

EXOTIC SYMMETRIC SPACES OF HIGHER LEVEL: SPRINGER CORRESPONDENCE FOR COMPLEX REFLECTION GROUPS

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Abstract. Let $G = \mathrm{GL}(V)$ for a $2n$ -dimensional vector space V , and θ an involutive automorphism of G such that $H = G^\theta \simeq \mathrm{Sp}(V)$. Let $G_{\mathrm{uni}}^{\theta}$ be the set of unipotent elements $g \in G$ such that $\theta(g) = g^{-1}$. For any integer $r \geq 2$, we consider the variety $G_{\mathrm{uni}}^{\theta} \times V^{r-1}$, on which H acts diagonally. Let $W_{n,r} = S_n \times (\mathbf{Z}/r\mathbf{Z})^n$ be a complex reflection group. In this paper, generalizing the known result for $r = 2$, we show that there exists a natural bijective correspondence (Springer correspondence) between the set of irreducible representations of $W_{n,r}$ and a certain set of H -equivariant simple perverse sheaves on $G_{\mathrm{uni}}^{\theta} \times V^{r-1}$. We also consider a similar problem for $G \times V^{r-1}$, on which G acts diagonally, where $G = \mathrm{GL}(V)$ for a finite-dimensional vector space V .

Introduction

Let V be a $2n$ -dimensional vector space over \mathbf{k} , where \mathbf{k} is an algebraically closed field with $\mathrm{char} \mathbf{k} \neq 2$. Let $G = \mathrm{GL}(V)$ and $\theta : G \rightarrow G$ an involutive automorphism such that $G^\theta \simeq \mathrm{Sp}(V)$. Let $\iota : G \rightarrow G$ be the anti-automorphism $g \mapsto g^{-1}$, and put $G^{\theta} = \{g \in G \mid \theta(g) = g^{-1}\}$. We consider the variety $\mathcal{X} = G^{\theta} \times V$, on which $H = G^\theta$ acts diagonally. Let $G_{\mathrm{uni}}^{\theta}$ be the set of unipotent elements in G^{θ} , and define a closed subvariety $\mathcal{X}_{\mathrm{uni}}$ of \mathcal{X} by $\mathcal{X}_{\mathrm{uni}} = G_{\mathrm{uni}}^{\theta} \times V$. $\mathcal{X}_{\mathrm{uni}}$ is nothing but the exotic nilpotent cone introduced by Kato [K]. It is known that $\mathcal{X}_{\mathrm{uni}}$ is H -stable, and the set of H -orbits in $\mathcal{X}_{\mathrm{uni}}$ is in bijection with the set $\mathcal{P}_{n,2}$ of double partitions of n ([K]). Let B be a θ -stable Borel subgroup of G , U the unipotent radical of B , and $(M_i)_{1 \leq i \leq n}$ be an isotropic flag in V whose stabilizer in H is B^θ . We define a variety $\tilde{\mathcal{X}}_{\mathrm{uni}}$ by

$$\tilde{\mathcal{X}}_{\mathrm{uni}} = \{(x, v, gB^\theta) \in G_{\mathrm{uni}}^{\theta} \times V \times H/B^\theta \mid g^{-1}xg \in U^{\theta}, g^{-1}v \in M_n\},$$

and define a map $\pi_1 : \tilde{\mathcal{X}}_{\mathrm{uni}} \rightarrow \mathcal{X}_{\mathrm{uni}}$ by the projection on the first two factors. In [K], [SS1], the Springer correspondence between the set of H -orbits in $\mathcal{X}_{\mathrm{uni}}$ and the

set of irreducible representations of the Weyl group $W_{n,2}$ of type C_n was studied. More precisely, it is stated as follows; $(\pi_1)_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\text{uni}}]$ is a semisimple perverse sheaf on \mathcal{X}_{uni} , equipped with $W_{n,2}$ -action and is decomposed as

$$(\pi_1)_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\text{uni}}] \simeq \bigoplus_{\lambda \in \tilde{\mathcal{P}}_{n,2}} V(\lambda) \otimes \text{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_\lambda], \tag{1}$$

where $V(\lambda)$ is the irreducible representation of $W_{n,2}$ and \mathcal{O}_λ is the H -orbit in \mathcal{X}_{uni} corresponding to $\lambda \in \mathcal{P}_{n,2}$.

In this paper, we consider the variety $G^{\iota\theta} \times V^{r-1}$ for a positive integer $r \geq 2$, with the diagonal action of H . We call it the exotic symmetric space of level r . Let $W_{n,r} = S_n \times (\mathbf{Z}/r\mathbf{Z})^n$ be the complex reflection group $G(r, 1, n)$, where S_n is the symmetric group of degree n . We will generalize the previous result to the correspondence between the set of irreducible representations of $W_{n,r}$ and a certain set of simple perverse sheaves on $G^{\iota\theta}_{\text{uni}} \times V^{r-1}$. Let $\mathcal{Q}_{n,r}$ be the set of $\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Z}_{\geq 0}^r$ such that $\sum m_i = n$, and $\mathcal{Q}_{n,r}^0$ the subset of $\mathcal{Q}_{n,r}$ consisting of \mathbf{m} such that $m_r = 0$. For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define varieties

$$\begin{aligned} \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} &= \{(x, \mathbf{v}, gB^\theta) \in G^{\iota\theta}_{\text{uni}} \times V^{r-1} \times H/B^\theta \mid g^{-1}xg \in U^{\iota\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}, \\ \mathcal{X}_{\mathbf{m},\text{uni}} &= \bigcup_{g \in H} g(U^{\iota\theta} \times \prod_{i=1}^{r-1} M_{p_i}), \end{aligned}$$

where $p_i = m_1 + \dots + m_i$ for each i . We define a map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ by the projection on the first two factors. In the case where $\mathbf{m} = (n, 0, \dots, 0)$, we write $\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}, \mathcal{X}_{\mathbf{m},\text{uni}}$ and $\pi_1^{(\mathbf{m})}$ simply by $\tilde{\mathcal{X}}_{\text{uni}}, \mathcal{X}_{\text{uni}}$ and π_1 . Note that even in this case, the map $\mathcal{X}_{\text{uni}} \rightarrow G^{\iota\theta}_{\text{uni}} \times V^{r-1}$ is not surjective if $r \geq 3$. For each $\mathbf{m} \in \mathcal{Q}_{n,r}$ we consider a map $\tilde{\pi}_{\mathbf{m},1} : \pi_1^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}}) \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$. Note that $\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \subset \pi_1^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}}) \subset \tilde{\mathcal{X}}_{\text{uni}}$. Let $\mathcal{P}_{n,r}$ be the set of r -tuples of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ such that $\sum_i |\lambda^{(i)}| = n$. For $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, let $\tilde{\mathcal{P}}(\mathbf{m})$ be the set of all $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$ such that $|\lambda^{(i)}| = m_i$ for $i = 1, \dots, r-2$ (hence $|\lambda^{(r-1)}| = k$ for $0 \leq k \leq m_{r-1}$). As a generalization of (1), we prove the following result (see Theorem 8.7 (iii)), which is regarded as the Springer correspondence for $W_{n,r}$. Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then the complex $(\tilde{\pi}_{\mathbf{m},1})_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathbf{m},\text{uni}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$, equipped with a $W_{n,r}$ -action, and is decomposed as

$$(\tilde{\pi}_{\mathbf{m},1})_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathbf{m},\text{uni}}] \simeq \bigoplus_{\lambda \in \tilde{\mathcal{P}}(\mathbf{m})} V(\lambda) \otimes \text{IC}(\overline{X}_\lambda, \bar{\mathbf{Q}}_l)[\dim X_\lambda], \tag{2}$$

where $V(\lambda)$ is an irreducible representation of $W_{n,r}$, and X_λ is a certain smooth irreducible subvariety of $\mathcal{X}_{\mathbf{m},\text{uni}}$ parametrized by $\lambda \in \tilde{\mathcal{P}}(\mathbf{m})$. Any irreducible representation of $W_{n,r}$ is realized in this way uniquely for a suitable choice of $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. We can determine the varieties X_λ explicitly. Note that in the case $r \geq 3$, $G^{\iota\theta}_{\text{uni}} \times V^{r-1}$ has infinitely many H -orbits. So the description of X_λ becomes more complicated compared to the case where $r = 2$.

In the course of the proof, we show a weaker version of the Springer correspondence. For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, we define a subgroup $\mathcal{W}_{\mathbf{m}}^{\natural}$ of $W_{n,2}$ by

$$\mathcal{W}_{\mathbf{m}}^{\natural} = S_{m_1} \times \cdots \times S_{m_{r-2}} \times W_{m_{r-1},2}.$$

For $\lambda \in \tilde{\mathcal{P}}(\mathbf{m})$, one can associate an irreducible representation ρ_{λ} of $\mathcal{W}_{\mathbf{m}}^{\natural}$ in a canonical way. We show that the complex $(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathbf{m},\text{uni}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$, equipped with $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action, and is decomposed as

$$(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathbf{m},\text{uni}}] \simeq \bigoplus_{\lambda \in \tilde{\mathcal{P}}(\mathbf{m})} \rho_{\lambda} \otimes \text{IC}(\bar{X}_{\lambda}, \bar{\mathbf{Q}}_l)[\dim X_{\lambda}] \tag{3}$$

(see Theorem 8.7 (ii)). Note that the group $W_{n,r}$ is not directly related to the geometry of H/B^{θ} , while $\mathcal{W}_{\mathbf{m}}^{\natural}$ behaves well since it is a subgroup of $W_{n,2}$. So first we show (3), and then prove (2) by making use of (3).

We also consider the variety $\mathcal{X} = G \times V$, where V is an n -dimensional vector space over \mathbf{k} (of any characteristic), and $G = \text{GL}(V)$. G acts diagonally on \mathcal{X} . Put $\mathcal{X}_{\text{uni}} = G_{\text{uni}} \times V$, where G_{uni} is the set of unipotent elements in G . The variety \mathcal{X}_{uni} is isomorphic to the enhanced nilpotent cone introduced by Achar–Henderson [AH]. It is known by [AH], [T] that \mathcal{X}_{uni} is G -stable, and the set of G -orbits is in bijection with $\mathcal{P}_{n,2}$. For each $\mathbf{m} = (m_1, m_2) \in \mathcal{Q}_{n,2}$, one can define a similar map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ as in the exotic case. Achar–Henderson [AH] and Finkelberg–Ginzburg–Travkin [FGT] proved the Springer correspondence between the set of irreducible representations of $S_{m_1} \times S_{m_2}$ and the set of simple perverse sheaves associated to the G -orbits in $\mathcal{X}_{\mathbf{m},\text{uni}}$, which are direct summands of $(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathbf{m},\text{uni}}]$. In this paper, we consider the variety $\mathcal{X} = G \times V^{r-1}$ with diagonal G -action, which we call the enhanced variety of level r . The arguments used to prove the Springer correspondence (3) in the exotic case can be applied also to the enhanced case, step by step, by a suitable modification. Actually, the argument becomes drastically simple. We show that the Springer correspondence holds for $\mathcal{W}_{\mathbf{m}}^{\natural} = S_{m_1} \times \cdots \times S_{m_r}$ for any $\mathbf{m} \in \mathcal{Q}_{n,r}$. (This result was announced by the author in 2009, but was not published.) In [Li], Li established the Springer correspondence for such $\mathcal{W}_{\mathbf{m}}^{\natural}$ in connection with certain perverse sheaves arising from the framed Jordan quiver. Considering the framed Jordan quiver is essentially the same as considering the enhanced variety. So in this case our result is regarded as an alternate approach for his result.

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1. Intersection cohomology on $G_{\text{reg}}^{u\theta} \times V^{r-1}$ (exotic case)

1.1. Let \mathbf{k} be an algebraically closed field. In this paper, we consider the following two cases.

(I) The exotic case

Let V be a $2n$ -dimensional vector space over \mathbf{k} (with $\text{char } \mathbf{k} \neq 2$), with basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$. Let $G = \text{GL}_{2n}$. Consider an involutive automorphism $\theta : G \rightarrow G$ given by

$$\theta(g) = J^{-1}({}^t g^{-1})J \quad \text{with } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

where 1_n is the identity matrix of degree n , and put $H = G^\theta$. Then H is the symplectic group Sp_{2n} with respect to the symplectic form $\langle u, v \rangle = {}^t u J v$ for $u, v \in V$ under the identification $V \simeq \mathbf{k}^{2n}$ via the basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$, which gives rise to a symplectic basis.

Let $\iota : G \rightarrow G$ be the anti-automorphism $g \rightarrow g^{-1}$. We consider the set $G^{u\theta} = \{g \in G \mid \theta(g) = g^{-1}\}$. It is known that $G^{u\theta} = \{g\theta(g)^{-1} \mid g \in G\}$, and so $G^{u\theta} \simeq G/H$. Let $T \subset B$ be a pair of a θ -stable maximal torus and a θ -stable Borel subgroup of G . Let $M_1 \subset \dots \subset M_n$ be an isotropic flag in V whose stabilizer in H coincides with B^θ . We assume that $M_i = \langle e_1, \dots, e_i \rangle$ for $i = 1, \dots, n$, and that e_i, f_j are weight vectors for T .

(II) The enhanced case

Let $\tilde{V} = V \oplus V$, where V is an n -dimensional vector space over \mathbf{k} , and $G = G_0 \times G_0$ a subgroup of $\text{GL}(\tilde{V})$ with $G_0 = \text{GL}(V)$. Let $\theta : G \rightarrow G$ be an involution defined by $\theta(g_1, g_2) = (g_2, g_1)$. Put $H = G^\theta \simeq G_0$. Then H acts naturally on V . Let $G^{u\theta}$ be a subset of G defined similarly to the case (I). Then $G^{u\theta} \simeq G_0$ (as a set) and $H \simeq G_0$ acts on $G^{u\theta}$ by conjugation. Let $T \subset B$ be a pair of a θ -stable maximal torus and a θ -stable Borel subgroup of G . We can write $T = T_0 \times T_0$ and $B = B_0 \times B_0$ so that $B^\theta \simeq B_0, T^\theta \simeq T_0$. Let $M_1 \subset \dots \subset M_n = V$ be a complete flag in V whose stabilizer in H coincides with B^θ . We fix a basis $\{e_1, \dots, e_n\}$ of V such that $M_i = \langle e_1, \dots, e_i \rangle$ for $i = 1, \dots, n$, and that e_i are weight vectors for T^θ . Let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a basis of \tilde{V} , where $f_i = e_i$ for each i , and define a symplectic form $\langle \cdot, \cdot \rangle$ on \tilde{V} so that $\{e_i, f_j\}$ gives a symplectic basis of \tilde{V} .

1.2. For an integer $r \geq 1$, we consider the variety $G^{u\theta} \times V^{r-1}$ on which H acts diagonally. We call $G^{u\theta} \times V^{r-1}$ the exotic symmetric space of level r in the case (I), and the enhanced space of level r in the case (II). Let $\mathcal{Q}_{n,r} = \{\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Z}_{\geq 0}^r \mid \sum_i m_i = n\}$. We define $\mathcal{Q}_{n,r}^0 = \{\mathbf{m} \in \mathcal{Q}_{n,r} \mid m_r = 0\}$. For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define $\mathbf{p}(\mathbf{m}) = (p_1, p_2, \dots, p_r)$ by $p_i = m_1 + \dots + m_i$ for each i . We define varieties

$$\tilde{\mathcal{X}}_{\mathbf{m}} = \left\{ (x, \mathbf{v}, gB^\theta) \in G^{u\theta} \times V^{r-1} \times H/B^\theta \mid g^{-1}xg \in B^{u\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\},$$

$$\mathcal{X}_{\mathbf{m}} = \bigcup_{g \in H} g(B^{u\theta} \times \prod_{i=1}^{r-1} M_{p_i}).$$

We define a map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow G^{\iota\theta} \times V^{r-1}$ by $\pi^{(\mathbf{m})}(x, \mathbf{v}, gB^\theta) = (x, \mathbf{v})$. Clearly $\mathcal{X}_{\mathbf{m}} = \text{Im } \pi^{(\mathbf{m})}$. Since $\tilde{\mathcal{X}}_{\mathbf{m}} \simeq H \times^{B^\theta} (B^{\iota\theta} \times \prod_i M_{p_i})$, $\tilde{\mathcal{X}}_{\mathbf{m}}$ is smooth and irreducible. Since $\pi^{(\mathbf{m})}$ is proper, $\mathcal{X}_{\mathbf{m}}$ is a closed irreducible subvariety of $G^{\iota\theta} \times V^{r-1}$. In the case where $\mathbf{m} = (n, 0, \dots, 0)$, namely, $\mathbf{p}(\mathbf{m}) = (n, \dots, n)$, we write $\tilde{\mathcal{X}}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}$ and $\pi^{(\mathbf{m})}$ by $\tilde{\mathcal{X}}, \mathcal{X}$ and π , respectively. Note that for any $\mathbf{m} \in \mathcal{Q}_{n,r}$, $\mathcal{X}_{\mathbf{m}}$ is contained in \mathcal{X} . The dimension of $\tilde{\mathcal{X}}_{\mathbf{m}}$ is computed as follows;

$$\dim \tilde{\mathcal{X}}_{\mathbf{m}} = \begin{cases} 2n^2 + \sum_{i=1}^r (r-i)m_i & \text{exotic case,} \\ n^2 + \sum_{i=1}^r (r-i)m_i & \text{enhanced case.} \end{cases} \tag{1.2.1}$$

In fact, in the exotic case, by [SS1; (3.1.1)], we have

$$\begin{aligned} \dim \tilde{\mathcal{X}}_{\mathbf{m}} &= \dim H/B^\theta + \dim B^{\iota\theta} + \sum_{i=1}^{r-1} \dim M_{p_i} \\ &= 2n^2 + \sum_{i=1}^{r-1} (m_1 + \dots + m_i) \\ &= 2n^2 + \sum_{i=1}^{r-1} (r-i)m_i. \end{aligned}$$

The computation for the enhanced case is similar (in this case, $\dim B^\theta = \dim B^{\iota\theta}$).

