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SPECIAL REDUCTIVE GROUPS OVER AN ARBITRARY FIELD

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Abstract. A linear algebraic group G defined over a field k is called special if every G-torsor over every field extension of k is trivial. In 1958 Grothendieck classified special groups in the case where the base field is algebraically closed. In this paper we describe the derived subgroup and the coradical of a special reductive group over an arbitrary field k. We also classify special semisimple groups, special reductive groups of inner type, and special quasisplit reductive groups over an arbitrary field k. Finally, we give an application to a conjecture of Serre.

1. Introduction

Let k be a base field and G an algebraic group defined over k, that is, a smooth k-group scheme of finite type (not necessarily connected). The group G is called **special** if every G-torsor defined over a field extension of k is trivial. In other words, if for every field extension K of k, the first fppf-cohomology set $H^1(K,G)$ contains only one element. Examples of special linear groups include the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, the general linear group GL_n, and more generally the group $\mathrm{GL}_1(A)$, where A is a central simple algebra over k, and the classical groups SL_n and Sp_{2n}. In contrast, the group SO_n is not special for $n \geqslant 3$. The special groups over an algebraically closed field were introduced by Serre in [Se1], reprinted in [Se4]. In this paper, Serre gave the basic properties of special groups; for example, he showed that they are linear and connected. The study of special groups over an algebraically closed field was then completed by Grothendieck in [Gr]. In the reductive case, his result can be stated as follows:

Theorem 1 (Grothendieck, 1958)**.** *Suppose that k is algebraically closed and G is reductive — that is, its unipotent radical is trivial. Then G is special if and only if its derived subgroup is isomorphic to a direct product*

$$G_1 \times G_2 \times \cdots \times G_r$$

where, for each i, the group G_i is isomorphic to SL_{n_i} or Sp_{2n_i} for some integer n_i.

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The result of Grothendieck naturally raises the problem of classifying special reductive groups over an arbitrary field k. The present paper is an attempt to solve this problem. Our most general classification result is the following; see Section 4, Theorem 9 below:

Theorem 2. *Let G be a reductive algebraic group over k. Then G is special if and only if the three following conditions hold:*

(1) *The derived subgroup of G is isomorphic to*

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each index i, the extension K_i of k is finite and separable; $R_{K_i|k}$ denotes the Weil scalar restriction functor — see, for example, [KMRT, Lem. 20.6]*; and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i.*

(2) *The coradical of G is a special torus.*

(3) *For every field extension K of k, the abelian group $\mathfrak{S}(K, G)$ is trivial — see Definition 1.*

Condition (1) above is explicit, as well as condition (2), by the classification of special tori due to Colliot-Thélène and recalled in Section 5 of the present paper. In contrast, condition (3) is not easy to check in general. However, under some additional assumptions on the group G, namely that G is semisimple, an inner form of a Chevalley group, or quasisplit, we are able to compute the groups $\mathfrak{S}(K, G)$ in condition (3), providing a full classification of special groups in these cases. We hope that a more explicit version of condition (3) will emerge in the future, unifying these cases and providing the classification of special reductive groups.

The paper is organized as follows. In Section 2 we gather some facts to be used in the following sections. In Section 3 we determine which algebraic groups can arise as derived subgroups of a special group and which can arise as coradicals, respectively, in Propositions 5 and 7. In Section 4 we prove our main classification result stated above and then derive from it the classification of special semisimple groups, special reductive groups of inner type, and special quasisplit groups in Proposition 10, 13, and 15, respectively. We recall in Section 5 the classification of special tori due to Colliot-Thélène. Finally, in Section 6 we prove the following theorem (see Section 6, Theorem 20) as an application of our main theorem. This proves a special case of a conjecture of Serre [Se3, 2.4, Quest. 2], as we explain at the end of Section 6.

Theorem 3. *Let G be a reductive group over a field k, and let $\{k_\alpha\}_{\alpha \in I}$ be a nonempty finite family of finite field extensions of k such that $\gcd_{\alpha \in I}[k_\alpha : k] = 1$. Then the following conditions are equivalent:*

(1) *G is a special group.*

(2) *For every index $\alpha \in I$, the group G_{k_α} is a special group.*

To finish this introduction, we say a word about special non-reductive groups. First, by [Sa, Lem. 1.13], if an algebraic group G over a field k possesses a k-split unipotent normal subgroup U, then G is special if and only if G/U is special. For example, if the field k is perfect, then G is special if and only if its quotient by the unipotent radical — which is a reductive group — is special, as every unipotent group over k is k-split. On a different note, Nguyen classifies the special unipotent groups over "reasonable fields" in [N]. It is a direct consequence of the fact that the additive group $\mathbb{G}_a$ is special that every k-split unipotent group is special. In [N], Nguyen proves conversely that a special unipotent group is k-split for certain fields k, for example, when k is finitely generated over a perfect field.

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2. Preliminary results

Let k be a base field and G a connected reductive algebraic group defined over k. Throughout the paper we denote by Z_G the *scheme-theoretic center* of G, which is a k-group scheme of multiplicative type, and by G_{ad} the *adjoint quotient* G/Z_G of G. We denote by G' the *derived subgroup* of G, which is a semisimple algebraic group. The adjoint quotients of G and G' are equal. Consequently, the inclusion of G' in G gives rise to a natural commutative diagram, where each row is exact in the fppf topology:

$$\begin{tikzcd}
1 \arrow[r] & Z_{G'} \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & (G')_{\mathrm{ad}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Z_G \arrow[r] & G \arrow[r] & G_{\mathrm{ad}} \arrow[r] & 1
\end{tikzcd} .$$

We refer the reader to [BFT, Appendix B] for the basics on the fppf cohomology of affine algebraic group schemes.

We denote by C_G the *coradical* of G, that is, the quotient of G by its derived subgroup G'. The group C_G is a torus. Using the natural isogeny $Z_G \times G' \to G$ given by the multiplication, we see that there is an exact sequence of algebraic group schemes in the fppf topology:

$$1 \to Z_{G'} \to Z_G \to C_G \to 1.$$

Let K be a field extension of k. The long exact sequences in fppf cohomology derived from this exact sequence and the commutative diagram above fit into the following diagram:

$$\begin{tikzcd}
 & C_G(K) \arrow[d, "\gamma_{G,K}"] & & & \\
 & H^1(K, Z_{G'}) \arrow[r] \arrow[dd, "\iota^1_{G,K}"] & H^1(K, G') \arrow[dd] \arrow[dr] & & H^2(K, Z_{G'}) \arrow[dd, "\iota^2_{G,K}"] \\
G_{\mathrm{ad}}(K) \arrow[ur, "\alpha_{G',K}"] \arrow[dr, "\alpha_{G,K}"'] & & & H^1(K, G_{\mathrm{ad}}) \arrow[ur, "\beta_{G',K}"] \arrow[dr, "\beta_{G,K}"'] & \\
 & H^1(K, Z_G) \arrow[r] \arrow[d] & H^1(K, G) \arrow[ur] & & H^2(K, Z_G) \\
 & H^1(K, C_G) \arrow[d] & & & \\
 & H^2(K, Z_{G'}) \arrow[d, "\iota^2_{G,K}"] & & & \\
 & H^2(K, Z_G) & & &
\end{tikzcd}$$

where the long vertical sequence and the rows are exact. In the following, we will refer to this diagram as the diagram $(*)$. Note that the map $\alpha_{G,K}$ (resp. $\alpha_{G',K}$) is a group homomorphism, because Z_G (resp. $Z_{G'}$) is central in G (resp. G').

Proposition 1. *The group G is special if and only if, for every field extension K of k, the map $\alpha_{G,K}$ is surjective and the map $\beta_{G,K}$ has trivial kernel.*

Proof. By the lower row in the diagram $(*)$, which is an exact sequence of pointed sets, we see that $H^1(K, G)$ is trivial if and only if $\alpha_{G,K}$ is surjective and $\beta_{G,K}$ has trivial kernel, which is what we want. $\square$

Proposition 2. *Let K be a field extension of k. If G is special then the following properties hold:*

(1) *The map*

$$\beta_{G',K} : H^1(K, G_{\mathrm{ad}}) \to H^2(K, Z_{G'})$$

has trivial kernel.

(2) *The image of the map $\beta_{G',K}$ intersects the kernel of the morphism*

$$\iota^2_{G,K} : H^2(K, Z_{G'}) \to H^2(K, Z_G)$$

trivially.

