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THE NORI FUNDAMENTAL GERBE OF TAME STACKS

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Abstract. Given an algebraic stack, we compare its Nori fundamental group with that of its coarse moduli space. We also study conditions under which the stack can be uniformized by an algebraic space.

Introduction

The aim here is to show that the results of [Noo04] concerning the étale fundamental group of algebraic stacks also hold for the Nori fundamental group.

Let us start by recalling Noohi's approach. Given a connected algebraic stack $\mathfrak{X}$, and a geometric point $x\colon \operatorname{Spec}\Omega \to \mathfrak{X}$, Noohi generalizes the definition of Grothendieck's étale fundamental group to get a profinite group $\pi_1(\mathfrak{X}, x)$ which classifies finite étale *representable* morphisms (coverings) to $\mathfrak{X}$. He then highlights a new feature specific to the stacky situation: for each geometric point x, there is a morphism $\omega_x\colon \operatorname{Aut} x \to \pi_1(\mathfrak{X}, x)$.

Noohi first studies the situation where $\mathfrak{X}$ admits a moduli space Y, and proceeds to show that if N is the closed normal subgroup of $\pi_1(\mathfrak{X}, x)$ generated by the images of ω_x, for x varying in all geometric points, then

$$\frac{\pi_1(\mathfrak{X}, x)}{N} \simeq \pi_1(Y, y)\ .$$

Noohi turns next to the issue of uniformizing algebraic stacks: he defines a Noetherian algebraic stack $\mathfrak{X}$ as uniformizable if it admits a covering, in the above sense, that is an algebraic space. His main result is that this happens precisely when $\mathfrak{X}$ is a Deligne–Mumford stack and for any geometric point x, the morphism ω_x is injective.

For our purpose, it turns out to be more convenient to use the Nori fundamental gerbe defined in [BV12]. For simplicity, we will assume in the rest of this introduction that $\mathfrak{X}$ is a proper, geometrically connected and reduced algebraic stack

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over a field k, so that a fundamental gerbe

$$\mathfrak{X} \to \pi_{\mathfrak{X}/k}$$

exists, and has a Tannakian interpretation. An essential role is played by residual gerbes at closed points x of $\mathfrak{X}$, denoted by $\mathcal{G}_x \to \mathfrak{X}$.

Let us first describe the content of Section 2. We assume that $\mathfrak{X}$ admits a good moduli space Y in the sense of Alper [Alp08] (this is the case, for instance, when $\mathfrak{X}$ is tame as defined in [AOV08]) and proceed to compare the fundamental gerbes $\pi_{\mathfrak{X}/k}$ and $\pi_{Y/k}$. We use a result of Alper relating vector bundles on $\mathfrak{X}$ and on Y to show that the morphism $\pi_{\mathfrak{X}/k} \to \pi_{Y/k}$ is universal with respect to the property that all composites

$$\mathcal{G}_x \to \mathfrak{X} \to \pi_{\mathfrak{X}/k} \to \pi_{Y/k}$$

are trivial, in a natural sense (see Corollary 2). Using Alper's theorem again, we also prove (Proposition 5) that a given G-torsor $\mathfrak{X}' \to \mathfrak{X}$ is the pullback of a G-torsor on Y via the morphism $\mathfrak{X} \to Y$ if and only if it is isovariant.

In Section 3, we work with stacks with finite inertia (again, tame stacks are examples). We say that such a stack $\mathfrak{X}$ over a field k is Nori-uniformizable if there exists a finite G-torsor $X' \to \mathfrak{X}$, where the total space X' is an algebraic space. Our main technical result (Proposition 11) states that $\mathfrak{X}$ is Nori-uniformizable if and only if all composite morphisms

$$\mathcal{G}_x \to \mathfrak{X} \to \pi_{\mathfrak{X}/k}$$

are representable. This is a "continuous" version of Noohi's main theorem, and this formulation also demonstrates how convenient it is to use Nori's fundamental gerbe instead of Nori's fundamental group scheme. Our main result, Theorem 13, is a Tannakian translation of Proposition 11 that gives a characterization of Nori-uniformizability in terms of restriction of essentially vector bundles on $\mathfrak{X}$ along all morphisms $\mathcal{G}_x \to \mathfrak{X}$. It states, morally, that $\mathfrak{X}$ is Nori-uniformizable if and only if for all x, any representation of $\mathcal{G}_x$ comes from an essentially finite vector bundle on $\mathfrak{X}$. We hope to be able to apply this result to certain orbifolds (called stack of roots) to relate Nori-uniformizability to parabolic bundles.

We conclude this introduction by pointing out that no properness assumption is needed to prove Proposition 11, while it is essential in our proofs of Corollary 2 and Proposition 5. Since Noohi's counterparts hold for any algebraic stack, it is an interesting question if it is possible to remove this hypothesis, but we have no idea of a proof avoiding Tannaka duality at the moment.

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1. Preliminaries

We work over a base field k, and denote $S = \operatorname{Spec} k$. We will mainly be interested in the case where the characteristic p of k is positive.

Concerning Nori fundamental gerbes, we use the terminology introduced in [BV12]. Let us sum up the conventions used and refer to [BV12] for more information.

A Tannakian gerbe (see [BV12, §3]) over S is a fpqc gerbe with affine diagonal and an affine chart. For such gerbes, Tannaka duality holds: our reference is [Saa72, III, §3] in the corrected formulation given in [Del90]. The Nori fundamental gerbe is a Tannakian gerbe.