Let $T_{\text{reg}}^{\iota\theta}$ be the set of regular semisimple elements in $T^{\iota\theta}$, namely, the set of elements in $T^{\iota\theta}$ such that all the eigenspaces in V have dimension 2 (resp. dimension 1) in the exotic case (resp. in the enhanced case). We put $G_{\text{reg}}^{\iota\theta} = \bigcup_{g \in H} gT_{\text{reg}}^{\iota\theta}g^{-1}$, $B_{\text{reg}}^{\iota\theta} = G_{\text{reg}}^{\iota\theta} \cap B$. We define varieties $\tilde{\mathcal{Y}}_{\mathbf{m}}, \mathcal{Y}_{\mathbf{m}}$ by

$$\begin{aligned} \tilde{\mathcal{Y}}_{\mathbf{m}} &= \left\{ (x, \mathbf{v}, gB^\theta) \in G_{\text{reg}}^{\iota\theta} \times V^{r-1} \times H/B^\theta \mid g^{-1}xg \in B_{\text{reg}}^{\iota\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\} \\ \mathcal{Y}_{\mathbf{m}} &= \bigcup_{g \in H} g(B_{\text{reg}}^{\iota\theta} \times \prod_{i=1}^{r-1} M_{p_i}), \end{aligned}$$

and a map $\psi^{(\mathbf{m})} : \tilde{\mathcal{Y}}_{\mathbf{m}} \rightarrow G^{\iota\theta} \times V^{r-1}$ by $\psi^{(\mathbf{m})}(x, \mathbf{v}, gB^\theta) = (x, \mathbf{v})$. Clearly $\text{Im } \psi^{(\mathbf{m})} = \mathcal{Y}_{\mathbf{m}}$. As in the case of $\tilde{\mathcal{X}}_{\mathbf{m}}$, we write $\tilde{\mathcal{Y}}_{\mathbf{m}}, \mathcal{Y}_{\mathbf{m}}$ and $\psi^{(\mathbf{m})}$ by $\tilde{\mathcal{Y}}, \mathcal{Y}, \psi$ in the case where $\mathbf{m} = (n, 0, \dots, 0)$. As in [SS1; (3.1.2)], $\tilde{\mathcal{Y}}_{\mathbf{m}}$ can be expressed as

$$\begin{aligned} \tilde{\mathcal{Y}}_{\mathbf{m}} &\simeq H \times^{B^\theta} \left(B_{\text{reg}}^{\iota\theta} \times \prod_i M_{p_i} \right) \\ &\simeq H \times^{B^\theta \cap Z_H(T^{\iota\theta})} \left(T_{\text{reg}}^{\iota\theta} \times \prod_i M_{p_i} \right). \end{aligned} \tag{1.2.2}$$

1.3. In the remainder of this section, we assume that \mathcal{X} is of exotic type. As in [SS1; 3.2], for each subset $I \subset [1, n]$, put $M_I = \{v \in M_n \mid \text{supp}(v) = I\}$, where for $v = \sum_{i=1}^n a_i e_i \in M_n$, $\text{supp}(v)$ is the set of $j \in [1, n]$ such that $a_j \neq 0$. M_I is

an open dense subset of the space spanned by $\{e_i \mid i \in I\}$, which we denote by \overline{M}_I . For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define $\mathcal{I}(\mathbf{m})$ as the set of $\mathbf{I} = (I_1, \dots, I_r)$ such that $[1, n] = \coprod_{i=1}^r I_i$ with $|I_i| = m_i$. For $\mathbf{I} = (I_1, \dots, I_r)$, put $I_{<i} = I_1 \cup I_2 \cup \dots \cup I_{i-1}$. Hence $|I_{<i}| = p_{i-1}$. For each $\mathbf{I} \in \mathcal{I}(\mathbf{m})$, we define a set $\mathbf{M}_{\mathbf{I}} \subset (M_n)^{r-1}$ by

$$\mathbf{M}_{\mathbf{I}} = \{\mathbf{v} = (v_1, \dots, v_{r-1}) \in (M_n)^{r-1} \mid v_i \in M_{I_i} + \overline{M}_{I_{<i}}\}.$$

and define a variety $\tilde{\mathcal{Y}}_{\mathbf{I}}$ by

$$\tilde{\mathcal{Y}}_{\mathbf{I}} = H \times^{B^\theta \cap Z_H(T^{i\theta})} (T_{\text{reg}}^{i\theta} \times \mathbf{M}_{\mathbf{I}}).$$

Note that $Z_H(T^{i\theta}) \simeq \text{SL}_2 \times \dots \times \text{SL}_2$ (n -times) and $B^\theta \cap Z_H(T^{i\theta})$ can be identified with the subgroup $B_2 \times \dots \times B_2$, where B_2 is a Borel subgroup of SL_2 . Since the action of $B^\theta \cap Z_H(T^{i\theta})$ on M_n is given by the action of the torus part T^θ , $\tilde{\mathcal{Y}}_{\mathbf{I}}$ is well-defined. Let $\psi_{\mathbf{I}} : \tilde{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}$ be the map induced from the map given by $(g, (t, \mathbf{v})) \mapsto (gtg^{-1}, g\mathbf{v}), H \times (T_{\text{reg}}^{i\theta} \times \mathbf{M}_{\mathbf{I}}) \rightarrow \mathcal{Y}$. Then $\text{Im } \psi_{\mathbf{I}}$ is independent of $\mathbf{I} \in \mathcal{I}(\mathbf{m})$, which we denote by $\mathcal{Y}_{\mathbf{m}}^0$. Hence, for $\mathbf{I} \in \mathcal{I}(\mathbf{m})$,

$$\mathcal{Y}_{\mathbf{m}}^0 = \bigcup_{g \in H} g(T_{\text{reg}}^{i\theta} \times M_{\mathbf{I}}). \tag{1.3.1}$$

For $\mathbf{I} \in \mathcal{I}(\mathbf{m})$, we define a parabolic subgroup $Z_H(T^{i\theta})_{\mathbf{I}}$ of $Z_H(T^{i\theta})$ by the condition that the i -th factor is SL_2 if $i \in I_r$ and is B_2 otherwise. Since $Z_H(T^{i\theta})_{\mathbf{I}}$ stabilizes $\mathbf{M}_{\mathbf{I}}$, one can define

$$\hat{\mathcal{Y}}_{\mathbf{I}} = H \times^{Z_H(T^{i\theta})_{\mathbf{I}}} (T_{\text{reg}}^{i\theta} \times M_{\mathbf{I}}).$$

Then the map $\psi_{\mathbf{I}}$ factors through $\hat{\mathcal{Y}}_{\mathbf{I}}$,

$$\psi_{\mathbf{I}} : \tilde{\mathcal{Y}}_{\mathbf{I}} \xrightarrow{\xi_{\mathbf{I}}} \hat{\mathcal{Y}}_{\mathbf{I}} \xrightarrow{\eta_{\mathbf{I}}} \mathcal{Y}_{\mathbf{m}}^0, \tag{1.3.2}$$

where $\xi_{\mathbf{I}}$ is the natural projection and $\eta_{\mathbf{I}}$ is the map induced from the map $(g, (t, \mathbf{v})) \mapsto (gtg^{-1}, g\mathbf{v})$. Then $\xi_{\mathbf{I}}$ is a locally trivial fibration with fibre isomorphic to

$$Z_H(T^{i\theta})_{\mathbf{I}} / (B^\theta \cap Z_H(T^{i\theta})) \simeq (\text{SL}_2 / B_2)^{I_r} \simeq \mathbf{P}_1^{I_r},$$

where $(\text{SL}_2 / B_2)^{I_r}$ denotes the direct product of SL_2 / B_2 with respect to the factors corresponding to I_r , and similarly for $\mathbf{P}_1^{I_r}$. Thus $\mathbf{P}_1^{I_r} = (\mathbf{P}_1)^{m_r}$.

Let $S_{\mathbf{I}} \simeq S_{I_1} \times \dots \times S_{I_r}$ be the stabilizer of (I_1, \dots, I_r) in S_n . Let $\mathcal{W} = N_H(T^{i\theta}) / Z_H(T^{i\theta}) \simeq S_n$, and $\mathcal{W}_{\mathbf{I}}$ the subgroup of \mathcal{W} corresponding to the subgroup $S_{\mathbf{I}}$. Then \mathcal{W} acts on $Z_H(T^{i\theta}) \simeq \text{SL}_2 \times \dots \times \text{SL}_2$ as the permutation of factors, and $\mathcal{W}_{\mathbf{I}}$ stabilizes the group $Z_H(T^{i\theta})_{\mathbf{I}}$. Since $\mathcal{W}_{\mathbf{I}}$ stabilizes $M_{\mathbf{I}}$, $\mathcal{W}_{\mathbf{I}}$ acts on $\tilde{\mathcal{Y}}_{\mathbf{I}}$ and on $\hat{\mathcal{Y}}_{\mathbf{I}}$. Now the map $\eta_{\mathbf{I}} : \hat{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ turns out to be a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$.

We define $\mathbf{I}(\mathbf{m}) = (I_1^\circ, \dots, I_r^\circ) \in \mathcal{I}(\mathbf{m})$ by $I_i^\circ = [p_{i-1} + 1, p_i]$ for $i = 1, \dots, r$. For $\mathbf{I} = \mathbf{I}(\mathbf{m})$, put $\tilde{\mathcal{Y}}_{\mathbf{I}} = \tilde{\mathcal{Y}}_{\mathbf{m}}^0$ and $\mathcal{W}_{\mathbf{I}} = \mathcal{W}_{\mathbf{m}}$. Note that $\tilde{\mathcal{Y}}_{\mathbf{m}}^0$ is an open dense subset of $\tilde{\mathcal{Y}}_{\mathbf{m}}$, hence irreducible. Put $\psi^{-1}(\mathcal{Y}_{\mathbf{m}}^0) = \tilde{\mathcal{Y}}_{\mathbf{m}}^+$. \mathcal{W} acts naturally on $\tilde{\mathcal{Y}}$ and the

map ψ is \mathcal{W} -equivariant with respect to the trivial action of \mathcal{W} on \mathcal{Y} . Hence it preserves the subset $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$, and the stabilizer of $\tilde{\mathcal{Y}}_{\mathbf{m}}^0$ in \mathcal{W} coincides with $\mathcal{W}_{\mathbf{m}}$. One can check that

$$\tilde{\mathcal{Y}}_{\mathbf{m}}^+ = \prod_{\mathbf{I} \in \mathcal{I}(\mathbf{m})} \tilde{\mathcal{Y}}_{\mathbf{I}} = \prod_{w \in \mathcal{W}/\mathcal{W}_{\mathbf{m}}} w(\tilde{\mathcal{Y}}_{\mathbf{m}}^0), \tag{1.3.3}$$

where $\tilde{\mathcal{Y}}_{\mathbf{I}}$ is an irreducible component of $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$.

We define a partial order on $\mathcal{Q}_{n,r}$ by $\mathbf{m}' \leq \mathbf{m}$ if $p'_i \leq p_i$ for each i , where $\mathbf{p}(\mathbf{m}) = (p_1, \dots, p_r)$ and $\mathbf{p}(\mathbf{m}') = (p'_1, \dots, p'_r)$, respectively. Then $\mathcal{Y}_{\mathbf{m}'} \subseteq \mathcal{Y}_{\mathbf{m}}$ and $\mathcal{X}_{\mathbf{m}'} \subseteq \mathcal{X}_{\mathbf{m}}$ if $\mathbf{m}' \leq \mathbf{m}$. Assume that $\mathbf{m}' \leq \mathbf{m}$. Then $\mathcal{X}_{\mathbf{m}'}$ is a closed subset of $\mathcal{X}_{\mathbf{m}}$, and since $\mathcal{Y}_{\mathbf{m}} = \mathcal{Y} \cap \mathcal{X}_{\mathbf{m}}$, $\mathcal{Y}_{\mathbf{m}'}$ is a closed subset of $\mathcal{Y}_{\mathbf{m}}$. Note that the partial order \leq is generated by $\mathbf{m}' < \mathbf{m}$, where $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{m}' = (m'_1, \dots, m'_r)$ with $m'_{i-1} = m_{i-1} - 1$, $m'_i = m_i + 1$ for some i . Then one can check that

$$\mathcal{Y}_{\mathbf{m}}^0 = \mathcal{Y}_{\mathbf{m}} \setminus \bigcup_{\mathbf{m}' < \mathbf{m}} \mathcal{Y}_{\mathbf{m}'}. \tag{1.3.4}$$

Thus $\mathcal{Y}_{\mathbf{m}}^0$ is an open dense subset of $\mathcal{Y}_{\mathbf{m}}$, and we have a partition $\mathcal{Y}_{\mathbf{m}} = \prod_{\mathbf{m}' \leq \mathbf{m}} \mathcal{Y}_{\mathbf{m}'}$. It follows that $\mathcal{Y}_{\mathbf{m}'} \subseteq \mathcal{Y}_{\mathbf{m}}$ if and only if $\mathbf{m}' \leq \mathbf{m}$. Also we have a partition $\mathcal{Y} = \prod_{\mathbf{m} \in \mathcal{Q}_{n,r}} \mathcal{Y}_{\mathbf{m}}^0$. We have the following lemma (cf. [SS1, Lem. 3.3]).

Lemma 1.4. *Assume that $r \geq 2$.*

- (i) $\mathcal{Y}_{\mathbf{m}}$ is open dense in $\mathcal{X}_{\mathbf{m}}$, and $\tilde{\mathcal{Y}}_{\mathbf{m}}$ is open dense in $\tilde{\mathcal{X}}_{\mathbf{m}}$.
- (ii) $\dim \tilde{\mathcal{X}}_{\mathbf{m}} = \dim \tilde{\mathcal{Y}}_{\mathbf{m}} = 2n^2 + \sum_{i=1}^r (r-i)m_i$.
- (iii) $\dim \mathcal{X}_{\mathbf{m}} = \dim \mathcal{Y}_{\mathbf{m}} = 2n^2 + \sum_{i=1}^r (r-i)m_i - m_r$.
- (iv) $\mathcal{Y} = \prod_{\mathbf{m} \in \mathcal{Q}_{n,r}} \mathcal{Y}_{\mathbf{m}}^0$ gives a stratification of \mathcal{Y} by smooth strata $\mathcal{Y}_{\mathbf{m}}^0$, and the map $\psi : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ is semismall with respect to this stratification.

Proof. Since $B_{\text{reg}}^{\iota\theta} \times \prod_i M_{p_i}$ is open dense in $B^{\iota\theta} \times \prod_i M_{p_i}$, $\tilde{\mathcal{Y}}_{\mathbf{m}}$ is open dense in $\tilde{\mathcal{X}}_{\mathbf{m}}$. Since $\pi^{(\mathbf{m})}$ is a closed map and $(\pi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}}) = \tilde{\mathcal{Y}}_{\mathbf{m}}$, $\mathcal{Y}_{\mathbf{m}}$ is open dense in $\mathcal{X}_{\mathbf{m}}$. So (i) holds. (ii) follows from (1.2.1). By using the decomposition $\psi_{\mathbf{I}} = \eta_{\mathbf{I}} \circ \xi_{\mathbf{I}}$ for $\mathbf{I} = \mathbf{I}(\mathbf{m})$, we see that $\dim \tilde{\mathcal{Y}}_{\mathbf{m}} = \dim \mathcal{Y}_{\mathbf{m}} + m_r$. Hence (iii) holds. By (1.3.1) and (1.3.2), $\dim \psi^{-1}(x, \mathbf{v}) = m_r$ for $(x, \mathbf{v}) \in \mathcal{Y}_{\mathbf{m}}^0$. Since

$$\begin{aligned} \dim \mathcal{Y} - \dim \mathcal{Y}_{\mathbf{m}}^0 &= (2n^2 + (r-1)n) - \left(2n^2 + \sum_{i=1}^r (r-i)m_i - m_r \right) \\ &= \sum_{i=1}^r (r-1)m_i - \sum_{i=1}^r (r-i)m_i + m_r \\ &= \sum_{i=1}^r (i-1)m_i + m_r \\ &\geq 2m_r. \end{aligned}$$

Hence $\dim \psi^{-1}(x, \mathbf{v}) \leq (\dim \mathcal{Y} - \dim \mathcal{Y}_{\mathbf{m}}^0)/2$ for $(x, \mathbf{v}) \in \mathcal{Y}_{\mathbf{m}}^0$, and so ψ is semismall.

