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DOUBLE BRUHAT CELLS AND SYMPLECTIC GROUPOIDS

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Abstract. Let G be a connected complex semisimple Lie group, equipped with a standard multiplicative Poisson structure $\pi_{\rm st}$ determined by a pair of opposite Borel subgroups (B, B_-). We prove that for each v in the Weyl group W of G, the double Bruhat cell $G^{v,v} = BvB \cap B_-vB_-$ in G, together with the Poisson structure $\pi_{\rm st}$, is naturally a Poisson groupoid over the Bruhat cell BvB/B in the flag variety G/B. Correspondingly, every symplectic leaf of $\pi_{\rm st}$ in $G^{v,v}$ is a symplectic groupoid over BvB/B. For $u, v \in W$, we show that the double Bruhat cell $(G^{u,v}, \pi_{\rm st})$ has a naturally defined left Poisson action by the Poisson groupoid $(G^{u,u}, \pi_{\rm st})$ and a right Poisson action by the Poisson groupoid $(G^{v,v}, \pi_{\rm st})$, and the two actions commute. Restricting to symplectic leaves of $\pi_{\rm st}$, one obtains commuting left and right Poisson actions on symplectic leaves in $G^{u,v}$ by symplectic leaves in $G^{u,u}$ and $G^{v,v}$ as symplectic groupoids.

1. Introduction and statements of results

1. Introduction

Let G be a connected complex semisimple Lie group, and let (B, B_-) be a pair of opposite Borel subgroups of G. It is well-known [CP], [ES], [HL], [HKKR], [KZ] that the choice of (B, B_-), together with that of a symmetric non-degenerate invariant bilinear form on the Lie algebra of G, determine a *standard multiplicative* Poisson structure $\pi_{\rm st}$ on G (see §1 for details), and that the complex Poisson Lie group $(G, \pi_{\rm st})$ is the semi-classical limit of the quantized function algebra $\mathbb{C}_q[G]$ of G. The Poisson structure $\pi_{\rm st}$ is invariant under left and right translation by elements of the maximal torus $T = B\cap B_-$ of G, and it is well-known [HL], [HKKR] that the *double Bruhat cells*

$$G^{u,v} = BuB \cap B_-vB_-, \quad u, v \in W,$$

where W is the Weyl group of (G, T), are precisely all the T*-leaves* of $(G, \pi_{\rm st})$, i.e., submanifolds of G of the form $\bigcup_{t\in T} \Sigma t$, where Σ is a symplectic leaf of $\pi_{\rm st}$ in G (see [LM2, §2] on some basic facts of $\mathbb{T}$-leaves, where $\mathbb{T}$ is any torus). Double Bruhat cells have been studied extensively and have served as motivating examples

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of the theories of total positivity and cluster algebras (see [BFZ], [FZ], [GY2] and references therein). When G is simply connected, symplectic leaves of $\pi_{\rm st}$ in each double Bruhat cell $G^{u,v}$ are explicitly described in [KZ].

The Poisson structure $\pi_{\rm st}$ on G projects to a well-defined Poisson structure π_1 on the flag variety G/B, and each *Bruhat cell* $BvB/B \subset G/B$, where $v \in W$, is a Poisson subvariety of $(G/B, \pi_1)$. In this paper, we show that for every $v \in W$ and any representative $\bar{v}$ of v in the normalizer subgroup $N_G(T)$ of T in G, the Poisson variety $(G^{v,v}, \pi_{\rm st})$ has a naturally defined groupoid structure over BvB/B, giving rise to a *Poisson groupoid* $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ over the Poisson variety $(BvB/B, \pi_1)$. The symplectic leaf $\Sigma^{\bar{v}}$ of $\pi_{\rm st}$ through $\bar{v}$ is then shown to be a Lie sub-groupoid of $G^{\bar{v},\bar{v}}$, becoming thus a symplectic groupoid over $(BvB/B, \pi_1)$. The groupoid structure on $G^{v,v}$ depends on the choice of $\bar{v} \in N_G(T)$ (thus the notation $G^{\bar{v},\bar{v}}$), but different choices give isomorphic Poisson groupoids. For $u, v \in W$ and respective representatives $\bar{u}, \bar{v} \in N_G(T)$, we show that the Bruhat cell $(G^{u,v}, \pi_{\rm st})$ has a left Poisson action by the Poisson groupoid $(G^{\bar{u},\bar{u}}, \pi_{\rm st})$ and a right Poisson action by the Poisson groupoid $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$, and the two actions commute. The two actions are then shown to restrict to commuting Poisson actions of the symplectic groupoids $(\Sigma^{\bar{u}}, \pi_{\rm st})$ and $(\Sigma^{\bar{v}}, \pi_{\rm st})$ on every symplectic leaf in $G^{u,v}$.

2. Statements of main results

Let $v \in W$, and let $\bar{v}$ be any representative of v in $N_G(T)$. Let $C_{\bar{v}} = N\bar{v} \cap \bar{v}N_-$, where N and N_- are respectively the uniradicals of B and B_-. One then has the unique decompositions $BvB = C_{\bar{v}}B$ and $B_-vB_- = B_-C_{\bar{v}}$ and the isomorphism

$$C_{\bar{v}} \xrightarrow{\sim} BvB/B, \; c \mapsto c.B, \quad c \in C_{\bar{v}}.$$

Writing an element $g \in G^{v,v}$ uniquely as $g = cb = b_-c'$, where $b \in B$, $b_- \in B_-$, and $c, c' \in C_{\bar{v}}$, the groupoid structure on $G^{v,v}$ over BvB/B is defined as follows:

$$\begin{aligned} \text{source map}: &\ \theta_{\bar{v}}(g) = g.B = c.B, \\ \text{target map}: &\ \tau_{\bar{v}}(g) = c'.B, \\ \text{inverse map}: &\ \iota_{\bar{v}}(g) = c'b^{-1} = b_-^{-1}c, \\ \text{identity bisection}: &\ \epsilon_{\bar{v}}(c.B) = c \in C_{\bar{v}} \subset G^{v,v}, \\ \text{multiplication}: &\ \mu_{\bar{v}}(g,h) = cbb' = b_-b'_-c'', \text{if } h = c'b' = b'_-c'', \\ &\qquad\qquad \text{where } b' \in B, b'_- \in B_-, c'' \in C_{\bar{v}}. \end{aligned}$$

We will denote by $G^{\bar{v},\bar{v}} \rightrightarrows BvB/B$, or simply $G^{\bar{v},\bar{v}}$, the double Bruhat cell $G^{v,v}$ with the groupoid structure thus defined. For another $u \in W$ and any representative $\bar{u} \in N_G(T)$ of u, define

$$\begin{aligned} \varpi: Guv \to BuB/B, &\quad \varpi(cb) = c.B, \quad b \in B, c \in C_{\bar{u}}, \\ \varpi': G^{u,v} \to BvB/B, &\quad \varpi'(b_-c') = c'.B, \quad b_- \in B_-, c' \in C_{\bar{v}}. \end{aligned}$$

The main results of the paper, Theorem 13, Theorem 15, and Theorem 24, can now be summarized as follows: let $u, v \in W$ and let $\bar{u}$ and $\bar{v}$ be any representatives of u and v in $N_G(T)$, respectively.

Main Theorems. (1) *The pair* $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ *is a Poisson groupoid over the Poisson manifold* $(BvB/B, \pi_1)$*, which, by restriction, also makes the symplectic leaf* $\Sigma^{\bar{v}}$ *of* $\pi_{\rm st}$ *through* $\bar{v}$ *into a symplectic groupoid over* $(BvB/B, \pi_1)$.

(2) *There is a natural left Poisson action of the Poisson groupoid* $(G^{\bar{u},\bar{u}}, \pi_{\rm st})$ *on* $(G^{u,v}, \pi_{\rm st})$ *with moment map* ϖ *and a natural right Poisson action of the Poisson groupoid* $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ *on* $(G^{u,v}, \pi_{\rm st})$ *with moment map* ϖ'*. The two actions commute, and they restrict to Poisson actions of the symplectic groupoids* $(\Sigma^{\bar{u}}, \pi_{\rm st})$ *and* $(\Sigma^{\bar{v}}, \pi_{\rm st})$ *on every symplectic leaf* $\Sigma^{u,v}$ *of* $\pi_{\rm st}$ *in* $G^{u,v}$.

We remark that for any symplectic leaf $\Sigma^{u,v}$ in $G^{u,v}$, the moment maps

$$\varpi|_{\Sigma^{u,v}} : (\Sigma^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \quad \text{and} \quad \varpi'|_{\Sigma^{u,v}} : (\Sigma^{u,v}, \pi_{\rm st}) \to (BvB/B, -\pi_1)$$

for the Poisson actions of the symplectic groupoids $(\Sigma^{\bar{u}}, \pi_{\rm st})$ and $(\Sigma^{\bar{v}}, \pi_{\rm st})$ on $(\Sigma^{u,v}, \pi_{\rm st})$ are symplectic realizations [W4], [X1] only in the sense that they are Poisson submersions, but in general are not surjective (see Lemma 7, Lemma 8, and Remark 14). More precisely, $\varpi(\Sigma^{u,v}) = BuB/B$ if and only if $u \leq v$ in the Bruhat order, and $\varpi'(\Sigma^{u,v}) = BvB/B$ if and only if $v \leq u$.

We in fact construct a Poisson groupoid $((G/B) \times B_-, \pi)$ over $(G/B, \pi_1)$, where the groupoid structure is that of the action groupoid defined by the right action of B_- on G/B given by

$$(g.B) \cdot b_- = (b_-^{-1} g).B, \quad g \in G, b_- \in B_-,$$

and the Poisson structure π is a *mixed product Poisson structure* in the sense of [LM1], or, more precisely, π is the sum of the product Poisson structure $\pi_1 \times (\pi_{\rm st}|_{B_-})$ on $(G/B) \times B_-$ and a certain mixed term determined by the action of B on G/B by left translation and by the action of B_- on itself by left translation. For each $v \in W$ and a representative $\bar{v}$ of v in $N_G(T)$, the Poisson groupoid $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ is then realized as a Poisson subgroupoid of the Poisson groupoid $((G/B) \times B_-, \pi)$ over $(G/B, \pi_1)$ via a Poisson embedding $I_{\bar{v}} : (B_- v B_-, \pi_{\rm st}) \to ((G/B) \times B_-, \pi)$ (see §2 and §4 for detail). Using the embeddings $I_{\bar{v}}$, we also interpret the Fomin–Zelevinsky twist map on double Bruhat cells [FZ], [KZ] in terms of the inverse map of the groupoid $(G/B) \times B_-$ over G/B. See Remark 11.

The Poisson groupoid $((G/B) \times B_-, \pi)$ over $(G/B, \pi_1)$ is a special case of a general construction of action Poisson groupoids associated to quasitriangular r-matrices (see §2). More precisely, given a Lie algebra $\mathfrak{g}$, a quasitriangular r-matrix r on $\mathfrak{g}$, and a Lie algebra action of $\mathfrak{g}$ on a manifold Y such that the stabilizer subalgebra of $\mathfrak{g}$ at each point of Y is coisotropic with respect to the symmetric part of r, Li-Bland and Meinrenken defined in [L-BM] an *action Courant algebroid* over Y with two transversal Dirac structures. In §2, we construct a pair of dual Poisson groupoids which integrate the two transversal Dirac structures in the sense that they have the two Dirac structures as their Lie bialgebroids (see Corollary 5 and Remark 5 for detail). Applying the general construction to the semi-simple Lie algebra $\mathfrak{g}$ and the standard quasitriangular r-matrix $r_{\rm st}$ on $\mathfrak{g}$ (see §1), we obtain the action Poisson groupoid $((G/B) \times B_-, \pi)$ over $(G/B, \pi_1)$.

Symplectic groupoids were introduced by Karasev [K] and Weinstein [W3] to study singular foliations in Poisson geometry, and are expected to play a key role in the problem of quantization of Poisson manifolds. While symplectic groupoids have been studied for almost three decades, not many explicit examples are known. This paper thus fills a gap in the literature by providing a large class of naturally defined *algebraic* symplectic groupoids. Relations between the symplectic groupoids of Bruhat cells described in this paper and quantum Bruhat cells [DeCKP], [LY], [Lus], [Y] will be investigated in the future.

The paper is organized as follows. Some basic facts on Poisson Lie groups and Lie bialgebras are recalled in §2. In §3 we construct a pair of dual action Poisson groupoids associated to quasitriangular r-matrices. Some properties of the standard complex semisimple Poisson Lie groups are reviewed and proved in §4. The main theorems of the paper are proved in §5 and §6, where we also generalize some results of [KZ] on the symplectic leaves of $\pi_{\rm st}$ in the double Bruhat cells to the case when G is not necessarily simply connected.

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3. Notation

Throughout this paper, vector spaces are understood to be real or complex. For a finite-dimensional vector space V, denote by $\langle\ ,\ \rangle$ the canonical pairing between V and its dual space V^*. If $r = \sum_i x_i \otimes y_i \in V \otimes V$, let $r^{21} = \sum_i y_i \otimes x_i \in V \otimes V$ and let $r^\sharp : V^* \to V$ be the linear map defined by

$$r^\sharp(\xi) = \sum_i \langle \xi, x_i\rangle y_i, \quad \xi \in V^*.$$

For a smooth (resp. complex) manifold X, denote by TX its smooth (resp. holomorphic) tangent bundle. If $k \geq 1$ is an integer and $\Phi : X \to Y$ a smooth (resp. holomorphic) map between smooth (resp. complex) manifolds X and Y, denote by the same symbol $\Phi : \wedge^k TX \to \wedge^k TY$ the differential of Φ. The space of smooth (resp. holomorphic) k-vector fields on X will be denoted by $\mathcal{V}^k(X)$, and if $V_X \in \mathcal{V}^k(X)$ and $V_Y \in \mathcal{V}^k(Y)$, denote by $(V_X, 0)$ and $(0, V_Y)$ the k-vector fields on $X \times Y$ whose values at $(x, y) \in X \times Y$ are respectively given by

$$(V_X, 0)(x, y) = i_y V_X(x) \quad \text{and} \quad (0, V_Y)(x, y) = i_x V_Y(y),$$

where $i_y : X \to X \times Y, x' \mapsto (x', y)$ for $x' \in X$, and $i_x : Y \to X \times Y, y' \mapsto (x, y')$ for $y' \in Y$. We also denote $(V_X, 0) + (0, V_Y)$ by $V_X \times V_Y$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *left action* of $\mathfrak{g}$ on a manifold Y is a Lie algebra anti-homomorphism $\lambda : \mathfrak{g} \to \mathcal{V}^1(Y)$, while a *right action* of $\mathfrak{g}$ on Y is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathcal{V}^1(Y)$. If $\lambda : G \times Y \to Y, (g, y) \mapsto gy$ is a left action of G on Y, one has the induced left action of $\mathfrak{g}$ on Y, also denoted by λ, given by

$$\lambda :\ \mathfrak{g} \to \mathcal{V}^1(Y),\ \lambda(x)_y = \frac{d}{dt}\Big|_{t=0} \exp(tx)y, \quad x \in \mathfrak{g}, y \in Y.$$

Similarly, a right Lie group action $\rho : Y \times G \to Y, (y, g) \to yg$ induces a right Lie algebra action

$$\rho :\ \mathfrak{g} \to \mathcal{V}^1(Y),\ \rho(x)_y = \frac{d}{dt}\Big|_{t=0} y \exp(tx), \quad x \in \mathfrak{g}, y \in Y.$$

For $g \in G$, the left and right translation on G by g, as well as their differentials, are respectively denoted by l_g and r_g. If $k \geq 0$ is an integer and $x \in \mathfrak{g}^{\otimes k}$, we denote by x^L and x^R the respective left- and right-invariant k-tensor fields on G whose value at the identity element e of G is x. If $\xi \in \wedge^k \mathfrak{g}^*$, we use similar notation for the left and right invariant k-forms with value ξ at e.

Throughout the paper, if (X, π) is a Poisson manifold and $X_1 \subset X$ a Poisson submanifold with respect to π, the restriction of π to X_1 will also be denoted by π unless otherwise specified.

2. Poisson Lie groups, r-matrices, and mixed product Poisson structures

We recall from [CP], [ES], [LM1] some basic facts on Poisson Lie groups and Lie bialgebras, and we refer to [LM1, §2] in particular on certain conventions on constants and signs.

1. Poisson Lie groups and Lie bialgebras

A *Lie bialgebra* is a pair $(\mathfrak{g}, \delta_\mathfrak{g})$, where $\mathfrak{g}$ is a (real or complex) finite-dimensional Lie algebra, and $\delta_\mathfrak{g} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ a linear map satisfying

$$\delta_\mathfrak{g}[x, y] = [x, \delta_\mathfrak{g}(y)] + [\delta_\mathfrak{g}(x), y], \quad x, y \in \mathfrak{g},$$

and such that the dual map $\delta_\mathfrak{g}^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$. Given a Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$, the pair $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ is also a Lie bialgebra, where $\mathfrak{g}^*$ is equipped with the Lie bracket dual to $\delta_\mathfrak{g}$, and $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ is the dual map of the Lie bracket on $\mathfrak{g}$. One calls $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ the *dual Lie bialgebra* of $(\mathfrak{g}, \delta_\mathfrak{g})$. If $(\mathfrak{g}', \delta_{\mathfrak{g}'})$ is any Lie bialgebra isomorphic to $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, we will call $((\mathfrak{g}, \delta_\mathfrak{g}), (\mathfrak{g}', \delta_{\mathfrak{g}'}))$ a *pair of dual Lie bialgebras.*

Given a Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$, the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ has a natural non-degenerate symmetric bilinear form $\langle \, , \, \rangle_\mathfrak{d}$ defined by

$$\langle x + \xi, y + \eta \rangle_\mathfrak{d} = \langle x, \eta \rangle + \langle y, \xi \rangle, \quad x,\, y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*, \tag{1}$$

and it is well-known that $\mathfrak{d}$ has a unique Lie bracket $[\, , \,]$ such that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie sub-algebras of $\mathfrak{d}$ and such that $\langle \, , \, \rangle_\mathfrak{d}$ is ad-invariant. One calls $\mathfrak{d}$ or $(\mathfrak{d}, \langle \, , \, \rangle_\mathfrak{d})$ the *double Lie algebra* of $(\mathfrak{g}, \delta_\mathfrak{g})$. Moreover, with $\delta_\mathfrak{d} : \mathfrak{d} \to \wedge^2 \mathfrak{d}$ defined by

$$\delta_\mathfrak{d}(x + \xi) = \delta_\mathfrak{g}(x) - \delta_{\mathfrak{g}^*}(\xi), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*,$$

the pair $(\mathfrak{d}, \delta_\mathfrak{d})$ is a Lie bialgebra, called the *double Lie bialgebra* of $(\mathfrak{g}, \delta_\mathfrak{g})$.

