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COTANGENT BUNDLE TO THE FLAG VARIETY–I

V. LAKSHMIBAI*

Department of Mathematics
Northeastern University
Boston, MA 02115, USA
lakshmibai@neu.edu

C. S. SESHADRI

Chennai Mathematical Institute
Siruseri, Kelambakkam, Chennai
Tamil Nadu 603103, India
css@cmi.ac.in

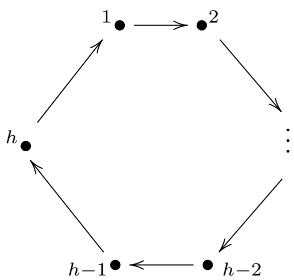
R. SINGH

Department of Mathematics
Northeastern University
Boston, MA 02115, USA
singh.rah@husky.neu.edu

Abstract. We show that there is a SL_n-stable closed subset of an affine Schubert variety in the infinite-dimensional flag variety (associated to the Kac–Moody group $\widehat{\mathrm{SL}_n}$) which is a natural compactification of the cotangent bundle to the finite-dimensional flag variety SL_n/B.

1. Introduction

Let the base field K be the field of complex numbers. Consider a cyclic quiver with h vertices and dimension vector $\underline{d} = (d_1, \ldots, d_h)$:



Denote $V_i = K^{d_i}$. Let

$$Z = \mathrm{Hom}(V_1, V_2) \times \cdots \times \mathrm{Hom}(V_h, V_1), \quad \mathrm{GL}_{\underline{d}} = \prod_{1 \le i \le h} \mathrm{GL}(V_i).$$

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Corresponding Author: V. Lakshmibai, e-mail: lakshmibai@neu.edu

We have a natural $\mathrm{GL}_{\underline{d}}$-action on Z: for $g=(g_1,\dots,g_h)\in\mathrm{GL}_{\underline{d}}$, $f=(f_1,\dots,f_h)\in Z$,

$$g\cdot f=(g_2f_1g_1^{-1},g_3f_2g_2^{-1},\dots,g_1f_hg_h^{-1}).$$

Let

$$\mathcal{N}=\{(f_1,\dots,f_h)\in Z\mid f_h\circ f_{h-1}\circ\cdots\circ f_1:V_1\to V_1\text{ is nilpotent}\}.$$

Note that $f_h\circ f_{h-1}\circ\cdots\circ f_1:V_1\to V_1$ being nilpotent is equivalent to $f_{i-1}\circ f_{i-2}\circ\cdots\circ f_1\circ f_h\circ\cdots\circ f_{i+1}f_i:V_i\to V_i$ being nilpotent. Clearly $\mathcal{N}$ is $\mathrm{GL}_{\underline{d}}$-stable. Lusztig (cf. [Lu1]) has shown that an orbit closure in $\mathcal{N}$ is canonically isomorphic to an open subset of a Schubert variety in $\widehat{\mathrm{SL}}_n/Q$, where $n=\sum_{1\le i\le h}d_i$, and Q is the parabolic subgroup of $\widehat{\mathrm{SL}}_n$ corresponding to omitting $\alpha_0,\alpha_{d_1},\alpha_{d_1+d_2}$, $\dots,\alpha_{d_1+\cdots+d_{h-1}}$ ($\alpha_i,0\le i\le n-1$ being the set of simple roots for $\widehat{\mathrm{SL}}_n$). Corresponding to $h=1$, we have that $\mathcal{N}$ is in fact the variety of nilpotent elements in $M_{d_1,d_1}(K)$, and thus the above isomorphism identifies $\mathcal{N}$ with an open subset of a Schubert variety $X_{\mathcal{N}}$ in $\widehat{\mathrm{SL}}_n/G_0$, G_0 being the maximal parabolic subgroup of $\widehat{\mathrm{SL}}_n$ corresponding to "omitting" α_0.

Let now $h=2$

$$Z_0=\{(f_1,f_2)\in Z\mid f_2\circ f_1=0,f_1\circ f_2=0\}.$$

Strickland (cf. [S]) has shown that each irreducible component of Z_0 is the conormal variety to a determinantal variety in $M_{d_1,d_2}(K)$. A determinantal variety in $M_{d_1,d_2}(K)$ being canonically isomorphic to an open subset in a certain Schubert variety in G_{d_2,d_1+d_2} (the Grassmannian variety of d_2-dimensional subspaces of $K^{d_1+d_2}$) (cf. [LS]), the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in the (finite-dimensional) flag variety and the affine Schubert varieties. This is the motivation for this article. Let $G=\mathrm{SL}_n$.

Inspired by Lusztig's embedding (cf. [Lu1], [Lu2]) of $\mathcal{N}$ in $\widehat{\mathrm{SL}_n}/Q$, we define a family of maps $\psi_p:T^*G/B\hookrightarrow\widehat{\mathrm{SL}}_n/\mathcal{B}$, parametrized by polynomials in one variable with coefficients in $\mathbb{C}((t))$, and with 1 as the constant term. For a particular map ϕ (analogous to Lusztig's map) in this family, we find a $\kappa_0\in\widehat{W}$ such that the affine Schubert variety $X(\kappa_0)$ is G_0-stable (G_0 being as above, the maximal parabolic subgroup of $\widehat{\mathrm{SL}}_n$ corresponding to "omitting" α_0) and show that ϕ gives an embedding $T^*G/B\hookrightarrow X(\kappa_0)\subset\widehat{\mathrm{SL}}_n/\mathcal{B}$. We thus obtain a SL_n-stable closed subvariety of $X(\kappa_0)$ as a natural compactification of T^*G/B (cf. Theorem 6.4). Let $\pi:\widehat{\mathrm{SL}}_n/\mathcal{B}\to\widehat{\mathrm{SL}}_n/G_0$ be the canonical projection. Then we show that $\pi(T^*G/B)=\mathcal{N}$, the variety of nilpotent matrices, and that $\pi\,|_{T^*G/B}:T^*G/B\to\mathcal{N}$ is in fact the Springer resolution.

Following the above ideas, Lakshmibai (cf. [L]) has obtained a stronger result for $T^*G_{d,n}$, the cotangent bundle to the Grassmannian variety $G_{d,n}$. She shows that there is an embedding χ (analogous to ϕ) of $T^*G_{d,n}$ inside a Schubert variety $X(\iota)\subset\widehat{\mathrm{SL}_n}/\mathcal{Q}_d$ (where $\mathcal{Q}_d$ is the two-step parabolic subgroup of $\widehat{\mathrm{SL}_n}$ corresponding to omitting α_0,α_d) such that $X(\iota)$ *is in fact a compactification of* $T^*G_{d,n}$. The result of [L] has been generalized to T^*G/P in [LRS], G/P being a cominuscule Grasssmannian variety.

It would be interesting to know if the result of [L] could be achieved replacing P with B, for a suitable generalization of χ. We show in §7 that this is not possible for any ψ_p in the above family, even when $n = 3$. We think that our result about the embedding $\phi : T^*G/B \hookrightarrow X(\kappa_0)$ identifying a certain SL_n-stable closed subvariety of $X(\kappa_0)$ as a natural compactification of T^*G/B is the best possible in relating T^*G/B and affine Schubert varieties in $\widehat{\mathrm{SL}_n}/\mathcal{B}$.

The results of this paper open up other related problems like the study of line bundles on $T^*G/B, G = \mathrm{SL}_n$ (using the embedding of T^*G/B into $X(\kappa_0)$, and realizing line bundles on T^*G/B as restrictions of suitable line bundles on $X(\kappa_0)$), establishing similar embeddings of the cotangent bundles to partial flag varieties G/Q (G semi-simple and Q a parabolic subgroup), etc. Further, the facts that conormal varieties to Schubert varieties in G/B are closed subvarieties of T^*G/B, and that the affine Schubert variety $X(\kappa_0)$ contains a G-stable closed subvariety which is a natural compactification of T^*G/B, suggest similar compactifications for conormal varieties to Schubert varieties in G/B (by suitable affine Schubert varieties in $\widehat{\mathrm{SL}_n}/\mathcal{B}$); such a realization could lead to important consequences such as a knowledge of the equations of the conormal varieties (to Schubert varieties) as subvarieties of the cotangent bundle. These problems will be dealt with in a subsequent paper.

