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ÉTALE REPRESENTATIONS FOR REDUCTIVE ALGEBRAIC GROUPS WITH FACTORS Sp_n OR SO_n

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Abstract. An étale module for a linear algebraic group G is a complex vector space V with a rational G-action on V that has a Zariski-open orbit and $\dim G = \dim V$. Such a module is called super-étale if the stabilizer of a point in the open orbit is trivial. Popov (2013) proved that reductive algebraic groups admitting super-étale modules are special algebraic groups. He further conjectured that a reductive group admitting a super-étale module is always isomorphic to a product of general linear groups. Our main result is a construction of counterexamples to this conjecture, namely, a family of super-étale modules for groups with a factor Sp_n for arbitrary $n \geq 1$. A similar construction provides a family of étale modules for groups with a factor SO_n, which shows that groups with étale modules with non-trivial stabilizer are not necessarily special. Both families of examples are somewhat surprising in light of the previously known examples of étale and super-étale modules for reductive groups. Finally, we show that the exceptional groups F_4 and E_8 cannot appear as simple factors in the maximal semisimple subgroup of an arbitrary Lie group with a linear étale representation.

Introduction

An *étale module* (G, ϱ, V) for an algebraic group G is a finite-dimensional complex vector space V together with a rational representation $\varrho : G \to \mathrm{GL}(V)$ such that $\varrho(G)$ has a Zariski-open orbit in V and $\dim G = \dim V$. In particular,

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the stabilizer H of any point in the open orbit is a finite subgroup of G. If H is the trivial group, the module is called *super-étale*. Similarly we call the representation ϱ étale or super-étale, respectively. More generally, one can study affine étale representations, but for rational representations of reductive algebraic groups these are equivalent to linear ones via affine changes of coordinates. As we are primarily interested in this case, we shall restrict ourselves to linear representations.

The existence of an affine étale representation for a given group G implies the existence of a left-invariant flat affine connection on G, and these structures appear in many different contexts in mathematics. For the specifics of this relationship and a survey of applications, see Burde [2], [3], Baues [1] and the references therein.

The primary motivation for the present work is Popov's study of linearizable subgroups of the Cremona group on affine n-space (those that are conjugate to a linear group within the Cremona group). Subgroups for which a super-étale module exists, called *flattenable* groups by Popov, allow particularly convenient criteria to decide their linearizability; compare the results in [9, Sect. 2]. Incidentally, a flattenable group G is precisely a group that admits a rational super-étale module. Popov [9, Lem. 2] proved (in our terminology):

A reductive algebraic group admitting a super-étale module is a special algebraic group.

By definition, G is *special* (in the sense of Serre) if every principal G-bundle is locally trivial in the étale topology. Serre [11, 4.1] showed that every special group is connected and linear, and that reductive groups with maximal connected semisimple subgroup

$$S(G) = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_k} \times \mathrm{Sp}_{m_1} \times \cdots \times \mathrm{Sp}_{m_j}$$

are special. A result of Grothendieck [6, Théorème 3] then implies that an affine algebraic group G is special if and only if a maximal connected semisimple subgroup is isomorphic to a group of this type. This result and the available examples lead Popov to make the following conjecture:

A reductive algebraic group G has a rational super-étale module if and only if

$$G \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}.$$

Clearly, every group $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ has a super-étale module $\mathrm{Mat}_{n_1} \oplus \cdots \oplus \mathrm{Mat}_{n_k}$ on which it acts factorwise by matrix multiplication. In previously available classification results on étale modules for reductive algebraic groups G, the only simple groups appearing as factors in G are SL_n and Sp_2 (see Burde and Globke [4, Sect. 5] for a summary). This suggests the more general questions of whether *in a reductive algebraic group with a rational super-étale module, all simple factors are either* Sp_2 *or* SL_n *for certain* $n \geq 2$. Somewhat surprisingly, this (and thus Popov's original conjecture) turns out to be false. Our main result is the existence of counterexamples to this conjecture, constructed and shown in Section 2.1 below. These examples consist of a family of super-étale modules for reductive groups $G = \mathrm{Sp}_n \times \mathrm{GL}_{2n-1} \times \cdots \times \mathrm{GL}_1$ for any $n \geq 1$. So in fact any factor SL_n or Sp_n for any $n \geq 1$ can appear in a group with a super-étale module. One might now be tempted to ask whether every special reductive algebraic group admits a super-étale module, but this can immediately be ruled out by comparison with classification results of reductive groups with few simple factors, see again [4, Sect. 5].