A *Poisson Lie group* is a pair (G, π_G), where G is a Lie group and π_G a Poisson bivector field on G that is *multiplicative* in the sense that the group multiplication $G \times G \to G$ is a Poisson map for the direct Poisson structure $\pi_G \times \pi_G$ on $G \times G$ and π_G on G. Given a Poisson Lie group (G, π_G), the bivector field π_G vanishes at the identity element e of G, and the linearization $d_e\pi_G : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ of π_G at e, defined by $d_e\pi_G(x) = [\widetilde{x}, \pi_G](e)$, where $\widetilde{x}$ is any local vector field such that $\widetilde{x}(e) = x$, is a Lie bialgebra structure on $\mathfrak{g}$, and one calls $(\mathfrak{g}, d_e\pi_G)$ the *Lie bialgebra of the Poisson Lie group* (G, π_G). If (G^*, π_{G^*}) is any Poisson Lie group whose Lie bialgebra is isomorphic to the dual Lie bialgebra of $(\mathfrak{g}, d_e\pi_G)$, one says that (G, π_G) and (G^*, π_{G^*}) form a *pair of dual Poisson Lie groups.*

Let (G, π_G) be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$, and let $\mathfrak{d}$ be the double Lie algebra of $(\mathfrak{g}, \delta_\mathfrak{g})$. Then $(G, \mathfrak{d})$ is a *Harish-Chandra pair* in the sense that the Lie algebra $\mathfrak{g}$ of G is a Lie subalgebra of $\mathfrak{d}$, and the Adjoint action Ad of G on $\mathfrak{g}$ extends to an action, still denoted by Ad, of G on $\mathfrak{d}$ by Lie algebra automorphisms. Indeed, one has [Dr]

$$\mathrm{Ad}_g\xi = r_{g^{-1}}\left(\pi_G^\#(g)(l_{g^{-1}}^*\xi)\right) + \mathrm{Ad}_{g^{-1}}^*\xi, \quad \xi \in \mathfrak{g}^*, \tag{2}$$

where $\mathrm{Ad}_{g^{-1}}^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ for $g \in G$.

For $\xi \in \mathfrak{g}^*$, the vector field $\mathbf{d}(\xi) = \pi_G^\#(\xi^R)$ on G is called the *dressing vector field* defined by ξ, where ξ^R is the right invariant 1-form on G with value ξ at e. By (2), one has

$$\mathbf{d}(\xi)(g) = -l_g p_\mathfrak{g}(\mathrm{Ad}_{g^{-1}}\xi), \quad \xi \in \mathfrak{g}^*, g \in G, \tag{3}$$

where $p_\mathfrak{g} : \mathfrak{d} \to \mathfrak{g}$ is the projection with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$.

A *left Poisson action of a Poisson Lie group* (G, π_G) on a Poisson manifold (Y, π_Y) is, by definition, a left Lie group action $\lambda : G \times Y \to Y$ which is also a Poisson map with respect to the product Poisson structure $\pi_G \times \pi_Y$ on $G \times Y$ and the Poisson structure π_Y on Y. Right Poisson actions of (G, π_G) are similarly defined. A *left Poisson action of a Lie bialgebra* $(\mathfrak{g}, \delta_\mathfrak{g})$ on a Poisson manifold (Y, π_Y) is a Lie algebra anti-homomorphism $\lambda : \mathfrak{g} \to \mathcal{V}^1(Y)$ such that

$$[\lambda(x), \pi_Y] = \lambda(\delta_\mathfrak{g}(x)), \quad x \in \mathfrak{g},$$

where λ also denotes the linear map $\wedge^2\mathfrak{g} \to \mathcal{V}^2(Y)$ by $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$ for $x, y \in \mathfrak{g}$. It is shown in [W2] that when a Poisson Lie group (G, π_G) is connected, a Lie group action $\lambda : G \times Y \to Y$ of G on a Poisson manifold (Y, π_Y) is a Poisson action of (G, π_G) on (Y, π_Y) if and only if the induced left Lie algebra action $\lambda : \mathfrak{g} \to \mathcal{V}^1(Y)$ is a Poisson action of the Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$ of (G, π_G) on (Y, π_Y). A similar statement holds for right Poisson Lie group actions.

2. Poisson structures defined by quasitriangular r-matrices

Recall that a quasitriangular r-matrix on a Lie algebra $\mathfrak{g}$ is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that its symmetric part $\frac{1}{2}(r + r^{21})$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ and that r satisfies the Classical Yang–Baxter Equation $\mathrm{CYB}(r) = 0$. Given a quasitriangular r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, one has the Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$, where $\delta_\mathfrak{g} : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is defined by

$$\delta_\mathfrak{g}(x) = \mathrm{ad}_x r, \quad x \in \mathfrak{g}. \tag{4}$$

A Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$ for which (4) holds for some quasitriangular r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ is said to be quasitriangular, and in such a case r is called a *quasitriangular structure* of $(\mathfrak{g}, \delta_\mathfrak{g})$.

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasitriangular r-matrix on a Lie algebra $\mathfrak{g}$, and let $\sigma : \mathfrak{g} \to \mathcal{V}^1(Y)$ be a right Lie algebra action of $\mathfrak{g}$ on a manifold Y. If $r = \sum_i x_i \otimes x_i' \in \mathfrak{g} \otimes \mathfrak{g}$, define

$$\sigma(r) = \sum_i \sigma(x_i) \otimes \sigma(x_i'),$$

which is a 2-tensor field on Y. The following observation was made in [LM1].

Lemma 1. *If the* 2*-tensor field* $\sigma(r) \in \Gamma(TY \otimes TY)$ *on* Y *is skew-symmetric, then it is Poisson, and* σ *is a (right) Poisson action of the Lie bialgebra* $(\mathfrak{g}, \delta_{\mathfrak{g}})$ *on the Poisson manifold* $(Y, \sigma(r))$*, where* $\delta_{\mathfrak{g}}$ *is defined in* (4)*.*

In the context of Lemma 1, when $\sigma(r)$ is skew-symmetric, the Poisson structure $\sigma(r)$ on Y is said to be *defined* by the Lie algebra action σ and the r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.

Remark 1. Let $s = \frac{1}{2}(r + r^{21})$ be the symmetric part of r. It is not hard to show ([LM1, §2.6]) that $\sigma(r)$ is skew-symmetric, i.e., $\sigma(s) = 0$, if and only if the stabilizer subalgebra of $\mathfrak{g}$ at every $y \in Y$ is coisotropic with respect to s. Here a subspace $\mathfrak{c}$ of $\mathfrak{g}$ is said to be *coisotropic with respect to* s if $s^{\#}(\mathfrak{c}^0) \subset \mathfrak{c}$, where $\mathfrak{c}^0 = \{\xi \in \mathfrak{g}^* : \langle \xi, \mathfrak{c} \rangle = 0\} \subset \mathfrak{g}^*$. □

Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a quasitriangular Lie bialgebra with quasitriangular r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$. Associated to r, one then [LM1, §2.3] has the Lie subalgebras

$$\mathfrak{f}_+ = \mathrm{Im}(r^\sharp) \quad \text{and} \quad \mathfrak{f}_- = \mathrm{Im}((r^{21})^\sharp) \tag{5}$$

of $\mathfrak{g}$ and the Lie bialgebras $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$ and $(\mathfrak{f}_+, -\delta_{\mathfrak{g}}|_{\mathfrak{f}_+})$, which are dual to each other under the pairing $\langle\ ,\ \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$ between $\mathfrak{f}_-$ and $\mathfrak{f}_+$ defined by

$$\langle (r^{21})^\sharp(\xi), r^\sharp(\eta) \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} = \langle \xi, r^\sharp(\eta) \rangle = \langle (r^{21})^\sharp(\xi), \eta \rangle, \quad \xi, \eta \in \mathfrak{g}^*. \tag{6}$$

If (G, π_G) is a connected Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and if F_+ and F_- are the connected Lie subgroups of G with respective Lie algebras $\mathfrak{f}_+$, $\mathfrak{f}_-$, then F_+ and F_- are Poisson Lie subgroups of (G, π_G). Moreover, denoting by the same symbol the restrictions of π_G to both F_- and F_+, $((F_-, \pi_G), (F_+, -\pi_G))$ is a pair of dual Poisson Lie groups, with $((\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-}), (\mathfrak{f}_+, -\delta_{\mathfrak{g}}|_{\mathfrak{f}_+}))$ as the corresponding pair of dual Lie bialgebras.

Example 1. The double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ of any Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is quasitriangular, with a quasitriangular structure defined by the quasitriangular r-matrix

$$r_{\mathfrak{d}} = \sum_{i=1}^{n} x_i \otimes \xi_i \in \mathfrak{d} \otimes \mathfrak{d}, \tag{7}$$

where $(x_i)_{i=1}^n$ is any basis of $\mathfrak{g}$ and $(\xi)_{i=1}^n$ the dual basis of $\mathfrak{g}^*$. In this example, the subalgebras $\mathfrak{f}_+$ and $\mathfrak{f}_-$ in (5) are respectively $\mathfrak{g}^*$ and $\mathfrak{g}$. □

Remark 2. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra with a quasitriangular structure $r \in \mathfrak{g} \otimes \mathfrak{g}$. Let $\mathfrak{d}_{\mathfrak{f}_-}$ be the double Lie algebra of $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$, and let $r_{\mathfrak{d}_{\mathfrak{f}_-}} \in \mathfrak{d}_{\mathfrak{f}_-} \otimes \mathfrak{d}_{\mathfrak{f}_-}$ be defined as in (7). Identifying $\mathfrak{f}_-^* \cong \mathfrak{f}_+$ via (6), the underlying vector space of $\mathfrak{d}_{\mathfrak{f}_-}$ is then $\mathfrak{f}_- \oplus \mathfrak{f}_+$, and the map

$$q \colon \mathfrak{d}_{\mathfrak{f}_-} \to \mathfrak{g}, \quad q(x_-, x_+) = x_- + x_+, \quad x_- \in \mathfrak{f}_-, x_+ \in \mathfrak{f}_+, \tag{8}$$

is a Lie algebra homomorphism. Moreover (see [ES, Lecture 4] and [LM1, §2.3]), $q(r_{\mathfrak{d}_{\mathfrak{f}_-}}) = r$. Thus if (Y, π_Y) is a Poisson manifold with a right Lie algebra action $\sigma : \mathfrak{g} \to \mathcal{V}^1(Y)$ such that π_Y is defined by σ and r, i.e., $\pi_Y = \sigma(r)$, then π_Y is also defined by the Lie algebra $\mathfrak{d}_{\mathfrak{f}_-}$-action $\sigma \circ q : \mathfrak{d}_{\mathfrak{f}_-} \to \mathcal{V}^1(Y)$ and the r-matrix $r_{\mathfrak{d}_{\mathfrak{f}_-}}$ on $\mathfrak{d}_{\mathfrak{f}_-}$. □

3. Mixed product Poisson structures

If $((\mathfrak{g}, \delta_{\mathfrak{g}}), (\mathfrak{g}^*, \delta_{\mathfrak{g}^*}))$ is a pair of dual Lie bialgebras and if (X, π_X) and (Y, π_Y) are Poisson manifolds with respective right and left Poisson actions

$$\rho\colon \mathfrak{g}^* \to \mathcal{V}^1(X) \quad \text{and} \quad \lambda\colon \mathfrak{g} \to \mathcal{V}^1(Y)$$

by Lie bialgebras, the bivector field $\pi_X \times_{(\rho,\lambda)} \pi_Y$ on the product manifold $X \times Y$ given by

$$\pi_X \times_{(\rho,\lambda)} \pi_Y = (\pi_X, 0) + (0, \pi_Y) - \sum_{i=1}^{n} (\rho(\xi_i), 0) \wedge (0, \lambda(x_i)), \tag{9}$$

is a Poisson structure on $X \times Y$, called the *mixed product of* π_X *and* π_Y *associated to* (ρ, λ), where $(x_i)_{i=1}^n$ is any basis for $\mathfrak{g}$ and $(\xi_i)_{i=1}^n$ the dual basis for $\mathfrak{g}^*$. We also refer to

$$-\sum_{i=1}^{n} (\rho(\xi_i), 0) \wedge (0, \lambda(x_i)) \in \mathcal{V}^2(X \times Y)$$

as *the mixed term* of $\pi_X \times_{(\rho,\lambda)} \pi_Y$. Mixed product Poisson structures of the form in (9) are studied in [LM1].

3. Action Poisson groupoids associated to quasitriangular r-matrices

1. Poisson groupoids

We recall from [MX], [W1], [X2] some basic facts on Poisson groupoids.

Let $\mathcal{G} \rightrightarrows Y$ be a Lie groupoid, with $\theta, \tau : \mathcal{G} \to Y$ its source and target maps, $\iota : \mathcal{G} \to \mathcal{G}$ the groupoid inverse map, and $\epsilon : Y \to \mathcal{G}$ the identity bisection. Let

$$\mathcal{G}_2 = \{(a, b) \in \mathcal{G} \times \mathcal{G} : \tau(a) = \theta(b)\}$$

be the submanifold of $\mathcal{G} \times \mathcal{G}$ of composable elements. A Poisson bivector field π on $\mathcal{G}$ is said to be *multiplicative* if the graph of the groupoid multiplication

$$\{(a, b, ab) : \ (a, b) \in \mathcal{G}_2\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$$

is a coisotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$, where $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is equipped with the Poisson structure $\pi \times \pi \times (-\pi)$. A *Poisson groupoid* is a pair $(\mathcal{G} \rightrightarrows Y, \pi)$, where $\mathcal{G} \rightrightarrows Y$ is a Lie groupoid and π is a multiplicative Poisson structure on $\mathcal{G}$. In such a case, $\iota(\pi) = -\pi$, and $\pi_Y = \theta(\pi) = -\tau(\pi)$ is a Poisson structure on Y, and one also says that $(\mathcal{G} \rightrightarrows Y, \pi)$ is a Poisson groupoid *over* (Y, π_Y). If in addition π is non-degenerate and $\dim \mathcal{G} = 2 \dim Y$, one says that $(\mathcal{G} \rightrightarrows Y, \pi)$ is a *symplectic groupoid* over (Y, π_Y).

Given a Lie groupoid $\mathcal{G} \rightrightarrows Y$, the left translation by $a \in \mathcal{G}$ is a smooth map

$$l_a\colon \ \theta^{-1}(\tau(a)) \to \theta^{-1}(\theta(a)).$$

Let $\ker \theta \to \mathcal{G}$ be the vector sub-bundle of the tangent bundle of $\mathcal{G}$ whose fiber over $a \in \mathcal{G}$ is the kernel of the differential of $\theta : \mathcal{G} \to Y$. A vector field V on $\mathcal{G}$ is said to be *left invariant* if it is everywhere tangent to $\ker\theta$ and is invariant under the left translation by every element in $\mathcal{G}$. The Lie algebroid of $\mathcal{G} \rightrightarrows Y$ is then the vector bundle $A = \epsilon^* \ker\theta$ over Y with $\tau : A \to TY$ as the anchor map and with the Lie bracket on the space $\Gamma(A)$ of its sections defined by

$$\overrightarrow{[s_1, s_2]} = [\overrightarrow{s_1}, \overrightarrow{s_2}],$$

where for $s \in \Gamma(A)$, $\overrightarrow{s}$ is the unique left invariant vector field on $\mathcal{G}$ which coincides with s on $\epsilon(Y) \cong Y$. As $T_{\epsilon(y)}\mathcal{G} = (\ker\theta)|_{\epsilon(y)} + T_{\epsilon(y)}\epsilon(Y)$ is a direct sum for every $y \in Y$, A can be identified with the normal bundle of $\epsilon(Y)$ in $\mathcal{G}$.

If $(\mathcal{G} \rightrightarrows Y, \pi)$ is a Poisson groupoid, then the identity section $\epsilon(Y)$ is a coisotropic submanifold with respect to the Poisson structure π, and the dual vector bundle A^* of A, identified with the co-normal bundle $N^*_{\epsilon(Y)}Y$ of $\epsilon(Y)$ in $\mathcal{G}$, is then a Lie sub-algebroid over $Y \cong \epsilon(Y) \hookrightarrow \mathcal{G}$ of the cotangent bundle Lie algebroid $T^*_\pi\mathcal{G}$ over $\mathcal{G}$ defined by the Poisson structure π. The pair of Lie algebroids (A, A^*) is then a Lie bialgebroid [MX] called the Lie bialgebroid of the Poisson groupoid $(\mathcal{G} \rightrightarrows Y, \pi)$. If $(\mathcal{G}' \rightrightarrows Y, \pi')$ is Poisson groupoid with Lie bialgebroid (A^*, A), one says that $((\mathcal{G} \rightrightarrows Y, \pi), (\mathcal{G}' \rightrightarrows Y, \pi'))$ is a *pair of dual Poisson groupoids.*

Recall also that if G is a Lie group and $\tau : Y \times G \to Y, (y, g) \to yg$, is a right Lie group action of G on a manifold Y, the product manifold $Y \times G$ then has the structure of an *action groupoid*, with $\tau : Y \times G \to Y$ as the target map, with

$$\theta(y, g) = y, \quad y \in Y, g \in G,$$

as the source map, and with the groupoid multiplication, inverse map ι, and the identity bisection ϵ respectively given by

$$(y_1, g_1)(y_2, g_2) = (y_1, g_1g_2), \quad \text{if} \quad y_1g_1 = y_2, \quad (y_1, g_1), (y_2, g_2) \in Y \times G,$$
$$\iota(y, g) = (yg, g^{-1}), \quad \epsilon(y) = (y, e), \quad y \in Y, g \in G.$$

Let $\mathfrak{g}$ be the Lie algebra of G. Identifying $\epsilon^* \ker\theta$ with the trivial vector bundle $A = Y \times \mathfrak{g}$ over Y, the Lie algebroid of the action groupoid $Y \times G \rightrightarrows Y$ is then the *action Lie algebroid* $Y \times \mathfrak{g}$ with anchor map, also denoted by τ, given by

$$\tau\colon Y \times \mathfrak{g} \to TY, \ \tau(y, x) = \frac{d}{dt}\Big|_{t=0} y\exp(tx), \quad y \in Y, x \in \mathfrak{g},$$

and the Lie bracket on its sections being the unique extending of the Lie bracket on $\mathfrak{g}$, identified with the space of constant sections. For $\varphi \in C^\infty(Y, \mathfrak{g}) \cong \Gamma(Y \times \mathfrak{g})$, the left-invariant vector field $\overrightarrow{\varphi}$ on the action groupoid $Y \times G \rightrightarrows Y$ is then given by

$$\overrightarrow{\varphi}\,(y, g) = (0, l_g\varphi(yg)), \quad y \in Y, g \in G. \tag{10}$$

By an *action Poisson groupoid* we mean a Poisson groupoid whose underlying groupoid structure is that of an action groupoid.

2. Action Poisson groupoids associated to quasitriangular r-matrices

Let (G, π_G) be a connected Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ be the dual Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Let (Y, π_Y) be a Poisson manifold, and assume that $\rho : \mathfrak{g}^* \to \mathcal{V}^1(Y)$ is a right Poisson action of the Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (Y, π_Y). One then has the mixed product Poisson structure π on the product manifold $Y \times G$ given by

$$\pi = \pi_Y \times_{(\rho, \lambda_G)} \pi_G, \tag{11}$$

where λ_G is the left Lie algebra action of $\mathfrak{g}$ on G generated by the left action of G on itself by left multiplication, i.e.,

$$\lambda_G(x) = x^R, \quad x \in \mathfrak{g},$$

where recall that for $x \in \mathfrak{g}$, x^R is the right invariant vector field on G with value x at the identity element e. Assume that G also acts on the right of Y by

$$\tau \colon Y \times G \to Y, \ (y, g) \mapsto yg, \quad y \in Y, g \in G.$$

Then $Y \times G$ has the corresponding structure of an action groupoid over Y. We review in this section a necessary and sufficient condition for the pair $(Y \times G \rightrightarrows Y, \pi)$ to be a Poisson groupoid.