□

1.5. Let $\psi_{\mathbf{m}} : \tilde{\mathcal{Y}}_{\mathbf{m}}^+ \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ be the restriction of ψ on $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$. Then $\psi_{\mathbf{m}}$ is \mathcal{W} -equivariant with respect to the natural action of \mathcal{W} on $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$ and the trivial action on $\mathcal{Y}_{\mathbf{m}}^0$. We consider the diagram

$$T^{\iota\theta} \xleftarrow{\alpha_0} \tilde{\mathcal{Y}} \xrightarrow{\psi} \mathcal{Y}, \tag{1.5.1}$$

where α_0 is given by $\alpha_0(x, \mathbf{v}, gB^\theta) = p_T(g^{-1}xg)$ ($p_T : B^{\iota\theta} \rightarrow T^{\iota\theta}$ is the natural projection). For each $\mathbf{I} \in \mathcal{I}(\mathbf{m})$, we have a similar diagram as (1.5.1) by replacing $\tilde{\mathcal{Y}}, \mathcal{Y}, \psi, \alpha_0$ by $\tilde{\mathcal{Y}}_{\mathbf{I}}, \mathcal{Y}_{\mathbf{m}}^0, \psi_{\mathbf{I}}, \alpha_{\mathbf{I}}$, where $\alpha_{\mathbf{I}}$ is the restriction of α_0 on $\tilde{\mathcal{Y}}_{\mathbf{I}}$. Let \mathcal{E} be a tame local system on $T^{\iota\theta}$. By (1.3.2), we have

$$(\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{m}}^+} \simeq \bigoplus_{\mathbf{I} \in \mathcal{I}(\mathbf{m})} (\psi_{\mathbf{I}})_! \alpha_{\mathbf{I}}^* \mathcal{E}. \tag{1.5.2}$$

We define a map $\beta_{\mathbf{I}} : \widehat{\mathcal{Y}}_{\mathbf{I}} \rightarrow T_{\text{reg}}^{\iota\theta}$ as the map induced from the projection $H \times (T_{\text{reg}}^{\iota\theta} \times M_{\mathbf{I}}) \rightarrow T_{\text{reg}}^{\iota\theta}$. Then $\alpha_{\mathbf{I}} = \beta_{\mathbf{I}} \circ \xi_{\mathbf{I}}$. Let $\mathcal{E}_{\mathbf{I}} = \beta_{\mathbf{I}}^* \mathcal{E}$ be a local system on $\widehat{\mathcal{Y}}_{\mathbf{I}}$. We have $\xi_{\mathbf{I}}^* \mathcal{E}_{\mathbf{I}} = \alpha_{\mathbf{I}}^* \mathcal{E}$. Let $\mathcal{W}_{\mathcal{E}_{\mathbf{I}}}$ be the stabilizer of $\mathcal{E}_{\mathbf{I}}$ in $\mathcal{W}_{\mathbf{I}}$. In the case where $\mathbf{I} = \mathbf{I}(\mathbf{m})$, we put $\mathcal{W}_{\mathcal{E}_{\mathbf{I}}} = \mathcal{W}_{\mathbf{m}, \mathcal{E}}$. In the case where $\mathbf{m} = (n, 0, \dots, 0)$, we put $\mathcal{W}_{\mathbf{m}, \mathcal{E}} = \mathcal{W}_{\mathcal{E}}$, which is the stabilizer of \mathcal{E} in \mathcal{W} . $\mathcal{W}_{\mathcal{E}}$ acts on $(\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{m}}^+}$ as automorphisms of complexes, and permutes each direct summand $(\psi_{\mathbf{I}})_! \alpha_{\mathbf{I}}^* \mathcal{E}$ according to the permutation action of S_n on $\mathcal{I}(\mathbf{m})$. Since $\eta_{\mathbf{I}}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$, $(\eta_{\mathbf{I}})_! \mathcal{E}_{\mathbf{I}}$ is a semisimple local system. As in [SS1, 3.4] the endomorphism algebra $\text{End}((\eta_{\mathbf{I}})_! \mathcal{E}_{\mathbf{I}})$ is canonically isomorphic to the group algebra $\bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_{\mathbf{I}}}]$, and $(\eta_{\mathbf{I}})_! \mathcal{E}_{\mathbf{I}}$ is decomposed as

$$(\eta_{\mathbf{I}})_! \mathcal{E}_{\mathbf{I}} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathcal{E}_{\mathbf{I}}}^\wedge} \rho \otimes \mathcal{L}_\rho, \tag{1.5.3}$$

where $\mathcal{L}_\rho = \text{Hom}(\rho, (\eta_{\mathbf{I}})_! \mathcal{E}_{\mathbf{I}})$ is a simple local system on $\mathcal{Y}_{\mathbf{m}}^0$.

1.6. Since $\psi_{\mathbf{m}}$ is proper and $\tilde{\mathcal{Y}}_{\mathbf{I}}$ is closed in $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$, $\psi_{\mathbf{I}}$ is proper. Hence $\xi_{\mathbf{I}}$ is also proper. We note that

$$(1.6.1) \quad R^i(\xi_{\mathbf{I}})_! \bar{\mathbf{Q}}_l \text{ is a constant sheaf for each } i.$$

In fact, we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{Y}}_{\mathbf{I}} & \xrightarrow{\xi_{\mathbf{I}}} & \widehat{\mathcal{Y}}_{\mathbf{I}} \\ \downarrow & & \downarrow \\ H/B^\theta \cap Z_H(T^{\iota\theta}) & \xrightarrow{\xi'_{\mathbf{I}}} & H/Z_H(T^{\iota\theta})_{\mathbf{I}}, \end{array}$$

where vertical maps are natural projections (see 1.3), and the map $\xi'_{\mathbf{I}}$ is the map induced from the inclusion $B^\theta \cap Z_H(T^{\iota\theta}) \hookrightarrow Z_H(T^{\iota\theta})_{\mathbf{I}}$. Since this diagram is cartesian, (1.6.1) is equivalent to the statement that

$$(1.6.2) \quad R^i(\xi'_{\mathbf{I}})_! \bar{\mathbf{Q}}_l \text{ is a constant sheaf for each } i.$$

We show (1.6.2). Since ξ'_I is a locally trivial fibration, $R^i(\xi'_I)_! \widetilde{\mathbf{Q}}_I$ is a locally constant sheaf on $H/Z_H(T^{i\theta})_I$. Since ξ'_I is H -equivariant, $R^i(\xi'_I)_! \widetilde{\mathbf{Q}}_I$ is an H -equivariant local system on $H/Z_H(T^{i\theta})_I$, hence it is a constant sheaf, as $Z_H(T^{i\theta})_I$ is connected. Thus (1.6.2), and so (1.6.1) holds.

Since ξ_I is a $\mathbf{P}_1^{I_r}$ -bundle, we see that

$$(\xi_I)_! \alpha_I^* \mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{I_r}) \otimes \mathcal{E}_I, \tag{1.6.3}$$

where $H^\bullet(\mathbf{P}_1^{I_r})$ denotes $\bigoplus_{i \geq 0} H^{2i}(\mathbf{P}_1^{I_r}, \widetilde{\mathbf{Q}}_I)$, which we regard as a complex of vector spaces (K_i) with $K_{\text{odd}} = 0$. It follows that

$$(\psi_I)_! \alpha_I^* \widetilde{\mathcal{E}} \simeq (\eta_I)_! (\xi_I)_! \alpha_I^* \mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{I_r}) \otimes (\eta_I)_! \mathcal{E}_I. \tag{1.6.4}$$

Let $W_{n,r} = S_n \times (\mathbf{Z}/r\mathbf{Z})^n$ be the complex reflection group $G(r, 1, n)$. We put $\widetilde{\mathcal{W}} = \mathcal{W} \times (\mathbf{Z}/r\mathbf{Z})^n$. We define a subgroup $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$ (resp. $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$) of $\widetilde{\mathcal{W}}$ by $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}} = \mathcal{W}_{\mathbf{m}, \mathcal{E}} \times (\mathbf{Z}/r\mathbf{Z})^n$ (resp. $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}} = \mathcal{W}_{\mathbf{m}, \mathcal{E}} \times (\mathbf{Z}/r\mathbf{Z})^n$). Let ζ be a primitive r -th root of unity in $\widetilde{\mathbf{Q}}_I$, and define a linear character $\tau_i : \mathbf{Z}/r\mathbf{Z} \rightarrow \widetilde{\mathbf{Q}}_I^*$ by $\tau_i(a) = \zeta^{i-1}$ for $i = 1, \dots, r$, where a is a generator of $\mathbf{Z}/r\mathbf{Z}$. Let ρ be an irreducible representation of $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$. Since $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$ is decomposed as $\mathcal{W}_{\mathbf{m}, \mathcal{E}} = \mathcal{W}_1 \times \dots \times \mathcal{W}_r$ with subgroups $\mathcal{W}_i \subset S_{m_i}$, ρ can be written as $\rho = \rho_1 \boxtimes \dots \boxtimes \rho_r$ with $\rho_i \in \mathcal{W}_i^\wedge$. Here $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$ is decomposed as $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}} = \widetilde{\mathcal{W}}_1 \times \dots \times \widetilde{\mathcal{W}}_r$ with $\widetilde{\mathcal{W}}_i = \mathcal{W}_i \times (\mathbf{Z}/r\mathbf{Z})^{m_i}$. We define an irreducible $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$ -module $\widetilde{\rho}$ (resp. $\widetilde{\rho}'$) by defining the action of $(\mathbf{Z}/r\mathbf{Z})^{m_i}$ on ρ_i via $\tau_i^{\otimes m_i}$ for $i = 1, \dots, r$ (resp. via $\tau_i^{\otimes m_i}$ for $i = 1, \dots, r-1$, and via the trivial action for $i = r$). Put $\widetilde{V}_\rho = \text{Ind}_{\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}}^{\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}} \widetilde{\rho}$. Then \widetilde{V}_ρ is an irreducible $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$ -module.

We regard $H^\bullet(\mathbf{P}_1^{m_r}) \simeq H^\bullet(\mathbf{P}_1)^{\otimes m_r}$ as a complex of $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$ -modules by the permutation of factors $H^\bullet(\mathbf{P}_1) = H^2(\mathbf{P}_1) \oplus H^0(\mathbf{P}_1) \simeq \widetilde{\mathbf{Q}}_I \oplus \widetilde{\mathbf{Q}}_I$. This makes $H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho$ a complex of $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$ -modules. In view of (1.5.2), (1.5.3) and (1.6.4), one can write

$$(\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}|_{\widetilde{\mathcal{Y}}_{\mathbf{m}}^+} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge} \text{Ind}_{\mathcal{W}_{\mathbf{m}, \mathcal{E}}}^{\mathcal{W}_{\mathbf{m}, \mathcal{E}}} (H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho) \otimes \mathcal{L}_\rho. \tag{1.6.5}$$

We define an action of $\mathbf{Z}/r\mathbf{Z}$ on $H^\bullet(\mathbf{P}_1) = H^2(\mathbf{P}_1) \oplus H^0(\mathbf{P}_1)$ by $\tau_r \oplus \tau_1$, and define an action of $(\mathbf{Z}/r\mathbf{Z})^{m_r}$ on $H^\bullet(\mathbf{P}_1^{m_r}) \simeq H^\bullet(\mathbf{P}_1)^{\otimes m_r}$ by $(\tau_r \oplus \tau_1) \boxtimes \dots \boxtimes (\tau_r \oplus \tau_1)$ (m_r -factors). Thus we can consider an extension $H^\bullet(\mathbf{P}_1^{m_r}) \otimes \widetilde{\rho}'$ of $H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho$, as a complex of $\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}$ -modules. It follows from (1.6.5) that

$$(\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}|_{\widetilde{\mathcal{Y}}_{\mathbf{m}}^+} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge} \text{Ind}_{\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}}^{\widetilde{\mathcal{W}}_{\mathbf{m}, \mathcal{E}}} (H^\bullet(\mathbf{P}_1^{m_r}) \otimes \widetilde{\rho}') \otimes \mathcal{L}_\rho. \tag{1.6.6}$$

Note that by our construction, (1.6.6) can be rewritten as

$$(\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}|_{\widetilde{\mathcal{Y}}_{\mathbf{m}}^+} \simeq \left(\bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \mathcal{L}_\rho \right) [-2m_r] + \mathcal{N}_{\mathbf{m}}, \tag{1.6.7}$$

where $\mathcal{N}_{\mathbf{m}}$ is a sum of various $\mathcal{L}_\rho[-2i]$ for $\rho \in \mathcal{W}_{\mathbf{m},\mathcal{E}}^\wedge$ with $0 \leq i < m_r$.

For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, let $\bar{\psi}_{\mathbf{m}}$ be the restriction of ψ on $\psi^{-1}(\mathcal{Y}_{\mathbf{m}})$. In what follows, we denote $\alpha_0^* \mathcal{E}|_{\psi^{-1}(\mathcal{Y}_{\mathbf{m}})}$ by $\alpha_0^* \mathcal{E}$ for short. Put $d_{\mathbf{m}} = \dim \mathcal{Y}_{\mathbf{m}}$. For $1 \leq j < r - 1$, $0 \leq k \leq m_j$, we define a subset $\mathcal{Q}(\mathbf{m}; j, k)$ of $\mathcal{Q}_{n,r}$ by

$$\mathcal{Q}(\mathbf{m}; j, k) = \{\mathbf{m}' \in \mathcal{Q}_{n,r} \mid \mathbf{m}' \leq \mathbf{m}, p_i = p'_i \ (1 \leq i \leq j - 1), p'_j = p_{j-1} + k\},$$

where $\mathbf{p}(\mathbf{m}) = (p_1, \dots, p_r)$ and $\mathbf{p}(\mathbf{m}') = (p'_1, \dots, p'_r)$. We also define $\mathbf{m}(j, k) \in \mathcal{Q}_{n,r}$ by $\mathbf{m}(j, k) = (m'_1, \dots, m'_r)$, where $m'_j = k, m'_{j+1} = m_{j+1} + (m_j - k)$ and $m'_i = m_i$ for $i \neq j, j + 1$. In the case where $j = r - 1$, we write $\mathbf{m}(j, k)$ simply as $\mathbf{m}(k)$. Hence if $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, $\mathbf{m}(k) = (m_1, \dots, m_{r-2}, k, k')$ with $k + k' = m_{r-1}$. We have the following proposition.

Proposition 1.7. *For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, $(\bar{\psi}_{\mathbf{m}})_! \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{Y}_{\mathbf{m}}$, equipped with $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action, and is decomposed as*

$$(\bar{\psi}_{\mathbf{m}})_! \alpha_0^* \mathcal{E}[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k),\mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \text{IC}(\mathcal{Y}_{\mathbf{m}(k)}, \mathcal{L}_\rho)[d_{\mathbf{m}(k)}].$$

Proof. $\bar{\psi}_{\mathbf{m}}$ is proper, and a similar argument as in the proof of Lemma 1.4 (iv) shows that $\bar{\psi}_{\mathbf{m}}$ is semismall with respect to the stratification $\mathcal{Y}_{\mathbf{m}} = \coprod_{\mathbf{m}' \leq \mathbf{m}} \mathcal{Y}_{\mathbf{m}'}$ (note that $m_r = 0$). It follows that $(\bar{\psi}_{\mathbf{m}})_! \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{Y}_{\mathbf{m}}$.

For a given $\mathbf{m} \in \mathcal{Q}_{n,r}$ (not necessarily in $\mathcal{Q}_{n,r}^0$) we define, for each integer $1 \leq j \leq r - 1$, and $0 \leq k \leq m_j$,

$$\begin{aligned} M^{(j,k)} &= \prod_{i=1}^{j-1} (M_{[p_{i-1}+1, p_i]} + M_{p_{i-1}}) \times (M_{[p_{j-1}+1, p_{j-1}+k]} + M_{p_{j-1}}) \times \prod_{i=j+1}^{r-1} M_{p_i}, \\ \mathcal{Y}_{j,k}^0 &= \bigcup_{g \in H} g(T_{\text{reg}}^{\iota\theta} \times M^{(j,k)}), \\ \widetilde{\mathcal{Y}}_{j,k}^+ &= \psi^{-1}(\mathcal{Y}_{j,k}^0), \end{aligned}$$

and let $\psi_{j,k} : \widetilde{\mathcal{Y}}_{j,k}^+ \rightarrow \mathcal{Y}_{j,k}^0$ be the restriction of ψ on $\widetilde{\mathcal{Y}}_{j,k}^+$. (As a convention, we also consider the case where $j = 0, k = 0$, in which case $M^{(0,0)} = \prod_{i=1}^{r-1} M_{p_i}$.) Then $\psi_{j,k}$ is a proper map. $\mathcal{Y}_{j,k}^0$ coincides with $\mathcal{Y}_{\mathbf{m}}^0$ in the case where $j = r - 1, k = m_{r-1}$, and coincides with $\mathcal{Y}_{\mathbf{m}}$ in the case where $j = 0$ and $k = 0$. We also consider the varieties

$$\begin{aligned} \overline{M}^{(j,k)} &= \prod_{i=1}^{j-1} (M_{[p_{i-1}+1, p_i]} + M_{p_{i-1}}) \times M_{p_{j-1}+k} \times \prod_{i=j+1}^{r-1} M_{p_i}, \\ \mathcal{Y}_{j,k} &= \bigcup_{g \in H} g(T_{\text{reg}}^{\iota\theta} \times \overline{M}^{(j,k)}), \\ \widetilde{\mathcal{Y}}_{j,k}^+ &= \psi^{-1}(\mathcal{Y}_{j,k}), \end{aligned}$$

and let $\overline{\psi}_{j,k} : \overline{\mathcal{Y}}_{j,k}^+ \rightarrow \mathcal{Y}_{j,k}$ be the restriction of ψ on $\overline{\mathcal{Y}}_{j,k}^+$. Then $\mathcal{Y}_{j,k}^0$ is open dense in $\mathcal{Y}_{j,k}$. We have

(1.7.1) $\mathcal{Y}_{j,k} \setminus \mathcal{Y}_{j,k-1} = \mathcal{Y}_{j,k}^0$ if $k \geq 1$, and $\mathcal{Y}_{j,0}(\mathbf{m}) = \mathcal{Y}_{j+1, m_j + m_{j+1}}(\mathbf{m}(j, 0))$. Moreover, $\mathcal{Y}_{j,k}^0(\mathbf{m})$ coincides with $\mathcal{Y}_{j+1, m_j + m_{j+1} - k}(\mathbf{m}(j, k))$. (Here we use the notation $\mathcal{Y}_{j,k}(\mathbf{m})$, etc., to indicate the dependence on \mathbf{m} .)