(3) *The morphism $\iota^1_{G,K}$ is surjective, hence there is an exact sequence of abelian groups:*

$$0 \to H^1(K, C_G) \to H^2(K, Z_{G'}) \to H^2(K, Z_G).$$

Proof. We will obtain these properties by looking at the diagram $(*)$. As G is special, by Proposition 1, the map $\beta_{G,K}$ has trivial kernel. This readily implies (1) and (2), because the triangle on the right of the diagram $(*)$ is commutative. To prove (3) we look at the commutative triangle on the left of the diagram $(*)$. As G is special, by Proposition 1 we see that $\alpha_{G,K}$ is surjective, forcing the map $\iota^1_{G,K}$ to be surjective as well. The last statement now follows from the fact that the vertical sequence in the diagram $(*)$ is exact. $\square$

Definition 1. Let K be a field extension of k. We denote by $\mathfrak{S}(K,G)$ the following abelian group:

$$\mathfrak{S}(K,G) = H^1(K, Z_{G'})/\langle \operatorname{Im} \alpha_{G',K}, \operatorname{Im} \gamma_{G,K} \rangle.$$

Proposition 3. *Suppose that the coradical C_G of G is special. Then G is special if and only if, for every field extension K of k, the following two conditions hold:*

(1) $\mathfrak{S}(K,G)$ *is trivial.*

(2) *The map*
$$\beta_{G',K} : H^1(K, G_{\mathrm{ad}}) \to H^2(K, Z_{G'})$$
has trivial kernel.

Proof. As the coradical C_G is special, the vertical exact sequence in the diagram $(*)$ shows that the morphism $\iota^1_{G,K}$ is surjective. Moreover, the kernel of $\iota^1_{G,K}$ is equal to the image of $\gamma_{G,K}$. By the commutative triangle on the left of the diagram $(*)$ we see that $\alpha_{G,K}$ is surjective if and only if

$$\langle \operatorname{Im} \alpha_{G',K}, \operatorname{Ker} \iota^1_{G,K} \rangle = H^1(K, Z_{G'}),$$

that is, if and only if $\mathfrak{S}(K,G)$ is trivial. Similarly, as the coradical C_G is special, we see that the morphism $\iota^2_{G,K}$ is injective. Therefore, by looking at the commutative triangle on the right of the diagram $(*)$, we see that (2) is equivalent to the fact that $\beta_{G,K}$ has a trivial kernel. $\square$

3. The derived subgroup and the coradical of a special reductive group

In this section we will determine which algebraic groups can arise as derived subgroups of a special reductive group and which can arise as coradicals respectively in Propositions 5 and 7 below.

3.1. A lemma on hermitian forms

In order to lighten the proof of Proposition 5, we start by proving Lemma 4 below about hermitian forms. We refer the reader to [KMRT, §4] for the definition of hermitian forms on a right module over an algebra D equipped with an involution.

Let D be a division algebra, k a subfield of its center and τ an involution of D. Let n be an integer and $t_1, \ldots, t_n$ be algebraically independent variables over k. We denote by K the field of rational functions $k(t_1, \ldots, t_n)$. We fix an integer m, a collection of scalars $\lambda_1, \ldots, \lambda_m$ in k^*, and, for every index i between 1 and m, we fix an element $a_i = (a_{i,1}, \ldots, a_{i,n})$ of $\mathbb{Z}^n$.

Lemma 4. *Suppose that the images of the a_is in $(\mathbb{Z}/2\mathbb{Z})^n$ are all different. Then, the hermitian form:*

$$h(x,y) = \sum_{i=1}^{m} \lambda_i t_1^{a_{i,1}} \dots t_n^{a_{i,n}} \tau(x_i) y_i$$

is anisotropic on $(D \otimes_k K)^m$.

Proof. The proof goes along the same line as [Pf, p.111]. Suppose that there exists an isotropic vector x. By clearing the denominator we can further assume that all the coordinates x_i of x belong to $D \otimes_k k[t_1, \dots, t_n]$. Now, as a consequence of our assumption, we see that the leading monomials of the Laurent polynomials

$$\lambda_i t_1^{a_{i,1}} \dots t_n^{a_{i,n}} \tau(x_i) x_i$$

with respect to the lexicographic order are all different when i ranges from 1 to m. Therefore they cannot cancel. $\square$

3.2. The derived subgroup of a special reductive group

We will use [KMRT, §26] as a basic reference for the classification of algebraic groups over non-algebraically closed fields. We will adopt the notations of [KMRT] throughout.

Proposition 5. *Let G be a special reductive algebraic group over k. The derived subgroup of G is isomorphic to*

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i.

Proof. By Theorem 1, the group $G'_{\bar{k}}$, where $\bar{k}$ is an algebraic closure of k, is a semisimple simply connected group whose simple components are of types A and C. Therefore, by [KMRT, Thm. 26.8], the group G' is isomorphic to a direct product

$$R_{K_1|k}(G_1) \times R_{K_2|k_2}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each index i, the extension K_i of k is finite and separable and the group G_i is an absolutely simple simply connected group over K_i of type A or C. For each index i, G_i is a direct factor of the derived subgroup of the special reductive group G_{K_i}. By Proposition 2 we get that the map $\beta_{G'_{K_i},K}$ has trivial kernel for every field extension K of K_i, which readily implies that the map $\beta_{G_i,K}$ has trivial kernel as well. This forces G_i to be of inner type A or split of type C, by Lemma 6 below, completing the proof of the proposition.

Lemma 6. *Let G be an absolutely simple simply connected group of type A or C over the field k. If, for every field extension K of k, the map $\beta_{G,K}$ has a trivial kernel, then G is either of inner type A or split of type C.*

Proof of Lemma 6. Suppose first that G is of outer type A. We will prove that the kernel of $\beta_{G,K}$ contains at least two elements for some field extension K of

k. Observe that to prove this property we can replace G by G_M for some scalar extension M of k. By [KMRT, §26], G is isomorphic to $\mathrm{SU}(A,\sigma)$, where A is a central simple algebra of degree n — at least 3, otherwise $\mathrm{SU}(A,\sigma)$ is of inner type — over a quadratic separable extension L of k equipped with an involution σ of the second kind.

We will now reduce to the case where A is split over L. To this aim, we denote by Y the Severi-Brauer variety of A, by X the Weil scalar restriction of Y from L to k, and by K be the function field of X. As X is geometrically integral, the field k is algebraically closed in K, and consequently $K \otimes_k L$ is a field. Moreover, the set $Y(K \otimes_k L)$ is not empty, as it is equal to $X(K)$. This implies that the field extension $K \otimes_k L$ of L is a splitting field for A. Now, we observe that the group G_K is isomorphic to $\mathrm{SU}(K \otimes_k A, \sigma_K)$, where $K \otimes_k A$ is a split central simple algebra over $K \otimes_k L$ equipped with an involution σ_K of the second kind. It is thus of outer type A and satisfies moreover the property that for every field extension M of K, the map $\beta_{G_K,M}$ has trivial kernel. Therefore, by replacing k by K and G by G_K, we are reduced to the case where the central simple algebra A is split over L.

Then A is isomorphic to $\mathrm{End}_L(L^n)$ for some integer n greater than or equal to three, and the involution σ is adjoint to a nonsingular hermitian form h on L^n, by [KMRT, §4.2]. The group G is therefore isomorphic to $\mathrm{SU}_L(n,h)$. Its center is the group $\mu_{n[L]}$; see [KMRT, §30B], the kernel of the norm map:

$$N_{L|k} : R_{L|k}(\mu_{n,L}) \to \mu_{n,k}.$$

Let K be the field $k(t_1, \dots, t_{n-1})$ where the t_is are algebraically independent variables over k.

We claim that the kernel of $\beta_{G,K}$ contains at least two elements. We have an exact sequence of pointed sets:

$$H^1(K, \mu_{n[L]}) \to H^1(K, G) \to H^1(K, G_{\mathrm{ad}}) \xrightarrow{\beta_{G,K}} H^2(K, \mu_{n[L]})$$

in the fppf-cohomology. As $\mu_{n[L]}$ is abelian and central in G, there is a natural action of $H^1(K, \mu_{n[L]})$ on $H^1(K, G)$, and the set of orbits for this action is precisely the kernel of $\beta_{G,K}$. By [KMRT, Example 29.19], the set $H^1(K, G)$ is in natural correspondence with the set of isometry classes of nonsingular hermitian forms on the vector space $(K \otimes_k L)^n$ with the same discriminant α as h. Moreover, by [KMRT, Prop. 30.13], the group $H^1(K, \mu_{n[L]})$ is the quotient of

$$\{(x,y) \in K^* \times (K \otimes_k L)^* \mid x^n = N_{K \otimes_k L|K}(y)\}$$

by the subgroup

$$\{(N_{K \otimes_k L|K}(z), z^n) \mid z \in (K \otimes_k L)^*\}.$$

Strictly speaking, the description above is given in [KMRT, Prop. 30.13] only when n is not divisible by the characteristic of the base field k. This comes from the fact that the cohomology considered there is the Galois cohomology. The same proof leads to the description in the fppf-cohomology, with no restriction on the integer

n. It is then easy to prove that the action of the class $[(x,y)]$ on the isometry class $[h']$ of the hermitian form h' is given as follows:

$$[(x,y)]\cdot[h'] = [xh'].$$

We will now prove that the set $H^1(K,G)$ contains the isometry class of an isotropic form and an anisotropic form. As these two classes cannot be in the same orbit under the action of $H^1(K,\mu_{n[L]})$, this proves the claim above.