Regarding results on similar compactifications, we mention Mirkovic–Vybornov's work (cf. [MV]), where the authors construct compactifications of Nakajima's quiver varieties of type A inside affine Grassmannians of type A. Manivel and Michalek ([MM]) have recently studied the local geometry of tangential varieties (which are compactifications of the tangent bundle) to cominuscule Grassmannians. Also of interest is the work of Achar, Henderson and Riche (see [AH], [AHR] for details) relating various results of Broer and Reeder to the Springer resolution via the geometric Satake correspondence.

The sections are organized as follows. In §2, we fix notation and recall *affine Schubert varieties*. In §3, we introduce the elements κ and κ_0 (in $\widehat{W}$, the affine Weyl group), and prove some properties of κ. In §4, we prove a crucial result on κ needed for realizing the embeddings of $\mathcal{N}$ and $T^*\mathrm{SL}_n/B$ inside $\widehat{\mathrm{SL}_n}/G_0$ and $\widehat{\mathrm{SL}_n}/\mathcal{B}$ respectively. In §5, we spell out Lusztig's isomorphism which identifies $\mathcal{N}$ with an open subset of $X_{G_0}(\kappa)$ (inside $\widehat{\mathrm{SL}_n}/G_0$). In §6, using the map $\phi : T^*G/B \to \widehat{\mathrm{SL}_n}/\mathcal{B}$ as above and the natural projection $\widehat{\mathrm{SL}_n}/\mathcal{B} \to \widehat{\mathrm{SL}_n}/G_0$, we recover the Springer resolution of $\mathcal{N}$; we also prove the main result that ϕ identifies an SL_n-stable closed subvariety of $X(\kappa_0)$ as a compactification of T^*G/B. In §7, we show that it is not possible, for any choice in the family ψ_p, to realize an affine Schubert variety (in $\widehat{\mathrm{SL}_3}/\mathcal{B}$) as a compactification of the cotangent bundle $T^*\mathrm{SL}_3/B$.

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2. Affine Schubert varieties

Let $K = \mathbb{C}, F = K((t))$, the field of Laurent series, $A = K[[t]]$. Let G be a semi-simple algebraic group over K, T a maximal torus in G, B a Borel subgroup, $B \supset T$, and let B^- be the Borel subgroup opposite to B. Let $\mathcal{G} = G(F)$. The natural inclusion $K \hookrightarrow A \hookrightarrow F$ induces an inclusion

$$G \hookrightarrow G(A) \hookrightarrow \mathcal{G}.$$

The natural projection $A \to K$ given by $t \mapsto 0$ induces a homomorphism

$$\pi : G(A) \to G.$$

The group $\mathcal{B} := \pi^{-1}(B)$ is a Borel subgroup of $\mathcal{G}$.

2.1. Bruhat decomposition

Let $\widehat{W} = N(K[t,t^{-1}])/T$, the *affine Weyl group* of G (here, N is the normalizer of T in G). The group $\widehat{W}$ is a Coxeter group (cf. [Kac]). We have that

$$G(F) = \dot{\bigcup}_{w \in \widehat{W}} \mathcal{B}w\mathcal{B}, G(F)/\mathcal{B} = \dot{\bigcup}_{w \in \widehat{W}} \mathcal{B}w\mathcal{B} (\operatorname{mod} \mathcal{B}).$$

For $w \in \widehat{W}$, let $X(w)$ be the *affine Schubert variety* in $G(F)/\mathcal{B}$:

$$X(w) = \dot{\bigcup}_{\tau \leq w} \mathcal{B}\tau\mathcal{B} (\operatorname{mod} \mathcal{B}).$$

It is a projective variety of dimension $\ell(w)$.

2.2. Affine flag variety, affine Grassmannian

Let $G = \mathrm{SL}(n)$, $\mathcal{G} = G(F)$, $G_0 = G(A)$. We say $g \in \mathcal{G}$ is integral if and only if $g \in G_0$, i.e., viewed as G-valued meromorphic function on $\mathbb{C}$, it has no poles at $t = 0$. The homogeneous space $\mathcal{G}/\mathcal{B}$ is the *affine flag variety*, and $\mathcal{G}/G_0$ is the *affine Grassmannian*. Further,

$$\mathcal{G}/G_0 = \dot{\bigcup}_{w \in \widehat{W}^{G_0}} \mathcal{B}w\, G_0 (\operatorname{mod} G_0)$$

where $\widehat{W}^{G_0}$ is the set of minimal representatives in $\widehat{W}$ of $\widehat{W}/W_{G_0}$.

Let

$$\widehat{\mathrm{Gr}(n)} = \{A\text{-lattices in } F^n\}.$$

Here, by an A-lattice in F^n, we mean a free A-submodule of F^n of rank n. Let E be the standard lattice, namely, the A-span of the standard F-basis $\{e_1, \ldots, e_n\}$ for F^n. For $V \in \widehat{\mathrm{Gr}(n)}$, define

$$\mathrm{vdim}(V) := \dim_K(V/V \cap E) - \dim_K(E/V \cap E).$$

One refers to $\mathrm{vdim}(V)$ as the *virtual dimension of* V. For $j \in \mathbb{Z}$ denote

$$\widehat{\mathrm{Gr}_j(n)} = \{V \in \widehat{\mathrm{Gr}(n)} \mid \mathrm{vdim}(V) = j\}.$$

Then $\widehat{\mathrm{Gr}_j(n)}, j \in \mathbb{Z}$ give the connected components of $\widehat{\mathrm{Gr}(n)}$. We have a transitive action of $\mathrm{GL}_n(F)$ on $\widehat{\mathrm{Gr}(n)}$ with $\mathrm{GL}_n(A)$ as the stabilizer at the standard lattice E. Further, let $\mathcal{G}_0$ be the subgroup of $\mathrm{GL}_n(F)$, defined as,

$$\mathcal{G}_0 = \{g \in \mathrm{GL}_n(F) \mid \mathrm{ord}(\det g) = 0\}$$

(here, for a $f \in F$, say $f = \sum a_i t^i$, *order* f is the smallest r such that $a_r \neq 0$). Then $\mathcal{G}_0$ acts transitively on $\widehat{\mathrm{Gr}_0(n)}$ with $\mathrm{GL}_n(A)$ as the stabilizer at the standard lattice E. Also, we have a transitive action of $\mathrm{SL}_n(F)$ on $\widehat{\mathrm{Gr}_0(n)}$ with $\mathrm{SL}_n(A)$ as the stabilizer at the standard lattice E. Thus we obtain the identifications:

$$\begin{gathered}\mathrm{GL}_n(F)/\mathrm{GL}_n(A) \simeq \widehat{\mathrm{Gr}(n)},\\ \mathcal{G}_0/\mathrm{GL}_n(A) \simeq \widehat{\mathrm{Gr}_0(n)}, \mathrm{SL}_n(F)/\mathrm{SL}_n(A) \simeq \widehat{\mathrm{Gr}_0(n)}.\end{gathered} \tag{$*$}$$

In particular, we obtain

$$\mathcal{G}_0/\mathrm{GL}_n(A) \simeq \mathrm{SL}_n(F)/\mathrm{SL}_n(A). \tag{$**$}$$

2.3. Generators for $\widehat{W}$

Recall the Weyl group $\widehat{W} = N(K[t, t^{-1}])/T$. Let R (resp. R^+) be the set of roots (resp. positive roots) of G relative to B, and let δ be the basic imaginary root of the affine Kac–Moody algebra of type $\widehat{\mathsf{A}}_{n-1}$ given by (cf. [Kac])

$$\delta = \alpha_0 + \theta = \alpha_0 + \cdots + \alpha_{n-1}.$$

The set of real roots of $\mathcal{G}$ is given by $\{r\delta + \beta \mid r \in \mathbb{Z}, \beta \in R\}$, and the set of positive roots of $\mathcal{G}$ is given by $\{r\delta + \beta \mid r > 0, \beta \in R\} \dot{\bigcup} R^+$ (cf. [Kac]). Following the notation in [Kac], we shall work with the set of generators for $\widehat{W}$ given by $\{s_0, s_1, \ldots, s_{n-1}\}$, where $s_i, 0 \leq i \leq n-1$, are the reflections with respect to $\alpha_i, 0 \leq i \leq n-1$. Note that $\{\alpha_i, 1 \leq i \leq n-1\}$ is simply the set of simple roots of SL_n (with respect to the Borel subgroup B). In particular, the Weyl group W of $\mathrm{SL}_n(\mathbb{C})$ is simply the subgroup of $\widehat{W}$ generated by $\{s_1, \ldots, s_{n-1}\}$.

