion vertex operator algebra of type $\mathfrak{g}$.

Example 1. For $\mathfrak{g} = \mathfrak{sl}_2$ and $k = 2$, the parafermion coset is the Virasoro minimal model $\mathsf{M}\big(3,4\big)$, also known as the Ising model. The decompositions (3.6) and (3.7) become

$$\mathsf{L}_2(\mathfrak{sl}_2) = \bigoplus_{\lambda \in 4\mathbb{Z}} \big[\mathsf{F}_\lambda \otimes \mathsf{K}_0 \oplus \mathsf{F}_{\lambda+2} \otimes \mathsf{K}_{1/2}\big] = \mathsf{W}_{[0]} \otimes \mathsf{K}_0 \oplus \mathsf{W}_{[2]} \otimes \mathsf{K}_{1/2}, \tag{3.13}$$

where K_h denotes the simple $\mathsf{M}\big(3,4\big)$-module of highest weight h, the lattice of H-weights of $\mathsf{L}_2(\mathfrak{sl}_2)$ is $\mathcal{L} = 2\mathbb{Z}$, and the sublattice of H-weights giving isomorphic coset modules is $\mathcal{N} = 4\mathbb{Z}$. The convention here for F_λ is that λ indicates the $\mathfrak{sl}_2$-weight so that the conformal dimension of this Heisenberg module is $\lambda^2/8$. The lattice vertex operator algebra W is thus obtained by extending H by the group of simple currents generated by F_4.

The representation theory of $\mathsf{L}_2(\mathfrak{sl}_2)$ is semisimple and it has three simple modules M^ω, $\omega = 0, 1, 2$, which are distinguished by the Dynkin labels $(k - \omega, \omega)$ of their highest weights. $\mathsf{L}_2(\mathfrak{sl}_2)$ is identified with M^0 and the decomposition corresponding to (3.13) for M^2 is obtained by swapping K_0 with $\mathsf{K}_{1/2}$. In particular, the $\mathcal{L}$-orbit for M^2 is also $\mathcal{M} = 2\mathbb{Z}$. The situation for M^1 is, however, slightly different:

$$\mathsf{M}^1 = \bigoplus_{\mu \in 2\mathbb{Z}+1} \mathsf{F}_\mu \otimes \mathsf{K}_{1/16} = \mathsf{W}_{[1]} \otimes \mathsf{K}_{1/16} \oplus \mathsf{W}_{[-1]} \otimes \mathsf{K}_{1/16}.$$

Here, $\mathcal{M} = 2\mathbb{Z} + 1$ and $\mathcal{N}' = 2\mathbb{Z} \neq \mathcal{N}$ (the non-isomorphic lattice modules are paired with isomorphic coset modules). In other words, this decomposition fails to be multiplicity-free.

To see that this is consistent with the criterion of Section 3.2, recall that $\widehat{\mathfrak{sl}}_2$ admits a family σ^ℓ, $\ell \in \mathbb{Z}$, of *spectral flow* automorphisms that lift to automorphisms of the corresponding affine vertex algebras. The latter may be used to twist the action on an $\mathsf{L}_k(\mathfrak{sl}_2)$-module M and thereby construct new modules $\sigma^\ell(\mathsf{M})$. Using the conventions of [97], the characters of M and $\sigma^\ell(\mathsf{M})$ are related by

$$\mathrm{ch}\big[\sigma^\ell(\mathsf{M})\big]\big(z;q\big) = z^{\ell k} q^{\ell^2 k/4} \mathrm{ch}\big[\mathsf{M}\big]\big(zq^{\ell/2};q\big).$$

For $k = 2$, spectral flow acts on the simple modules as $\sigma(\mathsf{M}^\omega) = \mathsf{M}^{2-\omega}$, $\omega = 0, 1, 2$. Identifying the weight space of $\mathfrak{sl}_2$ with $\mathbb{C}$ and noting that the scalar product on this space is then $\langle \lambda, \mu \rangle = \frac{1}{4}\lambda\mu$, the criterion of Section 3.2 asks us to check which $\lambda \in \mathbb{C}$ satisfy the relation

$$\mathrm{ch}\big[\mathsf{M}^\omega\big]\big(z;q\big) = z^\lambda q^{\lambda^2/8} \mathrm{ch}\big[\mathsf{M}^\omega\big]\big(zq^{\lambda/4};q\big) = \mathrm{ch}\big[\sigma^{\lambda/2}(\mathsf{M}^\omega)\big]\big(z;q\big), \tag{3.14}$$

for a given M^ω. Since σ^2 acts as the identity, this relation holds for each ω if $\lambda \in \mathcal{N} = 4\mathbb{Z}$. If $\omega \neq 1$, then it does not hold for $\lambda = 2$, hence $\mathcal{N}' = 4\mathbb{Z}$ and both M^0 and M^2 have multiplicity-free decompositions in terms of lattice modules. However, this relation does hold for $\omega = 1$ and $\lambda = 2$, so we cannot conclude that the lattice decomposition of M^1 is multiplicity-free (consistent with our explicit calculation that it is not).

With a little more work, we can also see how this failure is consistent with the criterion of Section 3.2. Let $\mathsf{X} = \mathsf{K}_{1/16}$ and let J be the simple current $\mathsf{K}_{1/2}$, so that X is a fixed point for J: $\mathsf{J} \boxtimes \mathsf{X} \cong \mathsf{X}$. Since $\mathsf{L}_2(\mathfrak{sl}_2)$ is a unitary vertex operator algebra, $\dim(\mathsf{J}) = 1$. Also, as recalled above, θ is given by $\mathsf{e}^{2\mathsf{i}\pi L_0}$, hence, in our notation, it acts on $\mathsf{K_t}$ by $\mathsf{e}^{2\mathsf{i}\pi t}$, where $t = 0, 1/2, 1/16$. Further, it is easy to check that the category $\mathscr{C}$ of $\mathsf{M}\big(3,4\big)$-modules has no non-trivial ideals except for $\mathscr{C}$ itself.

We now verify that for every indecomposable P in $\mathscr{C}$, either condition 1 or 2 of our Hopf link criterion (Theorem 3.10) is satisfied.

$\mathsf{P} = \mathsf{K}_0$: In this case, $\theta_{\mathsf{J}\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1}) = \theta_{\mathsf{K}_{1/2}} \circ (\theta_{\mathsf{K}_{1/2}}^{-1} \boxtimes \theta_{\mathsf{K}_0}^{-1}) = 1$.

$\mathsf{P} = \mathsf{K}_{1/2}$: In this case, $\theta_{\mathsf{J}\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1}) = \theta_{\mathsf{K}_0} \circ (\theta_{\mathsf{K}_{1/2}}^{-1} \boxtimes \theta_{\mathsf{K}_{1/2}}^{-1}) = 1$.

$\mathsf{P} = \mathsf{K}_{1/16}$: In this case, $\theta_{\mathsf{J}\boxtimes\mathsf{P}} \circ (\theta_{\mathsf{J}}^{-1} \boxtimes \theta_{\mathsf{P}}^{-1}) = \theta_{\mathsf{K}_{1/16}} \circ (\theta_{\mathsf{K}_{1/2}}^{-1} \boxtimes \theta_{\mathsf{K}_{1/16}}^{-1}) = -1$, but the modular S-matrix of $\mathsf{M}\big(3,4\big)$ has entry $S_{\mathsf{K}_{1/16},\mathsf{K}_{1/16}} = 0$.

So we see that in the first two cases condition (2) is satisfied while condition (1) holds in the last. This is, of course, consistent with the fact that the decomposition is not multiplicity-free. As an aside, we remark that if we had only known that $\mathsf{K}_{1/16}$ was a fixed-point of the simple current (which implies that the decomposition is not multiplicity-free), then we could have instead deduced that $S_{\mathsf{K}_{1/16},\mathsf{K}_{1/16}}$ must vanish, as above.

Example 2. A more interesting example is the parafermion coset with $\mathfrak{g} = \mathfrak{sl}_2$ at level $k = -4/3$. In [5], Adamović showed that the resulting coset vertex operator algebra is the (simple) singlet algebra $\mathsf{I}(1,3)$ of central charge $c = -7$. This is strongly generated by the energy-momentum tensor and a single conformal primary of weight 5. We can revisit and extend this study using the results of this section. We stress that at this point it is unknown if a large enough category for the parent vertex operator algebra $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ satisfies the conditions that would allow us to apply the theory of Huang–Lepowsky–Zhang. However, it was recently shown [29] that the category of *ordinary* modules for $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ (and more generally $\mathsf{L}_k(\widehat{\mathfrak{g}})$ for an admissible level k of $\widehat{\mathfrak{g}}$) does satisfy the necessary conditions and indeed forms a ribbon category. Nevertheless, we shall proceed with the analysis, assuming that this theory may be applied. The results suggest that this assumption is, in this case, not unreasonable.

Let Λ_0 and Λ_1 denote the fundamental weights of $\widehat{\mathfrak{sl}}_2$. The vertex operator algebra $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ admits precisely three highest-weight modules, namely the simple modules M^ω whose highest weights have the form $(k-\omega)\Lambda_0 + \omega\Lambda_1$, where $\omega \in \{0, -2/3, -4/3\}$, as well as an uncountable number of simple non-highest-weight modules [6], [59], [101]. In particular, $\mathsf{I}(1,3)$ is not a rational vertex operator algebra. As the level is negative and these highest-weight modules have conformal weights that are bounded below, the criterion of Section 3.2 applies and we conclude that their decompositions are multiplicity-free.

Explicitly, the decomposition (3.6) takes the form

$$\mathsf{L}_{-4/3}(\mathfrak{sl}_2) = \bigoplus_{\lambda \in 2\mathbb{Z}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda, \tag{3.15}$$

where C_λ is a simple highest-weight $\mathsf{I}(1,3)$-module whose highest-weight vector has conformal weight $\Delta_\lambda = |\lambda|\,(3\,|\lambda| + 8)/16$. The convention here for F_λ is again that λ indicates the $\mathfrak{sl}_2$-weight so that the conformal dimension of this Heisenberg module is $-3\lambda^2/16$. Of course, C_λ and $\mathsf{C}_{-\lambda}$ are not isomorphic for $\lambda \neq 0$ because the decomposition (3.15) is multiplicity-free — they must therefore be distinguished by the action of the zero mode of the weight 5 conformal primary.

The theory of Section 3.1 shows that the C_λ, with $\lambda \in 2\mathbb{Z}$, are all (non-isomorphic) simple currents. This had been previously deduced [100], [36] from the (conjectural) *standard* Verlinde formula of [40], [102] for non-rational vertex operator algebras. Noting that $\Delta_{\pm 4} = 5$, we remark [44], [100] that the simple current extension of $\mathsf{I}(1,3)$ by the C_λ, with $\lambda \in 4\mathbb{Z}$, is the triplet algebra $\mathsf{W}(1,3)$ of Kausch [72].

Consider now the $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$-modules $\sigma^{-2}(\mathsf{M}^{-2/3})$ and $\sigma(\mathsf{M}^{-2/3})$, obtained by twisting the action on $\mathsf{M}^{-2/3}$ by the spectral flow automorphisms σ^ℓ, $\ell \in \mathbb{Z}$. Whilst both these modules have conformal weights that are unbounded below, their decompositions into $\mathsf{H} \otimes \mathsf{I}(1,3)$-modules are nevertheless multiplicity-free:

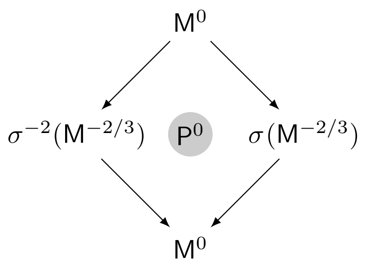
$$\sigma^{-2}(\mathsf{M}^{-2/3}) = \sum_{\mu \in 2\mathbb{Z}} \mathsf{F}_\mu \otimes \mathsf{D}_\mu^{(-2)}, \quad \sigma(\mathsf{M}^{-2/3}) = \sum_{\mu \in 2\mathbb{Z}} \mathsf{F}_\mu \otimes \mathsf{D}_\mu^{(1)}.$$

Here, the $\mathsf{D}_\mu^{(-2)}$ and $\mathsf{D}_\mu^{(1)}$ are simple highest-weight $\mathsf{I}(1,3)$-modules whose highest-weight vectors have conformal weights given by

$$\Delta_\mu^{(-2)} = \begin{cases} \dfrac{\mu(3\mu+8)}{16} & \text{if } \mu \le -2, \\ \dfrac{(\mu+4)(3\mu+4)}{16} & \text{if } \mu \ge -2 \end{cases} \quad \text{and} \quad \Delta_\mu^{(1)} = \begin{cases} \dfrac{(\mu-4)(3\mu-4)}{16} & \text{if } \mu \le 2, \\ \dfrac{\mu(3\mu-8)}{16} & \text{if } \mu \ge 2, \end{cases}$$

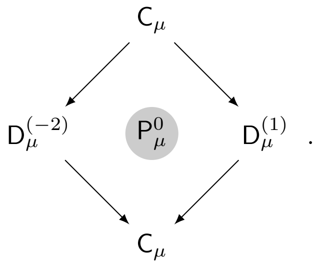
respectively.

The interesting thing about the $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$-modules $\sigma^{-2}(\mathsf{M}^{-2/3})$ and $\sigma(\mathsf{M}^{-2/3})$ is that they appear, together with two copies of the vacuum module M^0, as the composition factors of an indecomposable $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$-module P^0. This module was first constructed as a fusion product in [59] and was structurally characterised in [39] (see [8] for a construction and characterisation of a different indecomposable $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$-module). The action of the Virasoro zero mode L_0 on P^0 is non-semisimple. The Loewy diagram for P^0 has the form



,

where our convention is that the socle appears at the bottom. An immediate consequence of Theorem 3.8 is that there exists a countably-infinite number of mutually non-isomorphic indecomposable $\mathsf{I}(1,3)$-modules P^0_μ, $\mu \in 2\mathbb{Z}$, on which the $\mathsf{I}(1,3)$ Virasoro zero mode acts non-semisimply. The Loewy diagrams of these indecomposables are



The existence of such $\mathsf{I}(1,3)$-modules was predicted in [100] from the fact that similar indecomposables have been constructed [8], [106] for a simple current extension, the triplet algebra $\mathsf{W}(1,3)$.

4. Properties of Heisenberg cosets

Recall from the introduction that one of our main applications for Heisenberg cosets is to construct new, potentially C_2-cofinite, vertex operator algebras as extensions:

$$\mathsf{V} \xrightarrow{\mathsf{H}-\text{coset}} \mathsf{C} \xrightarrow{\text{extension}} \mathsf{E}.$$

So far, we understand how V-modules decompose as $\mathsf{H}\otimes\mathsf{C}$-modules. The remaining tasks are to identify when C may be extended by certain abelian intertwining algebras to a larger algebra E. This will be stated in Theorem 4.1. Since abelian intertwining algebra extensions are mild generalisations of simple current extensions, analogous arguments to [30] allow us to give precise criteria for the lifting of $\mathsf{H} \otimes \mathsf{C}$-modules to V-modules, see Theorem 4.3. An analogous criterion for the lifting of C-modules to E-modules is given in Theorem 4.4.

4.1. Extended algebras

If certain Fock modules involved in the vertex operator algebra decomposition yield a lattice vertex operator (super)algebra, then the corresponding coset modules form a vertex operator (super)algebra as well. Thus, we get extensions of the coset.