Let $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be the double Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$, where recall that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, and recall the quasitriangular r-matrix $r_{\mathfrak{d}}$ on $\mathfrak{d}$ given in (7). Let

$$\sigma \colon \mathfrak{d} \to \mathcal{V}^1(Y), \quad \sigma(x + \xi) = \tau(x) + \rho(\xi), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, \tag{12}$$

where τ also denotes the Lie algebra homomorphism $\mathfrak{g} \to \mathcal{V}^1(Y)$ induced by the group action $\tau : Y \times G \to Y$ (see notation in §3). The following Theorem 2 was proved in [L].

Theorem 2 ([L, Thm. 3.32]). *The pair $(Y \times G \rightrightarrows Y, \pi)$ is a Poisson groupoid if and only if $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ defined in* (12) *is a right Lie algebra action of $\mathfrak{d}$ on Y and $\pi_Y = -\sigma(r_{\mathfrak{d}})$.*

As [L] is not published, for the convenience of the reader, we give an outline of the proof of Theorem 2 given in [L]. We first prove a lemma which explains the main part of Theorem 2.

For $\alpha \in \Omega^1(Y)$, let $X_\alpha = \pi^\#(\tau^*\alpha) \in \mathcal{V}^1(Y \times G)$. By [X2, Prop. 2.7], if $(Y \times G \rightrightarrows Y, \pi)$ is a Poisson groupoid, X_α is necessarily a left invariant vector field on $Y \times G$ for every $\alpha \in \Omega^1(Y)$, i.e., $\theta(X_\alpha) = 0$ and $X_\alpha(ab) = l_a X_\alpha(b)$ for any composable pair (a, b) in $Y \times G$.

Lemma 3. 1) *One has $\theta(X_\alpha) = 0$ for all $\alpha \in \Omega^1(Y)$ if and only if $\pi_Y = -\sigma(r_{\mathfrak{d}})$.*

2) *Assume that $\pi_Y = -\sigma(r_{\mathfrak{d}})$. Then X_α is left invariant for all $\alpha \in \Omega^1(Y)$ if and only if $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ is a right Lie algebra action. In such a case, for $\alpha \in \Omega^1(Y)$, one has*

$$X_\alpha = \overset{\rightarrow}{\phi_\alpha},$$

where $\phi_\alpha \in C^\infty(Y, \mathfrak{g})$ is given by $\phi_\alpha(y) = -\rho_y^*(\alpha(y))$, with $\rho_y : \mathfrak{g}^* \to T_yY$ given by $\rho_y(\xi) = \rho(\xi)(y)$ for $y \in Y$ and $\xi \in \mathfrak{g}^*$.

Proof. For $g \in G$ and $y \in Y$, let

$$\tau_g \colon Y \to Y,\ y' \mapsto y'g, \quad y' \in Y, \text{ and } \tau_y :\ \ G \to Y,\ g' \mapsto yg',\ g' \in G.$$

Let $p_1 : Y \times G \to Y$ and $p_2 : Y \times G \to G$ be respectively the projections to the first and the second factors. Let $\alpha \in \Omega^1(Y)$ and let $y \in Y$ and $g \in G$. Then

$$(\tau^*\alpha)(y,g) = p_1^*\tau_g^*(\alpha(yg)) + p_2^* l_{g^{-1}}^*\tau_{yg}^*(\alpha(yg)) \in T^*_{(y,g)}(Y \times G).$$

Using the definition of π, one has

$$X_\alpha(y,g) = \big(\pi_Y^\#(y)(\tau_g^*(\alpha(yg))) + \rho_y(\tau_y^*\tau_g^*(\alpha(yg))),\\ \pi_G^\#(g)(l_{g^{-1}}^*\tau_{yg}^*(\alpha(yg))) - r_g\rho_y^*\tau_g^*(\alpha(yg))\big). \tag{13}$$

1) Let $\{x_i\}_{i=1}^n$ be any basis of $\mathfrak{g}$ and $\{\xi_i\}_{i=1}^n$ the dual basis of $\mathfrak{g}^*$, so that $r_\mathfrak{d} = \sum_{i=1}^n x_i \otimes \xi_i \in \mathfrak{d} \otimes \mathfrak{d}$. Then $\pi_Y = -\sigma(r_\mathfrak{d})$ if and only if $\pi_Y = -\sum_{i=1}^n \tau(x_i) \otimes \rho(\xi_i)$, which is equivalent to

$$\pi_Y^\#(y)(\alpha_y) = -\rho_y(\tau_y^*(\alpha_y)), \qquad y \in Y, \alpha_y \in T_y^*Y.$$

It is now clear from (13) that 1) holds.

2) Assume now that $\pi_Y = -\sigma(r_\mathfrak{d})$. By (13), X_α is left invariant if and only if

$$-l_g\rho_{yg}^*(\alpha(yg)) = \pi_G^\#(g)(l_{g^{-1}}^*\tau_{yg}^*(\alpha(yg))) - r_g\rho_y^*\tau_g^*(\alpha(yg)), \quad (y,g) \in Y \times G. \tag{14}$$

Pairing both sides of (14) with $l_{g^{-1}}^*\xi \in T_g^*G$, where $\xi \in \mathfrak{g}^*$, and using (2), one can rewrite (14) as

$$0 = \langle \alpha(yg),\ \rho_{yg}(\xi) - \tau_g\tau_y(p_\mathfrak{g}(\mathrm{Ad}_g\xi)) - \tau_g\rho_y(p_{\mathfrak{g}^*}(\mathrm{Ad}_g\xi))\rangle, \quad y \in Y,\ g \in G,$$

where recall that $p_\mathfrak{g} : \mathfrak{d} \to \mathfrak{g}$ and $p_{\mathfrak{g}^*} : \mathfrak{d} \to \mathfrak{g}^*$ are the projections with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. Therefore X_α is left-invariant for all $\alpha \in \Omega^1(Y)$ if and only if

$$\tau_{g^{-1}}(\rho(\xi)) = \sigma(\mathrm{Ad}_g\xi) \in \mathcal{V}^1(Y), \quad g \in G, \xi \in \mathfrak{g}^*. \tag{15}$$

Assuming (15) and differentiating $g \in G$ in the direction of $x \in \mathfrak{g}$ gives

$$[\tau(x), \rho(\xi)] = \sigma([x,\xi]), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, \tag{16}$$

so $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ is a Lie algebra homomorphism. Conversely, assume that σ is a Lie algebra homomorphism. The infinitesimal $\mathfrak{g}$-invariance of σ in (16) and the connectedness of G imply the G-equivariance of the σ, namely (15). It is also clear from (13) that in such a case, $X_\alpha = \overrightarrow{\varphi_\alpha}$ with φ_α as described. □

Proof of Theorem 2. Assuming that $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ is a Lie algebra homomorphism and that $\pi_Y = -\sigma(r_\mathfrak{d})$, we now show that $(Y \times G \rightrightarrows Y, \pi)$ is a Poisson groupoid, the other direction of Theorem 2 having been proved in Lemma 3.

Note first that $\pi_Y = -\sigma(r_{\mathfrak{d}})$ implies that σ is a right Poisson action of the double Lie bialgebra $(\mathfrak{d}, -\delta_{\mathfrak{d}})$ on (Y, π_Y), so τ is a right Poisson action of the Poisson Lie group $(G, -\pi_G)$ on (Y, π_Y). It follows by an easy calculation that the target map $\tau : (\mathcal{G}, \pi) \to (Y, \pi_Y)$ is anti-Poisson, where $\mathcal{G} = Y \times G$. As the source map $\theta : (\mathcal{G}, \pi) \to (Y, \pi_Y)$ is Poisson, the submanifold

$$\mathcal{G}_2 = \{(a, b) \in \mathcal{G} \times \mathcal{G} : \tau(a) = \theta(a)\} \subset \mathcal{G} \times \mathcal{G}$$

of composable pairs is coisotropic with respect to the product Poisson structure $\pi \times \pi$ on $\mathcal{G} \times \mathcal{G}$. Note that the graph $\{(a, b, ab) : (a, b) \in \mathcal{G}_2\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ of the groupoid multiplication is the graph of the map $\mu|_{\mathcal{G}_2} : \mathcal{G}_2 \to \mathcal{G}$, where

$$\mu\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}\colon (y_1, g_1, y_2, g_2) \mapsto (y_1, g_1 g_2), \quad y_i \in Y, g_i \in G.$$

Note also that the map μ is Poisson with respect to the product Poisson structures $\pi \times \pi$ on $\mathcal{G} \times \mathcal{G}$ and π on $\mathcal{G}$. Indeed, $\mu = \nu \circ (\mathrm{Id}_{\mathcal{G}} \times p)$, where the projection $p : (Y \times G, \pi) \to (G, \pi_G)$ to the second factor is Poisson, and the map

$$\nu :\ (Y \times G, \pi) \times (G, \pi_G) \to (Y \times G, \pi),\ (y, g, g_1) \mapsto (y, gg_1),\ y \in Y,\ g, g_1 \in G$$

is Poisson. It is a general fact, the proof of which is straightforward (see [L, Lem. 3.33]), that for a Poisson map $\Phi : (P, \pi_P) \to (Q, \pi_Q)$ and a coisotropic submanifold $P_1 \subset (P, \pi_P)$, the graph $\{(p, \Phi(p)) : p \in P_1\}$ of $\Phi|_{P_1} : P_1 \to Q$ is coisotropic in $(P \times Q, \pi_P \times (-\pi_Q))$ if and only if

$$\pi_P^{\#}(N^*_{P_1} P) \subset \ker \Phi,$$

where $N^*_{P_1} P \subset T^*P|_{P_1}$ is the co-normal space of P_1 in P, and the sub-bundle $\pi_P^{\#}(N^*_{P_1} P)$ of TP_1 is called the *characteristic distribution* of the coisotropic submanifold P_1 in P. Using Lemma 3, a direct calculation shows that the characteristic distribution of $\mathcal{G}_2$ in $\mathcal{G} \times \mathcal{G}$ at the point $(y_1, g_1, y_2, g_2) \in \mathcal{G}_2$ is given by

$$\{(0, -l_{g_1} x, -\tau_{y_2} x, r_{g_2} x) :\ x \in \rho^*_{y_2} T^*_{y_2} Y \subset \mathfrak{g}\},$$

which is easily seen to be contained in the kernel of the differential of μ at $(y_1, g_1, y_2, g_2) \in \mathcal{G}_2$. Thus the graph $\{(a, b, ab) : (a, b) \in \mathcal{G}_2\}$ is a coisotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ with respect to the Poisson structure $\pi \times \pi \times (-\pi)$, and hence $(\mathcal{G} \rightrightarrows Y, \pi)$ is a Poisson groupoid. This finishes the proof of Theorem 2. □

Remark 3. In the context of Theorem 2, it is easy to see that the Lie algebroid structure induced by π on the co-normal bundle of $\epsilon(Y)$ in $Y \times G$, identified with the trivial vector bundle $Y \times \mathfrak{g}^*$ over Y, is that of the action Lie algebroid defined by the right action ρ of $\mathfrak{g}^*$ on Y. Thus the Lie bialgebroid of the Poisson groupoid $(Y \times G \rightrightarrows Y, \pi)$ is the pair

$$(A = Y \times \mathfrak{g},\ A^* = Y \times \mathfrak{g}^*)$$

of action Lie algebroids. Their double, as a Courant Lie bialgebroid [LWX1], is the *action Courant algebroid* $Y \times \mathfrak{d}$ over Y defined by σ that has been studied in [L-BM]. □

Let $((G, \pi_G), (G^*, \pi_{G^*}))$ be a pair of dual Poisson Lie groups, with the corresponding pair of dual Lie bialgebras $((\mathfrak{g}, \delta_\mathfrak{g}), (\mathfrak{g}^*, \delta_{\mathfrak{g}^*}))$, and let $(\mathfrak{d}, \langle\ ,\ \rangle_\mathfrak{d})$ be their double Lie algebra. Let again $r_\mathfrak{d} = \sum_{i=1}^n x_i \otimes \xi_i$ be the quasitriangular r-matrix on $\mathfrak{d}$, where $\{x_i\}_{i=1}^n$ is any basis of $\mathfrak{g}$ and $\{\xi_i\}_{i=1}^n$ the dual basis of $\mathfrak{g}^*$. Assume that $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ is a right Lie algebra action of $\mathfrak{d}$ on a manifold Y such that the stabilizer subalgebra $\mathfrak{d}_y$ of $\mathfrak{d}$ at every $y \in Y$ is coisotropic with respect to $\langle\ ,\ \rangle_\mathfrak{d}$, which, by Remark 1, is equivalent to $\sigma(r_\mathfrak{d})$ being a Poisson structure on Y.

Corollary 4. 1) *Assume that* $\sigma|_\mathfrak{g} : \mathfrak{g} \to \mathcal{V}^1(Y)$ *integrates to a Lie group action* $Y \times G \to Y$. *Then one has the action Poisson groupoid* $(Y \times G \rightrightarrows Y, \pi_{Y\times G})$ *over* $(Y, -\sigma(r_\mathfrak{d}))$, *where* $Y \times G \rightrightarrows Y$ *is the action groupoid over* Y *defined by the group action of* G *on* Y, *and* $\pi_{Y\times G}$ *is the mixed product Poisson structure on* $Y \times G$ *given by*

$$\pi_{Y\times G} = (-\sigma(r_\mathfrak{d}), 0) + (0, \pi_G) - \sum_{i=1}^n \sigma(\xi_i), 0) \wedge (0, x_i^R).$$

2) *Assume that* $\sigma|_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathcal{V}^1(Y)$ *integrates to a Lie group action* $Y \times G^* \to Y$. *Then one has the action Poisson groupoid* $(Y \times G^* \rightrightarrows Y, \pi_{Y\times G^*})$ *over* $(Y, \sigma(r_\mathfrak{d}))$, *where* $Y \times G^* \rightrightarrows Y$ *is the action groupoid over* Y *defined by the group action of* G^* *on* Y, *and* $\pi_{Y\times G^*}$ *is the mixed product Poisson structure on* $Y \times G$ *given by*

$$\pi_{Y\times G^*} = (\sigma(r_\mathfrak{d}), 0) + (0, \pi_{G^*}) - \sum_{i=1}^n \sigma(x_i), 0) \wedge (0, \xi_i^R).$$

3) *When the assumptions in both 1) and 2) hold, the two action Poisson groupoids in* 1) *and* 2) *form a pair of dual Poisson groupoids.*

Proof. By Lemma 1, σ is a right Poisson action of the Lie bialgebra $(\mathfrak{d}, \delta_\mathfrak{d})$ on $(Y, \sigma(r_\mathfrak{d}))$, where recall that $\delta_\mathfrak{d}(v) = \mathrm{ad}_v r_\mathfrak{d}$ for $v \in \mathfrak{d}$. As $\delta_\mathfrak{g} = \delta_\mathfrak{d}|_\mathfrak{g}$ and $\delta_{\mathfrak{g}^*} = -\delta_\mathfrak{d}|_{\mathfrak{g}^*}$, $\sigma|_{\mathfrak{g}^*}$ is a right Poisson action of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}^*})$ on $(Y, -\sigma(r_\mathfrak{d}))$, and $\sigma|_\mathfrak{g}$ is a right Poisson action of the Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$ on $(Y, \sigma(r_\mathfrak{d}))$. Now 1) and 2) of Corollary 4 follow from Theorem 2 applied to the Poisson Lie groups (G, π_G) and (G^*, π_{G^*}) respectively, and 3) follows from Remark 3. □

Remark 4. When $\sigma|_\mathfrak{g} : \mathfrak{g} \to \mathcal{V}^1(Y)$ integrates to a Lie group action $\tau : Y \times G \to Y$, the pair (τ, σ) can be thought of as a (right) *action of the Harish-Chandra pair* $(G, \mathfrak{d})$ (see §1) on the manifold Y in the sense that τ is a right action of the Lie group G on Y and σ is a right action of the Lie algebra $\mathfrak{d}$ on Y such that $\sigma|_\mathfrak{g}$ coincides with the action of $\mathfrak{g}$ on Y induced by τ. □

Let (G, π_G) now be any connected Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$, and assume that $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasitriangular r-matrix for $(\mathfrak{g}, \delta_\mathfrak{g})$. Let Y be a manifold with a right G-action $\sigma : Y \times G \to Y$, and assume that the stabilizer subalgebra of $\mathfrak{g}$ at every $y \in Y$ is coisotropic with respect to the symmetric part of r. By Lemma 1 and Remark 1, $\sigma(r)$ is a Poisson structure on Y, where $\sigma : \mathfrak{g} \to \mathcal{V}^1(Y)$ also denotes the right Lie algebra action induced by σ.

Recall from §2 the pair of dual Lie subalgebras $((\mathfrak{f}_-, \delta_\mathfrak{g}|_{\mathfrak{f}_-}), (\mathfrak{f}_+, -\delta_\mathfrak{g}|_{\mathfrak{f}_+}))$. Let again F_- and F_+ be the connected subgroups of G with Lie algebras $\mathfrak{f}_-$ and $\mathfrak{f}_+$

respectively, so $(F_-, \pi_G|_{F_-})$ and $(F_+, -\pi_G|_{F_+})$ form a pair of dual Poisson Lie groups. Restricting the action σ of G on Y to actions of $F_\pm$ on Y, one then has the action groupoids

$$Y \times F_- \rightrightarrows Y \quad \text{and} \quad Y \times F_+ \rightrightarrows Y.$$

Let $\{x_i\}_{i=1}^n$ be a basis of $\mathfrak{f}_-$ and $\{\xi_i\}_{i=1}^n$ the dual basis of $\mathfrak{f}_+$ with respect to the pairing $\langle \, , \, \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$ between $\mathfrak{f}_-$ and $\mathfrak{f}_+$ given in (6).

Corollary 5. *With the notation as above, let*

$$\begin{aligned} \pi_{Y\times F_-} &= (-\sigma(r)) \times_{(\sigma|_{\mathfrak{f}_+}, \lambda_-)} \pi_G|_{F_-} \\ &= (-\sigma(r), 0) + (0, \pi_G|_{F_-}) - \sum_{i=1}^n (\sigma(\xi_i), 0) \wedge (0,\, x_i^R), \end{aligned} \tag{17}$$

$$\begin{aligned} \pi_{Y\times F_+} &= \sigma(r) \times_{(\sigma|_{\mathfrak{f}_-}, \lambda_+)} (-\pi_G|_{F_+}) \\ &= (\sigma(r), 0) + (0, -\pi_G|_{F_+}) - \sum_{i=1}^n (\sigma(x_i), 0) \wedge (0, \xi_i^R). \end{aligned} \tag{18}$$

Then $(Y \times F_- \rightrightarrows Y, \pi_{Y\times F_-})$ and $(Y \times F_+ \rightrightarrows Y, \pi_{Y\times F_+})$ form a pair of dual Poisson groupoids.

