Transformation Groups ©Springer Science+Business Media New York (2018)
Vol. 24, No. 3, 2019, pp. 721–739

JORDAN PROPERTIES OF AUTOMORPHISM GROUPS OF CERTAIN OPEN ALGEBRAIC VARIETIES

TATIANA BANDMAN

Department of Mathematics
Bar-Ilan University
Ramat Gan, 5290002, Israel
bandman@macs.biu.ac.il

YURI G. ZARHIN*

Department of Mathematics
Pennsylvania State University
University Park, PA 16802, USA
zarhin@math.psu.edu

Abstract. Let W be a quasiprojective variety over an algebraically closed field of characteristic zero. Assume that W is birational to a product of a smooth projective variety A and the projective line. We prove that if A contains no rational curves then the automorphism group $G := \mathrm{Aut}(W)$ of W is Jordan. That means that there is a positive integer $J = J(W)$ such that every finite subgroup $\mathcal{B}$ of G contains a commutative subgroup $\mathcal{A}$ such that $\mathcal{A}$ is normal in $\mathcal{B}$ and the index $[\mathcal{B} : \mathcal{A}] \leqslant J$.

1. Introduction

Throughout this paper k is an algebraically closed field of characteristic zero. All varieties, if not indicated otherwise, are irreducible, algebraic, and defined over k. If X is an algebraic variety over k then we write $\mathrm{Aut}(X)$ for its group of (biregular) k-automorphisms and $\mathrm{Bir}(X)$ for its group of birational k-automorphisms. As usual, $\mathbb{P}^n$ stands for the n-dimensional projective space and $\mathbb{A}^n$ ($\mathbb{A}^n_{x_1,\dots,x_n}$) for the n-dimensional affine space (with coordinates $x_1, \dots, x_n$, respectively).

The definition of a Jordan group was introduced in [23].

Definition 1. A group $\mathcal{G}$ is called *Jordan* [23] if there exists a positive integer J that enjoys the following property. Every finite subgroup $\mathcal{B}$ of $\mathcal{G}$ contains a commutative subgroup $\mathcal{A}$ such that $\mathcal{A}$ is normal in $\mathcal{B}$ and the index $[\mathcal{B} : \mathcal{A}] \leqslant J$. Such a smallest J is called the Jordan index of $\mathcal{G}$ and denoted by $J_{\mathcal{G}}$.

Definition 2. Let G be a group.

(a) G is called *bounded* [24], [26] if there is a positive integer $C = C_G$ such that the order of every finite subgroup of G does not exceed C.

DOI: 10.1007/s00031-018-9489-2

*During the preparation of this paper the second named author was partially supported by a grant from the Simons Foundation (#246625 to Yuri Zarkhin). Part of this work was done in May–June 2016 during his stay at the Max-Planck-Institut für Mathematik, whose hospitality and support are gratefully acknowledged.

Received June 26, 2017. Accepted January 27, 2018.
Published online August 11, 2018.
Corresponding Author: Yuri G. Zarhin, e-mail: zarhin@math.psu.edu

(b) G is called *quasi-bounded* if there is a nonnegative integer $a := a(G)$ such that each finite abelian subgroup of G is generated by at most A elements.
(c) G is called *strongly Jordan* [27], [3] if it is Jordan and quasi-bounded.

Remark 1. (i) If
$$\{0\} \to G_1 \to G \to G_2 \to \{0\}$$
is a short exact sequence of groups and both G_1 and G_2 are bounded (resp. quasi-bounded) then one may easily check that G is also bounded (resp. quasi-bounded). Indeed, let H be a finite (resp. finite abelian) subgroup of G. Let H_2 be the image of H in G_2 and H_1 the intersection of H and the kernel of $G \to G_2$. Then H sits in the short exact sequence
$$\{0\} \to H_1 \to H \to H_2 \to \{0\}$$
where H_i is a finite (resp. finite abelian) subgroup of $G_i,\ i = 1, 2$. If both G_1 and G_2 are bounded then the order of H does not exceed $C_{G_1} C_{G_2}$, i.e., G is also bounded. If both G_1 and G_2 are quasi-bounded then H is generated by, at most, $a(G_1) + a(G_2)$ elements [19, Lem. 2.3].

(ii) If both G_1 and G_2 are Jordan then G does *not* have to be Jordan.

(iii) Clearly, every subgroup of a (strongly) Jordan group is also (strongly) Jordan.

The group $\mathrm{GL}_n(\mathbb{Z})$ is bounded by Minkowski's Theorem ([31, Sect. 9.1]). The classical theorem of Jordan ([8, Sect. 36], [31, Sect. 9.2],[22]) asserts that $\mathrm{GL}(n, k)$ is strongly Jordan. An example of a non-Jordan group is given by $\mathrm{GL}(n, \overline{\mathbb{F}}_p)$ where $\overline{\mathbb{F}}_p$ is the algebraic closure of a finite field $\mathbb{F}_p$ and $n \geqslant 2$.

We refer the reader to [24] for references and a survey on this topic.

Let X be an algebraic variety over k. It is known that $\mathrm{Aut}(X)$ is Jordan if either $\dim(X) \leqslant 2$ [23], [2] or X is projective [19]. It is also known ([26] combined with [4]), that if X is an irreducible variety then $\mathrm{Bir}(X)$ is Jordan if either $q(X) = 0$ or X is not uniruled (in particular, Cremona groups $\mathrm{Bir}(\mathbb{P}^N)$ and groups $\mathrm{Aut}(\mathbb{A}^N)$ are Jordan).

On the other hand, $\mathrm{Bir}(X)$ is *not* Jordan if X is birational to a product $A \times \mathbb{P}^n$ where $n \geqslant 1$ and A is a positive-dimensional abelian variety over k [35].

Since $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Bir}(X)$, it is Jordan whenever $\mathrm{Bir}(X)$ is Jordan. But $\mathrm{Aut}(X)$ may be Jordan when $\mathrm{Bir}(X)$ is not. To the best of our knowledge, there is no example of an algebraic variety with non-Jordan automorphisms group. The aim of this paper is to prove the Jordan property of the group $\mathrm{Aut}(X)$ for open subsets of certain uniruled varieties.

Definition 3. We call a smooth projective variety A *rigid* if it is irreducible and contains no rational curves.

We prove the following

Theorem 4. *Let W be an irreducible quasiprojective variety that is birational to a product $A \times \mathbb{P}^1$ where A is a smooth rigid projective variety. Then* $\mathrm{Aut}(W)$ *is strongly Jordan.*

The case of $\dim(W) = 2$, $\dim(A) = 1$ was done in [23], [36], [2].

The case of $\dim(W) = 3$ was studied in [35], [25], [26], [3], [27]. Here is the final answer for $\mathrm{Bir}(X)$ [27]. Let X be a threefold. Then $\mathrm{Bir}(X)$ is not Jordan if and only if either X is birational to $E \times \mathbb{P}^2$, where E is an elliptic curve, or X is birational to $S \times \mathbb{P}^1$, where S is one of the following:

Case 1. an abelian surface;

Case 2. a bielliptic surface;

Case 3. a surface with Kodaira dimension $\varkappa(S) = 1$ such that the Jacobian fibration of the pluricanonical fibration is locally trivial in Zariski topology.