Theorem 4.1. *Let*

$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda.$$

If $\mathcal{E}$ is a sub-lattice of $\mathcal{L}$, such that $\bigoplus_{\lambda\in\mathcal{E}} \mathsf{F}_\lambda$ forms a lattice vertex operator (super)algebra, then $\mathsf{E} = \bigoplus_{\lambda\in\mathcal{E}} \mathsf{C}_\lambda$ has a natural vertex operator (super)algebra structure.

Moreover, assume that V is simple, the zeroth weight space of V is spanned by its vacuum, the C_λ are mutually inequivalent, and that the zeroth weight space of E is spanned by its vacuum. Then, E is simple.

Proof. The first statement is an immediate corollary of [81, Thms. 3.1, 3.2] with $\ell = 1$, see also [49]. These results in fact guarantee a generalised vertex algebra structure on $\bigoplus_{\lambda \in \mathcal{L}} \mathsf{C}_\lambda$. Note that no restrictions with regards to vertex tensor category theory are needed on V or C.

For the second statement, we first show that given any non-zero homogeneous $v \in \mathsf{V}$, there exists a homogeneous $w \in \mathsf{V}$ such that $Y_\mathsf{V}(w,x)v$ contains $\mathbf{1}_\mathsf{V}$ as a coefficient. Indeed, by [82, Cor. 4.5.10], there exist homogeneous $w^1, \dots, w^k \in \mathsf{V}$ and $n_1, \dots, n_k \in \mathbb{Z}$ such that $\mathbf{1}_\mathsf{V} = w^1_{(n_1)}v + \cdots + w^k_{(n_k)}v$. However, since $\mathsf{V}_{[0]} = \mathbb{C}\mathbf{1}_\mathsf{V}$, at least one of the summands is a non-zero scalar multiple of $\mathbf{1}_\mathsf{V}$. Now, fix $\lambda \in \mathcal{E}$ and let $c \in \mathsf{C}_\lambda$ be non-zero and homogeneous. Pick a non-zero homogeneous $f \in \mathsf{F}_\lambda$. Then, there exists a homogeneous $w = \sum_i f^i \otimes c^i \in \mathsf{V}$, with $f^i \in \mathsf{F}_{-\lambda}$ and $c^i \in \mathsf{C}_{-\lambda}$, such that $Y_\mathsf{V}(w,x)(f \otimes c)$ has $\mathbf{1}_\mathsf{V}$ as an expansion coefficient. Again, since $\mathsf{V}_{[0]} = \mathbb{C}\mathbf{1}_\mathsf{V}$, there must exist at least one i for which $Y_\mathsf{V}(f^i \otimes c^i, x)(f \otimes c)$ has the same property. However, by construction of the vertex operator algebra map for E, we can write $Y_\mathsf{V}(f^i \otimes c^i, x)(f \otimes c) = (Y_\mathsf{H}(f^i,x)f) \otimes (Y_\mathsf{E}(c^i,x)c)$, where Y_H and Y_E are the vertex operator maps for H and E, respectively. It follows that $\mathbf{1}_\mathsf{V} = \mathbf{1}_\mathsf{H} \otimes \mathbf{1}_\mathsf{E} = \sum_{n \in \mathbb{Z}} (f^i_{(n)}f) \otimes (c^i_{(K-n)}c)$, where K is some constant depending on the conformal weights of the elements involved. There must now exist n such that the L^E_0-eigenvalue of $c^i_{(K-n)}c$ is 0, hence, it must be a scalar multiple of $\mathbf{1}_\mathsf{E}$, since we have assumed that $\mathsf{E}_{[0]} = \mathbb{C}\mathbf{1}_\mathsf{E}$. This immediately gives the simplicity of E. □

For a more general scenario involving mirror extensions, see [84].

Example 3. Let $\mathfrak{g}$ be a simple simply-laced Lie algebra and choose a level $k = p/q$ that is non-zero and rational (take p and q coprime); we do not require k to be admissible. Then, $\mathsf{L}_k(\mathfrak{g})$ is graded by $(1/\sqrt{k})\mathcal{Q} = \sqrt{q/p}\,\mathcal{Q}$, where $\mathcal{Q}$ is the root lattice:

$$\mathsf{L}_k(\mathfrak{g}) = \bigoplus_{\lambda \in \sqrt{q/p}\,\mathcal{Q}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda.$$

The sublattice $p\sqrt{q/p}\,\mathcal{Q} = \sqrt{pq}\,\mathcal{Q}$ is even, so

$$\mathsf{V}_{\sqrt{pq}\mathcal{Q}} = \bigoplus_{\lambda \in \sqrt{pq}Q} \mathsf{F}_\lambda$$

is a lattice vertex operator algebra. It follows by Theorem 4.1 that

$$\mathsf{E}_{k,\mathfrak{g}} := \bigoplus_{\lambda \in \sqrt{pq}\mathcal{Q}} \mathsf{C}_\lambda$$

is also a vertex operator algebra.

We believe that these extended vertex operator algebras have a good chance to be C_2-cofinite. The main outcome of [21] is that in the case $\mathfrak{g} = \mathfrak{sl}_2$ and $k + 2 \in \mathbb{Q} \setminus \{1/n \mid n \in \mathbb{Z}_{>0}\}$, the characters of the modules of the extended vertex operator algebra are modular (when supplemented by pseudotraces). In two specific examples, C_2-cofiniteness is already known. One of them is $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$, which is thus a continuation of Example 2. The other is $\mathsf{L}_{-1/2}(\mathfrak{sl}_2)$, which will form a part of Example 4 below.

Recall that

$$\mathsf{L}_{-4/3}(\mathfrak{sl}_2) = \bigoplus_{\lambda \in 2\mathbb{Z}} \mathsf{F}_\lambda \otimes \mathsf{C}_\lambda,$$

where C_λ is a simple highest-weight $\mathsf{I}(1,3)$-module whose highest-weight vector has conformal weight $\Delta_\lambda = |\lambda|\,(3\,|\lambda| + 8)/16$ and the Heisenberg Fock module F_λ has conformal dimension $-3\lambda^2/16$. It follows that

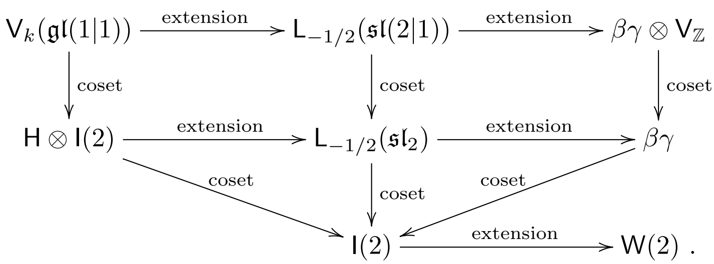
$$\mathsf{V}_{\mathcal{L}} = \bigoplus_{\lambda \in 4\mathbb{Z}} \mathsf{F}_\lambda$$

is the lattice vertex operator algebra with $\mathcal{L} = \sqrt{-6}\,\mathbb{Z}$ and hence

$$\mathsf{W}(1,3) = \bigoplus_{\lambda \in 4\mathbb{Z}} \mathsf{C}_\lambda$$

is also a vertex operator algebra. It is actually the $\mathsf{W}(1,3)$-triplet that is well known to be C_2-cofinite [7]. This relation between singlet vertex operator algebra and $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ was first realised by Adamović [5] and has a nice generalisation to a relation between singlet vertex operator algebras and certain W-algebras [44].

Example 4. We now illustrate how certain well-known, and somehow archetypal, logarithmic vertex operator superalgebras are related via simple current extensions and Heisenberg cosets, thus nicely illustrating the picture advocated in this work and [30]. For these examples, the picture is as follows:



Here, $\mathsf{I}(2)$ is the $p = 2$ singlet vertex operator algebra [6], [36], [40], [100] and $\mathsf{W}(2)$ is its C_2-cofinite, but non-rational, infinite order simple current extension, called the triplet, see [7] for example.

These and other extensions have been worked out in [42], [43], [11] while the coset picture has been part of [42], [45], [44]. Here, the situation of the singlet algebra $\mathsf{I}(2)$ is that the C_1-cofiniteness of the known admissible modules has been established [38], fusion coefficients are known [9], and the category of C_1-cofinite modules is a vertex tensor category in the sense of [67] provided that every C_1-cofinite $\mathbb{N}$-gradable module is of finite length [38, Thm. 17].

For references on $\mathsf{I}(2)$-modules, we refer to [6], [36], [40], [100]; for a reference on $\mathsf{V}_k(\mathfrak{gl}(1|1))$-modules, we refer to [42]. $\mathsf{I}(2)$ has simple highest-weight modules

F_λ of conformal weight $\lambda(\lambda-1)/2$, for $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. For $\lambda = 1 - r \in \mathbb{Z}$, we have instead the non-split short exact sequences

$$0 \to \mathsf{M}_r \to \mathsf{F}_{1-r} \to \mathsf{M}_{r+1} \to 0,$$

where M_r denotes a simple highest-weight module (with $r \in \mathbb{Z}$). Similarly, the affine vertex operator superalgebra $\mathsf{V}_k(\mathfrak{gl}(1|1))$ has simple highest-weight modules $\mathsf{V}_{n,e}$, where $n, e \in \mathbb{R}$ are weight labels and $e/k \notin \mathbb{Z}$. If $\ell = e/k \in \mathbb{Z}$, then we instead have the non-split short exact sequence

$$0 \to \mathsf{A}_{n-1,\ell k} \to \mathsf{V}_{n,\ell k} \to \mathsf{A}_{n,\ell k} \to 0,$$

where $\mathsf{A}_{n,\ell k}$ denotes a simple highest-weight module (with $n \in \mathbb{R}$ and $\ell \in \mathbb{Z}$). The $\mathsf{V}_{n,e}$ with $e/k \notin \mathbb{Z}$ are projective, while the projective cover $\mathsf{P}_{n,\ell k}$ of $\mathsf{A}_{n,\ell k}$ is characterised by the following non-split short exact sequence:

$$0 \to \mathsf{V}_{n+1,\ell k} \to \mathsf{P}_{n,\ell k} \to \mathsf{V}_{n,\ell k} \to 0.$$

The commutant of $\mathsf{I}(2)$ in $\mathsf{V}_k(\mathfrak{gl}(1|1))$ is a rank two Heisenberg vertex operator algebra and we denote the Fock modules of the latter by $\mathsf{F}_{n,e}$. Using the explicit realisation of $\mathsf{V}_k(\mathfrak{gl}(1|1))$-modules found in [45], we determine the decompositions of the simple $\mathsf{V}_k(\mathfrak{gl}(1|1))$-modules to be

$$\mathsf{V}_{n,e} = \bigoplus_{m \in \mathbb{Z}} \mathsf{F}_{m-n,e} \otimes \mathsf{F}_{-m+1-e/k}, \quad \mathsf{A}_{n,\ell k} = \bigoplus_{m \in \mathbb{Z}} \mathsf{F}_{m-n,\ell k} \otimes \mathsf{M}_{m+1+\ell}.$$

It now follows from Theorem 3.8 that

$$\mathsf{P}_{n,\ell k} = \bigoplus_{m \in \mathbb{Z}} \mathsf{F}_{m-n,\ell k} \otimes \mathsf{S}_{m+1+\ell}, \tag{4.1}$$

where S_m is an indecomposable $\mathsf{I}(2)$-module that has non-split short-exact sequence

$$0 \to \mathsf{F}_{1-m} \to \mathsf{S}_m \to \mathsf{F}_{2-m} \to 0.$$

In terms of Loewy diagrams, (4.1) becomes

$$\mathsf{P}_{n,\ell k} = \begin{array}{ccccc} & & \mathsf{A}_{n,\ell k} & & \\ & \swarrow & & \searrow & \\ \mathsf{A}_{n+1,\ell k} & & & & \mathsf{A}_{n-1,\ell k} \\ & \searrow & & \swarrow & \\ & & \mathsf{A}_{n,\ell k} & & \end{array} = \bigoplus_{m \in \mathbb{Z}} \mathsf{F}_{m-n,\ell k} \otimes \left[\begin{array}{ccccc} & & \mathsf{M}_{m+1+\ell} & & \\ & \swarrow & & \searrow & \\ \mathsf{M}_{m+\ell} & & & & \mathsf{M}_{m+2+\ell} \\ & \searrow & & \swarrow & \\ & & \mathsf{M}_{m+1+\ell} & & \end{array} \right].$$

The triplet algebra $\mathsf{W}(2)$ is known to be C_2-cofinite but non-rational. It is a simple current extension of $\mathsf{I}(2)$, namely,

$$\mathsf{W}(2) = \bigoplus_{m \in \mathbb{Z}} \mathsf{M}_{2m+1}.$$

4.2. Lifting coset modules

In this subsection, we show that the question of whether certain generalised C-modules D may be tensored with appropriate Fock modules so that the product can be induced (lifted) to a V-module is essentially answered by the monodromy

$$M_{\mathsf{C}_\lambda,\mathsf{D}} = R_{\mathsf{D},\mathsf{C}_\lambda} \circ R_{\mathsf{C}_\lambda,\mathsf{D}} \colon \mathsf{C}_\lambda \boxtimes \mathsf{D} \to \mathsf{C}_\lambda \boxtimes \mathsf{D}.$$

For the properties of the monodromy used here, we refer to [30].

The following lemma is easily proved as in [30] and will be used frequently below.

Lemma 4.2. *Let $\mathsf{X} \in \mathscr{C}$ be such that for any simple current $\mathsf{J}_i \in \mathscr{C}$, the monodromy satisfies $M_{\mathsf{J}_i,\mathsf{X}} = \lambda_{\mathsf{J}_i,\mathsf{X}} \operatorname{Id}_{\mathsf{J}\boxtimes\mathsf{X}}$, where $\lambda_{\mathsf{J}_1,\mathsf{X}} \in \mathbb{C}$ for $i = 1, 2$. Then, $\lambda_{\mathsf{J}_1,\mathsf{X}}\lambda_{\mathsf{J}_2,\mathsf{X}} = \lambda_{\mathsf{J}_1\boxtimes\mathsf{J}_2,\mathsf{X}}$.*

Theorem 4.3. *Let V, H, C and $\mathcal{L}$ be as in Theorem 3.5, let $\mathcal{L}'$ be the lattice dual to $\mathcal{L}$, let $U = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$, and let D be a generalised C-module that appears as a subquotient of the fusion product of some finite collection of simple C-modules. Then, there exists $\alpha \in U$ such that for all $\lambda \in \mathcal{L}$,*

$$M_{\mathsf{C}_\lambda,\mathsf{D}} = \mathsf{e}^{-2\pi\mathsf{i}\langle\alpha,\lambda\rangle} \operatorname{Id}_{\mathsf{C}_\lambda\boxtimes\mathsf{D}}$$

and $\mathsf{F}_\beta \otimes \mathsf{D}$ lifts to a V-module if and only if $\beta \in \alpha + \mathcal{L}'$.

Proof. Recall that we are working with categories of C and H that have real conformal weights. Additionally, recall that we are working with a semisimple category for H and a category for C in which each object has globally bounded Jordan blocks with respect to the L_0^{C}-action.

We know that $\mathcal{L}$ is equipped with a symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ and that this form takes real values (since the conformal weights with respect to H are real). By the non-degeneracy of $\langle\cdot,\cdot\rangle$, given a homomorphism $f\colon \mathcal{L} \to S^1$, there exists $\alpha \in U$ such that

$$f(\lambda) = \mathsf{e}^{2\mathsf{i}\pi\langle\alpha,\lambda\rangle}, \tag{4.2}$$

for all $\lambda \in \mathcal{L}$. Moreover, $\beta \in U$ also satisfies Equation (4.2) if and only if $\beta \in \alpha+\mathcal{L}'$.

