

A CASSELMAN–OSBORNE THEOREM FOR RATIONAL CHEREDNIK ALGEBRAS

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Abstract. We define Lie algebra cohomology associated with the half-Dirac operators for representations of rational Cherednik algebras and show that it has property described in the Casselman–Osborne Theorem by establishing a version of the Vogan’s conjecture for the half-Dirac operators. Moreover, we study the relationship between Lie algebra cohomology and Dirac cohomology in analogy of the representations for semisimple Lie algebras.

1. Introduction

The Dirac operator plays a pivotal role in mathematics and theoretical physics. In representation theory, the Dirac operator was used for geometric construction of discrete series representations by Parthasarathy [P], Atiyah and Schmid [AS]. In the later 1990s Vogan [V] formulated a conjecture on the Dirac operator in a Lie algebra setting that reveals an interesting algebraic nature of the Dirac operator. This conjecture was verified by Pandžić and the first named author [HP1]. Vogan’s conjecture has been generalized to Kostant’s cubic Dirac operator [K3], as well as various other settings in affine Lie algebras [KMP], Lie superalgebras [HP2], and in particular to graded affine Hecke algebras [BCT]. More recently, Ciubotaru has extended the definition of the Dirac operator and Vogan’s conjecture further to Drinfeld’s graded Hecke algebras including symplectic reflection algebras [C]. The Dirac cohomology $H_D(M)$ of an irreducible Harish-Chandra module M determines the infinitesimal character of M . Determination of Dirac cohomology $H_D(M)$ lends insight to many classical subjects such as Lie algebra cohomology [HP3], and it has applications in branching rules [HPZ] and harmonic analysis [H].

The work of Etingof and Ginzburg on symplectic reflection algebras [EG] has inspired interaction of representation theory with algebraic combinatorics, integr-

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able systems and algebraic geometry. Apart from a finite list of small rank exceptions, there are only two families of symplectic reflection algebras: one is associated with real or complex reflection groups, namely rational Cherednik algebras and the other is the wreath product. The representation theory of rational Cherednik algebras has more analogues of semisimple Lie algebras, and we focus on this case. Let \mathfrak{h} be a finite-dimensional \mathbb{C} -vector space and $W \subset GL(\mathfrak{h})$ a complex reflection group. The rational Cherednik algebras $\mathbf{H}_{t,c}(W, \mathfrak{h})$ with parameter $t \in \mathbb{C}$ and W -invariant functions c defined on the set of reflections of W is the symplectic reflection algebras with W acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with the naturally defined W -invariant symplectic form (cf. Section 2). We denote it by $\mathbf{H}_{t,c}$ if \mathfrak{h} and W are clear, and simply by \mathbf{H} if both t and c are also fixed.

The purpose of this paper is to study the \mathfrak{h}^* -cohomology $H^\bullet(\mathfrak{h}^*, M) := \bigoplus_i H^i(\mathfrak{h}^*, M)$ and the \mathfrak{h} -homology $H_\bullet(\mathfrak{h}, M) := \bigoplus_i H_i(\mathfrak{h}, M)$ (see Section 2 for the definitions) as W -modules. They are naturally associated with cohomology defined by the half-Dirac operators D_x and D_y with the twist of a genuine character χ of \widetilde{W} (see Proposition 2.12). We show that they have nice properties analogous to that of semisimple Lie algebras obtained in [HPR] and [HPR']; especially we prove an analogue of the Casselman–Osborne Theorem for semisimple modules and a Hodge decomposition theorem for unitarizable modules. Moreover, we obtain some results on relations between $H_D(M)$ and Lie algebra cohomology $H^\bullet(\mathfrak{u}^*, M)$ and homology $H_\bullet(\mathfrak{u}, M)$ analogous to the results on category \mathcal{O}^p obtained by Xiao and the first named author [HX].

We now describe our main results more precisely. Most of the necessary definitions are introduced in Sections 2 and 3. We note that Theorems A to C remain true upon replacing $H^\bullet(\mathfrak{h}^*, M)$ with $H_\bullet(\mathfrak{h}, M)$. Our first main result is an analogue of the Casselman–Osborne Theorem for $H^\bullet(\mathfrak{h}^*, M)$ and $H_\bullet(\mathfrak{h}, M)$.

Theorem A (Theorem 3.7). *Let \mathcal{B} be an abelian subalgebra of $\mathbf{H} \otimes C(V)$ defined before the statement of Theorem 3.6. Suppose $\sigma \otimes \beta$ is an isotypical component in $H^\bullet(\mathfrak{h}^*, M)$, where σ is an irreducible W -module and β is a \mathcal{B} -character. Then Vogan’s morphism*

$$\zeta_d^* : \text{Irr}(\widetilde{W}) \rightarrow \text{Spec } \mathcal{B}$$

satisfies the following condition

$$\zeta_d^*(\sigma \otimes \chi) = \beta.$$

Here $\chi : \widetilde{W} \rightarrow \mathbb{C}^\times$ is a genuine one-dimensional character of a double cover $p : \widetilde{W} \rightarrow W$ of W , satisfying $\chi^2(\tilde{w}) = \det_{\mathfrak{h}^}(p(\tilde{w}))$.*

Then we study the relation between Lie algebra cohomology $H^\bullet(\mathfrak{h}^*, M)$ and Dirac cohomology $H_D(M)$ introduced in [C]. More precisely, we have the following inclusion of \widetilde{W} -module homomorphisms.

Theorem B (Theorem 4.2). *Let M be a \mathbf{H} -module so that D^2 acts semisimply on $M \otimes S$, where S is the spinor module (Definition 2.8). Then there is an injective \widetilde{W} -module homomorphism: $H_D(M) \hookrightarrow H^\bullet(\mathfrak{h}^*, M) \otimes \chi$.*

If M is unitary, then the inclusion in Theorem B is an isomorphism. It follows from a Hodge decomposition theorem for unitary modules.

Theorem C (Theorem 5.1). *Let M be a $*$ -unitary module in the sense of [ES]. Then one has*

- (a) $H_D(M) = \ker D = \ker D^2$;
- (b) $M \otimes S = \ker D \oplus \operatorname{im} D_x \oplus \operatorname{im} D_y$;
- (c) $\ker D_x = \ker D \oplus \operatorname{im} D_x$, $\ker D_y = \ker D \oplus \operatorname{im} D_y$, where D_x, D_y are half Dirac operators defined in Definition 2.10 satisfying $D = D_x + D_y$. Consequently,

$$H_D(M) = \ker D = \ker D_x / \operatorname{im} D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi.$$

In the last two sections we study the Lie algebra cohomology of certain types of $\mathbf{H}_{t,c}$ -modules. Section 6 concerns the category \mathcal{O} of $\mathbf{H}_{1,c}$ -modules (see [GGOR]). For example, we show that if $M(\sigma) := \mathbf{H}_{1,c} \otimes_{S(\mathfrak{h}) \times \mathbb{C}[W]} \sigma$ is a standard object in category \mathcal{O} , then

$$H_D(M(\sigma)) \cong H^\bullet(\mathfrak{h}^*, M(\sigma)) \otimes \chi \cong H_\bullet(\mathfrak{h}, M(\sigma)) \otimes \chi \cong \sigma \otimes \chi^{-1}.$$

This is a refinement of Proposition 5.6 in [C]. Moreover, if $L(\operatorname{triv})$ be the finite-dimensional irreducible quotient of $M(\operatorname{triv})$ given in Proposition 1.3 of [BEG], then

$$H_D(L(\operatorname{triv})) \cong H^\bullet(\mathfrak{h}^*, L(\operatorname{triv})) \otimes \chi \cong H_\bullet(\mathfrak{h}, L(\operatorname{triv})) \otimes \chi \cong \wedge^\bullet \mathfrak{h} \otimes \chi^{-1}.$$

In other words, the inclusion in Theorem B is also an isomorphism for all standard modules and finite-dimensional modules in category \mathcal{O} .

Section 7 deals with the baby Verma modules for $\mathbf{H}_{0,c}$ (see [G1]). We show that if $\bar{L}(\sigma)$ is the irreducible head of a baby Verma module such that as W -modules,

$$H^\bullet(\mathfrak{h}^*, \bar{L}(\sigma)) \cong \bigoplus_{i=1}^k \nu_i$$

with all ν_i being irreducible, then the whole set $\{\nu_i \otimes \det_{\mathfrak{h}^*} \mid i = 1, \dots, k\}$ is contained in the same Calogero–Moser cell ([G1, Sect. 7]). Combining with Theorem B, this gives an alternative proof of [C, Cor. 5.10].

2. Preliminaries

We begin this section by recalling the definition of rational Cherednik algebras $\mathbf{H}_{t,c}$ given in [EG] and [GGOR].

Definition 2.1. *Let W be a complex reflection group acting on a complex vector space \mathfrak{h} , i.e., W is a group generated by the pseudo-reflections $s \in \mathcal{R}$ fixing a hyperplane $H_s \in \mathfrak{h}$. Let $\alpha_s \in \mathfrak{h}^*$ be a non-zero vector so that the W -invariant symmetric pairing $\langle \cdot, \cdot \rangle$ between \mathfrak{h} and \mathfrak{h}^* gives $\langle y, \alpha_s \rangle = 0$ for all $y \in H_s$. Similarly, we can define $\alpha_s^\vee \in \mathfrak{h}$ corresponding to the action of s on \mathfrak{h}^* . Set $V = \mathfrak{h} \oplus \mathfrak{h}^*$. The rational Cherednik algebra $\mathbf{H}_{t,c}$ associated to \mathfrak{h} , W , with parameters $t \in \mathbb{C}$ and*

W -invariant functions $c : \mathcal{R} \rightarrow \mathbb{C}$ is defined as the quotient of $S(V) \rtimes \mathbb{C}[W]$ by the relation

$$[y, x] = t\langle y, x \rangle - \sum_{s \in \mathcal{R}} c(s) \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s$$

for all $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$.

Let $\mathbf{H}_{t,c}$ be the rational Cherednik algebra corresponding to W, \mathfrak{h} with parameters t and c . Let $\{y_1, \dots, y_n\}$ be a basis of \mathfrak{h} , and $\{x_1, \dots, x_n\}$ be the corresponding dual basis of \mathfrak{h}^* . In [GGOR], a Casimir-type element \mathbf{h} is defined by

$$\mathbf{h} := \sum_i (x_i y_i + y_i x_i) = 2 \sum_i x_i y_i + nt - \sum_{s \in \mathcal{R}} c(s) s \in \mathbf{H}_{t,c}^W.$$

It gives a natural grading on $\mathbf{H}_{t,c}$ when $t = 1$. Note that the definition of \mathbf{h} does not depend on our choice of basis. Following [GGOR], we make a shift and define $\Omega_{\mathbf{H}_{t,c}}$ in the following.

Definition 2.2.

$$\Omega_{\mathbf{H}_{t,c}} := \mathbf{h} - \sum_{s \in \mathcal{R}} c(s) \frac{1 + \lambda_s}{1 - \lambda_s} s = 2 \sum_i x_i y_i + nt - \sum_{s \in \mathcal{R}} \frac{2c(s)}{1 - \lambda_s} s \in \mathbf{H}_{t,c}^W,$$

where $\lambda_s = \det_{\mathfrak{h}}(s) \in \mathbb{C}^\times$.