Knowing that algebraic groups with super-étale modules are special, one can further suspect that the same holds for groups with étale modules that have a non-trivial stabilizer. Again we find the surprising answer that this is not true. In Section 2.2 below we construct a family of étale modules for reductive groups $G = \mathrm{SO}_n \times \mathrm{GL}_{n-1} \times \cdots \times \mathrm{GL}_1$ for any $n \geq 2$. These are the first known examples of étale modules for groups with a simple factor SO_n for any number $n \geq 2$.

These two families are the first known examples of étale modules for reductive groups containing factors Sp_n or SO_n for arbitrary $n > 2$. This still leaves the question of whether there exist étale modules for reductive groups with exceptional simple groups as factors. In Section 4, we show in a much more general setting that a simple Lie group whose complexified Lie algebra is one of the exceptional algebras $\mathfrak{f}_4$ or $\mathfrak{e}_8$ cannot appear among the simple factors in a maximal semisimple subgroup of a Lie group with a linear étale representation, not necessarily algebraic (here, *étale* means that the action has an orbit that is open in the standard topology of the module). For the other exceptional groups, this question remains open.

A remark on the previously available classification results on étale modules is in order. As these results use the classification results on prehomogeneous modules due to Sato, Kimura and others (see Kimura's book [8, Chap. 7] and references therein), they very often rely on Lie algebraic methods. In most cases it is not immediately clear from their classifications whether the generic stabilizers are trivial, although many generic stabilizers (not just their identity component) are explicitely given in the appendix of [8].

Notations and conventions

All algebraic groups, such as GL_n, SL_n, SO_n and Sp_n, are considered over the complex numbers unless otherwise stated. We follow the convention that Sp_n means the symplectic group on $\mathbb{C}^{2n}$. The notation $\operatorname{Lie} G$ means the Lie algebra of a group G; we will also use the corresponding gothic letter $\mathfrak{g}$. The identity component of an algebraic group G is denoted by G°. $\mathrm{Mat}_{m,n}$ denotes the space of complex $m \times n$-matrices, and if $m = n$ we simply write Mat_n. The identity matrix in Mat_n is denoted by I_n. The transpose of a matrix A is denoted by $A^\top$. The canonical basis vectors of $\mathbb{C}^n$ are denoted by $e_1, \ldots, e_n$.

For any algebraic group G, let $\mathrm{Z}(G)$, $L(G)$, and $\mathrm{R_u}(G)$ denote the center, a maximal connected reductive subgroup, and the unipotent radical of G, respectively. Then G is the semidirect product $G = L(G) \cdot \mathrm{R_u}(G)$. Write $S(G)$ for a maximal connected semisimple subgroup of G, the commutator subgroup of $L(G)$. Note that $L(G)$ and $S(G)$ are unique up to conjugation.

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1. Preliminaries on prehomogeneous modules

A module (G, ϱ, V), or (G, V) for short, for an algebraic group G with a rational representation $\varrho : G \to \mathrm{GL}(V)$ on a finite-dimensional complex vector space V is called a *prehomogeneous module* if $\varrho(G)$ has a Zariski-open orbit in V. In this case, $\dim G \geq \dim V$. More precisely, if $x \in V$ is a *point in general position*, that is, it lies in the open orbit of G, and G_x is its stabilizer subgroup, then

$$\dim V = \dim G - \dim G_x.$$

The stabilizer $H = G_x$ of any point x in the open orbit is called the *generic stabilizer* of (G, ϱ, V). A prehomogeneous module is *étale* if $H^\circ = \{1\}$ (equivalently, if $\dim G = \dim V$). An étale module (G, V) is called *super-étale* if $H = \{1\}$.

See Burde and Globke [4, Prop. 4.1] for a proof of the following result which we will use frequently without further reference:

Proposition 1.1. *The following conditions are equivalent:*

(1) $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ *is an étale module.*
(2) (G, ϱ_1, V_1) *is prehomogeneous and* $(H^\circ, \varrho_2|_H, V_2)$ *is an étale module, where* H° *denotes the connected component of the generic stabilizer of* (G, ϱ_1, V_1).

Equivalence also holds if each "étale" is replaced by "prehomogeneous".

Two modules (G_1, ϱ_1, V_1) and (G_2, ϱ_2, V_2) are called *equivalent* if there exists an isomorphism of algebraic groups $\psi : \varrho_1(G_1) \to \varrho_2(G_2)$ and a linear isomorphism $\varphi : V_1 \to V_2$ such that $\psi(\varrho_1(g))\varphi(x) = \varphi(\varrho_1(g)x)$ for all $x \in V_1$ and $g \in G_1$.

Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V)$ be an m-dimensional rational representation of an algebraic group G, and let ϱ^* be the dual representation for ϱ. Then we say that the modules

$$\big(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V \otimes \mathbb{C}^n\big) \quad \text{and} \quad \big(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^* \otimes \mathbb{C}^{m-n}\big)$$

are *castling transforms* of each other. More generally, we say that two modules (G_1, ϱ_1, V_1) and (G_2, ϱ_2, V_2) are *castling-equivalent* if (G_1, ϱ_1, V_1) is equivalent to a module obtained after a finite number of castling transforms from (G_2, ϱ_2, V_2). A module (G, ϱ, V) is called *reduced* (or *castling-reduced*) if $\dim V \leq \dim V'$ for every castling transform (G, ϱ', V') of (G, ϱ, V). Sato and Kimura [10, §2] proved that prehomogeneity and generic stabilizers are preserved by castling transforms.

2. Étale modules for groups with factor Sp_n or SO_n

In this section we will construct two families of étale modules for reductive algebraic groups G. In the first family, G contains a simple factor Sp_n, $n \geq 1$, and theses modules are even super-étale, thus proving that groups with super-étale modules are not restricted to products of special linear groups. In the second family, G contains a factor SO_n, $n \geq 2$. This proves that groups with étale modules (but a possibly non-trivial stabilizer) do not have to be special in the sense of Serre. Moreover, these are the first known examples of étale modules for reductive algebraic groups that contain factors Sp_n or SO_n for arbitrary $n > 2$.

We need some preparations. Suppose G is an algebraic group of the form

$$G = G_m \times G_{m-1} \times \cdots \times G_1,$$

where $G_k \subseteq \mathrm{GL}_k$. The vector space

$$E_m = \mathrm{Mat}_{m,m-1} \oplus \mathrm{Mat}_{m-1,m-2} \oplus \ldots \oplus \mathrm{Mat}_{2,1} \tag{2.1}$$

becomes a G-module for the action defined as follows: If $A = (A_m, \ldots, A_1) \in G$ and $X = (X_{m-1}, \ldots, X_1) \in E_m$, then, by definition,

$$A.X = (A_m X_{m-1} A_{m-1}^\top,\ A_{m-1} X_{m-2} A_{m-2}^\top,\ \ldots,\ A_2 X_1 A_1^\top). \tag{2.2}$$

Note that

$$\dim E_m = \sum_{k=1}^{m-1} (k+1)k = \frac{m(m-1)}{2} + \sum_{k=1}^{m-1} k^2. \tag{2.3}$$

2.1. Super-étale modules for groups with factor Sp_n

We wish to construct a family of super-étale modules for the group

$$G = \mathrm{Sp}_n \times \mathrm{GL}_{2n-1} \times \cdots \times \mathrm{GL}_1.$$

We define a symplectic form ω in terms of the canonical basis of $\mathbb{C}^{2n}$ by

$$\omega(e_{2j-1}, e_{2j}) = 1 \quad \text{for } j = 1, \ldots, n,$$
$$\omega(e_{2j-1}, e_k) = 0 = \omega(e_{2j}, e_k) \quad \text{for } k \neq 2j, 2j-1.$$

Define subspaces $F_k = \mathrm{span}\{e_1, \ldots, e_k\}$ of $\mathbb{C}^{2n}$ for $k = 1, \ldots, 2n$.

Let $\mathrm{Sp}_n \subset \mathrm{GL}_{2n}$ denote the symplectic group that preserves the symplectic form ω. Then for every $A \in \mathrm{Sp}_n$ and $k = 1, \ldots, 2n$, we have $Ae_k^\perp \perp Ae_k$.

We can identify $F_{k+1} \otimes F_k$ with $\mathbb{C}^{k+1} \otimes \mathbb{C}^k \cong \mathrm{Mat}_{k+1,k}$. With E_{2n} from (2.1), introduce the G-module

$$V = \mathbb{C}^{2n} \oplus E_{2n},$$

where G acts on $\mathbb{C}^{2n}$ by the standard action of Sp_n and G acts on E_{2n} by (2.2), for $G_{2n} = \mathrm{Sp}_n$, $G_{2n-1} = \mathrm{GL}_{2n-1}, \ldots, G_1 = \mathrm{GL}_1$.

We have

$$\begin{aligned} \dim G &= 2n^2 + n + \sum_{k=1}^{2n-1} k^2 = 2n + \frac{2n(2n-1)}{2} + \sum_{k=1}^{2n-1} k^2 \\ &= 2n + \sum_{k=1}^{2n-1} k + \sum_{k=1}^{2n-1} k^2 = 2n + \dim E_{2n} = \dim V. \end{aligned} \tag{2.4}$$

We will prove by induction on n that V is super-étale for G. We only need to show that the generic stabilizer of the G-action is trivial, then it follows from (2.4) that G has an open orbit.