Since each C_λ is a simple current, the monodromy satisfies $M_{\mathsf{C}_\lambda,\,\mathsf{D}} = M_\lambda \operatorname{Id}_{\mathsf{C}_\lambda\boxtimes\mathsf{D}}$ for some scalar $M_\lambda \in \mathbb{C}^\times$ [30]. As $M_{\mathsf{C}_\lambda,\mathsf{D}}$ is semisimple and C_λ, D and $\mathsf{C}_\lambda \boxtimes \mathsf{D}$ have globally bounded L_0^{C}-Jordan blocks, proceeding as in the proof of [30, Eq. (3.10)], we gather that $M_{\mathsf{C}_\lambda,\mathsf{D}} = (\theta_{\mathsf{C}_\lambda\boxtimes\mathsf{D}})_{ss} \circ ((\theta_{\mathsf{C}_\lambda}^{-1})_{ss} \boxtimes (\theta_{\mathsf{D}}^{-1})_{ss})$, where ss denotes the semisimple part. Because each of the modules involved has real conformal weights, it follows that $M_\lambda = \mathsf{e}^{2\mathsf{i}\pi r_\lambda}$ for some $r_\lambda \in \mathbb{R}$. Using Theorem 4.2, we deduce that $\lambda \mapsto M_\lambda$ is a group homomorphism from $\mathcal{L}$ to S^1 and so is $\lambda \mapsto M_\lambda^{-1}$ (since S^1 is abelian).

In Equation (4.2), we now take the homomorphism $f(\lambda) = M_\lambda^{-1}$, deducing the existence of $\alpha \in U$ such that $M_\lambda^{-1} = \mathsf{e}^{2\mathsf{i}\pi\langle\alpha,\lambda\rangle} = M_{\mathsf{F}_\lambda,\mathsf{F}_\alpha}$. Using Theorem 3.3, it follows that $(\mathsf{F}_\lambda \otimes \mathsf{C}_\lambda) \boxtimes (\mathsf{F}_\alpha \otimes \mathsf{D}) \cong \mathsf{F}_{\lambda+\alpha} \otimes (\mathsf{C}_\lambda \boxtimes \mathsf{D})$ and therefore that the monodromy factors over the $\otimes$ tensorands. We conclude that $M_{\mathsf{F}_\lambda\otimes\mathsf{C}_\lambda,\,\mathsf{F}_\alpha\otimes D} = M_{\mathsf{F}_\lambda,\,\mathsf{F}_\alpha} \otimes M_{\mathsf{C}_\lambda,\,\mathsf{D}} = 1$. This means that $\mathsf{F}_\alpha \otimes \mathsf{D}$ lifts to a V-module. Moreover, the arguments given show that $\mathsf{F}_\beta \otimes \mathsf{D}$ lifts if and only if $\beta \in \alpha + \mathcal{L}'$. □

We now combine this with extensions of C, as in Theorem 4.1, to deduce the following result.

Corollary 4.4. *Assume the setup of* Theorem 4.3*. Let $\mathcal{E}$ be a sublattice of $\mathcal{L}$ such that $\mathsf{E} = \bigoplus_{\lambda \in \mathcal{E}} \mathsf{C}_\lambda$ has a vertex operator algebra structure inherited from V, as in Theorem* 4.1*. Then, D lifts to an E-module $\oplus_{\lambda \in \mathcal{E}} \mathsf{C}_\lambda \boxtimes \mathsf{D}$ if and only if the α of Theorem* 4.3 *belongs to $\mathcal{E}'$, where $\mathcal{E}'$ is the dual lattice of $\mathcal{E}$.*

Proof. Recall that each C_λ is a simple current for C. Therefore, using [66] (for the "if" direction) and [30] (for the "only if" direction), we know that $\oplus_{\lambda \in \mathcal{E}} \mathsf{C}_\lambda \boxtimes \mathsf{D}$ is an E-module if and only if $M_{\mathsf{C}_\lambda, \mathsf{C}_\mu \boxtimes \mathsf{D}} = \mathrm{Id}_{\mathsf{C}_\lambda \boxtimes (\mathsf{C}_\mu \boxtimes \mathsf{D})}$, for all $\lambda, \mu \in \mathcal{E}$. Since E is a vertex operator algebra, we know that $M_{\mathsf{C}_\lambda, \mathsf{C}_\mu} = \mathrm{Id}_{\mathsf{C}_\lambda \boxtimes \mathsf{C}_\mu}$ for all $\lambda, \mu \in \mathcal{E}$. Using standard properties of the monodromy, we gather that $M_{\mathsf{C}_\lambda, \mathsf{C}_\mu \boxtimes \mathsf{D}} = \mathrm{Id}_{\mathsf{C}_\lambda \boxtimes (\mathsf{C}_\mu \boxtimes \mathsf{D})}$, for $\lambda, \mu \in \mathcal{E}$, if and only if $M_{\mathsf{C}_\lambda, \mathsf{D}} = \mathrm{Id}_{\mathsf{C}_\lambda \boxtimes \mathsf{D}}$, for all $\lambda \in \mathcal{E}$, which in turn holds if and only if $\alpha \in \mathcal{E}'$. □

Remark 4.5. Since E is a simple current extension of C, we can utilise arguments similar to [74, Thm. 4.4] in order to analyse certain simple E-modules. Let X be a simple E-module such that there exists a simple C-module $\mathsf{X}_0 \subseteq \mathsf{X}$. (In the notation of [74], the role of the group $\mathcal{G}$ is played by $\mathcal{E}$ and the V^χ are identified with the C_λ, $\lambda \in \mathcal{E}$.) Then, $\oplus_{\lambda \in \mathcal{E}} \mathsf{C}_\lambda \boxtimes \mathsf{X}_0$ has a natural structure of an (induced) E-module and it surjects onto X.

Example 5. We now illustrate this lifting result using the unitary $N = 2$ minimal model vertex operator superalgebras. We refer the reader to [3], [4], [46], [104] for additional information on these minimal models.

We start with some well-known results whose proofs can be found, for example, in [32]. Let k be a positive integer, so that $\mathsf{L}_k(\mathfrak{sl}_2)$ contains the lattice vertex operator algebra $\mathsf{V}_{\mathcal{L}_\alpha}$, with $\mathcal{L}_\alpha = \alpha\mathbb{Z}$ and $\alpha^2 = 2k$, hence $\mathcal{L}_\alpha \cong \sqrt{2k}\mathbb{Z}$. The bc-ghost vertex operator algebra $\mathsf{E}(1)$ is isomorphic to $\mathsf{V}_{\mathcal{L}_\beta}$ with $\mathcal{L}_\beta = \beta\mathbb{Z}$ and $\beta^2 = 1$, hence $\mathcal{L}_\beta \cong \mathbb{Z}$. Then, the lattice $\mathcal{L}_\alpha \oplus \mathcal{L}_\beta$ contains the lattice $\mathcal{L}_\gamma = \gamma\mathbb{Z}$ with $\gamma = \alpha + k\beta$ as a sublattice. The orthogonal complement is $\mathcal{N} = \mu\mathbb{Z}$ with $\mu = \alpha - 2\beta$. In [32, Sect. 8] it is proved that

$$\mathsf{S}_k = \mathrm{Com}(\mathsf{V}_{\mathcal{L}_\mu}, \mathsf{L}_k(\mathfrak{sl}_2) \otimes \mathsf{E}(1))$$

is the simple rational $N = 2$ minimal model vertex operator superalgebra of central charge $c = 3k/(k+2)$.

We will now explain how to obtain simple S_k-modules. For this, let λ be an integer with $0 \le \lambda \le k$. Further, let Λ_0 and Λ_1 be the usual fundamental weights of $\widehat{\mathfrak{sl}}_2$. Then, the simple $\mathsf{L}_k(\mathfrak{sl}_2)$-modules are the integrable highest-weight modules $\mathsf{L}(\lambda)$ with highest weights $(k-\lambda)\Lambda_0 + \lambda\Lambda_1$. We note that $\mathsf{V}_{n\alpha/2k + \mathcal{L}_\alpha}$ appears in $\mathsf{L}(\lambda)$ if and only if $\lambda + n$ is even. This follows directly since $\mathsf{V}_{n\alpha/2k+\mathcal{L}_\alpha}$ appears in the decomposition of $\mathsf{L}_k(\mathfrak{sl}_2)$ if and only if n is even.

We now express the lattice vectors of $\mathcal{L}'_\alpha \oplus \mathcal{L}_\beta$ in terms of those of $\mathcal{L}'_\gamma \oplus \mathcal{L}'_\mu$, namely

$$\frac{a}{2k}\alpha + b\beta = (a + bk)\frac{\gamma}{k(k+2)} + (a - 2b)\frac{\mu}{2(k+2)} \quad (a, b \in \mathbb{Z}).$$

It follows that $\mathsf{V}_{n/2(k+2)+\mathcal{N}'}$ is contained in $\mathsf{L}(\lambda)\otimes\mathsf{V}_{\mathcal{L}_\beta}$ if and only if $\lambda+n$ is even as well. We thus get

$$\mathsf{L}(\lambda)\otimes\mathsf{V}_{\mathcal{L}_\beta}\cong\begin{cases}\displaystyle\bigoplus_{\nu\in 2\mathcal{N}'/\mathcal{N}}\mathsf{V}_{\nu+\mathcal{N}}\otimes\mathsf{M}(\lambda,\nu) & \text{if } \nu+\lambda \text{ is even,}\\ \displaystyle\bigoplus_{\nu\in 1/2(k+2)+2\mathcal{L}'/\mathcal{L}}\mathsf{V}_{\nu+\mathcal{L}}\otimes\mathsf{M}(\lambda,\nu) & \text{if } \nu+\lambda \text{ is odd,}\end{cases}$$

as $\mathsf{V}_{\mathcal{L}_\mu}\otimes\mathsf{S}_k$-modules. By Theorem 3.8, part 2, all of the $\mathsf{M}(\lambda,\nu)$ are simple S_k-modules. On the other hand, by Theorem 4.3, for every $\mathcal{L}_k$-module M, there exists a V_N-module $\mathsf{V}_{\nu+N}$ such that $\mathsf{V}_{\nu+\rho+N}\otimes\mathsf{M}$ lifts to a $\mathsf{V}_N\otimes\mathsf{S}_k$-module if and only if $\rho\in(2\mathcal{N}')'/\mathcal{N}=\frac{1}{2}\mathcal{N}/\mathcal{N}$.

Finally, we announce that the relation between the tensor category of a vertex operator algebra and its extensions can be made quite explicit [31] and that these results imply that every simple S_k-module appears in the decomposition of at least one of the $\mathsf{L}(\lambda)\otimes V_{\mathcal{L}_\beta}$. Moreover, one has

$$\mathsf{M}(\lambda,\nu)\cong\mathsf{M}(\lambda',\nu') \quad \text{if and only if} \quad \lambda'=k-\lambda \quad \text{and} \quad \nu'=\nu+\frac{\mu}{2} \mod \mathcal{L}_\mu.$$

4.3. Rationality

In this section, we prove an interesting rationality result. Let V be simple, rational, CFT-type and C_2-cofinite. Then, Theorem 4.12 below states that every grading-restricted generalised C-module is semisimple.

We work with the following setup: Let $\mathsf{C}=\mathrm{Com}(\mathsf{H},\mathsf{V})$ and assume that $\mathrm{Com}(\mathsf{C},\mathsf{V})=\mathsf{V}_{\mathcal{L}}$, where $\mathcal{L}$ is a positive-definite even lattice (possibly zero). With this, $\mathsf{V}_{\mathcal{L}}$ and C form a commuting pair and C is simple. We now collect several well-known results from the literature that guarantee the conditions under which we may invoke the vertex tensor category theory of [67] for C (under suitable assumptions on V).

Lemma 4.6. *If V is C_2-cofinite, then so is C. In particular, if $\mathsf{V}=\mathsf{L}_k(\widehat{\mathfrak{g}})$ with $k\in\mathbb{N}$, then C is C_2-cofinite.*

The proof of this statement can be found in [95]. For the special case in which $\mathsf{V}=\mathsf{L}_k(\widehat{\mathfrak{g}})$ with $k\in\mathbb{N}$, see [19].

Lemma 4.7. *If V is simple and CFT-type, then so is C.*

Proof. As $\mathsf{V}_{\mathcal{L}}$ and C form a commuting pair, there exists a non-zero map $\mathsf{V}_{\mathcal{L}}\otimes\mathsf{C}\to\mathsf{V}$. Since $\mathsf{V}_{\mathcal{L}}$ and C are both simple, so is $\mathsf{V}_{\mathcal{L}}\otimes\mathsf{C}$ and hence this map is an injection. Now, $\mathbf{1}\otimes\mathsf{C}_{[n]}\subseteq\mathsf{V}_{[n]}$ for any n, where we recall that $\mathsf{M}_{[n]}$ denotes the generalised eigenspace of L_0, acting on M, with eigenvalue n. In particular, we conclude that $\mathsf{C}_{[n]}=0$ for $n<0$ and $\mathsf{C}_{[0]}=\mathbb{C}\mathbf{1}_{\mathsf{C}}$. $\square$

Lemma 4.8. *If V is simple, CFT-type and self-contragredient, then so is C.*

Proof. As above, we have an injection $\mathsf{V}_{\mathcal{L}}\otimes\mathsf{C}\hookrightarrow\mathsf{V}$. Since $\mathsf{V}'\cong\mathsf{V}$, there exists a non-degenerate symmetric invariant bilinear form on V, see [58] or [80, Prop. 2.6]. Moreover, the space of symmetric invariant forms on V is naturally isomorphic to $(\mathsf{V}_0/L_1\mathsf{V}_1)^*$ [80, Thm. 3.1]. Since $\mathsf{V}_0 = \mathbb{C}\mathbf{1}$, we conclude that $L_1\mathsf{V}_1 = 0$. Now, $L_1\mathsf{V}_1 = 0$ implies that $L_1(\mathbf{1} \otimes \mathsf{C}_1) = \mathbf{1} \otimes ((L_{\mathsf{C}})_1\mathsf{C}_1) = 0$. This implies that $(L_{\mathsf{C}})_1\mathsf{C}_1 = 0$, which coupled with the simplicity of C implies that there exists a non-degenerate symmetric invariant bilinear form on C, by [80, Cor. 3.2]. In other words, $\mathsf{C}' \cong \mathsf{C}$. □

Lemma 4.9. *If V is simple, CFT-type and C_2-cofinite, then:*

(1) *The category of grading-restricted generalised modules for V and C satisfy the conditions needed to invoke Huang, Lepowsky and Zhang's tensor category theory.*
(2) *Denoting the finite abelian group $\mathcal{L}'/\mathcal{L}$ by $\mathcal{G}$, there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ such that*
$$\mathsf{V} = \bigoplus_{\lambda \in \mathcal{H}} \mathsf{V}_\lambda \otimes \mathsf{C}_\lambda.$$
(3) *Each C_λ appearing in the above decomposition is a simple current for C.*

Indeed, 1 follows from [65] and the previous lemmas; 2 and 3 follow from our results above.