Proof. Let $\mathfrak{d}_{\mathfrak{f}_-}$ be the double Lie algebra of $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$. Then $\sigma \circ q : \mathfrak{d}_{\mathfrak{f}_-} \to \mathcal{V}^1(Y)$ is a Lie algebra homomorphism, where $q : \mathfrak{d}_{\mathfrak{f}_-} \to \mathfrak{g}$ is the Lie algebra homomorphism given in (8). By Remark 2, $q(r_{\mathfrak{d}_{\mathfrak{f}_-}}) = r$. Thus $(\sigma \circ q)(r_{\mathfrak{d}_{\mathfrak{f}_-}}) = \sigma(r)$ is a Poisson structure on Y. Corollary 5 now follows by applying Corollary 4 to the pair of dual Poisson Lie groups $(F_-, \pi_G|_{F_-})$ and $(F_+, -\pi_G|_{F_+})$. □

Remark 5. The Lie algebra action $\sigma \circ q : \mathfrak{d}_{\mathfrak{f}_-} \to \mathcal{V}^1(Y)$ of $\mathfrak{d}_{\mathfrak{f}_-}$ on Y gives rise to the *action Courant algebroid* over Y as defined in [L-BM], with two transversal Dirac structures defined by the splitting $\mathfrak{d}_{\mathfrak{f}_-} = \mathfrak{f}_- + \mathfrak{f}_+$. The pair of dual Poisson groupoids in Corollary 5 then have the two transversal Dirac structures as their Lie bialgebroids. □

4. Review on standard complex semisimple Poisson Lie groups

1. The standard complex semisimple Poisson Lie group (G, π_{st})

For the rest of the paper, let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. We recall the so-called *standard multiplicative Poisson structures* on G and refer to [ES], [LM1], [LM2] for details.

Fix a pair (B, B_-) of opposite Borel subgroups of G and a non-degenerate symmetric ad-invariant bilinear form $\langle \, , \, \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, and let $T = B \cap B_-$. Denote the Lie algebras of B, B_- and T by $\mathfrak{b}, \mathfrak{b}_-$ and $\mathfrak{h}$ respectively. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let $\Delta_+ \subset \mathfrak{h}^*$ be the set of positive roots with respect to $\mathfrak{b}$. We will also write $\alpha > 0$ for $\alpha \in \Delta_+$. Let $\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}$, and let N, N_- be the connected subgroups of G with respective Lie algebras $\mathfrak{n}$ and $\mathfrak{n}_-$. For each $\alpha > 0$, let $E_\alpha \in \mathfrak{g}_\alpha$ and

$E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be such that $\langle E_\alpha, E_{-\alpha}\rangle_{\mathfrak{g}} = 1$. Denote by $\langle\ ,\ \rangle$ the bilinear form on both $\mathfrak{h}$ and $\mathfrak{h}^*$ induced by $\langle\ ,\ \rangle_{\mathfrak{g}}$, and let $\{h_i\}_{i=1}^r$, $r = \dim \mathfrak{h}$, be a basis of $\mathfrak{h}$ such that $\langle h_i, h_j\rangle = \delta_{ij}$. The *standard quasitriangular r-matrix* associated to the choice of the triple $(\mathfrak{b}, \mathfrak{b}_-, \langle\ ,\ \rangle_{\mathfrak{g}})$ is the element

$$r_{\rm st} = \frac{1}{2}\sum_{i=1}^{r} h_i \otimes h_i + \sum_{\alpha>0} E_{-\alpha} \otimes E_\alpha \in \mathfrak{g} \otimes \mathfrak{g}. \tag{19}$$

The bivector field on G defined by (see notation in §3)

$$\pi_{\rm st} = r_{\rm st}^L - r_{\rm st}^R \tag{20}$$

is a multiplicative Poisson structure on G, and $(G, \pi_{\rm st})$ is called a *standard semi-simple Poisson Lie group*. The Lie bialgebra of $(G, \pi_{\rm st})$ is $(\mathfrak{g}, \delta_{\rm st})$, where $\delta_{\rm st}(x) = {\rm ad}_x r_{\rm st}$ for $x \in \mathfrak{g}$. In the notation of §2, one has

$$\mathrm{Im}(r_{\rm st}^\sharp) = \mathfrak{b} \quad \text{and} \quad \mathrm{Im}((r_{\rm st}^{21})^\sharp) = \mathfrak{b}_-.$$

Thus B and B_- are Poisson Lie subgroups of $(G, \pi_{\rm st})$. Denoting the restrictions of $\pi_{\rm st}$ to B and to B_- by the same symbol, the pair $((B_-, \pi_{\rm st}), (B, -\pi_{\rm st}))$ is then a pair of dual Poisson Lie groups, with the pairing $\langle\ ,\ \rangle_{(\mathfrak{b}_-, \mathfrak{b})}$ in (6) given explicitly by

$$\langle x_- + x_0,\ y_+ + y_0\rangle_{(\mathfrak{b}_-, \mathfrak{b})} = \langle x_-, y_+\rangle_{\mathfrak{g}} + 2\langle x_0, y_0\rangle_{\mathfrak{g}}, \quad x_- \in \mathfrak{n}_-,\ x_0, y_0 \in \mathfrak{h}, y_+ \in \mathfrak{n}. \tag{21}$$

A basis for $\mathfrak{b}_-$ and its dual basis for $\mathfrak{b}_+$ with respect to the pairing $\langle\ ,\ \rangle_{(\mathfrak{b}_-, \mathfrak{b})}$ are now given by

$$\{h_i/\sqrt{2}\}_{i=1}^r \cup \{E_{-\alpha}\}_{\alpha>0} \subset \mathfrak{b}_- \quad \text{and} \quad \{h_i/\sqrt{2}\}_{i=1}^r \cup \{E_\alpha\}_{\alpha>0} \subset \mathfrak{b}. \tag{22}$$

The Poisson structure $\pi_{\rm st}$ is invariant under the action of T on G by left or right multiplication. Let $W = N_G(T)/T$ be the Weyl group of (G, T), where $N_G(T)$ is the normalizer subgroup of T in G. For $u, v \in W$, the *double Bruhat cell* (see [FZ])

$$G^{u,v} = BuB \cap B_-vB_-$$

is non-empty, and $\dim G^{u,v} = l(u) + l(v) + r$, where l is the length function on W and recall that $r = \dim \mathfrak{h}$. It is well-known [HL], [HKKR] that the T-leaves of $(G, \pi_{\rm st})$ are precisely the double Bruhat cells in G. In particular, for each $v \in W$, both BvB and B_-vB_- are Poisson submanifolds of G with respect to $\pi_{\rm st}$.

2. The Drinfeld double and the dressing vector fields of $(G, \pi_{\rm st})$

The double Lie algebra $(\mathfrak{d}, \langle\ ,\ \rangle_{\mathfrak{d}})$ of the Lie bialgebra $(\mathfrak{g}, \delta_{\rm st})$ can be identified with the quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}\oplus\mathfrak{g}})$, where $\mathfrak{g} \oplus \mathfrak{g}$ has the direct product Lie algebra structure, the invariant bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}\oplus\mathfrak{g}}$ is defined by

$$\langle (x_1, y_1), (x_2, y_2)\rangle_{\mathfrak{g}\oplus\mathfrak{g}} = \langle x_1, x_2\rangle_{\mathfrak{g}} - \langle y_1, y_2\rangle_{\mathfrak{g}}, \quad x_1, x_2, y_1, y_2 \in \mathfrak{g},$$

and $\mathfrak{g}$ is identified with $\mathfrak{g}_\Delta = \{(x,x) : x \in \mathfrak{g}\}$ and $\mathfrak{g}^*$ with

$$\mathfrak{g}^*_{\rm st} = \{(x_+ + x_0, -x_0 + x_-) : x_+ \in \mathfrak{n}, x_- \in \mathfrak{n}^-, x_0 \in \mathfrak{h}\} \tag{23}$$

(see [CP], [ES], [LM2]). Let $r^{(2)}_{\rm st} \in (\mathfrak{g} \oplus \mathfrak{g}) \otimes (\mathfrak{g} \oplus \mathfrak{g})$ be the r-matrix on $\mathfrak{g} \oplus \mathfrak{g}$ as the double Lie algebra of $(\mathfrak{g}, \delta_{\rm st})$ (see Example 1), and let

$$\Pi_{\rm st} = \left(r^{(2)}_{\rm st}\right)^L - \left(r^{(2)}_{\rm st}\right)^R$$

be the corresponding multiplicative Poisson structure on $G \times G$. Then the Poisson Lie group $(G \times G, \Pi_{\rm st})$ is a Drinfeld double of $(G, \pi_{\rm st})$, and the diagonal embedding

$$(G, \pi_{\rm st}) \hookrightarrow (G \times G, \Pi_{\rm st}), \quad g \mapsto (g, g), \quad g \in G, \tag{24}$$

realizes $(G, \pi_{\rm st})$ as a Poisson subgroup of $(G \times G, \Pi_{\rm st})$.

Let $B_-^{\rm op}$ be the Lie group which has the same underlying manifold as B_-, but with the opposite group structure. Then

$$(\widetilde{B}_-, \pi_{\widetilde{B}_-}) = (B_- \times B_-^{\rm op}, \pi_{\rm st} \times \pi_{\rm st}) \quad \text{and} \quad (\widetilde{B}, \pi_{\widetilde{B}}) = (B \times B, (-\pi_{\rm st}) \times \pi_{\rm st})$$

form a pair of dual Poisson Lie groups. Consider the respective right and left Poisson actions

$$\rho\colon (G, \pi_{\rm st}) \times (\widetilde{B}, \pi_{\widetilde{B}}) \to (G, \pi_{\rm st}),$$
$$\rho(g, (b_1, b_2)) = b_1^{-1} g b_2, \quad g \in G, b_1, b_2 \in B,$$

$$\lambda\colon (\widetilde{B}_-, \pi_{\widetilde{B}_-}) \times (G, \pi_{\rm st}) \to (G, \pi_{\rm st}),$$
$$\lambda((b_{-1}, b_{-2}), g) = b_{-1} g b_{-2}, \quad g \in G, b_{-1}, b_{-2} \in B_-.$$

It is proved in [LM1, §6.2 and §8] that $\Pi_{\rm st}$ is a mixed product Poisson structure on $G \times G$. Namely,

$$\Pi_{\rm st} = \pi_{\rm st} \times_{(\rho,\lambda)} \pi_{\rm st}. \tag{25}$$

We now present some explicit formulas for the dressing vector fields on $(G, \pi_{\rm st})$ which will be used in the proof of Lemma 7. Let $p_{\mathfrak{g}} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ the projection to $\mathfrak{g} \cong \mathfrak{g}_{\rm diag}$ with respect to the splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}^*_{\rm st}$. Note that for any $x \in \mathfrak{g}$, writing $x = [x]_- + [x]_0 + [x]_+$ with $[x]_- \in \mathfrak{n}^-$, $[x]_0 \in \mathfrak{h}$ and $[x]_+ \in \mathfrak{n}$, one has

$$p_{\mathfrak{g}}(0, x) = \tfrac{1}{2}[x]_0 + [x]_+ \in \mathfrak{b}, \quad p_{\mathfrak{g}}(x, 0) = \tfrac{1}{2}[x]_0 + [x]_- \in \mathfrak{b}^-. \tag{26}$$

Thus for $\eta \in \mathfrak{n}$, the dressing vector field $\mathbf{d}(\eta, 0)$ at $g \in G$ is given by

$$\begin{aligned}\mathbf{d}(\eta,0)(g) &= -l_g p_{\mathfrak{g}} \mathrm{Ad}_{(g^{-1},g^{-1})}(\eta, 0) = -l_g\left(\tfrac{1}{2}[\mathrm{Ad}_{g^{-1}}\eta]_0 + [\mathrm{Ad}_{g^{-1}}\eta]_-\right) \\ &= -r_g\eta + l_g\left(\tfrac{1}{2}[\mathrm{Ad}_{g^{-1}}\eta]_0 + [\mathrm{Ad}_{g^{-1}}\eta]_+\right) \in T_g(gB^-) \cap T_g(BgB).\end{aligned} \tag{27}$$

Similarly, for $\eta \in \mathfrak{n}_-$, and $x \in \mathfrak{h}$, one has

$$\begin{aligned}\mathbf{d}(0,\eta)(g) &= -l_g\left(\tfrac{1}{2}[\mathrm{Ad}_{g^{-1}}\eta]_0 + [\mathrm{Ad}_{g^{-1}}\eta]_+\right) \\ &= -r_g\eta + l_g\left(\tfrac{1}{2}[\mathrm{Ad}_{g^{-1}}\eta]_0 + [\mathrm{Ad}_{g^{-1}}\eta]_-\right) \in T_g(gB) \cap T_g(B^-gB^-),\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{d}(x,-x)(g) &= l_g\left([\mathrm{Ad}_{g^{-1}}x]_+ - [\mathrm{Ad}_{g^{-1}}x]_-\right) \\ &= r_g x - l_g\left([\mathrm{Ad}_{g^{-1}}x]_0 + 2[\mathrm{Ad}_{g^{-1}}x]_-\right) \\ &= -r_g x + l_g\left([\mathrm{Ad}_{g^{-1}}x]_0 + 2[\mathrm{Ad}_{g^{-1}}x]_+\right) \in T_g(TgB^-) \cap T_g(TgB).\end{aligned} \tag{29}$$

Remark 6. Note that it also follows from (27), (28), and (29) that all the (B, B)-double cosets and all the (B_-, B_-)-double cosets are Poisson submanifold of $(G, \pi_{\rm st})$. $\square$

3. Weak Poisson pairs

Consider the natural projections

$$\varpi\colon G \to G/B,\ g \mapsto g.B, \quad \varpi_- :\ \ G \to B_-\backslash G,\ g \mapsto B_-.g, \quad g \in G. \tag{30}$$

As both B and B_- are Poisson Lie subgroups of $(G, \pi_{\rm st})$,

$$\pi_1 \stackrel{\rm def}{=} \varpi(\pi_{\rm st}) \quad \text{and} \quad \pi_{-1} \stackrel{\rm def}{=} \varpi_-(\pi_{\rm st}) \tag{31}$$

are now well-defined Poisson structures on G/B and on $B_-\backslash G$, respectively. The Poisson structure π_1 is invariant under the action of T on G/B by left multiplication, and it is proven in [GY1] that the T-leaves of π_1 are precisely the so-called *open Richardson varieties*, i.e., non-empty intersections $(BuB/B) \cap (B_-wB/B)$, where $u, w \in W$. In particular, every *Bruhat cell* BuB/B, for $u \in W$, is a Poisson subvariety of $(G/B, \pi_1)$. Similarly, every Bruhat cell $B_-\backslash B_-uB_-$, for $u \in W$, is a Poisson subvariety of $(B_-\backslash G, \pi_{-1})$.

Definition 1 ([LM1, §8.6]). Two Poisson maps $\rho_Y : (X, \pi_X) \to (Y, \pi_Y)$ and $\rho_Z : (X, \pi_X) \to (Z, \pi_Z)$ are said to form a *Poisson pair* if the map

$$(\rho_Y, \rho_Z) :\ (X, \pi_X) \to (Y \times Z, \pi_Y \times \pi_Z), \quad (y, z) \mapsto (\rho_Y(y), \rho_Z(z)), \quad y \in Y, z \in Z,$$

is a Poisson map.

The following Lemma 6 is a special case of a fact proved in [LM1, §8.6], but for the convenience of the reader, we give a proof which is much simpler in our special case.

Lemma 6. *The two Poisson maps*

$$\varpi\colon (G, \pi_{\rm st}) \to (G/B, \pi_1) \quad \textit{and} \quad \varpi_-\colon (G, \pi_{\rm st}) \to (B_-\backslash G, \pi_{-1})$$

form a Poisson pair. Consequently, for $u, v \in W$ and for any symplectic leaf $\Sigma^{u,v} \subset G^{u,v}$, one has the Poisson pairs

$$\varpi|_{G^{u,v}}\colon (G^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \ \ \textit{and} \ \ \varpi_-|_{G^{u,v}}\colon (G^{u,v}, \pi_{\rm st}) \to (B_-\backslash B_-vB_-, \pi_{-1}),$$
$$\varpi|_{\Sigma^{u,v}}\colon (\Sigma^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \ \ \textit{and} \ \ \varpi_-|_{\Sigma^{u,v}}\colon (\Sigma^{u,v}, \pi_{\rm st}) \to (B_-\backslash B_-vB_-, \pi_{-1}).$$

Proof. Consider the projection $\Phi : G \times G \to (G/B) \times (B_-\backslash G)$ defined by

$$\Phi(g_1, g_2) = (g_1.B, B_-.g_2), \quad g_1, g_2 \in G.$$

Using (25) to write $\Pi_{\rm st} = (\pi_{\rm st}, 0) + (0, \pi_{\rm st}) + \pi_{\rm mix}$, it follows from the definition of the mixed term $\pi_{\rm mix}$ that $\Phi(\pi_{\rm mix}) = 0$. Thus

$$\Phi\colon (G \times G, \Pi_{\rm st}) \to ((G/B) \times (B_-\backslash G), \pi_1 \times \pi_{-1})$$

is Poisson. As the diagonal embedding $(G, \pi_{\rm st}) \hookrightarrow (G \times G, \Pi_{\rm st})$ is Poisson, ϖ and ϖ_- form a Poisson pair. As $G^{u,v}$ or any symplectic leaf in $G^{u,v}$ are Poisson submanifolds of $(G, \pi_{\rm st})$, the rest of Lemma 6 follows. $\square$

Note that in Definition 1 we do not require the two maps ρ_Y and ρ_Z in a Poisson pair to be surjective nor submersions. The next Lemma 7 and Lemma 8 say that the Poisson maps in the Poisson pairs in Lemma 6, although not necessarily surjective, are all submersions.

Lemma 7. *For any $u, v \in W$ and any symplectic leaf $\Sigma^{u,v}$ of $\pi_{\rm st}$ in $G^{u,v}$, the maps*

$$\varpi|_{\Sigma^{u,v}} : \Sigma^{u,v} \to BuB/B \quad \text{and} \quad \varpi_-|_{\Sigma^{u,v}} : \Sigma^{u,v} \to B_-\backslash B_-vB_-$$

are submersions.

Proof. Let $g \in \Sigma^{u,v}$. By definition, the value at g of every dressing vector field on $(G, \pi_{\rm st})$ is tangent to $\Sigma^{u,v}$. By (27) and (29), the differential of $\varpi|_{\Sigma^{u,v}}$ at g is a surjective linear map from $T_g\Sigma^{u,v}$ to $T_{g.B}(BuB/B)$. Thus $\varpi|_{\Sigma^{u,v}} : \Sigma^{u,v} \to BuB/B$ is a submersion. Similarly, $\varpi_-|_{\Sigma^{u,v}} : \Sigma^{u,v} \to B_-\backslash B_-vB_-$ is a submersion. □

Remark 7. Lemma 7 implies that for any $u, v \in W$, the maps

$$\varpi|_{G^{u,v}} : (G^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \quad \text{and} \quad \varpi_-|_{G^{u,v}} : (G^{u,v}, \pi_{\rm st}) \to (B_-\backslash B_-vB_-, \pi_{-1})$$

are also submersions, a fact one can in fact see directly without computing the dressing vector fields. Indeed, For any $g \in G$ and $x \in \mathfrak{b}$, the element

$$z_{g,x} \stackrel{\rm def}{=\!=} r_g x - l_g\left(\tfrac{1}{2}\left(\left[{\rm Ad}_{g^{-1}}x\right]_0\right) + \left[{\rm Ad}_{g^{-1}}x\right]_+\right) = l_g\left(\tfrac{1}{2}\left(\left[{\rm Ad}_{g^{-1}}x\right]_0\right) + \left[{\rm Ad}_{g^{-1}}x\right]_-\right)$$

lies in $T_g(BgB \cap B_-gB_-)$ and $\varpi(z_{g,x}) = \varpi(r_g x)$. It follows that the differential of ϖ restricts to a surjective linear map from $T_g(BgB \cap B_-gB_-)$ to $T_{g.B}(Bg.B)$ for every $g \in G$. This shows in particular that for any $u, v \in W$, the map $\varpi|_{G^{u,v}} : G^{u,v} \to BuB/B$ is a submersion. Similarly, one sees that $\varpi_-|_{G^{u,v}}$ is a submersion. □

Lemma 8. *For any $u, v \in W$ and for any symplectic leaf $\Sigma^{u,v}$ of $\pi_{\rm st}$ in $G^{u,v}$, one has*

$$\varpi(\Sigma^{u,v}) = \varpi(G^{u,v}) = \bigcup_{w \leq u, w \leq v} (BuB/B) \cap (B_-wB/B) \subset BuB/B,$$

$$\varpi_-(\Sigma^{u,v}) = \varpi_-(G^{u,v}) = \bigcup_{w \leq u, w \leq v} (B_-\backslash B_-wB) \cap (B_-\backslash B_-vB_-) \subset B_-\backslash B_-vB_-,$$

where $\leq$ is the Bruhat order on W defined by the choice of B.