For $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$, $\mathcal{Y}_{\mathbf{m}'}^0$ is contained in $\mathcal{Y}_{j,k}$. Hence one can define an intersection cohomology $\mathrm{IC}(\mathcal{Y}_{\mathbf{m}'}, \mathcal{L}_\rho)$ associated to the local system \mathcal{L}_ρ on $\mathcal{Y}_{\mathbf{m}'}$ (here $\mathcal{Y}_{\mathbf{m}'}$ denotes the closure of $\mathcal{Y}_{\mathbf{m}'}^0$ in $\mathcal{Y}_{j,k}$). Returning to the setting in the proposition, we consider $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. We show the following formulas. First assume that $j = r - 1$ and $0 \leq k \leq m_{r-1}$. Then we have

$$\begin{aligned} & (\overline{\psi}_{r-1,k})! \alpha_0^* \mathcal{E} \\ & \simeq \bigoplus_{0 \leq k' \leq k} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \mathrm{IC}(\mathcal{Y}_{r-1, k'}, \mathcal{L}_\rho)[-2(m_{r-1} - k')] + \mathcal{N}_{r-1, k}, \end{aligned} \quad (1.7.2)$$

where $\mathcal{N}_{r-1, k}$ is a sum of various $\mathrm{IC}(\mathcal{Y}_{r-1, k'}, \mathcal{L}_\rho)[-2i]$ for $0 \leq k' \leq k$ and $\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge$ with $0 \leq i < m_{r-1} - k'$. Next assume that $0 \leq j < r - 1$ and that $0 \leq k \leq m_j$. Then we have

$$\begin{aligned} & (\overline{\psi}_{j,k})! \alpha_0^* \mathcal{E} \\ & \simeq \bigoplus_{0 \leq k' \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \mathrm{IC}(\mathcal{Y}_{\mathbf{m}(k')}, \mathcal{L}_\rho)[-2(m_{r-1} - k')] + \mathcal{N}_{j,k}, \end{aligned} \quad (1.7.3)$$

where $\mathcal{N}_{j,k}$ is a sum of various $\mathrm{IC}(\mathcal{Y}_{\mathbf{m}'}, \mathcal{L}_\rho)[-2i]$ for $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$ and $\rho \in \mathcal{W}_{\mathbf{m}', \mathcal{E}}^\wedge$ with i such that $0 \leq 2i < d_{\mathbf{m}} - d_{\mathbf{m}'}$.

Note that (1.7.3) will imply the proposition. In fact, in the case where $j = 0, k = 0$, $\overline{\psi}_{j,k}$ coincides with $\overline{\psi}_{\mathbf{m}}$, and $\mathcal{Y}_{\mathbf{m}(k')}$ coincides with $\mathcal{Y}_{\mathbf{m}(k')}$. Take $\mathrm{IC}(\mathcal{Y}_{\mathbf{m}'}, \mathcal{L}_\rho)[-2i] \in \mathcal{N}_{0,0}$. Since $d_{\mathbf{m}} - d_{\mathbf{m}'} > 2i$, $\mathrm{IC}(\mathcal{Y}_{\mathbf{m}'}, \mathcal{L}_\rho)[d_{\mathbf{m}} - 2i]$ is not a perverse sheaf. Since $(\overline{\psi}_{\mathbf{m}})_* \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf, we conclude that $\mathcal{N}_{0,0} = 0$. By Lemma 1.4, (iii), we have $d_{\mathbf{m}} - d_{\mathbf{m}(k')} = 2(m_{r-1} - k')$. Thus the proposition follows from (1.7.3).

First we show (1.7.2) by induction on k . Put $j = r - 1$. In the case where $k = 0$, $\mathcal{Y}_{j,0}$ coincides with $\mathcal{Y}_{\mathbf{m}(j,0)}^0$. Thus (1.7.2) follows from (1.6.7). We assume that (1.7.2) holds for any $k' < k$. By (1.7.1), $\mathcal{Y}_{j,k} \setminus \mathcal{Y}_{j,k-1} = \mathcal{Y}_{j,k}^0 = \mathcal{Y}_{\mathbf{m}(k)}^0$, and $\mathcal{Y}_{\mathbf{m}(k)}^0$ is an open dense subset of $\mathcal{Y}_{j,k}$. Since $\overline{\psi}_{j,k}$ is proper, $(\overline{\psi}_{j,k})! \alpha_0^* \mathcal{E}$ is a semisimple complex on $\mathcal{Y}_{j,k}$. Here we note that $(\overline{\psi}_{j,k})! \alpha_0^* \mathcal{E}$ has a natural structure of $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -complex. In fact, $(\psi_{\mathbf{m}(k)})! \alpha_0^* \mathcal{E}$ has a $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action by (1.6.6). It induces a $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action on $(\iota_0 \circ \psi_{\mathbf{m}(k)})! \alpha_0^* \mathcal{E}$, where ι_0 is an open immersion $\mathcal{Y}_{\mathbf{m}(k)}^0 \hookrightarrow \mathcal{Y}_{j,k}$, and hence on its perverse cohomology ${}^p H^i((\iota_0 \circ \psi_{\mathbf{m}(k)})! \alpha_0^* \mathcal{E})$. On the other hand, by induction, $(\overline{\psi}_{j, k-1})! \alpha_0^* \mathcal{E}$ has a natural $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action, which induces a $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action on ${}^p H^i((\overline{\psi}_{j, k-1})! \alpha_0^* \mathcal{E})$. Thus, by using the perverse cohomology exact sequence, one

can define an action of $\widetilde{\mathcal{W}}_{\mathcal{E}}$ on ${}^p H^i((\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E})$. Since $(\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E}$ is a semisimple complex, in this way the action of $\widetilde{\mathcal{W}}_{\mathcal{E}}$ on $(\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E}$ can be defined.

Now, since $(\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E}$ is a semisimple complex, it is a direct sum of the form $A[s]$ for a simple perverse sheaf A . Suppose that $\text{supp } A$ is not contained in $\mathcal{Y}_{j,k-1}$. Then $\text{supp } A \cap \mathcal{Y}_{\mathbf{m}(k)}^0 \neq \emptyset$ and the restriction of A on $\mathcal{Y}_{\mathbf{m}(k)}^0$ is a simple perverse sheaf on $\mathcal{Y}_{\mathbf{m}(k)}^0$. The restriction of $(\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E}$ on $\mathcal{Y}_{\mathbf{m}(k)}^0$ is isomorphic to $(\psi_{\mathbf{m}(k)})_! \alpha_0^* \mathcal{E}$. Hence its decomposition is given by the formula in (1.6.6). It follows that $A|_{\mathcal{Y}_{\mathbf{m}(k)}^0} = \mathcal{L}_{\rho}[d_{j,k}]$ for some ρ (here $d_{j,k} = \dim \mathcal{Y}_{\mathbf{m}(k)}$). This implies that $A = \text{IC}(\mathcal{Y}_{j,k}, \mathcal{L}_{\rho})[d_{j,k}]$ and that the direct sum of $A[s]$ appearing in $(\overline{\psi}_{j,k})_! \alpha_0^* \mathcal{E}$ such that $\text{supp } A \cap \mathcal{Y}_{\mathbf{m}(k)}^0 \neq \emptyset$ is given, in view of (1.6.7), by

$$K_1 = \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^{\wedge}} \widetilde{V}_{\rho} \otimes \text{IC}(\mathcal{Y}_{j,k}, \mathcal{L}_{\rho})[-2(m_{r-1} - k)] + \mathcal{N}'_{\mathbf{m}(k)}, \tag{1.7.4}$$

where $\mathcal{N}'_{\mathbf{m}(k)}$ is a sum of various $\text{IC}(\mathcal{Y}_{j,k}, \mathcal{L}_{\rho})[-2i]$ with $0 \leq i < m_{r-1} - k$.

If $\text{supp } A$ is contained in $\mathcal{Y}_{j,k-1}$, then $A[s]$ appears as a direct summand of $(\overline{\psi}_{j,k-1})_! \alpha_0^* \mathcal{E}$, which is decomposed as in (1.7.2) by the induction hypothesis. Thus if we remove the contribution from the restriction of K_1 , such $A[s]$ is determined from $(\overline{\psi}_{j,k-1})_! \alpha_0^* \mathcal{E}$. So, we consider the restriction of K_1 on $\mathcal{Y}_{j,k-1}$. The summands $\text{IC}(\mathcal{Y}_{j,k}, \mathcal{L}_{\rho})[-2i]$ in $\mathcal{N}'_{\mathbf{m}(k)}$ are already contained in $\mathcal{N}_{r-1,k}$ if $0 \leq i < m_{r-1} - k$. So it is enough to consider $A = \text{IC}(\mathcal{Y}_{j,k}, \mathcal{L}_{\rho})[-2(m_{r-1} - k)]$. Note that the multiplicity space of A in K_1 is \widetilde{V}_{ρ} . Hence the multiplicity space of a simple perverse sheaf A' appearing in the decomposition of $A|_{\mathcal{Y}_{j,k-1}}$, up to shift, has a structure of $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -module which is a sum of \widetilde{V}_{ρ} . But by (1.7.2) applied for $k - 1$, the multiplicity space of a simple perverse sheaf B appearing in the first term of $(\overline{\psi}_{j,k-1})_! \alpha_0^* \mathcal{E}$ is a sum of $\widetilde{V}_{\rho'}$ with $\rho' \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^{\wedge}$ for $k' < k$. Thus $A|_{\mathcal{Y}_{j,k-1}}$ gives no contribution on those first terms. This proves (1.7.2) for k . Hence (1.7.2) holds.

We now prove (1.7.3) by backwards induction on j and induction on k . So assume that $j < r - 1$. By (1.7.1), $\mathcal{Y}_{j,0}(\mathbf{m}) = \mathcal{Y}_{j+1, m_j + m_{j+1}}(\mathbf{m}(j, 0))$. Hence by induction on j , we may assume that (1.7.3) holds for $\mathcal{Y}_{j,0}$. Take $k \geq 1$, and assume that (1.7.3) holds for $k - 1$. We have $\mathcal{Y}_{j,k} \setminus \mathcal{Y}_{j,k-1} = \mathcal{Y}_{j,k}^0$, and $\mathcal{Y}_{j,k}^0(\mathbf{m}) = \mathcal{Y}_{j+1, m_j + m_{j+1} - k}(\mathbf{m}(j, k))$ by (1.7.1). Thus by induction on j , $(\psi_{j,k})_! \alpha_0^* \mathcal{E}$ can be described by the formula in (1.7.3). In particular, $(\psi_{j,k})_! \alpha_0^* \mathcal{E}$ is a semisimple complex consisting of $\text{IC}(\mathcal{Y}'_{\mathbf{m}'}, \mathcal{L}_{\rho})$, up to shift, for various $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$. Similarly, by induction on k , $(\overline{\psi}_{j,k-1})_! \alpha_0^* \mathcal{E}$ is described by (1.7.3), and it is a semisimple complex consisting of $\text{IC}(\mathcal{Y}'_{\mathbf{m}'}, \mathcal{L}_{\rho})$, up to shift, for various $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k - 1)$. Let K_1 be a semisimple complex on $\mathcal{Y}_{j,k}$ obtained from $(\psi_{j,k})_! \alpha_0^* \mathcal{E}$ as in (1.7.4). It is described by the formula (1.7.3) by replacing \mathbf{m} by $\mathbf{m}'' = \mathbf{m}(j, k)$. Here we note the following:

(1.7.5) Assume that A is a direct summand of K_1 . Then A is contained in $\mathcal{N}_{j,k}$ unless $A = \text{IC}(\mathcal{Y}_{\mathbf{m}(k')}, \mathcal{L}_{\rho})[-2(m_{r-1} - k')]$.

Assume that A is a direct summand in the former part of K_1 . Then $A = \text{IC}(\mathcal{Y}_{\mathbf{m}''(k')}, \mathcal{L}_{\rho})[-2(m''_{r-1} - k')]$ for $0 \leq k' \leq m''_{r-1}$, where we write $\mathbf{m}'' =$

(m''_1, \dots, m''_r) . Thus $\mathbf{p}(\mathbf{m}''(k')) = (p'_1, \dots, p'_r)$, where $p_i = p'_i$ for $i \neq j, r - 1$, and $p'_j \leq p_j, p'_{r-1} \leq p_{r-1}$. Then by Lemma 1.4 (iii), we have

$$\begin{aligned} d_{\mathbf{m}} - d_{\mathbf{m}''(k')} &= (p_j - p'_j) + (p_{r-1} - p'_{r-1}) + m'_r \\ &\geq (p_j - p'_j) + 2m'_r, \end{aligned}$$

for $\mathbf{m}''(k') = (m'_1, \dots, m'_r)$. (Note that $m''_{r-1} - k' = m'_r$.) Hence $d_{\mathbf{m}} - d_{\mathbf{m}''(k')} = 2(m''_{r-1} - k')$ if and only if $p_j = p'_j$, i.e., $\mathbf{m}''(k') = \mathbf{m}(k')$. In that case $m''_{r-1} = m_{r-1}$. In all other cases, A is contained in $\mathcal{N}_{j,k}$. Next assume that A is a direct summand of the latter part of K_1 . Thus A is written as $A = \text{IC}(\mathcal{Y}'_{\mathbf{m}'}, \mathcal{L}_\rho)[-2i]$ for $\mathbf{m}' \leq \mathbf{m}''$ with $d_{\mathbf{m}''} - d_{\mathbf{m}'} > 2i$. But since $\mathbf{m}'' = \mathbf{m}(j, k)$, we have $\mathbf{m}'' \leq \mathbf{m}$ and $d_{\mathbf{m}} \geq d_{\mathbf{m}''}$. Hence $\mathbf{m}' \leq \mathbf{m}$ and $d_{\mathbf{m}} - d_{\mathbf{m}'} \geq d_{\mathbf{m}''} - d_{\mathbf{m}'} > 2i$. This implies that A is contained in $\mathcal{N}_{j,k}$. (1.7.5) is proved.

Now (1.7.5) shows that the former part of K_1 coincides with the former part of $(\overline{\psi}_{j,k-1})_! \alpha_0^* \mathcal{E}$. Hence by a similar argument as in the proof of (1.7.2), we obtain (1.7.3) for (j, k) . This proves (1.7.3), and so the proposition follows. \square

Remark 1.8. In Proposition 1.7, the condition $\mathbf{m} \in \mathcal{Q}_{n,r}^0$ is crucial. Since $\dim \psi^{-1}(z) = m_r$ for $z \in \mathcal{Y}_{\mathbf{m}}^0$, the map $\overline{\psi}_{\mathbf{m}}$ is not semismall if $m_r \neq 0$.

2. Intersection cohomology on $G^{\iota\theta} \times V^{r-1}$ (exotic case)

2.1. In this section we assume that \mathcal{X}, \mathcal{Y} are of exotic type. We keep the notation in Section 1. For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, we consider the complex $(\overline{\psi}_{\mathbf{m}})_! \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ as in Proposition 1.7. Under the notation there, $\mathcal{Y}_{\mathbf{m}(k)}^0$ is an open dense subset of $\mathcal{X}_{\mathbf{m}(k)}$. Hence one can consider the complex

$$K_{\mathbf{m},T,\mathcal{E}} = \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k),\mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \text{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_\rho)[d_{\mathbf{m}(k)}]. \tag{2.1.1}$$

We consider the diagram

$$T^{\iota\theta} \xleftarrow{\alpha} \widetilde{\mathcal{X}} \xrightarrow{\pi} \mathcal{X},$$

where $\alpha : \widetilde{\mathcal{X}} \rightarrow T^{\iota\theta}$ is defined by $\alpha(x, \mathbf{v}, gB^\theta) = p_T(g^{-1}xg)$. Let $\overline{\pi}_{\mathbf{m}} : \pi^{-1}(\mathcal{X}_{\mathbf{m}}) \rightarrow \mathcal{X}_{\mathbf{m}}$ be the restriction of π on $\pi^{-1}(\mathcal{X}_{\mathbf{m}})$. We consider the complex $(\overline{\pi}_{\mathbf{m}})_! \alpha^* \mathcal{E}[d_{\mathbf{m}}]$, where $\alpha^* \mathcal{E}$ is regarded as a local system on $\pi^{-1}(\mathcal{X}_{\mathbf{m}})$ by restriction. The following result is a generalization of [SS1, Thm. 4.2].