It is even an inverse limit of finite gerbes. Recall from [BV12, §4] that a finite gerbe is a fppf gerbe with finite diagonal and a finite flat chart. By Artin's theorem, this is an algebraic stack.

Given an algebraic stack $\mathfrak{X}/S$, we say that $\mathfrak{X}$ is inflexible if any morphism to a finite stack factors through a gerbe (see [BV12, Def. 5.3]). This condition is equivalent to the existence of a Nori fundamental gerbe $\pi_{\mathfrak{X}/S}$, i.e., a morphism to a profinite gerbe $\mathfrak{X} \to \pi_{\mathfrak{X}/S}$ which is universal. It is realized for instance when $\mathfrak{X}/S$ is of finite type, geometrically connected, and geometrically reduced.

We now turn to the Tannakian interpretation of the Nori fundamental gerbe. Recall that according to Nori a vector bundle $\mathcal{E}$ is called finite if there is a non-trivial relation between its tensor powers (see [Nor82]). Formally, this means that there are two distinct polynomials $f, g \in \mathbb{N}[t]$ such that $f(\mathcal{E}) \simeq g(\mathcal{E})$, when we replace $+$ by $\oplus$ and $\cdot$ by $\otimes$ when we evaluate a polynomial at a vector bundle. We adopt the definition of an essentially finite vector bundle given in [BV12], so essentially finite vector bundles are precisely the kernels of morphisms between two finite vector bundles.

We will say that an algebraic stack $\mathfrak{X}/S$ is pseudo-proper if for any vector bundle $\mathcal{E}$ on $\mathfrak{X}$, the space of global sections $\Gamma(X, \mathcal{E})$ is finite-dimensional over k (see [BV12, Def. 7.1] for the precise definition). If $\mathfrak{X}/S$ is inflexible and pseudo-proper, pull-back along $\mathfrak{X} \to \pi_{\mathfrak{X}/S}$ identifies representations of $\pi_{\mathfrak{X}/S}$ with essentially finite vector bundles on $\mathfrak{X}$ ([BV12, Thm. 7.9]), thus we get in this situation a Tannakian interpretation of the Nori fundamental gerbe.

Our main reference for stacks is the *Stacks Project* [Stacks]. If x is a point of an algebraic stack $\mathfrak{X}$, we will write $\mathcal{G}_x$ for the residual gerbe at x; see [Stacks, Tag 06ML]. This is a reduced stack with a single point, and there is a canonical monomorphism $\mathcal{G}_x \to \mathfrak{X}$ mapping this unique point to x. By closed point of a stack, we mean as usual a geometric point with closed image.

2. Generators of the Nori fundamental gerbe

In this section, we will deal with algebraic stacks $\mathfrak{X}$ with a *good moduli space*

$$\varphi\colon \mathfrak{X} \to Y$$

in the sense of [Alp08].

2.1. Characterization of essentially finite vector bundles coming from the moduli space

Proposition 1. *Assume $\mathfrak{X}$ is a locally Noetherian algebraic stack with good moduli space*

$$\varphi\colon \mathfrak{X} \to Y\,.$$

The functors φ^* and φ_* induce an equivalence of categories between the category of essentially finite vector bundles $\mathcal{F}$ on Y and the full subcategory of essentially finite vector bundles $\mathcal{E}$ on $\mathfrak{X}$ satisfying the condition that for any closed point x, the restriction $\mathcal{E}|_{\mathcal{G}_x}$ is trivial.

Proof. The fact that the same result holds for vector bundles is proved in [Alp08, Thm. 10.3]. Since φ^* is compatible with tensor products, so is the inverse equivalence φ_*, hence the equivalence holds for finite vector bundles. By definition of a good moduli space, the functor φ_* is exact, and so is the inverse equivalence φ^*. Since an essentially finite vector bundle is by definition the kernel of a morphism between two finite vector bundles, the equivalence holds for essentially finite vector bundles as well. □

Remark 1. It is unclear if the statement holds for *adequate* moduli spaces in the sense of [Alp10]. For vector bundles it is false according to [Alp10, Example 5.6.1].

2.2. Fundamental gerbe of the moduli space

We now use Tannaka duality to translate Proposition 1 in terms of fundamental gerbes. Morally, $\pi_{Y/S}$ is the quotient gerbe obtained from $\pi_{\mathfrak{X}/S}$ after dividing by the "normal sub-gerbe generated by the images of $\mathcal{G}_x \to \pi_{X/S}$". We must be careful since the $\mathcal{G}_x$ are not necessarily defined over the base field k, but only over the extension $k(\varphi(x))$. The precise definitions are as follows.

Definition 1. Let k'/k be a field extension, and $\mathcal{G}$ (respectively, $\mathcal{G}'$) a gerbe over $S = \operatorname{Spec} k$, (respectively, $S' = \operatorname{Spec} k'$). A $S' \to S$-morphism $\mathcal{G}' \to \mathcal{G}$ is *trivial* if there is a morphism $S' \to \mathcal{G}$ (shown below in dotted arrow)

$$\begin{array}{ccc} \mathcal{G}' & \longrightarrow & \mathcal{G} \\ \downarrow & \nearrow & \downarrow \\ S' & \longrightarrow & S \end{array}$$

making both triangles commute.