2.4. The affine presentation

The generators s_i, $1 \leq i \leq n-1$ have the following canonical lifts to $N(K[t, t^{-1}])$: s_i is the permutation matrix (a_{rs}), with $a_{jj} = 1, j \neq i, i+1$, $a_{i\,i+1} = 1, a_{i+1\,i} = -1$, and all other entries are 0. A canonical lift for s_0 is given by

$$\begin{pmatrix} 0 & 0 & \cdots & t^{-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ -t & 0 & 0 & 0 \end{pmatrix}.$$

Let $s_\theta \in W$ be the reflection with respect to the longest root θ in A_{n-1} given by $\theta = \alpha_1 + \cdots + \alpha_{n-1}$. Let L (resp. Q) be the root (resp. coroot) lattice of

$\mathfrak{sl}_n(=\mathrm{Lie}(\mathrm{SL}_n))$, and let $\langle\ ,\ \rangle$ be the canonical pairing on $L \times Q$. Consider $\theta^\vee \in Q$ given by $\theta^\vee = \alpha_1^\vee + \cdots + \alpha_{n-1}^\vee$. There exists (cf. [K], §13.1.6) a group isomorphism $\widehat{W} \to W \ltimes Q$ given by

$$\begin{aligned} s_i &\mapsto s_i \quad \text{for } 1 \leq i \leq n-1, \\ s_0 &\mapsto s_\theta \lambda_{-\theta^\vee}, \end{aligned}$$

where we write λ_q for $(\mathrm{id}, q) \in W \ltimes Q$. In particular, we get $s_0 s_\theta \mapsto \lambda_{\theta^\vee}$, which we use to compute a lift of $\lambda_{\theta^\vee}$ to $N(K[t,t^{-1}])$:

$$\begin{pmatrix} 0 & 0 & \cdots & t^{-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ -t & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Consider the element $w \in W$ corresponding to $(1,i)(i+1,n) \in S_n$, and observe that $w(\theta^\vee) = \alpha_i^\vee$, the i^{th} simple coroot. It follows that a lift of $\lambda_{\alpha_i^\vee} = w\lambda_{\theta^\vee} w^{-1}$ is given by

$$w \begin{pmatrix} t^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} w^{-1} = \begin{pmatrix} \ddots & & & \\ & t^{-1} & & \\ & & t & \\ & & & \ddots \end{pmatrix}$$

where in the matrix on the right-hand side, the dots are 1, and the off-diagonal entries are 0, i.e., the matrix on the right-hand side is the diagonal matrix with i, $(i+1)$-th entries being t^{-1}, t respectively, and all other diagonal entries being 1.

The (Coxeter) length of λ_q is given by the following formula (cf. [K, §13.1.E(3)]):

$$l(\lambda_q) = \sum_{\alpha \in R^+} |\alpha(q)|, \quad q \in Q$$

where $\alpha(q) := \langle \alpha, q \rangle$. The action of λ_q on the root system of $\mathcal{G}$ is determined by the following formulae (cf. [K, §13.1.6]):

$$\begin{aligned} \lambda_q(\alpha) &= \alpha - \alpha(q)\delta \quad \text{for } \alpha \in R, q \in Q, \\ \lambda_q(\delta) &= \delta. \end{aligned}$$

In particular, for $\alpha \in R^+$, $\lambda_q(\alpha) > 0$ if and only if $\alpha(q) \leq 0$.

Corollary 2.5. *For $\alpha \in R^+$, $q \in Q$, $l(\lambda_q s_\alpha) > l(\lambda_q)$ if and only if $\alpha(q) \leq 0$.*

Proof. Follows from the equivalence $ws_\alpha > w$ if and only if $w(\alpha) > 0$, applied to $w = \lambda_q$. □

3. The element κ_0

Our goal is to give a compactification of the cotangent bundle T^*G/B as a (left) SL_n stable subvariety of the affine Schubert variety $X(\kappa_0)$, where κ_0 is as defined below:

$$\begin{aligned}\tau &:= s_{n-1}\cdots s_2 s_1 s_0,\\ \kappa &:= \tau^{n-1},\\ \kappa_0 &:= w'\tau^{n-1}\end{aligned}$$

where w' is the longest element in the Weyl group generated by $s_1, \dots s_{n-2}$. We first prove some properties of κ and τ which are consequences of the braid relations

$$\begin{aligned}s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1},\ \ 0 \le i \le n-2,\\ s_0 s_{n-1} s_0 &= s_{n-1} s_0 s_{n-1}\end{aligned}$$

and the commutation relations:

$$s_i s_j = s_j s_i,\ 1 \le i, j \le n-1, |i-j| > 1, \quad s_0 s_i = s_i s_0,\ 2 \le i \le n-2.$$

3.1. Some facts

Fact 1: $\tau(\delta) = \delta$.
Fact 2: $\tau(\alpha_1 + \cdots + \alpha_{n-1}) = 2\delta + \alpha_{n-1}$.
Fact 3: $\tau(r\delta + \alpha_i + \cdots + \alpha_{n-1}) = (r+1)\delta + \alpha_{i-1} + \alpha_i + \cdots + \alpha_{n-1}, 2 \le i \le n-1, r \in \mathbb{Z}_+$.
Fact 4: $s_{n-1}\cdots s_{j+1}(\alpha_j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1},\ j \ne 0, n-1$.
Fact 5: $s_{n-1}\cdots s_1(\alpha_0) = \delta + \alpha_{n-1}$.
Fact 6: $\tau(\alpha_{n-1}) = \delta + \alpha_{n-2} + \alpha_{n-1}$ (a special case of Fact 3 with $r = 0, i = n-1$).
Fact 7: $\tau(\alpha_1) = \alpha_0 + \alpha_{n-1}$.
Fact 8: $\tau(\alpha_i) = \alpha_{i-1}, i \ne 1, n-1$.
Fact 9: $\tau(\alpha_0 + \alpha_{n-1}) = \alpha_{n-2}$.

Remark 3.2. Facts $7, 8, 9$ imply that $(\alpha_{n-1} + \alpha_0, \alpha_{n-2}, \alpha_{n-3}, \dots, \alpha_1)$ is a cycle of order $n-1$ for τ. In particular, each of these roots is fixed by κ.

3.3. A reduced expression for κ

Let κ be the element in $\widehat{W}$ defined as above. We may write $\kappa = \tau_1 \cdots \tau_{n-1}$, where τ_i's are equal, and equal to $\tau (= s_{n-1}\cdots s_2 s_1 s_0)$ (we have a specific purpose behind writing κ as above).

Lemma 3.4. *The expression $\tau_1 \cdots \tau_{n-1}$ for κ is reduced.*

Proof.

Claim: $\tau_1 \cdots \tau_i s_{n-1} \cdots s_{j+1}(\alpha_j),\ 1 \le i \le n-2,\ 0 \le j \le n-2,\ \tau_1 \cdots \tau_i(\alpha_{n-1})$, $1 \le i \le n-2$ are positive real roots.

Note that the Claim implies the required result. We divide the proof of the Claim into the following three cases.

Case 1: To show: $\tau_1 \cdots \tau_i(\alpha_{n-1}), 1 \leq i \leq n-2$ is a positive real root.

We have

$$\begin{aligned}
&\tau_1 \cdots \tau_i(\alpha_{n-1})\\
&\quad = \tau_1 \cdots \tau_{i-1}(\delta + \alpha_{n-2} + \alpha_{n-1}) \quad \text{(cf. §3.1, Fact 6)}\\
&\quad = \tau_1 \cdots \tau_{i-2}(2\delta + \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}) \quad \text{(cf. §3.1, Fact 3)}\\
&\quad = \tau_1 \cdots \tau_{i-k}(k\delta + \alpha_{n-k-1} + \cdots + \alpha_{n-1}), 0 \leq k \leq i-1 \quad \text{(cf. §3.1, Fact 3).}
\end{aligned}$$

Note that $k \leq i-1$ implies that $n-k-1 \geq n-i \geq 2$, and hence we can apply §3.1, Fact 3. Corresponding to $k = i-1$, we obtain $\tau_1 \cdots \tau_i(\alpha_{n-1}) = \tau_1((i-1)\delta + \alpha_{n-i} + \cdots + \alpha_{n-1})$. Hence once again using §3.1, Fact 3, we obtain

$$\tau_1 \cdots \tau_i(\alpha_{n-1}) = i\delta + \alpha_{n-i-1} + \cdots + \alpha_{n-1},\ 1 \leq i \leq n-2$$

(note that for $1 \leq i \leq n-2$, $n-i-1 \geq 1$).