First, as n is greater than 2, $H^1(K,G)$ contains the isometry class of an isotropic hermitian form, namely the one with matrix

$$\operatorname{diag}\Big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1, \cdots, 1, -\alpha\Big).$$

Moreover, by Lemma 4 above, the hermitian form:

$$h'(x_1,\dots,x_n) = t_1\sigma(x_1)x_1 + \cdots + t_{n-1}\sigma(x_{n-1})x_{n-1} + \alpha t_1^{-1}\cdots t_{n-1}^{-1}\sigma(x_n)x_n$$

which has discriminant α, is anisotropic over the field $K\otimes_k L$.

Suppose now that G is of type C and not split. Again here, we want to see that the kernel of $\beta_{G,K}$ contains at least two elements for some field extension K of k. By [KMRT, §26], G is isomorphic to $\mathrm{Sp}(A,\sigma)$, where A is a nonsplit central simple algebra of degree $2n$ — at least 4, otherwise $\mathrm{Sp}(A,\sigma)$ is of type A — over k equipped with an involution σ of symplectic type. The center of G is isomorphic to μ_2. Let K be a field extension of k. By [KMRT, 29.22] the kernel of $\beta_{G,K}$ is in bijection with the conjugacy classes of involutions of symplectic type on A_K.

If A is a division algebra, then by [L, Thm. 3.2], there is more than one conjugacy class of involution of symplectic type on A_K, for some field extension K of k. From now on, we suppose that A is not a division algebra. Let D be the division algebra Brauer equivalent to A. It is not k, as A is nonsplit. Therefore, D carries an involution τ of symplectic type, by [KMRT, Thm. 3.1] and [KMRT, Cor. 2.8], and, by Wedderburn's theorem [KMRT, Thm. 1.1], A is isomorphic to $M_n(D)$ for some integer n greater than or equal to 2.

Let K be the field of rational functions $k(t_1,\cdots,t_n)$ on n indeterminates. The algebra D_K is a division algebra, and is therefore the division algebra Brauer equivalent to A_K. Let M be a simple right A_K-module, isomorphic to D_K^n — thought of as row vectors. We will make use of the correspondence between involutions of symplectic type on A_K and hermitian forms on M, as explained in [KMRT, Thm. 4.2]. We refer the reader to [KMRT, §4] for the notion of singular hermitian form and alternating hermitian form in characteristic 2. We define two hermitian forms h and h' on M in the following way:

$$h(x,y) = -\tau(x_1)y_1 + \sum_{i=2}^{n}\tau(x_i)y_i$$

and

$$h'(x,y) = \sum_{i=1}^{n} t_i\tau(x_i)y_i.$$

These forms are easily seen to be nonsingular. Furthermore, if the characteristic of k is 2, then h and h' are alternating. Indeed, by [KMRT, Prop. 2.6], as τ is an involution of symplectic type on D_K we know that K is contained in $\mathrm{Symd}(D_K, \tau)$, which is enough to prove that for every x in M, the elements $h(x,x)$ and $h'(x,x)$ both belong to $\mathrm{Symd}(D_K, \tau)$.

By [KMRT, Thm. 4.2], the hermitian forms h and h' give rise to two involutions τ_h and $\tau_{h'}$ on A_K which are both of symplectic type. If these two involutions were conjugate, then there would exist an element u of $\mathrm{GL}_n(D_K)$ such that the hermitian forms h' and the hermitian form:

$$M \times M \mapsto D, \quad (x,y) \mapsto h(u(x), u(y))$$

are proportional by a factor in K^*. But h is isotropic, for instance, $(1,1,0,\cdots,0)$ is an isotropic element, and h is not by Lemma 4. This provides a contradiction, proving that the involutions τ_h and $\tau_{h'}$ are not conjugate. □

Observe that every group admitting a direct factor decomposition as in Proposition 5 occurs as the derived subgroup of a special reductive group. Indeed, a semi-simple group

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i, is the derived subgroup of the special reductive group

$$R_{K_1|k}(H_1) \times R_{K_2|k}(H_2) \times \cdots \times R_{K_r|k}(H_r)$$

where, for each index i, H_i is equal to $\mathrm{GL}_1(A_i)$ if G_i is isomorphic to $\mathrm{SL}_1(A_i)$, and H_i is equal to G_i otherwise.

3.3. The coradical of a special reductive group

We prove now that the coradical, as defined in Section 2, C_G of a special reductive group G is a special torus. The classification of special tori, due to Colliot-Thélène, will be recalled in Section 5 below.

Proposition 7. *Let G be a special reductive algebraic group defined over k. The coradical C_G of G is a special torus.*

Proof. We say that a reductive algebraic group G defined over a field — which is not necessarily k — satisfies property (P) if, for every field extension K of the field of definition of G and every nontrivial element x in $H^2(K, Z_G)$, there exists a field extension L of K such that x_L is not trivial in $H^2(L, Z_G)$ and belongs to the image of $\beta_{G,L}$. We will prove as a consequence of Lemma 8 below that the derived subgroup G' of G — and more generally any group admitting a direct factor decomposition as in Proposition 5 — satisfies property (P).

Before proving this fact let us show how it implies the proposition. Suppose that C_G is not special. There exists a field extension K of k and a nontrivial element x in $H^1(K, C_G)$. By Proposition 2, the image of x in $H^2(K, Z_{G'})$ — still denoted x — is nonzero, and is mapped to zero in $H^2(K, Z_G)$. As the group G' satisfies property (P), we can even assume, after possibly extending scalars, that there exists y in $H^1(K, G_{\mathrm{ad}})$ such that $x = \beta_{G',K}(y)$. We get that y is not trivial and is in the kernel of $\beta_{G,K}$, a contradiction to Proposition 1.

Now, the fact that any group admitting a direct factor decomposition as in Proposition 5 satisfies property (P) is a direct consequence of Lemma 8 below.

Lemma 8. *The following properties hold:*

(1) *for every central simple algebra A over a field k, the group $\mathrm{SL}_1(A)$ obeys property (P);*
(2) *for every integer n and every field k, the group Sp_{2n} obeys property (P);*
(3) *if G_1 and G_2, both defined over a field k, satisfy property (P), then $G_1 \times G_2$ satisfies it as well;*
(4) *if G is defined over a finite separable field extension M of k and satisfies property (P), then the group $R_{M|k}(G)$ satisfies property (P) as well.*

Proof of Lemma 8. Let k be a field and A a central simple algebra over k. The center of $\mathrm{SL}_1(A)$ is μ_n, where n is the degree of A. Let K be a field extension of k. Let x be a nontrivial element of $H^2(K, \mu_n)$. By [BFT, Appendix B], the element x can be seen as the Brauer class of a central simple algebra B over K of period d dividing n, d being greater than 1. By the Schofield-Van den Bergh index reduction formula [SVdB, Thm. 2.5], there exists a field extension L of K such that B_L is a central simple algebra over L of index d. This proves that the class x_L is not trivial in $H^2(L, \mu_n)$ because d is not 1, and belongs to the image of $\beta_{\mathrm{SL}_1(A),L}$, this image being precisely the classes of index dividing n. We have proved (1).

The proof of (2) is similar. Let k be a field and n an integer. The center of Sp_{2n} is μ_2. Let K be a field extension of k and x a nontrivial element of $H^2(K, \mu_2)$. The element x is the Brauer class of a central simple algebra B over K of period 2. Applying the index reduction formula once again, there exists a field extension L of K such that B_L is a central simple algebra over L of index 2. This proves that the class x_L is not trivial in $H^2(L, \mu_2)$, and belongs to the image of $\beta_{\mathrm{Sp}_{2n},L}$, this image being precisely the classes of index dividing $2n$.