If the gerbes are Tannakian, this means by duality that the pullback functor

$$\operatorname{Vect} \mathcal{G} \to \operatorname{Vect} \mathcal{G}'$$

sends any object to a trivial one.

Definition 2. Let $(k_i/k)_{i\in I}$ be a family of field extensions, and for each $i \in I$, we are given a k_i/k-morphism

$$\alpha_i \colon \mathcal{G}_i \to \mathcal{G}$$

from a k_i-gerbe $\mathcal{G}_i$ to a fixed k-gerbe $\mathcal{G}$. We say that a morphism of k-gerbes $\mathcal{G} \to \widetilde{\mathcal{G}}$ is a quotient by the "normal sub-gerbe generated by the images of the α_i's" if all composite morphisms $\mathcal{G}_i \to \widetilde{\mathcal{G}}$ are trivial, and $\mathcal{G} \to \widetilde{\mathcal{G}}$ is universal for this property.

It is clear that the quotient gerbe, if it exists, is unique. The existence follows, when all gerbes are Tannakian, from duality; indeed, it is enough to define $\widetilde{\mathcal{G}}$ as the Tannaka dual of the full subcategory of $\operatorname{Vect} \mathcal{G}$ generated by objects that are trivialized by all pullback functors $\alpha_i^* \colon \operatorname{Vect} \mathcal{G} \to \operatorname{Vect} \mathcal{G}_i$.

Remark 2. Even if the quotient makes sense, the "normal sub-gerbe generated by the images of the α_i's" does not always exist, and even if it exists, it is not uniquely defined (see [Mil07]).

Corollary 2. *Let $\mathfrak{X}$ be a locally Noetherian algebraic stack with good moduli space $\varphi\colon \mathfrak{X} \to Y$. Assume that both $\mathfrak{X}$ and Y are inflexible (i.e., admit fundamental gerbes) and are pseudo-proper. Then the fundamental gerbe $\pi_{Y/S}$ is the quotient of $\pi_{\mathfrak{X}/S}$ by the normal sub-gerbe generated by the images of the morphisms $\mathcal{G}_x \to \pi_{\mathfrak{X}/S}$.*

Proof. This follows from the Tannakian interpretation of fundamental gerbes (see [BV12, Thm. 7.9]) and Proposition 1 by duality. □

Example 1. We assume that k is of positive characteristic p. Consider the standard action of $\mu_p \subset \mathbb{G}_m$ on $\mathbb{P}^1$ and put

$$\mathfrak{X} = [\mathbb{P}^1/\mu_p]\,.$$

Then $\varphi\colon \mathfrak{X} \to \mathbb{P}^1$ is a good moduli space (because $\mathfrak{X}$ is in fact tame; see [AOV08]), and Corollary 2 applies: $\pi_{\mathfrak{X}/S}$ is generated by $\mathcal{G}_0$ and $\mathcal{G}_\infty$. In fact, it is easy to show directly that $\pi_{\mathfrak{X}/S} = \mathrm{B}\mu_p$.

2.3. Characterization of torsors coming from the moduli space

Definition 3. Let $f\colon \mathfrak{X} \to \mathcal{G}$ be a morphism from an algebraic stack to a finite gerbe. We say that f *is trivial on inertia* if the morphism $\mathcal{I}_{\mathfrak{X}} \to \mathcal{I}_{\mathcal{G}}$ induced by f factors through the unit morphism $\mathcal{G} \to \mathcal{I}_{\mathcal{G}}$.

Clearly, f is trivial on inertia if and only if for any section $\sigma\colon T \to \mathfrak{X}$, the induced morphism of T-group spaces $\mathbf{Aut}_T\,\sigma \to \mathbf{Aut}_T\,f(\sigma)$ is trivial.

The following corollary of Proposition 1 provides us with an interpretation of

$$\mathfrak{X} \to \pi_{Y/S}$$

as the limit over all morphisms $\mathfrak{X} \to \mathcal{G}$ that are trivial on inertia.

Corollary 3. *With the hypothesis of Corollary 2, a given morphism to a finite gerbe*

$$f\colon \mathfrak{X} \to \mathcal{G}$$

factors through $\mathfrak{X} \to Y$ if and only if f is trivial on inertia.

Proof. The "only if" direction is obvious, thus we assume f is trivial on inertia. By Tannaka duality, we must prove that the functor

$$f^*\colon \operatorname{Vect}\mathcal{G} \to \operatorname{EFVect}\mathfrak{X}$$

factors through $\operatorname{EFVect} Y$; or in other words, according to Proposition 1, that for any representation V of $\mathcal{G}$, and any closed point x of $\mathfrak{X}$, the restriction $f^*V|_{\mathcal{G}_x}$ is trivial. This follows from the fact that for any geometric point $x\colon \operatorname{Spec}\Omega \to \mathfrak{X}$, the morphism

$$\mathbf{Aut}_{\mathfrak{X}}\,x \to \mathbf{Aut}_{\mathcal{G}}\,x$$

is trivial by hypothesis, and by the following lemma.