We now define the Lie algebra (co)homology of a $\mathbf{H}_{t,c}$ -module, analogous to the case of the \mathfrak{n} and $\bar{\mathfrak{n}}$ -(co)homology of a \mathfrak{g} -module studied in [K1].

Definition 2.3. Let M be an $\mathbf{H}_{t,c}$ -module. The p^{th} \mathfrak{h}^* -cohomology group $H^p(\mathfrak{h}^*, M)$ of M arises as the p^{th} derived functor of the covariant, left exact functor

$$M \mapsto M^{\mathfrak{h}^*} = H^0((\mathfrak{h}^*, M))$$

and can be identified with the p^{th} cohomology group of the cochain complex

$$0 \rightarrow \text{Hom}_{\mathfrak{h}^*}(\wedge^0 \mathfrak{h}^*, M) \xrightarrow{d_0} \text{Hom}_{\mathfrak{h}^*}(\wedge^1 \mathfrak{h}^*, M) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \text{Hom}_{\mathfrak{h}^*}(\wedge^n \mathfrak{h}^*, M) \rightarrow 0,$$

where the differential is defined by

$$d_p f(x_{i_1} \wedge \dots \wedge x_{i_p}) := \sum_j (-1)^j x_{i_j} \cdot f(x_{i_1} \wedge \dots \wedge \widehat{x_{i_j}} \wedge \dots \wedge x_{i_p}).$$

If we identify $\text{Hom}_{\mathfrak{h}^*}(\wedge^p \mathfrak{h}^*, M)$ with $M \otimes \wedge^p \mathfrak{h}$, then $H^p(\mathfrak{h}^*, M)$ is also identified with the p^{th} cohomology of the complex

$$0 \rightarrow M \otimes \wedge^0 \mathfrak{h} \xrightarrow{d_0} M \otimes \wedge^1 \mathfrak{h} \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} M \otimes \wedge^n \mathfrak{h} \rightarrow 0, \quad (1)$$

where the differential is defined by

$$d_p(m \otimes y_{i_1} \wedge \dots \wedge y_{i_p}) := \sum_j x_j \cdot m \otimes y_j \wedge y_{i_1} \wedge \dots \wedge y_{i_p}.$$

The boundary map d_p is well defined because of the following proposition.

Proposition 2.4.

- (a) The definition of d_p is independent of the choice of basis of \mathfrak{h} and \mathfrak{h}^* .
 (b) Treating M and $\wedge^p \mathfrak{h}$ as W -modules, then d_p commutes with the action of W on $M \otimes \wedge^p \mathfrak{h}$.
 (c) $d_{p+1}d_p = 0$.

Consequently, $H^p(\mathfrak{h}^*, M)$ is a W -module for all p .

Proof. (a) Suppose we have another basis y'_i of \mathfrak{h} given by $y'_i = \sum_j A_{ji}y_j$ for some invertible $A = (A_{ij})$. Then the corresponding dual basis x'_i of \mathfrak{h}^* must be given by $x'_i = \sum_k B_{ki}x_k$, where $B = (A^{-1})^T$. Then it is easy to check that

$$\sum_j x'_j \cdot m \otimes y'_j \wedge y_{i_1} \wedge \cdots \wedge y_{i_p} = \sum_j x_j \cdot m \otimes y_j \wedge y_{i_1} \wedge \cdots \wedge y_{i_p}.$$

Hence, d_p is independent of the basis of \mathfrak{h} .

- (b) For all $w \in W$,

$$\begin{aligned} w \cdot d_p(m \otimes y_{i_1} \wedge \cdots \wedge y_{i_p}) &= w \sum_j x_j \cdot m \otimes y_j \wedge y_{i_1} \wedge \cdots \wedge y_{i_p} \\ &= \sum_j wx_j \cdot m \otimes w(y_j) \wedge w(y_{i_1}) \wedge \cdots \wedge w(y_{i_p}) \\ &= \sum_j w(x_j)w \cdot m \otimes w(y_j) \wedge w(y_{i_1}) \wedge \cdots \wedge w(y_{i_p}) \\ &= d_p(w \cdot m \otimes w(y_{i_1}) \wedge \cdots \wedge w(y_{i_p})) \\ &= d_p(w \cdot (m \otimes y_{i_1} \wedge \cdots \wedge y_{i_p})), \end{aligned}$$

where the third equality follows from the definition of $\mathbf{H}_{t,c}$, and the fourth equality follows from (a).

- (c) By the definition of d_p ,

$$\begin{aligned} d_{p+1}d_p(m \otimes y_{i_1} \wedge \cdots \wedge y_{i_p}) &= \sum_{i,j} x_i x_j \cdot m \otimes y_i \wedge y_j \wedge y_{i_1} \wedge \cdots \wedge y_{i_p} \\ &= \sum_{i < j} x_i x_j \cdot m \otimes (y_i \wedge y_j + y_j \wedge y_i) \wedge y_{i_1} \wedge \cdots \wedge y_{i_p} \\ &\quad + \sum_i x_i^2 \cdot m \otimes (y_i \wedge y_i) \wedge y_{i_1} \wedge \cdots \wedge y_{i_p}, \end{aligned}$$

which is equal to 0 since $y_i \wedge y_j = -y_j \wedge y_i$ for all i, j . \square

Definition 2.5. Let M be an $\mathbf{H}_{t,c}$ -module. The p^{th} \mathfrak{h} -homology group $H_p(\mathfrak{h}, M)$ of M arises as the p^{th} derived functor of the covariant, right exact functor

$$M \mapsto M/\mathfrak{h}M = H_0(\mathfrak{h}, M)$$

on the category of \mathfrak{h} -modules. It can be calculated as the p^{th} homology group of the chain complex

$$0 \rightarrow M \otimes \wedge^n \mathfrak{h} \xrightarrow{\partial_n} M \otimes \wedge^{n-1} \mathfrak{h} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M \otimes \mathfrak{h} \xrightarrow{\partial_1} M \rightarrow 0, \quad (2)$$

where the differential is defined by

$$\begin{aligned} \partial_p(m \otimes y_{i_1} \wedge \cdots \wedge y_{i_p}) &:= \sum_k (-1)^k y_{i_k} \cdot m \otimes y_{i_1} \wedge \cdots \wedge \widehat{y_{i_k}} \wedge \cdots \wedge y_{i_p} \\ &= \sum_k \sum_j (-1)^k \langle x_j, y_{i_k} \rangle y_j \cdot m \otimes y_{i_1} \wedge \cdots \wedge \widehat{y_{i_k}} \wedge \cdots \wedge y_{i_p}. \end{aligned}$$

We have the following proposition whose proof is similar to that of \mathfrak{h}^* -cohomology.

Proposition 2.6.

- (a) *The definition of ∂_p is independent of the choice of basis of \mathfrak{h} and \mathfrak{h}^* .*
- (b) *Treating M and $\wedge^p \mathfrak{h}$ as W -modules, then ∂_p commutes with the action of W on $M \otimes \wedge^p \mathfrak{h}$.*
- (c) $\partial_{p-1} \partial_p = 0$.

Consequently, $H_p(\mathfrak{h}, M)$ is a W -module.

From now on, we write

$$H^\bullet(\mathfrak{h}^*, M) := \bigoplus_i H^i(\mathfrak{h}^*, M) \quad \text{and} \quad H_\bullet(\mathfrak{h}, M) := \bigoplus_i H_i(\mathfrak{h}, M)$$

as the ungraded sum of the (co)homology spaces. Similarly, we can define \mathfrak{h}^* -homology $H_\bullet(\mathfrak{h}^*, M)$ and \mathfrak{h} -cohomology $H^\bullet(\mathfrak{h}, M)$. They are related to \mathfrak{h}^* -cohomology and \mathfrak{h} -homology respectively by the Poincaré duality:

Proposition 2.7 (Poincaré duality). *Let $\mathfrak{c} = \mathfrak{h}$ or \mathfrak{h}^* , then the perfect pairing $\wedge^p \mathfrak{c} \times \wedge^{n-p} \mathfrak{c} \rightarrow \wedge^n \mathfrak{c}$ gives a Poincaré duality between the $H^i(\mathfrak{c}, M)$ and the homology $H_{n-i}(\mathfrak{c}, M)$ defined above. More precisely, for all $i = 1, \dots, n$,*

$$H_i(\mathfrak{c}, M) \cong H^{n-i}(\mathfrak{c}, M) \otimes \wedge^n \mathfrak{c}, \quad \text{or} \quad H^i(\mathfrak{c}, M) \cong H_{n-i}(\mathfrak{c}, M) \otimes \wedge^n \mathfrak{c}^*$$

as W -modules.

We now recall the Clifford algebra and spinor module of $V = \mathfrak{h} \oplus \mathfrak{h}^*$. Define a W -invariant bilinear product on V by $\langle \cdot, \cdot \rangle$ by $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 0$, $\langle x_i, y_j \rangle = \delta_{ij}$ (this is the same as the pairing given in Definition 2.1). The Clifford algebra $C(V)$ with respect to $\langle \cdot, \cdot \rangle$ is the tensor algebra of V subject to the relations

$$x_i x_j + x_j x_i = y_i y_j + y_j y_i = 0, \quad x_i y_j + y_j x_i = -2\delta_{ij}.$$

There is a natural injection $\widetilde{W} \hookrightarrow \text{Pin}(V) \hookrightarrow C(V)^\times$, where $p: \widetilde{W} \rightarrow W$ is the double cover of W given by the pull-back of the Pin cover $p: \text{Pin}(V) \rightarrow \text{O}(V)$. For any $s \in \mathcal{R}$, let

$$\mu_s = \frac{\sqrt{\lambda_s} - 1/\sqrt{\lambda_s}}{2\langle \alpha_s^\vee, \alpha_s \rangle} \alpha_s \alpha_s^\vee + \sqrt{\lambda_s} \in C(V), \quad (3)$$

where $\sqrt{\lambda_s}$ is a choice of the square root of $\lambda_s = \det_{\mathfrak{h}}(s)$. Then the calculations in Lemma 4.6 of [C] show that $\{\pm \mu_s\} = p^{-1}(s) \subset \widetilde{W}$, and $\{\pm \mu_s \mid s \in \mathcal{R}\}$ generate \widetilde{W} .

Definition 2.8. The *spinor module* S corresponding to the Clifford algebra $C(V)$ can be realized as $S \cong \wedge^\bullet \mathfrak{h}$ as vector spaces. The action of $C(V)$ on S is defined by

$$\begin{aligned} x(y_{i_1} \wedge \cdots \wedge y_{i_p}) &= 2 \sum_j (-1)^j \langle x, y_{i_j} \rangle y_{i_1} \wedge \cdots \wedge \widehat{y_{i_j}} \wedge \cdots \wedge y_{i_p}, \quad x \in \mathfrak{h}^*; \\ y(y_{i_1} \wedge \cdots \wedge y_{i_p}) &= y \wedge y_{i_1} \wedge \cdots \wedge y_{i_p}, \quad y \in \mathfrak{h}. \end{aligned}$$

The following proposition describes S as a \widetilde{W} -module:

Proposition 2.9. *Every $\widetilde{w} \in \widetilde{W}$ preserves every $\wedge^l \mathfrak{h} \subset S$. In particular, for every $v_i \in \mathfrak{h}$,*

$$\widetilde{w} \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_l = \chi(\widetilde{w}) \cdot (p(\widetilde{w})(v_1) \wedge p(\widetilde{w})(v_2) \wedge \cdots \wedge p(\widetilde{w})(v_l))$$

with χ being a genuine one-dimensional \widetilde{W} -module satisfying $\chi^2(\widetilde{w}) = \det_{\mathfrak{h}^*}(p(\widetilde{w}))$. Therefore, as \widetilde{W} -modules,

$$S \cong \chi \otimes \wedge^\bullet \mathfrak{h}, \tag{4}$$

where $\wedge^\bullet \mathfrak{h}$ is considered as a \widetilde{W} -module that factors through $p: \widetilde{W} \rightarrow W$.