Let k be a field. We will now prove that if G_1 and G_2 are reductive algebraic groups both satisfying property (P), then the direct product $G_1 \times G_2$ satisfies property (P) as well. Let K be a field extension of k and x be a nontrivial element of

$$H^2(K, Z_{G_1 \times G_2}) = H^2(K, Z_{G_1}) \times H^2(K, Z_{G_2}).$$

We write $x = (x_1, x_2)$. As G_1 satisfies property (P), there exists a field extension L_1 of K such that $(x_1)_{L_1}$ is not trivial and belongs to the image of β_{G_1,L_1}. If $(x_2)_{L_1}$ is trivial then we are done. Otherwise, as G_2 satisfies property (P), there exists a field extension L_2 of L_1 such that $(x_2)_{L_2}$ is not trivial and belongs to the image of β_{G_2,L_2}. As $(x_1)_{L_2}$ belongs to the image of β_{G_1,L_2}, we see that x_{L_2} is not trivial and belongs to the image of $\beta_{G_1 \times G_2,L_2}$. This completes the proof of (3).

Let k be a field, M a finite separable field extension of k, and let G be a reductive algebraic group over M which satisfies property (P). We will now prove that the group $R_{M|k}(G)$ satisfies property (P) as well. We denote by d the degree of the field extension M of k. Let K be a field extension of k. We can write

$$K \otimes_k M = K_1 \times \cdots \times K_s$$

where the K_is are finite separable extensions of K and M. Let x be a nontrivial element in

$$H^2(K, Z_{R_{M|k}(G)}) = H^2(K_1, Z_G) \times \cdots \times H^1(K_s, Z_G).$$

We write $x = (x_1, \dots, x_s)$, and we define d_x to be the sum of the degrees of the K_is over k such that x_i belongs to the image of β_{G,K_i}. The integer d_x is obviously less than or equal to d. We prove the desired conclusion by a decreasing induction on d_x, the case where d_x is equal to d being obvious. Suppose that d_x is strictly less than d. After permuting the K_is, we can assume for example that x_1 is not in the image of β_{G,K_1}. In particular, x_1 is not trivial. As G satisfies property (P), there exists a field extension L_1 of K_1 such that $(x_1)_{L_1}$ is not trivial and belongs to the image of β_{G,L_1}. One then easily proves that x_{L_1} is not trivial and $d_{x_{L_1}}$ is strictly greater than d_x. By the induction hypothesis there is a field extension L of L_1 such that x_L is not trivial and belongs to the image of $\beta_{G,L}$, completing the proof of (4). $\square$

Remark 1. The radical of a special reductive group does not need to be special. Suppose that K is a separable quadratic extension of k. Recall that the torus $R^1_{K|k}(\mathbb{G}_m)$ is defined as the kernel of the norm map from $R_{K|k}(\mathbb{G}_m)$ to $\mathbb{G}_m$. We denote by R the direct product $R^1_{K|k}(\mathbb{G}_m) \times \mathbb{G}_m$. There is an exact sequence of groups of multiplicative type:

$$1 \to \mu_2 \xrightarrow{\varphi} R \to R_{K|k}(\mathbb{G}_m) \to 1$$

corresponding to the following exact sequence of Γ-modules, where Γ is the Galois group of K over k:

$$0 \to \mathbb{Z}^2 \xrightarrow{(x,y)\mapsto(x-y,x+y)} \mathbb{Z}^2 \xrightarrow{(x,y)\mapsto[x+y]} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Here the nontrivial element of Γ acts on $\mathbb{Z}^2$ on the left by permuting the coordinates, on $\mathbb{Z}^2$ in the center by multiplying the first coordinate by -1 and the second by 1, and on $\mathbb{Z}/2\mathbb{Z}$ as the identity. We define G to be the quotient:

$$(\mathrm{SL}_2 \times R)/\mu_2,$$

where μ_2 is embedded diagonally in SL_2 and in R by using the morphism φ above. It is readily seen that the derived group of G is SL_2 and its coradical is $R_{K|k}(\mathbb{G}_m)$. An easy argument then shows that G is special; see, for instance, Proposition 15 below. However, the radical of G is equal to $R = R^1_{K|k}(\mathbb{G}_m) \times \mathbb{G}_m$ and is therefore not special, as it can be seen directly or from the classification of special tori recalled in Theorem 18 below.

4. Classification results

We start by classifying special reductive groups over the field k in Theorem 9 below. This classification is obtained as a straightforward consequence of the results from Sections 2 and 3. However, conditions (1) and (2) in Theorem 9 are very explicit, unlike condition (3). Under the additional assumption that the group G is semisimple, reductive of inner type, or quasisplit, we will make condition (3) explicit as well, providing an explicit classification in these cases.

Theorem 9. *Let G be a reductive algebraic group over k. Then G is special if and only if the following three conditions hold:*

(1) *The derived subgroup of G is isomorphic to*

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i.

(2) *The coradical C_G of G is a special torus.*

(3) *For every field extension K of k, the group $\mathfrak{S}(K,G)$ is trivial.*

Proof. If G is special, then (1) is satisfied by Proposition 5, (2) by Proposition 7, and (3) by Proposition 3. Suppose now that G satisfies the three conditions. By (1), for every field extension K of k, the map $\beta_{G',K}$ has trivial kernel. This can be seen as follows. First of all, the property for an algebraic group H over k to satisfy that $\beta_{H,K}$ has trivial kernel for every field extension K of k is preserved by direct products and scalar restriction. Therefore it suffices to check this property for $H = \mathrm{SL}_1(A)$, where A is a central simple algebra over k, and for $H = \mathrm{Sp}_{2n}$. In the first case the map $\beta_{H,K}$ is even injective, because two central simple algebras of the same degree and Brauer class are isomorphic. In the second case, the map has trivial kernel because there is only one symplectic involution on the split central simple algebra of degree $2n$, namely the split one. This proves our claim. Together with (2) and (3), it implies that G is special, by Proposition 3. □

4.1. The classification of special semisimple groups

We provide now the classification of special semisimple groups over the field k.

Proposition 10. *Let G be a semisimple algebraic group over k. Then G is special if and only if it is isomorphic to*

$$R_{K_1|k}(G_1) \times R_{K_2|k_2}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to SL_{n_i} or Sp_{2n_i} for some integer n_i.

Proof. The "if part" of the proposition follows directly from Shapiro's lemma and the fact that the split groups SL_n and Sp_{2n} are special for every integer n. For the "only if part", we use Proposition 5. As G is its own derived subgroup, we find that G is isomorphic to

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i. Now, for every index i, G_i is a direct factor of G_{K_i}, and, as such, is a special group. If G_i is isomorphic over K_i to $\mathrm{SL}_1(A_i)$ then Lemma 11 below shows that A_i is split, completing the proof of the proposition.

Lemma 11. *Let A be a central simple algebra over the field k. If $\mathrm{SL}_1(A)$ is a special group, then A is split.*

This result is part of the folklore; see, for example, [GS, Chapter 2, Exercise 6]. We sketch a proof for the convenience of the reader. Let $k((t))$ be the field of formal Laurent series. By [KMRT, Cor. 29.4] we know that the set $H^1(k((t)), \mathrm{SL}_1(A))$ is naturally identified with the cokernel of the reduced norm:

$$\mathrm{Nrd} : (k((t)) \otimes_k A)^* \to k((t))^*.$$

We claim that the class $[t]$ of t in $H^1(k((t)), \mathrm{SL}_1(A))$ is not trivial if A is not split, proving that $\mathrm{SL}_1(A)$ is not special in that case.

To see this, we will prove below that the image of the composite:

$$v \circ \mathrm{Nrd} : (k((t)) \otimes_k A)^* \to k((t))^* \to \mathbb{Z}$$

where v is the valuation given by t, is the ideal spanned by the index $\mathrm{ind}(A)$ of A. Let us first show how it implies the result. If A is not split, then the index of A is not 1, and we see that t, whose valuation is 1, is not in the image of Nrd, proving that the class $[t]$ in $H^1(k((t)), \mathrm{SL}_1(A))$ is not trivial.