Lemma 4.10. *Let $(\mathcal{L}, \langle \cdot, \cdot \rangle)$ be a positive-definite even lattice, $\mathcal{L}'$ be the dual lattice and let $\mathcal{G} = \mathcal{L}'/\mathcal{L}$. Then, the map $f \colon \mu \mapsto Q_\mu$, where $Q_\mu(\nu) = \mathsf{e}^{2\pi i \langle \mu, \nu \rangle}$ and $\mu, \nu \in \mathcal{G}$, is an isomorphism of $\mathcal{G}$ with its dual group $\widehat{\mathcal{G}}$.*

Proof. It is clear that the image of f is in $\widehat{\mathcal{G}}$. Let λ be in the kernel of f. Then, we see that $\langle \lambda, \mathcal{L}' \rangle \subseteq \mathbb{Z}$, hence that $\lambda \in \mathcal{L}'' = \mathcal{L}$ and so $\lambda = 0$ in $\mathcal{G}$. □

Lemma 4.11. *Let C be C_2-cofinite and CFT-type. Then, the endomorphism space of any grading-restricted generalised module for C is finite-dimensional. Moreover, each grading-restricted generalised module has finite length and has L_0-Jordan blocks of bounded length.*

These are the results [65, Thm. 3.24, Prop. 4.1 and Prop. 4.7]. In fact, the conclusions hold under somewhat weaker hypotheses.

Theorem 4.12. *Let V be simple, rational, C_2-cofinite and CFT-type. Then, every grading-restricted generalised C-module is semisimple.*

Proof. We shall freely use the lemmas above as well as the notation they introduce. Let W be a grading-restricted generalised C-module. We know that W decomposes as a finite direct sum of indecomposable modules. Therefore, without loss of generality, assume that W is indecomposable.

Since W is indecomposable and the C_λ are finite-order simple currents for every $\lambda \in \mathcal{H}$, we know that $M_{\mathsf{C}_\lambda,\mathsf{W}}$ is a scalar multiple $M_\lambda \in \mathbb{C}^\times$ of the identity morphism, by [30, Lem. 3.17]. Let us assume that W is such that for some non-zero C-modules R and S, we have an exact sequence

$$0 \to \mathsf{R} \to \mathsf{W} \to \mathsf{S} \to 0.$$

We know from [30, Lem. 3.19(b)] that $M_{\mathsf{C}_\lambda,\mathsf{R}} = M_\lambda \,\mathrm{id}_{\mathsf{C}_\lambda \boxtimes \mathsf{R}}$ and $M_{\mathsf{C}_\lambda,\mathsf{S}} = M_\lambda \,\mathrm{id}_{\mathsf{C}_\lambda \boxtimes \mathsf{S}}$. From Theorem 4.2, we know that $\lambda \mapsto M_\lambda^{\pm 1}$ are group homomorphisms $\mathcal{H} \to S^1$.

We now seek $\mu \in \mathcal{L}'$ such that, for all $\lambda \in \mathcal{H}$, the $\mathsf{V}_{\mathcal{L}}$-module $\mathsf{V}_{\mu+\mathcal{L}}$ is such that the monodromy of $\mathsf{V}_{\lambda+\mathcal{L}} \otimes \mathsf{C}_\lambda$ with $\mathsf{V}_{\mu+\mathcal{L}} \otimes \mathsf{X}$ is trivial, for $\mathsf{X} = \mathsf{R}$, S and W. In other words, we want to find μ such that for all $\lambda \in \mathcal{H}$,

$$M_{\mathsf{V}_{\mu+\mathcal{L}}, \mathsf{V}_{\lambda+\mathcal{L}}} = M_\lambda^{-1}.$$

Since $\mathcal{H}$ and $\mathcal{G}$ are finite abelian groups, every character of $\mathcal{H}$ can be extended to a character of $\mathcal{G}$. Choose $\chi \in \hat{\mathcal{G}}$ extending $\lambda \mapsto M_\lambda^{-1}$. We will be done if we can find $\mu \in \mathcal{L}'$ such that for each $\lambda \in \mathcal{G} = \mathcal{L}'/\mathcal{L}$, we have

$$Q_\mu(\lambda) = \mathsf{e}^{2\pi i \langle \mu, \lambda \rangle} = M_{\mathsf{V}_{\mu+\mathcal{L}}, \mathsf{V}_{\lambda+\mathcal{L}}} = \chi(\lambda).$$

But, this is guaranteed by Theorem 4.10.

For $\mathsf{X} = \mathsf{R}, \mathsf{S}, \mathsf{W}$, denote $V_{\mu+\mathcal{L}} \otimes \mathsf{X}$ by $\widetilde{\mathsf{X}}$ and let

$$\widetilde{\mathsf{X}}_e = \bigoplus_{\lambda \in \mathcal{H}} (\mathsf{V}_{\lambda+\mathcal{L}} \boxtimes \mathsf{V}_{\mu+\mathcal{L}}) \otimes (\mathsf{C}_\lambda \boxtimes \mathsf{X}) = \bigoplus_{\lambda \in \mathcal{H}} \mathsf{V}_{\lambda+\mu+\mathcal{L}} \otimes (\mathsf{C}_\lambda \boxtimes \mathsf{X}).$$

We now invoke [66, Thm. 3.4] to see that $\widetilde{\mathsf{X}}_e$ is indeed a generalised (untwisted) module for V.

Using the exactness of simple currents (Theorem 2.8, part 4), we deduce the following exact sequence of V-modules:

$$0 \to \widetilde{\mathsf{R}}_e \to \widetilde{\mathsf{W}}_e \to \widetilde{\mathsf{S}}_e \to 0.$$

However, every such exact sequence splits by the rationality of V. As any morphism of V-modules preserves Heisenberg weights, we conclude that $0 \to \mathsf{R} \to \mathsf{W} \to \mathsf{S} \to 0$ splits. □

Now we can combine our results with those of [64, 65] to obtain the following corollary.

Corollary 4.13. *If V is simple, rational, C_2-cofinite, CFT-type and self-contragredient, then we have the following:*

(1) *Every C-module is semisimple.*
(2) *There exist finitely many inequivalent simple modules.*
(3) *Fusion coefficients amongst simple modules are finite.*
(4) *Every finitely generated generalised C-module is an ordinary C-module.*

In particular, C is rational and the category of finitely generated C-modules has the structure of a modular tensor category.

Example 6. The Bershadsky–Polyakov algebra [25, 96] is the quantum Hamiltonian reduction of $\mathsf{L}_{\ell-3/2}(\mathfrak{sl}_3)$ for the minimal embedding of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3$. This vertex operator algebra is strongly generated by four fields of conformal dimensions 1, 2, 3/2 and 3/2. We denote its simple quotient by $\mathcal{W}_\ell$. This quotient is rational provided that ℓ is a positive integer [12].

When $\ell \in \mathbb{Z}_{>0}$, $\mathcal{W}_\ell$ contains the lattice vertex operator algebra $\mathsf{V}_{\mathcal{L}}$ of the lattice $\mathcal{L} = \sqrt{6(\ell-1)}\mathbb{Z}$ as a vertex operator subalgebra [16]. Furthermore, its Heisenberg coset is rational, since it is isomorphic to the principal W-algebra $\mathcal{W}(\mathfrak{sl}_{2\ell})$ at level $k = -2\ell + (2\ell+3)/(2\ell+1)$ and central charge $c = -3(2\ell-1)^2/(2\ell+3)$ [16], which is known to be rational [13]. Our results give thus another, more direct, proof of the rationality of this coset.

A second example are the Heisenberg cosets of the subregular $\mathcal{W}$-algebra of $\mathfrak{sl}_4$ at levels $k = -4 + (m+4)/3$ for m a positive integer greater than two and $m+1$ not divisible by three. These are also rational [35].

5. Heisenberg cosets inside free field algebras and $\mathsf{L}_{-1}(\mathfrak{sl}(m|n))$

We take the opportunity to prove that $\mathsf{L}_{-1}(\mathfrak{sl}(m|n))$ arises as a certain Heisenberg coset inside a free field algebra, specifically inside a certain tensor product of bc- and $\beta\gamma$-ghost systems. It has been known for a while that this affine vertex operator superalgebra is a subalgebra of the Heisenberg coset [70]. Identifying this coset precisely is not only of interest in its own right, but it also gives a different proof to a recent result for the case $n = 0$ and $m \geq 3$ [10]. As simple affine vertex operator superalgebras are poorly understood at present, we hope that one can use this realisation to clarify the structure of $\mathsf{L}_{-1}(\mathfrak{sl}(m|n))$-modules.

Let S denote the $\beta\gamma$-system, which has even generators β and γ and defining operator product expansions

$$\beta(z)\beta(w) \sim 0, \quad \beta(z)\gamma(w) \sim \frac{1}{z-w}, \quad \gamma(z)\gamma(w) \sim 0.$$

Let H be the Heisenberg vertex operator subalgebra generated by $h = {:}\beta\gamma{:}$ and let $\mathsf{C} = \mathrm{Com}(\mathsf{H}, \mathsf{S})$. By a theorem of Wang [108], C is isomorphic to the singlet algebra $\mathsf{I}(2)$. The explicit generators, suitably normalised, are as follows:

$$\begin{aligned} L &= {:}\beta\beta\gamma\gamma{:} + 2\,{:}\beta\partial\gamma{:} - 2\,{:}\partial\beta\gamma{:}\,, \\ W &= {:}\beta\beta\beta\gamma\gamma\gamma{:} + 3\,{:}\beta\beta\partial\gamma\gamma{:} - 6\,{:}\partial\beta\beta\gamma\gamma{:} - 6\,{:}\partial\beta\partial\gamma{:} + 3\,{:}\partial^2\beta\gamma{:}\,. \end{aligned} \tag{5.1}$$

Now, let $\mathsf{S}(n)$ denote the rank n $\beta\gamma$-system, generated by the even elements β^i and γ^j, for $i = 1, \dots, n$, subject to

$$\beta^i(z)\beta^j(w) \sim 0, \quad \beta^i(z)\gamma^j(w) \sim \frac{\delta_{i,j}}{z-w}, \quad \gamma^i(z)\gamma^j(w) \sim 0.$$

In this case, H is the Heisenberg vertex operator subalgebra with generator $h = \sum_{i=1}^n {:}\beta^i\gamma^i{:}$ and $\mathsf{C}(n)$ is the coset $\mathrm{Com}(\mathsf{H}, \mathsf{S}(n))$. Note that $\mathsf{C}(n)$ contains n commuting copies of $\mathsf{I}(2)$ with generators L^i and W^i, obtained from (5.1) by replacing β and γ with β^i and γ^i, respectively. Moreover, $\mathsf{C}(n)$ contains the fields

$$\begin{aligned} X^{jk} &= -\,{:}\beta^j\gamma^k{:}\,, \quad j,k = 1,\dots,n, \quad j \neq k, \\ H^\ell &= -\,{:}\beta^1\gamma^1{:} + {:}\beta^{\ell+1}\gamma^{\ell+1}{:}\,, \quad 1 \leq \ell < n, \end{aligned}$$

which generate a homomorphic image of the universal affine vertex operator algebra $\mathsf{V}_{-1}(\mathfrak{sl}_n)$.

A consequence of [85, Thm. 7.3] is the following result:

Lemma 5.1. $\mathsf{C}(n)$ *is generated as a vertex algebra by the* L^i, W^i, X^{jk} *and* H^ℓ, *for* i, j, k, ℓ *as above.*

A recent theorem of Adamović and Perše [10] states that the map $\mathsf{V}_{-1}(\mathfrak{sl}_n) \to \mathsf{C}(n)$ is surjective, for $n \geq 3$, hence that $\mathsf{C}(n)$ is isomorphic to the simple affine vertex operator algebra $\mathsf{L}_{-1}(\mathfrak{sl}_n)$. Using Theorem 5.1, we now provide a much shorter proof of this result. It suffices to show that all the L^i and W^i lie in the image of the map $\mathsf{V}_{-1}(\mathfrak{sl}_n) \to \mathsf{C}(n)$ and, by symmetry, it is enough to prove this for L^1 and W^1. However, this is immediate from the following identifications:

$$\begin{aligned} L^1 &= \,:H^1H^2: + :X^{12}X^{21}: + :X^{13}X^{31}: - :X^{23}X^{32}: - \partial H^1, \\ W^1 &= - :H^1H^2H^2: - :X^{12}X^{21}H^2: - :X^{13}X^{31}H^1: - :X^{13}X^{31}H^2: \\ &\quad + :X^{23}X^{32}H^2: - :X^{13}X^{32}X^{21}: + \tfrac{1}{2} :X^{12}\partial X^{21}: - \tfrac{3}{2} :\partial X^{12}X^{21}: \\ &\quad + \tfrac{7}{2} :X^{13}\partial X^{31}: - \tfrac{9}{2} :\partial X^{13}X^{31}: - \tfrac{1}{2} :X^{23}\partial X^{32}: \\ &\quad + \tfrac{3}{2} :\partial X^{23}X^{32}: - \tfrac{1}{2} :H^1\partial H^2: + \tfrac{1}{2} :\partial H^1H^2: + \tfrac{1}{2}\partial^2H^1. \end{aligned}$$

Next, we find a minimal strong generating set for the remaining case $\mathsf{C}(2)$. In this case, it is readily verified that L^1 and W^1 do not lie in the affine vertex operator algebra generated by X^{12}, X^{21} and H^1. However, consider the following elements of $\mathsf{C}(2)$:

$$\begin{aligned} P &= -\tfrac{1}{2}L^2_{(0)}X^{12} + \tfrac{1}{3} :H^1X^{12}: + \tfrac{2}{3}\partial X^{12} \\ &= \,:\beta^1\partial\gamma^2: - :\partial\beta^1\gamma^2: + \tfrac{1}{3} :\beta^1\beta^1\gamma^1\gamma^2: + \tfrac{2}{3} :\beta^1\beta^2\gamma^2\gamma^2:, \\ Q &= -\tfrac{1}{2}L^1_{(0)}X^{21} - \tfrac{2}{3} :H^1X^{21}: + \tfrac{1}{3}\partial X^{21} \\ &= \,:\beta^2\partial\gamma^1: - :\partial\beta^2\gamma^1: + \tfrac{1}{3} :\beta^1\beta^2\gamma^1\gamma^1: + \tfrac{2}{3} :\beta^2\beta^2\gamma^1\gamma^2:, \\ R &= L^1 - L^2, \quad L = \,:X^{12}X^{21}: + \tfrac{1}{4} :H^1H^1: - \tfrac{1}{2}\partial H^1. \end{aligned}$$

Here, L is the Sugawara–Virasoro field of $\mathsf{V}_{-1}(\mathfrak{sl}_2)$, which has central charge -3, and X^{12}, X^{21} and H^1 are primary with respect to L of conformal weight 1. It is easily verified that P, Q and R are primary of weight 2 with respect to L and that the generators X^{12}, X^{21}, H^1, P, Q and R close under operator product expansion. They therefore strongly generate a vertex operator subalgebra $\mathsf{C}'(2) \subseteq \mathsf{C}(2)$. Moreover, we have

$$\begin{aligned} L^1 &= \tfrac{1}{2}R + :X^{12}X^{21}: + \tfrac{1}{2} :H^1H^1: - \tfrac{1}{2}\partial H^1, \\ L^2 &= -\tfrac{1}{2}R + :X^{12}X^{21}: + \tfrac{1}{2} :H^1H^1: - \tfrac{1}{2}\partial H^1, \\ W^1 &= -\tfrac{1}{2} :RH^1: - :PX^{21}: - \tfrac{1}{2} :H^1H^1H^1: - \tfrac{5}{3} :X^{12}X^{21}H^1: \\ &\quad - \tfrac{13}{3} :\partial X^{12}X^{21}: + \tfrac{10}{3} :X^{12}\partial X^{21}: - \tfrac{1}{6} :\partial H^1H^1: + \tfrac{1}{3}\partial^2H^1, \\ W^2 &= -\tfrac{1}{2} :RH^1: - :PX^{21}: + \tfrac{1}{2} :H^1H^1H^1: + \tfrac{4}{3} :X^{12}X^{21}H^1: \\ &\quad + \tfrac{19}{6} :\partial X^{12}X^{21}: - \tfrac{25}{6} :X^{12}\partial X^{21}: - \tfrac{5}{3} :\partial H^1H^1: + \tfrac{3}{4}\partial R + \tfrac{7}{12}\partial^2H^1. \end{aligned}$$