Proof. We only need to prove the proposition for all generators $\pm\mu_s \in \widetilde{W}$. Let $\alpha_s^\vee \in \mathfrak{h}$, $\alpha_s \in \mathfrak{h}^*$ be as in Definition 2.1, and fix a basis $\{y_1, \dots, y_n\}$ of \mathfrak{h} with $y_1 = \alpha_s^\vee$, $y_2, \dots, y_n \in \ker(\text{Id} - s)|_{\mathfrak{h}}$. Then

$$s(y_1) = \lambda_s y_1, \quad s(y_i) = y_i \text{ for } i > 1.$$

For $1 \leq l_1 < \cdots < l_k \leq n$, it is easy to check that

$$\alpha_s \alpha_s^\vee \cdot y_{l_1} \wedge \cdots \wedge y_{l_k} = \begin{cases} 0, & \text{if } l_1 = 1; \\ -2\langle \alpha_s^\vee, \alpha_s \rangle y_{l_1} \wedge \cdots \wedge y_{l_k}, & \text{otherwise.} \end{cases}$$

Using (3), $\pm\mu_s$ acts on $y_{l_1} \wedge \cdots \wedge y_{l_k}$ by $\pm\mu_s \cdot y_{l_1} \wedge \cdots \wedge y_{l_k} = (\pm 1/\sqrt{\lambda_s}) s(y_{l_1}) \wedge \cdots \wedge s(y_{l_k})$ in both cases. Hence the result follows from the fact that $(\pm 1/\sqrt{\lambda_s})^2 = \lambda_s^{-1} = \det_{\mathfrak{h}^*}(s)$. \square

Definition 2.10. Let D_x, D_y be elements in $\mathbf{H} \otimes C(V)$ given by

$$D_x := \sum_i x_i \otimes y_i, \quad D_y := \sum_i y_i \otimes x_i.$$

If M is a $\mathbf{H}_{t,c}$ -module, then D_x, D_y acts on $M \otimes S$ by

$$D_x(m \otimes s) = \sum_i x_i \cdot m \otimes y_i s; \quad D_y(m \otimes s) = \sum_i y_i \cdot m \otimes x_i s.$$

Theorem 2.11 ([C, Lem. 2.4, Prop. 4.9]). *We have*

- (a) D_x and D_y are independent of the choice of basis of \mathfrak{h} .
- (b) Let $\Delta : \mathbb{C}[\widetilde{W}] \rightarrow \mathbf{H}_{t,c} \otimes C(V)$ be the diagonal map $\tilde{w} \mapsto p(\tilde{w}) \otimes \tilde{w}$. Then D_x and D_y commute with $\Delta(\mathbb{C}[\widetilde{W}])$.
- (c) $D_x^2 = D_y^2 = 0$.

Consequently, $\ker D_x / \text{im } D_x$ and $\ker D_y / \text{im } D_y$ are naturally \widetilde{W} -modules.

Recall from Proposition 2.9 that we can identify S with $\wedge^\bullet \mathfrak{h} \otimes \chi$ as \widetilde{W} -modules. With this identification,

$$D_x = d \otimes \chi, \quad D_y = 2\partial \otimes \chi.$$

Thus, we have the following proposition relating the above operators with the Lie algebra (co)homology.

Proposition 2.12. *There are \widetilde{W} -module isomorphisms:*

$$\ker D_x / \text{im } D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi \quad \text{and} \quad \ker D_y / \text{im } D_y \cong H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

3. An analogue of Casselman–Osborne Theorem

In this section, we prove a version of the Casselman–Osborne Theorem for Lie algebra cohomology of $\mathbf{H}_{t,c}$ and associated with D_x and D_y . It relates the ‘central character’ of a $\mathbf{H}_{t,c}$ -module M (denoted as \mathcal{B} -character below) to the central characters of $\mathbb{C}[W]$ -modules $H^\bullet(\mathfrak{h}^*, M)$ and $H_\bullet(\mathfrak{h}, M)$.

We set $\mathbf{H} = \mathbf{H}_{t,c}$. Recall that we have the following Poincaré–Birkhoff–Witt decomposition $\mathbf{H} \cong S(\mathfrak{h}) \otimes \mathbb{C}[W] \otimes S(\mathfrak{h}^*)$ as vector spaces. We define a \mathbb{C}^* -action on \mathbf{H} by

$$\lambda \cdot x = \lambda^{-1}x, \quad \lambda \cdot y = \lambda y, \quad \lambda \cdot w = w$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$. We also define a \mathbb{C}^* -action on $C(V) \cong \wedge^\bullet \mathfrak{h} \otimes \wedge^\bullet \mathfrak{h}^*$ by

$$\lambda \cdot x = \lambda^{-1}x, \quad \lambda \cdot y = \lambda y$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$.

Definition 3.1. *We define a subalgebra \mathbf{A} of $\mathbf{H} \otimes C(V)$ by setting*

$$\mathbf{A} := (\mathbf{H} \otimes C(V))^{\mathbb{C}^*}.$$

It is easy to check that we have the following $\Delta(\mathbb{C}[\widetilde{W}])$ -module isomorphism

$$\mathbf{A} \cong \bigoplus_{k_1+l_1=k_2+l_2} (S^{k_1}(\mathfrak{h}) \otimes \mathbb{C}[W] \otimes S^{k_2}(\mathfrak{h}^*)) \otimes (\wedge^{l_1} \mathfrak{h} \otimes \wedge^{l_2} \mathfrak{h}^*) \subset \mathbf{H} \otimes C(V).$$

We have a filtration $\mathbf{H}^0 \subset \mathbf{H}^1 \subset \mathbf{H}^2 \subset \dots$ on \mathbf{H} by taking $\deg(x) = \deg(y) = 1$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $\deg(w) = 0$ for all $w \in W$. Then the graded algebra is given by $\text{gr}(\mathbf{H}) \cong \mathbf{H}_{0,0} \cong S(V) \rtimes \mathbb{C}[W]$. With the filtration on \mathbf{H} , define the filtration $\mathbf{A}^0 \subset \mathbf{A}^1 \subset \dots$ of \mathbf{A} , where $\mathbf{A}^n = \mathbf{A} \cap \mathbf{H}^n \otimes C(V)$. So we have a graded algebra

$$\text{gr}(\mathbf{A}) \cong \bigoplus_{k_1+l_1=k_2+l_2} (S^{k_1}(\mathfrak{h}) \otimes S^{k_2}(\mathfrak{h}^*)) \rtimes \mathbb{C}[W] \otimes (\wedge^{l_1} \mathfrak{h} \otimes \wedge^{l_2} \mathfrak{h}^*) \subset \mathbf{H}_{0,0} \otimes C(V).$$

By definition, we have $\Delta(\mathbb{C}[\widetilde{W}])$, D_x , D_y are all contained in \mathbf{A} . Therefore, \widetilde{W} acts on \mathbf{A} by conjugation. We denote by $\mathbf{A}^{\widetilde{W}}$ the subalgebra of \widetilde{W} -invariants in \mathbf{A} . By Proposition 2.12(b), D_x , $D_y \in \mathbf{A}^{\widetilde{W}}$. Define the maps $\delta_d, \delta_\partial : \mathbf{A}^{\widetilde{W}} \rightarrow \mathbf{A}^{\widetilde{W}}$ by

$$\delta_d a = D_x a - \epsilon(a)D_x, \quad \delta_\partial a = D_y a - \epsilon(a)D_y, \quad (5)$$

where $\epsilon(a) = a$ if $a \in \mathbf{H} \otimes C^{\text{even}}(V)$ and $\epsilon(a) = -a$ if $a \in \mathbf{H} \otimes C^{\text{odd}}(V)$.

The main theorem of this section is the following:

Theorem 3.2. *For $\delta_d, \delta_\partial : \mathbf{A}^{\widetilde{W}} \rightarrow \mathbf{A}^{\widetilde{W}}$ defined in (5), we have*

$$\ker \delta_d = \text{im } \delta_d \oplus \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

The results hold analogously for δ_∂ .

For the rest of this section, we will only prove the theorem for δ_d . The proof of δ_∂ is analogous to that of δ_d .

Lemma 3.3. *We have*

- (a) $\delta_d^2 = 0$ and consequently $\text{im } \delta_d \subset \ker \delta_d$.
- (b) *The map δ_d is an odd derivation, i.e., $\delta_d(ab) = \delta_d(a)b + \epsilon(a)\delta_d(b)$. Therefore, if $a, b \in \ker \delta_d$, $ab \in \ker \delta_d$ and $\ker \delta_d$ is a subalgebra of $\mathbf{A}^{\widetilde{W}}$.*

Proof. (a) By the definition of δ_d , $\delta_d^2 a = D_x^2 a - aD_x^2 = 0 - 0 = 0$.

(b) $\delta_d(ab) = D_x(ab) - \epsilon(ab)D_x = D_x ab - \epsilon(a)D_x b + \epsilon(a)D_x b - \epsilon(a)\epsilon(b)D_x = \delta_d(a)b + \epsilon(a)\delta_d(b)$. \square

Note that the action δ_d increases the filtration by 1, while the action of $\Delta(\widetilde{w})$ preserves the filtration. So $\text{gr}(\mathbf{A})^{\widetilde{W}} = \text{gr}(\mathbf{A}^{\widetilde{W}})$, and δ_d descend to the map

$$\overline{\delta}_d : \text{gr}(\mathbf{A})^{\widetilde{W}} \rightarrow \text{gr}(\mathbf{A})^{\widetilde{W}}.$$

Lemma 3.4. *Let $\overline{\delta}_d : \text{gr}(\mathbf{A})^{\widetilde{W}} \rightarrow \text{gr}(\mathbf{A})^{\widetilde{W}}$ be defined as in Equation (5). Then the following holds:*

- (a) $\Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}) \subset \ker \overline{\delta}_d$.
- (b) $\text{im } \overline{\delta}_d \cap \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}) = 0$, hence

$$\text{im } \overline{\delta}_d \oplus \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}) \subset \ker \overline{\delta}_d.$$

Proof. (a) For all $\widetilde{w} \in \widetilde{W}$, Theorem 2.11(b) says that $\Delta(\widetilde{w})D_x - D_x\Delta(\widetilde{w}) = 0$. Also, recall \widetilde{w} is generated by $\pm\mu_s \in C^{\text{even}}(V)$. Hence $\epsilon(\Delta(\widetilde{w})) = \Delta(\widetilde{w})$ and $\Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}) \subset \ker \overline{\delta}_d$.