Thus, Theorem 4 leads to the following

Corollary 5. *Assume that W is a quasiprojective irreducible variety of dimension $d \leqslant 3$. Assume that W is not birational to $E \times \mathbb{P}^2$, where E is an elliptic curve. Then* $\mathrm{Aut}(W)$ *is Jordan.*

Remark 2. Let W be a (nonempty) irreducible algebraic variety over k and let $W^{\mathrm{ns}} \subset W$ be the open dense (sub)set of its nonsingular points. Then $u(W^{\mathrm{ns}}) \subset W^{\mathrm{ns}}$ for each $u \in \mathrm{Aut}(W)$. This gives rise to the natural group homomorphism $\mathrm{Aut}(W) \to \mathrm{Aut}(W^{\mathrm{ns}})$, which is injective, since W^{ns} is dense in W in Zariski topology. This implies that in the course of the proof of Theorem 4 and Corollary 5 we may assume that W is *smooth*.

The paper is organized as follows. Section 2 contains notation and auxiliary results about fiberwise automorphisms of fibered varieties. In Section 3 we discuss automorphism groups of varieties that are birational to a product $A \times \mathbb{P}^1$ where A is a smooth rigid projective variety. Section 4 contains the proof of Theorem 4 and Corollary 5

Acknowledgements. We are grateful to Shulim Kaliman, Michel Brion, and Vladimir Berkovich for helpful discussions. Our thanks go to the referees, whose comments helped to improve the exposition.

2. Preliminaries

If X is an irreducible algebraic variety over k then

• We write $k[X]$ for the ring of regular functions on X and $k(X)$ for its field of rational functions. In this case one may view $\mathrm{Bir}(X)$ as the group of all k-linear automorphisms of $k(X)$ and $\mathrm{Aut}(X)$ as a certain subgroup of $\mathrm{Bir}(X)$. We write id_X (or simply id) for the identity automorphism of X, which may be viewed as the identity element of groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$.

• By points of X (unless otherwise stated) we always mean k-points. A *general point* means a point of an open dense subset of X.

• If X is smooth then K_X and $q(X)$ stand for the canonical class of X and irregularity $h^{0,1}(X)$ of X, respectively.

• $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ stand for fields of complex numbers, the rationals, and ring of integers, respectively.

• If F is a field then we write $\overline{F}$ for its algebraic closure.

• Let X, Y, T be irreducible varieties, $p : X \to T,\ q : Y \to T$ morphisms. We say that a rational map $f : X \dashrightarrow Y$ is p, q-fiberwise if there exists morphism $g_f : T \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ T & \xrightarrow{g_f} & T \end{array} .$$

• If $X = Y,\ p = q,\ f \in \mathrm{Bir}(X)$, then we say that f is p-fiberwise and denote by $\mathrm{Bir}_p(X)$ the group of all p-fiberwise birational automorphisms of X. We write $\mathrm{Aut}_p(X)$ for the intersection of $\mathrm{Bir}_p(X)$ and $\mathrm{Aut}(X)$ in $\mathrm{Bir}(X)$, which is the group of all p-fiberwise automorphisms of X.

• Recall that if a smooth projective variety A is rigid, then any rational map from a smooth variety to A is a morphism ([9, Cor. 1.44]). In particular, $\mathrm{Bir}(A) = \mathrm{Aut}(A)$. Abelian varieties and bielliptic surfaces are rigid.

We start with an auxiliary

Lemma 6. *Assume that U, V are smooth irreducible quasiprojective varieties endowed by a surjective morphism $p\colon U \to V$ such that the fiber $P_v := p^{-1}(v)$ is projective and irreducible for every point $v \in V$. Assume that $C \subset U$ is a closed subset and that $C \cap P_v$ is a finite set for every point $v \in V$. Assume that $f \in \mathrm{Aut}_p(U \setminus C)$. Then $f \in \mathrm{Aut}_p(U)$.*

Remark 3. In loose language this Lemma asserts that every fiberwise automorphism $f \in \mathrm{Aut}(U \setminus C)$ may be extended to an automorphism of U if C has only "p-horizontal" components over V.

Proof. Take any smooth projective closure $\overline{V}$ of V and choose such a smooth projective closure $\overline{U}$ of U that the rational extension $\overline{p} : \overline{U} \to \overline{V}$ of p is a morphism. Since all the fibers of p are projective and irreducible, we have $\overline{p}^{-1}(V) = U$, $\overline{p}^{-1}(\overline{V} \setminus V) = \overline{U} \setminus U$ and $\overline{p}\,\big|_U = p$ (see, for example, [18, Sect. 2.6]). Let $\overline{f} : \overline{U} \dashrightarrow \overline{U}$ be the rational extension of f. Let $(\widetilde{U}', \widetilde{f}, \pi)$ be a resolution of indeterminacy of $\overline{f}$. Let $\overline{g}_f \in \mathrm{Bir}(\overline{V})$ be an extension of g_f.

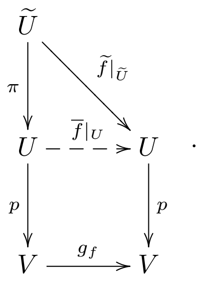
We have a commutative diagram

$$\begin{array}{ccc} \widetilde{U}' & & \\ \big\downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \widetilde{f}} & \\ \overline{U} & \overset{\overline{f}}{\dashrightarrow} & \overline{U} \\ \big\downarrow{\scriptstyle \overline{p}} & & \big\downarrow{\scriptstyle \overline{p}} \\ \overline{V} & \overset{\overline{g}_f}{\dashrightarrow} & \overline{V} \end{array} .$$

Since g_f is an automorphism of V, we have

$$\widetilde{U} := (\overline{p} \circ \pi)^{-1}(V) = (\overline{p} \circ \widetilde{f})^{-1}(V) = \pi^{-1}(U)$$

and we may restrict the maps π, $\widetilde{f}$, $\overline{f}$, $\overline{p}$, $\overline{g}_f$ to quasiprojective varieties $\widetilde{U}, U$, and V and obtain the following commutative diagram:



Here

- π and $\widetilde{f} := \widetilde{f}|_{\widetilde{U}}$ are morphisms;
- $f := \overline{f}|_U \in \mathrm{Aut}(U \setminus C) \cap \mathrm{Bir}(U)$;
- π is an isomorphism of $U_1 := \pi^{-1}(U \setminus C)$ to $U \setminus C$.

We have to show that f is defined at all points of C. For this, we need to check that $\widetilde{f}(\pi^{-1}(c))$ is a point for every point $c \in C$. Since π and $\widetilde{f}$ are birational morphisms, the sets $\widetilde{f}^{-1}(a)$ and $\pi^{-1}(a)$ are connected for every point $a \in U$ by the Zariski Main Theorem (see [20, Chap. III, §9]). Take $a \in U \setminus C$. Then $\widetilde{f}^{-1}(a)$ contains an isolated point $\pi^{-1}(f^{-1}(a)) \in U_1$, which (by the Zariski Main Theorem) is the only connected component of $\widetilde{f}^{-1}(a)$. Thus $\widetilde{f}^{-1}(U \setminus C) = U_1$, $\widetilde{f}^{-1}(C) = \pi^{-1}(C)$, or $\widetilde{f}(\pi^{-1}(C)) = C$, $\widetilde{f}(U_1) = U \setminus C$. Hence for every point $c \in C$ we have

$$\widetilde{f}(\pi^{-1}(c)) \subset C \cap P_{g_f(p(v))}$$

and the latter is a finite set. Since $\widetilde{f}(\pi^{-1}(c))$ has to be connected, it is a single point. Thus $f = \widetilde{f} \circ \pi^{-1}$ is defined at every point of U. □

Lemma 7. *Assume that a group G sits in the short exact sequence*

$$\{0\} \to G_1 \to G \to G_2 \to \{0\}.$$

Suppose that one of the following two condition holds.