Case 2: To show: $\tau_1 \cdots \tau_i s_{n-1} \cdots s_1(\alpha_0), 1 \leq i \leq n-2$ is a positive real root.

We have

$$\begin{aligned}
&\tau_1 \cdots \tau_i s_{n-1} \cdots s_1(\alpha_0)\\
&\quad = \tau_1 \cdots \tau_i(\delta + \alpha_{n-1}) \quad \text{(cf. §3.1, Fact 5)}\\
&\quad = \tau_1 \cdots \tau_{i-1}(2\delta + \alpha_{n-2} + \alpha_{n-1}) \quad \text{(cf. §3.1, Fact 6)}\\
&\quad = \tau_1 \cdots \tau_{i-k}((k+1)\delta + \alpha_{n-k-1} + \cdots + \alpha_{n-1}),\ 0 \leq k \leq i-1 \quad \text{(cf. §3.1, Fact 3).}
\end{aligned}$$

Note that as in Case 1, for $k \leq i-1$, we have $n-k-1 \geq 2$, and therefore §3.1, Fact 3 holds. Corresponding to $k = i-1$, we have $\tau_1 \cdots \tau_i s_{n-1} \cdots s_1(\alpha_0) = \tau_1(i\delta + \alpha_{n-i} + \cdots + \alpha_{n-1})$. Hence once again using §3.1 Fact 3, we obtain

$$\tau_1 \cdots \tau_i s_{n-1} \cdots s_1(\alpha_0) = (i+1)\delta + \alpha_{n-i-1} + \cdots + \alpha_{n-1}, 1 \leq i \leq n-2$$

(note that for $1 \leq i \leq n-2$, $n-i-1 \geq 1$).

Case 3: To show: $\tau_1 \cdots \tau_i s_{n-1} \cdots s_{j+1}(\alpha_j), 1 \leq i \leq n-2, j \neq 0, n-1$ is a positive real root.

We have $\tau_1 \cdots \tau_i s_{n-1} \cdots s_{j+1}(\alpha_j) = \tau^i(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1})$ (cf. §3.1, Fact 4) $= \tau^i(\alpha_j) + \ldots + \tau^i(\alpha_{n-2}) + \tau^i(\alpha_{n-1})$ which is positive because each term is positive (cf. Case 1 and Remark 3.2). $\square$

Corollary 3.5. $\ell(\kappa) = n(n-1)$.

3.6. Minimal representative-property for κ

Lemma 3.7. *$\kappa(\alpha_i)$ is a real positive root for all $i \neq 0$.*

Proof. For $1 \leq i \leq n-2$, $\kappa(\alpha_i) = \alpha_i$ is positive from Remark 3.2. Further,

$$\begin{aligned}
&\tau_1 \cdots \tau_{n-1}(\alpha_{n-1})\\
&\quad = \tau_1 \cdots \tau_{n-2}(\delta + \alpha_{n-2} + \alpha_{n-1}) \quad \text{(cf. §3.1, Fact 6))}\\
&\quad = \tau_1 \cdots \tau_{n-k}((k-1)\delta + \alpha_{n-k} + \cdots + \alpha_{n-1}), 1 \leq k \leq n-1 \quad \text{(cf. §3.1, Fact 3)).}
\end{aligned}$$

Note that for $1 \leq k \leq n-2, n-k \geq 2$ and hence §3.1, Fact 3 holds. Corresponding to $k = n-1$, we get

$$\begin{aligned}
\tau_1 \cdots \tau_{n-1}(\alpha_{n-1}) &= \tau_1((n-2)\delta + \alpha_1 + \cdots + \alpha_{n-1})\\
&= n\delta + \alpha_{n-1} \quad \text{(cf. §3.1, Facts 1,2).} \qquad \square
\end{aligned}$$

Corollary 3.8. *κ is a minimal representative in $\widehat{W}/\widehat{W}_{G_0}$.*

For $w \in \widehat{W}$, we shall denote the Schubert variety in $\mathcal{G}/G_0$ by $X_{G_0}(w)$.

Lemma 3.9. *$X_{G_0}(\kappa)$ is stable for multiplication on the left by G_0.*

Proof. It suffices to show that

$$s_i\kappa \leq \kappa(\operatorname{mod} \widehat{W}_{G_0}), \quad 1 \leq i \leq n-1. \tag{$*$}$$

The assertion $(*)$ is clear if $i = n-1$. Observe that $ws_\alpha = s_{w(\alpha)}w$. In particular, since κ fixes α_i, $1 \leq i \leq n-2$, it follows $s_i\kappa = \kappa s_i = \kappa(\operatorname{mod} \widehat{W}_{G_0})$, for $1 \leq i \leq n-2$. □

Lemma 3.10. *Let $\mathcal{P}$ be the parabolic subgroup of $\mathcal{G}$ corresponding to the choice of simple roots $\{\alpha_1, \ldots \alpha_{n-2}\}$. The element κ is a minimal representative in $\widehat{W}_{\mathcal{P}}\backslash\widehat{W}$.*

Proof. It is enough to show that $s_i\kappa > \kappa$, or equivalently, $\kappa^{-1}(\alpha_i) > 0$ for $1 \leq i \leq n-2$. This follows from Remark 3.2. □

Remark 3.11. For the discussion in §3.3, §3.6, concerning reduced expressions, minimal-representative property and G_0-stability, we have used the expression for elements of $\widehat{W}$, $\widehat{W}$ being considered as a Coxeter group. One may as well carry out the discussion using the permutation presentations for elements of $\widehat{W}$.

Theorem 3.12 (A reduced expression for κ_0)**.** *The element $\kappa_0(= w'\tau^{n-1})$ is the maximal representative of κ in $\widehat{W}_{G_0}\backslash\widehat{W}$, i.e., the unique element in $\widehat{W}$ such that*

$$X(\kappa_0) = \overline{G_0\kappa\mathcal{B}}(\operatorname{mod} \mathcal{B}).$$

In particular, $X(\kappa_0)$ is (left) G_0-stable. Let $\underline{w}'$ be a reduced expression for the longest element w' in $\widehat{W}_{\mathcal{P}}$ and $\underline{\tau}$ the reduced expression $s_{n-1}\cdots s_1s_0$. Then $\underline{w}'\underline{\tau}^{n-1}$ is a reduced expression for κ_0.

Proof. Observe that $\underline{w} = \underline{w}'s_{n-1}\cdots s_1$ is a reduced expression for the longest element w in $\widehat{W}_{G_0}$, and so $w'\kappa = ws_0\tau^{n-2}$. Lemma 3.10 implies that $\underline{w}'\underline{\tau}^{n-1}$ is a reduced expression. In particular,

$$l(\kappa_0) = l(w'\kappa) = l(w's_{n-1}\cdots s_1) + l(s_0\tau^{n-2}) = l(w) + l(s_0\tau^{n-2}).$$

It remains to show that $w'\kappa$ is a maximal representative in $\widehat{W}_{G_0}\backslash\widehat{W}$, i.e., $s_iw'\kappa < w'\kappa$, or equivalently $l(s_iw'\kappa) < l(w'\kappa)$ for $1 \leq i \leq n-1$. First note that

$$l(s_iw'\kappa) = l(s_iws_0\tau^{n-2}) \leq l(s_iw) + l(s_0\tau^{n-2}).$$

Now, since w is the longest element in $\widehat{W}_{G_0}$, it follows $l(s_iw) < l(w)$ and further

$$l(s_iw'\kappa) < l(w) + l(s_0\tau^{n-2}) = l(w'\kappa). \qquad \square$$

4. The main lemma

In this section, we prove one crucial result involving κ, which we then use to prove the main result.