(b) We have already seen that $\overline{\delta}_d^2 = 0$. So $\text{im } \overline{\delta}_d \subset \ker \overline{\delta}_d$. On the other hand, every summand in $\text{im } \delta_d$ must have an $x_i \in \mathfrak{h}^*$ in its $S(V)$ -component, while every element in $\Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}})$ does not contain any \mathfrak{h}^* factor. Hence they must be mutually disjoint. \square

Proposition 3.5. *We have*

$$\ker \bar{\delta}_d = \text{im } \bar{\delta}_d \oplus \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

Proof. By the above lemma, we just need to show

$$\ker \bar{\delta}_d \subset \text{im } \bar{\delta}_d \oplus \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

Suppose $a \in \ker \bar{\delta}_d$, write $a = a_1 + \cdots + a_{|W|} \in \text{gr}(\mathbf{A})^{\widetilde{W}}$ where a_i is the sum of the elements of the form $fw_i \otimes g$ for $f \in S^{k_1}(\mathfrak{h}) \otimes S^{k_2}(\mathfrak{h}^*)$, $w_i \in W$ and $g \in \wedge^{l_1} \mathfrak{h} \otimes \wedge^{l_2} \mathfrak{h}^*$ with $k_1 + l_1 = k_2 + l_2$.

For simplicity of notation, we write $\bar{\delta}_d(\alpha) = D_x \alpha - \epsilon(\alpha) D_x$ for all elements $\alpha \in \text{gr}(\mathbf{A})$. Then $\bar{\delta}_d(a) = \bar{\delta}_d(a_1) + \cdots + \bar{\delta}_d(a_{|W|}) = 0$. Note that $\bar{\delta}_d$ does not change the w_i component on each a_i . Hence, we must have

$$\bar{\delta}_d(a_i) = 0 \quad \text{for all } i.$$

The summands in a_i are of the form

$$fw_i \otimes g = (f \otimes g \cdot \widetilde{w}_i^{-1})(w_i \otimes \widetilde{w}_i) = [(fw_i \otimes g) \cdot \Delta(\widetilde{w}_i^{-1})] \Delta(\widetilde{w}_i)$$

for some $\widetilde{w}_i \in \widetilde{W}$ satisfying $p(\widetilde{w}_i) = w_i$. Now $\bar{\delta}_d(a_i) = 0$ means

$$0 = \bar{\delta}_d[a_i \cdot \Delta(\widetilde{w}_i^{-1}) \cdot \Delta(\widetilde{w}_i)] = \bar{\delta}_d(a_i \cdot \Delta(\widetilde{w}_i^{-1})) \cdot \Delta(\widetilde{w}_i) \pm (a_i \cdot \Delta(\widetilde{w}_i^{-1})) \cdot \bar{\delta}_d(\Delta(\widetilde{w}_i)),$$

but we know $\bar{\delta}_d(\Delta(\widetilde{w}_i)) = 0$ since D_x commutes with $\Delta(\mathbb{C}[\widetilde{W}])$, hence

$$\bar{\delta}_d(a_i \Delta(\widetilde{w}_i^{-1})) = 0, \quad a_i \Delta(\widetilde{w}_i^{-1}) \in (S^{k_1}(\mathfrak{h}) \otimes S^{k_2}(\mathfrak{h}^*)) \otimes (\wedge^{l_1} \mathfrak{h} \otimes \wedge^{l_2} \mathfrak{h}^*),$$

with $k_1 + l_1 = k_2 + l_2$.

It follows that $\bar{\delta}_d$ is the differential in the Koszul complex

$$\bigoplus_m \bigoplus_{k_2+l_2=m} S^{k_2}(\mathfrak{h}^*) \otimes \wedge^{l_2} \mathfrak{h}^*,$$

which has cohomology \mathbb{C} on degree $m = 0$ and zero at other degrees. Therefore, by restricting $\bar{\delta}_d$ to

$$\bigoplus_m \bigoplus_{k_1+l_1=k_2+l_2=m} S^{k_1}(\mathfrak{h}) \otimes \wedge^{l_1} \mathfrak{h}^* \otimes S^{k_2}(\mathfrak{h}^*) \otimes \wedge^{l_2} \mathfrak{h}^*,$$

we have

$$\ker \bar{\delta}_d = \text{im } \bar{\delta}_d \oplus \bigoplus_{k_1+l_1=0} S^{k_1}(\mathfrak{h}) \otimes \wedge^{l_1}(\mathfrak{h}) \otimes \mathbb{C}(1 \otimes 1) = \text{im } \bar{\delta}_d \oplus \mathbb{C}(1 \otimes 1).$$

In particular, we have

$$a_i \Delta(\widetilde{w}_i^{-1}) = \bar{\delta}_d z_i + \beta_i (1 \otimes 1),$$

where $\beta_i \in \mathbb{C}$ and $a_i = \bar{\delta}_d(z_i \Delta(\widetilde{w}_i)) + \beta_i \Delta(\widetilde{w}_i)$. Hence,

$$a = \sum_i a_i = \bar{\delta}_d \left(\sum_i z_i \Delta(\widetilde{w}_i) \right) + \sum_i \beta_i \Delta(\widetilde{w}_i).$$

Therefore, $\sum_i \beta_i \Delta(\widetilde{w}_i)$ must be \widetilde{W} -invariant, i.e., $\sum_i \beta_i \Delta(\widetilde{w}_i) \in \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}})$ and the proposition is proved. \square

To finish the proof of Theorem 3.2, one needs to remove the $\bar{\cdot}$'s in Proposition 3.5:

Proof of Theorem 3.2. We have already shown in Lemma 3.4 that $\text{im } \delta_d \oplus \mathbb{C}[\widetilde{W}]^{\widetilde{W}} \subset \ker \delta_d$ so we check the reverse inclusion. Suppose $a \in \mathbf{A}^{\widetilde{W},n}$ is in $\ker \delta_d$, then $\bar{a} \in \ker \bar{\delta}_d$ and $\bar{\delta}_d(\bar{a}) = 0$. By Proposition 3.5, there exists $\bar{b} \in \text{gr}(\mathbf{A}^{\widetilde{W}})_{n-1}$ and $s \in \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ such that

$$\bar{a} = \bar{\delta}_d \bar{b} + \Delta(s).$$

Pick $b \in \mathbf{A}^{\widetilde{W},n-1}$ such that $\text{gr}(b) = \bar{b}$. Then

$$\overline{a - \delta_d b - \Delta(s)} = 0$$

and hence $a - \delta_d b - \Delta(s) \in \mathbf{A}^{\widetilde{W},n-1}$. Note that

$$a - \delta_d b - \Delta(s) \in \ker \delta_d.$$

By induction on n , we have $a - \delta_d b - \Delta(s) \in \text{im } \delta_d \oplus \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$. Thus, $a \in \text{im } \delta_d \oplus \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ as required. \square

Let \mathcal{B} be an abelian subalgebra of $\ker \delta_d \cap (\mathbf{H} \otimes \mathbb{C}^{\text{even}}(V))$. Then, by definition of δ_d , \mathcal{B} commutes with D_x and $\Delta(\mathbb{C}[\widetilde{W}])$. So $\ker D_x / \text{im } D_x$ is naturally a $\mathbb{C}[\widetilde{W}] \otimes \mathcal{B}$ -module.

Theorem 3.6. *Let $\zeta_d : \mathcal{B} \rightarrow \Delta(\mathbb{C}[\widetilde{W}]^{\widetilde{W}})$ be the restriction to \mathcal{B} of the projection map given by Theorem 3.2. Then we have*

- (a) ζ_d is an algebra homomorphism.
- (b) Suppose that $\tilde{\sigma} \otimes \beta$ is an isotypical component in $\ker D_x / \text{im } D_x$, where $\tilde{\sigma}$ is an irreducible \widetilde{W} -module and β is a \mathcal{B} -character. Then the morphism

$$\zeta_d^* : \text{Irr}(\widetilde{W}) \rightarrow \text{Spec } \mathcal{B}$$

satisfies the following condition

$$\zeta_d^*(\tilde{\sigma}) = \beta.$$

The results hold analogously by replacing D_x with D_y .

Proof. We only present the proof for δ_d below:

- (a) Let $b_1, b_2 \in \mathcal{B}$, then $b_i = \delta_d(a_i) + \zeta_d(b_i)$ for some $a_i \in \mathbf{A}$, so

$$\begin{aligned} b_1 b_2 &= \delta_d(a_1) \delta_d(a_2) + \delta_d(a_1) \zeta_d(b_2) + \zeta_d(b_1) \delta_d(a_2) + \zeta_d(b_1) \zeta_d(b_2) \\ &= \delta_d(a_1 \delta_d(a_2) + a_1 \zeta_d(b_2) + \zeta_d(b_1) a_2) + \zeta_d(b_1) \zeta_d(b_2). \end{aligned}$$

The second equality comes from the facts that $\delta_d^2 = 0$, $\delta_d(\Delta(\tilde{w})) = 0$ and δ_d is a derivation. Hence by the uniqueness of the decomposition in Theorem 3.2,

$$\zeta_d(b_1 b_2) = \zeta_d(b_1) \zeta_d(b_2)$$

and the result follows.

(b) Let $0 \neq z \in \mathcal{B}$. By Theorem 3.2, we have

$$z = \zeta_d(z) + D_x a - \epsilon(a)D_x$$

for some $a \in \mathbf{H} \otimes C(V)$. Let $\tilde{\alpha} \neq 0$ be an element in the $\tilde{\sigma} \otimes \beta$ component of $\ker D_x / \text{im } D_x$. Then

$$\begin{aligned} z \cdot \tilde{\alpha} &= \zeta_d(z) \cdot \tilde{\alpha} + D_x a \cdot \tilde{\alpha} - \epsilon(a)D_x \cdot \tilde{\alpha} \\ \beta(z)\tilde{\alpha} &= \tilde{\sigma}(\zeta_d(z))\tilde{\alpha} + D_x a \cdot \tilde{\alpha} \\ (\beta(z) - \tilde{\sigma}(\zeta_d(z)))\tilde{\alpha} &= D_x a \cdot \tilde{\alpha}, \end{aligned}$$

where the second equality comes from the fact that $\tilde{\alpha}$ is an $\tilde{\sigma} \otimes \beta$ isotypic element in $\ker D_x / \text{im } D_x$. Therefore, the left-hand side of the equality is in $\ker D_x / \text{im } D_x$, and the right-hand side is in $\text{im } D_x$ and hence it must be equal to zero, i.e.,

$$(\beta(z) - \tilde{\sigma}(\zeta_d(z)))\tilde{\alpha} = 0.$$

However, $\tilde{\alpha} \neq 0$, and it follows that $\beta(z) - \tilde{\sigma}(\zeta_d(z)) = 0$ for all $z \in \mathcal{B}$ and the result holds. \square

With the identification $\ker D_x / \text{im } D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi$, the following result is straightforward:

Theorem 3.7. *Retain the notations in Theorem 3.6. Suppose $\sigma \otimes \beta$ is an isotypical component in $H^\bullet(\mathfrak{h}^*, M)$, where σ is a W -module and β is a \mathcal{B} -character. Then*

$$\zeta_d^*(\sigma \otimes \chi) = \beta.$$

The similar statement holds for $H_\bullet(\mathfrak{h}, M)$ and ζ_δ^ . (Note that $\sigma \otimes \chi$ is an irreducible \widetilde{W} -module for any irreducible W -module σ).*

4. Embedding of Dirac cohomology into Lie algebra cohomology

In this section, we give a criterion such that the Dirac cohomology $H_D(M)$ of M can be embedded into its Lie algebra (co)homology.