In the case $n = 1$, $G \cong \mathrm{SL}_2 \times \mathrm{GL}_1$ and $V = \mathrm{Mat}_2$, where SL_2 acts by matrix multiplication and GL_1 by scalar multiplication of the second column of a 2×2-matrix. One verifies directly that this is a super-étale module, and so this confirms the initial case for the induction:

Lemma 2.1. *For $n = 1$, the given action of $G = \mathrm{Sp}_1 \times \mathrm{GL}_1$ on $V = \mathbb{C}^2 \oplus \mathbb{C}^2$ is étale and has trivial stabilizer at the point $(e_1, e_2) \in V$.*

For the induction step, consider the action of $\mathrm{Sp}_n \times \mathrm{GL}_{2n-1}$ on $\mathbb{C}^{2n} \oplus (F_{2n} \otimes F_{2n-1})$ first. We can identify this space with Mat_{2n}, the action of $(A, B) \in \mathrm{Sp}_n \times \mathrm{GL}_{2n-1}$ given by

$$(A, B).X = AX \begin{pmatrix} B^\top & 0 \\ 0 & 1 \end{pmatrix}, \quad X \in \mathrm{Mat}_{2n,2n}.$$

As a point in general position, choose the identity matrix $X_0 = I_n$. Then, if

$$AI_n \begin{pmatrix} B^\top & 0 \\ 0 & 1 \end{pmatrix} = I_n,$$

it follows that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}_n$$

with $A_1 = (B^\top)^{-1} \in \mathrm{GL}_{2n-1}$. Recall that $Ae_{2n} = e_{2n}$ implies $Ae_{2n}^\perp = e_{2n}^\perp$, and the form of the matrix A thus requires $AF_{2n-2} = F_{2n-2}$. Also, $Ae_{2n-1} = e_{2n-1}$ since A also preserves $F_{2n-2}^\perp$. Hence

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & I_2 \end{pmatrix} \in \mathrm{Sp}_n, \quad A_0 \in \mathrm{Sp}_{n-1}.$$

This proves:

Lemma 2.2. *The stabilizer H of the $\mathrm{Sp}_n \times \mathrm{GL}_{2n-1}$-action on $\mathbb{C}^{2n} \oplus (\mathbb{C}^{2n} \otimes \mathbb{C}^{2n-1})$ at the point X_0 is given by*

$$H = \left\{ \left(\begin{pmatrix} A_0 & 0 \\ 0 & I_2 \end{pmatrix}, \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \;\middle|\; A_0 \in \mathrm{Sp}_{n-1} \right\} \cong \mathrm{Sp}_{n-1}.$$

Hence the stabilizer H_{2n-1} of the G-action on the submodule $\mathbb{C}^{2n} \oplus (\mathbb{C}^{2n} \otimes \mathbb{C}^{2n-1})$ of V at the point X_0 is

$$H_{2n-1} = \mathrm{Sp}_{n-1} \times \mathrm{GL}_{2n-2} \times \cdots \times \mathrm{GL}_1,$$

with the embedding of Sp_{n-1} in G given as above.

Consider the first summand W in E_{2n-1},

$$W = F_{2n-1} \otimes F_{2n-2} = \mathrm{Mat}_{2n-1,2n-2}$$

where the H_{2n-1}-action is given by the action of the factor $\mathrm{Sp}_{n-1} \times \mathrm{GL}_{2n-2}$. Here, Sp_{n-1} is identified with the projection of the stabilizer of $\mathrm{Sp}_n \times \mathrm{GL}_{2n-1}$ to

GL_{2n-1} (see Lemma 2.2), and this projection acts on the subspace $F_{2n-2} \subset F_{2n-1}$ and trivially on its complement in F_{2n-1}. Thus we can rewrite the module W as

$$\begin{aligned} W &= (F_{2n-2} \oplus \mathbb{C}e_{2n-1}) \otimes F_{2n-2} = W_1 \oplus W_2, \\ W_1 &= F_{2n-2} \otimes F_{2n-2} \cong \mathrm{Mat}_{2n-2}, \\ W_2 &= \mathbb{C} \otimes F_{2n-2} = F_{2n-2} \cong \mathbb{C}^{2n-2}, \end{aligned}$$

where $(A, B) \in \mathrm{Sp}_{n-1} \times \mathrm{GL}_{2n-2}$ acts on $X \in W_1$ by $X \mapsto AXB^\top$ and on $y \in W_2$ by $y \mapsto By$.

Choose $X_1 = I_{2n-2}$ as a point in general position for the action on W_1. The stabilizer of this action is again a diagonally embedded copy of Sp_{n-1} in $\mathrm{Sp}_{n-1} \times \mathrm{GL}_{2n-2}$. Identifying this copy once again with its projection to GL_{n-2}, we have an Sp_{n-1}-action on $W_2 \cong \mathbb{C}^{n-2}$ by left multiplication.