(1) *G_1 is bounded and G_2 is strongly Jordan.*
(2) *G_1 is strongly Jordan and G_2 is bounded.*

Then G is strongly Jordan.

Proof. Suppose (1) holds. Then a lemma of Anton Klyachko [3, Lem. 2.1] implies that G is strongly Jordan.

Suppose (2) holds. Then both G_1 and G_2 are quasi-bounded. By Remark 1, G is also quasi-bounded. It follows from [19, Lem. 2.3(1)] that G is Jordan. This implies that G is strongly Jordan. □

Remark 4. Let X be a projective variety. It is actually proven in [19] that $\mathrm{Aut}(X)$ is strongly Jordan (not just Jordan): the assertion follows readily from the combination of [19, Thm. 1.4] and [19, Lem. 2.5].

In the next Proposition we consider the group $\mathrm{Aut}_p(X)$ where $p : W \to A$ is a morphism from a smooth quasiprojective variety W with projective fibers and A is a smooth rigid projective variety.

Proposition 8. *Suppose that A is a smooth rigid projective variety of positive dimension. Let X be a smooth irreducible projective variety and $p : X \to A$ a surjective morphism such that the generic fiber (and, hence, the fiber over every point $a \in A$) is connected. Let $S \subsetneq A$ be a closed subset of A. Put $Z = p^{-1}(S)$ and $W = X \setminus Z$. Then the group $H = \mathrm{Aut}_p(W)$ is strongly Jordan.*

Remark 5. Let A_r be the largest open subset of all points $a \in A \setminus S$ such that the fiber $p^{-1}(a)$ is smooth (hence, irreducible). Then $W_r := p^{-1}(A_r)$ is evidently H-invariant and H is embedded in $\mathrm{Aut}(W_r)$. Thus while proving the Proposition we may assume that for every point $a \notin S$ the fiber $p^{-1}(a)$ is irreducible.

Proof. If $S = \varnothing$ then $W = X$ is projective and the desired result follows from results of [19] (and Remark 4). Thus we assume that $S \neq \varnothing$. Then

- We denote by $G(A) := \mathrm{Aut}(A)$ the group of automorphisms of A.
- We denote by $G(S) \subset G(A)$ the subgroup of all elements $g \in G(A)$ such that $g(S) = S$.
- The identity component $G(A)_0$ of $G(A)$ is a connected algebraic group ([17, Cor. 2]).
- The intersection $G_S = G(S) \cap G(A)_0$ is a closed subgroup of $G(A)_0$, because S is a closed subset of A.
- The identity component G_0 of G_S is a closed subgroup in G_S, thus it is a connected algebraic group, and has finite index in G_S.
- The quotient group $G(S)/G_S$ is bounded ([19, Lem. 2.5]).
- Hence, the group $G(S)/G_0$ is bounded.
- Since G_0 acts on a non-uniruled projective variety A, it contains no non-trivial connected linear algebraic subgroup (otherwise, the open dense subset of A would be covered by rational orbits). Thus it is isomorphic to an abelian variety by Chevalley's Theorem ([6]).

By definition, for every automorphism $f \in \mathrm{Aut}_p(W)$ there is $g_f \in \mathrm{Aut}(A \setminus S)$ that may be included into the following commutative diagram :

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & W \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ A \setminus S & \xrightarrow{\ g_f\ } & A \setminus S \end{array}.$$

We denote by $H_0 \subset H$ the subgroup of all $f \in H$ such that $g_f \in G_0$. The group $H = \mathrm{Aut}_p(W)$ sits in the following exact sequence

$$0 \to H_i \to H \to H_a \to 0,$$

where

— $H_i = \{f \in H \mid g_f = \mathrm{id}_A\}$ is a subgroup of the automorphism group of the generic fiber $\mathcal{W}_p$ of p;

— $H_a = \{g \in \mathrm{Aut}(A \setminus S) \mid g = g_f \text{ for some} f \in H\}$.

Note that we have

- $H_a \subset G(S)$, since $\mathrm{Bir}(A) = \mathrm{Aut}(A)$.
- Every $g \in H_a \subset G(S)$ moves a G_0-orbit (in A) to a G_0-orbit, since G_0 is a closed normal subgroup of $G(S)$.
- The orbit $G_0(z)$ of a point $z \notin S$ is a projective subset of A, since G_0 is an abelian variety.
- The orbit $G_0(z)$ of a point $z \notin S$ does not meet S. Hence, if $z \notin S$ then $p^{-1}(G_0(z)) \cap Z = p^{-1}(G_0(z) \cap S) = \varnothing$, i.e $p^{-1}(G_0(z))$ is a closed irreducible projective subset of W. Indeed it is a fibration with irreducible projective fibers over a projective orbit $G_0(z)$ ([32, Chap. 1, n.6.3, Thm, 8]).

By a theorem of M. Rosenlicht [28], there exist a dense open G_0-invariant subset $U \subset A$, a quasiprojective variety V and a morphism $\pi\colon U \to V$ such that a fiber $\pi^{-1}(v)$ is precisely an orbit of G_0 for every $v \in V$. That means that V is a geometric quotient of U by the G_0-action. Since S is G_0-invariant, we may assume that $U \subset A \setminus S$. Since every $g \in H_a \subset G(S)$ moves a G_0-orbit (in A) to a G_0-orbit, the map $h_g := \pi \circ g \circ \pi^{-1} : V \to V$ is defined at every point of V, hence is a morphism (see [12, Lem. 10.7, pp. 314–315]). Moreover, the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ f\ } & p^{-1}(U) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ g_f\ } & U \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\ h_{g_f}\ } & V \end{array} .$$

Let $\tau = \pi \circ p$. Since the general fiber $T_v = \tau^{-1}(v), v \in V$ is a projective irreducible variety, the generic fiber $\mathcal{T}$ of τ is projective and irreducible as well ([11, Prop. 9.7.8.]). If $f \in H_0$, then $f \in \mathrm{Aut}_\tau(p^{-1}(U))$ and $h_{g_f} = \mathrm{id}_V$. Thus $H_0 \subset \mathrm{Aut}(\mathcal{T})$ is strongly Jordan, according to [19, Thm. 1.4, Lem. 2.5].

Moreover, we have the following exact sequence of groups

$$0 \to H_0 \to H \to H_V \to 0,$$

where $H_V = H/H_0$ is isomorphic to a subgroup of $G(S)/G_0$, hence is bounded. Therefore, by Lemma 7, H is strongly Jordan. (In the particular case of $G_0 = \{\mathrm{id}_A\}$ we have $V = A \setminus S$, $\pi = \mathrm{id}_A$, $h_{g_f} = g_f$, and $H_V = H_a \subset G_S$.) $\square$

Remark 6. The proof given for $S \neq \varnothing$ remains valid in case $S = \varnothing$, but then it reduces to double implementation of results of [19] (and Remark 4), instead of direct one (as is done in the beginning of the proof).