Proof. For $w \in W$, $B_-wB \subset B_-vB_-B$ if and only if $B_-wB \cap B_-vB_- \neq \emptyset$, which, by [De, Cor. 1.2], is equivalent to $w \leq v$. Thus $B_-vB_-B = \bigcup_{w \leq v} B_-wB$. It follows that

$$\varpi(G^{u,v}) = \varpi(BuB) \cap \varpi(B_-vB_-B) = \bigcup_{w \leq u, w \leq v} (BuB/B) \cap (B_-wB/B).$$

Since $G^{u,v} = \Sigma^{u,v}T$, one has $\varpi(\Sigma^{u,v}) = \varpi(G^{u,v})$. The claims on $\varpi_-(\Sigma^{u,v})$ and $\varpi_-(G^{u,v})$ are proved similarly. □

Remark 8. For $u, v \in W$ and a symplectic leaf $\Sigma^{u,v}$ of $\pi_{\rm st}$ in $G^{u,v}$, the Poisson pair

$$\varpi|_{\Sigma^{u,v}} : (\Sigma^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \quad \text{and} \quad \varpi_-|_{\Sigma^{u,v}} : (\Sigma^{u,v}, \pi_{\rm st}) \to (B_-\backslash B_-vB_-, \pi_{-1})$$

in Lemma 6 is in general not a *symplectic dual pair* [W4] which requires the two Poisson maps to be surjective submersions and their fibers to be mutual symplectic orthogonals of each other. □

5. The double Bruhat cells $G^{v,v}$ as Poisson groupoids

Let the notation be as in §4. In this section, we apply the results in §3 to the Poisson Lie group $(G, \pi_{\rm st})$ to construct an action Poisson groupoid $((G/B)\times B_-, \pi)$ over $(G/B, \pi_1)$. For $v \in W$, the choice of a representative $\bar{v}$ of v in $N_G(T)$ is used to identify $(G^{v,v}, \pi_{\rm st})$ with a Poisson subgroupoid of $((G/B) \times B_-, \pi)$ through a Poisson embedding $I_{\bar{v}} : (B_-vB_-, \pi_{\rm st}) \hookrightarrow ((G/B) \times B_-, \pi)$.

1. The action Poisson groupoid $((G/B) \times B_-, \pi)$ over $(G/B, \pi_1)$

Let G act on the flag variety G/B from the *right* by

$$(G/B) \times G \to G/B,\ (g.B, g_1) \mapsto g_1^{-1}g.B, \quad g, g_1 \in G,$$

and let $\sigma : \mathfrak{g} \to \mathcal{V}^1(G/B)$ be the induced *right* Lie algebra action of $\mathfrak{g}$ on G/B given by

$$\sigma(x) = -\varpi(x^R) \quad \text{or} \quad \sigma(x)(g.B) = \frac{d}{dt}\Big|_{t=0} \exp(-tx)g.B \quad x \in \mathfrak{g}, g \in G, \tag{32}$$

where recall that $\varpi : G \to G/B$ is the projection. Restricting the G-action on G/B to one of B_- on G/B, one then has the action groupoid $(G/B)\times B_- \rightrightarrows G/B$, with the source map θ, the target map τ, the groupoid multiplication μ, the inverse map ι, and the identity bisection ϵ respectively given by

$$\theta(g.B, b_-) = g.B, \quad \tau(g.B, b_-) = (b_-^{-1}g).B, \tag{33}$$

$$\mu(g.B, b_-, b_-^{-1}g.B, b'_-) = (g.B, b_-b'_-), \tag{34}$$

$$\iota(g.B, b_-) = (b_-^{-1}g.B, b_-^{-1}),\ \epsilon(g.B) = (g.B, e),\ b_-, b'_- \in B_-, g \in G. \tag{35}$$

Consider the Poisson structure $\pi_1 = \varpi(\pi_{\rm st})$ on G/B. As $\pi_{\rm st} = r_{\rm st}^L - r_{\rm st}^R$ and $\varpi(r_{\rm st}^L) = 0$, one has

$$\pi_1 = -\varpi(r_{\rm st}^R) = -\sigma(r_{\rm st}). \tag{36}$$

Let $\lambda_- : \mathfrak{b}_- \to \mathcal{V}^1(B_-)$ be given by $\lambda_-(x) = x^R$ for $x \in \mathfrak{b}_-$. As $\sigma|_{\mathfrak{b}} : \mathfrak{b} \to \mathcal{V}^1(G/B)$ is a right Poisson action of the Lie bialgebra $(\mathfrak{b}, -\delta_{\rm st}|_{\mathfrak{b}})$ on $(G/B, \pi_1)$, one has the mixed product Poisson structure π on $(G/B) \times B_-$ given by

$$\pi = \pi_1 \times_{(\sigma|_{\mathfrak{b}}, \lambda_-)} \pi_{\rm st} = (\pi_1, 0) + (0, \pi_{\rm st}) - \sum_{i=1}^n (\sigma(\xi_i), 0) \wedge (0, x_i^R), \tag{37}$$

where $\{x_i\}_{i=1}^n$ is any basis of $\mathfrak{b}_-$ and $\{\xi_i\}_{i=1}^n$ the dual basis of $\mathfrak{b}$ with respect to the pairing $\langle\ ,\ \rangle_{(\mathfrak{b}_-, \mathfrak{b})}$ between $\mathfrak{b}_-$ and $\mathfrak{b}$ given in (21). By Corollary 5 and (36),

$$((G/B) \times B_- \rightrightarrows G/B,\ \pi)$$

is an action Poisson groupoid over the Poisson manifold $(G/B, \pi_1)$. Note that the bases for $\mathfrak{b}_-$ and $\mathfrak{b}$ in (37) can be taken to be the ones in (21).

2. The Poisson embedding of $(B_-vB_-, \pi_{\rm st})$ into $((G/B)\times B_-, \pi)$

Recall that $N_G(T)$ is the normalizer subgroup of T in G. In this section, fix $v \in W$ and let $\bar{v} \in N_G(T)$ be any representative of v in $N_G(T)$. Let

$$C_{\bar{v}} = N\bar{v} \cap \bar{v}N_- \subset G. \tag{38}$$

It is well-known that the multiplication maps

$$\begin{array}{lll} C_{\bar{v}} \times B \to BvB, & (c,b) \mapsto cb, & c \in C_{\bar{v}}, b \in B, \\ B_- \times C_{\bar{v}} \to B_-vB_-, & (b_-,c) \mapsto b_-c, & b_- \in B_-, c \in C_{\bar{v}}, \end{array}$$

are algebraic isomorphisms. Consider now the embedding

$$I_{\bar{v}}\colon B_-vB_- \to (G/B)\times B_-,\ I_{\bar{v}}(b_-c) = (b_-c.B, b_-), \quad b_- \in B_-, c \in C_{\bar{v}}. \tag{39}$$

The goal of §2 is to prove the following Proposition 9.

Proposition 9. *The embedding $I_{\bar{v}} : (B_-vB_-, \pi_{\rm st}) \to ((G/B)\times B_-, \pi)$ is Poisson.*

To prepare for the proof of Proposition 9, we first prove some properties of $C_{\bar{v}}$.

Lemma 10. *The submanifold $C_{\bar{v}}$ of G is coisotropic with respect to the Poisson structure $\pi_{\rm st}$.*

Proof. Consider first the subgroup $N_v = N \cap (\bar{v}N_-\bar{v}^{-1})$ with Lie algebra $\mathfrak{n}_v = \mathfrak{n} \cap \mathrm{Ad}_{\bar{v}}\mathfrak{n}_-$. We first show that $N_v \subset G$ is coisotropic with respect to $\pi_{\rm st}$. With $\mathfrak{g}^* \cong \mathfrak{g}^*_{\rm st}$, where the pairing between $\mathfrak{g} \cong \mathfrak{g}_{\rm diag}$ and $\mathfrak{g}^*_{\rm st}$ is via the bilinear form $\langle\, ,\, \rangle_{\mathfrak{g}\oplus\mathfrak{g}}$ on $\mathfrak{g}\oplus\mathfrak{g}$, the annihilator subspace $\mathfrak{n}_v^0 = \{\xi \in \mathfrak{g}^* : \xi|_{\mathfrak{n}_v} = 0\}$ of $\mathfrak{n}_v$ in $\mathfrak{g}^*_{\rm st}$ is

$$\{(x_+ + x_0, -x_0 + x_-) :\ x_+ \in \mathfrak{n}, x_0 \in \mathfrak{h}, x_- \in \mathfrak{n}_- \cap \mathrm{Ad}_{\bar{v}}\mathfrak{n}_-\},$$

which is a Lie subalgebra of $\mathfrak{g}^*_{\rm st}$. It follows [LW, STS] that N_v is a coisotropic subgroup of $(G, \pi_{\rm st})$.

Let $c \in C_{\bar{v}}$ and write $c = n\bar{v}$, where $n \in N_v$. By the multiplicativity of $\pi_{\rm st}$, one has

$$\pi_{\rm st}(c) = \pi_{\rm st}(n\bar{v}) = l_n\pi_{\rm st}(\bar{v}) + r_{\bar{v}}\pi_{\rm st}(n).$$

As N_v is coisotropic with respect to $\pi_{\rm st}$, $\pi_{\rm st}(n) \in (T_nG)\wedge(T_nN_v)$, so $r_{\bar{v}}\pi_{\rm st}(n) \in (T_cG)\wedge(T_cC_{\bar{v}})$. On the other hand, it is easy to see that

$$\pi_{\rm st}(\bar{v}) = -r_{\bar{v}}\Bigg(\sum_{\alpha>0, v^{-1}\alpha<0} E_{-\alpha}\wedge E_\alpha\Bigg). \tag{40}$$

It follows that $l_n\pi_{\rm st}(\bar{v}) \in (T_cG)\wedge(T_cC_{\bar{v}})$. Thus $C_{\bar{v}}$ is a coisotropic submanifold of $(G, \pi_{\rm st})$. □

Lemma 11. *The map*

$$q_{\bar{v}}\colon (B_-vB_-, \pi_{\rm st}) \to (B_-, \pi_{\rm st}), \quad q_{\bar{v}}(b_-c) = b_-, \quad b_- \in B_-, c \in C_{\bar{v}},$$

is Poisson.

Proof. (See also [GSV, Thm. 3.1]) Let $b_- \in B_-$ and $c \in C_{\bar{v}}$. By the multiplicativity of $\pi_{\rm st}$, one has $\pi_{\rm st}(b_-c) = l_{b_-}\pi_{\rm st}(c) + r_c\pi_{\rm st}(b_-)$. As $C_{\bar{v}}$ is a coisotropic submanifold of $(B_-vB_-, \pi_{\rm st})$, one has $\pi_{\rm st}(c) \in T_cC_{\bar{v}} \wedge T_c(B_-vB_-)$. As $q_{\bar{v}}(l_{b_-}T_cC_{\bar{v}}) = 0$, one has $q_{\bar{v}}l_{b_-}\pi_{\rm st}(c) = 0$. Using the fact that $\pi_{\rm st}(b_-) \in \wedge^2 T_{b_-}B_-$, one sees that

$$q_{\bar{v}}(\pi_{\rm st}(b_-c)) = (q_{\bar{v}}r_c)(\pi_{\rm st}(b_-)) = \pi_{\rm st}(b_-). \qquad \square$$

Proof of Proposition 9. Let $(B_-vB_-)_{\rm diag} = \{(g,g) : g \in B_-vB_-\}$. Then $I_{\bar{v}}$ is the restriction to $(B_-vB_-)_{\rm diag} \subset G \times (B_-vB_-)$ of the map

$$K_{\bar{v}}\colon G \times (B_-vB_-) \to (G/B) \times B_-,\ (g, b_-c) \mapsto (g.B, b_-), \quad g\in G, b_-\in B_-, c\in C_{\bar{v}}.$$

By §2 and in particular (25), both $(B_-vB_-)_{\rm diag}$ and $G \times (B_-vB_-)$ are Poisson submanifolds of $G \times G$ with respect to the Poisson structure $\Pi_{\rm st}$. It is thus enough to show that

$$K_{\bar{v}}\colon (G \times (B_-vB_-), \Pi_{\rm st}) \to ((G/B) \times B_-, \pi)$$

is Poisson. Let again $(x_i)_{i=1}^n$ be any basis of $\mathfrak{b}_-$ and $(\xi_i)_{i=1}^n$ the basis of $\mathfrak{b}$ dual to $(x_i)_{i=1}^n$ under the pairing $\langle\ ,\ \rangle_{(\mathfrak{b}_-,\mathfrak{b})}$ in (21). By (25), one has $\Pi_{\rm st} = (\pi_{\rm st}, 0) + (0, \pi_{\rm st}) + \mu_1 + \mu_2$, where

$$\mu_1 = \sum_{i=1}^n (\xi_i^R, 0) \wedge (0, x_i^R) \quad \text{and} \quad \mu_2 = -\sum_{i=1}^n (\xi_i^L, 0) \wedge (0, x_i^L).$$

By the definition of π_1, $K_{\bar{v}}(\pi_{\rm st}, 0) = (\pi_1, 0)$. By Lemma 11, $K_{\bar{v}}(0, \pi_{\rm st}) = (0, \pi_{\rm st})$. Since for any $\xi \in \mathfrak{b}$, the vector field ξ^L on G vanishes when projected to G/B, one has $K_{\bar{v}}(\mu_2) = 0$. It is also clear from the definitions that $K_{\bar{v}}(\mu_1)$ coincides with the mixed term of π. Thus $K_{\bar{v}}$ is Poisson.

This finishes the proof of Proposition 9. $\square$

Remark 9 (The Poisson structure $\pi_{\rm st}$ on B_-vB_- as a mixed product). Define

$$\Psi\colon\ (G/B) \times B_- \to B_- \times (G/B),\ \ \Psi(g.B, b_-) = (b_-^{-1}, g.B),$$

and consider the Poisson structure $\pi' = -\Psi(\pi)$ on $B_- \times (G/B)$. It is easy to see that

$$\pi' = \pi_{\rm st} \times_{(\rho_-,\lambda_+)} (-\pi_1),$$

where ρ_- and λ_+ denote the Poisson Lie group actions as well as the induced Lie bialgebra actions, respectively given by

$$\begin{aligned} &(B_-, \pi_{\rm st}) \times (B_-, \pi_{\rm st}) \to (B_-, \pi_{\rm st}), && (b_-, b'_-) \mapsto b_-b'_-, && b_-, b'_- \in B_-, \\ &(B, -\pi_{\rm st}) \times (G/B, -\pi_1) \to (G/B, -\pi_1), && (b, g.B) \mapsto bg.B, && b \in B, g \in G. \end{aligned}$$

One then has the Poisson embedding

$$\Psi\circ\iota\circ I_{\bar{v}}\colon (B_-vB_-, \pi_{\rm st}) \to (B_-\times(G/B), \pi'),\ b_-c \mapsto (b_-, c.B),\ \ b_-\in B_-, c\in C_{\bar{v}}, \quad (41)$$

where ι is the inverse map of the Poisson groupoid $((G/B) \times B_- \rightrightarrows G/B, \pi))$. Note the image of B_-vB_- under $\Psi \circ \iota \circ I_{\bar{v}}$ is the Poisson submanifold $B_- \times (BvB)/B$ of $(B_- \times (G/B), \pi')$. We have thus identified the restriction of $\pi_{\rm st}$ to B_-vB_- as the mixed product Poisson structure π' on the product manifold $B_- \times (BvB/B)$ via the map in (41). $\square$

Remark 10. Consider also the Poisson embedding

$$J_{\bar{v}} \stackrel{\text{def}}{=} \iota \circ I_{\bar{v}} \colon (B_-vB_-, -\pi_{\rm st}) \to ((G/B) \times B_-, \pi),$$
$$J_{\bar{v}}(b_-c) = (c.B, b_-^{-1}), \;\; b_- \in B_-, c \in C_{\bar{v}}.$$

Then $J_{\bar{v}}(B_-vB_-) = (BvB/B) \times B_-$. As v runs over W, one has the respective disjoint unions

$$G = \bigsqcup_{v \in W} B_-vB_- \quad \text{and} \quad (G/B) \times B_- = \bigsqcup_{v \in W} (BvB/B) \times B_-$$

of the Poisson varieties $(G, -\pi_{\rm st})$ and $((G/B) \times B_-, \pi)$ into Poisson subvarieties, together with piecewise Poisson isomorphisms $\{J_{\bar{v}} : v \in W\}$, but these piecewise Poisson isomorphisms do not patch together to define a smooth map from G to $(G/B) \times B_-$. $\square$

Example 2. Let $G = \mathrm{SL}(2, \mathbb{C})$ and let B and B_- be the subgroups of G consisting of upper and lower triangular matrices respectively. Let $s \in W$ be the non-trivial element, so that

$$B_-sB_- = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, b \neq 0 \right\}.$$

Identify the flag variety G/B with the complex projective space $\mathbb{CP}^1$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.B \mapsto [a, c]$. For $\bar{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the map $J_{\bar{s}} : B_-sB_- \to \mathbb{CP}^1 \times B_-$ is given by

$$J_{\bar{s}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left([a, -b], \begin{pmatrix} -b^{-1} & 0 \\ d & -b \end{pmatrix} \right),$$

which does not extend to a smooth map from G to $\mathbb{CP}^1 \times B_-$. $\square$

3. Poisson embeddings of $(G^{u,v}, \pi_{\rm st})$ into $((G/B) \times B_-, \pi)$

Recall that θ and τ are respectively the source and target maps of the action groupoid $(G/B) \times B_-$ over G/B, and note that the image of B_-vB_- under the embedding $I_{\bar{v}}$ is

$$I_{\bar{v}}(B_-vB_-) = \tau^{-1}(BvB/B) = \iota((BvB/B) \times B_-).$$

For $u \in W$, restricting $I_{\bar{v}}$ to $G^{u,v} = BuB \cap B_-vB_- \subset B_-vB_-$, one has the embedding

$$I_{\bar{v}}|_{G^{u,v}} \colon \; G^{u,v} \hookrightarrow (G/B) \times B_-. \tag{42}$$