Lemma 4.1. *Let* $Y = \sum_{1\le i<j\le n} a_{ij}E_{ij}$*, where* E_{ij} *is the elementary* $n\times n$ *matrix with* 1 *at the* (i,j)*-th place and 0's elsewhere. Let* $\underline{Y} = \mathrm{Id}_{n\times n} + \sum_{1\le i\le n-1} t^{-i}Y^i$ *(note that* $Y^n = 0$*). Assume that* $a_{ii+1} \neq 0, 1 \le i \le n-1$*. There exist* $g \in G_0, h \in \mathcal{B}$ *such that* $g\kappa = \underline{Y}h$*.*

Proof. Choose g to be the matrix

$$g = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & g_{2n} \\ 0 & -1 & 0 & \cdots & g_{3n} \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & g_{nn} \end{pmatrix}.$$

Note that the lower left corner submatrix (i.e.,the $n-1\times n-1$ submatrix with rows 2-nd through the $(n-1)$-th of g, and the first $n-1$ columns of g) is $-\mathrm{Id}_{n-1\times n-1}$, and that the determinant of g equals 1. Hence, we may take $g_{in}, 2 \le i \le n$ as elements in $K[[t]]$ so that $g \in G_0$. We shall now show that there exist $g_{in}, 2 \le i \le n$, and $h_{ij}, 1 \le i,j \le n$ such that $h(\in \mathcal{B})$, and $g\kappa = \underline{Y}h$. We have $\underline{Y}^{-1} = \mathrm{Id}_{n\times n} - t^{-1}Y$.

Set

$$h = (\mathrm{Id}_{n\times n} - t^{-1}\underline{Y})g\kappa.$$

We have (by definition of κ (§3), and the choice of lifts for s_i (cf. §2.3))

$$\kappa = \mathrm{diag}(t, \ldots, t, t^{-(n-1)}).$$

Note that since we want h to belong to $\mathcal{B}$, each diagonal entry in h (as an element of $K[[t]]$) should have order 0, $h_{ij}, i > j$ should have order > 0, and $h_{ij}, i < j$ should have order ≥ 0 (since $h(0)$ should belong to B). Now the diagonal entries in h are given by

$$h_{ii} = a_{ii+1},\ \ 1 \le i \le n-1, h_{nn} = t^{-(n-1)}g_{nn}.$$

Hence choosing g_{nn} such that order $g_{nn} = n-1$ (note that since $g \in G_0$, order $g_{ij} \ge 0, 1 \le i,j \le n$, so this choice for g is allowed), we obtain that each diagonal entry in h is in $K[[t]]$, with order equal to 0. Also, we have

$$\begin{aligned} h_{i+1i} &= -t,\ \ 1 \le i \le n-1, \\ h_{ik} &= 0,\ \ k \le i-2, 3 \le i \le n-1, \\ h_{ik} &= a_{ik+1}, 1 \le i < k \le n-1. \end{aligned}$$

Thus the entries $h_{ik}, k \le n-1$ satisfy the order conditions mentioned above. Let us then consider $h_{jn}, 1 \le j \le n$. We have

$$h_{jn} = t^{-(n-1)}g_{jn} - \sum_{j+1\le k\le n} t^{-n}a_{jk}g_{kn},\ \ 1 \le j \le n. \qquad (*)$$

We shall choose g_{in} (in $K[[t]]$) so that the order of g_{in} equals $i-1$ (note that this agrees with the above choice of g_{nn} in the discussion of the diagonal entries in h). Let us write

$$g_{in} = \sum g_{in}^{(k)} t^k.$$

We shall show that with the above choice of g_{in}, the integrality condition on the h_{in}'s imposes conditions on $g_{in}^{(k)}, i-1 \leq k \leq n, 1 \leq i \leq n$, leading to a linear system in these $g_{in}^{(k)}$'s (note that, the integrality condition on the h_{in}'s, $1 \leq i \leq n$, implies that h_{in}'s should belong to $K[[t]]$, with the additional condition that h_{nn} should have order 0, the latter condition having already been accommodated, since g_{nn} has been chosen to have order $n-1$). Treating $g_{in}^{(k)}$'s as the unknowns, we show that the resulting linear system has a unique solution, thus proving the choice of g, h with the said properties. We shall now describe this linear system. The linear system will involve $\binom{n}{2}$ equations in $\binom{n}{2}$ unknowns, namely, $g_{in}^{(k)}, i-1 \leq k \leq n, 2 \leq i \leq n$. The linear system is obtained as follows. The lowest power of t appearing on the right-hand side of $(*)$ above is $-(n-j)$ (note that order of g_{kn} equals $k-1$). Hence equating the coefficients of $t^{-(n-i)}, j \leq i \leq n-1$ on the right-hand side of $(*)$ to 0, we obtain

$$g_{jn}^{(i-1)} - \sum_{j+1 \leq k \leq n} a_{jk} g_{kn}^{(i)} = 0, \quad j \leq i \leq n-1, 1 \leq j \leq n-1. \tag{$**$}$$

Note that, corresponding to h_{nn}, we do not have any conditions, since by our choice of g_{nn} (the order of g_{nn} is $n-1$), we have that h_{nn} $(= t^{-(n-1)} g_{nn})$ is integral. Also, corresponding to g_{1n} (which is equal to 1, by our choice of g), we have $g_{1n}^{(i)} = 0, i \geq 1$, and this occurs just in one equation, namely, the equation corresponding to the coefficient of $t^{-(n-1)}$ in h_{1n}:

$$g_{1n} - a_{12} g_{2n}^{(1)} = 0.$$

Rewriting this equation as

$$-a_{12} g_{2n}^{(1)} = -1$$

(there is a purpose behind retaining the negative sign in $-a_{12} g_{2n}^{(1)}$), we arrive at the linear system

$$A_n X = B$$

where A_n is a square matrix of size $\binom{n}{2}$, X is the $\binom{n}{2}$ column matrix $(g_{jn}^{k}, j-1 \leq k \leq n, 2 \leq j \leq n)$, and B is the $\binom{n}{2}$ column matrix with the first entry equal to -1, and all other entries equal to 0.

Claim: A_n is invertible, and $|A_n| = (-1)^{\binom{n}{2}} \prod_{1 \leq i \leq n-1} a_{i\,i+1}^{n-i}$.

Note that the Claim implies that ($g_{jn}^{(k)}$'s, $j-1 \leq k \leq n, 2 \leq j \leq n$) are uniquely determined, and therefore we may choose g_{jn} as elements in $K[[t]]$ with $g_{jn}^{(k)}$'s, $j-1 \leq k \leq n$, as the solutions of the above linear system, with $g_{jn}^{(k)}, k > n$, being arbitrary.

We prove the Claim by induction on n. We shall first show that A_{n-1} can be identified in a natural way as a submatrix of A_n. We want to think of the rows of A_n forming $(n-1)$ blocks (referred to as *row-blocks* in the sequel) of size $n-1, n-2, \ldots, n-j, \ldots, 1$, namely, the j-th block consists of $n-j$ rows given by the coefficients occurring on the left-hand side of $(**)$ for $j \geq 2$, and for $j = 1$, the first block consists of $n-1$ rows given by the coefficients occurring on the left-hand side of the following $n-1$ equations:

$$-a_{12}g_{2n}^{(1)} = -1, \ -g_{2n}^{(i)} - \sum_{3 \leq k \leq n} a_{2k}g_{kn}^{(i)} = 0, \quad 2 \leq i \leq n-1.$$

Similarly, we want to think of the columns of A_n forming $(n-1)$ blocks (referred to as *column-blocks* in the sequel) of size $n-1, n-2, \ldots, n-j, \ldots, 1$, namely, the j-th block consisting of $n-j$ columns indexed by $g_{jn}^{(i)}, j-1 \leq i \leq n$. Then indexing the $n-j$ rows in the j-th row-block as $j, j+1, \ldots, n-1$, the entries in the rows of the j-th row-block have the following description:

The non-zero entries in the i-th row in the jth row-block $(j \geq 2)$ are $1, -a_{23}, -a_{24}, \ldots, -a_{2\,i+1}$ respectively, occurring at the columns indexed by $g_{2n}^{(i-1)}, g_{3n}^{(i)}, \ldots, g_{i+1\,n}^{(i)}$.

The non-zero entries in the i-th row in the first row-block $(j \geq 2)$ are $-a_{12}, -a_{13}, \ldots, -a_{2\,i+1}$ respectively, occurring at the columns indexed by $g_{2n}^{(i)}, g_{3n}^{(i)}, \ldots, g_{i+1\,n}^{(i)}$.

From this it follows that A_{n-1} is obtained from A_n by deleting the first row in each row-block and the first column in each column-block. For instance, we describe below A_5 and A_4; for convenience of notation, we denote $b_{ij} = -a_{ij}$. We have

$$A_5 = \begin{pmatrix} b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & 0 & 0 & b_{13} & 0 & b_{14} & 0 & 0 \\ 0 & 0 & 0 & b_{12} & 0 & 0 & b_{13} & 0 & b_{14} & b_{15} \\ 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & b_{24} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & b_{24} & b_{25} \\ 0 & 0 & 0 & & 1 & 0 & 0 & b_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_{34} & b_{35} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & b_{45} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & b_{13} & 0 & 0 \\ 0 & 0 & b_{12} & 0 & b_{13} & b_{14} \\ 1 & 0 & 0 & b_{23} & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 1 & 0 & b_{34} \end{pmatrix}.$$

As rows (respectively columns) of A_5, the positions of the first row (respectively, the first column) in each of the four row-blocks (respectively columns-blocks) in A_5 are given by $1, 5, 8, 10$; deleting these rows and columns in A_5, we get A_4. These rows and columns are highlighted in A_5.