3. Admissible triples and related exact sequences

Let n be a positive integer and A be a n-dimensional irreducible smooth rigid projective variety (e.g., an abelian variety or a product of curves of positive genus). We write $\mathcal{K}$ for $k(A)$.

Let us define an A-*admissible triple* as a triple (X, ϕ, Z) that consists of a smooth irreducible projective variety X, a birational isomorphism $\phi : X \dashrightarrow A \times \mathbb{P}^1$ and a closed subset $Z \subsetneq X$. We denote by W the open subset

$$W = X \setminus Z \subset X.$$

We will freely use the following notation and properties of admissible A-triples.

(a) Let $p_A : A \times \mathbb{P}^1 \to A$ be the projection map on the first factor. Then the composition $p := p_A \circ \phi : X \to A$ is a morphism, since A is rigid. We say that p is induced by ϕ.

(b) Since X is birational to $A \times \mathbb{P}^1$, there is an open non-empty subset $B \subset A$ such that ϕ induces an isomorphism between $X_B = p^{-1}(B)$ and $B \times \mathbb{P}^1$. (This follows from the fact that the indeterminancy locus of ϕ has codimension $\leqslant 2$ in X, thus it is mapped by p into a proper closed subset of A.)

Moreover, ϕ is p, p_A-fiberwise: the following diagram commutes.

$$\begin{array}{ccc} X_B & \xrightarrow{\phi} & B \times \mathbb{P}^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_A} \\ B & \xrightarrow{\text{id}} & B \end{array} \quad ; \tag{1}$$

(c) It follows from (1) that the general fiber $P_x := p^{-1}(x)$ (i.e., fiber over a point x of a certain open dense subset of A) is isomorphic to $\mathbb{P}^1$.

(d) Let us put:

- $r(Z)$ is the number of irreducible over $\mathcal{K}$ components of Z that are mapped dominantly onto A; we will call such components "horizontal";
- $m(Z)$ is the degree of the restriction of p to Z, i.e., the number of $\overline{\mathcal{K}}$-points in $p^{-1}(a) \bigcap Z$ for a general point $a \in A$.

(e) The generic fiber $\mathcal{X}_p$ of p is isomorphic to the projective line $\mathbb{P}^1_{\mathcal{K}}$ over $\mathcal{K}$; the generic fiber $\mathcal{W}_p$ of the restriction $p|_W \to A$ (of p to W) is isomorphic to $\mathbb{P}^1_{\mathcal{K}} \setminus M$, where M is a finite set that is defined over $\mathcal{K}$ and consists of $m(Z)$ points that are defined over a finite algebraic extension of $\mathcal{K}$. In other words, the $\mathcal{K}$-variety $\mathcal{W}_p$ is isomorphic to the projective line over $\mathcal{K}$ with $m(Z)$ punctures. In particular, the group $\mathrm{Aut}_{\mathcal{K}}(\mathcal{W}_p)$ is finite if $m(Z) > 2$. On the other hand, $r(Z)$ is the number of Galois orbits in the set of punctures. In particular,

$$1 \leqslant r(Z) \leqslant m(Z) \quad \text{or} \quad 0 = r(Z) = m(Z).$$

(f) We may choose B in such a way that $Z_B := Z \cap X_B$ meets every fiber P_b, $b \in B$, at precisely $m(Z)$ $\overline{\mathcal{K}}$- points. In particular, Z_B is a finite cover of B.

(g) Every birational map $f \in \mathrm{Bir}(X)$ is p-fiberwise : we denote by $g_f \in \mathrm{Bir}(A)$ the corresponding automorphism $g_f \colon A \dashrightarrow A$ (see [3, Lem. 3.4]). Since A is rigid, g_f actually belongs to $\mathrm{Aut}(A)$. This implies that

$$\mathrm{Aut}(W) = \mathrm{Aut}_p(W).$$

(Here p denotes the restriction of $p : X \to A$ to $W \subset X$.)

(h) Let us consider the subgroups

$$H_i = \{f \in \mathrm{Aut}(W) \mid g_f = \mathrm{id}_A\} \subset \mathrm{Aut}(W)$$

and

$$H_a = \{g \in \mathrm{Aut}(A) \mid g = g_f \text{ for some } f \in \mathrm{Aut}(W)\} \subset \mathrm{Aut}(A).$$

We have the following short exact sequence of groups.

$$0 \to H_i \to \mathrm{Aut}(W) \to H_a \to 0. \tag{2}$$

(i) Group H_i is isomorphic to a subgroup of $\mathrm{Aut}_{\mathcal{K}}(\mathcal{W}_p)$. Thus it is Jordan; it is finite if $m(Z) > 2$.

Remark 7. Let W be a smooth quasiprojective irreducible variety that is birational to $A\times\mathbb{P}^1$. Then there is an A-admissible triple (X, ϕ, Z) such that $X\setminus Z$ is biregular to W. Indeed, one may take as X any smooth projective closure of W and put $Z = X \setminus W$.

Lemma 9. *Suppose that A is an irreducible smooth projective variety that is not uniruled (e.g., A is rigid). Then $\mathrm{Bir}(A \times \mathbb{P}^1)$ is quasi-bounded.*

Proof. Let $p_A : A \times \mathbb{P}^1 \to A$ be the projection map. Its generic fiber $\mathcal{X}$ is the projective line $\mathbb{P}^1_{k(A)}$ over $k(A)$. Each $u \in \mathrm{Bir}(A \times \mathbb{P}^1)$ is a p_A-fiberwise; see [3, Lem. 3.4 and Cor. 3.6]. By [3, Cor. 3.6], $\mathrm{Bir}(A \times \mathbb{P}^1)$ sits in an exact sequence

$$\{0\} \to \mathrm{Bir}_{k(A)}(\mathcal{X}) \to \mathrm{Bir}(A \times \mathbb{P}^1) \to \mathrm{Bir}(A).$$

Actually, $\mathrm{Bir}(A \times \mathbb{P}^1) \to \mathrm{Bir}(A)$ is surjective, because one may lift any birational automorphism of A to a birational automorphism of $A \times \mathbb{P}^1$. On the other hand, since $\mathcal{X}$ is the projective line, $\mathrm{Bir}_{k(A)}(\mathcal{X})$ is the projective linear group $\mathrm{PGL}(2, k(A))$. This gives us a short exact sequence

$$\{0\} \to \mathrm{PGL}(2, k(A)) \to \mathrm{Bir}(A \times \mathbb{P}^1) \to \mathrm{Bir}(A) \to \{0\}. \tag{3}$$

The theorem of Jordan implies that the linear group $\mathrm{PGL}(2, k(A))$ is strongly Jordan. In particular, it is quasi-bounded. On the other hand, since A is *not* uniruled, $\mathrm{Bir}(A)$ is also quasi-bounded ([26, Rem. 6.9], [3, Proof of Cor. 3.8 on p. 236]. It follows from (3) and Remark 1 that $\mathrm{Bir}(A \times \mathbb{P}^1)$ is also quasi-bounded. □

Lemma 10. *Assume that $m(Z) = 2$ and $r(Z) = 1$. Let $\mathrm{Tor}(H_i)$ be the set of all elements of H_i of finite order. Then the following conditions hold.*

(i) $\mathrm{Tor}(H_i)$ *consists of, at most,* 4 *elements.*
(ii) *Every element of finite order in* H_i *has order* 1 *or* 2.
(iii) *Every finite subgroup of* H_i *is abelian and its order divides* 4 *while its exponent divides* 2.