For $u, v \in W$, set

$$F^{u,v} \stackrel{\text{def}}{=} \theta^{-1}(BuB/B) \cap \tau^{-1}(BvB/B) \subset (G/B) \times B_-, \quad u, v \in W. \tag{43}$$

It is clear from the definitions that

$$I_{\bar{v}}(G^{u,v}) = F^{u,v}, \qquad u \in W. \tag{44}$$

Let T act on $(G/B) \times B_-$ via

$$t \cdot (g.B, b_-) = (tg.B, tb_-), \quad t \in T, g \in G, b_- \in B_-. \tag{45}$$

Proposition 12. *The mixed product Poisson structure π on $(G/B)\times B_-$ is invariant under the T-action, and its T-leaves are precisely the intersections $F^{u,v}$, where $u, v \in W$.*

Proof. For each $v \in W$, choose a representative $\bar{v}$ of v in $N_G(T)$. Let T act on B_-vB_- by left translation. Clearly, $I_{\bar{v}}\colon B_-vB_- \to (G/B)\times B_-$ is T-equivariant. The statement of Proposition 12 now follows from the T-equivariant Poisson isomorphisms $I_{\bar{v}}$, $v \in W$, and the fact that the T-leaves of $\pi_{\rm st}$ in B_-vB_- are the $G^{u,v}$'s for $u \in W$. □

Remark 11 (The Fomin–Zelevinsky twist map). Let $u, v \in W$ and let $\bar{u}$ and $\bar{v}$ be any representatives of u and v in $N_G(T)$ respectively. Recall that the inverse map ι of the Poisson groupoid $((G/B)\times B_-, \pi)$ satisfies $\iota(\pi) = -\pi$. As $\iota(F^{u,v}) = F^{v,u}$, by Proposition 9,

$$\iota^{\bar{u},\bar{v}} \stackrel{\rm def}{=} (I_{\bar{u}}|_{G^{v,u}})^{-1}\circ\iota\circ(I_{\bar{v}}|_{G^{u,v}})\colon (G^{u,v},\pi_{\rm st}) \to (G^{v,u},\pi_{\rm st}) \tag{46}$$

is anti-Poisson. Explicitly, the map $\iota^{\bar{u},\bar{v}} : G^{u,v} \to G^{v,u}$ is given by

$$\iota^{\bar{u},\bar{v}}(g) = b_-^{-1}c = c'b^{-1},$$
$$\text{if } \ g = cb = b_-c' \in G^{u,v}, \text{where } \ c \in C_{\bar{u}}, b \in B, b_- \in B_-, c' \in C_{\bar{v}},$$

or, if for $h \in N_-TN$, we write $h = [h]_-[h]_0[h]_+$ with $[h]_- \in N_-$, $[h]_0 \in T$, $[h]_+ \in N$, then

$$\iota^{\bar{u},\bar{v}}(g) = \left([\bar{u}^{-1}g]_-^{-1}\bar{u}^{-1}g\bar{v}^{-1}[g\bar{v}^{-1}]_+^{-1}\right)^{-1}, \quad g \in G^{u,v}. \tag{47}$$

In [FZ, §1.5], Fomin and Zelevinsky introduced a twist map $G^{u,v} \to G^{u^{-1},v^{-1}}$ (for certain special ways of choosing $\bar{u}$ and $\bar{v}$). By (47), the Fomin–Zelevinsky twist map is the composition of $\iota^{\bar{u},\bar{v}}$ with the group inverse $G \to G, g \mapsto g^{-1}$, of G and with an involutive automorphism $x \to x^\theta$ of G (see [FZ, Formula (1.11)], while the latter two involutions are easily seen to be both anti-Poisson with respect to $\pi_{\rm st}$. It follows that the Fomin–Zelevinsky twist $(G^{u,v},\pi_{\rm st}) \to (G^{u^{-1},v^{-1}},\pi_{\rm st})$ is anti-Poisson, a fact already proved in [GSV, Thm. 3.1]. □

Remark 12. Consider the two disjoint union decompositions

$$G = \bigsqcup_{u,v\in W} G^{u,v}, \qquad (G/B)\times B_- = \bigsqcup_{u,v\in W} F^{u,v}. \tag{48}$$

Let T act on G by left multiplication and on $(G/B)\times B_-$ by (45). Then the two decompositions in (48) are respectively that of T-leaves of $(G,\pi_{\rm st})$ and $((G/B)\times B_-,\pi)$. Any choice $\{\bar{v} \in N_G(T) : v \in W\}$ gives rise to piecewise T-equivariant Poisson isomorphisms

$$I_{\bar{v}}\colon (B_-vB_- = \bigsqcup_{u\in W} G^{u,v},\pi_{\rm st}) \to (\tau^{-1}(BvB/B) = \bigsqcup_{u\in W} F^{u,v},\pi)$$

but the maps $\{I_{\bar{v}} : v \in W\}$ do not patch together to define a smooth map from G to $(G/B)\times B_-$. See also Remark 10. □

4. The double Bruhat cell $G^{v,v}$ as Poisson groupoids

Observe that for any $v \in W$,

$$F^{v,v} = \theta^{-1}(BvB/B) \cap \tau^{-1}(BvB/B) \subset (G/B) \times B_-$$

is the subgroupoid of $(G/B) \times B_- \rightrightarrows G/B$ over the subset BvB/B of G/B.

Definition 2. For $v \in W$ and any representative $\bar{v}$ of v in $N_G(T)$, denote by $G^{\bar{v},\bar{v}} \rightrightarrows BvB/B$ the double Bruhat cell $G^{v,v}$, equipped with the groupoid structure induced by the isomorphism $I_{\bar{v}} : G^{v,v} \to F^{v,v}$. In details, the groupoid structure is defined as follows: for $g = cb = b_-c' \in G^{v,v}$, where $b \in B$, $b_- \in B_-$, and $c, c' \in C_{\bar{v}}$,

$$\begin{aligned} \text{source map}: \ & \theta_{\bar{v}}(g) = g.B = c.B, \\ \text{target map}: \ & \tau_{\bar{v}}(g) = c'.B, \\ \text{inverse map}: \ & \iota_{\bar{v}}(g) = c'b^{-1} = b_-^{-1}c, \\ \text{identity bisection}: \ & \epsilon_{\bar{v}}(c.B) = c \in C_{\bar{v}} \subset G^{v,v}. \end{aligned}$$

If $h \in G^{v,v}$ is such that $\tau_{\bar{v}}(g) = \theta_{\bar{v}}(h)$, so $h = c'b' = b'_-c''$, with $b' \in B$, $b'_- \in B_-$, and $c'' \in C_{\bar{v}}$, the groupoid product of g and h is given by

$$\mu_{\bar{v}}(g, h) = cbb' = b_-b'_-c''. \tag{49}$$

The following Theorem 13, which follows directly from Proposition 9, is the first main result of this paper.

Theorem 13. *For any $v \in W$ and $\bar{v} \in N_G(T)$, the pair $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ is a Poisson groupoid over the Poisson manifold $(BvB/B, \pi_1)$.*

Proof. It is clear that all the structure maps of the groupoid $G^{\bar{v},\bar{v}} \rightrightarrows BvB/B$ are smooth. As $C_{\bar{v}} \subset G^{\bar{v},\bar{v}}$, the source map $\theta_{\bar{v}}$ is surjective. By Lemma 7, $\theta_{\bar{v}}$ is a submersion. Thus $G^{\bar{v},\bar{v}}$ is a Lie groupoid over BvB/B. As $I_{\bar{v}}(G^{v,v})$ is a Poisson submanifold of $(G/B) \times B_-$ with respect to π, $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ is a Poisson groupoid over $(BvB/B, \pi_1)$. □

Remark 13. If $\bar{v}, \widetilde{v}$ are two representatives of $v \in W$ and if $t \in T$ is such that $\bar{v} = t\widetilde{v}$, then the left translation $l_t : (G^{\widetilde{v},\widetilde{v}}, \pi_{\rm st}) \to (G^{\bar{v},\bar{v}}, \pi_{\rm st})$ is a Poisson groupoid isomorphism covering the Poisson isomorphism $l_t : (BvB/B, \pi_1) \to (BvB/B, \pi_1)$. Hence the isomorphism class of $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ as a Poisson groupoid is independent of the choice of the representative $\bar{v}$. □

Recall that $\varpi_- : G \to B_-\backslash G$ is the projection, and for each $v \in W$, $B_-\backslash B_-vB_-$ is a Poisson submanifold of $B_-\backslash G$ with respect to the Poisson structure $\pi_{-1} = \varpi_-(\pi_{\rm st})$. For $v \in W$ and any representative $\bar{v}$ of v in $N_G(T)$, define

$$\Phi_{\bar{v}} \colon B_-\backslash B_-vB_- \to BvB/B, \quad B_{-.}c \mapsto c.B, \quad c \in C_{\bar{v}}. \tag{50}$$

Lemma 14. *For $v \in W$ and any representative $\bar{v}$ of v in $N_G(T)$,*

$$\Phi_{\bar{v}} : \quad (B_-\backslash B_-vB_-, \pi_{-1}) \to (BvB/B, \pi_1)$$

is an anti-Poisson isomorphism.

Proof. It is proved in [EL, Appendix A] that if $\rho_Y : (X, \pi_X) \to (Y, \pi_Y)$ and $\rho_Z : (X, \pi_X) \to (Z, \pi_Z)$ form a Poisson pair and if X' is a coisotropic submanifold of (X, π_X) such that $\rho_Y|_{X'} : X' \to Y$ is a diffeomorphism, then $\Phi = \rho_Z \circ (\rho_Y|_{X'})^{-1} : (Y, \pi_Y) \to (Z, \pi_Z)$ is an anti-Poisson map. Applying the above statement to the Poisson pair $(\varpi_-|_{G^{v,v}}, \varpi|_{G^{v,v}})$ in Lemma 6 and the coisotropic submanifold $C_{\bar{v}}$ of $(G^{v,v}, \pi_{\rm st})$, one proves Lemma 14. □

Remark 14. With $\Phi_{\bar{v}}$ defined in (50), for $u \in W$, let

$$\varpi_{\bar{v}}^{u,v} = \Phi_{\bar{v}} \circ (\varpi_-|_{G^{u,v}}) \colon G^{u,v} \to BvB/B, \; b_-c \mapsto c.B, \quad b_- \in B_-, c \in C_{\bar{v}}. \quad (51)$$

It follows from Lemma 14 that $\varpi_{\bar{v}}^{u,v} : (G^{u,v}, \pi_{\rm st}) \to (BvB/B, \pi_1)$ is anti-Poisson. Consequently, by Lemma 6, one has the Poisson pairs

$$\varpi|_{G^{u,v}} \colon (G^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \quad \text{and} \quad \varpi_{\bar{v}}^{u,v} \colon (G^{u,v}, \pi_{\rm st}) \to (BvB/B, -\pi_1),$$
$$\varpi|_{\Sigma^{u,v}} \colon (\Sigma^{u,v}, \pi_{\rm st}) \to (BuB/B, \pi_1) \quad \text{and} \quad \varpi_{\bar{v}}^{u,v} \colon (\Sigma^{u,v}, \pi_{\rm st}) \to (BvB/B, -\pi_1),$$

where $\Sigma^{u,v}$ is any symplectic leaf of $\pi_{\rm st}$ in $G^{u,v}$. Note that when $u = v$, $\varpi|_{G^{v,v}} = \theta_{\bar{v}}$ and $\varpi_{\bar{v}}^{v,v} = \tau_{\bar{v}}$, the source and target maps of the Poisson groupoid $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ over $(BvB/B, \pi_1)$. □

5. Commuting Poisson actions of $(G^{\bar{u},\bar{u}}, \pi_{\rm st})$ and $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ on $(G^{u,v}, \pi_{\rm st})$

Recall that if $(\mathcal{G} \rightrightarrows Y, \pi_{\mathcal{G}})$ is a Poisson groupoid over a Poisson manifold (Y, π_Y) with target map $\tau : \mathcal{G} \to Y$, a *left Poisson action* of $(\mathcal{G}, \pi_{\mathcal{G}})$ on a Poisson manifold (X, π_X) is a left Lie groupoid $\mathcal{G}$-action on X with a moment map $\nu : X \to Y$ and an action map

$$\mathbf{a} \colon \mathcal{G} * X \stackrel{\rm def}{=} \{(\gamma, x) \in \mathcal{G} \times X : \tau(\gamma) = \nu(x)\} \to X$$

such that $\text{Graph}(\mathbf{a}) \stackrel{\rm def}{=} \{(\gamma, x, \mathbf{a}(\gamma, x)) : (\gamma, x) \in \mathcal{G} * X\}$ is a coisotropic submanifold of the Poisson manifold $(\mathcal{G} \times X \times X, \pi_{\mathcal{G}} \times \pi_X \times (-\pi_X))$. In such a case, the moment map $\nu : (X, \pi_X) \to (Y, \pi_Y)$ is automatically Poisson [LWX2]. Note that the moment map ν is required to be a submersion to ensure that $\mathcal{G} * X$ is a smooth submanifold of $\mathcal{G} \times X$. Right Poisson actions of Poisson groupoids are similarly defined, where the moment maps are necessarily anti-Poisson.

Let now $u, v \in W$ and let $\bar{u}, \bar{v}$ be any respective representatives of u and v in $N_G(T)$. Then it is straightforward to check that the groupoid $G^{\bar{u},\bar{u}}$ acts on $G^{u,v}$ on the left with the moment map $\varpi|_{G^{u,v}} : G^{u,v} \to BuB/B$, where the action of $g \in G^{\bar{u},\bar{u}}$ on $x \in G^{u,v}$ with $\tau_{\bar{u}}(g) = \varpi(x)$ is the element $g \triangleright x \in G^{u,v}$ given by

$$g \triangleright x \stackrel{\rm def}{=} cbb' = b_-b'_-c'' \quad \text{if } \; g = cb = b_-c', \; x = c'b' = b'_-c'', \quad (52)$$

with $c, c' \in C_{\bar{u}}, c'' \in C_{\bar{v}}, b, b' \in B$ and $b_-, b'_- \in B_-$. Similarly the groupoid $G^{\bar{v},\bar{v}}$ acts on $G^{u,v}$ on the right with the moment map $\varpi_{\bar{v}}^{u,v} : G^{u,v} \to BvB/B$ (see (51)), and the action of $h \in G^{\bar{v},\bar{v}}$ on $x \in G^{u,v}$ with $\varpi_{\bar{v}}^{u,v}(x) = \theta_{\bar{v}}(h)$ is the element $x \triangleleft h \in G^{u,v}$ given by

$$x \triangleleft h \stackrel{\rm def}{=} c'b'b'' = b'_-b''_-c''', \quad \text{if } x = c'b' = b'_-c'', \text{ and } \; h = c''b'' = b''_-c''', \quad (53)$$

with $c' \in C_{\bar{u}}$, $c'', c''' \in C_{\bar{v}}, b', b'' \in B$ and $b'_-, b''_- \in B_-$. One can also check directly that the two groupoid actions commute.

Theorem 15. *For any $u, v \in W$ and respective representatives $\bar{u}, \bar{v} \in N_G(T)$,* (52) *and* (53) *are respectively left and right Poisson actions of the Poisson groupoids $(G^{\bar{u},\bar{u}}, \pi_{\rm st})$ and $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ on $(G^{u,v}, \pi_{\rm st})$.*

Proof. Consider first the right action of $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ on $(G^{u,v}, \pi_{\rm st})$. Under the Poisson embedding $I_{\bar{v}} : (B_-vB_-, \pi_{\rm st}) \to ((G/B) \times B_-, \pi_1)$, one has $I_{\bar{v}}(G^{u,v}) = F^{u,v}$ and $I_{\bar{v}}(G^{v,v}) = F^{v,v}$ (see (43)), and the right action of $G^{\bar{v},\bar{v}}$ on $G^{u,v}$ corresponds to the right action of $F^{v,v}$ on $F^{u,v}$ by restricting the right Poisson action of the Poisson groupoid $((G/B) \times B_-, \pi)$ on itself by right multiplication with the target map τ as the moment map. As $F^{v,v}$ and $F^{u,v}$ are both Poisson submanifolds of $(G/B) \times B_-$ with respect to π, the right action of $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ on $(G^{u,v}, \pi_{\rm st})$ is Poisson.

By replacing (u, v) by (v, u) in the above arguments, one has a right Poisson action

$$G^{v,u} \times G^{\bar{u},\bar{u}} \ni (x, g) \mapsto x \triangleleft g \in G^{v,u} \quad \text{if} \quad \varpi_{\bar{u}}^{v,u}(x) = \theta_{\bar{u}}(g). \tag{54}$$

One now checks directly that under the Poisson isomorphisms

$$(\iota^{\bar{u},\bar{v}})^{-1} \colon (G^{v,u}, \pi_{\rm st}) \to (G^{u,v}, -\pi_{\rm st}) \quad \text{and} \quad \iota_{\bar{u}} \colon (G^{\bar{u},\bar{u}}, \pi_{\rm st}) \to (G^{\bar{u},\bar{u}}, -\pi_{\rm st}),$$

where the Poisson isomorphism $\iota^{\bar{u},\bar{v}} : (G^{u,v}, \pi_{\rm st}) \to (G^{v,u}, -\pi_{\rm st})$ is given in (46), the right groupoid action of $G^{\bar{u},\bar{u}}$ on $G^{v,u}$ in (54) becomes precisely the left action of the groupoid $G^{\bar{u},\bar{u}}$ on $G^{u,v}$ given in (52). This shows that the groupoid action in (52) is Poisson. $\square$

6. Symplectic groupoids associated to double Bruhat cells

1. Symplectic leaves in $G^{u,v}$

To describe the symplectic leaves of $\pi_{\rm st}$ in G, it is enough to describe the symplectic leaves in the double Bruhat cells, as the latter are the T-orbits of symplectic leaves of $\pi_{\rm st}$ in G. For $u, v \in W$, and for any symplectic leaf Σ of $\pi_{\rm st}$ in $G^{u,v}$, let $T_\Sigma = \{t \in T : \Sigma t = \Sigma\}$. As T acts transitively on the set of all symplectic leaves of $\pi_{\rm st}$ in $G^{u,v}$, T_Σ is independent of $\Sigma \subset G^{u,v}$. We define the *leaf-stabilizer of T in $G^{u,v}$* to be

$$T^{u,v}_{\rm stab} = T_\Sigma, \tag{55}$$

where Σ is any symplectic leaf of $\pi_{\rm st}$ in $G^{u,v}$. In particular, one has

$$\dim(\Sigma) = l(u) + l(v) + \dim(T^{u,v}_{\rm stab}).$$

When G is simply connected, symplectic leaves of $\pi_{\rm st}$ in each $G^{u,v}$ are determined by Kogan and Zelevinsky in [KZ] using specially chosen representatives in $N_G(T)$ of elements in W. In this section, for G simply connected, we adapt the results in [KZ] to describe the symplectic leaves of $\pi_{\rm st}$ in G using arbitrary choices of representatives of elements in W, and we describe the leaf-stabilizers of T in the double Bruhat cells. We also extend some results from [KZ] to the case when G is not necessarily simply connected.