Lemma 4 ([Alp08, Rem. 10.2])**.** *Let $\mathcal{F}$ be a vector bundle on an algebraic stack $\mathfrak{X}$, and $x\colon \operatorname{Spec}\Omega \to \mathfrak{X}$ be a geometric point with closed image. Then $\mathcal{F}|_{\mathcal{G}_x}$ is trivial if and only if $\mathbf{Aut}_{\mathfrak{X}}\, x$ acts trivially on $\mathcal{F}\bigotimes_{\mathcal{O}_{\mathfrak{X},x}}\Omega$.*

Proof. The last assertion means that $\mathcal{F}|_{\mathrm{B}\,\mathbf{Aut}_{\mathfrak{X}}\, x}$ is trivial. If $p_x\colon \mathcal{G}_x \to \operatorname{Spec} k(x)$ is the structure morphism, then $\mathcal{F}|_{\mathcal{G}_x}$ is trivial if and only if the morphism

$$p_x{}^* p_{x*}\mathcal{F}|_{\mathcal{G}_x} \to \mathcal{F}|_{\mathcal{G}_x}$$

is an isomorphism. Since this property is local, it can be checked on the cover $\mathrm{B}\,\mathbf{Aut}_{\mathfrak{X}}\, x \to \mathcal{G}_x$. □

Let us now specialize the previous discussion to neutral finite gerbes. We first recall a definition due to Joshua [Jos03, Def. 3.1(i)].

Definition 4. A morphism of algebraic stacks $\mathfrak{X}' \to \mathfrak{X}$ is *isovariant* if the following diagram is Cartesian:

$$\begin{array}{ccc} I_{\mathfrak{X}'/S} & \longrightarrow & I_{\mathfrak{X}/S} \\ \downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & \mathfrak{X} \end{array} .$$

Remark 3.

(1) In [Noo04], the alternative name "fixed points reflecting morphism" is used.
(2) Monomorphisms of algebraic stacks are isovariant (Proposition 17). However, of course, there are many more examples; in particular, any morphism between algebraic spaces is isovariant.
(3) It is easy to see that the property of being isovariant is stable by base change, but is not local.

Corollary 5. *Let G/S be a finite group scheme. With the hypothesis of Corollary 2, a G-torsor $\mathfrak{X}' \to \mathfrak{X}$ descends to the moduli space if and only if it is isovariant.*

Proof. According to Corollary 3, the corresponding morphism

$$\mathfrak{X} \to \mathrm{B}G$$

factors through $\mathfrak{X} \to Y$ if and only if it is trivial on inertia. But the sequence

$$\mathfrak{X}' \to \mathfrak{X} \to \mathrm{B}G$$

induces an exact sequence

$$1 \to I_{\mathfrak{X}'/S} \to \left(I_{\mathfrak{X}/S}\right)|_{\mathfrak{X}'} \to \left(I_{\mathrm{B}G/S}\right)|_{\mathfrak{X}'}$$

and the result follows. □

3. Nori-uniformization of stacks with finite inertia

In this section, we will restrict ourselves to *algebraic stacks with finite inertia*, that is, the inertia stack $I_{\mathfrak{X}} \to \mathfrak{X}$ is a finite group space. In particular, tame stacks in the sense of [AOV08] are of this class.

3.1. Nori-uniformizable stack

Definition 5. Let $\mathfrak{X}$ be a stack over a field k. We will say that $\mathfrak{X}$ is *Nori-uniformizable* if there exists a representable k-morphism $\mathfrak{X} \to \mathcal{G}$, where $\mathcal{G}/S$ is a finite gerbe.

Example 2. Assume that k is of positive characteristic p and put

$$\mathfrak{X} = [\mathbb{P}^1/\mu_p]$$

as in Example 1. Then $\mathfrak{X}$ is Nori-uniformizable, but it is not uniformizable by an algebraic space in the sense of [BN06], since it is evident that the pro-étale fundamental gerbe $\pi^{et}_{\mathfrak{X}/S}$ is trivial.

Clearly, if there exists a finite k-group scheme G and a G-torsor $X' \to \mathfrak{X}$, where X' is an algebraic space, then $\mathfrak{X}$ is Nori-uniformizable. As A. Vistoli indicated to us, it turns out that the converse is true. The key point is the following:

Proposition 6. *Let $\mathcal{G}/S$ be an algebraic stack that is a fppf gerbe. Then $\mathcal{G}/S$ is smooth.*

Proof. See [Ber14, Prop. A.2]. □

Lemma 7. *Let $\mathcal{G}/S$ be a finite gerbe, and k'/k be a finite separable extension so that $\mathcal{G}(k') \neq \varnothing$. Denote by $\mathrm{R}_{k'/k}\cdot$ the Weil restriction along k'/k. Then*

1. *$\mathrm{R}_{k'/k}\mathcal{G}_{k'}$ is a finite neutral gerbe over k.*
2. *The canonical morphism $\mathcal{G} \to \mathrm{R}_{k'/k}\mathcal{G}_{k'}$ is representable.*

Proof. We fix a separable closure k^{sep} of k. Then if $n = [k' : k]$, we have:

$$\left(\mathrm{R}_{k'/k}\mathcal{G}_{k'}\right)_{k^{\mathrm{sep}}} \simeq \mathcal{G}^{\times n}_{k^{\mathrm{sep}}} .$$