Since $\mathsf{C}(2)$ is generated by L^1, L^2, W^1, W^2, X^{12}, X^{21} and H^1, this shows that $\mathsf{C}'(2) = \mathsf{C}(2)$. We have therefore proved the following:

Theorem 5.2. $\mathsf{C}(2)$ *is of type* $\mathsf{W}(1,1,1,2,2,2)$*. In fact, it is the simple quotient of an algebra of type* $\mathsf{W}(1,1,1,2,2,2,2)$*, where the Virasoro field of weight* 2 *coincides with the Sugawara field.*

Remark 5.3. Recall that each embedding of $\mathfrak{sl}_2$ inside a reductive Lie superalgebra $\mathfrak{g}$ gives an associated affine W-superalgebra from the level k affine vertex operator superalgebra associated with $\mathfrak{g}$ [71]. Denote by $\mathsf{W}^k(\mathfrak{sl}_4)$ the universal affine $\mathcal{W}$-algebra of $\mathfrak{sl}_4$ for the embedding of $\mathfrak{sl}_2$ such that $\mathfrak{sl}_4$ decomposes into four copies of the adjoint module of $\mathfrak{sl}_2$ plus three copies of the trivial one. This implies that $\mathsf{W}^k(\mathfrak{sl}_4)$ is of type $(1,1,1,2,2,2,2)$ and, in fact, one can check that the three primaries of conformal weight 1 generate the vertex operator subalgebra $\mathsf{V}_{2k+4}(\mathfrak{sl}_2)$. Specialising to $k=-5/2$, so that the central charge of $\mathsf{W}^k(\mathfrak{sl}_4)$ is -3, we see that it contains $\mathsf{L}_{-1}(\mathfrak{sl}_2)$ as a vertex operator subalgebra.

Proposition 5.4. $\mathsf{C}(2)$ *is isomorphic to the simple quotient* $\mathsf{W}_{-5/2}(\mathfrak{sl}_4)$*.*

Proof. At level $k=-5/2$, $\mathsf{W}^{-5/2}(\mathfrak{sl}_4)$ has a singular vector in weight 2 given by the difference between the Virasoro field and the Sugawara field for $\mathsf{V}^{-1}(\mathfrak{sl}_2)$. Therefore in the simple quotient $\mathsf{W}_{-5/2}(\mathfrak{sl}_4)$, $\mathsf{L}_{-1}(\mathfrak{sl}_2)$ is conformally embedded and $\mathsf{W}_{-5/2}(\mathfrak{sl}_4)$ is of type $W(1,1,1,2,2,2)$. Using the free field realization of $\mathsf{W}_{-5/2}(\mathfrak{sl}_4)$ given in [20, Ex. 3.3], a straightforward computation reveals that $\mathsf{C}(2)$ and $\mathsf{W}_{-5/2}(\mathfrak{sl}_4)$ have the same OPE algebra. Since $\mathsf{C}(2)$ is simple, the claim follows. □

Next, we consider Heisenberg cosets inside bc- and $bc\beta\gamma$-ghost systems. First, consider the rank n bc-system $\mathsf{E}(n)$ with odd generators b^i, c^i satisfying

$$b^i(z)b^j(w)\sim 0,\quad b^i(z)c^j(w)\sim \frac{\delta_{i,j}}{z-w},\quad c^i(z)c^j(w)\sim 0.$$

The Heisenberg subalgebra H has generator $h=-\sum_{i=1}^n :b^ic^i:$ and the coset $\mathrm{Com}(\mathsf{H},\mathsf{E}(n))$ is well-known to be trivial, for $n=1$, and isomorphic to $\mathsf{L}_1(\mathfrak{sl}_n)$, for $n\geq 2$.

We therefore turn to the Heisenberg subalgebra H inside $\mathsf{S}(n)\otimes\mathsf{E}(m)$ with generator

$$h=\sum_{i=1}^n :\beta^i\gamma^i: -\sum_{j=1}^m :b^ic^i:.$$

Let $\mathsf{C}(n,m)=\mathrm{Com}(\mathsf{H},\mathsf{S}(n)\otimes\mathsf{E}(m))$ denote the coset. It is easy to verify that $\mathsf{C}(n,m)$ contains the following fields:

$$\begin{aligned}
X^{jk} &= -:\beta^j\gamma^k:, \quad j,k=1,\dots,n,\quad j\neq k,\\
H^{\ell} &= -:\beta^1\gamma^1: + :\beta^{\ell+1}\gamma^{\ell+1}:,\quad 1\leq \ell < n,\\
\bar{X}^{rs} &= :b^rc^s:,\quad r,s=1,\dots,m,\quad r\neq s,\\
\bar{H}^{u} &= :b^1c^1: - :b^{u+1}c^{u+1}:,\quad 1\leq u<m,\\
J^{i,r} &= :\beta^i\gamma^i: - :b^rc^r:,\quad 1\leq i\leq n,\quad 1<r<m,\\
\phi^{r,k} &= :b^r\gamma^k:,\quad \psi^{j,s} = :\beta^jc^s:,\quad j,k=1,\dots,n,\quad r,s=1,\dots,m.
\end{aligned}$$

Moreover, these generate a homomorphic image of $\mathsf{V}_1(\mathfrak{sl}(n|m))$.

Lemma 5.5. *For all $m, n \geq 1$, $\mathsf{C}(n,m)$ is generated as a vertex algebra by the L^i and W^i, with $i = 1, \dots, n$, together with the image of the map $\mathsf{V}_1(\mathfrak{sl}(n|m)) \to \mathsf{C}(n,m)$ referred to above.*

The proof is similar to that of Theorem 5.1.

Theorem 5.6. *For all $m, n \geq 1$, $\mathsf{C}(n,m)$ is isomorphic to the simple affine vertex superalgebra $\mathsf{L}_1(\mathfrak{sl}(n|m))$.*

Proof. Since $\mathsf{C}(n,m)$ is simple, it suffices to show that the L^i and W^i lie in the image of the map $\mathsf{V}_1(\mathfrak{sl}(n|m)) \to \mathsf{C}(n,m)$. By symmetry, it is enough to show this for L^1 and W^1. Consider the following fields in the image of $\mathsf{V}_1(\mathfrak{sl}(n|m))$:

$$J^{1,1} = \,:\beta^1\gamma^1:\, - \,:b^1c^1:\,, \quad \psi^{1,1} = \,:\beta^1 c^1:\,, \quad \phi^{1,1} = \,:b^1\gamma^1:\,.$$

A straightforward calculation shows that

$$\begin{aligned} L^1 &= \,:J^{1,1}J^{1,1}:\, - 2\,:\psi^{1,1}\phi^{1,1}:\, + \partial J^{1,1}, \\ W^1 &= \,:J^{1,1}J^{1,1}J^{1,1}:\, - 3\,:J^{1,1}\psi^{1,1}\phi^{1,1}:\, + 3\,:\partial\psi^{1,1}\phi^{1,1}:\, - \tfrac{1}{2}\partial^2 J^{1,1}, \end{aligned}$$

which completes the proof. □

6. Some C_1-cofiniteness results

In this section, we show that the simple parafermion algebras of $\mathfrak{sl}_2$, as well as the cosets by the Heisenberg subalgebras of the Bershadsky–Polyakov algebras, each admit large categories of C_1-cofinite modules.

6.1. The $\mathfrak{sl}_2$ parafermion algebra

We work with the usual generating set $\{X \equiv E, Y \equiv F, H\}$ for the universal affine vertex operator algebra $\mathsf{V}_k(\mathfrak{sl}_2)$. Let $\mathsf{I}_k \subset \mathsf{V}_k(\mathfrak{sl}_2)$ denote the maximal proper ideal, graded by conformal weight, so that the simple affine vertex operator algebra $\mathsf{L}_k(\mathfrak{sl}_2)$ is isomorphic to $\mathsf{V}_k(\mathfrak{sl}_2)/\mathsf{I}_k$. By abuse of notation, we use the same symbols X, Y and H for the generators of $\mathsf{L}_k(\mathfrak{sl}_2)$. Let $\mathsf{N}_k(\mathfrak{sl}_2) = \mathrm{Com}(\mathsf{H}, \mathsf{L}_k(\mathfrak{sl}_2))$ denote the simple parafermion vertex operator algebra of $\mathfrak{sl}_2$. We will prove the following.

Theorem 6.1. *For all $k \neq 0, -2$, every simple $\mathsf{N}_k(\mathfrak{sl}_2)$-module appearing in the coset decomposition of $\mathsf{L}_k(\mathfrak{sl}_2)$ has the C_1-cofiniteness property.*

Here, we note that we are assuming Miyamoto's definition for C_1-cofiniteness (see [94]): A module M for a vertex algebra V is *C_1-cofinite* if $\mathsf{M}/C_1(\mathsf{M})$ is finite-dimensional, where $C_1(\mathsf{M})$ is spanned by the elements $A_{(j)}m$, where $A \in \mathsf{V}$ has positive conformal weight, $A(z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$, $j < 0$ and $m \in \mathsf{M}$.

When k is a positive integer, $\mathsf{N}_k(\mathfrak{sl}_2)$ is rational, so the C_1-cofiniteness of the above modules is already known. Therefore, we shall assume from here on that k is not a non-negative integer. As the zeroth Dynkin label of the highest weight of the vacuum module of $\mathsf{V}_k(\mathfrak{sl}_2)$ is k, it follows now that I_k does not contain the normally ordered powers $:X^n:$ or $:Y^n:$, for any n.

Recall that the invariant subalgebra $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$ is isomorphic to $\mathsf{H} \otimes \mathsf{N}_k(\mathfrak{sl}_2)$, where the $\mathcal{U}_1$-action is infinitesimally generated by the zero mode H_0. Since each simple $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-module M appearing in the decomposition of $\mathsf{L}_k(\mathfrak{sl}_2)$ is isomorphic to $\mathsf{H}\otimes\mathsf{N}$, for some simple $\mathsf{N}_k(\mathfrak{sl}_2)$-module N, it suffices to prove the C_1-cofiniteness of the simple modules M.

For all $k \neq -2$, the invariant subalgebra $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$ has a strong generating set $\{H, U_{1,j} \mid j \geq 0\}$, where $U_{i,j} = \, :\partial^i X \partial^j Y:$. If $k \neq 0, -2$ and $i \geq 4$, then there is a relation of weight $i+2$ of the form

$$U_{0,i} = P_i(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}), \tag{6.1}$$

where P_i is a normally ordered polynomial in the given fields and their derivatives. Therefore, $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$ is strongly generated by $\{H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}\}$ and hence is of type $\mathsf{W}(1,2,3,4,5)$, for all $k \neq 0, -2$. Moreover, since the map $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1} \to \mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$ is surjective, this strong generating set descends to one for $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$.

Since $\mathcal{U}_1$ is compact and $\mathsf{L}_k(\mathfrak{sl}_2)$ is simple, we have a decomposition

$$\mathsf{L}_k(\mathfrak{sl}_2) = \bigoplus_{n\in\mathbb{Z}} \mathsf{L}_n \otimes \mathsf{M}_n,$$

where L_n is the simple one-dimensional $\mathcal{U}_1$-module, indexed by $n \in \mathbb{Z}$, and the M_n are inequivalent simple $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-modules. More precisely, M_n consists of elements on which H_0 has eigenvalue $2n$. Since $:X^n:$ and $:Y^n:$ are non-zero in $\mathsf{L}_k(\mathfrak{sl}_2)$ (since $k \notin \mathbb{N}$), but these elements lie in M_n and M_{-n}, respectively and have minimal conformal weight n in these eigenspaces, it follows that M_n and M_{-n} are generated as $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-modules by $:X^n:$ and $:Y^n:$, respectively. Note that we have a similar decomposition

$$\mathsf{V}_k(\mathfrak{sl}_2) = \bigoplus_{n\in\mathbb{Z}} \mathsf{L}_n \otimes \widetilde{\mathsf{M}}_n,$$

where the $\widetilde{\mathsf{M}}_n$ are $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-modules which are no longer simple when $\mathsf{V}_k(\mathfrak{sl}_2)$ is not simple.

To prove that the M_n are C_1-cofinite as $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-modules, it suffices to prove the C_1-cofiniteness of $\mathsf{M}_{\pm 1}$. In fact, we shall prove a stronger statement: $\widetilde{\mathsf{M}}_{\pm 1}$ is C_1-cofinite as a $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$-module. Since the map $\widetilde{\mathsf{M}}_{\pm 1} \to \mathsf{M}_{\pm 1}$ is surjective and compatible with the actions of $\mathsf{V}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$ and $\mathsf{L}_k(\mathfrak{sl}_2)^{\mathcal{U}_1}$, this implies the C_1-cofiniteness of $\mathsf{M}_{\pm 1}$. We only prove the C_1-cofiniteness of $\widetilde{\mathsf{M}}_{-1}$; the proof for $\widetilde{\mathsf{M}}_1$ is the same.

Since $\mathsf{V}_k(\mathfrak{sl}_2)$ is *freely* generated by X, Y and H, it has a good increasing filtration

$$\mathsf{V}_k(\mathfrak{sl}_2)_{(0)} \subseteq \mathsf{V}_k(\mathfrak{sl}_2)_{(1)} \subseteq \cdots, \quad \mathsf{V}_k(\mathfrak{sl}_2)_{(0)} = \bigcup_{d\geq 0} \mathsf{V}_k(\mathfrak{sl}_2)_{(d)},$$

where $\mathsf{V}_k(\mathfrak{sl}_2)_{(d)}$ is spanned by normally ordered products of X, Y, H and their derivatives, whose length is at most d. Then, $\widetilde{\mathsf{M}}_{-1}$ inherits this filtration and $(\widetilde{\mathsf{M}}_{-1})_{(d)}$ has a basis consisting of the elements

$$:\partial^{i_1} H \cdots \partial^{i_r} H \partial^{j_1} X \cdots \partial^{j_s} X \partial^{k_1} Y \cdots \partial^{k_s} Y \partial^{k_{s+1}} Y:, \tag{6.2}$$

where $r, s \geq 0$, $i_1 \geq \cdots \geq i_r \geq 0$, $j_1 \geq \cdots \geq j_s \geq 0$, $k_1 \geq \cdots \geq k_s \geq k_{s+1} \geq 0$ and $r + 2s + 1 \leq d$. In particular, $(\widetilde{\mathsf{M}}_{-1})_{(1)}$ has basis $\{\partial^j Y \mid j \geq 0\}$.

Lemma 6.2. *Any* $\omega \in \widetilde{\mathsf{M}}_{-1}$ *of conformal weight* $m > 0$ *is equivalent to a scalar multiple of* $\partial^{m-1}Y$*, modulo* $C_1(\widetilde{\mathsf{M}}_{-1})$.