As above, let $b_{ij} = -a_{ij}$. Now expanding A_n along the first row, we have that $|A_n|$ equals $b_{12}|M_1|, M_1$ being the submatrix of A_n obtained by deleting the first row and first column in A_n (i.e., deleting the first row (respectively, the first column) in the first row-block (respectively, the first column-block)). Now in M_1, in the first row in the second row-block the only non-zero entry is b_{23}, and it is a diagonal entry in M_1. Hence expanding M_1 through this row, we get that $|A_n|$ equals $b_{12}b_{23}|M_2|, M_2$ being the submatrix of A_n obtained by deleting the first rows (respectively, the first columns) in the first two row-blocks (respectively, the first two column-blocks) in A_n. Now in M_2, in the first row in the third row-block, the only non-zero entry is b_{34}, and it is a diagonal entry in M_2. Hence expanding M_2 along this row, we get that $|A_n|$ equals $b_{12}b_{23}b_{34}|M_3|, M_3$ being the submatrix of A_n obtained by deleting the first rows (respectively, the first columns) in the first three row-blocks (respectively, the first three column-blocks) in A_n. Thus proceeding, at the $(n-1)$-th step, we get that $|A_n|$ equals $b_{12}b_{23}\cdots b_{n-1\,n}|A_{n-1}|$. By induction, we have $|A_{n-1}| = (-1)^{\binom{n-1}{2}} \prod_{1\leq i\leq n-2} a_{i\,i+1}^{n-1-i}$. Substituting back for b_{ij}'s, we obtain $|A_n| = (-1)^{\binom{n}{2}} \prod_{1\leq i\leq n-1} a_{i\,i+1}^{n-i}$. It remains to verify the statement of the claim when $n = 2$ (starting point of induction). In this case, we have

$$g = \begin{pmatrix} 0 & 1 \\ -1 & g_{22} \end{pmatrix}, \qquad \kappa = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$
$$\underline{Y}^{-1} = \begin{pmatrix} 1 & -t^{-1}a_{12} \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} a_{12} & t^{-1} - t^{-2}a_{12}g_{22} \\ -t & -t^{-1}g_{22} \end{pmatrix}.$$

Hence the linear system consists of the single equation

$$-a_{12}g_{22}^{(1)} = -1.$$

Hence A_2 is the 1×1 matrix $(-a_{12})$, and $|A_2| = -a_{12}$, as required. $\square$

5. Lusztig's map

Consider $\mathcal{N}$, the variety of nilpotent elements in $\mathfrak{g}$ (the Lie algebra of G). In this section, we spell out (Lusztig's) isomorphism which identifies $X_{G_0}(\kappa)$ as a compactification of $\mathcal{N}$.

5.1. The map ψ

Consider the map

$$\psi : \mathcal{N} \to \mathcal{G}/G_0, \quad \psi(N) = (\mathrm{Id} + t^{-1}N + t^{-2}N^2 + \cdots)(\mathrm{mod}\, G_0), N \in \mathcal{N}.$$

Note that the sum on the right-hand side is finite, since N is nilpotent. We now list some properties of ψ.

(i) *ψ is injective:*

Let $\psi(N_1) = \psi(N_2)$. Denoting $\lambda_i := \psi(N_i), i = 1, 2$, we get that $\lambda_2^{-1}\lambda_1$ belongs to G_0. On the other hand,

$$\lambda_2^{-1}\lambda_1 = (\mathrm{Id} - t^{-1}N_2)(Id + t^{-1}N + t^{-2}N^2 + \cdots).$$

Now $\lambda_2^{-1}\lambda_1$ is integral. It follows that both sides of the above equation equal Id. This implies $\lambda_1 = \lambda_2$, which in turn implies that $N_1 = N_2$. Hence we obtain the injectivity of ψ.

(ii) *ψ is G-equivariant:*

We have

$$\begin{aligned}\psi(g \cdot N) &= \psi(gNg^{-1}) \\ &= (\mathrm{Id} + t^{-1}gNg^{-1} + t^{-2}gN^2g^{-1} + \cdots)(\mathrm{mod}\, G_0) \\ &= g(\mathrm{Id} + t^{-1}N + t^{-2}N^2 + \cdots)g^{-1}(\mathrm{mod}\, G_0) \\ &= g(\mathrm{Id} + t^{-1}N + t^{-2}N^2 + \cdots)(\mathrm{mod}\, G_0) \quad (\text{since } g^{-1} \in G_0) \\ &= g\psi(N).\end{aligned}$$

Proposition 5.2. *For $N \in \mathcal{N}, \psi(N)$ belongs to $X_{G_0}(\kappa)$.*

Proof. We divide the proof into two cases.

Case 1: Let N be upper triangular, say,

$$N = (n_{ij})_{1 \le i,j \le n}$$

where $n_{ij} = 0$, for $i \ge j$; note that $N \in \underline{b}_u, \underline{b}_u$ being the Lie algebra of B_u, the unipotent radical of B. We may work in the open subset $x_{ii+1} \neq 0, 1 \le i \le n-1$ in $\underline{b}_u$, $\sum_{1 \le i < j \le n} x_{ij}E_{ij}$ being a generic element in $\underline{b}_u$. Hence we may suppose that $n_{i\,i+1} \neq 0, 1 \le i \le n-1$. In this case, in view of Lemma 4.1, we have that there exist $g \in G_0, h \in \mathcal{B}$ such that $g\kappa = \psi(N)h$. This implies, in view of the G_0-stability for $X_{G_0}(\kappa)$ (cf. Lemma 3.9), $\psi(N)$ belongs to $X_{G_0}(\kappa)$.

Case 2: Let M be an arbitrary nilpotent matrix. Then there exists an upper triangular matrix N in the G-orbit through N. Hence there exists a $g \in G$ such that $M = gNg^{-1} (= g \cdot N)$ with N upper triangular. Now by G-equivariance of ψ (cf. (ii) above), we have $\psi(M) = g \cdot \psi(N)$. By case 1, $\psi(N) \in X_{G_0}(\kappa)$; this together with the G_0-stability for $X_{G_0}(\kappa)$ implies that $\psi(M)$ belongs to $X_{G_0}(\kappa)$. □

Theorem 5.3. *$X_{G_0}(\kappa)$ is a compactification of $\mathcal{N}$.*

Proof. Let $\overline{\mathcal{N}}$ be the closure of $\mathcal{N}$ in $\mathcal{G}/G_0$. Combining the above Proposition with §5.1, (i) and the facts that $\dim \mathcal{N} = n(n-1) = \dim X_{G_0}(\kappa)$ (cf. Corollaries 3.5,3.8), we obtain $\overline{\mathcal{N}} = X_{G_0}(\kappa)$. □

6. Cotangent bundle

In this section, we first recall the Springer resolution. We then construct a family ψ_p, parametrized by polynomials p in one variable with coefficients in $\mathbb{C}((t))$ and constant term 1, of maps $\psi_p : T^*G/B \to \mathcal{G}/\mathcal{B}$. We show that for a particular choice ϕ in the family, we get an embedding of T^*G/B inside $\mathcal{G}/\mathcal{B}$. Using the natural projection $\mathcal{G}/\mathcal{B} \to \mathcal{G}/G_0$ and the results of §5, we recover the Springer resolution. We then show that ϕ identifies an SL_n-stable closed subvariety of $X(\kappa_0)$ as a compactification of T^*G/B.