We now prove the result above. Let D be the division algebra over k which is Brauer equivalent to A. Observe that the valuation v extends to $k((t)) \otimes_k D$, the valuation of

$$d_r t^r + d_{r+1} t^{r+1} + \cdots$$

being r if d_r is not zero. This implies actually that the $k((t))$-algebra $k((t)) \otimes_k D$ is a division algebra, and is thus the division algebra Brauer-equivalent to $k((t)) \otimes_k A$. By [GS, Cor. 2.8.10], the image in $k((t))^*$ of the reduced norms from $(k((t)) \otimes_k A)^*$ and $(k((t)) \otimes_k D)^*$ are the same. Therefore, in order to prove the result above, we can replace A by D. By extending the scalars to $k_s((t))$ — which is contained in a separable closure of $k((t))$ — where the reduced norm becomes the determinant, we see that the valuation of the reduced norm of

$$d_r t^r + d_{r+1} t^{r+1} + \cdots$$

where d_r is not zero, is $r \dim_k D$, which is equal to $r\,\mathrm{ind}(A)$. This completes the proof of the result above. □

Proposition 10 can be found (without proof) in [CS2, 4.2], where its connection to the rationality problem of Noether is explained. Namely, the authors notice the following:

Corollary 12 (Colliot-Thélène, Sansuc)**.** *Let G be a special semisimple group and V a generically free linear representation of G. Then the field of invariants $k(V)^G$ is stably pure.*

Before giving the proof of this corollary, we recall a few definitions. A linear representation V of an algebraic group G is called *generically free* if there exists an open subscheme U of V such that the scheme-theoretic stabilizer of every point of U is trivial. By [SGA3, Exp.V, Thm. 8.1], it is equivalent to the existence of an open subscheme X and a G-torsor $X \to Y$, where Y is a variety over k. The field extension $k(Y)$ of k is then independent of the choice of X or Y up to isomorphism, and is called the field of invariants $k(V)^G$. A field extension K of k is called *stably pure* if one can find algebraically independent variables $t_1, \dots, t_s$ over k such that the field extension $K(t_1, \dots, t_s)$ is purely transcendental over k; see [CS2, p. 3].

Proof of Corollary 12. Let X be an open subscheme of V such that there exists a G-torsor $\pi : X \to Y$, with Y a k-variety. The generic fiber of π is a G-torsor defined over the field $k(V)^G$. As G is a special group, this torsor is trivial. Consequently there exists a rational section to the morphism π, proving that X is k-birational to $G \times Y$. Now X is an open subscheme of V, hence is k-rational, and G is also a k-rational variety by Proposition 10 (a split group is a rational variety over the ground field by the Bruhat decomposition, and a Weil restriction of a rational variety is a rational variety). We have proved that Y is a stably k-rational variety in the sense of [CS2], which is to say, that the field $k(Y) = k(V)^G$ is stably pure. □

We will prove an analog of Corollary 12 for special quasisplit reductive algebraic groups in Corollary 16 below.

4.2. The classification of special reductive groups of inner type

The *split form* of a reductive algebraic group G defined over k is the unique Chevalley group G_{split} over k which is isomorphic to G over the algebraic closure $\bar{k}$ of k. The existence and uniqueness of the split form is guaranteed by Chevalley's classification of split reductive groups (see, for instance, [Sp]) and the fact that every reductive group is split over an algebraically closed field.

A reductive algebraic group G is called of *inner type* if it is an inner form of its split form, that is, if it is obtained by twisting G_{split} by a cocycle with values in the group of inner automorphisms of G_{split}; see for instance [KMRT, §31]. If G is a reductive group of inner type, then:

$$Z_G = Z_{G_{\text{split}}}, \quad Z_{G'} = Z_{(G_{\text{split}})'} \text{ and } \mathrm{R_G} = \mathrm{R_{G_{split}}},$$

where R_G is the *radical* of G, that is, the identity component of the center Z_G. Consequently, we see that Z_G and $Z_{G'}$ are split diagonalizable groups and R_G is a split torus. We provide now the classification of special reductive algebraic groups which are of inner type.

Proposition 13. *Let G be a reductive algebraic group over k of inner type. The intersection R'_G of R_G with $Z_{G'}$ is a finite split diagonalizable group. We fix an* *isomorphism:*

$$R'_G \simeq \mu_{m_1} \times \cdots \times \mu_{m_q}$$

for some integers m_j. Then G is special if and only if the following two conditions are satisfied:

(1) *The derived subgroup G' of G is isomorphic to a direct product:*

$$G_1 \times \cdots \times G_s \times \cdots \times G_r \qquad (**)$$

where, for each index i from 1 to s, the group G_i is equal to $\mathrm{SL}_1(A_i)$, with A_i a nonsplit central simple algebra of degree n_i and index d_i over k, and, for i from $s+1$ to r, the group G_i is equal to either SL_{n_i} or Sp_{2n_i} for some integer n_i.

(2) *The projection onto the first s factors in the direct product decomposition $(**)$ leads to a morphism:*

$$\begin{aligned} R'_G \simeq \mu_{m_1} \times \cdots \times \mu_{m_q} &\to Z_{G_1\times\cdots\times G_s} = \mu_{n_1} \times \cdots \times \mu_{n_s}, \\ (x_1, \ldots, x_q) &\mapsto (x_1^{a_{1,1}} \cdots x_q^{a_{1,q}}, \ldots, x_1^{a_{s,1}} \cdots x_q^{a_{s,q}}) \end{aligned}$$

for some integers $a_{i,j}$. We set $b_{i,j} = a_{i,j}n_i/m_j$. Then the rows of the following matrix:

$$\begin{bmatrix} d_1 & 0 & \ldots & 0 & b_{1,1} & \ldots & b_{1,q} \\ 0 & d_2 & \ldots & 0 & b_{2,1} & \ldots & b_{2,q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_s & b_{s,1} & \ldots & b_{s,q} \end{bmatrix}$$

span a saturated sublattice of $\mathbb{Z}^{s+q}$ (i.e., such that the quotient is torsion-free).

Proof. First we prove that if G is special then it satisfies (1). As G is of inner type, it is obtained by twisting the split form G_{split} of G by a cocycle whose class is in $H^1(k, (G_{\mathrm{split}})_{\mathrm{ad}})$. As the last set is equal to $H^1(k, (G'_{\mathrm{split}})_{\mathrm{ad}})$, and the group G'_{split} is a direct product of split absolutely simple simply connected groups — because it is the derived subgroup of the special split reductive group G_{split} — we see that G', which is obtained from G'_{split} by the same twisting procedure, is a direct product of absolutely simple simply connected groups of types A and C. By Proposition 5, the factors are either of inner type A or split of type C, proving that G satisfies (1).

We suppose now that G satisfies (1), and we claim that G is special if and only if it satisfies (2). It is readily seen that G satisfies the first assertion of Theorem 9, and also the second, as the coradical of G is a split torus — it is the coradical of the split form G_{split} of G. Therefore, to prove the claim, it suffices to show that (2) is satisfied if and only if the third assertion of Theorem 9 is satisfied.

Let K be a field extension of k. First, we claim that the image of the map $\gamma_{G,K}$ in $H^1(K, Z_{G'})$ is equal to the image of the morphism:

$$\iota : H^1(K, R'_G) \to H^1(K, Z_{G'})$$

induced by the inclusion. To see this, we look at the following two exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & R'_G \arrow[r] \arrow[d] & R_G \arrow[r] \arrow[d] & C_G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & Z_{G'} \arrow[r] & Z_G \arrow[r] & C_G \arrow[r] & 1 \end{tikzcd}.$$

As the torus R_G is split, by Hilbert's theorem 90, we see that the map

$$\delta : C_G(K) \to H^1(K, R'_G)$$

induced by the top exact sequence in cohomology is surjective. As the following diagram:

$$\begin{tikzcd} & H^1(K, R'_G) \arrow[d, "\iota"] \\ C_G(K) \arrow[ur, "\delta"] \arrow[r, "\gamma_{G,K}"'] & H^1(K, Z_{G'}) \end{tikzcd}$$

is commutative, we have proved the claim above.

Now, we have:

$$H^1(K, Z_{G'}) = H^1(K, Z_{G_1\times\cdots\times G_s}) \oplus \bigoplus_{i=s+1}^{r} H^1(K, Z_{G_i})$$

The map $\alpha_{G',K}$ is the direct product of the maps $\alpha_{G_i,K}$, where i ranges from 1 to r. As the group G_i is special for i between $s+1$ and r we know by Proposition 1 that $\alpha_{G_i,K}$ is surjective. In other words, the group $H^1(K, Z_{G_i})$ is contained in the image of $\alpha_{G',K}$ for every i from $s+1$ to r. Therefore, we see that the group $\mathfrak{S}(K, G)$ is trivial if and only if:

$$H^1(K, Z_{G_1\times\cdots\times G_s}) = \langle \operatorname{Im} \alpha_{G_1\times\cdots\times G_s}, \operatorname{Im}(H^1(K, R'_G) \to H^1(K, Z_{G_1\times\cdots\times G_s}))\rangle,$$

where the morphism

$$H^1(K, R'_G) \to H^1(K, Z_{G_1\times\cdots\times G_s})$$

is induced by the composite φ of the inclusion of R'_G in $Z_{G'}$ followed by the projection on $Z_{G_1\times\cdots\times G_s}$. The morphism φ has the following explicit description:

$$\begin{aligned} \varphi : \mu_{m_1} \times \cdots \times \mu_{m_q} &\to \mu_{n_1} \times \cdots \times \mu_{n_r}, \\ (x_1, \ldots, x_q) &\mapsto (x_1^{a_{1,1}} \cdots x_q^{a_{1,q}}, \ldots, x_1^{a_{r,1}} \cdots x_q^{a_{r,q}}). \end{aligned}$$