Theorem 2.2. *For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, $(\overline{\pi}_{\mathbf{m}})_! \alpha^* \mathcal{E}[d_{\mathbf{m}}] \simeq K_{\mathbf{m},T,\mathcal{E}}$ as perverse sheaves on $\mathcal{X}_{\mathbf{m}}$.*

2.3. The remainder of this section is devoted to the proof of the theorem. As in the case of $\mathcal{Y}_{\mathbf{m}}^0$, consider $\mathcal{X}_{\mathbf{m}}^0 = \mathcal{X}_{\mathbf{m}} \setminus \bigcup_{\mathbf{m}' < \mathbf{m}} \mathcal{X}_{\mathbf{m}'}$ for each $\mathbf{m} \in \mathcal{Q}_{n,r}$. We shall describe the set $\mathcal{X}_{\mathbf{m}}^0$ explicitly. Put $\mathcal{X}_{\text{uni}} = \mathcal{X} \cap (G_{\text{uni}}^{\iota\theta} \times V^{r-1})$ and we define $\mathcal{X}_{\mathbf{m},\text{uni}}, \mathcal{X}_{\mathbf{m},\text{uni}}^0$, etc., as the intersection of $\mathcal{X}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}^0$, etc., with \mathcal{X}_{uni} . For

$(x, v) \in G_{\mathbf{uni}}^{u\theta} \times V$, we denote by $\mathbf{k}[x]v$ the subspace of V spanned by v, xv, x^2v, \dots . First we note the following.

(2.3.1) For each $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{uni}}$ with $\mathbf{v} = (v_1, \dots, v_{r-1})$, define a sequence $W_1 \subset W_2 \subset \dots \subset W_{r-1}$ of subspaces of V by $W_i = \mathbf{k}[x]v_1 + \dots + \mathbf{k}[x]v_i$ for $i = 1, \dots, r-1$. Then $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}, \mathbf{uni}}^0$ if and only if $\dim W_i = p_i$ for each i .

In fact, if $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}, \mathbf{uni}}$, there exists an x -stable isotropic flag $(V_i)_{1 \leq i \leq n}$ such that $v_i \in V_{p_i}$. Hence we have $\dim W_i \leq p_i$ for each i . This implies that (x, \mathbf{v}) satisfying the condition on (W_i) is contained in $\mathcal{X}_{\mathbf{m}, \mathbf{uni}}^0$. Conversely, assume that $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}, \mathbf{uni}}^0$. Take an x -stable isotropic flag (V_i) such that $v_i \in V_{p_i}$. Suppose there exists k such that $\dim W_i = p_i$ for $i < k$ and that $\dim W_k < p_k$. Then $W_i = V_{p_i}$ for $i = 1, \dots, k-1$, and $\mathbf{k}[x]v_k + V_{p_{k-1}}$ is an x -stable proper subspace of V_{p_k} . One can find an x -stable flag $V_{p_{k-1}} \subset V'_{p_{k-1}+1} \subset \dots \subset V'_{p_k-1} \subset V_{p_k}$, and $v_k \in V'_j \subsetneq V_{p_k}$ for some j . This implies that $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}'}$ for $\mathbf{m}' < \mathbf{m}$, a contradiction. Hence (2.3.1) holds.

More generally, we consider $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}$. Let $x = su = us$ be the Jordan decomposition of $x \in G^{u\theta}$, where $s \in G^{u\theta}$ is semisimple, $u \in G^{u\theta}$ is unipotent. We consider the decomposition $V = V_1 \oplus \dots \oplus V_t$ into eigenspaces of s . Then $Z_G(s) \simeq \mathrm{GL}_{2n_1} \times \dots \times \mathrm{GL}_{2n_t}$ with $\dim V_j = 2n_j$. Put $G_j = \mathrm{GL}_{2n_j}$ for each j . Then $Z_G(s)$ is θ -stable, and θ stabilizes each factor so that $Z_H(s) \simeq G_1^\theta \times \dots \times G_t^\theta$ with $G_j^\theta \simeq \mathrm{Sp}_{2n_j}$. Take $\mathbf{v} = (v_1, \dots, v_{r-1}) \in V^{r-1}$. For $j = 1, \dots, t$, we define $\mathbf{v}_j = (v_{1,j}, \dots, v_{r-1,j}) \in V_j^{r-1}$, where v_{ij} is the projection of $v_i \in V$ on V_j . Let u_j be the restriction of u on V_j . Then $(u_j, \mathbf{v}_j) \in (G_j)_{\mathbf{uni}}^{u\theta} \times V_j^{r-1}$. We denote by $(\mathcal{X}^{G_j})_{\mathbf{m}_j, \mathbf{uni}}^0$ the subvariety of $(G_j)_{\mathbf{uni}}^{u\theta} \times V_j^{r-1}$ defined in a similar way as $\mathcal{X}_{\mathbf{m}, \mathbf{uni}}^0$. The following property is checked easily.

(2.3.2) Assume that (x, \mathbf{v}) is contained in $\mathcal{X}_{\mathbf{m}}^0$. Then there exist unique $\mathbf{m}_1, \dots, \mathbf{m}_t$ such that $(u_j, \mathbf{v}_j) \in (\mathcal{X}^{G_j})_{\mathbf{m}_j, \mathbf{uni}}^0$, where $\mathbf{m}_j = (m_{1,j}, \dots, m_{r,j})$ with $\sum_{j=1}^t m_{ij} = m_i$ for $1 \leq i \leq r$. Conversely, if $(u_j, \mathbf{v}_j) \in (\mathcal{X}^{G_j})_{\mathbf{m}_j, \mathbf{uni}}^0$ for each j , then $(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}^0$ for \mathbf{m} determined from $\mathbf{m}_1, \dots, \mathbf{m}_t$.

For each $(x, \mathbf{v}) \in \mathcal{X}$, let (u_j, \mathbf{v}_j) be defined as above. For $j = 1, \dots, t$, we define a flag $W_{1,j} \subset \dots \subset W_{r-1,j}$ of V_j with respect to (u_j, \mathbf{v}_j) as in (2.3.1). Put $W_i = W_{i,1} \oplus \dots \oplus W_{i,t}$ for $i = 1, \dots, r-1$; then we obtain a sequence $W_1 \subset \dots \subset W_{r-1}$ of subspaces in V . We put $W_i(x, \mathbf{v}) = W_i$ for each i . Then (2.3.2) can be rewritten as

$$\mathcal{X}_{\mathbf{m}}^0 = \{(x, \mathbf{v}) \in \mathcal{X} \mid \dim W_i(x, \mathbf{v}) = p_i \ (1 \leq i \leq r-1)\}. \quad (2.3.3)$$

Recall the map $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$. For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define $\tilde{\mathcal{X}}_{\mathbf{m}}^+ = \pi^{-1}(\mathcal{X}_{\mathbf{m}}^0)$, and let $\pi_{\mathbf{m}} : \tilde{\mathcal{X}}_{\mathbf{m}}^+ \rightarrow \mathcal{X}_{\mathbf{m}}^0$ be the restriction of π on $\tilde{\mathcal{X}}_{\mathbf{m}}^+$. Since $\mathcal{Y}_{\mathbf{m}}^0$ is open in $\mathcal{X}_{\mathbf{m}}^0$, $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$ is an open subset of $\tilde{\mathcal{X}}_{\mathbf{m}}^+$. For $(x, \mathbf{v}, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}^+$, we shall associate $\mathbf{I} \in \mathcal{I}(\mathbf{m})$ as follows; assume that $(x, \mathbf{v}) \in B^{u\theta} \times M_n^{r-1}$, and that $x = su$ is the Jordan decomposition of x . Then M_n is s -stable, and is decomposed as $M_n = M_{n,1} \oplus \dots \oplus M_{n,t}$, where $M_{n,j} = M_n \cap V_j$ is a maximal isotropic subspace of V_j . Here $M_n = \langle e_1, \dots, e_n \rangle$. Since $s \in B^\theta$, $M_{n,j}$ determines a set $\{e_{k_1}, \dots, e_{k_{n_j}}\}$ with $k_1 < k_2 < \dots < k_{n_j}$, where $n_j = \dim M_{n,j}$ (if $s_1 \in T^{u\theta}$ is the projection of s , $\{e_{k_1}, \dots, e_{k_{n_j}}\}$ are eigenvectors

of s_1 on $M_{n,j}$). Let (u_j, \mathbf{v}_j) be as before. We define $\mathbf{I} = (I_1, \dots, I_r) \in \mathcal{I}(\mathbf{m})$ as follows; let $(W_i(u_j, \mathbf{v}_j))$ be as in (2.3.1) and put $p_{ij} = \dim W_i(u_j, \mathbf{v}_j)$. Then $W_i(u_j, \mathbf{v}_j)$ is a subspace of $M_{n,j}$. We define a subset $\tilde{I}_{i,j}$ of $\{k_1, \dots, k_{n_j}\}$ as the first $p_{i,j}$ numbers in $\{k_1, \dots, k_{n_j}\}$, and define $I_{i,j}$ by $I_{i,j} = \tilde{I}_{i,j} \setminus \tilde{I}_{i-1,j}$. Thus $|I_{i,j}| = m_{i,j}$ for $\mathbf{m}_j = (m_{1,j}, \dots, m_{r,j})$. Put $I_i = I_{i,1} \amalg \dots \amalg I_{i,t}$, and $\mathbf{I} = (I_1, \dots, I_r)$. Then $\mathbf{I} \in \mathcal{I}(\mathbf{m})$. Note that the attachment $(x, \mathbf{v}) \mapsto \mathbf{I}$ depends only on the B^θ -conjugacy class of (x, \mathbf{v}) . Thus we have a well-defined map $(x, \mathbf{v}, gB^\theta) \mapsto \mathbf{I}$. We define a subvariety $\tilde{\mathcal{X}}_{\mathbf{I}}$ of $\tilde{\mathcal{X}}_{\mathbf{m}}^+$ by

$$\tilde{\mathcal{X}}_{\mathbf{I}} = \{(x, \mathbf{v}, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}^+ \mid (x, \mathbf{v}, gB^\theta) \mapsto \mathbf{I}\}.$$

We show the following lemma (cf. [SS1, Lem. 4.4]).

Lemma 2.4. $\tilde{\mathcal{X}}_{\mathbf{m}}^+$ is decomposed as

$$\tilde{\mathcal{X}}_{\mathbf{m}}^+ = \coprod_{\mathbf{I} \in \mathcal{I}(\mathbf{m})} \tilde{\mathcal{X}}_{\mathbf{I}},$$

where $\tilde{\mathcal{X}}_{\mathbf{I}}$ is an irreducible component of $\tilde{\mathcal{X}}_{\mathbf{m}}^+$ for each \mathbf{I} .

Proof. It is clear from the definition that $\tilde{\mathcal{X}}_{\mathbf{m}}^+$ is a disjoint union of various $\tilde{\mathcal{X}}_{\mathbf{I}}$, and that $\tilde{\mathcal{X}}_{\mathbf{I}}$ contains $\tilde{\mathcal{Y}}_{\mathbf{I}}$ as an open dense subset. Since $\tilde{\mathcal{Y}}_{\mathbf{m}}^+ = \coprod_{\mathbf{I}} \tilde{\mathcal{Y}}_{\mathbf{I}}$, $\tilde{\mathcal{Y}}_{\mathbf{m}}^+$ is open dense in $\tilde{\mathcal{X}}_{\mathbf{m}}^+$. Hence $\tilde{\mathcal{X}}_{\mathbf{m}} = \bigcup_{\mathbf{I}} \overline{\tilde{\mathcal{Y}}_{\mathbf{I}}}$ gives a decomposition into irreducible components, where $\overline{\tilde{\mathcal{Y}}_{\mathbf{I}}}$ is the closure of $\tilde{\mathcal{Y}}_{\mathbf{I}}$ in $\tilde{\mathcal{X}}_{\mathbf{m}}^+$. Thus in order to show the lemma, it is enough to see that $\tilde{\mathcal{X}}_{\mathbf{I}}$ is closed in $\tilde{\mathcal{X}}_{\mathbf{m}}^+$ for each \mathbf{I} . But the closure $\mathcal{Z}_{\mathbf{I}}$ of $\tilde{\mathcal{X}}_{\mathbf{I}}$ in \mathcal{X} is contained in the set $\tilde{\mathcal{X}}_{\mathbf{I}} \cup \bigcup_{\mathbf{m}' < \mathbf{m}} \tilde{\mathcal{X}}_{\mathbf{m}'}$. Hence $\tilde{\mathcal{X}}_{\mathbf{I}} = \mathcal{Z}_{\mathbf{I}} \cap \tilde{\mathcal{X}}_{\mathbf{m}}^+$ is closed in $\tilde{\mathcal{X}}_{\mathbf{m}}^+$. \square

2.5. We fix $\mathbf{m} \in \mathcal{Q}_{n,r}$. Let us consider the spaces $V_0 = M_{m_1}$ and $\overline{V}_0 = V_0^\perp/V_0$. We put $G_1 = \mathrm{GL}(V_0)$ and $G_2 = \mathrm{GL}(\overline{V}_0)$. Then \overline{V}_0 has a natural symplectic structure, and G_2 is identified with a θ -stable subgroup of G . We consider the variety $\mathcal{X}' \subset G_2^\theta \times \overline{V}_0^{r-2}$ as in the case of $G^\theta \times V^{r-1}$. Put $\mathbf{m}' = (m_2, \dots, m_r)$. Thus $\mathbf{m}' \in \mathcal{Q}_{n',r-1}$, where $n' = \dim \overline{V}_0/2$. The subvariety $\mathcal{X}'_{\mathbf{m}'}$ of \mathcal{X}' with respect to \mathbf{m}' is defined similarly to $\mathcal{X}_{\mathbf{m}}$. Let G_1^0 be the set of regular elements in G_1 (namely, the set of $x \in G_1$ such that u is regular unipotent in $Z_{G_1}(s)$ for the Jordan decomposition $x = su$). For each $z = (x, \mathbf{v}) \in \mathcal{X}'_{\mathbf{m}'}$, put $W_z = W_1(x, \mathbf{v})$. Note that W_z is an x -stable subspace of V with $\dim W_z = m_1$, and that $x|_{W_z}$ is a regular element in $\mathrm{GL}(W_z)$. Moreover, W_z is the unique x -stable subspace of V containing v_1 with dimension m_1 .

We define a variety

$$\begin{aligned} \mathcal{K}_{\mathbf{m}} = \{ & (z, \phi_1, \phi_2) \mid z = (x, \mathbf{v}) \in \mathcal{X}'_{\mathbf{m}'} \\ & \phi_1 : W_z \xrightarrow{\sim} V_0, \phi_2 : W_z^\perp/W_z \xrightarrow{\sim} \overline{V}_0 \text{ (symplectic isom.)} \}, \end{aligned} \tag{2.5.1}$$

and morphisms

$$\begin{aligned} q : \mathcal{K}_{\mathbf{m}} & \rightarrow \mathcal{X}'_{\mathbf{m}'}, & (x, \mathbf{v}, \phi_1, \phi_2) & \mapsto (x, \mathbf{v}), \\ \sigma : \mathcal{K}_{\mathbf{m}} & \rightarrow G_1^0 \times \mathcal{X}'_{\mathbf{m}'}, & (x, \mathbf{v}, \phi_1, \phi_2) & \mapsto (\phi_1(x|_{W_z})\phi_1^{-1}, \phi_2(x|_{W_z^\perp/W_z})\phi_2^{-1}, \phi_2(\overline{\mathbf{v}})), \end{aligned}$$

where $\bar{\mathbf{v}} = (\bar{v}_2, \dots, \bar{v}_{r-1}) \in (W_z^\perp/W_z)^{r-2}$, and \bar{v}_i is the image of $v_i \in W_z^\perp$ to W_z^\perp/W_z . Put $H_0 = G_1 \times G_2^\theta$. Then $H \times H_0$ acts on $\mathcal{K}_{\mathbf{m}}$ by

$$(g, (h_1, h_2)) : (x, \mathbf{v}, \phi_1, \phi_2) \mapsto (gxg^{-1}, g\mathbf{v}, h_1\phi_1g^{-1}, h_2\phi_2g^{-1})$$

for $g \in H, (h_1, h_2) \in H_0$. Moreover, σ is $H \times H_0$ -equivariant with respect to the natural action of H_0 and the trivial action of H on $G_1^0 \times \mathcal{X}_{\mathbf{m}}^{\prime 0}$. We have

(2.5.2) The map q is a principal bundle with fibre isomorphic to H_0 .