Lemma 2.3. *The stabilizer of the H_{2n-1}-action at the point $X_1 = I_{2n-2}$ in the module W_1 is the group*

$$H_{2n-2} = \mathrm{Sp}_{n-1} \times \mathrm{GL}_{2n-3} \times \cdots \times \mathrm{GL}_1,$$

where the Sp_{n-1}-action on $W_2 = \mathbb{C}^{2n-2}$ is by left multiplication.

In order for $E_{2n-1} = W \oplus E_{2n-2}$ to be étale for the H_{2n-1}-action (and thus the original module V to be étale for the G-action), the stabilizer H_{2n-2} must have an étale action on

$$W_2 \oplus E_{2n-2} = \mathbb{C}^{2n-2} \oplus E_{2n-2}.$$

Observe now that $\mathbb{C}^{2n-2} \oplus E_{2n-2}$ is of the same form as the original module V, and H_{2n-2} is of the same form as the original group G, with n replaced by $n-2$. Now we can apply the induction hypothesis to conclude that the H_{2n-2}-action on V_{2n-2} and thus the G-action on V is super-étale (where we assume that all points in general position are chosen similarly to X_0, X_1 above).

Theorem 2.4. *The module $(\mathrm{Sp}_n \times \mathrm{GL}_{2n-1} \times \cdots \times \mathrm{GL}_1, \mathbb{C}^{2n} \oplus E_{2n})$ with the action given above is a super-étale module.*

Remark 2.5. For $n = 2$, (G, ϱ, V) in Theorem 2.4 can be viewed as a variation of an example given by Helmstetter [7, p. 1090], which is the module

$$G = \mathrm{Sp}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1, \quad V = \mathbb{C}^4 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3) \oplus (\mathbb{C}^3 \otimes \mathbb{C}^2) \oplus \mathbb{C}^3$$

where the last copy of $\mathbb{C}^3$ is identified with the space of traceless 2×2-matrices, and the action of G is given by

$$(A, B, C, \alpha, \beta).(x, Y, Z, U) = (\alpha Ax, AYB^\top, BZC^\top, \beta CUC^{-1}).$$

This module is étale, but it is not super-étale, since the action of GL_2 on the last copy of $\mathbb{C}^3$ has a non-connected generic stabilizer.

Remark 2.6. A second family of super-étale modules appears in the construction of this section, namely the group $\mathrm{Sp}_n \times \mathrm{GL}_{2n} \times \cdots \times \mathrm{GL}_1$ acting on the module E_{2n}. This group appears as the stabilizer in Lemma 2.2 (for $n-1$), where the module is the module complement of $\mathbb{C}^{2n} \oplus (\mathbb{C}^{2n} \otimes \mathbb{C}^{2n-1})$ in this lemma.

2.2. Étale modules for groups with factor SO_n

We wish to construct a family of étale modules for the group

$$G = \mathrm{SO}_n \times \mathrm{GL}_{n-1} \times \cdots \times \mathrm{GL}_1,$$

where we take SO_n to be the subgroup of SL_n preserving the bilinear form represented by the identity matrix I_n.

Let $n \geq 2$. Consider the G-module $V = E_n$ with the action given by (2.2), where $G_n = \mathrm{SO}_n$, $G_{n-1} = \mathrm{GL}_{n-1}, \ldots, G_1 = \mathrm{GL}_1$. We have

$$\dim G = \tfrac{1}{2}n(n-1) + \sum_{k=1}^{n-1} k^2 = \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} k^2 = \dim E_n. \tag{2.5}$$

In order to verify that V is an étale module for G, we only need to show that the connected component H° of the generic stabilizer H is trivial. Then it follows from (2.5) that G has an open orbit and the action is étale.

Lemma 2.7. *The stabilizer H_1 of $\mathrm{SO}_n \times \mathrm{GL}_{n-1}$ on the module $\mathrm{Mat}_{n,n-1}$ at the point in general position $X_1 = \left(\begin{smallmatrix} I_{n-1} \\ 0 \ldots 0 \end{smallmatrix}\right)$ consists of the elements*

$$(A, A_0) \in \mathrm{SO}_n \times \mathrm{GL}_{n-1} \quad \textit{with} \;\; A_0 \in \mathrm{O}_{n-1} \;\; \textit{and} \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & \alpha \end{pmatrix} \in \mathrm{SO}_n,$$

where $\alpha = \det(A_0)^{-1}$. In particular,

$$H_1 \cong \mathrm{O}_{n-1}.$$

Proof. Let $(A, B) \in \mathrm{SO}_n \times \mathrm{GL}_{n-1}$, and let A_0 be the upper left $(n-1) \times (n-1)$-block of A and a_n the first $n-1$ entries in the last row of A. Then $AX_1B^\top = X_1$ is equivalent to $A_0^{-1} = B^\top$, $a_n = 0$, and as A_0 is orthogonal, this gives the required form of the stabilizer H_1 of X_1. □