Its corresponding morphism in fppf cohomology is given by:

$$\begin{aligned} K^*/(K^*)^{(m_1)} \times \cdots \times K^*/(K^*)^{(m_q)} &\to K^*/(K^*)^{(n_1)} \times \cdots \times K^*/(K^*)^{(n_s)}, \\ ([x_1], \ldots, [x_q]) &\mapsto ([x_1^{b_{1,1}} \cdots x_q^{b_{1,q}}], \ldots, [x_1^{b_{s,1}} \cdots x_q^{b_{s,q}}]) \end{aligned}$$

where $b_{i,j} = a_{i,j} n_i / m_j$, and $(K^*)^{(n)}$ denotes the group of nth power of elements of K^*. Furthermore, for each index i from 1 to s, the map $\alpha_{G_i,K}$:

$$\mathrm{PSL}_1(A_i)(K) = (A_i)_K^*/K^* \to H^1(K, \mu_{n_i}) = K^*/(K^*)^{(n_i)}$$

maps the class of an element g of $(A_i)_K^*$ to the class of its reduced norm. Therefore, Lemma 14 below completes the proof of the proposition.

Lemma 14. *Let s and q be two positive integers. For every i from 1 to s, let A_i be a central simple algebra over k of indice d_i. For every i from 1 to s and j from 1 to q, let $b_{i,j}$ be an integer. The following conditions are equivalent:*

(1) *the rows of the following matrix:*

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & b_{1,1} & \dots & b_{1,q} \\ 0 & d_2 & \dots & 0 & b_{2,1} & \dots & b_{2,q} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_s & b_{s,1} & \dots & b_{s,q} \end{bmatrix}$$

span a saturated sublattice of $\mathbb{Z}^{s+q}$;

(2) *for every field extension K of k, the map:*

$$\gamma_K : \prod_{i=1}^{s} \mathrm{Nrd}((A_i)^*_K) \times (K^*)^q \to (K^*)^s,$$
$$(y_1, \dots, y_s, x_1, \dots, x_q) \mapsto (x_1^{b_{1,1}} \cdots x_q^{b_{1,q}} y_1, \dots, x_1^{b_{r,1}} \cdots x_q^{b_{r,q}} y_s)$$

is surjective.

Proof of Lemma 14. Suppose that (1) holds. Let M be the matrix in (1). First, as the rows of M are linearly independent, the morphism of algebraic tori:

$$\mathbb{G}_m^{q+s} \to \mathbb{G}_m^s,$$
$$(y_1, \dots, y_s, x_1, \dots, x_q) \mapsto (x_1^{b_{1,1}} \cdots x_q^{b_{1,q}} y_1^{d_1}, \dots, x_1^{b_{s,1}} \cdots x_q^{b_{s,q}} y_s^{d_s})$$

is surjective. Its kernel is precisely the subgroup of multiplicative type $\mathbb{G}_m^{s+q}$ whose character lattice is the quotient of $\mathbb{Z}^{s+q}$ by the rows of M. By assumption, this kernel is therefore a split torus. By Hilbert's theorem 90, for every field extension K of k, the map:

$$(K^*)^{s+q} \to (K^*)^s,$$
$$(y_1, \dots, y_s, x_1, \dots, x_q) \mapsto (x_1^{b_{1,1}} \cdots x_q^{b_{1,q}} y_1^{d_1}, \dots, x_1^{b_{s,1}} \cdots x_q^{b_{s,q}} y_s^{d_s})$$

induced on the K-points is surjective. As for each index i between 1 and s the subgroup $\mathrm{Nrd}((A_i)^*_K)$ of K^* contains $(K^*)^{(d_i)}$, we see that the map γ_K is surjective.

Suppose now that (1) fails. There exists a primitive element $(c_1, \dots, c_s)$ of $\mathbb{Z}^s$ such that the element

$$\sum_{i=1}^{s} c_i(0, \dots, d_i, \dots, 0, b_{i,1}, \dots, b_{i,q})$$

is divisible, say by d, in $\mathbb{Z}^{s+q}$. Let K be the field of Laurent series $k((t))$ and v the valuation defined by t. As the element $(c_1, \dots, c_s)$ is primitive, the map

$$(K^*)^s \to K^*$$

$$(z_1, \dots, z_s) \mapsto z_1^{c_1} \dots z_s^{c_s}$$

is surjective. We claim now that if $(z_1, \dots, z_s)$ belongs to the image of γ_K then the valuation of $z_1^{c_1} \dots z_s^{c_s}$ is divisible by d, proving that γ_K is not surjective.

To prove the claim, let:

$$(y_1, \dots, y_s, x_1, \dots, x_q) \in \prod_{i=1}^{s} \mathrm{Nrd}((A_i)_K^*) \times (K^*)^q,$$

and

$$(z_1, \dots, z_s) = \gamma_K(y_1, \dots, y_s, x_1, \dots, x_q).$$

We have:

$$z_1^{c_1} \cdots z_s^{c_s} = y_1^{c_1} \cdots y_s^{c_s} x_1^{\sum_{i=1}^s c_i b_{i,1}} \cdots x_q^{\sum_{i=1}^s c_i b_{i,q}}.$$

For every i from 1 to s the integer $v(y_i)$ is divisible by d_i, as y_i is a reduced norm of the central simple algebra $(A_i)_K$ over K and the index of A_i over k is d_i. Therefore $v(y_i^{c_i})$ is divisible by $c_i d_i$, hence by d. Moreover, for every index j between 1 and q, the sum $\sum_{i=1}^s c_i b_{i,j}$ is also divisible by d, completing the proof of the claim. $\square$

Here is an example of a situation where condition (2) in Proposition 13 is easy to work out. Suppose that the group R'_G decomposes along the direct factor decomposition $(**)$ in (1). That is,

$$R'_G \simeq \mu_{m_1} \times \cdots \times \mu_{m_r}$$

where, for each index i, μ_{m_i} is a subgroup of Z_{G_i}. In this setting, condition (2) in Proposition 13 is equivalent to the fact that for every i from 1 to s the integers d_i and n_i/m_i are relatively prime.

4.3. The classification of special quasisplit groups

Recall that a reductive group G over a field k is called *quasisplit* if it possesses a Borel subgroup defined over k (see for instance [Se2, III, 2.2]) or, in other words, if the variety of Borel subgroups of G has a rational point. We show now that a quasisplit group is special if and only if its derived subgroup and coradical are special.

Proposition 15. *Let G be a reductive algebraic group over k. Then G is quasisplit and special if and only if there exists an exact sequence of algebraic groups:*

$$1 \to D \to G \to C \to 1$$

where D is isomorphic to a direct product:

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for every index i, K_i is a finite separable extension of k, G_i is equal to either SL_{n_i} or Sp_{2n_i} for some integer n_i, and the group C is a special torus over k. In that case, D is the derived subgroup of G and C is the coradical of G.

Proof. If such an exact sequence exists, then, as D and C are special, it follows readily from the derived exact sequence of pointed sets in fppf-cohomology that G is special as well. Moreover, as C is commutative, the derived subgroup G' is contained in D. Now, as D is semisimple, it is equal to its own derived subgroup, and is in particular contained in G'. Finally, we see that D is equal to G', and the fact that C is the coradical of G follows readily. Now, as G and G' share the same variety of Borel subgroups, and G' is quasisplit, we see that G is quasisplit as well.

Suppose now that G is quasisplit and special. By the same argument as above, the derived subgroup G' is quasisplit. Moreover, by Proposition 5, G' is isomorphic to

$$R_{K_1|k}(G_1) \times R_{K_2|k}(G_2) \times \cdots \times R_{K_r|k}(G_r)$$

where, for each index i, the extension K_i of k is finite and separable and the group G_i is isomorphic over K_i to either $\mathrm{SL}_1(A_i)$, where A_i is a central simple algebra over K_i, or Sp_{2n_i} for some integer n_i. For each index i such that G_i is isomorphic to $\mathrm{SL}_1(A_i)$, we see that $\mathrm{SL}_1(A_i)$ is a direct factor of G_{K_i}. As G_{K_i} is quasisplit, this forces $\mathrm{SL}_1(A_i)$ to be quasisplit as well, implying that A_i is split. By Proposition 7, the coradical of G is special. We thus have an exact sequence as in the proposition, with D the derived subgroup G' of G and C the coradical C_G. □

We give now an application to the rationality problem of Noether, in the spirit of Corollary 12.