(2.5.3) The map σ is a locally trivial fibration with smooth fibre of dimension $\dim H + (r - 2)m_1$.

In fact, (2.5.2) is clear. We show (2.5.3). For a fixed $z = (x', (x'', \mathbf{v}')) \in G_1^0 \times \mathcal{X}_{\mathbf{m}}^{\prime 0}$ with $\mathbf{v}' = (v'_2, \dots, v'_{r-1})$, the fibre $\sigma^{-1}(z)$ is determined by the following procedure.

- (i) Choose an isotropic subspace W_1 of V with $\dim W_1 = m_1$.
- (ii) For such W_1 , choose an isomorphism $\phi_1 : W_1 \rightarrow V_0$ and a symplectic isomorphism $\phi_2 : W_1^\perp/W_1 \rightarrow \bar{V}_0$.
- (iii) Choose $x \in G^{\iota\theta}$ such that x stabilizes W_1 and that $\phi_1(x|_{W_1})\phi_1^{-1} = x', \phi_2(x|_{W_1^\perp/W_1})\phi_2^{-1} = x''$.
- (iv) Choose $v_1 \in W_1$ and $v_i \in W_1^\perp$ such that $\mathbf{k}[x]v_1 = W_1$ and that $\phi_2(\bar{v}_i) = v'_i$ for $i = 2, \dots, r - 1$.

Let P be the stabilizer of the flag $(V_0 \subset V_0^\perp)$ in G . Then P is θ -stable, and is decomposed as $P = LU_P$, where L is a θ -stable Levi subgroup of P containing T and U_P is the unipotent radical of P . For (i), such W_1 are parametrized by H/P^θ . For (ii), they are parametrized by $G_1 \times G_2^\theta$. For (iii), x should be contained in $P^{\iota\theta}$, but x', x'' determines the part corresponding to $L^{\iota\theta}$. Hence the choice of x is parametrized by $U_P^{\iota\theta}$. Finally, v_1 form an open dense subset of W_1 , and v_2, \dots, v_{r-1} are determined uniquely by v'_2, \dots, v'_{r-1} modulo W_1 . One can check that the thus obtained (x, \mathbf{v}) is contained in $\mathcal{X}_{\mathbf{m}}^0$. It follows that each fibre $\sigma^{-1}(z)$ is smooth with dimension $\dim H + (r - 2)m_1$. Hence (2.5.3) holds.

Let B_1 be a Borel subgroup of G_1 which is the stabilizer of the flag $(M_k)_{0 \leq k \leq m_1}$ in G_1 , and B_2 a θ -stable Borel subgroup of G_2 which is the stabilizer of the flag $(M_{m_1+1}/M_{m_1} \subset \dots \subset M_n/M_{m_1})$ in G_2 . Put

$$\tilde{G}_1 = \{(x, gB_1) \in G_1 \times G_1/B_1 \mid g^{-1}xg \in B_1\},$$

and define the map $\pi^1 : \tilde{G}_1 \rightarrow G_1$ by $(x, gB_1) \mapsto x$. Put $\tilde{G}_1^0 = (\pi^1)^{-1}(G_1^0)$, and let $\varphi^1 : \tilde{G}_1^0 \rightarrow G_1^0$ be the restriction of π^1 . We define $\tilde{\mathcal{X}}'$ as the subvariety of $G_2^{\iota\theta} \times \bar{V}_0^{r-2} \times G_2^\theta/B_2^\theta$ as in the case of \mathcal{X}' , and let $\pi^2 : \tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$ be the projection $(x, \mathbf{v}', gB_2^\theta) \mapsto (x, \mathbf{v}')$. We put $\tilde{\mathcal{X}}_{\mathbf{m}'}^{\prime+} = (\pi^2)^{-1}(\mathcal{X}'^0_{\mathbf{m}'})$, and let $\pi_{\mathbf{m}'}^2$ be the restriction of π^2 on $\tilde{\mathcal{X}}_{\mathbf{m}'}^{\prime+}$. We define a variety

$$\begin{aligned} \tilde{\mathcal{X}}_{\mathbf{m}}^+ &= \{(x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mid (x, \mathbf{v}, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}^+, \\ &\quad \phi_1 : W_z \xrightarrow{\sim} V_0, \phi_2 : W_z^\perp/W_z \xrightarrow{\sim} \bar{V}_0 \text{ for } z = (x, \mathbf{v})\}, \end{aligned}$$

and define a map $\tilde{q} : \tilde{\mathcal{Z}}_{\mathbf{m}}^+ \rightarrow \tilde{\mathcal{X}}_{\mathbf{m}}^+$ by the natural projection. We define a map $\tilde{\sigma} : \tilde{\mathcal{Z}}_{\mathbf{m}}^+ \rightarrow \tilde{G}_1^0 \times \tilde{\mathcal{X}}_{\mathbf{m}'}^+$ as follows; take $(x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \in \tilde{\mathcal{Z}}_{\mathbf{m}}^+$. Since $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}^0$, W_z coincides with $g(M_{m_1})$. Let g_1B_1 be the element corresponding to the flag $\phi_1(g(M_i))_{0 \leq i \leq m_1}$, and $g_2B_2^\theta$ be the element corresponding to the isotropic flag $\phi_2(g(M_i)/g(M_{m_1}))_{i \geq m_1}$. Then

$$\tilde{\sigma} : (x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mapsto ((\phi_1(x|_{W_z})\phi_1^{-1}, g_1B_1), (\phi_2(x|_{W_z^\perp/W_z})\phi_2^{-1}, \phi_2(\bar{\mathbf{v}}), g_2B_2^\theta)).$$

We also define a map $\tilde{\pi}_{\mathbf{m}} : \tilde{\mathcal{Z}}_{\mathbf{m}}^+ \rightarrow \mathcal{K}_{\mathbf{m}}$ by $(x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mapsto (x, \mathbf{v}, \phi_1, \phi_2)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} T_1 \times T_2^{\iota\theta} & \xleftarrow{f} & T^{\iota\theta} & \xrightarrow{\text{id}} & T^{\iota\theta} \\ \alpha^1 \times \alpha' \uparrow & & \tilde{\alpha} \uparrow & & \alpha \uparrow \\ \tilde{G}_1^0 \times \tilde{\mathcal{X}}_{\mathbf{m}'}^+ & \xleftarrow{\tilde{\sigma}} & \tilde{\mathcal{Z}}_{\mathbf{m}}^+ & \xrightarrow{\tilde{q}} & \tilde{\mathcal{X}}_{\mathbf{m}}^+ \\ \varphi^1 \times \pi_{\mathbf{m}'} \downarrow & & \downarrow \tilde{\pi}_{\mathbf{m}} & & \downarrow \pi_{\mathbf{m}} \\ G_1^0 \times \mathcal{X}_{\mathbf{m}'}^0 & \xleftarrow{\sigma} & \mathcal{K}_{\mathbf{m}} & \xrightarrow{q} & \mathcal{X}_{\mathbf{m}}^0 \end{array}$$

where the map $\tilde{\alpha}$ is defined naturally. Note that $T^{\iota\theta}$ can be written as $T^{\iota\theta} \simeq T_1 \times T_2^{\iota\theta}$, where T_1 is a maximal torus of G_1 , and T_2 is a θ -stable maximal torus of G_2 . We fix an isomorphism $f : T^{\iota\theta} \rightarrow T_1 \times T_2^{\iota\theta}$. The map $\alpha^1 : \tilde{G}_1 \rightarrow T_1$ is defined as in 2.1, by ignoring v . The maps $\alpha', \pi_{\mathbf{m}'}$ are defined similarly to $\alpha, \pi_{\mathbf{m}}$.

2.6. Let \mathcal{E} be a tame local system on $T^{\iota\theta}$. Under the isomorphism $f : T^{\iota\theta} \rightarrow T_1 \times T_2^{\iota\theta}$, \mathcal{E} can be written as $\mathcal{E} \simeq \mathcal{E}_1 \boxtimes \mathcal{E}_2$, where \mathcal{E}_1 (resp. \mathcal{E}_2) is a tame local system on T_1 (resp. $T_2^{\iota\theta}$). Then we have $\mathcal{W}_{\mathbf{m}, \mathcal{E}} \simeq \mathcal{W}_1 \times \mathcal{W}'_{\mathbf{m}', \mathcal{E}_2}$, where \mathcal{W}_1 is the stabilizer of \mathcal{E}_1 in $S_{m_1} \simeq N_{G_1}(T_1)/T_1$, and $\mathcal{W}'_{\mathbf{m}', \mathcal{E}_2}$ is defined similarly to $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$ with respect to $\mathcal{W}' = N_{G_2^\theta}(T_2^{\iota\theta})/Z_{G_2^\theta}(T_2^{\iota\theta})$. As in 1.6, $\mathcal{W}_{\mathbf{m}, \mathcal{E}}$ is decomposed as $\mathcal{W}_{\mathbf{m}, \mathcal{E}} \simeq \mathcal{W}_1 \times \cdots \times \mathcal{W}_r$ with subgroups $\mathcal{W}_i \subset S_{m_i}$. Then $\mathcal{W}'_{\mathbf{m}', \mathcal{E}_2} \simeq \mathcal{W}_2 \times \cdots \times \mathcal{W}_r$. For each $\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge$, we construct a simple perverse sheaf A_ρ on $\mathcal{X}_{\mathbf{m}}^0$ as follows: The decomposition of the complex $\pi_1^1(\alpha^1)^*\mathcal{E}_1[\dim G_1]$ into simple summands is well known. Let $G_{1, \text{reg}}$ be the set of regular semisimple elements in G_1 . Since G_1^0 is an open dense subset of G_1 containing $G_{1, \text{reg}}$, the decomposition of $\varphi_1^1(\alpha^1)^*\mathcal{E}_1$ is described similarly. Namely, we have

$$\varphi_1^1(\alpha^1)^*\mathcal{E}_1 \simeq \bigoplus_{\rho_1 \in \mathcal{W}_1^\wedge} \rho_1 \otimes \text{IC}(G_1^0, \mathcal{L}_{\rho_1}), \tag{2.6.1}$$

where \mathcal{L}_{ρ_1} is a simple local system on $G_{1, \text{reg}}$. Write ρ as $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_r$ with $\rho_i \in \mathcal{W}_i^\wedge$. Then $\rho' = \rho_2 \boxtimes \cdots \boxtimes \rho_r \in \mathcal{W}'_{\mathbf{m}', \mathcal{E}_2}^\wedge$. Suppose that a simple perverse sheaf $A_{\rho'}$ on $\mathcal{X}_{\mathbf{m}'}^0$ was constructed. Put $A_1 = \text{IC}(G_1^0, \mathcal{L}_{\rho_1})[\dim G_1]$. Then $A_1 \boxtimes A_{\rho'}$ is an H_0 -equivariant simple perverse sheaf on $G_1^0 \times \mathcal{X}_{\mathbf{m}'}^0$, and so $\sigma^*(A_1 \boxtimes A_{\rho'})[\beta_1]$ is an H_0 -equivariant simple perverse sheaf on $\mathcal{K}_{\mathbf{m}}$ by (2.5.3), where $\beta_1 = \dim H + (r-2)m_1$.

Since q is a principal bundle with group H_0 by (2.5.2), there exists a unique simple perverse sheaf A_ρ on $\mathcal{X}_{\mathbf{m}}^0$ such that

$$q^* A_\rho[\beta_2] \simeq \sigma^*(A_1 \boxtimes A_{\rho'})[\beta_1],$$

where $\beta_2 = \dim H_0$. Note that in the case where $r = 2$, A_ρ coincides with the simple perverse sheaf A_ρ^0 constructed in [SS1, 4.6] (\mathcal{G}_m^0 in [loc. cit.] corresponds to our $\mathcal{X}_{\mathbf{m}}^0$ with $\mathbf{m} = (m, n - m)$).

Let \mathcal{L}_ρ be a simple local system on $\mathcal{Y}_{\mathbf{m}}^0$ as appeared in (1.5.3). Since $\mathcal{Y}_{\mathbf{m}}^0$ is an open dense smooth subset of $\mathcal{X}_{\mathbf{m}}^0$, one can consider the intersection cohomology $\mathrm{IC}(\mathcal{X}_{\mathbf{m}}^0, \mathcal{L}_\rho)$ on $\mathcal{X}_{\mathbf{m}}^0$. We have the following lemma.

Lemma 2.7. $A_\rho \simeq \mathrm{IC}(\mathcal{X}_{\mathbf{m}}^0, \mathcal{L}_\rho)[d_{\mathbf{m}}]$.

Proof. We prove the lemma by induction on r . The case where $r = 2$ comes from [SS1, Lemma 4.7]. In order to prove the lemma, it is enough to see that

$$\mathcal{H}^{-d_{\mathbf{m}}} A_\rho|_{\mathcal{Y}_{\mathbf{m}}^0} \simeq \mathcal{L}_\rho. \tag{2.7.1}$$

We consider the following commutative diagram:

$$\begin{array}{ccccc}
 T_1 \times T_2^{\iota\theta} & \xleftarrow{f} & T^{\iota\theta} & \xrightarrow{\mathrm{id}} & T^{\iota\theta} \\
 \alpha_0^1 \times \alpha_0' \uparrow & & \uparrow \tilde{\alpha}_0 & & \uparrow \alpha_0 \\
 \tilde{G}_{1,\mathrm{reg}} \times \tilde{\mathcal{Y}}_{\mathbf{m}'}^{\prime 0} & \xleftarrow{\tilde{\sigma}_0} & \tilde{Z}_{\mathbf{m}}^0 & \xrightarrow{\tilde{q}_0} & \tilde{\mathcal{Y}}_{\mathbf{m}}^0 \\
 \xi^1 \times \xi_0' \downarrow & & \downarrow \tilde{\xi}_0 & & \downarrow \xi_0 \\
 (G_1/T_1 \times T_{1,\mathrm{reg}}) \times \hat{\mathcal{Y}}_{\mathbf{m}'}^{\prime 0} & \xleftarrow{\hat{\sigma}_0} & \hat{Z}_{\mathbf{m}}^0 & \xrightarrow{\hat{q}_0} & \hat{\mathcal{Y}}_{\mathbf{m}}^0 \\
 \eta^1 \times \eta_0' \downarrow & & \downarrow \tilde{\eta}_0 & & \downarrow \eta_0 \\
 G_{1,\mathrm{reg}} \times \mathcal{Y}'_{\mathbf{m}'} & \xleftarrow{\sigma_0} & \mathcal{K}_{\mathbf{m},\mathrm{reg}} & \xrightarrow{q_0} & \mathcal{Y}_{\mathbf{m}}^0,
 \end{array} \tag{2.7.2}$$

where $\tilde{\mathcal{Y}}_{\mathbf{m}}^0 = \tilde{\mathcal{Y}}_{\mathbf{I}}$, $\hat{\mathcal{Y}}_{\mathbf{m}}^0 = \hat{\mathcal{Y}}_{\mathbf{I}}$ for $\mathbf{I} = \mathbf{I}(\mathbf{m})$ (see 1.3), and $\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\prime 0} = \tilde{\mathcal{Y}}_{\mathbf{I}'}$, $\hat{\mathcal{Y}}_{\mathbf{m}'}^{\prime 0} = \hat{\mathcal{Y}}_{\mathbf{I}'}$ are defined similarly with respect to $G_2^{\iota\theta} \times \overline{V}_0^{r-2}$ with $\mathbf{I}' = \mathbf{I}(\mathbf{m}')$. Moreover,

$$\begin{aligned}
 \tilde{G}_{1,\mathrm{reg}} &= (\pi^1)^{-1}(G_{1,\mathrm{reg}}), \\
 \mathcal{K}_{\mathbf{m},\mathrm{reg}} &= q^{-1}(\mathcal{Y}_{\mathbf{m}}^0), \\
 \tilde{Z}_{\mathbf{m}}^0 &= \tilde{q}^{-1}(\tilde{\mathcal{Y}}_{\mathbf{m}}^0),
 \end{aligned}$$

and $\hat{Z}_{\mathbf{m}}^0$ is defined as the quotient of $\tilde{Z}_{\mathbf{m}}^0$ under the natural action of the group $Z_H(T^{\iota\theta})_{\mathbf{I}}/(B^\theta \cap Z_H(T^{\iota\theta}))$. The maps $\tilde{q}_0, q_0, \tilde{\sigma}_0, \sigma_0$ are defined as the restriction of the corresponding maps $\tilde{q}, q, \tilde{\sigma}, \sigma$. The map ξ_0 is $\xi_{\mathbf{I}}$ for $\mathbf{I} = \mathbf{I}(\mathbf{m})$. The maps $\tilde{\xi}_0, \tilde{\eta}_0$ are defined according to ξ_0, η_0 . ξ_0', η_0' are defined similarly to ξ_0, η_0 with respect to

$G_2^{u\theta} \times \overline{V}_0^{r-2}$. ξ^1, η^1 are standard maps in the groups case (ξ^1 is an isomorphism). The map $\widehat{\sigma}_0$ is naturally induced from $\widetilde{\sigma}_0$.