Assume first that G is connected but not necessarily simply connected. The action of the Weyl group on T will be denoted as $t^v = \bar{v}^{-1} t \bar{v}$, where $v \in W, t \in T$, and $\bar{v}$ is any representative of v in $N_G(T)$. For $u, v \in W$, let

$$T^{u,v} = \{(t^u)^{-1} t^v : t \in T\}.$$

Fix $u, v \in W$ and let $\bar{u}, \bar{v}$ be any representatives of u and v in $N_G(T)$, respectively. Note that

$$\bar{u}^{-1} B u B = \bar{u}^{-1} C_{\bar{u}} B \subset N_- T N \quad \text{and} \quad B_- v B_- \bar{v}^{-1} = B_- C_{\bar{v}} \bar{v}^{-1} \subset N_- T N,$$

and recall that for $g \in N_- T N$, we write $g = [g]_- [g]_0 [g]_+$, where $[g]_- \in N_-, [g]_0 \in T, [g]_+ \in N$. For $t \in T$, define

$$S_{[t]}^{\bar{u},\bar{v}} = \left\{ g \in G^{u,v} : \; \left[\bar{u}^{-1} g\right]_0 \left[g \, \bar{v}^{-1}\right]_0^v \in t T^{u,v} \right\}, \tag{56}$$

where $[t]$ denotes the image of t in $T/T^{u,v}$. Define the map

$$\chi \colon G^{u,v} \to T/T^{u,v}, \;\; \chi(g) = [\bar{u}^{-1} g]_0 [g \, \bar{v}^{-1}]_0^v \, T^{u,v} \in T/T^{u,v}, \qquad g \in G^{u,v}. \tag{57}$$

Then clearly $S_{[t]}^{\bar{u},\bar{v}} = \chi^{-1}([t])$ for $t \in T$, a level set of χ. One also has

$$\chi(ga) = [a]^2 \chi(g), \quad g \in G^{u,v}, a \in T. \tag{58}$$

The following Lemma 16 is proved in [KZ, Prop. 3.1] (neither the assumption that G be simply-connected nor the special way of choosing representatives of Weyl group elements in $N_G(T)$ made in [KZ] is needed in its proof).

Lemma 16 ([KZ, Prop. 3.1]). *The symplectic leaves of $\pi_{\rm st}$ in $G^{u,v}$ are the connected components of the sets $S_{[t]}^{\bar{u},\bar{v}}$, $t \in T$. Moreover, for any $t_1, t_2, t \in T$, $S_{[t_1]}^{\bar{u},\bar{v}} = S_{[t_2]}^{\bar{u},\bar{v}}$ if and only if $[t_1] = [t_2]$, and $S_{[t_1]}^{\bar{u},\bar{v}} t = S_{[t_1 t^2]}^{\bar{u},\bar{v}}$.*

Assume now that G is simply-connected, and let $\Gamma \subset \Delta_+$ be the set of simple roots. For $\alpha \in \Gamma$, let $\omega_\alpha \in \mathrm{Hom}(T, \mathbb{C}^\times)$ be the corresponding fundamental weight, and let Δ_α be the corresponding generalized principal minor [FZ], [KZ], which is a regular function on G whose restriction to $N_- T N$ is given by $\Delta_\alpha(g) = [g]_0^{\omega_\alpha}$. For $u, v \in W$, let $I(u,v) = I(u) \cap I(v)$, where

$$I(u) = \{\alpha \in \Gamma : \; u(\omega_\alpha) = \omega_\alpha\} = \Gamma \backslash \{\alpha_1, \ldots, \alpha_l\}$$

for any reduced word $u = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$, and define the maps $\delta, \delta^2 : G \to \mathbb{C}^{|I(u,v)|}$ by

$$\delta(g) = \{\Delta_\alpha(g) : \; \alpha \in I(u,v)\} \quad \text{and} \quad \delta^2(g) = \{(\Delta_\alpha(g))^2 : \; \alpha \in I(u,v)\}.$$

We now modify the results from [KZ] to give a description of the connected components of $S_{[t]}^{\bar{u},\bar{v}}$, and thus also of the symplectic leaves of $\pi_{\rm st}$ in $G^{u,v}$.

Proposition 17. *Assume that G is simply connected. Let $u, v \in W$ and let $\bar{u}$ and $\bar{v}$ be any respective representatives of u and v in $N_G(T)$. Then for any $t \in T$, the restriction of δ^2 to $S_{[t]}^{\bar{u},\bar{v}}$ is a constant map, or, more precisely,*

$$(\Delta_\alpha(g))^2 = \Delta_\alpha(\bar{u})\Delta_\alpha(\bar{v})\, t^{\omega_\alpha}, \qquad \forall\ g \in S_{[t]}^{\bar{u},\bar{v}}. \tag{59}$$

The connected components of $S_{[t]}^{\bar{u},\bar{v}}$ are the $2^{|I(u,v)|}$ (all of which non-empty) level sets of the map $\delta : S_{[t]}^{\bar{u},\bar{v}} \to (\mathbb{C}^\times)^{|I(u,v)|}$.

Proof. By first choosing a set $\{e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, \alpha^\vee \in \mathfrak{h} : \alpha \in \Gamma\}$ of Chevalley generators of $\mathfrak{g}$ which determines Lie group homomorphisms $\phi_\alpha : \mathrm{SL}(2,\mathbb{C}) \to G$ for each $\alpha \in \Gamma$, one can choose the representative $\widetilde{s}_\alpha$ of s_α in $N_G(T)$ to be $\widetilde{s}_\alpha = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for each $\alpha \in \Gamma$. For $w \in W$ and any reduced word $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$ of w, the element $\widetilde{w} = \widetilde{s}_{\alpha_1}\widetilde{s}_{\alpha_2} \cdots \widetilde{s}_{\alpha_l}$ is then a representative of w in $N_G(T)$ independent of the choice of the reduced word. Moreover [MR, Lem. 6.1], $\Delta_\alpha(\widetilde{w}) = 1$ if $\alpha \in I(w)$. Define

$$S_e^{u,v} = \left\{ g \in G^{u,v} :\ \left[\widetilde{u}^{-1} g\right]_0 \left[g\, \widetilde{v^{-1}}\right]_0^v \in T^{u,v} \right\}.$$

By [KZ, Thm. 2.3, Cor. 2.5, Lem. 3.2], $[\widetilde{u}^{-1} g]_0^{\omega_\alpha} = \pm 1$ for all $g \in S_e^{u,v}$ and $\alpha \in I(u,v)$, and $S_e^{u,v}$ has $2^{|I(u,v)|}$ connected components $S_e^{u,v} = \bigsqcup_\epsilon S_e^{u,v}(\epsilon)$, where ϵ runs over the set of all sign functions $\epsilon : I(u,v) \to \{\pm 1\}$ on $I(u,v)$, and

$$S_e^{u,v}(\epsilon) = \left\{ g \in S_e^{u,v} :\quad \left[\widetilde{u}^{-1} g\right]_0^{\omega_\alpha} = \epsilon(\alpha), \forall\, \alpha \in I(u,v) \right\}.$$

Let $t_0, t_1 \in T$ be such that $\widetilde{u} = t_0 \bar{u}$ and $\widetilde{v^{-1}}\bar{v} = t_1$. One checks directly that for $g \in G^{u,v}$,

$$\left[\widetilde{u}^{-1} g\right]_0 = \left[\bar{u}^{-1} g\right]_0 (t_0^{-1})^u \quad \text{and} \quad \left[g\, \widetilde{v^{-1}}\right]_0^v = \left[g\, \bar{v}^{-1}\right]_0^v t_1.$$

It follows that for any $t \in T$ and $a \in T$ with $a^2 = (t_0^{-1})^u t_1 t$, one has

$$S_{[t]}^{\bar{u},\bar{v}} = \left\{ g \in G^{u,v} :\ \left[\widetilde{u}^{-1} g\right]_0 \left[g\, \widetilde{v^{-1}}\right]_0^v \in (t_0^{-1})^u t_1 t T^{u,v} \right\} = S_e^{u,v} a.$$

Note now (see [KZ, (3.12)]) that $\left[\widetilde{u}^{-1} g\right]_0^{\omega_\alpha} = \Delta_\alpha(g)$ for any $\alpha \in I(u)$ and $g \in BuB$. It follows that $\Delta_\alpha(g) = \pm 1$ for all $g \in S_e^{u,v}$ and $\alpha \in I(u,v)$. Consequently, for all $g \in S_{[t]}^{\bar{u},\bar{v}}$ and $\alpha \in I(u,v)$,

$$(\Delta_\alpha(g))^2 = \Delta_\alpha(a^2) = t_0^{-\omega_\alpha} t_1^{\omega_\alpha} t^{\omega_\alpha} = \Delta_\alpha(\bar{u})\Delta_\alpha(\bar{v}) t^{\omega_\alpha}.$$

As there is one connected component of $S_e^{u,v}$ for each sign function ϵ on $I(u,v)$, the connected components of $S_{[t]}^{\bar{u},\bar{v}} = S_e^{u,v} a$ are precisely the $2^{|I(u,v)|}$ level sets, all of which non-empty, of the map $\delta : S_{[t]}^{\bar{u},\bar{v}} \to (\mathbb{C}^\times)^{|I(u,v)|}$. $\square$

Recall the map $\chi : G^{u,v} \to T/T^{u,v}$ defined in (57). The following Corollary 18 is also proved in [Y, Cor. 4.5].

Corollary 18. *For any $g_0 \in G^{u,v}$, the symplectic leaf Σ^{g_0} of $\pi_{\rm st}$ through g_0 is given by*

$$\Sigma^{g_0} = \{g \in G^{u,v} : \ \chi(g) = \chi(g_0) \ \text{and} \ \delta(g) = \delta(g_0)\}. \tag{60}$$

Remark 15. Using the decompositions $BuB = C_{\bar{u}}B$ and $B_-vB_- = B_-C_{\bar{v}}$, one can describe the maps χ and δ on $G^{u,v}$ more explicitly. Indeed, writing an element $g \in G^{u,v}$ as $g = ctn = n_-t_-c'$, where $c \in C_{\bar{u}}$, $c' \in C_{\bar{v}}$, $t, t_- \in T$, $n \in N$ and $n_- \in N_-$, one has $\chi(g) = [tt_-^v] \in T/T^{u,v}$ and $\Delta_\alpha(g) = t^{\omega_\alpha}\Delta_\alpha(\bar{u})$ for all $\alpha \in I(u)$. □

When $u = v$, one has $T^{u,v} = \{e\}$. As a special case of Corollary 18, one has

Corollary 19. *Assume that G is simply connected. Let $v \in W$ and let $\bar{v}$ be any representative of v in $N_G(T)$. Then the symplectic leaf $\Sigma^{\bar{v}}$ of $\pi_{\rm st}$ in G through $\bar{v}$ is given by*

$$\begin{aligned}\Sigma^{\bar{v}} &= \left\{g \in G^{v,v} : \ \left[\bar{v}^{-1}g\right]_0 \left(\left[g\,\bar{v}^{-1}\right]_0\right)^v = e, \Delta_\alpha(g) = \Delta_\alpha(\bar{v})\, \forall \alpha \in I(v)\right\} \\ &= \left\{g \in G^{v,v} : \ \left[\bar{v}^{-1}g\right]_0 \left(\left[g\,\bar{v}^{-1}\right]_0\right)^v = e, \left[\bar{v}^{-1}g\right]_0^{\omega_\alpha} = 1\, \forall \alpha \in I(v)\right\}\end{aligned}$$

Still assuming that G is simply connected, let

$$\widetilde{T}^{u,v} = \{t \in T : \ t^{\omega_\alpha} = 1 \ \forall \ \alpha \in I(u,v)\}.$$

It is clear that $T^{u,v} \subset \widetilde{T}^{u,v}$. As a direct consequence of Corollary 18 and (58), one has

Corollary 20. *Assume that G is simply connected. Then for $u, v \in W$, the leaf-stabilizer of T in $G^{u,v}$ is given by $T^{u,v}_{\rm stab} = \{t \in \widetilde{T}^{u,v} : \ t^2 \in T^{u,v}\}$.*

Returning now to the connected semisimple complex Lie group G which may not be simply connected, let $\widehat{G}$ be the connected and simply connected cover of G, and let $\kappa : \widehat{G} \to G$ be the covering map with $\ker \kappa = Z$, a subgroup of the center of $\widehat{G}$. Denoting by $\widehat{\pi}_{\rm st}$ the multiplicative Poisson structure on $\widehat{G}$ defined by the same r-matrix $r_{\rm st} \in \mathfrak{g} \otimes \mathfrak{g}$, the map $\kappa : (\widehat{G}, \widehat{\pi}_{\rm st}) \to (G, \pi_{\rm st})$ is then Poisson. For $\widehat{g} \in \widehat{G}$ and $g \in G$, let again $\Sigma^{\widehat{g}} \subset \widehat{G}$ and $\Sigma^g \subset G$ respectively denote the symplectic leaves of $\widehat{\pi}_{\rm st}$ and $\pi_{\rm st}$ through $\widehat{g}$ and g. Let $\widehat{T} = \kappa^{-1}(T)$, a maximal torus of $\widehat{G}$. By Corollary 20, the leaf-stabilizer of $\widehat{T}$ in $\widehat{G}^{u,v}$ is

$$\widehat{T}^{u,v}_{\rm stab} = \{\widehat{a} \in \widehat{T} : \ \widehat{a}^{\omega_\alpha} = 1 \ \forall \ \alpha \in I(u,v) \ \text{ and } \ \widehat{a}^2 \in \widehat{T}^{u,v}\}.$$

Let $Z_{u,v} = Z \cap \widehat{T}^{u,v}_{\rm stab} = \{z \in Z : \ z^{\omega_\alpha} = 1 \ \forall \ \alpha \in I(u,v) \ \text{ and } \ z^2 \in \widehat{T}^{u,v}\}$.

Lemma 21. *For any $\widehat{g} \in \widehat{G}^{u,v}$, one has $\kappa(\Sigma^{\widehat{g}}) = \Sigma^g$, where $g = \kappa(\widehat{g}) \in G^{u,v}$, and $\kappa : \Sigma^{\widehat{g}} \to \Sigma^g$ is the quotient map $\Sigma^{\widehat{g}} \to \Sigma^{\widehat{g}}/Z_{u,v}$, where $Z_{u,v}$ acts on $\Sigma^{\widehat{g}}$ by multiplication.*

Proof. As $\kappa : (\widehat{G}, \widehat{\pi}_{\rm st}) \to (G, \pi_{\rm st})$ is a local Poisson diffeomorphism, and as $\Sigma^{\widehat{g}}$ is connected, we have $\kappa(\Sigma^{\widehat{g}}) \subset \Sigma^g$. To show that $\Sigma^g \subset \kappa(\Sigma^{\widehat{g}})$, let $h \in \Sigma^g$ and let $\gamma : [0,1] \to \Sigma^g$ be any smooth path in Σ^g such that $\gamma(0) = g$ and $\gamma(1) = h$. Let $\widehat{\gamma} : [0,1] \to \widehat{G}$ be the unique lifting of γ such that $\widehat{\gamma}(0) = \widehat{g}$. Again as κ is a local Poisson diffeomorphism, $\widehat{\gamma}$ is tangent to the symplectic leaf through $\widehat{\gamma}(x)$ for every $x \in [0,1]$. Thus $\widehat{\gamma}([0,1]) \subset \Sigma^{\widehat{g}}$. This shows that $\kappa\left(\Sigma^{\widehat{g}}\right) = \Sigma^g$.

Clearly the $Z_{u,v}$-orbits in $\Sigma^{\widehat{g}}$ are contained in the fibers of $\kappa : \Sigma^{\widehat{g}} \to \Sigma^g$. Suppose that $\widehat{h}, \widehat{k} \in \Sigma^{\widehat{g}}$ are in the same fiber of $\kappa : \Sigma^{\widehat{g}} \to \Sigma^g$. Then $\widehat{h}z = \widehat{k}$ for some $z \in Z$. As $\Sigma^{\widehat{g}}z$ and $\Sigma^{\widehat{g}}$ are both symplectic leaves of $\widehat{\pi}_{\rm st}$ and have now a non-empty intersection, $\Sigma^{\widehat{g}}z = \Sigma^{\widehat{g}}$, and thus $z \in Z_{u,v}$. □

Remark 16. The same arguments as in the proof of Lemma 21 show that if $\kappa : (X, \pi_X) \to (Y, \pi_Y)$ is a covering map that is also Poisson, then the images under κ of the symplectic leaves of (X, π_X) are precisely all the symplectic leaves of (Y, π_Y). □

Lemma 22. *For any $u, v \in W$, the leaf-stabilizer of T in $G^{u,v}$ is given by $T^{u,v}_{\rm stab} = \kappa\left(\widehat{T}^{u,v}_{\rm stab}\right)$.*

Proof. Let $\widehat{\Sigma}$ be a symplectic of $\widehat{\pi}_{\rm st}$ in $\widehat{G}^{u,v}$, and let $\Sigma = \kappa(\widehat{\Sigma})$. If $\widehat{a} \in \widehat{T}^{u,v}_{\rm stab}$, then it follows from $\widehat{\Sigma}\widehat{a} = \widehat{\Sigma}$ that $\Sigma\kappa(\widehat{a}) = \Sigma$, so $\kappa(\widehat{a}) \in T^{u,v}_{\rm stab}$. Conversely, let $a \in T^{u,v}_{\rm stab}$ and choose any $\widehat{a} \in \kappa^{-1}(a)$. Let $\widehat{g} \in \widehat{\Sigma}$. Then $\kappa(\widehat{g}\widehat{a}) \in \Sigma a = \Sigma$, so $\kappa(\widehat{g}\widehat{a}) = \kappa(\widehat{g}')$ for some $\widehat{g}' \in \widehat{\Sigma}$. Let $z \in Z$ be such that $\widehat{g}\widehat{a}z = \widehat{g}'$. As $\widehat{\Sigma}\widehat{a}z$ and $\widehat{\Sigma}$ are two symplectic leaves of $\widehat{\pi}_{\rm st}$ and have a non-empty intersection, one must have $\widehat{\Sigma}\widehat{a}z = \widehat{\Sigma}$, and thus $a = \kappa(\widehat{a}z) \in \kappa\left(\widehat{T}^{u,v}_{\rm stab}\right)$. □

Recall from Lemma 16 that symplectic leaves of $\pi_{\rm st}$ in $G^{u,v}$ are the connected components of the sets $S^{\bar{u},\bar{v}}_{[t]}$ given in (56), where $t \in T$. Define

$$T^{(2)} = \{a \in T : \ a^2 = e\}.$$

It is clear that for each $t \in T$, $S^{\bar{u},\bar{v}}_{[t]}$ is invariant under left translation by elements in $T^{(2)}$.