Proof. It suffices to assume that ω is a monomial of the form (6.2) with $r+2s > 0$, which has filtration degree $r+2s+1$. Let $\nu = \,:\partial^{i_1}H \cdots \partial^{i_r}H\, U_{i_1,j_1} \cdots U_{i_s,j_s}\, \partial^{s+1}Y:$ and observe that ν has conformal weight m and lies in $C_1(\widetilde{\mathsf{M}}_{-1})$, while $\omega - \nu$ has filtration degree $r+2s$. Therefore, by induction on filtration degree, ω is equivalent to an element of filtration degree 1 and weight m. The only such element, up to scalar multiples, is $\partial^{m-1}Y$. □

Now we are ready to prove Theorem 6.1. By the preceding lemma, it is enough to prove that $\partial^i Y \in C_1(\widetilde{\mathsf{M}}_{-1})$, for i sufficiently large. For this purpose, we compute

$$(U_{0,4})_{(0)}(\partial^i Y) = \Big(k + \frac{2}{5}\Big)\partial^{i+5}Y + \cdots,$$

where the remaining terms are of the form $:\partial^r H \partial^{i+4-r}Y:$, for $0 \le r \le i$, and hence lie in $C_1(\widetilde{\mathsf{M}}_{-1})$. Recall that for all $k \neq 0, -2$, we have the relation (6.1) expressing $U_{0,4}$ as a normally ordered polynomial P_4 in H and the $U_{0,j}$ with $j \le 3$. We claim that

$$(U_{0,4})_{(0)}(\partial^i Y) = P_4(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3})_{(0)}(\partial^i Y) \in C_1(\widetilde{\mathsf{M}}_{-1}).$$

To see this, let ω be a term appearing in $P_4(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3})$ of the form $:\alpha_1 \dots \alpha_t:$, where $t > 1$ and α_j is either H, $U_{0,0}$, $U_{0,1}$, $U_{0,2}$, $U_{0,3}$ or one of their derivatives. Then, $\omega_{(0)}(\partial^i Y) \in C_1(\widetilde{\mathsf{M}}_{-1})$ because the zero mode of such an operator cannot only consist of non-negative modes of the α_j. If $t = 1$, then ω is a total derivative by weight considerations, so $\omega_{(0)}(\partial^i Y) = 0$. It follows that for all $k \neq -2/5$, $\partial^i Y \in C_1(\widetilde{\mathsf{M}}_{-1})$ for all $i \ge 5$.

Finally, suppose that $k = -2/5$. A similar computation shows that

$$(U_{0,5})_{(0)}(\partial^i Y) = -\frac{1}{15}\partial^{i+6}Y + \cdots,$$

where the remaining terms are of the form $:\partial^r H \partial^{i+5-r}Y:$, for $0 \le r \le i$, and hence lie in $C_1(\widetilde{\mathsf{M}}_{-1})$. The same argument using the relation $U_{0,5} = P_5(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3})$ shows that $\partial^i Y \in C_1(\widetilde{\mathsf{M}}_{-1})$ for all $i \ge 6$.

6.2. Bershadsky–Polyakov algebras

Let W^k denote the universal Bershadsky–Polyakov algebra, freely generated by fields J, T, G^+ and G^- of conformal weights 1, 2, 3/2 and 3/2, respectively. We refer to [57] for the defining operator product expansions. This algebra appeared originally in [25], [96], and it coincides with the Feigin–Semikhatov algebra $\mathsf{W}_3^{(2)}$ [57] and also with the minimal universal W-algebra associated to $\mathfrak{sl}_3$ [71]. Let $\mathsf{I}_k \subset \mathsf{W}^k$ denote the maximal proper ideal, graded by conformal weight, and let $\mathsf{W}_k = \mathsf{W}^k/\mathsf{I}_k$ be the simple quotient.

The field J generates a Heisenberg algebra H and we define $\mathsf{C}^k = \mathrm{Com}(\mathsf{H}, \mathsf{W}^k)$ and $\mathsf{C}_k = \mathrm{Com}(\mathsf{H}, \mathsf{W}_k)$. In [16], it was shown that C^k is of type $\mathsf{W}(2,3,4,5,6,7)$, for all k except for $\{-1, -3/2, -3\}$. As there is a projection $\mathsf{C}^k \to \mathsf{C}_k$, the strong generators of C^k descend to give strong generators for C_k as well.

Theorem 6.3. *For all $k \neq -1, -3/2, -3$, every simple C_k-module appearing in the coset decomposition of W_k has the C_1-cofiniteness property.*

The proof of this result is similar to that of Theorem 6.1. First, suppose that $k = p/2-3$, for $p = 5, 7, 9, \dots$. It was shown in [16] that $\mathsf{C}_{p/2-3}$ is isomorphic to a simple rational W-algebra associated with $\mathfrak{sl}_{p-3}$ of central charge $c = -3(p-4)^2/p$. Moreover, $\mathsf{W}_{p/2-3}$ is a simple current extension of $\mathsf{C}_{p/2-3} \otimes \mathsf{V}_{\mathcal{L}}$, where $\mathsf{V}_{\mathcal{L}}$ is the lattice vertex algebra for $\mathcal{L} = \sqrt{3p-9}\mathbb{Z}$. From this result, we see that Theorem 6.3 holds for these cases, so from now on we assume that k does not have this form. One consequence of this restriction is that I_k does not contain $:(G^{\pm})^n:$, for any $n \geq 0$.

Recall that $\mathsf{W}_k^{\mathcal{U}_1} \cong \mathsf{H} \otimes \mathsf{C}_k$, where the $\mathcal{U}_1$ action is infinitesimally generated by the zero mode of J. Since each simple $(\mathsf{W}_k)^{\mathcal{U}_1}$-module M appearing in W_k is isomorphic to $\mathsf{H} \otimes \mathsf{N}$, for some simple C_k-module N, it suffices to prove the C_1-cofiniteness of the simple modules M.

For all $k \neq -1, -3/2, -3$, $(\mathsf{W}^k)^{\mathcal{U}_1}$ has [16, Thm. 5.3] a strong generating set $\{J, L, U_{0,j} \mid j \geq 0\}$, where $U_{i,j} = \ :\partial^i G^+ \partial^j G^-:$. Given $i \geq 5$, there is a relation of weight $i+3$ of the form

$$U_{0,i} = P_i(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}), \tag{6.3}$$

where P_i is a normally ordered polynomial in the given fields and their derivatives. Therefore, $(\mathsf{W}_k)^{\mathcal{U}_1}$ is strongly generated by $\{J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}\}$ and hence is of type $\mathsf{W}(1,2,3,4,5,6,7)$ for all $k \neq -1, -3/2, -3$. Since the map $(\mathsf{W}^k)^{\mathcal{U}_1} \to (\mathsf{W}_k)^{\mathcal{U}_1}$ is surjective, this strong generating set descends to a set of strong generators for $(\mathsf{W}_k)^{\mathcal{U}_1}$.

We have the decomposition

$$\mathsf{W}_k = \bigoplus_{n \in \mathbb{Z}} \mathsf{L}_n \otimes \mathsf{M}_n, \tag{6.4}$$

where L_n is the simple one-dimensional $\mathcal{U}_1$-module indexed by $n \in \mathbb{Z}$ and the M_n are inequivalent simple $(\mathsf{W}_k)^{\mathcal{U}_1}$-modules on which J_0 has eigenvalue n. We note that M_n contains a unique, up to scalar, element ω_n of minimal conformal weight $3n/2$. Indeed, $\omega_0 = 1$, $\omega_n = \ :(G^-)^{-n}:$, for $n < 0$, and $\omega_n = \ :(G^+)^n:$, for $n > 0$. It follows that M_n is generated as a $(\mathsf{W}_k)^{\mathcal{U}_1}$-module by ω_n.

As usual, to prove the C_1-cofiniteness of M_n as a $(\mathsf{W}_k)^{\mathcal{U}_1}$-module for all n, it suffices to prove the C_1-cofiniteness of $\mathsf{M}_{\pm 1}$. For this purpose, it is enough to prove that $\widetilde{\mathsf{M}}_{\pm 1}$ is C_1-cofinite as a $(\mathsf{W}^k)^{\mathcal{U}_1}$-module, where the $\widetilde{\mathsf{M}}_n$ are defined by the decomposition of W^k analogous to (6.4). We only prove the C_1-cofiniteness of $\widetilde{\mathsf{M}}_{-1}$ as the proof for $\widetilde{\mathsf{M}}_1$ is virtually identical.

Recall from [16] that W^k has a weak filtration

$$(\mathsf{W}^k)_{(0)} \subseteq (\mathsf{W}^k)_{(1)} \subseteq \cdots, \quad (\mathsf{W}^k) = \bigcup_{d \geq 0} (\mathsf{W}^k)_{(d)},$$

where $(\mathsf{W}^k)_{(d)}$ is spanned by the normally ordered products of J, L, $G^{\pm}$ and their derivatives, where at most d of the fields $G^{\pm}$ and their derivatives appear. Then, $\widetilde{\mathsf{M}}_{-1}$ inherits this filtration and $(\widetilde{\mathsf{M}}_{-1})_{(d)}$ has a basis consisting of the

$$:\partial^{a_1} L \cdots \partial^{a_i} L \partial^{b_1} J \cdots \partial^{b_j} J \partial^{c_1} G^+ \cdots \partial^{c_r} G^+ \partial^{d_1} G^- \cdots \partial^{d_{r+1}} G^-:, \tag{6.5}$$

with $i,j,r \geq 0$, $0 \leq a_1 \leq \cdots \leq a_i$, $0 \leq b_1 \leq \cdots \leq b_j$, $0 \leq c_1 \leq \cdots \leq c_r$, $0 \leq d_1 \leq \cdots \leq d_{r+1}$ and $2r+1 \leq d$.

Lemma 6.4. *Any $\omega \in \widetilde{\mathsf{M}}_{-1}$ of weight $m+3/2 > 0$ is equivalent to a scalar multiple of $\partial^m G^-$, modulo $C_1(\widetilde{\mathsf{M}}_{-1})$.*

Proof. ω is equivalent modulo $C_1(\widetilde{\mathsf{M}}_{-1})$ to a linear combination of terms of the form $:\partial^{a_1} L \cdots \partial^{a_i} L \partial^{b_1} J \cdots \partial^{b_j} J \partial^m G^-:$ by the same argument that was used to prove Theorem 6.2. All such terms, except possibly $\partial^m G^-$, clearly lie in $C_1(\widetilde{\mathsf{M}}_{-1})$. □

To prove Theorem 6.3, it is enough to show that $\partial^i G^- \in C_1(\widetilde{\mathsf{M}}_{-1})$, for i sufficiently large. For this purpose, we compute

$$(U_{0,5})_{(0)}(\partial^i G^-) = \Big(k^2 + \frac{2}{21}k + \frac{1}{28}\Big)\partial^{i+7}G^- + \cdots ,$$

where the remaining terms lie in $C_1(\widetilde{\mathsf{M}}_{-1})$. Recall that for all $k \neq -1, -3/2, -3$, we have the relation (6.3) expressing $U_{0,5}$ as a normally ordered polynomial P_5 in J, L and the $U_{0,j}$ with $j \leq 4$. We claim that

$$(U_{0,5})_{(0)}(\partial^i G^-) = P_5(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4})_{(0)}(\partial^i G^-) \in C_1(\widetilde{\mathsf{M}}_{-1}).$$

To see this, let ω be a term appearing in $P_5(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4})$ of the form $:\alpha_1 \dots \alpha_t:$, where $t > 1$ and α_j is either J, L, $U_{0,0}$, $U_{0,1}$, $U_{0,2}$, $U_{0,3}$, $U_{0,4}$ or one of their derivatives. Then, $\omega_{(0)}(\partial^i G^-) \in C_1(\widetilde{\mathsf{M}}_{-1})$ because the zero mode of such an operator cannot consist only of non-negative modes of the α_j. If $t = 1$, then ω is a total derivative by weight considerations, so $\omega_{(0)}(\partial^i G^-) = 0$. It follows that if k is not a root of $x^2 + 2x/21 + 1/28$, then $\partial^i G^- \in C_1(\widetilde{\mathsf{M}}_{-1})$ for all $i \geq 7$.

Finally, suppose that k is a root of $x^2 + 2x/21 + 1/28$. A similar computation shows that

$$(U_{0,6})_{(0)}(\partial^i G^-) = \Big(k^2 + \frac{1}{56}k + \frac{3}{112}\Big)\partial^{i+8}G^- + \cdots ,$$

where the remaining terms lie in $C_1(\widetilde{\mathsf{M}}_{-1})$. Since k is not a root of $x^2 + x/56 + 3/112$, the same argument using the relation $U_{0,6} = P_6(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4})$ shows that $\partial^i G^- \in C_1(\widetilde{\mathsf{M}}_{-1})$ for all $i \geq 8$.

A. A proof of Theorem 3.1

Let V be a simple vertex operator algebra and let $\mathcal{G}$ be a finitely generated abelian group of semisimple automorphisms of V. Assume that $\mathsf{V} = \bigoplus_{\lambda \in \mathcal{L}} \mathsf{V}_\lambda$ for some subgroup $\mathcal{L}$ of $\widehat{\mathcal{G}}$. Assume also that we are working with a category of V_0-modules that satisfies the conditions required to invoke Huang, Lepowsky and Zhang's tensor category theory (see Theorem 2.3 above); under these conditions we have the following fundamental result [67, Thm 9.23, Cor. 9.24] that we shall need below.

Theorem A.1 ([67, Thm. 9.23, Cor. 9.24]). *Assume the setting of Theorem* 2.3. *Given* $z_1, z_2 \in \mathbb{C}^\times$ *with* $|z_1| > |z_2| > |z_1 - z_2| > 0$, *modules* W_1, W_2, W_3, W_4, M_1 *and logarithmic intertwining operators* $\mathcal{Y}_1$, $\mathcal{Y}_2$ *of types* $\binom{\mathsf{W}_4}{\mathsf{W}_1\,\mathsf{M}_1}$ *and* $\binom{\mathsf{M}_1}{\mathsf{W}_2\,\mathsf{M}_3}$, *respectively, there exists a module* M_2 *and logarithmic intertwining operators* $\mathcal{Y}^1$, $\mathcal{Y}^2$ *of types* $\binom{\mathsf{W}_4}{\mathsf{M}_2\,\mathsf{W}_3}$ *and* $\binom{\mathsf{M}_2}{\mathsf{W}_1\,\mathsf{W}_2}$, *respectively, such that for* $w_4' \in \mathsf{W}_4'$, $w_1 \in \mathsf{W}_1$, $w_2 \in \mathsf{W}_2$, $w_3 \in \mathsf{W}_3$, *the following equality holds:*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3\rangle.$$

Conversely, given $\mathcal{Y}^1$, $\mathcal{Y}^2$ *as above, there exist* M_1, $\mathcal{Y}_1$, $\mathcal{Y}_2$ *satisfying the above equality.*

Put simply, a product of logarithmic intertwining operators may be written as an iterate, and vice versa.

We denote the vertex operator map of V by Y. Fix an $i \in \mathcal{L}$. We shall prove that $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i \cong \mathsf{V}_0$. In other words, we shall prove that V_i is a simple current. The proof we provide below is essentially a detailed version of the proof given in [93], [26].

We break the proof into several steps.

(1) Let us think of Y as a V-intertwining operator of type $\binom{\mathsf{V}}{\mathsf{V}\,\mathsf{V}}$. We have already assumed that V is simple, so V is simple as a V-module. Using [49, Prop. 11.9], we see that for any $t_1, t_2 \in \mathsf{V}$, $Y(t_1, x)t_2 \neq 0$. This implies that the coefficients of $Y(t_1, x)t_2$, as t_1 runs over V_j and t_2 runs over V_k, span a non-zero V_0-submodule of V_{j+k}. Since V_{j+k} is a simple V_0-module, it follows that the coefficients of $Y(t_1, x)t_2$, for $t_1 \in \mathsf{V}_j$ and $t_2 \in \mathsf{V}_k$, span V_{j+k}.