(1) From [BV12, Lemma 6.2] we know that $\mathrm{R}_{k'/k}\mathcal{G}_{k'}$ is a finite stack. To prove that is it a finite gerbe, according to [BV12, Proposition 4.3], it is enough to prove that it is geometrically connected and geometrically reduced. But if $\mathcal{G}_{k^{\mathrm{sep}}} \simeq \mathrm{B}G$, it follows from the displayed formula that $\left(\mathrm{R}_{k'/k}\mathcal{G}_{k'}\right)_{k^{\mathrm{sep}}} \simeq \mathrm{B}\left(G^{\times n}\right)$; hence $\left(\mathrm{R}_{k'/k}\mathcal{G}_{k'}\right)_{k^{\mathrm{sep}}}$ is a gerbe, and so is geometrically connected and geometrically reduced. To conclude, by definition of Weil restriction, $\mathrm{R}_{k'/k}\mathcal{G}_{k'}(k) = \mathcal{G}_{k'}(k') \neq \varnothing$, that is, $\mathrm{R}_{k'/k}\mathcal{G}_{k'}$ is a neutral gerbe over k.

(2)Proposition 15(5) and the fact that being a monomorphism is local on the base for the fppf topology [Stacks, Tag 02YK], together show that being representable is also local on the base for the fppf topology. So it is enough to prove that $\mathcal{G}_{k^{\mathrm{sep}}} \to \left(\mathrm{R}_{k'/k}\mathcal{G}_{k'}\right)_{k^{\mathrm{sep}}}$ is representable. But if $\mathcal{G}_{k^{\mathrm{sep}}} \simeq \mathrm{B}G$, this morphism identifies with $\mathrm{B}G \to \mathrm{B}\left(G^{\times n}\right)$, which is representable since the diagonal morphism $G \to G^{\times n}$ is a monomorphism. □

Proposition 8. *Let $\mathfrak{X}$ be a stack over a field k. Then $\mathfrak{X}$ is Nori-uniformizable if and only if there exists a finite k-group scheme G and a G-torsor $X' \to \mathfrak{X}$, where X' is an algebraic space.*

Proof. It is enough to prove that any finite gerbe $\mathcal{G}/S$ has this last property. Since surjective and smooth morphisms have sections étale locally, it follows from Proposition 6 that there exists k'/k a finite separable extension so that $\mathcal{G}(k') \neq \varnothing$. Then according to Lemma 7, the canonical morphism $\mathcal{G} \to \mathrm{R}_{k'/k}\mathcal{G}_{k'}$ is a representable morphism to a neutral gerbe. □

Proposition 9. *The Noetherian inflexible stack $\mathfrak{X}$ with finite inertia is Nori-uniformizable if and only if the morphism*

$$\mathfrak{X} \to \pi_{\mathfrak{X}/S}$$

is representable.

Proof. The "only if" part is clear. Indeed, if $\mathfrak{X} \to \mathcal{G}$ is a representable morphism to a finite gerbe, it factors through the morphism $\mathfrak{X} \to \pi_{\mathfrak{X}/S}$, that must then be representable by Proposition 15(2).

We will now prove the "if" part. The morphism $\mathfrak{X} \to \pi_{\mathfrak{X}/S}$ is the projective limit over the directed set $D_{\mathfrak{X}}$ of all Nori-reduced morphisms $\mathfrak{X} \to \mathcal{G}$ to a finite gerbe (see [BV12, proof of Thm. 5.7]). It follows by commutation of limits that for relative inertia stacks

$$I_{\mathfrak{X}/\pi_{\mathfrak{X}/S}} \simeq \varprojlim_{\mathfrak{X}\to\mathcal{G}} I_{\mathfrak{X}/\mathcal{G}}\,.$$

By Proposition 15(3), the assumption is equivalent to the fact that $I_{\mathfrak{X}/\pi_{\mathfrak{X}/S}}$ is trivial as a group space over $\mathfrak{X}$. We have to prove that there exists a Nori-reduced morphism

$$f_0 \colon \mathfrak{X} \to \mathcal{G}_0$$

such that $I_{\mathfrak{X}/\mathcal{G}_0}$ is the trivial group space.

More generally, we can consider, for any closed sub-stack $\mathfrak{X}' \subset \mathfrak{X}$, the issue of finding such a morphism $f_0 \colon \mathfrak{X} \to \mathcal{G}_0$ satisfying the condition that $I_{\mathfrak{X}'/\mathcal{G}_0}$ is trivial. We proceed by Noetherian induction, and fix a closed sub-stack

$$\mathfrak{X}' \subset \mathfrak{X}\,,$$

assuming that the problem has a solution for any strict closed sub-stack $\mathfrak{X}'' \subset \mathfrak{X}'$. Using the fact that $D_{\mathfrak{X}}$ is directed, we can suppose that $\mathfrak{X}'$ is irreducible. The same fact shows that it is enough to prove that there exists a non-empty open sub-stack $\mathfrak{U}$ of $\mathfrak{X}'$ for which there exists $f_0 \colon \mathfrak{X} \to \mathcal{G}_0$ such that $I_{\mathfrak{U}/\mathcal{G}_0}$ is trivial.