The identity component $H_1^\circ \cong \mathrm{SO}_{n-1}$ of the generic stabilizer of (G, V) acts on the next summand $\mathrm{Mat}_{n-1,n-2}$ in E_n via its injective projection to the GL_{n-1}-factor. But this is identical to the left multiplication of SO_{n-1} on $\mathrm{Mat}_{n-1,n-2}$. So we are now looking at the action of

$$\mathrm{SO}_{n-1} \times \mathrm{GL}_{n-1} \times \cdots \times \mathrm{GL}_1$$

given by (2.2) on E_{n-1}. When choosing a point in general position for this action as in Lemma 2.7, we can apply induction on n to conclude that this module is étale. Moreover, Lemma 2.7 for $n = 2$ takes care of the initial case, that is, the action of the abelian group $\mathrm{SO}_2 \times \mathrm{GL}_1$ on $V = \mathbb{C}^2$ given by $(A, \lambda) \mapsto \lambda Ax$, $x \in \mathbb{C}^2$, is étale with generic stabilizer $H \cong \mathbb{Z}_2$.

So we have shown:

Theorem 2.8. *Let $n \geq 2$. The module $(\mathrm{SO}_n \times \mathrm{GL}_{n-1} \times \cdots \times \mathrm{GL}_1, E_n)$ with the action given by (2.2) is an étale module.*

3. Étale Lie algebras over fields of characteristic 0

Let $\mathbb{k}$ be a field of characteristic 0. Recall that a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{k})$ is called *algebraic* if there is a $\mathbb{k}$-defined linear algebraic group $G \subset \mathrm{GL}_n$ such that $\mathfrak{g} = (\operatorname{Lie} G)(\mathbb{k})$. The Lie algebra $\mathfrak{g}$ is called *prehomogeneous* if there is a point $o \in \mathbb{k}^n$ such that the map $\beta : \mathfrak{g} \to \mathbb{k}^n$, $X \mapsto Xo$ is a surjective homomorphism of vector spaces, and $\mathfrak{g}$ is called *étale* if β is an isomorphism.

Proposition 3.1. *Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{k})$ be a prehomogeneous Lie algebra with generic stabilizer $\mathfrak{h}$. Then there is a prehomogeneous algebraic Lie algebra $\widetilde{\mathfrak{g}} \subset \mathfrak{gl}_n(\mathbb{k})$ with generic stabilizer $\widetilde{\mathfrak{h}}$ such that $[\mathfrak{g},\mathfrak{g}] = [\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}]$ and $\widetilde{\mathfrak{h}} = \mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}]$.*

Proof. Let $\mathfrak{g}^{\mathrm{a}} \subset \mathfrak{gl}_n(\mathbb{k})$ denote the algebraic hull of $\mathfrak{g}$ (the smallest algebraic subalgebra containing $\mathfrak{g}$). We have $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{g}^{\mathrm{a}},\mathfrak{g}^{\mathrm{a}}]$, cf. Chevalley [5, Prop. 1]. Let $\mathfrak{h}' \subset \mathfrak{g}^{\mathrm{a}}$ stand for the annihilator of o. Then $\mathfrak{h}'$ is an algebraic subalgebra of $\mathfrak{g}^{\mathrm{a}}$.

Let $\mathfrak{a} = \mathfrak{g}^{\mathrm{a}}/[\mathfrak{g},\mathfrak{g}]$. Consider the canonical map $\pi : \mathfrak{g}^{\mathrm{a}} \to \mathfrak{a}$. Since an algebraic subalgebra of a commutative algebraic Lie algebra has a complementary algebraic subalgebra, also defined over $\mathbb{k}$, there is an algebraic subalgebra $\mathfrak{h}_1 \subset \mathfrak{a}$ such that $\mathfrak{a} = \pi(\mathfrak{h}') \oplus \mathfrak{h}_1$. Set $\widetilde{\mathfrak{g}} = \pi^{-1}(\mathfrak{h}_1)$ and $\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{g}} \cap \mathfrak{h}'$. We have

$$\widetilde{\mathfrak{h}} = [\mathfrak{g},\mathfrak{g}] \cap \mathfrak{h}' = [\mathfrak{g},\mathfrak{g}] \cap \mathfrak{h}.$$