Corollary 16. *Let G be a special reductive quasisplit group. Then G is a stably k-rational variety (in the sense of* [CS2, p. 3]*) if and only if the coradical C_G is a stably k-rational variety. In this case, for every generically free linear representation V of G, the field of invariants $k(V)^G$ is stably pure.*

Note that there is an explicit description of the stably k-rational tori; see, for example, [V, 4.7, Thm. 2]. Namely, if T is a torus defined over k, then T is a stably k-rational variety if and only if there exist two tori S and R whose character lattices are permutation Galois modules (i.e., having a basis that is permuted by the action of the absolute Galois group of k) and a short exact sequence:

$$1 \to R \to S \to T \to 1.$$

Proof. By Proposition 15, the derived subgroup G' of G is a special group. By using the exact sequence:

$$1 \to G' \to G \to C_G \to 1$$

and an argument explained in the proof of Corollary 12, this implies that G is k-birational to $C_G \times G'$. Still by Proposition 15, we see that G' is a k-rational variety. This implies that G is a stably k-rational variety if and only if C_G is. The assertion about the generically free representations of G is proved exactly as in Corollary 12. □

Let G be an arbitrary reductive group over the field k. There is a unique inner form of G that is quasisplit, called the *quasisplit form* G_{qsplit} of G; see, for instance, [Sp]. In the following proposition, we prove that the quasisplit form of G is special if G is special. This is very reasonable, as we expect G_{qsplit} to be less "twisted" than G.

Proposition 17. *Let G be a special reductive group over k. The quasisplit form of G is special as well.*

Proof. The groups G and G_{qsplit} share the same coradical, and $(G_{\text{qsplit}})'$ is the quasisplit form of G'. Therefore, by Propositions 5 and 7, the derived subgroup and coradical of G_{qsplit} are special, proving that G_{qsplit} is special. □

5. Special tori

In this section we give the classification of special tori after Colliot-Thélène. This classification is implicitely contained in [CS] and explicitely given in the first version of [BR] on the arXiv, but not in the published version. For this reason we thought that it would be a good idea to include it in the present paper. We actually reproduce the proof from [BR]. The relevance of the classification of special tori for our problem of classifying reductive groups is twofold: firstly, tori are reductive groups and secondly, by Proposition 7, the coradical of a special reductive group is a special torus.

Let k be a base field, k_s a fixed separable closure and Γ the absolute Galois group $\mathrm{Gal}(k_s|k)$ of k. A continuous Γ-module is called a *permutation* Γ*-module* if it is a free $\mathbb{Z}$-module possessing a basis which is permuted by the action of Γ. A continuous Γ-module is called *invertible* if it is a direct factor of a permutation Γ-module.

Theorem 18 (Colliot-Thélène)**.** *Let T be a torus defined over a field k. The torus T is special if and only if the character lattice of T_{k_s} is invertible.*

Proof. If the character lattice of T_{k_s} is invertible, then T is a direct factor of a finite product of tori of type $R_{K|k}(\mathbb{G}_{m,K})$, where K is a finite separable extension of k and $\mathbb{G}_{m,K}$ is the multiplicative group over K. By Hilbert's theorem 90 and Shapiro's lemma, it follows that T is special.

Conversely, assume that T is special. Let K be a finite separable field extension of k. By Lemma 19 below,

$$H^1(K((t)),T) \simeq H^1(K,T) \oplus H^1(K,N)$$

where N is the cocharacter lattice of T_{k_s} and $K((t))$ is the field of formal Laurent series over K. Since the torus T is special, we see that $H^1(K,N)$ is trivial. As this property holds for every finite separable field extension K of k, it means that the torus T is flasque. By [CS, Prop. 7.4] a flasque torus is special if and only if the character lattice of T_{k_s} is invertible, which completes the proof, modulo the following lemma:

Lemma 19. *For any torus over the field k, there is an isomorphism:*

$$H^1(k((t)),T) \simeq H^1(k,T) \oplus H^1(k,N)$$

where N is the cocharacter lattice of T_{k_s}.

Proof of Lemma 19. Set $K = k((t))$. Let L be the union of the fields $k'((t))$ for all finite extensions of k inside k_s. Then the Galois group $\mathrm{Gal}(L|K)$ is equal to Γ. We have the inflation-restriction exact sequence; see, for example, [Se2, I.2.6(b)]:

$$1 \to H^1(\Gamma, T(L)) \to H^1(K, T) \to H^1(L, T) \to 1.$$

The torus T is split over L, hence, by Hilbert's theorem 90, we see that the sets $H^1(\Gamma, T(L))$ and $H^1(K, T)$ are equal. By [CGP, Lem. 5.17(3)], we have

$$H^1(\Gamma, T(L)) \simeq H^1(\Gamma, T(k_s[t, t^{-1}])).$$

Now, we write:

$$T(k_s[t, t^{-1}]) = N \otimes_{\mathbb{Z}} k_s[t, t^{-1}]^* = N \otimes_{\mathbb{Z}} (k_s^* \oplus \mathbb{Z}) = T(k_s) \oplus N$$

because T splits over k_s and $k_s[t, t^{-1}]^*$ is equal to $k_s^* \oplus \mathbb{Z}$. Finally, we obtain:

$$\begin{aligned} H^1(K, T) &\simeq H^1(\Gamma, T(L)) \simeq H^1(\Gamma, T(k_s[t, t^{-1}])) \\ &\simeq H^1(\Gamma, T(k_s)) \oplus H^1(\Gamma, N) \simeq H^1(k, T) \oplus H^1(k, N). \qquad \square \end{aligned}$$

6. An application

This section is devoted to the proof of the following result. At the end of the section, we explain how this result is connected to a conjecture of Serre in [Se3].

Theorem 20. *Let G be a reductive group over a field k, and let $\{k_\alpha\}_{\alpha \in I}$ be a nonempty finite family of finite field extensions of k such that the degrees $[k_\alpha : k]$ are relatively prime. Then the following conditions are equivalent:*

(1) *G is a special group.*
(2) *For every index $\alpha \in I$, the group G_{k_α} is a special group.*

Proof. If G is special then for every index $\alpha \in I$, the group G_{k_α} is special, as the property of being special is preserved by scalar extension. In the following, we will thus prove that (2) implies (1). To this aim, we prove that G satisfies the conditions of Theorem 9.

As the group G satisfies (2), the group $G_{\bar{k}}$ is special. As in the proof of Proposition 5, we see that

$$G' \simeq R_{K_1|k}(G_1) \times ... \times R_{K_r|k}(G_r)$$

where for each i, K_i is a finite separable extension of k and G_i is an absolutely simple simply connected group of type A or C. We want to prove that for each index i, G_i is either of inner type A, or split of type C.

For each index i and α, the k_α-algebra $k_\alpha \otimes_k K_i$ is étale, hence is a direct product $K_{\alpha,i,1} \times ... \times K_{\alpha,i,s_{\alpha,i}}$, where the $K_{\alpha,i,j}$s are finite separable extensions of K_i. We have the following equality:

$$G'_{k_\alpha} = (G_{k_\alpha})' \simeq \prod_{i=1}^{r} \prod_{j=1}^{s_{\alpha,i}} R_{K_{\alpha,i,j}|k_\alpha}(G_i \times_{K_i} K_{\alpha,i,j}).$$

As the group G_{k_α} is special, we know by Proposition 5 that each $G_i \times_{K_i} K_{\alpha,i,j}$ is either of inner type A, or split of type C.

Suppose now that there exists an index i such that G_i is of outer type A. That is, $G_i = \mathrm{SU}(A_i, \sigma)$ where A_i is a central simple algebra over a quadratic field extension L_i of K_i, and σ is an involution of the second kind on A_i. Let $\alpha \in I$ such that $[k_\alpha : k]$ is odd. Such an index α exists by assumption. As the dimension of $k_\alpha \otimes_k K_i$ over K_i is equal to $[k_\alpha : k]$, it is clear that there is an index j such that $[K_{\alpha,i,j} : K_i]$ is odd. Consequently, $K_{\alpha,i,j} \otimes_{K_i} L_i$ is a field, and the group $G_i \times_{K_i} K_{\alpha,i,j}$ is of outer type A, a contradiction.