It follows from the diagram (2.7.2) that

$$\widetilde{\sigma}_0^*((\alpha_0^1)^*\mathcal{E}_1 \boxtimes (\alpha_0^2)^*\mathcal{E}_2) \simeq \widetilde{q}_0^*\alpha_0^*\mathcal{E}. \tag{2.7.3}$$

It is easy to check that the squares in the middle row and in the bottom row are all cartesian squares. Here $\psi^1 = \eta^1 \circ \xi^1 : \widetilde{G}_{1,\text{reg}} \rightarrow G_{1,\text{reg}}$ is a finite Galois covering with group S_{m_1} , and $\psi_!^1(\alpha_0^1)^*\mathcal{E}_1$ is decomposed as

$$\psi_!^1(\alpha_0^1)^*\mathcal{E}_1 \simeq \bigoplus_{\rho_1 \in \mathcal{W}_1^\wedge} \rho_1 \otimes \mathcal{L}_{\rho_1}. \tag{2.7.4}$$

On the other hand, by (1.5.3) and (1.6.3), we have

$$(\eta'_0 \circ \xi'_0)_!(\alpha_0^2)^*\mathcal{E}_2 \simeq \bigoplus_{\rho' \in (\mathcal{W}'_{\mathbf{m}', \mathcal{E}_2})^\wedge} H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho' \otimes \mathcal{L}_{\rho'}. \tag{2.7.5}$$

Similarly, the map $\eta_0 \circ \xi_0$ coincides with $\psi_{\mathbf{I}} : \widetilde{\mathcal{Y}}_{\mathbf{I}} = \widetilde{\mathcal{Y}}_{\mathbf{m}}^0 \rightarrow \mathcal{Y}_{\mathbf{m}}^0$. Hence we have

$$(\eta_0 \circ \xi_0)_!\alpha_0^*\mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge} H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho \otimes \mathcal{L}_\rho. \tag{2.7.6}$$

Since the Galois covering is compatible with σ_0 and q_0 thanks to the diagram (2.7.2) (it corresponds to the squares in the bottom row), we have $\sigma_0^*(\mathcal{L}_{\rho_1} \boxtimes \mathcal{L}_{\rho'}) \simeq q_0^*\mathcal{L}_\rho$ under the identification $\rho = \rho_1 \boxtimes \rho'$ for $\mathcal{W}_{\mathbf{m}, \mathcal{E}} = \mathcal{W}_1 \times \mathcal{W}'_{\mathbf{m}', \mathcal{E}_2}$. This implies, by applying the induction hypothesis for $A_{\rho'}$, that $A_\rho|_{\mathcal{Y}_{\mathbf{m}}^0} \simeq \mathcal{L}_\rho[d_{\mathbf{m}}]$. Hence (2.7.1) holds and the lemma follows. \square

By using Lemma 2.7, we show the following:

Proposition 2.8. *Under the notation in Lemma 2.7, $(\pi_{\mathbf{m}})_!\alpha^*\mathcal{E}$ is decomposed as*

$$(\pi_{\mathbf{m}})_!\alpha^*\mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{m_r}) \otimes \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge} \widetilde{V}_\rho \otimes \text{IC}(\mathcal{X}_{\mathbf{m}}^0, \mathcal{L}_\rho),$$

where \widetilde{V}_ρ is regarded as a vector space, ignoring the $\widetilde{\mathcal{W}}_{\mathcal{E}}$ -action.

Proof. We prove the proposition by induction on r . In the case where $r = 2$, the proposition holds by Proposition 4.8 in [SS1]. We assume that the proposition holds for $r' < r$. We fix $\mathbf{I} = (I_1, \dots, I_r) \in \mathcal{I}(\mathbf{m})$, and put $\mathbf{I}' = (I_2, \dots, I_r) \in \mathcal{I}(\mathbf{m}')$. Put $\widetilde{\mathcal{Z}}_{\mathbf{I}} = \widetilde{q}^{-1}(\widetilde{\mathcal{X}}_{\mathbf{I}})$. We have the following commutative diagram

$$\begin{array}{ccccc} \widetilde{G}_1^0 \times \widetilde{\mathcal{X}}_{\mathbf{I}'} & \longleftarrow & \widetilde{\mathcal{Z}}_{\mathbf{I}} & \longrightarrow & \widetilde{\mathcal{X}}_{\mathbf{I}} \\ \varphi^1 \times \pi_{\mathbf{I}'} \downarrow & & \downarrow & & \downarrow \pi_{\mathbf{I}} \\ G_1^0 \times \mathcal{X}_{\mathbf{m}'}^0 & \xleftarrow{\sigma} & \mathcal{K}_{\mathbf{m}} & \xrightarrow{q} & \mathcal{X}_{\mathbf{m}}^0, \end{array} \tag{2.8.1}$$

where $\pi_{\mathbf{I}}$ is the restriction of $\pi_{\mathbf{m}}$ on $\tilde{\mathcal{X}}_{\mathbf{I}}$, and similarly for $\pi_{\mathbf{I}'}$. Other maps are determined correspondingly. We note that both squares are cartesian squares.

We show the following.

(2.8.2) Any simple summand (up to shift) of the semisimple complex $(\pi_{\mathbf{I}})! \alpha^* \mathcal{E}$ is contained in the set $\{A_\rho \mid \rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge\}$.

Put $K_1 = \varphi_{\mathbf{I}}^1(\alpha^1)^* \mathcal{E}_1$, and $K_2 = (\pi_{\mathbf{m}'})_!(\alpha')^* \mathcal{E}_2$. By (2.6.1) and the induction hypothesis, we have

$$\begin{aligned} K_1 &\simeq \bigoplus_{\rho_1 \in \mathcal{W}_1^\wedge} \rho_1 \otimes \mathrm{IC}(G_1^0, \mathcal{L}_{\rho_1}) \\ K_2 &\simeq H^\bullet(\mathbf{P}_1^{m_r}) \otimes \bigoplus_{\rho' \in \mathcal{W}_{\mathbf{m}', \mathcal{E}_2}^\wedge} \tilde{V}_{\rho'} \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}'}^0, \mathcal{L}_{\rho'}). \end{aligned}$$

Put $K_{\mathbf{I}'} = (\pi_{\mathbf{I}'})_!(\alpha')^* \mathcal{E}$. Since $\tilde{\mathcal{X}}_{\mathbf{I}'}^+$ is a connected component of $\tilde{\mathcal{X}}_{\mathbf{m}'}^+$, any simple summand of $K_{\mathbf{I}'}$ is contained in K_2 , up to shift, thus it is of the form $\mathrm{IC}(\mathcal{X}_{\mathbf{m}'}^0, \mathcal{L}_{\rho'})$. A simple perverse sheaf on $\mathcal{X}_{\mathbf{m}'}^0$ obtained from $\mathrm{IC}(G_1^0, \mathcal{L}_{\rho_1})$ and $\mathrm{IC}(\mathcal{X}_{\mathbf{m}'}^0, \mathcal{L}_{\rho'})$ by the procedure in 2.6 actually coincides with A_ρ . On the other hand, since the squares in the diagram (2.8.1) are both cartesian, we have $\sigma_1^*(K_1 \boxtimes K_{\mathbf{I}'}) \simeq q_1^*((\pi_{\mathbf{I}})! \alpha^* \mathcal{E})$. (2.8.2) follows from this.

By Lemma 2.4 and Lemma 2.7, (2.8.2) implies that

(2.8.3) Any simple summand (up to shift) of the semisimple complex $(\pi_{\mathbf{m}})! \alpha^* \mathcal{E}$ is contained in the set $\{\mathrm{IC}(\mathcal{X}_{\mathbf{m}}^0, \mathcal{L}_\rho) \mid \rho \in \mathcal{W}_{\mathbf{m}, \mathcal{E}}^\wedge\}$.

(2.8.3) implies, in particular, that any simple summand of $K = (\pi_{\mathbf{m}})! \alpha^* \mathcal{E}$ has its support $\mathcal{X}_{\mathbf{m}}^0$. Since the restriction of K on $\mathcal{Y}_{\mathbf{m}}^0$ coincides with $K_0 = (\psi_{\mathbf{m}})_! \alpha_0^* \mathcal{E}$, the decomposition of K into simple summands is determined by the decomposition of K_0 . Hence the proposition follows from (1.6.6). \square

Remark 2.9. Proposition 2.8 is a generalization of Proposition 4.8 in [SS1]. But the argument here is much simpler than that of [SS1].

2.10. For $\mathbf{m} \in \mathcal{Q}_{n,r}$, and for each j, k , we consider $M^{(j,k)}$ and $\overline{M}^{(j,k)}$ as in the proof of Proposition 1.7. Put

$$\begin{aligned} \mathcal{X}_{j,k}^0 &= \bigcup_{g \in H} g(B^{\iota\theta} \times M^{(j,k)}), \\ \tilde{\mathcal{X}}_{j,k}^+ &= \pi^{-1}(\mathcal{X}_{j,k}^0), \end{aligned}$$

and let $\pi_{j,k} : \tilde{\mathcal{X}}_{j,k}^+ \rightarrow \mathcal{X}_{j,k}^0$ be the restriction of π on $\tilde{\mathcal{X}}_{j,k}^+$. Then $\pi_{j,k}$ is a proper map, and $\mathcal{Y}_{j,k}^0$ is open dense in $\mathcal{X}_{j,k}^0$. Moreover, $\mathcal{X}_{j,k}^0$ coincides with $\mathcal{X}_{\mathbf{m}}^0$ in the case where $j = r - 1, k = m_{r-1}$, and coincides with $\mathcal{X}_{\mathbf{m}}$ in the case where $j = 0$ and $k = 0$. Also put

$$\begin{aligned} \mathcal{X}_{j,k} &= \bigcup_{g \in H} g(B^{\iota\theta} \times \overline{M}^{(j,k)}), \\ \overline{\tilde{\mathcal{X}}_{j,k}^+} &= \pi^{-1}(\mathcal{X}_{j,k}), \end{aligned}$$

and let $\overline{\pi}_{j,k} : \overline{\mathcal{X}}_{j,k}^+ \rightarrow \mathcal{X}_{j,k}$ be the restriction of π on $\overline{\mathcal{X}}_{j,k}^+$. Then $\mathcal{X}_{j,k}^0$ is open dense in $\mathcal{X}_{j,k}$. As in (1.7.1), we have

$$(2.10.1) \quad \mathcal{X}_{j,k} \setminus \mathcal{X}_{j,k-1} = \mathcal{X}_{j,k}^0 \text{ if } k \geq 1, \text{ and } \mathcal{X}_{j,0}(\mathbf{m}) = \mathcal{X}_{j+1,m_j+m_{j+1}}(\mathbf{m}(j,0)).$$

Moreover, $\mathcal{X}_{j,k}^0(\mathbf{m})$ coincides with $\mathcal{X}_{j+1,m_j+m_{j+1}-k}(\mathbf{m}(j,k))$.

For $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$, $\mathcal{Y}_{\mathbf{m}'}$ is contained in $\mathcal{Y}_{j,k}$, hence in $\mathcal{X}_{j,k}$. One can define an intersection cohomology $\text{IC}(\mathcal{X}'_{\mathbf{m}'}, \mathcal{L}_\rho)$ associated to the local system \mathcal{L}_ρ on $\mathcal{Y}_{\mathbf{m}'}$ (here $\mathcal{X}'_{\mathbf{m}'}$ denotes the closure of $\mathcal{Y}_{\mathbf{m}'}$ in $\mathcal{X}_{j,k}$). We show the following formulas. First assume that $j = r - 1$ and $0 \leq k \leq m_{r-1}$.

$$\begin{aligned} & (\overline{\pi}_{r-1,k})_! \alpha^* \mathcal{E} \\ & \simeq \bigoplus_{0 \leq k' \leq k} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k')}, \mathcal{E}} \widetilde{V}_\rho \otimes \text{IC}(\mathcal{X}_{r-1,k'}, \mathcal{L}_\rho)[-2(m_{r-1} - k')] + \mathcal{M}_{r-1,k}, \end{aligned} \quad (2.10.2)$$

where $\mathcal{M}_{r-1,k}$ is a sum of various $\text{IC}(\mathcal{X}_{r-1,k'}, \mathcal{L}_\rho)[-2i]$ for $0 \leq k' \leq k$ and $\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}$ with $0 \leq i < m_{r-1} - k'$. Next assume that $0 \leq j < r - 1$ and that $0 \leq k \leq m_j$. Then we have

$$\begin{aligned} & (\overline{\pi}_{j,k})_! \alpha^* \mathcal{E} \\ & \simeq \bigoplus_{0 \leq k' \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k')}, \mathcal{E}} \widetilde{V}_\rho \otimes \text{IC}(\mathcal{X}'_{\mathbf{m}(k')}, \mathcal{L}_\rho)[-2(m_{r-1} - k')] + \mathcal{M}_{j,k}, \end{aligned} \quad (2.10.3)$$

where $\mathcal{M}_{j,k}$ is a sum of various $\text{IC}(\mathcal{X}'_{\mathbf{m}'}, \mathcal{L}_\rho)[-2i]$ for $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$ and $\rho \in \mathcal{W}_{\mathbf{m}', \mathcal{E}}$ with i such that $0 \leq 2i < d_{\mathbf{m}} - d_{\mathbf{m}'}$.

As in the proof of Proposition 1.7, one can define an action of $\widetilde{\mathcal{W}}_\mathcal{E}$ on $(\overline{\pi}_{j,k})_! \alpha^* \mathcal{E}$. Then (2.10.2) and (2.10.3) can be proved by a similar argument as in the proof of Proposition 1.7.

Now apply (2.10.3) to the case where $j = 0, k = 0$. In this case, $\overline{\pi}_{j,k}$ coincides with $\overline{\pi}_{\mathbf{m}}$. (2.10.3) shows that any simple perverse sheaf $A[s]$ appearing in the semisimple complex $(\overline{\pi}_{\mathbf{m}})_! \alpha^* \mathcal{E}$ has the property that $\text{supp } A \cap \mathcal{Y}_{\mathbf{m}} \neq \emptyset$. $\mathcal{Y}_{\mathbf{m}}$ is open dense in $\mathcal{X}_{\mathbf{m}}$, and the restriction of $(\overline{\pi}_{\mathbf{m}})_! \alpha^* \mathcal{E}$ on $\mathcal{Y}_{\mathbf{m}}$ coincides with $(\overline{\psi}_{\mathbf{m}})_! \alpha_0^* \mathcal{E}$. Thus the theorem follows from Proposition 1.7. This completes the proof of Theorem 2.2.

3. A variant of Theorem 2.2

3.1. In this section, we assume that \mathcal{X} is of exotic type. We keep the notation in Section 1 and Section 2. For $\mathbf{m} = (m_1, \dots, m_{r-1}, 0) \in \mathcal{Q}_{n,r}^0$, put $\mathcal{W}_{\mathbf{m}}^{\natural} = S_{m_1} \times \dots \times S_{m_{r-2}} \times W_{m_{r-1}}$, where W_n is the Weyl group of type C_n , and let $\mathcal{W}_{\mathbf{m}, \mathcal{E}}^{\natural}$ be the stabilizer of \mathcal{E} in $\mathcal{W}_{\mathbf{m}}^{\natural}$. (Note that $\mathcal{W}_{\mathbf{m}}^{\natural}$ is not a subgroup of $W_{n,r}$ if $r \geq 3$.) Recall that $\mathbf{m}(k) = (m_1, \dots, m_{r-2}, k, k')$ with $k + k' = m_{r-1}$ for $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Hence $\mathcal{W}_{\mathbf{m}(k)} \simeq S_{m_1} \times \dots \times S_{m_{r-2}} \times S_k \times S_{k'}$. For $\rho = \rho_1 \boxtimes \dots \boxtimes \rho_r \in \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^{\natural}$, we

define an irreducible $\mathcal{W}_{\mathbf{m},\mathcal{E}}^{\natural}$ -module V_{ρ}^{\natural} by $V_{\rho}^{\natural} = \rho_1 \boxtimes \cdots \boxtimes \rho_{r-2} \boxtimes \tilde{\rho}_{r-1}$, where $\tilde{\rho}_{r-1}$ is an irreducible $W_{m_{r-1}}$ -module obtained from $\rho_{r-1} \boxtimes \rho_r \in (S_k \times S_{k'})^{\wedge}$ (apply 1.6 for the case $r = 2$). Recall the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$ as in 1.2 and consider $\alpha|_{\tilde{\mathcal{X}}_{\mathbf{m}}} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow T^{i\theta}$, which we denote by the same symbol α . The following result is a variant of Theorem 2.2.