The fact that $\widetilde{\mathfrak{g}}$ is prehomogeneous follows from

$$\dim \widetilde{\mathfrak{g}} = \dim \mathfrak{g}^{\mathrm{a}} - \dim \pi(\mathfrak{h}') = n + \dim \mathfrak{h}' - \dim \pi(\mathfrak{h}') = n + \dim \widetilde{\mathfrak{h}}. \qquad \square$$

Corollary 3.2. *For every étale Lie algebra there exists an algebraic étale Lie algebra over $\mathbb{k}$ with the same derived subalgebra (and the same maximal semisimple subalgebra).*

If $X \mapsto Xo$ is an isomorphism over $\mathbb{k}$ then it is such over any extension field of $\mathbb{k}$. Hence:

Proposition 3.3. *Let $\overline{\mathbb{k}}$ denote an extension field of $\mathbb{k}$. A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{k})$ is étale if and only if $\mathfrak{g} \otimes_{\mathbb{k}} \overline{\mathbb{k}} \subset \mathfrak{gl}_n(\overline{\mathbb{k}})$ is étale.*

4. Non-existence of étale modules for groups with simple factors $\mathbf{F_4}$ or $\mathbf{E_8}$

For an arbitrary Lie group G to have a (real, finite-dimensional) *étale* module V means that G has an open orbit in V in the standard topology of V and $\dim G = \dim V$. We use the results of the previous section and the Sato–Kimura classification of algebraic prehomogeneous modules to establish the following non-existence result:

Theorem 4.1. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and a linear action on a finite-dimensional real vector space V. If the module (G,V) is étale, then a maximal semisimple subalgebra of $\mathbb{C} \otimes \mathfrak{g}$ does not contain simple factors $\mathfrak{f}_4$ or $\mathfrak{e}_8$.*

The proof needs some preparations.

Proposition 4.2. *Let G be a linear algebraic group. Given a short exact sequence of G-modules*

$$0 \to U \to V \xrightarrow{\pi} W \to 0 \tag{4.1}$$

where V is prehomogeneous with a point o in general position, let G' be the stabilizer in G of the line spanned by $\pi(o) \in W$. Then G' preserves $U' := U + \langle o \rangle$ and has an open orbit on it. Moreover, the stabilizer H of o in (G, V) is also the stabilizer of o in (G', U').

Proof. Note that $o \notin U$ since o is in general position. The fact that G' preserves U' follows immediately from definitions and the property $\operatorname{Ker} \pi = U$. Note that $H \subset G'$. It remains to show that the orbit $G'o \subset U'$ is open. Since (G, V) is prehomogeneous, so is (G, W). Hence, the action of G on the projective space $\mathrm{P}W$ over W has an open orbit and its generic stabilizer is conjugate to G'. We conclude $\dim G - \dim G' = \dim W - 1$, and therefore

$$\begin{aligned} \dim G'o &= \dim G' - \dim H = \dim V - (\dim G - \dim G') \\ &= \dim V - \dim W + 1 = \dim U'. \qquad \square \end{aligned}$$

Lemma 4.3. *Let (G, V) be a prehomogeneous module for an algebraic group G with solvable radical R. Assume there exists an irreducible submodule U of codimension 1 in V that is not a direct summand of V. Then (R, V) is prehomogeneous.*

Proof. Let $W = V/U$ denote the one-dimensional quotient module for G. Note that W is prehomogeneous since V is. Let $\mathrm{R_u}(G)$ denote the unipotent radical of G, and A the center of $L(G)$, so that $R = A \cdot \mathrm{R_u}(G)$ and $L(G) = A \cdot S(G)$. Let $\mathfrak{r} = \operatorname{Lie} R$, $\mathfrak{r}_{\mathrm{u}} = \operatorname{Lie} \mathrm{R_u}(G)$ and $\mathfrak{a} = \operatorname{Lie} A$. Let x be a non-zero point in a one-dimensional $L(G)$-invariant complementary subspace W' to U in V. We will show that $Rx \subset V$ is open. It suffices to show that $\mathfrak{r}x = V$.

Note that $\mathrm{R_u}(G)$ acts trivially on U, as the eigenspace for eigenvalue 1 of $\mathrm{R_u}(G)$ is G-invariant and non-zero, hence all of U by irreducibility. It follows that U is $L(G)$-irreducible. Also, both $\mathrm{R_u}(G)$ and $S(G)$ act trivially on the one-dimensional module W.