Proof. Let $(u_0 : u_1)$ be homogeneous coordinates in $\mathbb{P}^1_{\mathcal{K}}$. Since $m(Z) = 2$, and Z has only one irreducible component Z_1 over $\mathcal{K}$, we may assume that Z_1 is defined by the equation $(u_0 - \mu_1 u_1)(u_0 - \mu_2 u_1) = 0$, where μ_1, μ_2 are distinct elements of a quadratic extension $\mathcal{K}_2$ of $\mathcal{K}$ that are conjugate over $\mathcal{K}$.

Every automorphism $f \in \mathrm{Tor}(H_i)$ of $\mathcal{W}_p$ may be extended uniquely to a periodic automorphism $\overline{f}$ of $\mathcal{X}_p \cong \mathbb{P}^1_{\mathcal{K}}$. The 2-element subset

$$\{\mu_1, \mu_2\} \subset \mathbb{A}^1(\mathcal{K}_2) \subset \mathbb{P}^1(\mathcal{K}_2)$$

is $\overline{f}$-invariant for all $f \in H_i$. This means that $\overline{f}$ either leaves invariant both μ_i or permutes them.

Put $z = (u_0 - \mu_1 u_1)/(u_0 - \mu_2 u_1) \in \mathcal{K}_2(A)$. The extension $\overline{f}$ leaves the set $\{z = 0\} \cup \{z = \infty\}$ invariant for all $f \in H_i$. Thus, $\overline{f}(z) = \lambda z$ or $\overline{f}(z) = \lambda/z$ for suitable nonzero $\lambda \in \mathcal{K}_2$. In both cases $(\overline{f})^2(z) := (\overline{f} \circ \overline{f})(z) = \lambda^2 z$, This implies that λ is a *root of unity* if $\overline{f}$ is *periodic*, i.e., if $f \in \mathrm{Tor}(H_i)$; in particular $\lambda \in \mathcal{K}$. Suppose that $\overline{f}(u_0 : u_1) = (u_0' : u_1')$. Then one may easily check that either (in the former case)

$$\frac{(u_0' - \mu_1 u_1')}{(u_0' - \mu_2 u_1')} = \lambda \frac{(u_0 - \mu_1 u_1)}{(u_0 - \mu_2 u_1)}, \tag{4}$$

or (in the latter case)

$$\frac{(u_0' - \mu_1 u_1')}{(u_0' - \mu_2 u_1')} = \lambda \frac{(u_0 - \mu_2 u_1)}{(u_0 - \mu_1 u_1)}. \tag{5}$$

In order for these maps to be defined over $\mathcal{K}$ the matrices (respectively)

$$\begin{pmatrix} \mu_1 - \lambda\mu_2 & \mu_1\mu_2(\lambda - 1) \\ (1 - \lambda) & \lambda\mu_1 - \mu_2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_1 - \lambda\mu_2 & \lambda\mu_2^2 - \mu_1^2 \\ (1 - \lambda) & \lambda\mu_2 - \mu_1 \end{pmatrix}$$

should be defined (up to multiplication by a nonzero element of $\mathcal{K}_2$) over $\mathcal{K}$ as well. Since $\lambda \in \mathcal{K}$, it may happen only if $\lambda = \pm 1$. This implies that $(\overline{f})^2 : z \mapsto \lambda^2 z$ is the identity map, i.e., the order of $\overline{f}$ is either 1 or 2. In addition, there are, at most, four elements in $\mathrm{Tor}(H_i)$. Namely, (written in z-coordanate):

$$\overline{f}(z) = z, \quad \overline{f}(z) = -z, \quad \overline{f}(z) = \frac{1}{z}, \quad \overline{f}(z) = -\frac{1}{z}. \qquad \square$$

One may see Lemma 10 in a more general way. Let $\mathcal{K}$ be a field of characteristic zero that contains all roots of unity. Let $n \geqslant 2$ be an integer.

The following assertion is an easy application of Kummer theory [15, Chap. VI, Sect. 8].

Theorem 11. *Let u be a matrix in $\mathrm{GL}(n,\mathcal{K})$, whose image $\bar{u}$ in $\mathrm{PGL}(n,\mathcal{K})$ has finite order. Suppose that u has an eigenvalue that does not belong to $\mathcal{K}$. Then there is a positive integer d such that $d \mid n$ and all eigenvalues of u^d lie in $\mathcal{K}$. In addition, if n is a prime, then $\bar{u}$ has order n.*

Proof. We know that there are a positive integer m and a nonzero element $a \in \mathcal{K}$ such that $u^m = a \cdot \mathbf{1}_n$ where $\mathbf{1}_n$ is the identity square matrix of size n. Clearly, the order of $\bar{u}$ is strictly greater than 1 and divides n.

Let α be an eigenvalue of u that does not belong to $\mathcal{K}$. Then $\alpha^m = a$. Let us consider the finite algebraic field extension $\mathcal{K}' = \mathcal{K}(\alpha)$ of $\mathcal{K}$ and denote by d its degree $[\mathcal{K}' : \mathcal{K}]$. Clearly, $d > 1$. The Kummer theory tells us that $\mathcal{K}'/\mathcal{K}$ is a cyclic extension and $d \mid m$. In other words, $\mathcal{K}'/\mathcal{K}$ is Galois and its Galois group G is cyclic of order d. If β is another eigenvalue of u then

$$\beta^m = a = \alpha^m$$

and therefore the ratio β/α is an mth root of unity and therefore lies in $\mathcal{K}$. This implies that

$$\mathcal{K}(\beta) = \mathcal{K}(\alpha) = \mathcal{K}';$$

in particular, none of the eigenvalues of u lies in $\mathcal{K}$.

Recall that the cardinality of G coincides with d. Since β generates $\mathcal{K}'$ over $\mathcal{K}$, the set $\{\sigma(\beta) \mid \sigma \in G\}$ consists of d distinct elements, each of which is an eigenvalue of u and has the same multiplicity. Since the spectrum of u is a disjoint union of G-orbits, d divides n.

Take an element τ of (abelian group) G. Then $\tau(\beta) = \zeta\beta$ where ζ is a root of unity that lies in $\mathcal{K}$. The norms of conjugate β and $\tau(\beta)$ (with repect to $\mathcal{K}'/\mathcal{K}$) do coincide. This means that

$$\prod_{\sigma \in G} \sigma(\beta) = \prod_{\sigma \in G} \sigma(\tau\beta) = \zeta^d \prod_{\sigma \in G} \sigma(\beta).$$

It follows that

$$\zeta^d = 1, \ \tau(\beta^d) = (\tau\beta)^d = \beta^d$$

for all $\tau \in G$. This implies that $\beta^d \in \mathcal{K}$ for all eigenvalues β of u and therefore all eigenvalues of u^d lie in $\mathcal{K}$.

Now assume that n is a prime. Then $d = n$ and counting arguments imply that the spectrum of u consists of exactly one G-orbit, say, $G\beta$. Then all the eigenvalues of $u^n = u^d$ coincide with

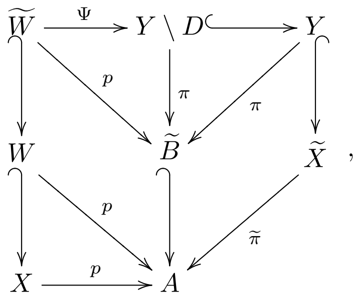
$$\beta^n = \beta^d \in \mathcal{K}.$$

This implies that u^n is a scalar and therefore the order of $\bar{u}$ divides n. One has only to recall that this order is greater than 1 and n is a prime. $\square$

The next lemmas show that the case $m(Z) \leqslant 2$ may be reduced to the case $m(Z) = 0$.