The cotangent bundle T^*G/B is a vector bundle over G/B, with the fiber at any point $x \in G/B$ being the cotangent space to G/B at x; the dimension of T^*G/B equals $2\dim G/B$. Also, T^*G/B is the fiber bundle over G/B associated to the principal B-bundle $G \to G/B$, for the adjoint action of B on $\underline{b}_u$ (the Lie algebra of the unipotent radical B_u of B). Thus

$$T^*G/B = G \times^B \underline{b}_u = G \times \underline{b}_u/\sim$$

where the equivalence relation $\sim$ is given by

$$(g, Y) \sim (gb, b^{-1}Yb), \quad g \in G, Y \in \underline{b}_u, b \in B.$$

6.1. Springer resolution

Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}$, the Lie algebra of G. Consider the map

$$\theta : G \times^B \underline{b}_u \to G/B \times \mathcal{N}, \quad \theta((g, Y)) = (gB, gYg^{-1}), g \in G, Y \in \underline{b}_u$$

We observe the following on the map θ:

(i) *θ is well defined:*
Let $b \in B$. Consider $(gb, b^{-1}Yb)(\sim (g, b))$. We have,

$$\theta((gb, b^{-1}Yb)) = (gB, gb(b^{-1}Yb)b^{-1}g^{-1}) = (gB, gYg^{-1}) = \theta((g, Y)).$$

(ii) *θ is injective:*
Suppose $\theta((g_1, Y_1)) = \theta((g_2, Y_2))$. Then $(g_1B, g_1Y_1g_1^{-1}) = (g_2B, g_2Y_2g_2^{-1})$. This implies

$$g_1B = g_2B, g_1Y_1g_1^{-1} = g_2Y_2g_2^{-1}.$$

Hence we obtain

$$\begin{aligned} g_1^{-1}g_2 =: b \in B, Y_2 = g_2^{-1}g_1Y_1g_1^{-1}g_2, \\ \therefore g_2 = g_1b, Y_2 = b^{-1}Y_1b, \\ \therefore (g_1, Y_1) = (g_1b, b^{-1}Y_1b) = (g_2, Y_2). \end{aligned}$$

Thus we get an embedding

$$\theta : T^*G/B \hookrightarrow G/B \times \mathcal{N}.$$

The second projection

$$T^*G/B \to \mathcal{N}, (g, Y) \mapsto gYg^{-1}$$

is proper and birational and is the celebrated *Springer resolution.*

6.2. The maps ψ_p

Let $p(Y)$ be a polynomial in Y with coefficients in F, and constant term 1. We write

$$p(Y) = 1 + \sum_{i \geq 1} p_i(t) Y^i.$$

It is clear that $p(Y) \in \mathcal{G}$. Define the map $\psi_p : G \times^B \underline{b}_u \to \mathcal{G}/\mathcal{B}$ by

$$\psi_p(g, Y) = gp(Y)(\operatorname{mod} \mathcal{B}), g \in G, Y \in \underline{b}_u.$$

The following calculation shows that ψ_p is well defined: Let $g \in G,\ b \in B,\ Y \in \underline{b}_u$. Then

$$\begin{aligned}
\psi_p((gb, b^{-1}Yb)) &= gb\left(\mathrm{Id} + p_1(t)b^{-1}Yb + p_2(t)b^{-1}Y^2b + \cdots\right)(\operatorname{mod} \mathcal{B}) \\
&= g\left(\mathrm{Id} + p_1(t)Yb + p_2(t)Y^2b + \cdots\right)(\operatorname{mod} \mathcal{B}) \\
&= g\left(\mathrm{Id} + p_1(t)Y + p_2(t)Y^2 + \cdots\right)(\operatorname{mod} \mathcal{B}) \\
&= \psi_p(g, Y).
\end{aligned}$$

Also, it is clear that ψ_p is G-equivariant.

6.3. Embedding of T^*G/B into $\mathcal{G}/\mathcal{B}$

We consider one particular member ϕ of the family ψ_p: namely $\phi = \psi_p$ where $p(Y)$ is the polynomial $(1 - t^{-1}Y)^{-1}$; observe that for nilpotent Y, the function

$$\begin{aligned}
p(Y) &= (1 - t^{-1}Y)^{-1} \\
&= 1 + t^{-1}Y + t^{-2}Y^2 + \cdots
\end{aligned}$$

is a polynomial, since the sum on the right-hand side is finite. In particular, $\phi : G \times^B \underline{b}_u \to \mathcal{G}/\mathcal{B}$ is given by

$$\phi(g, Y) = g(\mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots)(\operatorname{mod} \mathcal{B}).$$

In the sequel, we shall denote

$$\underline{Y} := \mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots$$

We now list some facts on the map ϕ:

(i) ϕ *is well-defined.*

(ii) ϕ *is injective:*

Let $\phi((g_1, Y_1)) = \phi((g_2, Y_2))$. This implies that $g_1\underline{Y_1} \equiv g_2\underline{Y_2}(\operatorname{mod} \mathcal{B})$, where recall that for $Y \in \underline{b}_u, \underline{Y} = \mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots$. Hence, $g_1\underline{Y_1} = g_2\underline{Y_2}x$, for some $x \in \mathcal{B}$. Denoting $h =: g_2^{-1}g_1$, we have $h\underline{Y_1} = \underline{Y_2}x$, and therefore,

$$x = \underline{Y_2}^{-1}h\underline{Y_1} = \underline{Y_2}^{-1}(h\underline{Y_1}h^{-1})h = \underline{Y_2}^{-1}\underline{Y_1'}h$$

where $\underline{Y_1'} = h\underline{Y_1}h^{-1}$. Hence

$$xh^{-1} = \underline{Y_2}^{-1}\underline{Y_1'} = (\mathrm{Id} - t^{-1}Y_2)(\mathrm{Id} + t^{-1}hY_1h^{-1} + t^{-2}hY_1^2h^{-1} + \cdots).$$

Now, since $x \in \mathcal{B}, h(= g_2^{-1}g_1) \in G$, the left-hand side is integral, i.e., it does not involve negative powers of t. Hence both sides equal Id. This implies

$$\underline{Y_2} = \underline{Y_1'}, \quad x = h.$$

The fact that $x = h$ together with the facts that $x \in \mathcal{B}, h \in G$ implies that

$$h \in \mathcal{B} \cap G(= B). \tag{$*$}$$

Further, the fact that $\underline{Y_2} = \underline{Y_1'}$ implies that $\underline{Y_1} = h^{-1}\underline{Y_2}h$. Hence

$$\mathrm{Id} + t^{-1}Y_1 + t^{-2}Y_1^2 + \cdots = \mathrm{Id} + t^{-1}h^{-1}Y_2h + t^{-2}h^{-1}Y_2^2h + \cdots$$

From this it follows that

$$Y_1 = h^{-1}Y_2h. \tag{$**$}$$

Now $(*)$, $(**)$ together with the fact that $h = g_2^{-1}g_1$ imply that

$$(g_1, Y_1) = (g_2h, h^{-1}Y_2h) \sim (g_2, Y_2).$$

From this, injectivity of ϕ follows.

(iii) *G-equivariance:* It is clear that ϕ is G-equivariant.

(iv) *Springer resolution:* Consider the projection $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/G_0$. Let $x \in T^*G/B$, say, $x = (g, Y), g \in G, Y \in \underline{b}_u$. We have

$$\begin{aligned}\pi \circ \phi((g, Y)) &= \phi((g, Y))(\mathrm{mod}\, G_0)\\ &= g(\mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots)(\mathrm{mod}\, G_0)\\ &= g(\mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots)g^{-1}(\mathrm{mod}\, G_0)\\ &= (\mathrm{Id} + t^{-1}N + t^{-2}N^2 + \cdots)(\mathrm{mod}\, G_0)\end{aligned}$$

where $N = gYg^{-1}$ is nilpotent. Hence, in view of Lusztig's isomorphism (cf. Proposition 5.3), we recover the Springer resolution as

$$\pi\,|_{T^*G/B} : T^*G/B \to \mathcal{N} \hookrightarrow \mathcal{G}/G_0.$$

Theorem 6.4 (Compactification of T^*G/B**).** *Let* $G = \mathrm{SL}_n(\mathbb{C})$ *and* $\phi : T^*G/B \to \mathcal{G}/\mathcal{B}$ *be as in Section* 6.3. *Then* ϕ *identifies* $\overline{T^*G/B}$ (*the closure being in* $\mathcal{G}/\mathcal{B}$) *with a G-stable closed subvariety of the affine Schubert variety* $X(\kappa_0)$.