Let $f_1 \colon \mathfrak{X} \to \mathcal{G}_1$ be an arbitrary element of $D_{\mathfrak{X}}$. By generic flatness (see Proposition 18), there exists a non-empty open sub-stack $\mathfrak{U}_1$ of $\mathfrak{X}'$ such that $I_{\mathfrak{U}_1/\mathcal{G}_1}$ is flat. Being also finite, this group has a well-defined order. If this order is not 1, as shown below, we can produce an element

$$f_2 \colon \mathfrak{X} \to \mathcal{G}_2$$

of $D_{\mathfrak{X}}$ and a non-empty open sub-stack $\mathfrak{U}_2$ of $\mathfrak{X}'$ such that $I_{\mathfrak{U}_2/\mathcal{G}_2}$ is flat, and $\#I_{\mathfrak{U}_2/\mathcal{G}_2} < \#I_{\mathfrak{U}_1/\mathcal{G}_1}$. This completes the proof of the proposition by induction.

To prove the above claim, assume that $I_{\mathfrak{U}_1/\mathcal{G}_1}$ is not trivial. Since by assumption $\varprojlim_{\mathfrak{X}\to\mathcal{G}} I_{\mathfrak{X}/\mathcal{G}}$ is trivial, there exists a morphism $f_2\colon \mathfrak{X}\to\mathcal{G}_2$ mapping to f_1 in $D_{\mathfrak{X}}$ such that the induced monomorphism $I_{\mathfrak{U}_1/\mathcal{G}_2}\to I_{\mathfrak{U}_1/\mathcal{G}_1}$ is not an isomorphism. Let $\mathfrak{U}_2$ be a nonempty subset of $\mathfrak{U}_1$ such that $I_{\mathfrak{U}_2/\mathcal{G}_2}$ is flat. Since the cokernel of $I_{\mathfrak{U}_1/\mathcal{G}_2}\to I_{\mathfrak{U}_1/\mathcal{G}_1}$, namely $f_2^* I_{\mathcal{G}_2/\mathcal{G}_1}$, is flat, it remains non-trivial after restriction to $\mathfrak{U}_2$. Hence we have $\#I_{\mathfrak{U}_2/\mathcal{G}_2} < \#I_{\mathfrak{U}_1/\mathcal{G}_1}$. □

Remark 4. We used mainly two aspects: the fact that $D_{\mathfrak{X}}$ is directed, and the fact that when $\mathfrak{X}$ is Noetherian and reduced, and $\mathfrak{X}\to\mathcal{G}$ is a morphism, the relative inertia stack $I_{\mathfrak{X}/\mathcal{G}}$ is flat over a non-empty open subset $\mathfrak{U}$ of $\mathfrak{X}$. This fact can be interpreted in the following way: $\mathfrak{U}$ is a gerbe over its coarse sheafification $\pi_0(\mathfrak{U})$ over $\mathcal{G}$. When $\mathcal{G}=S$, this is the core of the classical result called "stratification by gerbes". For the relative version of this result, see appendix C.

Notice, however, that the flatness of the relative inertia stack is $I_{\mathfrak{X}/\mathcal{G}}$ over a non-empty open subset does not follow from the flatness of the absolute inertia stack $I_{\mathfrak{X}/S}$, since the kernel of a morphism between two finite and flat group spaces is not necessarily flat. This is the main difference between our situation and the one considered in [Noo04]. Since the kernel of a morphism between two finite and étale group spaces is finite and étale, Noohi can directly use stratification by gerbes over S.

Corollary 10. *Let k'/k be a finite separable extension. Then the stack $\mathfrak{X}/S$ is Nori-uniformizable if and only if $\mathfrak{X}_{k'}$ is Nori-uniformizable.*

Proof. Since being representable by algebraic spaces is a local property, this follows from Proposition 9 and [BV12, Prop. 6.1] (which asserts that the fundamental gerbe commutes with finite separable base change). □

3.2. Nori-uniformization and residual gerbes

The following proposition generalizes Theorem 6.2 of [Noo04].

Proposition 11. *Let $\mathfrak{X}/S$ be an inflexible stack with finite inertia and of finite type. Then $\mathfrak{X}$ is Nori-uniformizable if and only if for any closed point x, the canonical morphism*

$$\mathcal{G}_x \to \pi_{\mathfrak{X}/S}$$

is representable.

Proof. This follows from Proposition 9 and Lemma 12 below. □

Lemma 12. *Let $\mathfrak{X}$ be a stack of finite type over a field k and $f\colon \mathfrak{X}\to\mathfrak{Y}$ be a morphism to an algebraic stack. Then f is representable if and only if for any closed point $x\in|\mathfrak{X}|_0$, the induced morphism $\mathcal{G}_x\to\mathfrak{Y}$ is representable.*

Proof. By Proposition 17, for any closed point $x\colon \operatorname{Spec}\Omega\to\mathfrak{X}$, we have that $\left(I_{\mathfrak{X}/\mathfrak{Y}}\right)|_{\mathcal{G}_x}\simeq I_{\mathcal{G}_x/\mathfrak{Y}}$, hence the statement follows from Proposition 15(3) and the fact that the set of closed points is dense. □

We recall that, using the terminology of [BV12], if $\mathfrak{X}$ is pseudo-proper, then the pull-back along $\mathfrak{X} \to \pi_{\mathfrak{X}/S}$ identifies representations of $\pi_{\mathfrak{X}/S}$ with the category $\mathrm{EFVect}(\mathfrak{X})$ of essentially finite vector bundles on $\mathfrak{X}$. We can now state our main theorem.