(2) Given generalised V_0-modules A and B, we denote by $\mathcal{Y}^{\boxtimes}_{\mathsf{A},\mathsf{B}}$ the universal intertwining operator of type $\binom{\mathsf{A}\boxtimes\mathsf{B}}{\mathsf{A}\,\mathsf{B}}$ furnished by the universal property of fusion. If V_0 is a direct summand of A, then we assume that $\mathcal{Y}^{\boxtimes}_{\mathsf{A},\mathsf{B}}$ is normalised so that $\mathcal{Y}^{\boxtimes}_{\mathsf{A},\mathsf{B}}(v_0, x)b = Y_{\mathsf{B}}(v_0, x)b$ for all $v_0 \in \mathsf{V}_0$ and $b \in \mathsf{B}$, where Y_{B} is the module map for the V_0-module B. Moreover, for finite direct sums, $\mathsf{A} = \bigoplus \mathsf{A}_i$, we will assume that $\mathcal{Y}^{\boxtimes}_{\mathsf{A},\mathsf{B}}\big|_{\mathsf{A}_i} = \mathcal{Y}^{\boxtimes}_{\mathsf{A}_i,\mathsf{B}}$, for all B.

We mention that in what follows, we will often make the identification $\mathsf{V}_0 \boxtimes \mathsf{V}_i = \mathsf{V}_i$, for simplicity.

(3) Recall that we have fixed an $i \in \mathcal{L}$. By [67, Thm. 9.23, Cor. 9.24], we have the associativity of intertwining operators and hence there exists a logarithmic intertwining operator $\mathcal{Y}_{r,s;i}$ of type $\binom{\mathsf{V}_{r+s}\boxtimes\mathsf{V}_i}{\mathsf{V}_r\ \mathsf{V}_s\boxtimes\mathsf{V}_i}$ such that for complex numbers x, y with $|x| > |y| > |x - y| > 0$, we have

$$\langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_{r+s},\mathsf{V}_i}(Y(u_r, x-y)u_s, y)v_i\rangle = \langle w', \mathcal{Y}_{r,s;i}(u_r, x)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_s,\mathsf{V}_i}(u_s, y)v_i\rangle, \qquad (A.1)$$

for any $u_r \in \mathsf{V}_r$, $u_s \in \mathsf{V}_s$, $v_i \in \mathsf{V}_i$ and $w' \in (\mathsf{V}_{r+s} \boxtimes \mathsf{V}_i)'$.

(4) Taking $u_r = \mathbf{1}$ in (A.1) now gives

$$\langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_s,\mathsf{V}_i}(u_s, y)v_i\rangle = \langle w', \mathcal{Y}_{0,s;i}(\mathbf{1}, x)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_s,\mathsf{V}_i}(u_s, y)v_i\rangle.$$

Combining this with the observation that the coefficients of $\mathcal{Y}^{\boxtimes}_{\mathsf{V}_s,\mathsf{V}_i}(t_s,y)v_i$ span $\mathsf{V}_s \boxtimes \mathsf{V}_i$, we see that

$$\mathcal{Y}_{0,s;i}(\mathbf{1},x)v^e = v^e,$$

for all $v^e \in \mathsf{V}_s \boxtimes \mathsf{V}_i$. Using the Jacobi identity, it now follows that $\mathcal{Y}_{0,s;i}(u_0,x)v^e$, where $u_0 \in \mathsf{V}_0$ and $v^e \in \mathsf{V}_s \boxtimes \mathsf{V}_i$, coincides with the action of u_0 on $\mathsf{V}_s \boxtimes \mathsf{V}_i$ by the V_0-module map.

(5) Taking $u_s = \mathbf{1}$ in (A.1), we instead arrive at

$$\begin{aligned}\langle w', &\mathcal{Y}_{r,0;i}(u_r,x)v_i\rangle \\ &= \langle w', \mathcal{Y}_{r,0;i}(u_r,x)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_0,\mathsf{V}_i}(\mathbf{1},y)v_i\rangle = \langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_r,\mathsf{V}_i}(Y(u_r,x-y)\mathbf{1},y)v_i\rangle \\ &= \langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_r,\mathsf{V}_i}(\mathsf{e}^{(x-y)L_{-1}}u_r,y)v_i\rangle = \langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_r,\mathsf{V}_i}(u_r,y+x-y)v_i\rangle,\end{aligned}$$

where all the equalities hold for complex numbers x,y with $|x|>|y|>|x-y|>0$. We may now choose $y = 2x/3$, as this satisfies the required constraints, and deduce that

$$\mathcal{Y}_{r,0;i}(u_r,x)v_i = \mathcal{Y}^{\boxtimes}_{\mathsf{V}_r,\mathsf{V}_i}(u_r,x)v_i, \tag{A.2}$$

for all $u_r \in \mathsf{V}_r$ and $v_i \in \mathsf{V}_i$.

(6) For complex numbers $|x|>|y|>|z|>|x-z|>|y-z|>|x-y|>0$, we have

$$\begin{aligned}\langle w', \mathcal{Y}_{r,s+t;i}(u_r,x)&\mathcal{Y}_{s,t;i}(u_s,y)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_t,\mathsf{V}_i}(u_t,z)v_i\rangle \\ &= \langle w', \mathcal{Y}_{r,s+t;i}(u_r,x)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_{s+t},\mathsf{V}_i}(Y(u_s,y-z)u_t,z)v_i\rangle \\ &= \langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_{r+s+t},\mathsf{V}_i}(Y(u_r,x-z)Y(u_s,y-z)u_t,z)v_i\rangle \\ &= \langle w', \mathcal{Y}^{\boxtimes}_{\mathsf{V}_{r+s+t},\mathsf{V}_i}(Y(Y(u_r,x-y)u_s,y-z)u_t,z)v_i\rangle \\ &= \langle w', \mathcal{Y}_{r+s,t;i}(Y(u_r,x-y)u_s,y)\mathcal{Y}^{\boxtimes}_{\mathsf{V}_t,\mathsf{V}_i}(u_t,z)v_i\rangle.\end{aligned}$$

Again, because the coefficients of $\mathcal{Y}^{\boxtimes}_{\mathsf{V}_t,\mathsf{V}_i}$ span $\mathsf{V}_t \boxtimes \mathsf{V}_i$, it follows that for all $u_r \in \mathsf{V}_r$, $u_s \in \mathsf{V}_s$ and $v^e \in \mathsf{V}_t \boxtimes \mathsf{V}_i$,

$$\mathcal{Y}_{r,s+t;i}(u_r,x)\mathcal{Y}_{s,t;i}(u_s,y)v^e = \mathcal{Y}_{r+s,t;i}(Y(u_r,x-y)u_s,z)v^e. \tag{A.3}$$

(7) Now we consider $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$. Since the vertex operator map Y for V furnishes a V_0-intertwining operator of type $\binom{\mathsf{V}_0}{\mathsf{V}_{-i}\ \mathsf{V}_i}$, there exists a morphism from $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ to V_0, by the universal property of fusion. As the coefficients of $Y(u_{-i},x)u_i$, for $u_{-i} \in \mathsf{V}_{-i}$ and $u_i \in \mathsf{V}_i$, span V_0, $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ surjects onto V_0. But, the latter is simple, so proving the simplicity of $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ will give $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i \cong \mathsf{V}_0$, as desired.

(8) Let B be a non-zero V_0-submodule of $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ (we recall that $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ is non-zero because it surjects onto V_0) and let

$$\mathsf{E} = \operatorname{span}\left\{\text{coefficients of } \mathcal{Y}_{i,-i;i}(u_i,x)b \mid u_i \in \mathsf{V}_i,\ b \in \mathsf{B}\right\}. \tag{A.4}$$

Since the type of $\mathcal{Y}_{i,-i;i}$ is $\binom{\mathsf{V}_i}{\mathsf{V}_i\ \mathsf{V}_{-i}\boxtimes\mathsf{V}_i}$, E can be regarded as a V_0-submodule of V_i.

(9) E is in fact a non-zero submodule of V_i. Indeed, if it were 0, then the left-hand side of (A.3), with $r = t = -i$ and $s = i$, would vanish and this would imply that $\mathcal{Y}_{0,-i;i}(Y(u_{-i}, x-y)u_i, y)b = 0$ for all $u_{-i} \in \mathsf{V}_{-i}$, $u_i \in \mathsf{V}_i$ and $b \in \mathsf{B}$. However, the coefficients of $Y(u_{-i}, x-y)u_i$ would then span V_0 and thus $\mathcal{Y}_{0,-i;i}(u_0, x)b$ would equal $Y_{\mathsf{B}}(u_0, x)b$, for all $u_0 \in \mathsf{V}_0$, where Y_{B} is the module map for the V_0-module B. Since the coefficients of the module map span the entire module, we have a contradiction.

(10) Since $0 \subsetneq \mathsf{E} \subseteq \mathsf{V}_i$ and V_i is simple, we conclude that $\mathsf{E} = \mathsf{V}_i$. Combining this with Equation (A.2) now gives

$$\begin{aligned}
&\operatorname{span}\{\text{coefficients of } \mathcal{Y}_{-i,0;i}(v_{-i}, x)\mathcal{Y}_{i,-i;i}(v_i, y)b \mid v_{-i} \in \mathsf{V}_{-i}, v_i \in \mathsf{V}_i, b \in \mathsf{B}\} \\
&\quad = \operatorname{span}\{\text{coefficients of } \mathcal{Y}_{-i,0;i}(v_{-i}, x)\epsilon \mid v_{-i} \in \mathsf{V}_{-i}, \epsilon \in \mathsf{E}\} \\
&\quad = \operatorname{span}\{\text{coefficients of } \mathcal{Y}_{-i,0;i}(v_{-i}, x)v_i \mid v_{-i} \in \mathsf{V}_{-i}, v_i \in \mathsf{V}_i\} \\
&\quad = \operatorname{span}\{\text{coefficients of } \mathcal{Y}^{\boxtimes}_{\mathsf{V}_{-i},\mathsf{V}_i}(v_{-i}, x)v_i \mid v_{-i} \in \mathsf{V}_{-i}, v_i \in \mathsf{V}_i\} \\
&\quad = \mathsf{V}_{-i} \boxtimes \mathsf{V}_i.
\end{aligned}$$

However, using the right-hand side of Equation (A.3) instead gives

$$\begin{aligned}
&\operatorname{span}\{\text{coefficients of } \mathcal{Y}_{-i,0;i}(v_{-i}, x)\mathcal{Y}_{i,-i;i}(v_i, y)b \mid v_{-i} \in \mathsf{V}_{-i}, v_i \in \mathsf{V}_i, b \in \mathsf{B}\} \\
&\quad = \operatorname{span}\{\text{coefficients of } \mathcal{Y}_{0,-i;i}(Y(v_{-i}, x-y)v_i, y)b \mid v_{-i} \in \mathsf{V}_{-i}, v_i \in \mathsf{V}_i, b \in \mathsf{B}\} \\
&\quad = \operatorname{span}\{\text{coefficients of } \mathcal{Y}_{0,-i;i}(v_0, x-y)b \mid v_0 \in \mathsf{V}_0, b \in \mathsf{B}\} \\
&\quad = \operatorname{span}\{\text{coefficients of } Y_{\mathsf{B}}(v_0, x-y)b \mid v_0 \in \mathsf{V}_0, b \in \mathsf{B}\} \\
&\quad = \mathsf{B}.
\end{aligned}$$

This shows that $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i = \mathsf{B}$ for any non-zero submodule B of $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$. We conclude that $\mathsf{V}_{-i} \boxtimes \mathsf{V}_i$ is simple. Hence, it equals V_0.

Remark A.2. In the proof above, it is clear that we never switched the order of the vertex operator maps. It follows that the statement also holds when V is a vertex operator superalgebra.

References

[1] T. Abe, *A $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra*, Math. Z. **255** (2007), no. 4, 755–792.

[2] T. Abe, C. Dong, H. Li, *Fusion rules for the vertex operator algebra $M(1)$ and V_L^+*, Comm. Math. Phys. **253** (2005), no. 1, 171–219.

[3] D. Adamović, *Representations of the $N = 2$ superconformal vertex algebra*, Int. Math. Res. Not. **1999**, no. 2, 61–79.

[4] D. Adamović, *Vertex algebra approach to fusion rules for $N = 2$ superconformal minimal models*, J. Algebra **239** (2001), no. (2), 549–572.

[5] D. Adamović, *A construction of admissible $A_1^{(1)}$-modules of level $-\frac{4}{3}$*, J. Pure Appl. Algebra **196** (2005), no. 2–3, 119–134.

[6] D. Adamović, A. Milas, *Vertex operator algebras associated to modular invariant representations for* $A_1^{(1)}$, Math. Res. Lett. **2** (1995), no. 5, 563–575.

[7] D. Adamović, A. Milas, *On the triplet vertex algebra* $\mathcal{W}(p)$, Adv. Math. **217** (2008), no. 6, 2664–2699.

[8] D. Adamović, A. Milas, *Lattice construction of logarithmic modules for certain vertex algebras*, Selecta Math. (N.S.) **15** (2009), no. 4, 535–561.

[9] D. Adamović, A. Milas, *Some applications and constructions of intertwining operators in LCFT*, Contemp. Math. **695** (2017), 15–27.

[10] D. Adamović, O. Perše, *Fusion rules and complete reducibility of certain modules for affine Lie algebras*, J. Algebra Appl. **13** (2014), no. 1, 1350062.

[11] C. Alfes, T. Creutzig, *The mock modular data of a family of superalgebras*, Proc. Amer. Math. Soc. **142** (2014), no. 7, 2265–2280.

[12] T. Arakawa, *Rationality of Bershadsky–Polyakov vertex algebras*, Comm. Math. Phys. **323** (2013), no. 2, 627–633.

[13] T. Arakawa, *Rationality of W-algebras: principal nilpotent cases*, Ann. Math. (2) **182** (2015), no. 2, 565–604.

[14] T. Arakawa, *Rationality of admissible affine vertex algebras in the category* $\mathcal{O}$, Duke Math. J. **165** (2016), no. 1, 67–93.

[15] T. Arakawa, T. Creutzig, K. Kawasetsu, A. R. Linshaw, *Orbifolds and cosets of minimal* $\mathcal{W}$*-algebras*, Comm. Math. Phys. **355** (2017), no. 1, 339–372.

[16] T. Arakawa, T. Creutzig, A. R. Linshaw, *Cosets of Bershadsky–Polyakov algebras and rational* $\mathcal{W}$*-algebras of type* A, Selecta Math. (N.S.) **23** (2017), no. 4, 2369–2395.

[17] T. Arakawa, T. Creutzig, A. R. Linshaw, *W-algebras as coset vertex algebras*, `https://arxiv.org/abs/1801.03822`.

[18] T. Arakawa, V. Futorny, L. E. Ramirez, *Weight representations of admissible affine vertex algebras*, Comm. Math. Phys. **353** (2017), no. 3, 1151–1178.

[19] T. Arakawa, C.-H. Lam, H. Yamada, *Zhu's algebra,* C_2*-algebra and* C_2*-cofiniteness of parafermion vertex operator algebras*, Adv. Math. **264** (2014), 261–295.