Suppose now that there exists an index i such that G_i of type C, that is, $G_i = \mathrm{Sp}(A_i, \sigma)$, where A_i is a central simple algebra over K_i, and σ is a symplectic involution on A_i. By the remark above, the field $K_{\alpha,i,j}$ splits A_i, where α runs over I and j runs from 1 to $s_{\alpha,i}$. Observe now that the degrees $[K_{\alpha,i,j} : K_i]$ are relatively prime, as a common divisor would divide $[k_\alpha : k]$ for every α in I, hence must be 1. By a restriction-corestriction argument, we claim that A_i is split over K_i. Indeed, for every finite field extension K of K_i, there exists a norm morphism $H^2(K, \mathbb{G}_m) \to H^2(K_i, \mathbb{G}_m)$ such that the composite

$$H^2(K_i, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m) \to H^2(K_i, \mathbb{G}_m)$$

is the multplication by the degree $[K : K_i]$, where the morphism on the left is induced by extending the scalars from K_i to K. For the existence of the norm; see [Gi, 0.4]. As the field $K_{\alpha,i,j}$ splits A_i, we then see that $[K_{\alpha,i,j} : K_i][A_i] = 0$ in $H^2(K_i, \mathbb{G}_m)$. Now, using this equality for all j and α, and using a Bezout identity among the degrees $[K_{\alpha,i,j} : K_i]$, we get that $[A_i] = 0$, that is, A_i is split.

We prove now that $H^1(k, C_G)$ is trivial. To this aim, let α be an element of I. By Proposition 7 the torus $C_G \times_k k_\alpha$ is special, hence the group $H^1(k_\alpha, C_G)$ is trivial. By a restriction-corestriction argument, we get that $H^1(k, C_G)$ is annihilated by $[k_\alpha : k]$. As the degrees $[k_\alpha : k]$ are relatively prime where α runs over I, we obtain that $H^1(k, C) = 1$.

By using a similar restriction-corestriction argument, we will prove now that $\mathfrak{S}(k, G)$ is trivial. To perform this argument, the only thing that needs to be checked is that there exists a well-defined norm map

$$N_{L|k} : \mathfrak{S}(L, G) \to \mathfrak{S}(k, G)$$

for any finite field extension L of k, or, in other words, that the norm map

$$N_{L|k} : H^1(L, Z_{G'}) \to H^1(k, Z_{G'})$$

maps $\operatorname{Im} \alpha_{G',L}$ into $\operatorname{Im} \alpha_{G',k}$ (as it maps automatically $\operatorname{Im} \gamma_{G,L}$ into $\operatorname{Im} \gamma_{G,k}$). But this fact follows from [Gi, Cor. II.3.3]: as G'_{ad} is a k-rational variety, the isogeny $G' \to G_{\mathrm{ad}}$ satisfies the norm principle [Gi, Def. I, p. 206] with respect to any finite field extension L of k, which is exactly what we want.

To complete the proof of the theorem, we use:

Lemma 21. *Let K be a field extension of k. If G satisfies condition (2) then there exist a finite set J and a family of finite field extensions $\{K_\beta\}_{\beta \in J}$ of K such that the degrees $[K_\beta : K]$ are relatively prime and for every index $\beta \in J$, the group G_{K_β} is special.*

Before proving the lemma, let us show how it implies the theorem. We have already seen that G satisfies the first condition of Theorem 9. Now, let K be a field extension of k. By Lemma 21, the group G_K satisfies the second condition of Theorem 20. Consequently, by the above, we have $H^1(K, C_G) = 1$ and $\mathfrak{S}(K, G) = 1$. That is, G satisfies the second and third conditions of Theorem 9. Hence G is special.

Proof of Lemma 21. We take for the family $\{K_\beta\}_{\beta \in J}$ the family of all composite fields of K and the k_αs. More precisely, let J be the following set:

$$\{(\alpha, \mathfrak{m}), \quad \alpha \in I \text{ and } \mathfrak{m} \text{ is a maximal ideal of } A_\alpha\},$$

where A_α denotes $k_\alpha \otimes_k K$. For $\beta = (\alpha, \mathfrak{m})$ in J, we set $K_\beta = A_\alpha/\mathfrak{m}$. It is a field extension of K and of k_α. Hence the group G_{K_β} is special. We claim that J is a finite set, that for every β in J the field extension K_β of K is finite, and that the degrees $[K_\beta : K]$ are relatively prime.

Let $\beta = (\alpha, \mathfrak{m})$ in J. We denote by $A_{\alpha,\mathfrak{m}}$ the localization of A_α with respect to the maximal ideal $\mathfrak{m}$. Observe that A_α is a finite-dimensional vector space over K. In particular, it is an artinian K-algebra, a fact that has many consequences. First of all, the field extension K_β is finite. We let d be a common divisor to all the degrees $[K_\gamma : K]$, where γ runs over J. Secondly, by [AM, Prop. 8.1, 8.3] there are only finitely many prime ideals in A_α and each of them is maximal. This forces J to be finite. In addition, by [AM, Prop. 8.4] the unique maximal ideal of $A_{\alpha,\mathfrak{m}}$, also denoted $\mathfrak{m}$, satisfies $\mathfrak{m}^N = 0$ for some integer N. Now, in the following filtration:

$$\mathfrak{m}^N = 0 \subseteq \mathfrak{m}^{N-1} \subseteq \ldots \subseteq \mathfrak{m} \subseteq A_{\alpha,\mathfrak{m}}$$

each successive quotient is a finite dimensional K_β-vector space. In particular, the dimension of $A_{\alpha,\mathfrak{m}}$ over K is divisible by $[K_\beta : K]$, hence by d. The localization map:

$$A_\alpha \to \prod_{\mathfrak{m} \in \mathrm{Spec}(A_{\alpha,K})} A_{\alpha,\mathfrak{m}}$$

is an isomorphism by [AM, Prop. 8.7]. Therefore, we see that d divides the dimension of A_α over K, which is equal to $[k_\alpha : k]$. As the $[k_\alpha : k]$ are relatively prime when α runs over I, we see that d is equal to 1, proving the lemma. $\square$

This completes the proof of Theorem 20. $\square$

To finish this section, we explain the connection of Theorem 20 with a conjecture of Serre. The following question is a special case of [Se3, 2.4, Quest. 2]:

Question 22. *Let K be a field extension of k and X a G-torsor defined over K. Let $\{K_\alpha\}_{\alpha \in I}$ be a nonempty finite family of finite field extensions of K such that* *the degrees $[K_\alpha : K]$ are relatively prime. Suppose that $X(K_\alpha) \neq \varnothing$ for every index $\alpha \in I$. Is it true that $X(K) \neq \varnothing$?*

In other words, if X possesses a 0-cycle of degree 1, does it have a rational point? We refer the interested reader to the work of Bayer-Fluckiger and Lenstra [BL] and Black [B], who proved that the question has a positive answer for many classical groups. Note that the analogous question, due to Totaro [T], with X a homogeneous space instead of a G-torsor, has a negative answer in general. Indeed, examples of homogeneous spaces under connected linear groups having a 0-cycle of degree 1 but no rational point were constructed by Florence in [F] and Parimala in [Pa]. For X a G-torsor the question remains widely open. Our approach is different from the one in [BL] or [B]. Namely, instead of considering a specific group G, we consider any connected reductive group G, but restrict our attention to a certain type of G-torsor.

Let V be a versal generically free G-variety (k-variety = geometrically integral k-scheme, see [M, 3d] for the definition of versal and generically free) defined over k: for example, a generically free representation of G defined over k, see [M, Prop. 3.10]. By definition, there exists a G-invariant open k-subscheme U in V, a k-scheme Y defined over k and a morphism $\pi : U \to Y$ that gives U the structure of a G-torsor over Y. As π is faithfully flat, the fact that U is geometrically integral implies that the scheme Y is geometrically integral as well. We denote by K the function field of Y and by $X \to \operatorname{Spec} K$ the generic fiber of π, called the generic G-torsor of V. By [M, Cor. 3.12] and [M, Prop. 3.16], the group G is special if and only if X is trivial, that is $X(K) \neq \varnothing$. Moreover, the same assertion holds after a finite field extension l of k. More precisely, as $Y_l := Y \times_k l$ is integral, the ring $L := l \otimes_k K$ is a field, namely the function field of Y_l. As V_l is a versal generically free G_l-variety defined over l with generic torsor $X_L \to \operatorname{Spec} L$, we see, again by [M, Cor. 3.12] and [M, Prop. 3.16] that G_l is special if and only if $X(L) \neq \varnothing$.

Now let $\{k_\alpha\}_{\alpha \in I}$ be a family of finite field extensions of k as in the statement of Theorem 20. We denote by K_α the field $k_\alpha \otimes_k K$. We see that Theorem 20 gives a positive answer to Question 22 for the torsor $X \to \operatorname{Spec} K$ and the family $\{K_\alpha\}_{\alpha \in I}$: indeed, if $X(K_\alpha) \neq \varnothing$ for every index α, then the group G_{k_α} is special for every index α, hence the group G is special by Theorem 20, and finally we see that $X(K) \neq \varnothing$.

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