To this end we find an open H_a-invariant subset $\widetilde{B} \subset A$ of A such that $\widetilde{W} = p^{-1}(\widetilde{B})$ is a complement of exactly two (respectively 1) "horizontal" components . Namely, we build a rank two vector bundle E over $\widetilde{B}$ such that $\widetilde{W}$ appears to be isomorphic to the complement $Y \setminus D$ of two (respectively, one) disjoint sections in $Y := P(E)$.

More precisely, we are going to build the following chain of maps and inclusions of smooth irreducible quasiprojective varieties:



such that:

- $\widetilde{B}$ is an open dense subset of A invariant under the H_a-action;
- $\widetilde{W} = p^{-1}(\widetilde{B}) \subset W$ is invariant under the action of $\mathrm{Aut}(W)$;
- Ψ is an isomorphism;
- every fiber of $\pi : Y \to \widetilde{B}$ is projective;
- D is a closed subset of Y that meets every fiber of π at no more than two points;
- $\widetilde{X}$ is projective;
- $\widetilde{\pi}(\widetilde{X} \setminus Y) = A \setminus \widetilde{B}$.

According to Lemma 6, $\mathrm{Aut}(Y \setminus D) \subset \mathrm{Aut}(Y)$. Thus, instead of $\mathrm{Aut}(W)$ we may study $\mathrm{Aut}(Y)$ where Y is fibered over $\widetilde{B} \subset A$ with projective fibers (hence $m(\widetilde{X} \setminus Y) = 0$).

The building of this construction is done in the following lemmas.

Lemma 12. *If (X, ϕ, Z) is an A-admissible triple, $W := X \setminus Z$ and $m(Z) = 1$, then there exists an A-admissible triple $(\widetilde{X}, \widetilde{\phi}, \widetilde{Z})$ with $m(\widetilde{Z}) = 0$ and a group embedding* $\mathrm{Aut}(W) \hookrightarrow \mathrm{Aut}(\widetilde{W})$, *where $\widetilde{W} := \widetilde{X} \setminus \widetilde{Z}$.*

Proof. Let $(w_0 : w_1)$ be homogeneous coordinates in $\mathbb{P}^1$. We may choose them in such a way that $\phi(Z_B) = B \times \{(0 : 1)\}$.

Let

- $W_B = W \cap X_B = X_B \setminus Z_B$;
- $B_g = g(B)$ for an automorphism $g \in H_a$;
- $W_g = W \cap p^{-1}(B_g)$;
- $\widetilde{B} = \cup B_g,\ g \in H_a$;
- $\widetilde{W} = p^{-1}(\widetilde{B}) = \cup W_g,\ g \in H_a$.

We have $\phi(W_B) = B \times (\mathbb{P}^1 \setminus \{w_0 = 0\})$. Thus, the rational function $t = w_1/w_0$ is defined on $\phi(W_B)$ and the rational function $\tau = \phi^*(t)$ is defined on W_B. It establishes an isomorphism of the fiber $p^{-1}(b) \cap W$ with $\mathbb{A}^1_t$ if b is a point of B.

For every $g \in H_a$ there exists $f_g \in \mathrm{Aut}(W)$ such that $g = g_{f_g}$ and $f_g(W_B) = W_g$. We define $\tau_g = \tau \circ f_g^{-1}$, which is a regular function on W_g. (Note that a priori the choice of f_g is not unique. A different choice of f_g will change τ_g by a p-fiberwise automorphism of W_B, which becomes a nondegenerate affine transformation of the generic fiber $\mathcal{W}_p \sim \mathbb{A}^1_{\mathcal{K}}$.)

We introduce the isomorphisms $\psi_g : W_g \to B_g \times \mathbb{A}^1$ by $\psi_g(w) = (p(w), \tau_g(w))$. Actually, ψ_g are compositions of the chain of automorphisms

$$W_g \xrightarrow{f_g^{-1}} W_B \xrightarrow{\phi} B \times \mathbb{A}^1_t \xrightarrow{(g^{-1},\mathrm{id})} B_g \times \mathbb{A}^1_t.$$

Note that in this chain f_g^{-1} is p-fiberwise, ϕ is p, p_A-fiberwise, and (g^{-1}, id) is p_A-fiberwise, thus ψ_g is p, p_A-fiberwise. It may be included into the following commutative diagram:

$$\begin{array}{ccc} W_g & \longrightarrow & B_g \times \mathbb{A}^1_t \\ \Big\downarrow p & & \Big\downarrow p_A \\ B_g & \xrightarrow{\mathrm{id}} & B_g \end{array} .$$

If $g, h \in H_a$ and $w \in W_g \cap W_h$ then:

- $b = p(w) \in (B_g \cap B_h)$;
- functions τ_g and τ_h provide an isomorphism of the fiber $P_b = p^{-1}(b) \cap W$ with $\mathbb{A}^1_t$ hence

$$\tau_g = \tau \circ f_g^{-1} = \tau \circ f_h^{-1} \circ f_h \circ f_g^{-1} = \tau_h \circ f_h \circ f_g^{-1} = \tau_h \alpha + \beta,$$

  where $\alpha := \alpha_{gh}(b), \beta := \beta_{gh}(b)$ are regular in $B_g \cap B_h$, constant along P_b, and α_{gh} does *not* vanish in $(B_g \cap B_h)$;
- $\Psi_{gh} = \psi_g \circ \psi_h^{-1}$ is a p_A-fiberwise automorphism of $(B_g \cap B_h) \times \mathbb{A}^1$ defined by $\Psi_{gh}(b, \tau_h) = (b, \tau_g) = (b, \alpha_{gh}(b)\tau_h + \beta_{gh}(b))$.

It follows that $\widetilde{W}$ is the total body of an $\mathbb{A}^1$-bundle on $\widetilde{B}$: the latter is defined by transition functions Ψ_{gh}.

We define a rank two vector bundle by the following data:

- the covering of $\widetilde{B}$ by the open subsets $B_g, g \in H_a$;
- natural projection $\pi_E : B_g \times \mathbb{A}^2_{(u_0,u_1)} \to B_g$;
- transition matrices on $B_g \cap B_h$

$$M_{gh} = \begin{pmatrix} 1 & 0 \\ \beta_{gh} & \alpha_{gh} \end{pmatrix}.$$

The maps

$$\overline{\psi}_g\colon W_g \to \mathbb{P}(E), \quad \overline{\psi}_g(w) = (p(w), (1 : \tau_g(w))$$

glue together to an isomorphism

$$\Psi : \widetilde{W} \cong \mathbb{P}(E) \setminus \{u_0 = 0\}.$$

We denote by D the divisor (image of the section) $\{u_0 = 0\}$ in $\mathbb{P}(E)$ and by π the induced by π_E projection map $P(E) \to \widetilde{B}$.