[20] T. Arakawa, A. Molev, *Explicit generators in rectangular affine* $\mathcal{W}$*-algebras of type* A, Lett. Math. Phys. **107** (2017), no. 1, 47–59.

[21] J. Auger, T. Creutzig, D. Ridout, *Modularity of logarithmic parafermion vertex algebras*, D. Lett Math Phys (2018), `https://doi.org/10.1007/s11005-018-1098-4`.

[22] A. Babichenko, T. Creutzig, *Harmonic analysis and free field realization of the Takiff supergroup of* GL(1|1), SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), Paper 067.

[23] A. Babichenko, D. Ridout, *Takiff superalgebras and conformal field theory,* J. Phys. A **46** (2013), no. 12, 125204.

[24] D. J. Benson, *Representations and Cohomology.* I. *Basic Representation Theory of Finite Groups and Associative Algebras*, Cambridge Studies in Advanced Mathematics, Vol. 30, Cambridge University Press, Cambridge, 1991.

[25] M. Bershadsky, *Conformal field theories via Hamiltonian reduction*, Comm. Math. Phys. **139** (1991), no. 1, 71–82.

[26] S. Carnahan, M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras*, arXiv:1603.05645v4 (2018).

[27] T. Creutzig, T. Gannon, *Logarithmic conformal field theory, log-modular tensor categories and modular forms*, J. Phys. A **50** (2017), no. 40, 404004.

[28] T. Creutzig, T. Gannon, *The theory of C_2-cofinite VOAs*, in preparation.

[29] T. Creutzig, Y.-Z. Huang, J. Yang, *Braided tensor categories of admissible modules for affine Lie algebras*, Commun. Math. Phys. **362** (2018), no. 3, 827–854.

[30] T. Creutzig, S. Kanade, A. R. Linshaw, *Simple current extensions beyond semi-simplicity*, arXiv:1511.08754 (2015).

[31] T. Creutzig, S. Kanade, R. McRae, *Tensor categories for vertex operator superalgebra extensions*, arXiv:1705.05017 (2017).

[32] T. Creutzig, A. R. Linshaw, *Cosets of affine vertex algebras inside larger structures*, to appear in J. Algebra, arXiv:1407.8512v5 (2018).

[33] T. Creutzig, A. R. Linshaw, *The super $\mathcal{W}_{1+\infty}$ algebra with integral central charge*, Trans. Amer. Math. Soc. **367** (2015), no. 8, 5521–5551.

[34] T. Creutzig, A. R. Linshaw, *Orbifolds of symplectic fermion algebras*, Trans. Amer. Math. Soc. **369** (2017), no. 1, 467–494.

[35] T. Creutzig, A. R. Linshaw, *Cosets of the $W^k(sl_4, f_{subreg})$-algebra*, Contemp. Math. **711** (2018), 105–117.

[36] T. Creutzig, A. Milas, *False theta functions and the Verlinde formula*, Adv. Math. **262** (2014), 520–545.

[37] T. Creutzig, A. Milas, *Higher rank partial and false theta functions and representation theory*, Adv. Math. **314** (2017), 203-227.

[38] T. Creutzig, A. Milas, M. Rupert, *Logarithmic link invariants of $\overline{U}_q^H(\mathfrak{sl}_2)$ and asymptotic dimensions of singlet vertex algebras*, J. Pure Appl. Algebra **222** (2016), no. 10, 3224–3247.

[39] T. Creutzig, D. Ridout, *Modular data and Verlinde formulae for fractional level WZW models* I, Nucl. Phys. B **865** (2012), no. 1, 83–114.

[40] T. Creutzig, D. Ridout, *Logarithmic conformal field theory: Beyond an introduction*, J. Phys. A **46** (2013), no. 49, 494006.

[41] T. Creutzig, D. Ridout, *Modular data and Verlinde formulae for fractional level WZW models* II, Nucl. Phys. B **875** (2013), no. 2, 423–458.

[42] T. Creutzig, D. Ridout, *Relating the archetypes of logarithmic conformal field theory*, Nucl. Phys. B **872** (2013), no. 3, 348–391.

[43] T. Creutzig, D. Ridout, *W-algebras extending* $\widehat{\mathfrak{gl}}(1|1)$, in: *Lie Theory and its Applications in Physics*, Springer Proc. Math. Stat., Vol. 36, Springer, Tokyo, 2013, pp. 349–367.

[44] T. Creutzig, D. Ridout, S. Wood, *Coset constructions of logarithmic* $(1,p)$ *models*, Lett. Math. Phys. **104** (2014), no. 5, 553–583.

[45] T. Creutzig, P. B. Rønne, *The* GL(1|1)*-symplectic fermion correspondence*, Nucl. Phys. B **815** (2009), no. 1–2, 95–124.

[46] P. Di Vecchia, J. L. Petersen, M. Yu, H. B. Zheng, *Explicit construction of unitary representations of the $N = 2$ superconformal algebra*, Phys. Lett. B **174** (1986), no. 3, 280–284.

[47] C. Dong, C.-H. Lam, Q. Wang, H. Yamada, *The structure of parafermion vertex operator algebras*, J. Algebra **323** (2010), no. 2, 371–381.

[48] C. Dong, C.-H. Lam, H. Yamada, *W-algebras related to parafermion algebras*, J. Algebra **322** (2009), no. 7, 2366–2403.

[49] C. Dong, J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Mathematics, Vol. 112, Birkhäuser, Boston, 1993.

[50] C. Dong, H. Li, G. Mason, *Compact automorphism groups of vertex operator algebras*, Int. Math. Res. Not. **1996**, no. 18, 913–921.

[51] C. Dong, H. Li, G. Mason, *Simple currents and extensions of vertex operator algebras*, Comm. Math. Phys. **180** (1996), no. 3, 671–707.

[52] C. Dong, H. Li, G. Mason, *Vertex operator algebras and associative algebras*, J. Algebra **206** (1998), no. 1, 67–96.

[53] C. Dong, L. Ren, *Representations of the parafermion vertex operator algebras*, Adv. Math. **315** (2017), 88–101.

[54] C. Dong, Q. Wang, *On C_2-cofiniteness of parafermion vertex operator algebras*, J. Algebra **328** (2011), 420–431.

[55] C. Dong, Q. Wang, *Quantum dimensions and fusion rules for parafermion vertex operator algebras*, Proc. Amer. Math. Soc. **144** (2016), no. 4, 1483–1492.

[56] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories*, Mathematical Surveys and Monographs, Vol. 205, American Mathematical Society, Providence, 2015.

[57] B. L. Feigin, A. M. Semikhatov, *$\mathcal{W}_n^{(2)}$ algebras*, Nucl. Phys. B **698** (2004), no. 3, 409–449.

[58] I. B. Frenkel, Y.-Z. Huang, J. Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Mem. Amer. Math. Soc., Vol. 494, 1993.

[59] M. R. Gaberdiel, *Fusion rules and logarithmic representations of a WZW model at fractional level*, Nucl. Phys. B **618** (2001), no. 3, 407–436.

[60] N. Geer, J. Kujawa, B. Patureau-Mirand, *Ambidextrous objects and trace functions for nonsemisimple categories*, Proc. Amer. Math. Soc. **141** (2013), no. 9, 2963–2978.

[61] N. Genra, *Screening operators for $\mathcal{W}$-algebras*, Selecta Math. (N.S.) **23** (2017), no. 3, 2157–2202.

[62] D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, Nucl. Phys. B **290** (1987), no. 1, 10–24.

[63] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570.

[64] Y.-Z. Huang, *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. **10** (2008), suppl. 1, 871–911.

[65] Y.-Z. Huang, *Cofiniteness conditions, projective covers and the logarithmic tensor product theory*, J. Pure Appl. Algebra **213** (2009), no. 4, 458–475.

[66] Y.-Z. Huang, A. Kirillov Jr., J. Lepowsky, *Braided tensor categories and extensions of vertex operator algebras*, Comm. Math. Phys. **337** (2015), no. 3, 1143–1159.

[67] Y.-Z. Huang, J. Lepowsky, L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra I–VIII, `arXiv:1012.4193v7` (2013),

arXiv:1012.4196v2 (2012), arXiv:1012.4197v2 (2012), arXiv:1012.4198v2 (2012), arXiv:1012.4199v3 (2012), arXiv:1012.4202v3 (2012), arXiv:1110.1929v2 (2012), arXiv:1110.1931v2 (2012).

[68] V. Kac, A. Radul, *Representation theory of the vertex algebra* $W_{1+\infty}$, Transform. Groups **1** (1996), no. 1–2, 41–70.

[69] V. G. Kac, M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), no. 14, 4956–4960.

[70] V. G. Kac, M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell's function*, Comm. Math. Phys. **215** (2001), no. 3, 631–682.

[71] V. G. Kac, M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*, Adv. Math. **185** (2004), no. 2, 400–458.

[72] H. G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. B **259** (1991), no. 4, 448–455.

[73] M. Krauel, M. Miyamoto, *A modular invariance property of multivariable trace functions for regular vertex operator algebras*, J. Algebra **444** (2015), 124–142.

[74] C.-H. Lam, *Induced modules for orbifold vertex operator algebras*, J. Math. Soc. Japan **53** (2001), no. 3, 541–557.

[75] J. Lepowsky, M. Primc, *Structure of the Standard Modules for the Affine Lie Algebra* $A_1^{(1)}$, *Contemp. Math.* American Mathematical Society, Vol. 46, Providence, 1985.

[76] J. Lepowsky, R. L. Wilson, *A new family of algebras underlying the Rogers–Ramanujan identities and generalizations*, Proc. Nat. Acad. Sci. USA **78** (1981), no. 12, part 1, 7254–7258.

[77] J. Lepowsky, R. L. Wilson, *A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities*, Adv. Math. **45** (1982), no. 1, 21–72.

[78] J. Lepowsky, R. L. Wilson, *The structure of standard modules.* I. *Universal algebras and the Rogers–Ramanujan identities*, Invent. Math. **77** (1984), no. 2, 199–290.

[79] J. Lepowsky, R. L. Wilson, *The structure of standard modules.* II. *The case* $A_1^{(1)}$, *principal gradation*, Invent. Math. **79** (1985), no. 3, 417–442.

[80] H. Li, *Symmetric invariant bilinear forms on vertex operator algebras*, J. Pure Appl. Algebra **96** (1994), no. 3, 279–297.

[81] H. Li, *On abelian coset generalized vertex algebras*, Commun. Contemp. Math. **3** (2001), no. 2, 287–340.

[82] H. Li, J. Lepowsky, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, Vol. 227, Birkhäuser, Boston, 2004.

[83] H. Li, X. Xu, *A characterization of vertex algebras associated to even lattices*, J. Algebra **173** (1995), no. 2, 253–270.

[84] X. Lin, *Mirror extensions of rational vertex operator algebras*, Trans. Amer. Math. Soc. **369** (2017), no. 6, 3821–3840.

[85] A. R. Linshaw, *Invariant chiral differential operators and the* $\mathcal{W}_3$ *algebra*, J. Pure Appl. Algebra **213** (2009), no. 5, 632–648.

[86] A. R. Linshaw, *A Hilbert theorem for vertex algebras*, Transform. Groups **15** (2010), no. 2, 427–448.

[87] A. R. Linshaw, *Invariant theory and the $\mathcal{W}_{1+\infty}$ algebra with negative integral central charge*, J. Eur. Math. Soc. **13** (2011), no. 6, 1737–1768.

[88] A. R. Linshaw, *Invariant theory and the Heisenberg vertex algebra*, Int. Math. Res. Not. **2012**, no. 17, 4014–4050.

[89] A. R. Linshaw, *Invariant subalgebras of affine vertex algebras*, Adv. Math. **234** (2013), 61–84.

[90] A. R. Linshaw, *The structure of the Kac-Wang-Yan algebra*, Comm. Math. Phys. **345** (2016), no. 2, 545–585.

[91] A. Milas, *Logarithmic intertwining operators and vertex operators*, Comm. Math. Phys. **277** (2008), no. 2, 497–529.

[92] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C_2-cofiniteness*, Duke Math. J. **122** (2004), no. 1, 51–91.

[93] M. Miyamoto, *Flatness and semi-rigidity of vertex operator algebras*, `arXiv:1104.4675` (2011).

[94] M. Miyamoto, *C_1-cofiniteness and fusion products for vertex operator algebras*, in: *Conformal Field Theories and Tensor Categories*, Proceedings of a workshop held at Beijing International Center for Mathematical Research, Beijing, China, June 13–17, 2011, Springer, Heidelberg, 2014, pp. 271–279.

[95] M. Miyamoto, *C_2-cofiniteness of cyclic-orbifold models*, Comm. Math. Phys. **335** (2015), no. 3, 1279–1286.

[96] A. M. Polyakov, *Gauge transformations and diffeomorphisms*, Internat. J. Modern Phys. A **5** (1990), no. 5, 833–842.

[97] D. Ridout, $\widehat{\mathfrak{sl}}(2)_{-1/2}$*: a case study*, Nucl. Phys. B 814 (2009), no. 3, 485521.

[98] D. Ridout, $\widehat{\mathfrak{sl}}(2)_{-1/2}$ *and the triplet model*, Nucl. Phys. B **835** (2010), no. 3, 314-342.

[99] D. Ridout, *Fusion in fractional level $\widehat{\mathfrak{sl}}(2)$-theories with $k = -\frac{1}{2}$*, Nucl. Phys. B **848** (2011), no. 1, 216–250.

[100] D. Ridout, S. Wood, *Modular transformations and Verlinde formulae for logarithmic (p_+, p_-)-models*, Nucl. Phys. B **880** (2014), 175–202.

[101] D. Ridout, S. Wood, *Relaxed singular vectors, Jack symmetric functions and fractional level $\widehat{\mathfrak{sl}}(2)$ models*, Nucl. Phys. B **894** (2015), 621–664.

[102] D. Ridout, S. Wood, *The Verlinde formula in logarithmic CFT*, Journal of Physics: Conference Series **597** (2015), no. 1, 012065.

[103] I. Runkel, *A braided monoidal category for free super-bosons*, J. Math. Phys. **55** (2014), no. 4, 041702.

[104] R. Sato, *Equivalences between weight modules via $\mathcal{N} = 2$ coset constructions*, `arXiv:1605.02343`(2016).

[105] A. N. Schellekens, S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, Nucl. Phys. B **327** (1989), no. 3, 673–703.

[106] A. Tsuchiya, S. Wood, *The tensor structure on the representation category of the $\mathcal{W}_p$ triplet algebra*, J. Phys. A **46** (2013), no. 44, 445203.

[107] A. Tsuchiya, S. Wood, *On the extended W-algebra of type $\mathfrak{sl}_2$ at positive rational level*, Int. Math. Res. Not. **2015**, no. 14, 5357–5435.

[108] W. Wang, *$\mathcal{W}_{1+\infty}$ algebra, $\mathcal{W}_3$ algebra, and Friedan–Martinec–Shenker bosonization*, Comm. Math. Phys. **195** (1998), no. 1, 95–111.

[109] H. Weyl, *The Classical Groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1997.

[110] А. Б. Замолодчиков, В. А. Фатеев, *Нелокальные* (*парафермионные*) *токи в двумерной конформной квантовой теории поля и самодуальные критические точки в Z_N-симметричных статистических системах*, ЖЭТФ **89** (1985), 380–399. Engl. transl.: A. B. Zamolodchikov, V. A. Fateev, *Nonlocal* (*parafermion*) *currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems*, Soviet Phys. JETP **62** (1985), no. 2, 215–225.