Definition 4.1 ([C]). *Let $D := D_x + D_y \in \mathbf{H}_{t,c} \otimes C(V)$. The Dirac cohomology of an $\mathbf{H}_{t,c}$ -module M is defined by*

$$H_D(M) := \ker D / \text{im } D,$$

where $D : M \otimes S \rightarrow M \otimes S$ is defined as in Definition 2.10.

The main theorem of this section is the following:

Theorem 4.2. *If M is an $\mathbf{H}_{t,c}$ -module so that D^2 acts semisimply on $M \otimes S$, then we have the following \widetilde{W} -module injective homomorphisms*

$$H_D(M) \hookrightarrow H^\bullet(\mathfrak{h}^*, M) \otimes \chi, \quad H_D(M) \hookrightarrow H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

We now prove the first inclusion of Theorem 4.2. The second inclusion can be proved in an analogous way.

Lemma 4.3. *If we identify S with $\wedge^\bullet \mathfrak{h}$ as vector spaces (Definition 2.8) and define an increasing \widetilde{W} -invariant filtration on $\ker D$ in $M \otimes S$ by*

$$0 = (\ker D)_{-1} \subset (\ker D)_0 \subset \cdots \subset (\ker D)_{n-1} \subset (\ker D)_n = \ker D,$$

with $(\ker D)_i = \ker D \cap (\bigoplus_{p=0}^i M \otimes \wedge^p \mathfrak{h})$. Then we have an injective W -module homomorphism:

$$f : \text{gr}(\ker D) \hookrightarrow \ker D_x \cap \ker D^2.$$

Remark 4.4. Since the action of $\Delta(\mathbb{C}[\widetilde{W}])$ on $M \otimes S$ preserves $M \otimes \wedge^l \mathfrak{h}$ for each l , we have an isomorphism of \widetilde{W} -modules $\ker D \cong \text{gr}(\ker D)$ and hence the injection of \widetilde{W} -modules in the above lemma can be rewritten as:

$$f' : \ker D \hookrightarrow \ker D_x \cap \ker D^2.$$

Proof. Suppose $\widetilde{m}_0 + \cdots + \widetilde{m}_i \in (\ker D)_i$, with $\widetilde{m}_p \in M \otimes \wedge^p \mathfrak{h}$, then

$$0 = D(\widetilde{m}_0 + \cdots + \widetilde{m}_i) = (D_x + D_y)(\widetilde{m}_0 + \cdots + \widetilde{m}_i) = \sum_{p=0}^i D_y \widetilde{m}_p + \sum_{p=0}^{i-1} D_x \widetilde{m}_p + D_x(\widetilde{m}_i).$$

Note that the last term is the only term in $M \otimes \wedge^{i+1} \mathfrak{h}$, hence $\widetilde{m}_i \in \ker D_x$ and we can define a map

$$f_i : (\ker D)_i / (\ker D)_{i+1} \rightarrow \ker D_x; \\ \widetilde{m}_0 + \cdots + \widetilde{m}_i \mapsto \widetilde{m}_i.$$

To check the image is in $\ker D^2$, we note that $D^2 = D_x D_y + D_y D_x$ preserves the degrees. Then

$$0 = D^2(\widetilde{m}_0 + \cdots + \widetilde{m}_i) = D^2(\widetilde{m}_0) + \cdots + D^2(\widetilde{m}_i),$$

and it follows that $D^2(\widetilde{m}_i) = 0$. Now we show that f_i is injective. If $\widetilde{m}' = \widetilde{m}'_0 + \cdots + \widetilde{m}'_i \in (\ker D)_i$ such that $f_i(\widetilde{m}'_0 + \cdots + \widetilde{m}'_i) = 0$, then $\widetilde{m}'_i = 0$. Hence, $\widetilde{m}' \in (\ker D)_{i-1}$. Thus, we have an injective map

$$f := \bigoplus_i f_i : \text{gr}(\ker D) \hookrightarrow \ker D_x \cap \ker D^2. \quad \square$$

Proof of Theorem 4.2. By hypothesis, we can decompose $M \otimes S$ into $M \otimes S = \ker D^2 \oplus \text{im } D^2$. Let $U := \ker D^2$ and $V := \text{im } D^2$, then it is obvious that D maps U to U and V to V . Also, we have

$$D^2 D_x = (D_x D_y + D_y D_x) D_x = D_x (D_x D_y + D_y D_x) = D_x D^2,$$

therefore D_x also maps U to U and maps V to V . We write D' and D'_x as restrictions of D and D_x to U , and similarly write D'' and D''_x as restrictions of D and D_x to V .

Note that $H_D(M)$ is a quotient of $\ker D \subset \ker D^2 = U$, so we focus on our study to U . Since $(D')^2 = 0$, so $\text{im } D' \subset \ker D'$ and $H_D(M) = \ker D' / \text{im } D'$. Also, $(D'_x)^2 = 0$ implies $\text{im } D'_x \subset \ker D'_x$. Now for any irreducible \widetilde{W} -module ν :

$$[\nu : \ker D']_{\widetilde{W}} + [\nu : \text{im } D']_{\widetilde{W}} = [\nu : U]_{\widetilde{W}} = [\nu : \ker D'_x]_{\widetilde{W}} + [\nu : \text{im } D'_x]_{\widetilde{W}}.$$

(Note that $[\nu : U]$ is finite, since U is finite-dimensional by Lemma 3.13 of [C])

By Lemma 4.3, $[\nu : \ker D']_{\widetilde{W}} \leq [\nu : \ker D'_x]_{\widetilde{W}}$ for all ν . Hence, $[\nu : \text{im } D']_{\widetilde{W}} \leq [\nu : \text{im } D'_x]_{\widetilde{W}}$ and consequently we have an inclusion of W -modules:

$$H_D(M) = \ker D' / \text{im } D' \hookrightarrow \ker D'_x / \text{im } D'_x$$

Finally, note that

$$\ker D_x / \text{im } D_x = \ker D'_x / \text{im } D'_x \oplus \ker D''_x / \text{im } D''_x,$$

so $\ker D'_x / \text{im } D'_x$ is naturally a subspace of the following space

$$\ker D_x / \text{im } D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi.$$

Thus, the theorem is proved. \square

Remark 4.5. In fact, the hypothesis of Theorem 4.2 is satisfied when $\Omega_{\mathbf{H}_{t,c}}$ (Definition 2.2) acts semisimply on M . This can be seen from the formula of D^2 given by Equation (4.24) of [C]. For the rest of the manuscript, we will apply Theorem 4.2 in this setting.

We have an alternative proof of Theorem 3.14 in [C] for \mathbf{H} -modules M which $\Omega_{\mathbf{H}}$ acts semisimply upon:

Corollary 4.6. *Let $\mathbf{H} = \mathbf{H}_{t,c}$ with $t \neq 0$, and M be an $\Omega_{\mathbf{H}}$ -semisimple module. Suppose that $\tilde{\sigma} \otimes \beta$ is an isotypical component in $H_D(M)$, where $\tilde{\sigma}$ is an irreducible \widetilde{W} -module and β is a \mathcal{B} -character. Then the morphism $\zeta_d^* : \text{Irr}(\widetilde{W}) \rightarrow \text{Spec } \mathcal{B}$ satisfies the following condition*

$$\zeta_d^*(\tilde{\sigma}) = \beta.$$

Proof. By Equation (4.12) of [C], we have

$$[\Omega_{\mathbf{H}}, x] = 2tx, \quad [\Omega_{\mathbf{H}}, y] = -2ty. \quad (6)$$

Then it is easy to see our definition of \mathbf{A} in Section 3 is the same as that in Equations (3.4)–(3.5) of loc. cit. when $\mathbf{H}_{t,c}$ with $t \neq 0$. Suppose $\tilde{\sigma} \otimes \beta \in H_D(M)$, then by the above Remark, we can apply Theorem 4.2 so that $\tilde{\sigma} \otimes \beta \in \ker D_x / \text{im } D_x$. Hence the result follows from Theorem 3.6. \square

5. Hodge decomposition for unitarizable modules

In this section, we show that Dirac cohomology is isomorphic to Lie algebra cohomology up to a twist of a character for unitarizable modules. This follows from a Hodge decomposition theorem for the Dirac operators.

We set $\mathbf{H} := \mathbf{H}_{t,c}$ with $t, c(s) \in \mathbb{R}$ for all $s \in \mathcal{R}$. Recall that a $*$ -action is defined on \mathbf{H} in [ES]. For a suitable choice of the basis x_i, y_i in \mathfrak{h}^* and \mathfrak{h} respectively, $*$ has the property that $x_i^* = y_i, y_i^* = x_i$. A \mathbf{H} -module M has a Hermitian $*$ -invariant form if

$$(z \cdot m_1, m_2)_M = (m_1, z^* \cdot m_2)_M$$

for all $z \in \mathbf{H}$ and $m_1, m_2 \in M$. Furthermore, if the form is positive definite, we call M a $*$ -unitary module.

The main theorem of this section is the following:

Theorem 5.1. *Suppose the M is a $*$ -unitary \mathbf{H} -module. Then the injection in Theorem 4.2 is an isomorphism of \widehat{W} -modules, i.e.,*

$$H_D(M) \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi.$$

The same result holds for $H_\bullet(\mathfrak{h}, M)$.

In order to prove Theorem 5.1, we need a Hermitian form on $M \otimes S$. In fact, we can endow S with a positive definite Hermitian form $(\cdot, \cdot)_S$ by the following: Let $\{y_I := y_{i_1} \wedge \cdots \wedge y_{i_k} \mid I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}\}$ be a basis of $\wedge^\bullet \mathfrak{h} \cong S$. Then the Hermitian form on S is defined by $(y_I, y_J) := \delta_{I,J}$. One can verify that $(x_i \cdot s_1, s_2) = -(s_1, y_i \cdot s_2)$ for all $s_1, s_2 \in S$ and all $i = 1, \dots, n$.