Theorem 13. *Let $\mathfrak{X}/S$ be an inflexible and pseudo-proper stack with finite inertia and of finite type. Then $\mathfrak{X}$ is Nori-uniformizable if and only if for any closed point x, any representation V of $\mathcal{G}_x$ is a subquotient of the restriction of an essentially finite vector bundle on $\mathfrak{X}$ along $\mathcal{G}_x \to \mathfrak{X}$.*

Proof. According to Proposition 11 and Proposition 15(2), the stack $\mathfrak{X}$ is Nori-uniformizable if and only if for any closed point x, the morphism

$$\mathcal{G}_x \to \pi_{\mathfrak{X}/S} \otimes_k k(x)$$

is representable. According to Proposition 21, this is equivalent to the fact that any representation V of $\mathcal{G}_x$ is a subquotient of the restriction of a representation of $\pi_{\mathfrak{X}/S} \otimes_k k(x)$. Now the following lemma completes the proof.

Lemma 14. *Let $\mathcal{G}/S$ be a Tannakian gerbe, k'/k an extension, and $f\colon \mathcal{G}_{k'} \to \mathcal{G}$ the canonical morphism. Then for any representation V' of $\mathcal{G}_{k'}$, the canonical morphism $f^*f_*V \to V$ is an epimorphism.*

Proof. The morphism f is affine, and in particular it is quasi-affine, and hence the result follows (see [AE12, Prop. 6.2]). □

A. Representable morphisms

We start by recalling a characterization of representable morphisms.

Proposition 15. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of S-stacks. The following properties are equivalent:*

1. *The morphism f is representable by algebraic spaces.*
2. *For any section $\sigma\colon T \to \mathfrak{X}$, the canonical morphism of T-group spaces*
$$\mathbf{Aut}_T \sigma \to \mathbf{Aut}_T f(\sigma)$$
has trivial kernel.
3. *The relative inertia stack $I_{\mathfrak{X}/\mathfrak{Y}} = \mathfrak{X} \times_{\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X}} \mathfrak{X}$ is trivial (as a group space over $\mathfrak{X}$).*
4. *The group morphism $I_{\mathfrak{X}/S} \to f^*I_{\mathfrak{Y}/S}$ is a monomorphism.*
5. *The diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a monomorphism.*

Proof. See [Stacks, Tag 04YY] for the equivalence of the first three statements. The fourth statement is equivalent to the third one, considering the following 2-Cartesian diagram:

$$\begin{array}{ccc} I_{\mathfrak{X}/\mathfrak{Y}} & \longrightarrow & I_{\mathfrak{X}/S} \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \longrightarrow & I_{\mathfrak{Y}/S} \end{array} .$$

The fifth statement is a reformulation of the third one; see Proposition 16. □

B. Monomorphisms of algebraic stacks

A morphism $f : \mathfrak{X}' \to \mathfrak{X}$ of algebraic stacks is a monomorphism if it is representable by a morphism of algebraic spaces that is a monomorphism (see [Stacks, Tag 04ZV] for details).

For the convenience of the reader, we recall the following characterization.

Proposition 16. *Let $f\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is a monomorphism,*
(2) *f is fully faithful,*
(3) *the diagonal $\Delta_f\colon \mathfrak{X}' \to \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}'$ is an isomorphism.*

Proof. See [Stacks, Tag 04ZZ]. □

Proposition 17. *Let $\mathcal{S}/S$ be a base stack, and let*

$$\mathfrak{X}' \to \mathfrak{X}$$

be a $\mathcal{S}$-monomorphism of $\mathcal{S}$-algebraic stacks. Then the following diagram is 2-Cartesian:

$$\begin{array}{ccc} I_{\mathfrak{X}'/\mathcal{S}} & \longrightarrow & I_{\mathfrak{X}/\mathcal{S}} \\ \downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & \mathfrak{X} \end{array} .$$

Proof. This follows from the absolute statement ($\mathcal{S} = S$, [Stacks, Tag 06R5]) and the following 2-Cartesian diagram:

$$\begin{array}{ccc} I_{\mathfrak{X}/\mathcal{S}} & \longrightarrow & I_{\mathfrak{X}/S} \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & I_{\mathcal{S}/S} \end{array} . \qquad \square$$

Remark 5. With the terminology introduced in Definition 4, Proposition 17 means exactly that monomorphisms are isovariant.

C. Stratification by gerbes over a base stack

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks over some base S. We assume for simplicity that the diagonal $\Delta_f\colon \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-compact (equivalently, it is of finite type). Then the relative inertia stack $I_{\mathfrak{X}/\mathfrak{Y}} \to \mathfrak{X}$ is a group space of finite type and, if we further assume that $\mathfrak{X}$ is Noetherian and reduced, then we can apply the classical generic flatness theorem, [Gro65, Thm. 6.9.1], to get the following.

Proposition 18. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks with quasi-compact diagonal, and assume that $\mathfrak{X}$ is Noetherian and reduced. Then there exists a dense open sub-stack $\mathcal{U} \subset \mathcal{X}$ such that $I_{\mathfrak{U}/\mathfrak{Y}} \to \mathfrak{U}$ is flat.*

Proof. See [Stacks, Tag 06RC], for the absolute version. □

The flatness of the inertia stack has a standard interpretation. We start by giving the definition of an "absolute" gerbe in this relative setting.

Definition 6. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks. We say that $\mathfrak{X}$ *is a gerbe in* $\mathfrak{Y}$*-stacks* if there exists a factorization $\mathfrak{X} \to \mathfrak{Z} \to \mathfrak{Y}$ of f such that $\mathfrak{X} \to \mathfrak{Z}$ is a gerbe, and $\mathfrak{Z} \to \mathfrak{Y}$ is representable by algebraic spaces.