We have $\mathrm{Aut}(W) \subset \mathrm{Aut}(\widetilde{W})$, since the (sub)set $\widetilde{B} \subset A$ is invariant under the action of H_a. On the other hand, according to Lemma 6, $\mathrm{Aut}(\widetilde{W}) \subset \mathrm{Aut}(Y)$ where $Y := \mathbb{P}(E)$. Take any smooth projective closure $\widetilde{X}$ of Y and extend π to the rational map $\widetilde{\pi} : \widetilde{X} \to A$. Since A contains no rational curves, $\widetilde{\pi}$ is a morphism, which is obviously projective. Let $\widetilde{D}$ be the closure of D in $\widetilde{X}$. Note that $\widetilde{D} \cap Y = D$, $\widetilde{\pi}^{-1}(\widetilde{B}) = Y$, and $\widetilde{p}^{-1}(A \setminus \widetilde{B}) = \widetilde{X} \setminus Y$, in light of the "maximality" property of projective (and therefore proper) morphism π [18, Sect. 2.6, pp. 95–96]).

Let $\widetilde{Z} = \widetilde{X} \setminus Y$. Let $\widetilde{\phi} : \widetilde{X} \dashrightarrow A \times \mathbb{P}^1$ be the rational extension of $\phi \circ \Psi^{-1} : Y \dashrightarrow A \times \mathbb{P}^1$. Then $m(\widetilde{Z}) = 0$, and the A-admissible triple $(\widetilde{X}, \widetilde{\phi}, \widetilde{Z})$ is the one we were looking for. □

Remark 8. This lemma may be derived from general results in [13] and [33], [34]; but we prefer an explicit construction which we use in the next lemma.

Lemma 13. *Assume that a triple (X, ϕ, Z) is A-admissible, $m(Z)=2$ and $r(Z)=2$. Then there exists an A-admissible triple $(\widetilde{X}, \widetilde{\phi}, \widetilde{Z})$ with $m(\widetilde{Z}) = 0$ and a group embedding $\mathrm{Aut}(W) \hookrightarrow \mathrm{Aut}(\widetilde{W})$, where $\widetilde{W} := \widetilde{X} \setminus \widetilde{Z}$.*

Proof. Since Z_B contains two disjoint irreducible over $\mathcal{K}$ horizontal components we may choose homogeneous coordinates $(w_0 : w_1)$ in $\mathbb{P}^1$ in such a way that $\phi(Z_B) = B \times \{w_0 w_1 = 0\}$. Thus this is the special case of Lemma 12 when (in the notation of Lemma 12) $\tau_g = 0$ whenever $t = 0$ for all $g \in H_a$. Thus this lemma follows from Lemma 12. Note that in this case $\beta_{gh} \equiv 0$ and instead of the $\mathbb{A}^1$-bundle we have a line bundle. □

It follows that the case $m(Z) \leqslant 2$ may be reduced to the case $m(Z) = 0$.

Lemma 14. *If a triple (X, ϕ, Z) is A-admissible, $W := X \setminus Z$ and $m(Z) = 0$, then there exists an A-admissible triple $(\widetilde{X}, \widetilde{\phi}, \widetilde{Z})$ such that:*

1) *There is a group embedding $\mathrm{Aut}(W) \hookrightarrow \mathrm{Aut}(\widetilde{W})$ where $\widetilde{W} = \widetilde{X} \setminus \widetilde{Z}$.*
2) *If $\widetilde{p}$ is the projection map from $\widetilde{X}$ onto A induced by $\widetilde{\phi}$, then $\widetilde{Z} = \widetilde{p}^{-1}(S)$ for a certain closed subset S of A; in addition, for every point $b \in B := A \setminus S$ the fiber $P_b = \widetilde{p}^{-1}(b)$ is an irreducible reduced curve isomorphic to $\mathbb{P}^1$.*

Remark 9. In loose words it means that we can add to Z all the singular fibers of p without reducing an automorphism group.

Proof. Since $m(Z) = 0$, we have $Z_A = p(Z) \neq A$ is a closed subset of A. Let a be a point of Z_A. Then the fiber $P_a \cap W = P_a \setminus (Z \cap P_a)$ either has a non-projective irreducible component or is empty. Let S_s be the set of all points $a \in A \setminus Z_A$ such that $P_a = p^{-1}(a)$ is singular (namely, has several irreducible components or a non-reduced component). Let $S := S_s \cup Z_A \subset A$, i.e., $B := A \setminus S$ is the set of all points $a \in A$ such that the fiber $P_a \subset W$ is a reduced irreducible smooth curve isomorphic to $\mathbb{P}^1$. Then sets B and S are invariant under the action of H_a (see (2)), thus $\widetilde{W} := p^{-1}(B)$ is invariant under the action of $\mathrm{Aut}(W)$, i.e., $\mathrm{Aut}(W) \subset \mathrm{Aut}(\widetilde{W})$.

Thus, the A-admissible triple

$$\widetilde{X} := X, \widetilde{\phi} := \phi, \widetilde{Z} := p^{-1}(S)$$

enjoys the desired properties. □

4. Proof of Theorem 4

In this section we prove Theorem 4 and Corollary 5.

Proof of Theorem 4. By Remark 2 we may assume that W is smooth. When W is projective, the desired result follows from [19]. So, we may assume that quasiprojective W is *not* projective. By Remark 7 we may choose such an A-admissible (X, ϕ, Z) that $W = X \setminus Z$. We use the exact sequence (2).

It is proven in [26, Sect. 6] (see also [3, Cor. 3.8]) that for an irreducible non-uniruled variety A the group $\mathrm{Bir}(A)$ (and, hence, $\mathrm{Aut}(A)$) is *strongly* Jordan.

If $m(Z) > 2$, then the (sub)group H_i in the short exact sequence (2) is finite. If $m(Z) = 2$, and $r(Z) = 1$, then, according to Lemma 10, H_i is bounded. It follows from Lemma 7 that in both cases $\mathrm{Aut}(W)$ is Jordan. (See also [26, Lem. 2.8].)

According to Lemma 13, Lemma 12, and Lemma 14, in all other cases one may assume that conditions of Proposition 8 are satisfied, hence, $\mathrm{Aut}(W)$ is strongly Jordan. □

Proof of Corollary 5. Assume that W is a quasiprojective irreducible variety of dimension $d \leqslant 3$. The case $\dim(W) \leq 2$ was done in [23], [36], [2]. Assume that W is not birational to $E \times \mathbb{P}^2$, where E is an elliptic curve.

If $\dim(W) = 3$ and $\mathrm{Bir}(W)$ is Jordan, then its subgroup $\mathrm{Aut}(W)$ is also Jordan. If $\mathrm{Bir}(W)$ is *not* Jordan, then according to [27], the variety W has to be birational to $S \times \mathbb{P}^1$, where S is a surface that enjoys one of the following three properties.

Case 1. S is an abelian surface. Since S contains no rational curves, it is rigid. Thus, $\mathrm{Aut}(W)$ is Jordan by Theorem 4.

Case 2. S is a bielliptic surface. Since S contains no rational curves, it is rigid. Thus, $\mathrm{Aut}(W)$ is Jordan by Theorem 4.

Case 3. S is a surface with Kodaira dimension $\varkappa(S) = 1$ such that the Jacobian fibration of the pluricanonical fibration is locally trivial in Zariski topology.

Consider Case 3. Further on we assume that $k = \mathbb{C}$. (See Remark 10 below.)