Now one can define Hermitian form $(\cdot, \cdot)_{M \otimes S}$ on $M \otimes S$ by:

$$(m_1 \otimes s_1, m_2 \otimes s_2)_{M \otimes S} = (m_1, m_2)_M (s_1, s_2)_S.$$

Lemma 5.2. *The adjoint of the half-Dirac operator D_x is $D_x^* = -D_y$.*

Proof.

$$\begin{aligned} (D_x(m_1 \otimes s_1), m_2 \otimes s_2)_{M \otimes S} &= \sum_i (x_i \cdot m_1 \otimes y_i \cdot s_1, m_2 \otimes s_2)_{M \otimes S} \\ &= \sum_i (x_i \cdot m_1, m_2)_M (y_i \cdot s_1, s_2)_S \\ &= - \sum_i (m_1, y_i \cdot m_2)_M (s_1, x_i \cdot s_2)_S \\ &= (m_1 \otimes s_1, - \sum_i (y_i \otimes x_i) \cdot (m_2 \otimes s_2))_{M \otimes S}. \end{aligned}$$

Thus, $D_x^* = -D_y$. \square

Similarly, $D_y^* = -D_x$ and we have

$$D^* = (D_x + D_y)^* = -D_y - D_x = -D.$$

Proposition 5.3. *Let M be an irreducible \mathbf{H} -module satisfying the hypothesis given in Theorem 5.1. Then we have*

$$\ker D = \ker D^2 = \ker D_x \cap \ker D_y.$$

Proof. It is easy to see $\ker D \subset \ker D^2$. Suppose $\tilde{m} \in \ker D^2$. Then

$$0 = (D^2\tilde{m}, \tilde{m}) = (D\tilde{m}, D^*\tilde{m}) = -(D\tilde{m}, D\tilde{m}).$$

It follows that $D\tilde{m} = 0$ and hence $\tilde{m} \in \ker D$. Also, since $\tilde{m} \in \ker D = \ker D^2$,

$$0 = D^2\tilde{m} = D_x D_y \tilde{m} + D_y D_x \tilde{m}$$

and $D_x D_y \tilde{m} = -D_y D_x \tilde{m}$. Applying D_x to both sides, we have $D_x D_y D_x \tilde{m} = 0$ and

$$0 = (D_x D_y D_x \tilde{m}, D_x \tilde{m}) = -(D_y D_x \tilde{m}, D_y D_x \tilde{m}).$$

It follows that $D_y D_x \tilde{m} = 0$. Similarly, $0 = (D_y D_x \tilde{m}, \tilde{m}) = -(D_x \tilde{m}, D_x \tilde{m})$ and hence $D_x \tilde{m} = 0$, i.e., $\tilde{m} \in \ker D_x$. A similar argument also gives $\tilde{m} \in \ker D_y$. Therefore, we conclude that $\ker D^2 \subset \ker D_x \cap \ker D_y$. For the other inclusion, we note that $D_x \tilde{m} = D_y \tilde{m} = 0$ implies $D^2\tilde{m} = D_x D_y \tilde{m} + D_y D_x \tilde{m} = 0 + 0 = 0$. \square

We now prove a Hodge decomposition theorem for $M \otimes S$. First of all, we notice the following two facts:

Fact (i): We have $\operatorname{im} D_x \cap \operatorname{im} D_y = \{0\}$. If $\tilde{m} \in \operatorname{im} D_x \cap \operatorname{im} D_y$, i.e., $\tilde{m} = D_x \tilde{n} = D_y \tilde{n}'$, then $D_x \tilde{m} = D_x^2 \tilde{n} = 0 = D_x D_y \tilde{n}'$. By the same argument as in Proposition 5.3, $D_x D_y \tilde{n}' = 0$ implies $D_y \tilde{n}' = \tilde{m} = 0$.

Fact (ii): We have $\ker D_x \perp \operatorname{im} D_y$. If $\tilde{m} \in \ker D_x$, then $0 = (D_x \tilde{m}, \tilde{n}) = -(\tilde{m}, D_y \tilde{n})$. Similarly, we have $\ker D_y \perp \operatorname{im} D_x$.

Theorem 5.4 (Hodge decomposition). *Let M be a $*$ -unitary \mathbf{H} -module. Then we have the following:*

- (a) $M \otimes S = \ker D \oplus \operatorname{im} D_x \oplus \operatorname{im} D_y$.
- (b) $\ker D_x = \ker D \oplus \operatorname{im} D_x$, $\ker D_y = \ker D \oplus \operatorname{im} D_y$.

Proof. (a) By Proposition 5.3, $\ker D = \ker D^2$ and hence we only need to prove $\operatorname{im} D^2 = \operatorname{im} D_x \oplus \operatorname{im} D_y$ (recall $\operatorname{im} D_x \cap \operatorname{im} D_y = \{0\}$ by Fact (i)).

One inclusion is simple: suppose $\tilde{m} = D_x D_y \tilde{n} + D_y D_x \tilde{n} \in \operatorname{im} D^2$, then it is automatically in $\operatorname{im} D_x \oplus \operatorname{im} D_y$.

Now suppose $\tilde{m} \in \operatorname{im} D_x$, then $\tilde{m} \perp \ker D_y$. But $\ker D^2 \subset \ker D_y$ by Proposition 5.3, hence $\tilde{m} \in (\ker D^2)^\perp$. Since $M \otimes S = \ker D^2 \oplus \operatorname{im} D^2$, and our Hermitian product is positive definite, $(\ker D^2)^\perp = \operatorname{im} D^2$ and consequently $\tilde{m} \in \operatorname{im} D^2$. Similarly, one can prove $\operatorname{im} D_y \subset \operatorname{im} D^2$ and therefore (a) is proved.

(b) By Fact (ii) above, $\ker D_x \subset (\operatorname{im} D_y)^\perp$. Now (a) says $(\operatorname{im} D_y)^\perp = \ker D \oplus \operatorname{im} D_x$. So we have the inclusion $\ker D_x \subset \ker D \oplus \operatorname{im} D_x$.

For the other inclusion, note that by Proposition 5.3 we have $\ker D \subset \ker D_x$, and $\operatorname{im} D_x \subset \ker D_x$ since $D_x^2 = 0$ so the inclusion must be an equality.

The second part of the statement is analogous to the first part, and we omit the proof of it. \square

Proof of Theorem 5.1. Since $\ker D = \ker D^2$, $\ker D \cap \operatorname{im} D = \{0\}$ and hence $H_D(M) = \ker D$. By the first equality of Theorem 5.4(b),

$$H_D(M) = \ker D \cong \ker D_x / \operatorname{im} D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi.$$

Similarly,

$$H_D(M) = \ker D \cong \ker D_y / \operatorname{im} D_y \cong H_\bullet(\mathfrak{h}, M) \otimes \chi$$

by the second equality of Theorem 5.4(b). \square

6. Lie algebra cohomology for $\mathbf{H}_{t,c}$ with $t = 1$

We note that the mapping $x \mapsto \lambda x, y \mapsto \lambda y, w \mapsto w$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$ induces an algebra isomorphism

$$\mathbf{H}_{t,c} \cong \mathbf{H}_{\lambda^2 t, \lambda^2 c}.$$

Therefore, we need to consider only two cases, $t = 1$ and $t = 0$, up to equivalence.

In this section, we assume $t = 1$ and set $\mathbf{H} := \mathbf{H}_{1,c}$. Recall from [GGOR], the category \mathcal{O} for $\mathbf{H} := \mathbf{H}_{1,c}$ is defined. For any irreducible W -module σ , the standard module $M(\sigma)$ is defined to be

$$M(\sigma) = \mathbf{H} \otimes_{S(\mathfrak{h}) \rtimes \mathbb{C}[W]} \sigma.$$

As $S(\mathfrak{h}^*) \rtimes \mathbb{C}[W]$ -module, it is isomorphic to $S(\mathfrak{h}^*) \otimes \sigma$, and it has a unique irreducible quotient $L(\sigma)$. For most values of c , the standard module is irreducible, i.e., $M(\sigma) = L(\sigma)$.

The standard module $M(\sigma) \cong S(\mathfrak{h}^*) \otimes \sigma$ has an $\Omega_{\mathbf{H}}$ -eigenspace decomposition. More precisely, $\Omega_{\mathbf{H}}$ acts semisimply on $M(\sigma)$ with lowest weight vectors being of the form $1 \otimes v_\sigma \in M(\sigma)$ and $\Omega_{\mathbf{H}}$ acting by the scalar a_0 . By Equation (6), if $m \in M$ is an eigenvector of $\Omega_{\mathbf{H}}$ of eigenvalue r , then $x \cdot m$ is also an eigenvector of eigenvalue $r + 2$ for any $x \in \mathfrak{h}^*$. Therefore, by letting $a_k = a_0 + 2k$, we have an $\Omega_{\mathbf{H}}$ -eigenspace decomposition

$$M(\sigma) = \bigoplus M_{a_i} \quad \text{with} \quad M_{a_i} = S^k(\mathfrak{h}^*) \otimes \sigma.$$

since every submodule of $J \subset M(\sigma)$ is graded, i.e., $J = \bigoplus J_{a_i}$ with $J_{a_i} \subset M_{a_i}$, we conclude that $\Omega_{\mathbf{H}}$ acts semisimply on every subquotient L of $M(\sigma)$ and Theorem 4.2 applies.

Proposition 6.1. *Suppose $M \in \mathcal{O}$ has a BGG resolution, i.e., there is a exact sequence of \mathbf{H} -modules*

$$0 \rightarrow \bigoplus_{j=1}^{j_n} M(\sigma_{n,j}) \rightarrow \bigoplus_{j=1}^{j_{n-1}} M(\sigma_{n-1,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{j_0} M(\sigma_{0,j}) \rightarrow M \rightarrow 0. \quad (7)$$

Then we have inclusions of W -modules for all $i \geq 0$:

$$H_i(\mathfrak{h}^*, M) \leq \bigoplus_{j=1}^{j_i} \sigma_{i,j}.$$

Moreover, if $\sigma_{i,j} \not\cong \sigma_{i+1,j'}$ for all i, j, j' , then the above inclusions are isomorphisms.

Proof. We restrict our attention to the $R = S(\mathfrak{h}^*)$ -module structure of a \mathbf{H} -module M . By (7), there is a free R -resolution of M by:

$$0 \rightarrow R \otimes \left(\bigoplus_{j=1}^{j_n} \sigma_{n,j} \right) \rightarrow R \otimes \left(\bigoplus_{j=1}^{j_{n-1}} \sigma_{n-1,j} \right) \rightarrow \cdots \rightarrow R \otimes \left(\bigoplus_{j=1}^{j_0} \sigma_{0,j} \right) \rightarrow M \rightarrow 0.$$

Therefore, $\mathrm{Tor}_{\bullet}^R(\mathbb{C}, M)$ can be computed by tensoring $\mathbb{C} \otimes_R \bullet$ to the above sequence, which gives a complex of W -modules

$$0 \rightarrow \bigoplus_{j=1}^{j_n} \sigma_{n,j} \rightarrow \bigoplus_{j=1}^{j_{n-1}} \sigma_{n-1,j} \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{j_0} \sigma_{0,j} \rightarrow 0. \quad (8)$$

Hence, $\mathrm{Tor}_i^R(\mathbb{C}, M) \leq \bigoplus_{j=1}^{j_i} \sigma_{i,j}$ and by the usual homological algebra argument, we have $\mathrm{Tor}_i^R(\mathbb{C}, M) \cong H_i(\mathfrak{h}^*, M)$ and the first part of the proposition is proved. Assume now that $\sigma_{i,j} \neq \sigma_{i+1,j'}$ for all i, j, j' . Then the differentials in the complex (8) must be all zeros. So in this case

$$H_i(\mathfrak{h}^*, M) \cong \mathrm{Tor}_i^R(\mathbb{C}, M) \cong \bigoplus_{j=1}^{j_i} \sigma_{i,j}.$$

Therefore, the proposition is proved. \square

It follows from the Poincaré duality (Proposition 2.7) for $\mathfrak{c} = \mathfrak{h}^*$ that

$$H^\bullet(\mathfrak{h}^*, M) \otimes \chi \cong H_\bullet(\mathfrak{h}^*, M) \otimes \chi^{-1} \leq \bigoplus_{i=0}^n \bigoplus_{j=1}^{j_i} \sigma_{i,j} \otimes \chi^{-1}, \quad (9)$$

and the above inclusion is an isomorphism if $\sigma_{i,j} \not\cong \sigma_{i+1,j'}$ for all i, j, j' .