Since U is not a direct summand in V, $\mathfrak{r}_{\mathrm{u}}x$ is a non-zero subspace of U. Moreover, $\mathfrak{r}_{\mathrm{u}}x$ is $L(G)$-invariant, and hence coincides with U, since the latter is $L(G)$-irreducible. Since $\mathrm{R_u}(G)$ and $S(G)$ act trivially on W, the prehomogeneity of W requires that A acts non-trivially on W and hence on x. So $\mathfrak{a}x = W'$, and it follows that $\mathfrak{r}x = \mathfrak{a}x + \mathfrak{r}_{\mathrm{u}}x = W' + U = V$. $\square$

Given a linear algebraic group G and a rational G-module V, we call (G, V) *casual*[1] if it is equivalent to $(G' \times \mathrm{GL}_n, V' \otimes \mathbb{C}^n)$ for an algebraic subgroup $G' \subset \mathrm{GL}(V')$ and $n \geq \dim V'$. All such modules are prehomogeneous with generic stabilizer H satisfying $L(H) \cong L(G') \times \mathrm{GL}_{n - \dim V'}$, and the irreducible ones are given by cases I(1) and III(1) in the Sato–Kimura classification [10, §7].

[1]It is called *trivial* in [10, Def. 5, p. 43]. We decided to use another term to avoid confusions.

Remark 4.4. A module that is equivalent to a casual irreducible étale module is necessarily equivalent to $(F \times \mathrm{GL}_n, \mathbb{C}^n \otimes \mathbb{C}^n)$ for some finite group F acting irreducibly on $\mathbb{C}^n$. If (G, V) is castling-equivalent to such a module, then it follows immediately that all simple factors of $S(G)$ are special linear groups.

Proposition 4.5. *Let (G, V) be an étale module for a linear algebraic group G, and let Q be a simple factor of $S(G)$ not isomorphic to SL_n for any n. There exists an étale module $(\widetilde{G}, \widetilde{V})$ with a simple factor $\widetilde{Q} \cong Q$ in $S(\widetilde{G})$ and an irreducible quotient module $\widetilde{W}$ of $\widetilde{V}$ such that $\widetilde{Q}$ acts non-trivially on $\widetilde{W}$ and $\widetilde{W}$ is not castling-equivalent to a casual module.*

Proof. We prove the claim by induction on $\dim V$. Note that since $\dim Q > 1$ the module cannot be étale in the case $\dim V = 1$, so the claim holds trivially.

Suppose now that $\dim V \geq 2$. If V is irreducible, then $(G, V) = (\widetilde{G}, \widetilde{V})$ satisfies the claim in light of Remark 4.4. So we may further assume that V is not irreducible.

Assume that there is an irreducible quotient $W = V/U$ with $\dim W \geq 2$. If Q acts non-trivially on W and (G, W) is not castling-equivalent to a casual module, we can put $(\widetilde{G}, \widetilde{V}) := (G, V)$ and $\widetilde{Q} := Q$. Otherwise, either Q acts trivially on W or (G, W) is castling-equivalent to a casual module. Then $S(G_x)$ contains a factor isomorphic to Q, where $x \in W$ is a point in general position. In this case, if (G', U') is as in Proposition 4.2, then (G', U') is étale and G' contains a conjugate of Q. Since $\dim U' = 1 + \dim U < \dim V$, the claim now follows by induction on $\dim V$.

Suppose now that all irreducible quotients of V are one-dimensional, and let $W = V/U$ be one of them. There exists a maximal proper submodule $U_0 \subset U$, so that $W_0 := U/U_0$ is irreducible, and for $W_1 := V/U_0$ we have the exact sequence

$$0 \to W_0 \to W_1 \to W \to 0.$$

Note that W_1 is prehomogeneous since V is. We claim that the solvable radical R of G has an open orbit in W_1. If W_0 is a direct summand in W_1, then by the assumption that all quotients of V are one-dimensional, $\dim W_0 = 1$, and therefore $S(G)$ acts trivially on W_1, implying that the open G-orbit is also an open R-orbit. Suppose W_0 is not a direct summand in W_1. Since W and W_0 are both irreducible, we can apply Lemma 4.3 (with V replaced by W_1) to conclude that R has an open orbit in W_1. Therefore, $S(G)$ belongs to the stabilizer of a point in general position in W_1. We can now use Proposition 4.2 (with W, U replaced by W_1, U_0) and induction to derive the statement. $\square$

Proof of Theorem 4.1. If (G, V) is a real étale module, then by Proposition 3.3 there exists a complex étale module $(G_{\mathbb{C}}, V_{\mathbb{C}})$ where $G_{\mathbb{C}}$ is a Lie group with Lie algebra $\mathbb{C} \otimes \mathfrak{g}$. So by Proposition 3.1, we may assume that (G, V) is a complex algebraic étale module. According to the classification of irreducible prehomogeneous modules for reductive algebraic groups [10, §7], all irreducible prehomogeneous modules for reductive algebraic groups with F_4 or E_8 as a simple factor are castling-equivalent to a casual module. It remains to apply Proposition 4.5 to (G, V). $\square$

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