Remark 6.

(1) This definition is the direct generalization of the absolute version given in [Stacks, Tag 06QC].

(2) The condition that $\mathfrak{X} \to \mathfrak{Z}$ is a gerbe roughly means that $\mathfrak{X}$ is a gerbe if we endow $\mathfrak{Z}$ from the topology inherited from the base S; see [Stacks, Tag 06P2] for details.

(3) The stack $\mathfrak{Z}$, if it exists, is unique, and it is obtained by sheafifying, over $\mathfrak{Y}$ endowed with its topology inherited from the base S, the presheaf $U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$ (see [Stacks, Tag 06QD]).

Proposition 19. *The stack $\mathfrak{X}$ is a gerbe in $\mathfrak{Y}$-stacks if and only if $I_{\mathfrak{X}/\mathfrak{Y}} \to \mathfrak{X}$ is flat and locally of finite presentation.*

Proof. See [Stacks, Tag 06QJ], for the absolute version. □

From Proposition 18 we have the following:

Theorem 20. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks with quasi-compact diagonal, and assume that $\mathfrak{X}$ is Noetherian. Then there exists a finite decomposition $\mathfrak{X} = \coprod_{i\in I} \mathfrak{X}_i$ of $\mathfrak{X}$ by locally closed sub-stacks such that, for all $i \in I$, the stack $\mathfrak{X}_i$, endowed with the reduced structure, is a gerbe in $\mathfrak{Y}$-stacks.*

Remark 7. Our formulation of the statement, based on the classical generic flatness theorem (see [Gro65, Thm. 6.9.1]) is rather restrictive, even if it is more than enough for our purposes (in fact we only need Proposition 18). For a more general version, based on a more powerful generic flatness theorem, see [Stacks, Tag 06RF].

D. Morphisms of Tannakian gerbes

The following proposition is well known for *neutral* gerbes (see [Saa72, II 4.3.2]); it is included here due to the lack of a reference for the more general statement.

Proposition 21. *Let $\phi\colon \mathcal{G} \to \mathcal{G}'$ be a morphism between Tannakian gerbes, and let $\phi^*\colon \mathrm{Vect}\,\mathcal{G}' \to \mathrm{Vect}\,\mathcal{G}$ be the corresponding Tannakian functor.*

1. *The morphism ϕ is representable if and only if any object of $\mathrm{Vect}\,\mathcal{G}$ is a subquotient of the image by ϕ^* of an object of $\mathrm{Vect}\,\mathcal{G}'$.*
2. *The morphism ϕ is a (relative) gerbe if and only the functor ϕ^* is fully faithful, and the essential image of ϕ^* is stable by subobject.*

Proof.

(1) We recall that given a base S, there is a canonical morphism

$$\mathrm{bd}\colon \mathrm{Gr}_S \to \mathrm{Bd}_S$$

from the stack of groups to the stack of bands. A group morphism $\Phi\colon G \to G'$ is a monomorphism if and only if the corresponding band morphism

$$\mathrm{bd}(\Phi)\colon \mathrm{bd}(G) \to \mathrm{bd}(G')$$

is injective (indeed, by definition a morphism of bands is injective if it is locally represented by a group monomorphism).

Moreover, each gerbe $\varphi\colon \mathcal{G} \to S$ admits a well-defined S-band $\mathrm{bd}(\mathcal{G})$, and there is a natural isomorphism $\varphi^*(\mathrm{bd}(\mathcal{G})) \simeq \mathrm{bd}(I_{\mathcal{G}})$. To check this, we recall that the association $\mathcal{G} \mapsto \mathrm{bd}(\mathcal{G})$ is characterized by three properties: it is functorial, compatible with localization, and $\mathrm{bd}(\mathrm{B}G) = \mathrm{bd}(G)$. But when we base change $\mathcal{G} \to S$ by itself, it is easy to check that we get the neutral gerbe $\mathrm{B}I_{\mathcal{G}} \to \mathcal{G}$.

According to [Saa72, III 3.3.3], any object of $\mathrm{Vect}\,\mathcal{G}$ is a subquotient of the image by ϕ^* of an object of $\mathrm{Vect}\,\mathcal{G}'$ if and only if the morphism

$$\mathrm{bd}(\phi)\colon \mathrm{bd}(\mathcal{G}) \to \mathrm{bd}(\mathcal{G}')$$

is injective.

Since the structural morphism $\varphi\colon \mathcal{G} \to S$ is a covering, this is equivalent to the assertion that $\varphi^*\mathrm{bd}(\phi)\colon \varphi^*\mathrm{bd}(\mathcal{G}) \to \varphi^*\mathrm{bd}(\mathcal{G}')$ is injective; in other words, equivalent to the assertion that the natural morphism $\mathrm{bd}(I_{\mathcal{G}}) \to \mathrm{bd}(\phi^* I_{\mathcal{G}'})$ is injective. This is in turn equivalent to the fact that the morphism $I_{\mathcal{G}} \to \phi^* I_{\mathcal{G}'}$ is a monomorphism, and we conclude by Proposition 15(4).

(2) Since this is similar to the proof of the first part, we omit the details. Also, this is not used in the present article. This completes the proof. □

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