The following corollary gives precisely the Dirac cohomology of a standard module $M(\sigma)$. This refines Proposition 5.6 of [C]:

Corollary 6.2. *The standard module $M(\sigma)$ has Dirac cohomology*

$$H_D(M(\sigma)) \cong \sigma \otimes \chi^{-1}.$$

Proof. The BGG resolution of $M = M(\sigma)$ is obviously

$$0 \rightarrow M(\sigma) \rightarrow M \rightarrow 0.$$

By Proposition 6.1, we have

$$H_i(\mathfrak{h}^*, M(\sigma)) = 0 \quad \text{for } i > 0, \quad H_0(\mathfrak{h}^*, M(\sigma)) = \sigma.$$

Since $M(\sigma)$ is $\Omega_{\mathbf{H}}$ -semisimple, we can apply Theorem 4.2 and Equation (9) to get $H_D(M(\sigma)) \leq H_\bullet(\mathfrak{h}^*, M(\sigma)) \otimes \chi^{-1} = \sigma \otimes \chi^{-1}$. On the other hand, Proposition 5.6 of [C] says $\sigma \otimes \chi^{-1} \in H_D(M(\sigma))$. Therefore the result follows. \square

As another application of Proposition 6.1, we consider the finite-dimensional \mathbf{H} -modules $L(\text{triv})$ studied in [BEG]. In fact, Proposition 1.6 of [BEG] gives the BGG resolution of $L(\text{triv})$ by

$$0 \rightarrow M(\wedge^n \mathfrak{h}^*) \rightarrow M(\wedge^{n-1} \mathfrak{h}^*) \rightarrow \cdots \rightarrow M(\wedge^1 \mathfrak{h}^*) \rightarrow M(\text{triv}) \rightarrow L(\text{triv}) \rightarrow 0.$$

For any Weyl group W , all $\wedge^i \mathfrak{h}^* \cong \wedge^i \mathfrak{h}$ are distinct irreducible modules, so Proposition 6.1 applies and

$$H_i(\mathfrak{h}^*, L(\text{triv})) \cong \wedge^i \mathfrak{h}.$$

One would like to relate the above result with Dirac cohomology $H_D(L(\text{triv}))$. Indeed, by Theorem 4.2 and Equation (9), we have

$$H_D(L(\text{triv})) \leq H^\bullet(\mathfrak{h}^*, L(\text{triv})) \otimes \chi = H_\bullet(\mathfrak{h}^*, L(\text{triv})) \otimes \chi^{-1} = \wedge^\bullet \mathfrak{h} \otimes \chi^{-1}.$$

To see the inclusion is an isomorphism, we need the following:

Theorem 6.3. *Let M be an $\Omega_{\mathbf{H}}$ -semisimple \mathbf{H} -module with an eigenspace decomposition $M = \bigoplus_\lambda M_\lambda$. If M satisfies the parity condition, i.e.,*

$$[H^{\text{even}}(\mathfrak{h}^*, M) : H^{\text{odd}}(\mathfrak{h}^*, M)]_W = 0,$$

then $H_D(M) \cong H^\bullet(\mathfrak{h}^, M) \otimes \chi$. The analogous statement holds also for $H_\bullet(\mathfrak{h}, M)$.*

Applying Theorem 6.3 to the finite-dimensional modules $L(\text{triv})$, we get the following:

Corollary 6.4. *Let $M = L(\text{triv})$ be a finite-dimensional \mathbf{H} -module in [BEG], then*

$$H_D(L(\text{triv})) \cong \wedge^\bullet \mathfrak{h} \otimes \chi^{-1}$$

Proof. Since all $H^i(\mathfrak{h}^*, M) = H_{n-i}(\mathfrak{h}^*, M) \otimes \det_{\mathfrak{h}} = \wedge^{n-i} \mathfrak{h} \otimes \wedge^n \mathfrak{h} \cong \wedge^i \mathfrak{h}$ are distinct W -modules, the hypothesis of Theorem 6.3 is satisfied. Hence the result follows. \square

As before, we only prove Theorem 6.3 for $H^\bullet(\mathfrak{h}^*, M)$. The proof is analogous to that of [HX]. However, the proof given there requires the fact that weight spaces of $M \otimes S$ are finite-dimensional, yet in our setting the multiplicities of irreducible W -modules can be infinite. So we have to decompose $M \otimes S$ into a direct sum of finite-dimensional W -modules, where D_x, D_y and therefore D will act on each summand individually.

Identify S with $\wedge^\bullet \mathfrak{h}$ as vector spaces. Under the hypothesis, we have the $\Omega_{\mathbf{H}}$ -eigenspace decomposition of $M = \bigoplus_a M_a$. Define a bi-grading on $M \otimes S = \bigoplus_{a,j} (M \otimes S)_{a,j}$ by

$$(M \otimes S)_{a,l} = M_a \otimes \wedge^l \mathfrak{h}. \tag{10}$$

By (6) and the definitions of D_x and D_y , D_x maps $(M \otimes S)_{a,l}$ to $(M \otimes S)_{a+2,l+1}$, and D_y maps $(M \otimes S)_{a,l}$ to $(M \otimes S)_{a-2,l-1}$.

Definition 6.5. Given the bi-grading of $M \otimes S$ in (10), set

$$U_r := \bigoplus_{a-2l=r} M_{a,l}, \quad U_r^+ := \bigoplus_{a-4k=r} M_{a,2k}, \quad U_r^- := \bigoplus_{a-4k+2=r} M_{a,2k-1}.$$

We can express U_r as a complex :

$$0 \rightarrow M_{r,0} \xrightarrow{D_x} M_{r+2,1} \xrightarrow{D_x} \dots \xrightarrow{D_x} M_{r+2n,n} \rightarrow 0$$

We define

$$H^i(U_r, D_x) := \ker(D_x : M_{r+2i,i} \rightarrow M_{r+2i+2,i+1}) / \text{im}(D_x : M_{r+2i-2,i-1} \rightarrow M_{r+2i,i}).$$

Then we have

$$\ker D_x / \text{im } D_x = \bigoplus_r H^\bullet(U_r, D_x) = \bigoplus_r (H^{\text{even}}(U_r, D_x) \oplus H^{\text{odd}}(U_r, D_x)).$$

Replacing D_x with D_y , we define $H_i(U_r, D_y)$ analogously.

As for D , let

$$D_r : U_r \rightarrow U_r, \quad D_r^\pm : U_r^\pm \rightarrow U_r^\mp$$

be the restriction of D on U_r , and

$$H_D(U_r) := \ker D_r / (\ker D_r \cap \text{im } D_r), \quad H_D^\pm(U_r) := \ker D_r^\pm / (\ker D_r^\pm \cap \text{im } D_r^\mp).$$

Then

$$H_D(M) = \bigoplus_r H_D(U_r) = \bigoplus_r (H_D^+(U_r) \oplus H_D^-(U_r)).$$

So we can work on the finite-dimensional space U_r instead of the whole space $M \otimes S$.

Lemma 6.6. *In the above setting, we have the following:*

(a) *inclusions of \widetilde{W} -modules*

$$H_D^+(U_r) \hookrightarrow H^{\text{even}}(U_r, D_x), \quad H_D^-(U_r) \hookrightarrow H^{\text{odd}}(U_r, D_x); \quad \text{and}$$

(b) *identity for virtual \widetilde{W} -modules,*

$$H_D^+(U_r) - H_D^-(U_r) = H^{\text{even}}(U_r, D_x) - H^{\text{odd}}(U_r, D_x).$$

Proof. (a) Follows directly from Theorem 4.2 when we restrict to U_r .

(b) By hypothesis and Theorem 2.11(e), $D_r^+ \circ D_r^-$ and $D_r^- \circ D_r^+$ are both semisimple. Then using linear algebra, e.g., Proposition 5.2 of [HX], one can conclude that $H_D^+(U_r) - H_D^-(U_r)$ is equal to $U_r^+ - U_r^-$ as W -modules. However, $U_r^+ - U_r^-$ is the Euler characteristic of the half-Dirac complex in U_r , hence it is also equal to $H^{\text{even}}(U_r, D_x) - H^{\text{odd}}(U_r, D_x)$. \square

Proof of Theorem 6.3. Recall that

$$\begin{aligned} \ker D_x / \operatorname{im} D_x &= \bigoplus_r (H^{\text{even}}(U_r, D_x) \oplus H^{\text{odd}}(U_r, D_x)) \\ &\cong (H^{\text{even}}(\mathfrak{h}^*, M) \otimes \chi) \oplus (H^{\text{odd}}(\mathfrak{h}^*, M) \otimes \chi). \end{aligned}$$

By hypothesis, the two summands on the right-hand side are disjoint. Also,

$$\bigoplus_r H^{\text{even}}(U_r, D_x) = H^{\text{even}}(\mathfrak{h}^*, M) \otimes \chi, \quad \bigoplus_r H^{\text{odd}}(U_r, D_x) = H^{\text{odd}}(\mathfrak{h}^*, M) \otimes \chi$$

implies $H^{\text{even}}(U_r, D_x)$ and $H^{\text{odd}}(U_r, D_x)$ are disjoint as well. By Lemma 6.6(b), we have

$$H_D^+(U_r) \supset H^{\text{even}}(U_r, D_x) \text{ and } H_D^-(U_r) \supset H^{\text{odd}}(U_r, D_x).$$

It follows from Lemma 6.6(a), the inclusions must be equalities and we have

$$H_D^+(U_r) = H^{\text{even}}(U_r, D_x), \quad H_D^-(U_r) = H^{\text{odd}}(U_r, D_x).$$

Hence, $H_D(U_r) = H^\bullet(U_r, D_x)$. Adding up the r 's, we have $H_D(M) = \ker D_x / \operatorname{im} D_x$ and the result follows. \square

To conclude this section, we make the following:

Conjecture 6.7. *For any irreducible modules $L(\sigma)$, we have the \widetilde{W} -module isomorphisms*

$$H_D(L(\sigma)) \cong H^\bullet(\mathfrak{h}^*, L(\sigma)) \otimes \chi \cong H_\bullet(\mathfrak{h}, L(\sigma)) \otimes \chi.$$

7. Lie algebra cohomology for $\mathbf{H}_{t,c}$ with $t = 0$

In this section, we set $\mathbf{H} := \mathbf{H}_{0,c}$. We study the center $Z(\mathbf{H})$ of \mathbf{H} first of all. As opposed to the $t = 1$ case whose center only contains the constants, \mathbf{H} has a large center as shown in the following lemma:

Lemma 7.1. *The center $Z(\mathbf{H})$ of \mathbf{H} is a free $S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W$ -module of rank $|W|$. More precisely, there exists $\gamma_i \in \mathbf{H}^{\mathbb{C}^*}$ (recall the \mathbb{C}^* -action on \mathbf{H} in Definition 3.1) with $\gamma_1 = 1$ such that*

$$Z(\mathbf{H}) \cong \bigoplus_{i=1}^{|W|} (S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W) \cdot \gamma_i.$$

Proof. The first statement of the lemma is in Proposition 4.15 of [EG]. For the last statement, note that by the commutation relations of \mathbf{H} , $a \in \mathbf{H}^{\mathbb{C}^*} \Leftrightarrow \operatorname{gr}(a) \in \operatorname{gr}(\mathbf{H})^{\mathbb{C}^*}$, where the grading is the one given in Section 3. So we only need to prove $\operatorname{gr}(\gamma_i) \in \operatorname{gr}(\mathbf{H})^{\mathbb{C}^*}$ for all i . In this setting, the proof of Proposition 4.15 [EG] gives the structure of $\operatorname{gr}(\gamma_i)$ precisely, with all $\operatorname{gr}(\gamma_i)$ being chosen to be in $\operatorname{gr}(\mathbf{H})^{\mathbb{C}^*}$. Hence, the Lemma follows. \square

Let \mathfrak{m}_+ be the maximal ideal of $S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W$ consisting of all non-constant polynomials. The *restricted rational Cherednik algebra* $\overline{\mathbf{H}}$ is defined to be the quotient $\mathbf{H}/\mathfrak{m}_+\mathbf{H}$. Section 6 of [G2] says that $\overline{\mathbf{H}}$ is of rank $|W|^3$, and the center of $\overline{\mathbf{H}}$ is $Z(\overline{\mathbf{H}}) = Z(\mathbf{H})/\mathfrak{m}_+Z(\mathbf{H})$. Therefore, Lemma 7.1 says $\dim Z(\overline{\mathbf{H}}) = |W|$.