Proof. Let $(g_0, Y), g_0 \in G, Y \in \underline{b}_u$. Then $\phi(g_0, Y) = g_0(\mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots)(\mathrm{mod}\, \mathcal{B}) = g_0\underline{Y}(\mathrm{mod}\, \mathcal{B})$, where $\underline{Y} = \mathrm{Id} + t^{-1}Y + t^{-2}Y^2 + \cdots$. Writing $Y = \sum_{1\le i<j\le n} a_{ij}E_{ij}$ with E_{ij} as in Lemma 4.1, we may work in the open subset $x_{ii+1} \neq 0, 1 \le i \le n-1$ in $\underline{b}_u$, $\sum_{1\le i<j\le n} x_{ij}E_{ij}$ being a generic element in $\underline{b}_u$. Then Lemma 4.1 implies that there exist $g \in G_0, h \in \mathcal{B}$ such that $g\kappa = \underline{Y}h$. Hence $\underline{Y}$ belongs to $X(\kappa_0)(= \overline{G_0\kappa\mathcal{B}}(\mathrm{mod}\, \mathcal{B}))$; hence $g_0\underline{Y}$ is also in $X(\kappa_0)$ (since g_0 is clearly in G_0). $\square$

7. Consequences of ψ_p for T^*G/B

In this section, we show that for any polynomial p, the map ψ_p as defined in §6.2 cannot realize an affine Schubert variety (in $\widehat{\mathrm{SL}_3}/\mathcal{B}$) as a compactification of the cotangent bundle $T^*\mathrm{SL}_3(K)/B$.

Proposition 7.1. *Let G be the group $\mathrm{SL}_3(K)$ and the B the Borel subgroup of upper triangular matrices in G. Let p be a polynomial as in §6.2. Suppose that the associated map $\psi_p : T^*G/B \to \mathcal{G}/\mathcal{B}$ is injective. Then there exist $g \in G$, $w \in \widehat{W}$ and $Y \in \underline{b}_u$ such that $\psi_p(g, Y) \in \mathcal{B}w\mathcal{B}$ and $l(w) > 6$.*

Proof. From §6.2, we may assume $p(Y) = 1 + \sum_{i \geq 1} p_i(t)Y^i$. We first claim that $p_1(t) \notin A$. Assume the contrary. For

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we see that $Z^2 = 0$, and so $p(Z) = 1 + p_1(t)Z \in \mathcal{B}$. In particular, $\psi_p(Z) = \psi_p(0)$, contradicting the injectivity of ψ_p.

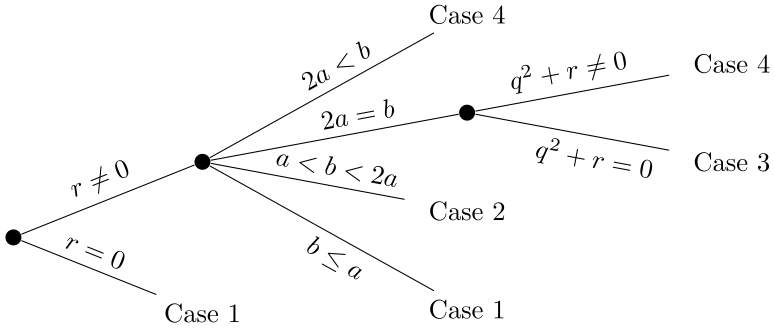
We now write $p(Y) = 1 - t^{-a}qY - t^{-b}rY^2$ where

- $q, r \in A$.
- $q(0) \neq 0$.
- $a \geq 1$.
- Either $r = 0$ or $r(0) \neq 0$.

We now fix $Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, so that

$$gp(Y) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & t^{-a}q \\ -1 & t^{-a}q & t^{-b}r \end{pmatrix}.$$

Our strategy is to find elements $C, D \in \mathcal{B}$ such that $Cgp(Y)D \in N(K[t, t^{-1}])$. We can then identify the Bruhat cell containing $gp(Y)$, and so identify the minimal Schubert variety containing $\psi_p(g, Y)$. The choice of C, D depends on the values of certain inequalities, which we divide into 4 cases. We draw here a decision tree showing the relationship between the inequalities and the choice C, D:



A rational function in t is implicitly equated with its Laurent power series at 0. In particular, a rational function f belongs to A if and only if f has no poles at 0, i.e. its denominator is not divisible by t.

(1) If $r = 0$ or $b \leq a$, let

$$C = \begin{pmatrix} rt^{2a-b} + q^2 & qt^a & t^{2a} \\ 0 & -\dfrac{q}{rt^{2a-b}+q^2} & -\dfrac{t^a}{rt^{2a-b}+q^2} \\ 0 & 0 & -\dfrac{1}{q} \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{qt^a}{rt^{2a-b}+q^2} & 1 & -\dfrac{rt^{a-b}}{q} \\ \dfrac{t^{2a}}{rt^{2a-b}+q^2} & 0 & 1 \end{pmatrix}.$$

We compute

$$Cgp(Y)D = \begin{pmatrix} -t^{2a} & 0 & 0 \\ 0 & 0 & t^{-a} \\ 0 & t^{-a} & 0 \end{pmatrix}.$$

It follows that $gp(Y) \in \mathcal{B}\lambda_q s_2 \mathcal{B}$, where $q = -2a\alpha_1^\vee - a\alpha_2^\vee$. We calculate

$$\begin{aligned} l(\lambda_q) &= |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)| \\ &= 3a + 0 + 3a \\ &= 6a. \end{aligned}$$

It follows from lemma 2.5 that $l(\lambda_q s_2) > l(\lambda_q) = 6a \geq 6$.

(2) Suppose $a < b < 2a$. In particular, $a \geq 2, b \geq 3$. Let

$$C = \begin{pmatrix} -rt^{2a-b} + q^2 & qt^a & t^{2a} \\ 0 & -\dfrac{r}{rt^{2a-b}+q^2} & \dfrac{-qt^{b-a}}{rt^{2a-b}+q^2} \\ 0 & 0 & \dfrac{1}{r} \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{t^a q}{q^2 + t^{2a-b}r} & 1 & 0 \\ \dfrac{t^{2a}}{q^2 + t^{2a-b}r} & -\dfrac{qt^{b-a}}{r} & 1 \end{pmatrix}.$$

We compute

$$Cgp(Y)R = \begin{pmatrix} t^{2a} & 0 & 0 \\ 0 & t^{b-2a} & 0 \\ 0 & 0 & t^{-b} \end{pmatrix}.$$

It follows that $gp(Y) \in \mathcal{B}\lambda_q\mathcal{B}$, where $q = -2a\alpha_1^\vee - b\alpha_2^\vee$. We calculate

$$\begin{aligned} l(\lambda_q) &= |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)| \\ &= (4a - b) + (2b - 2a) + (2a + b) \\ &= 4a + 2b \geq 14. \end{aligned}$$

(3) If $b = 2a$ and $q^2 + r = 0$, let

$$C = \begin{pmatrix} q & t^a & 0 \\ 0 & q & t^a \\ 0 & 0 & \dfrac{1}{q^2} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{t^{2a}}{q^2} & \dfrac{t^a}{q} & 1 \end{pmatrix}.$$

We compute

$$Cgp(Y)D = \begin{pmatrix} 0 & -t^a & 0 \\ -t^a & 0 & 0 \\ 0 & 0 & -t^{-2a} \end{pmatrix}.$$

It follows that $gp(Y) \in \mathcal{B}\lambda_q s_1\mathcal{B}$, where $q = -a\alpha_1^\vee - 2a\alpha_2^\vee$. Similar to the first case, we see that $l(\lambda_q s_1) > 6$.

(4) Suppose either $b > 2a$, or $b = 2a$ and $r + q^2 \neq 0$. In particular, $r + q^2 t^{b-2a} \neq 0$ and $b \geq 2$. Let

$$C = \begin{pmatrix} -r - q^2 t^{b-2a} & -qt^{b-a} & -t^b \\ 0 & -\dfrac{r}{r + q^2 t^{b-2a}} & \dfrac{qt^{b-a}}{r + q^2 t^{b-2a}} \\ 0 & 0 & \dfrac{1}{r} \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{qt^{b-a}}{q^2 t^{b-2a} + r} & 1 & 0 \\ \dfrac{t^b}{q^2 t^{b-2a} + r} & -\dfrac{qt^{b-a}}{r} & 1 \end{pmatrix}.$$

We compute

$$Cgp(Y)D = \begin{pmatrix} t^b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-b} \end{pmatrix}.$$

It follows that $gp(Y) \in \mathcal{B}\lambda_q\mathcal{B}$, where $q = -b\alpha_1^\vee - b\alpha_2^\vee$. We calculate

$$\begin{aligned} l(\lambda_q) &= |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)| \\ &= b + b + 2b \\ &= 4b \geq 8. \qquad \square \end{aligned}$$

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