Lemma 23. *For any $t \in T$, the induced action of $T^{(2)}$ on the set of all symplectic leaves of $\pi_{\rm st}$ in $S^{\bar{u},\bar{v}}_{[t]}$ is transitive.*

Proof. Let Σ and Σ' be any two symplectic leaves of $\pi_{\rm st}$ in $S^{\bar{u},\bar{v}}_{[t]}$, and let $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ be two symplectic leaves of $\widehat{\pi}_{\rm st}$ in $\widehat{G}^{u,v}$ such that $\kappa(\widehat{\Sigma}) = \Sigma$ and $\kappa(\widehat{\Sigma}') = \Sigma'$. Let $[\kappa] : \widehat{T}/\widehat{T}^{u,v} \to T/T^{u,v}$ be the group homomorphism induced by $\kappa : \widehat{T} \to T$. Then the fibers of $[\kappa]$ are the Z-orbits in $\widehat{T}/\widehat{T}^{u,v}$ by multiplication. Let $\widehat{u}$ and $\widehat{v}$ be any respective representatives of u and v in $N_{\widehat{G}}(\widehat{T}) \subset \widehat{G}$. Recalling the map $\widehat{\chi} : \widehat{G}^{\widehat{u},\widehat{v}} \to \widehat{T}/\widehat{T}^{u,v}$ defined as in (57), one has

$$[\kappa](\widehat{\chi}(\widehat{\Sigma})) = [\kappa](\widehat{\chi}(\widehat{\Sigma}')) = [t].$$

Thus there exists $z \in Z$ such that $z\widehat{\chi}(\widehat{\Sigma}) = \widehat{\chi}(\widehat{\Sigma}')$. Let $\widehat{a} \in \widehat{T}$ be such that $\widehat{a}^2 = z$. Then $\widehat{\chi}(\widehat{\Sigma}\widehat{a}) = \widehat{\chi}(\widehat{\Sigma}')$. By Proposition 17 (see also the proof of [KZ, Thm. 2.3]), the group $\widehat{T}^{(2)} = \{\widehat{x} \in \widehat{T} : \widehat{x}^2 = e\}$ acts transitively on the set of the symplectic leaves of $\widehat{\pi}_{\rm st}$ in any level set of $\widehat{\chi}$. Thus there exists $\widehat{x} \in \widehat{T}^{(2)}$ such that $\widehat{\Sigma}\widehat{a}\widehat{x} = \widehat{\Sigma}'$. Let $a = \kappa(\widehat{a}\widehat{x}) \in T$. Then $a \in T^{(2)}$ and $\Sigma a = \Sigma'$. □

Remark 17. It follows from Lemma 23 and Lemma 22 that for $t \in T$, the number of symplectic leaves of $\pi_{\rm st}$ in $S^{\bar{u},\bar{v}}_{[t]}$ is equal to $\left|T^{(2)}/T^{(2)} \cap T^{u,v}_{\rm stab}\right|$. As $T^{(2)}$ is a 2-group, the number of symplectic leaves of $\pi_{\rm st}$ in $S^{\bar{u},\bar{v}}_{[t]}$ is always a power of 2. □

2. The symplectic leaf $\Sigma^{\bar{v}}$ as a symplectic groupoid

Let now $(G, \pi_{\rm st})$ be any standard complex semisimple Poisson Lie group, where G is connected but not necessarily simply connected. Let $v, u \in W$ and let $\bar{u}$ and $\bar{v}$ be any respective representatives of u and v in $N_G(T)$. One then has the Poisson groupoid $(G^{\bar{u},\bar{u}}, \pi_{\rm st})$ over $(BuB/B, \pi_1)$ and the Poisson groupoid $(G^{\bar{v},\bar{v}}, \pi_{\rm st})$ over $(BvB/B, \pi_1)$. Recall their commuting (left and right) Poisson actions on $(G^{u,v}, \pi_{\rm st})$, respectively given in (52) and (53).

Theorem 24. 1) *The symplectic leaf* $\Sigma^{\bar{v}}$ *of* $\pi_{\rm st}$ *through* $\bar{v}$ *is a Lie subgroupoid of* $G^{\bar{v},\bar{v}}$. *Consequently,* $(\Sigma^{\bar{v}}, \pi_{\rm st})$ *is a symplectic groupoid over* $(BvB/B, \pi_1)$.

2) *For any symplectic leaf* $\Sigma^{u,v}$ *of* $\pi_{\rm st}$ *in* $G^{u,v}$, *the two commuting Poisson actions in* (52) *and* (53) *restrict to Poisson actions of the symplectic groupoids* $(\Sigma^{\bar{u}}, \pi_{\rm st})$ *and* $(\Sigma^{\bar{v}}, \pi_{\rm st})$ *on the symplectic manifold* $(\Sigma^{u,v}, \pi_{\rm st})$.

Proof. Assume first that G is simply connected. Consider the action in (52). Assume that $g \in \Sigma^{\bar{u}}$ and $x \in \Sigma^{u,v}$ be such that $\tau_{\bar{u}}(g) = \varpi(x)$, and write $g = ctn = n_-t_-c'$ and $x = c't'n' = n'_-t'_-c''$, where $c, c' \in C_{\bar{u}}, c'' \in C_{\bar{v}}$, $t, t_-, t', t'_- \in T$, $n, n' \in N$, and $n_-, n'_- \in N_-$. Then $g \triangleright x = ctnt'n' = n_-t_-n'_-t'_-c''$. By Proposition 17, $tt^u_- = e$, $t^{\omega_\alpha} = 1$ for all $\alpha \in I(u,v)$, and $\Sigma^{u,v} = \{h \in G^{u,v} : \chi(h) = \chi(x), \Delta_\alpha(h) = \Delta_\alpha(x) \ \forall \ \alpha \in I(u,v)\}$. By the definitions of the map χ and the functions Δ_α (see Remark 15),

$$\chi(g \triangleright x) = [tt'(t_-t'_-)^v] = [tt^u_-t'(t'_-)^v(t^u_-)^{-1}t^v_-] = [t'(t'_-)^v] = \chi(x) \in T/T^{u,v},$$

and for every $\alpha \in I(u,v)$, $\Delta_\alpha(g \triangleright x) = (tt')^{\omega_\alpha}\Delta_\alpha(\bar{u}) = (t')^{\omega_\alpha}\Delta_\alpha(\bar{u}) = \Delta_\alpha(x)$. Thus $g \triangleright x \in \Sigma^{u,v}$. Similarly, one shows that for all $x \in \Sigma^{u,v}$ and $h \in \Sigma^{\bar{v}}$ with $\varpi^{u,v}_{\bar{v}}(x) = \theta_{\bar{v}}(h)$ one has $x \triangleleft h \in \Sigma^{u,v}$. Applying to the special case of $u = v, \bar{u} = \bar{v}$ and $\Sigma^{u,v} = \Sigma^{\bar{v}}$, it shows in particular that $\Sigma^{\bar{v}}$ is closed under the groupoid multiplication of $G^{\bar{v},\bar{v}}$. It is easy to see that $\Sigma^{\bar{v}}$ is closed under the groupoid inverse of $G^{\bar{v},\bar{v}}$. By Lemma 7, both $\varpi|_{\Sigma^{u,v}} : \Sigma^{u,v} \to BuB/B$ and $\varpi^{u,v}_{\bar{v}}|_{\Sigma^{u,v}} : \Sigma^{u,v} \to BvB/B$ are submersions. Thus $\Sigma^{\bar{v}}$ is a Lie subgroupoid of $G^{\bar{v},\bar{v}}$, and since $\dim(\Sigma^{\bar{v}}) = 2l(v)$ by Corollary 20, it is a symplectic groupoid over $(BvB/B, \pi_1)$. Furthermore, the two actions in (52) and (53) restrict to Poisson actions of the symplectic groupoids $(\Sigma^{\bar{u}}, \pi_{\rm st})$ and $(\Sigma^{\bar{v}}, \pi_{\rm st})$ on the symplectic manifold $(\Sigma^{u,v}, \pi_{\rm st})$.

For an arbitrary G, let $\widehat{G}$ be the simply connected cover of G with $\kappa : \widehat{G} \to G$ the covering map and multiplicative Poisson structure $\widehat{\pi}_{\rm st}$, and choose any $\widehat{u}, \widehat{v} \in \widehat{G}$ such that $\kappa(\widehat{u}) = \bar{u}$ and $\kappa(\widehat{v}) = \bar{v}$. Let $Z = \ker \kappa$, and let $\widehat{\Sigma}^{u,v}$ be any symplectic leaf of $\widehat{\pi}_{\rm st}$ such that $\kappa(\widehat{\Sigma}^{u,v}) = \Sigma^{u,v}$. By Lemma 21, the symplectic groupoids

$(\Sigma^{\bar{u}}, \pi_{\rm st})$ and $(\Sigma^{\bar{v}}, \pi_{\rm st})$ are the respective quotients of the symplectic groupoids $(\Sigma^{\widehat{u}}, \widehat{\pi}_{\rm st})$ and $(\Sigma^{\widehat{v}}, \widehat{\pi}_{\rm st})$ by $Z_{u,u}$ and $Z_{v,v}$, and that $\kappa : \widehat{\Sigma}^{u,v} \to \Sigma^{u,v}$ is the quotient map by $Z_{u,v}$. It is easy to see that $Z_{u,u} \subset Z_{u,v}$ and $Z_{v,v} \subset Z_{u,v}$. Statements 1) and 2) for G now follow from the corresponding statements for $\widehat{G}$. $\square$

Remark 18. Let $u, v \in W$, and let $\Sigma^u \subset G^{u,u}, \Sigma^{u,v} \subset G^{u,v}$, and $\Sigma^v \subset G^{v,v}$ be arbitrary symplectic leaves of $\pi_{\rm st}$. As $\Sigma^u = \Sigma^{\bar{u}}$ and $\Sigma^v = \Sigma^{\bar{v}}$ for some representatives of $\bar{u}$ and $\bar{v}$, we conclude that Σ^u and Σ^v are symplectic groupoids, respectively over (BuB, π_1) and $(BvB/B, \pi_1)$, acting by commuting Poisson actions from the left and right on the symplectic groupoid $(\Sigma^{u,v}, \pi_{\rm st})$. $\square$

Example 3. Let $G = \mathrm{SL}(2, \mathbb{C})$, where the pair (B, B_-) consists of the subgroups of respectively upper and lower triangular matrices, and where $\langle x_1, x_2\rangle_{\mathfrak{g}} = \mathrm{tr}(x_1x_2)$, $x_1, x_2 \in \mathfrak{sl}(2, \mathbb{C})$. Writing $g \in G$ as $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, the Poisson brackets between the coordinate functions are

$$\begin{aligned} &\{g_{11}, g_{12}\} = g_{11}g_{12}, \quad \{g_{11}, g_{21}\} = g_{11}g_{21}, \quad \{g_{12}, g_{22}\} = g_{12}g_{22}, \\ &\{g_{21}, g_{22}\} = g_{21}g_{22}, \quad \{g_{11}, g_{22}\} = 2g_{12}g_{21}, \ \{g_{12}, g_{21}\} = 0. \end{aligned}$$

Let $\bar{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $C_{\bar{s}} = \left\{ \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} : z \in \mathbb{C} \right\}$. Then

$$G^{s,s} = \left\{ \begin{pmatrix} az & a^{-1}(abz-1) \\ a & b \end{pmatrix} : \ a, b, z \in \mathbb{C}, a \neq 0, abz - 1 \neq 0 \right\},$$

with the Poisson structure given by $\{z, a\} = za$, $\{z, b\} = a^{-1}(abz-2)$, $\{a, b\} = ab$. Let $\chi = a^2(1 - abz)^{-1}$. The groupoid structure on $G^{\bar{s},\bar{s}}$ over $\mathbb{C}$ is given by

$$\begin{aligned} \text{source map}: &\ \theta_{\bar{s}}(z, a, b) = z, \\ \text{target map}: &\ \tau_{\bar{s}}(z, a, b) = \chi z, \\ \text{inverse map}: &\ \iota_{\bar{s}}(z, a, b) = (\chi z, a^{-1}, -b), \\ \text{identity bisection}: &\ z \mapsto (z, 1, 0), z \in \mathbb{C}, \\ \text{multiplication}: &\ \mu_{\bar{s}}((z_1, a_1, b_1), (z_2, a_2, b_2)) = (z_1, a_1a_2, a_1b_2 + b_1a_2^{-1}) \\ &\qquad\qquad \text{if } z_2 = \tau_{\bar{s}}(z_1, a_1, b_1). \end{aligned}$$

Note that χ is a Casimir function on $G^{s,s}$ and the symplectic leaves in $G^{s,s}$ are precisely given by the (non-zero) level sets of χ. Hence the symplectic leaf $\Sigma_{\bar{s}}$ of $\pi_{\rm st}$ through $\bar{s} \in G^{\bar{s},\bar{s}}$ is $\Sigma_{\bar{s}} = \left\{ \begin{pmatrix} az & -a \\ a & b \end{pmatrix} : \ a, b, z \in \mathbb{C}, a \neq 0, a^2 = 1 - abz \right\}$. Identify $\Sigma_{\bar{s}}$ with

$$\Sigma = \left\{ \begin{pmatrix} pt & -t \\ t & -qt \end{pmatrix} : (p, q, t) \in \mathbb{C}^3, t^2(1 - pq) = 1 \right\}. \tag{61}$$

The induced (non-degenerate) Poisson structure on Σ is given by

$$\{p, q\} = 2(1 - pq), \quad \{p, t\} = pt, \quad \{q, t\} = -qt, \tag{62}$$

and the induced symplectic groupoid structure on Σ is given by

$$\begin{aligned}
\text{source map}: &\ \theta(p,q,t)=p,\\
\text{target map}: &\ \tau(p,q,t)=p,\\
\text{inverse map}: &\ \iota(p,q,t)=(p,-qt^2,\,t^{-1}),\\
\text{identity section}: &\ \epsilon(p)=(p,0,1),\\
\text{multiplication}: &\ \mu((p_1,q_1,t_1),(p_2,q_2,t_2))=(p_1,q_2+q_1t_2^{-2},t_1t_2)\text{ when }p_1=p_2.
\end{aligned}$$

Note that $\theta^{-1}(0)=\tau^{-1}(0)$ is isomorphic to the non-connected abelian Lie group $\mathbb{C}\times\mathbb{Z}_2$.

Consider now the group $P\mathrm{SL}(2,\mathbb{C})$, and write its elements as $[g]$, where $g\in\mathrm{SL}(2,\mathbb{C})$. Then the symplectic leaf of π_{st} through $[\bar{s}]\in P\mathrm{SL}(2,\mathbb{C})$ is parametrized by the surface

$$\Sigma_0=\{(p,q)\in\mathbb{C}^2:1-pq\neq 0\}\cong\left\{\left[\begin{pmatrix}p & -1\\ 1 & -q\end{pmatrix}\right]:(p,q)\in\mathbb{C}^2,1-pq\neq 0\right\},$$

with the Poisson structure $\{p,q\}=2(1-pq)$ and the groupoid structure given by

$$\begin{aligned}
\text{source map}: &\ \theta(p,q)=p,\\
\text{target map}: &\ \tau(p,q)=p,\\
\text{inverse map}: &\ \iota(p,q,t)=(p,q(pq-1)^{-1}),\\
\text{identity bisection}: &\ \{(p,0):p\in\mathbb{C}\},\\
\text{multiplication}: &\ \mu((p_1,q_1),(p_2,q_2))=(p_1,q_2+q_1(1-p_2q_2))\ \text{ if }p_1=p_2.
\end{aligned}$$

Note the Lie group isomorphisms $\theta^{-1}(0)=\tau^{-1}(0)\cong\mathbb{C}$ and $\theta^{-1}(p)=\tau^{-1}(p)\cong\mathbb{C}^\times$ for $p\neq 0$. □

Example 4. Let $G=\mathrm{SL}(3,\mathbb{C})$, with B, B_- respectively the subgroups of upper and lower triangular matrices and the bilinear form $\langle x_1,x_2\rangle_{\mathfrak{g}}=\mathrm{tr}(x_1x_2)$ on $\mathfrak{sl}(3,\mathbb{C})$. Let s_1,s_2 be the two generators of the Weyl group W, identified with the symmetric group S_3. Let $v=s_1s_2$,

$$\bar{s}_1=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \bar{s}_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\quad\text{and}\quad \bar{v}=\bar{s}_1\bar{s}_2.$$

Let $\Sigma_{\bar{s}_1},\Sigma_{\bar{s}_2},\Sigma_{\bar{v}}$ be the symplectic leaves of π_{st} through respectively $\bar{s}_1,\bar{s}_2,\bar{v}$. The group multiplication $(G^{s_1,s_1},\pi_{\mathrm{st}})\times(G^{s_2,s_2},\pi_{\mathrm{st}})\to(G^{v,v},\pi_{\mathrm{st}})$ is a Poisson morphism, and one can check that its restriction gives a Poisson isomorphism $(\Sigma_{\bar{s}_1},\pi_{\mathrm{st}})\times(\Sigma_{\bar{s}_2},\pi_{\mathrm{st}})\cong(\Sigma_{\bar{v}},\pi_{\mathrm{st}})$. One thus has

$$\begin{aligned}
\Sigma_{\bar{v}}&=\left\{\begin{pmatrix}p_1t_1 & -t_1 & 0\\ t_1 & -q_1t_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & p_2t_2 & -t_2\\ 0 & t_2 & -q_2t_2\end{pmatrix}:t_1^2(1-p_1q_1)=1,t_2^2(1-p_2q_2)=1\right\}\\
&\cong\Sigma\times\Sigma=\{(p_1,q_1,t_1,p_2,q_2,t_2):\ (p_j,q_j,t_j)\in\Sigma,j=1,2\},
\end{aligned}$$

where Σ is given in (61), and $\pi_{\rm st}$ is identified with the direct Poisson bracket given in (62). On the other hand, parametrize $BvB/B \subset G/B$ by

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto [z_1, z_2] \stackrel{\rm def}{=} \begin{pmatrix} z_1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z_2 & -1 \\ 0 & 1 & 0 \end{pmatrix} B \in BvB/B.$$

The Poisson structure π_1 on BvB/B is then given by $\{z_1, z_2\} = -z_1z_2$. One checks that the groupoid structure on $\Sigma^{\bar{v}}$ over BvB/B is given as follows:

$$\begin{aligned} \text{source map}: &\ \theta(p_1,q_1,t_1,p_2,q_2,t_2) = [p_1, p_2t_1^{-1}], \\ \text{target map}: &\ \tau(p_1,q_1,t_1,p_2,q_2,t_2) = [p_1t_2^{-1}, p_2], \\ \text{inverse map}: &\ \iota(p_1,q_1,t_1,p_2,q_2,t_2) = (p_1t_2^{-1}, -q_1t_1^2t_2, t_1^{-1}, p_2t_1^{-1}, -q_2t_1t_2^2, t_2^{-1}), \\ \text{identity bisection}: &\ \epsilon(z_1,z_2) = (z_1, 0, 1, z_2, 0, 1), \end{aligned}$$

and the groupoid multiplication is given by

$$\mu(\gamma, \gamma') = (p_1,\, q_1't_2^{-1} + q_1(t_1')^{-2},\, t_1t_1',\, p_2',\, q_2' + q_2(t_1')^{-1}(t_2')^{-2},\, t_2t_2'),$$

if $\gamma = (p_1, q_1, t_1, p_2, q_2, t_2)$ and $\gamma' = (p_1', q_1', t_1', p_2', q_2', t_2')$ with $p_1t_2^{-1} = p_1'$ and $p_2 = p_2'(t_1')^{-1}$. $\square$

Remark 19. For any $v \in W$ and any representative $\bar{v}$ of v in $N_G(T)$, the symplectic groupoid $(\Sigma^{\bar{v}}, \pi_{\rm st})$ over $(BvB/B, \pi_1)$ is algebraic in the sense that $(\Sigma^{\bar{v}}, \pi_{\rm st})$ is an algebraic symplectic variety and that the structure maps for the groupoid are all algebraic morphisms. However, as one can already see in the example of $\mathrm{SL}(2, \mathbb{C})$, the source fibers of these groupoids are not necessarily connected. It would be interesting to understand how source-fiber connected symplectic groupoids can be constructed from the ones in this paper. $\square$.

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