Following to [G1], a family of \mathbf{H} -modules called *baby Verma modules* is defined as follows. For each $\sigma \in \text{Irr}(W)$, set

$$\overline{M}(\sigma) := \mathbf{H} \otimes_{(S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W) \rtimes \mathbb{C}[W]} \sigma,$$

where $S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W$ acts by evaluating at $(0, 0)$. Since \mathfrak{m}_+ annihilates $\overline{M}(\sigma)$, so $\overline{M}(\sigma)$ becomes an $\overline{\mathbf{H}}$ -module. By the results in [G1], it has a unique irreducible head $\overline{L}(\sigma)$. Clearly, the dimension of $\overline{L}(\sigma)$ is bounded above by the dimension of $\overline{M}(\sigma)$ which is equal to $|W| \cdot \dim \sigma$. Also, $\overline{L}(\sigma) \cong \overline{L}(\lambda)$ if and only if $\sigma \cong \lambda$, and $\{\overline{L}(\sigma) \mid \sigma \in \text{Irr}(W)\}$ is a complete set of irreducible $\overline{\mathbf{H}}$ -modules up to equivalence.

The central character of all such $\overline{L}(\sigma)$ defines a map

$$\Theta : \text{Irr}(W) \rightarrow \text{Spec } Z(\overline{\mathbf{H}}).$$

The elements in the pre-image of Θ of an element in $\text{Spec } Z(\overline{\mathbf{H}})$ form a *Calogero–Moser cell*. This defines a partition of W . The Calogero–Moser cell carries information on the smoothness of the variety $X_c := \text{Spec } Z(\mathbf{H})$. More precisely, if X_c is smooth, then all $\overline{L}(\sigma)$ has dimension $|W| \cdot \dim \sigma$, and every Calogero–Moser cell is a singleton.

By Remark 4.10 of [C], all $\overline{L}(\sigma)$ are $\Omega_{\mathbf{H}}$ -semisimple, and Theorem 4.2 of inclusion of Dirac cohomology into Lie algebra cohomology can be applied to M (regarded as an \mathbf{H} -module). We set

$$\mathcal{B} := Z(\mathbf{H})^{\mathbb{C}^*} \otimes 1 = \left(\bigoplus_{i=1}^{|W|} \bigoplus_{k \geq 0} (S^k(\mathfrak{h})^W \otimes S^k(\mathfrak{h}^*)^W) \cdot \gamma_i \right) \otimes 1.$$

Then \mathcal{B} is in $(Z(\mathbf{H}) \otimes 1) \cap \mathbf{A}$, so $\mathcal{B} \subset \ker \delta_d \subset \mathbf{A}^W$ (or $\mathcal{B} \subset \ker \delta_{\partial} \subset \mathbf{A}^W$) satisfies the hypothesis of Theorem 3.7. So we can define the homomorphism

$$\zeta_d : Z(\mathbf{H})^{\mathbb{C}^*} \otimes 1 \rightarrow \Delta(\mathbb{C}[\widetilde{W}^{\vee}]).$$

Proposition 7.2. *We have*

$$(\mathfrak{m}_+Z(\mathbf{H}))^{\mathbb{C}^*} \otimes 1 \in \text{im } \delta_d,$$

hence ζ_d descends to the homomorphism:

$$\zeta_d : Z(\mathbf{H})^{\mathbb{C}^*} \otimes 1 / (\mathfrak{m}_+Z(\mathbf{H}))^{\mathbb{C}^*} \otimes 1 \cong Z(\overline{\mathbf{H}}) \rightarrow \Delta(\mathbb{C}[\widetilde{W}^{\vee}]).$$

Proof. By Lemma 7.1, all elements in $(\mathfrak{m}_+Z(\mathbf{H}))^{\mathbb{C}^*}$ are of the form $\sum_i f_i g_i \gamma_i$, where $f_i \in S(\mathfrak{h})_+^W$, $g_i \in S(\mathfrak{h}^*)_+^W$ are of the same (positive) degree. We first show that $g \otimes 1 \in \text{im } \delta_d$ for any $g \in S(\mathfrak{h}^*)_+^W$: Consider the map

$$\delta_d : (S(\mathfrak{h}^*) \otimes \wedge^{\bullet} \mathfrak{h}^*)^W \rightarrow (S(\mathfrak{h}^*) \otimes \wedge^{\bullet} \mathfrak{h}^*)^W$$

given by $\delta_d(a) = D_x a - \epsilon(a)D_x$. Since $g \in Z(\mathbf{H})$, it is easy to see that $g \otimes 1 \in \ker \delta_d$. Also, from the proof of Proposition 3.5, we have seen that the map δ_d is an exact differential on non-zero degrees. Since g is of positive degree, $g \otimes 1 \in \ker \delta_d = \text{im } \delta_d$ as follows.

If $g_i \otimes 1 = D_x a - \epsilon(a)D_x$, then for $f_i g_i \gamma_i \otimes 1 \in (\mathfrak{m}_+ Z(\mathbf{H}))^{\mathbb{C}^*}$, and

$$\begin{aligned} f_i g_i \gamma_i \otimes 1 &= (f_i \otimes 1)(g_i \otimes 1)(\gamma_i \otimes 1) \\ &= (f_i \otimes 1)(D_x a - \epsilon(a)D_x)(\gamma_i \otimes 1) \\ &= D_x(f_i \otimes 1)a(\gamma_i \otimes 1) - \epsilon((f_i \otimes 1)a(\gamma_i \otimes 1))D_x, \end{aligned}$$

where the last equality comes from that fact that both $f_i \otimes 1, \gamma_i \otimes 1$ commute with D_x . Hence, $f_i g_i \gamma_i \otimes 1 \in \text{im } \delta_d$ as required. \square

The morphism $\zeta_d^* : \text{Irr}(\widetilde{W}) \rightarrow \text{Spec} Z(\overline{\mathbf{H}})$ relates the central characters of an irreducible $\overline{\mathbf{H}}$ -module and its Lie algebra cohomology.

Theorem 7.3. *Let M be an irreducible $\overline{\mathbf{H}}$ -module with central character $\beta \in \text{Spec } Z(\overline{\mathbf{H}})$. Suppose ν is an irreducible W -module appearing in $H^\bullet(\mathfrak{h}^*, M)$. Then we have*

$$\beta = \zeta_d^*(\nu \otimes \chi).$$

Proof. The proof is similar to Theorem 3.7. \square

We can now relate the maps Θ and ζ_d^* . Combined with Theorem 4.2, part (b) of the following Corollary gives an alternative proof of Corollary 5.10 of [C].

Corollary 7.4.

(a) For any $\sigma \in \text{Irr}(W)$,

$$\Theta(\sigma) = \zeta_d^*(\sigma \otimes \chi^{-1}).$$

(b) If $H^\bullet(\mathfrak{h}^*, \overline{L}(\sigma)) \cong \bigoplus_i \nu_i$ as W -modules with $\nu_i \in \text{Irr}(W)$, then all such $\nu_i \otimes \det_{\mathfrak{h}^*}$'s belong to the same Calogero–Moser cell.

Proof. (a) We claim that the W -module $\sigma \otimes \det_{\mathfrak{h}} \cong (1 \otimes \sigma) \otimes \wedge^{\dim \mathfrak{h}} \mathfrak{h}$ in $\overline{L}(\sigma) \otimes \wedge^\bullet \mathfrak{h}$ is in $H^\bullet(\mathfrak{h}^*, M)$. Indeed, since $\wedge^{\dim \mathfrak{h}} \mathfrak{h}$ is in its top degree, $D_x((1 \otimes \sigma) \otimes \wedge^{\dim \mathfrak{h}} \mathfrak{h}) = 0$. Also, noting that $(1 \otimes \sigma)$ has zero degree on its \mathfrak{h} and \mathfrak{h}^* factor, it cannot be in the image of $\text{im } D_x$. So $(1 \otimes \sigma) \otimes \wedge^{\dim \mathfrak{h}} \mathfrak{h}$ must be non-zero in $\ker D_x / \text{im } D_x = H^\bullet(\mathfrak{h}^*, M)$.

Taking $M = \overline{L}(\sigma)$ and $\nu = \sigma \otimes \det_{\mathfrak{h}}$ in Theorem 7.3, we have

$$\Theta(\sigma) = \zeta_d^*(\sigma \otimes \det_{\mathfrak{h}} \otimes \chi) = \zeta_d^*(\sigma \otimes \chi^{-1}).$$

(b) Suppose $\nu \in H^\bullet(\mathfrak{h}^*, \overline{L}(\sigma))$. By Theorem 7.3,

$$\Theta(\sigma) = \zeta_d^*(\nu \otimes \chi).$$

On the other hand, (a) says $\zeta_d^*(\nu \otimes \chi) = \zeta_d^*((\nu \otimes \det_{\mathfrak{h}^*}) \otimes \chi^{-1}) = \Theta(\nu \otimes \det_{\mathfrak{h}^*})$. Hence $\nu \otimes \det_{\mathfrak{h}^*}, \sigma$ are in the preimage of the same element in $\text{Spec } Z(\overline{\mathbf{H}})$, i.e., they are in the same Calogero–Moser cell. \square

Since Theorem 3.7 holds for both D_x and D_y , the above results also hold if we replace $H^\bullet(\mathfrak{h}^*, M)$ with $H_\bullet(\mathfrak{h}, M)$. We skip the proofs here.

As in the case of $t = 1$, we make the following conjecture:

Conjecture 7.5. *For irreducible $\overline{\mathbf{H}}$ -module $\overline{L}(\sigma)$, there are \widetilde{W} -module isomorphisms*

$$H_D(\overline{L}(\sigma)) \cong H^\bullet(\mathfrak{h}^*, \overline{L}(\sigma)) \otimes \chi \cong H_\bullet(\mathfrak{h}, \overline{L}(\sigma)) \otimes \chi.$$

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