We have to prove that S is rigid. Since Jacobian fibration of the pluricanonical fibration is locally trivial in Zariski topology, all fibers (even the multiple ones) of the pluricanonical fibration are smooth elliptic curves ([32, Chap. VII, Sect. 7, Cor. 2], [7, Thm. 5.3.1]).

Lemma 15. *Assume that A is a smooth irreducible surface endowed with a morphism $\pi\colon A \to C$ such that*

- *C is a smooth curve of genus g.*
- *Every fiber $F_c = \pi^{-1}(c), c \in C$ is a smooth elliptic curve.*
- *Kodaira dimension $\varkappa(A) = 1$.*
- *Morphism π is a pluricanonical fibration, i.e., for some N and every effective divisor $D \in |NK_A|$ there are positive numbers $\nu_1, \dots, \nu_n$ and fibers $F_1, \dots, F_n$ of π such that $D = \sum_1^n \nu_i F_i$.*

Then surface A contains no rational curves.

Proof. The surface A enjoys the following properties:

- Euler characteristics $e(A) = 0$ (see [32, Chap. IV, Sect. 4, Thm. 6]).
- Since $K_A^2 = 0$, we have $\chi(A) = \chi(A, \mathcal{O}_A) = 0$ (since $\chi(A) = (K_A^2 + e(A))/12$, see [32, Introduction, Basic formulas]).
- If fibration π has precisely k multiple fibers $F_1, \dots, F_k$ with multiplicities $m_1, \dots, m_k$, respectively, then

$$\delta(\pi) := 2g - 2 + \sum_{i=1}^{i=k} \Big(1 - \frac{1}{m_i}\Big) > 0 \tag{6}$$

  (see [1, Chap. V, Prop. 12.5]).
- In particular, $2g - 2 + k > 0$.
- Since π is a pluricanonical fibration every automorphism $\phi \in \mathrm{Aut}(A)$ is π-fiberwise.
- For every automorphism $\phi \in \mathrm{Aut}(A)$ the subset $F_{\mathrm{sing}} = F_1 \cup \dots \cup F_k$ is invariant since multiple fibers go to multiple fibers.

Let $B \subset A$ be a rational curve. Since it cannot be contained in a fiber of π, it is mapped by π onto C with some degree $m \geq 1$. Hence C is rational. Assume that B intersects F_i at points $a_1^i, \dots, a_{n_i}^i$ that are ramification points of restriction $\widetilde{\pi}$ of π onto B, of orders $r_{i,1} \dots, r_{i,n_i}$, respectively. Then

- $r_{i,1} + \dots + r_{i,n_i} = m$,
- $r_{i,j} \geqslant m_i \ j = 1, \dots, n_i$,
- $n_i \leqslant m/m_i$.

Assume that $\widetilde{\pi}$ has also ramification points $b_1, \dots, b_r$ of orders $p_1, \dots, p_r$ respectively, (including nodes of B) outside F_{sing}.

By the Hurwitz formula we have

$$\begin{aligned} 2 &= 2m - \sum_{i=1}^{i=k} \sum_{j=1}^{j=n_i} (r_{i,j} - 1) - \sum_{l=1}^{l=r} (p_l - 1) \\ &= 2m - mk + \sum n_i - \nu, \end{aligned}$$

where $\nu := \sum_{l=1}^{l=r} (p_l - 1)$ is a non-negative number.

Thus, dividing by m we get

$$k - 2 = \frac{1}{m}\Big(- 2 + \sum n_i - \nu\Big) \leqslant \sum \frac{1}{m_i},$$

and

$$\delta(\pi) := -2 + \sum_{i=1}^{i=k}\Big(1 - \frac{1}{m_i}\Big) = -2 + k - \sum \frac{1}{m_i} \leqslant 0$$

which contradicts to (6). □

Thus, in Case 3 surface S is rigid as well, and therefore $\mathrm{Aut}(W)$ is Jordan by Theorem 4. □

Remark 10. In the course of the proof of Theorem 4 and Corollary 5 it suffices to consider the case when the ground field is the field $\mathbb{C}$ of complex numbers. Indeed, suppose we know that the Theorems hold true when the ground field is $\mathbb{C}$. Let k be any algebraically closed field of characteristic 0 and an algebraic variety W over k satisfies the conditions either of Theorem 4 or Corollary 5. Let us assume that $\mathrm{Aut}(W)$ is *not* Jordan. We need to arrive at a contradiction.

The variety W is defined over a subfield k_0 (of k) such that k_0 is finitely generated over the field $\mathbb{Q}$ of rational numbers, i.e., there is a quasiprojective variety W_0 over k_0 such that $W = W_0 \times_{k_0} k$. (Clearly, k_0 is a countable field.) Replacing if necessary k_0 by its finitely generated extension, we may assume that there is a surface A_0 over k_0 and a k_0-birational map between W and $A_0 \times \mathbb{P}^1$. Moreover, we may choose k_0 in such a way that

- if A is bielliptic, the same is valid for A_0 (the bielliptic structure would be defined over k_0);
- if $\varkappa(A) = 1$, the same is valid for A_0 (the pluricanonical fibration would be defined over k_0);
- if a pluricanonical fibration of A has smooth irreducible elliptic fibers, the same is valid for A_0 (smoothness is preserved under base change [16, Prop. 3.38, Chap. 4]);
- if A contains no rational curves the same is valid for A_0. Indeed, if A_0 contained a rational curve, then for some integer d one of the irreducible quasiprojective components of the variety $RatCurves^n_d(A)$ (see [14, Def. 2.11]) would have a point over $\mathbb{C}$. But then it would have a point over k as well, since k is algebraically closed.

The non-Jordanness of $\mathrm{Aut}(W)$ means that there exists an infinite sequence of finite subgroups $\{G_i \subset \mathrm{Aut}(W)\}_{i=1}^{\infty}$, whose Jordan indices J_{G_i} tend to infinity. For each positive i there is a subfield k_i of k that contains k_0 and is finitely generated over k_0, and such that all automorphisms from G_i are defined over k_i. Clearly, all k_i are countable fields. The compositum of all k_i's (in k) is countably generated over k_0 and therefore is also a countable field. Let us consider the algebraic closure k_∞ of this compositum in k. Clearly, k_∞ is an algebraically closed countable subfield of k that contains all k_i. Let us consider the quasiprojective variety $W_\infty = W_0 \times_{k_0} k_\infty$. Clearly, there exist group embeddings $G_i \hookrightarrow \mathrm{Aut}_{k_\infty}(W_\infty)$ for all positive i. This implies that $\mathrm{Aut}_{k_\infty}(W_\infty)$ is *not* Jordan.

Since k_∞ is countable, there is a field embedding $k_\infty \hookrightarrow \mathbb{C}$. Let us consider the complex quasiprojective variety $W_{\mathbb{C}} = W_\infty \times_{k_\infty} \mathbb{C}$, which is birational to $A_{\mathbb{C}} \times \mathbb{P}^1$

where $A_{\mathbb{C}} = A \times_{k_0} \mathbb{C}$ is a complex variety meeting conditions of Theorem 4. In particular, $\mathrm{Aut}(W_{\mathbb{C}})$ is Jordan. On the other hand, there is a group embedding $\mathrm{Aut}(W_\infty) \hookrightarrow \mathrm{Aut}(W_{\mathbb{C}})$. This implies that $\mathrm{Aut}(W_{\mathbb{C}})$ is *not* Jordan as well, which gives us the desired contradiction.

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