Theorem 3.2. *For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, $\pi_1^{(\mathbf{m})} \alpha^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}}$ equipped with $\mathcal{W}_{\mathbf{m},\mathcal{E}}^{\natural}$ -action, and is decomposed as*

$$\pi_1^{(\mathbf{m})} \alpha^* \mathcal{E}[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k),\mathcal{E}}^{\wedge}} V_{\rho}^{\natural} \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_{\rho})[d_{\mathbf{m}(k)}].$$

3.3. The theorem can be proved in a similar way as in the proof of Theorem 2.2. We will give an outline of the proof below. We follow the notation in 1.3. We fix $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. We also consider the map $\psi^{(\mathbf{m})} : \tilde{\mathcal{Y}}_{\mathbf{m}} \rightarrow \mathcal{Y}_{\mathbf{m}}$. For each $\mathbf{m}' \leq \mathbf{m}$ ($\mathbf{m}' \in \mathcal{Q}_{n,r}$), put $\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger} = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}'}^0)$. For each $\mathbf{I} \in \mathcal{I}(\mathbf{m}')$, the variety $\tilde{\mathcal{Y}}_{\mathbf{I}} \subset \tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger}$ is defined as in 1.3. Put $\mathbf{I}(\mathbf{m}) = (I_1^{\circ}, \dots, I_r^{\circ}) \in \mathcal{I}(\mathbf{m})$ with $I_i^{\circ} = [p_{i-1} + 1, p_i]$. In particular, $I_r^{\circ} = \emptyset$. Put $\mathcal{W}^{(\mathbf{m})} = S_{m_1} \times \cdots \times S_{m_{r-1}}$ ($= \mathcal{W}_{\mathbf{m}}$), and we denote by $\mathcal{W}_{\mathbf{m}'}^{(\mathbf{m})}$ the subgroup of $\mathcal{W}^{(\mathbf{m})}$ which is the stabilizer of $\mathbf{I}(\mathbf{m}')$. Put

$$\mathcal{I}^{(\mathbf{m})}(\mathbf{m}') = \{\mathbf{I} \in \mathcal{I}(\mathbf{m}') \mid I_{\leq i} \subset I_{\leq i}^{\circ} \ (1 \leq i \leq r)\},$$

where $I_{\leq i}$ is defined similarly to $I_{< i}$ in 1.3. Then as in (1.3.2), $\mathcal{W}^{(\mathbf{m})}$ acts naturally on $\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger}$, and

$$\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger} = \coprod_{\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}')} \tilde{\mathcal{Y}}_{\mathbf{I}} = \coprod_{w \in \mathcal{W}^{(\mathbf{m})}/\mathcal{W}_{\mathbf{m}'}^{(\mathbf{m})}} w(\tilde{\mathcal{Y}}_{\mathbf{m}'}^0), \tag{3.3.1}$$

where $\mathcal{Y}_{\mathbf{m}'}^0 = \mathcal{Y}_{\mathbf{I}(\mathbf{m}')}^0$ is as in 1.3. For a tame local system \mathcal{E} on $T^{i\theta}$, we denote by $\mathcal{W}_{\mathcal{E}}^{(\mathbf{m})}$ (resp. $\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})}$) the stabilizer of \mathcal{E} in $\mathcal{W}^{(\mathbf{m})}$ (resp. in $\mathcal{W}_{\mathbf{m}'}^{(\mathbf{m})}$). Put $\tilde{\mathcal{W}}^{(\mathbf{m})} = \mathcal{W}^{(\mathbf{m})} \times (\mathbf{Z}/2\mathbf{Z})^{m_{r-1}}$, which coincides with $\mathcal{W}_{\mathbf{m}}^{\natural}$. We define a subgroup $\tilde{\mathcal{W}}_{\mathcal{E}}^{(\mathbf{m})}$ of $\tilde{\mathcal{W}}^{(\mathbf{m})}$ by $\tilde{\mathcal{W}}_{\mathcal{E}}^{(\mathbf{m})} = \mathcal{W}_{\mathcal{E}}^{(\mathbf{m})} \times (\mathbf{Z}/2\mathbf{Z})^{m_{r-1}}$. Thus $\tilde{\mathcal{W}}_{\mathcal{E}}^{(\mathbf{m})} = \mathcal{W}_{\mathbf{m},\mathcal{E}}^{\natural}$ in the notation of 3.1. Let $\psi_{\mathbf{m}'}^{(\mathbf{m})}$ be the restriction of $\psi^{(\mathbf{m})}$ on $\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger}$. As an analogue of (1.5.2), we have

$$(\psi_{\mathbf{m}'}^{(\mathbf{m})})! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger}} \simeq \bigoplus_{\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}')} (\psi_{\mathbf{I}})! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{I}}}. \tag{3.3.2}$$

Put $\mathbf{m}' = (m'_1, \dots, m'_r)$. The action of $(\mathbf{Z}/2\mathbf{Z})^{m'_r}$ on $H^{\bullet}(\mathbf{P}_1^{m'_r})$ is defined as in 1.6 by considering the case where $r = 2$. By a similar argument as in 1.5 and 1.6, we see that $(\psi_{\mathbf{m}'}^{(\mathbf{m})})! \alpha_0^* \mathcal{E}$ is equipped with $(\mathbf{Z}/2\mathbf{Z})^{m'_r} \times \mathcal{W}_{\mathcal{E}}^{(\mathbf{m})}$ -action, and is decomposed as

$$(\psi_{\mathbf{m}'}^{(\mathbf{m})})! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{m}'}^{\dagger}} \simeq H^{\bullet}(\mathbf{P}_1^{m'_r}) \otimes \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})})^{\wedge}} \mathrm{Ind}_{\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})}}^{\mathcal{W}_{\mathcal{E}}^{(\mathbf{m})}} \rho \otimes \mathcal{L}_{\rho}, \tag{3.3.3}$$

where \mathcal{L}_{ρ} is a simple local system on $\mathcal{Y}_{\mathbf{m}'}^0$ obtained from the Galois covering $\widehat{\mathcal{Y}}_{\mathbf{I}(\mathbf{m}')} \rightarrow \mathcal{Y}_{\mathbf{m}'}^0$ as in (1.5.3).

As in (1.6.7), (3.3.3) can be rewritten in the following form:

$$(\psi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}|_{\widetilde{\mathcal{Y}}_{\mathbf{m}'}} \simeq \left(\bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^\wedge} V_\rho^{\natural} \otimes \mathcal{L}_\rho \right) [-2m'_r] + \mathcal{N}_{\mathbf{m}'} \quad (3.3.4)$$

if $\mathbf{m}' = \mathbf{m}(k)$ for some k , and $(\psi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}|_{\widetilde{\mathcal{Y}}_{\mathbf{m}'}} \simeq \mathcal{N}_{\mathbf{m}'}$ otherwise, where $\mathcal{N}_{\mathbf{m}'}$ is a sum of various $\mathcal{L}_\rho[-2i]$ for $\rho \in (\mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})})^\wedge$ with $2i < d_{\mathbf{m}} - d_{\mathbf{m}'}$.

In fact, put $\mathbf{p}(\mathbf{m}') = (p'_1, \dots, p'_r)$. Since $\mathbf{m} \geq \mathbf{m}'$, $p_i \geq p'_i$ for each i . Moreover, $p_{r-1} - p'_{r-1} = m'_r$ since $p_{r-1} = n$. Then by Lemma 1.4 we have

$$\begin{aligned} d_{\mathbf{m}} - d_{\mathbf{m}'} &= \sum_{i=1}^{r-1} (p_i - p'_i) + m'_r \\ &= \sum_{i=1}^{r-2} (p_i - p'_i) + 2m'_r \\ &\geq 2m'_r, \end{aligned}$$

and the equality holds only when $p_i = p'_i$ for $i = 1, \dots, r-2$, namely when $\mathbf{m}' = \mathbf{m}(k)$ for some k . By (3.3.3), $K = (\psi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}$ is a semisimple complex and each direct summand is of the form $\mathcal{L}_\rho[-2i]$ with $i \leq m'_r$. Hence $K \simeq \mathcal{N}_{\mathbf{m}'}$ if \mathbf{m}' is not of the form $\mathbf{m}(k)$. Now assume that $\mathbf{m}' = \mathbf{m}(k)$. In this case, $\mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})} = \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}$, and it follows from (3.3.3) that

$$K \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^\wedge} \text{Ind}_{\widetilde{\mathcal{W}}_{\mathbf{m}(k), \mathcal{E}}^{(\mathbf{m})}}^{\widetilde{\mathcal{W}}_{\mathcal{E}}^{(\mathbf{m})}} (H^\bullet(\mathbf{P}_1^{m'_r}) \otimes \rho) \otimes \mathcal{L}_\rho, \quad (3.3.5)$$

where $\widetilde{\mathcal{W}}_{\mathbf{m}(k), \mathcal{E}}^{(\mathbf{m})} = \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^{(\mathbf{m})} \times (\mathbf{Z}/2\mathbf{Z})^{m_{r-1}}$, and $H^\bullet(\mathbf{P}_1^{m'_r}) \otimes \rho$ is regarded as an $\widetilde{\mathcal{W}}_{\mathbf{m}(k), \mathcal{E}}^{(\mathbf{m})}$ -module by the trivial action of $(\mathbf{Z}/2\mathbf{Z})^k$, and by the action of $(\mathbf{Z}/2\mathbf{Z})^{m'_r}$ through $H^\bullet(\mathbf{P}_1^{m'_r})$ (here $k + m'_r = m_{r-1}$). The direct summand $\mathcal{L}_\rho[-2i]$ of K satisfies the relation $d_{\mathbf{m}} - d_{\mathbf{m}'} = 2i$ only when $i = m'_{r-1}$. Hence the first assertion of (3.3.4) follows from (3.3.5).

3.4. For $1 \leq j \leq r-1$ and $0 \leq k \leq m_j$, $M^{(j,k)}, \overline{M}^{(j,k)}, \mathcal{Y}_{j,k}^0, \mathcal{Y}_{j,k}$ are defined as in the proof of Proposition 1.7. Put $\widetilde{\mathcal{Y}}_{j,k}^\dagger = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{j,k}^0)$ and $\overline{\widetilde{\mathcal{Y}}}_{j,k}^\dagger = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{j,k})$. Let $\psi_{j,k}^{(\mathbf{m})} : \widetilde{\mathcal{Y}}_{j,k}^\dagger \rightarrow \mathcal{Y}_{j,k}^0$ be the restriction of $\psi^{(\mathbf{m})}$ on $\widetilde{\mathcal{Y}}_{j,k}^\dagger$, and $\overline{\psi}_{j,k}^{(\mathbf{m})} : \overline{\widetilde{\mathcal{Y}}}_{j,k}^\dagger \rightarrow \mathcal{Y}_{j,k}$ the restriction of $\psi^{(\mathbf{m})}$ on $\overline{\widetilde{\mathcal{Y}}}_{j,k}^\dagger$. By using a similar argument as in the proof of (1.7.2) and (1.7.3), we can show the following formulas. First assume that $j = r-1$ and $0 \leq k \leq m_{r-1}$.

$$\begin{aligned} &(\overline{\psi}_{r-1,k}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E} \\ &\simeq \bigoplus_{0 \leq k' \leq k} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge} V_\rho^{\natural} \otimes \text{IC}(\mathcal{Y}_{r-1,k'}, \mathcal{L}_\rho) [-2(m_{r-1} - k')] + \mathcal{N}_{r-1,k}, \end{aligned} \quad (3.4.1)$$

where $\mathcal{N}_{r-1,k}$ is a sum of various $\mathrm{IC}(\mathcal{Y}_{r-1,k'}, \mathcal{L}_\rho)[-2i]$ for $0 \leq k' \leq k$ and $\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge$ with $i < m_{r-1} - k'$. Next assume that $0 \leq j < r-1$ and that $0 \leq k \leq m_j$. Then we have

$$\begin{aligned} & \overline{(\psi_{j,k}^{(\mathbf{m})})_* \alpha_0^* \mathcal{E}} \\ & \simeq \bigoplus_{0 \leq k' \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k'), \mathcal{E}}^\wedge} V_\rho^\natural \otimes \mathrm{IC}(\mathcal{Y}'_{\mathbf{m}(k')}, \mathcal{L}_\rho)[-2(m_{r-1} - k')] + \mathcal{N}_{j,k}, \end{aligned} \quad (3.4.2)$$

where $\mathcal{N}_{j,k}$ is a sum of various $\mathrm{IC}(\mathcal{Y}'_{\mathbf{m}'}, \mathcal{L}_\rho)[-2i]$ for $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$ and $\rho \in (\mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})})^\wedge$ with i such that $2i < d_{\mathbf{m}} - d_{\mathbf{m}'}$.

Note that in the proof of (3.4.1), the role of the irreducible $\widetilde{\mathcal{W}}_\mathcal{E}$ -module \widetilde{V}_ρ is replaced by the irreducible $\mathcal{W}_{\mathbf{m}, \mathcal{E}}^\natural$ -module V_ρ^\natural .

By a similar argument as in the proof of Lemma 1.4 (iv), one can show that $\psi^{(\mathbf{m})}$ is semismall for $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then, as in the proof of Proposition 1.7 (see the paragraph after (1.7.3)), we obtain the following proposition from (3.4.2).

Proposition 3.5. *For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, $\psi_!^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{Y}_{\mathbf{m}}$ equipped with $\mathcal{W}_{\mathbf{m}, \mathcal{E}}^\natural$ -action, and is decomposed as*

$$\psi_!^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}(k), \mathcal{E}}^\wedge} V_\rho^\natural \otimes \mathrm{IC}(\mathcal{Y}_{\mathbf{m}(k)}, \mathcal{L}_\rho)[d_{\mathbf{m}(k)}].$$

3.6. We follow the notation in 2.3. For $\mathbf{m}' \in \mathcal{Q}_{n,r}$ such that $\mathbf{m}' \leq \mathbf{m}$, put $\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger = (\pi^{(\mathbf{m})})^{-1}(\mathcal{X}_{\mathbf{m}'}^0)$. Then $\widetilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger$ is an open dense subset of $\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$. For each $\mathbf{I} \in \mathcal{I}(\mathbf{m}')$, the subvariety $\widetilde{\mathcal{X}}_{\mathbf{I}}^\dagger$ of $\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$ is defined as in 2.3. Then, as in Lemma 2.4, we have

$$\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger = \prod_{\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}')} \widetilde{\mathcal{X}}_{\mathbf{I}}^\dagger,$$

where $\widetilde{\mathcal{X}}_{\mathbf{I}}^\dagger$ is an irreducible component of $\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$.

Let $\pi_{\mathbf{m}'}^{(\mathbf{m})} : \widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger \rightarrow \mathcal{X}_{\mathbf{m}'}^0$ be the restriction of $\pi^{(\mathbf{m})}$ on $\widetilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$. The following result is an analogue of Proposition 2.8, and is proved in a similar way.

Proposition 3.7. *Assume that $\mathbf{m}' \leq \mathbf{m}$. Then $(\pi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha^* \mathcal{E}$ is decomposed as*

$$(\pi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha^* \mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{m'_r}) \otimes \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})}^\wedge} \mathrm{Ind}_{\mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})}}^{\mathcal{W}_\mathcal{E}^{(\mathbf{m})}} \rho \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}'}^0, \mathcal{L}_\rho),$$

where $\mathrm{Ind}_{\mathcal{W}_{\mathbf{m}', \mathcal{E}}^{(\mathbf{m})}}^{\mathcal{W}_\mathcal{E}^{(\mathbf{m})}} \rho$ is regarded as a vector space ignoring the $\mathcal{W}_\mathcal{E}^{(\mathbf{m})}$ -module structure.

3.8. By making use of Proposition 3.5 and 3.7, the theorem can be proved in a similar way as in 2.9, 2.10.

4. Intersection cohomology on $G^{\iota\theta} \times V^{r-1}$ (enhanced case)

4.1. In this section we assume that \mathcal{X} is of enhanced type. We fix $\mathbf{m} \in \mathcal{Q}_{n,r}$ (note that we do not assume $\mathbf{m} \in \mathcal{Q}_{n,r}^0$), and consider the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$. Here, in order to emphasize a similarity with the exotic case, we follow the notation in Section 1. But of course a simpler expression is possible for the enhanced case. For example, if we write $G = G_0 \times G_0$ and $B = B_0 \times B_0$ for a Borel subgroup B_0 of G_0 , $\tilde{\mathcal{X}}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}$ are given by

$$\tilde{\mathcal{X}}_{\mathbf{m}} = \left\{ (x, \mathbf{v}, gB_0) \in G_0 \times V^{r-1} \times G_0/B_0 \mid g^{-1}xg \in B_0, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}$$

$$\mathcal{X}_{\mathbf{m}} = \bigcup_{g \in G_0} g(B_0 \times \prod_{i=1}^{r-1} M_{p_i}).$$

The map $\psi^{(\mathbf{m})} : \tilde{\mathcal{Y}}_{\mathbf{m}} \rightarrow \mathcal{Y}_{\mathbf{m}}$ is defined as in 1.2. The subset $\mathcal{Y}_{\mathbf{m}'}^0$ is defined for each $\mathbf{m}' \in \mathcal{Q}_{n,r}$ as in 1.3. For each $\mathbf{m}' \leq \mathbf{m}$, put $\tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}'}^0)$. For each $\mathbf{I} \in \mathcal{I}(\mathbf{m}')$, the subvariety $\tilde{\mathcal{Y}}_{\mathbf{I}}$ of $\tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger$ and the map $\psi_{\mathbf{I}}$ are defined as in 1.3. Note that in the enhanced case, if we write $T = T_0 \times T_0$, then $Z_H(T^{\iota\theta}) = Z_{G_0}(T_0) = T_0$. Hence $B^\theta \cap Z_H(T^{\iota\theta}) = T_0$, and $T_{\text{reg}}^{\iota\theta}$ is the set of regular semisimple elements in T_0 . Hence $\tilde{\mathcal{Y}}_{\mathbf{I}}$ is written as

$$\tilde{\mathcal{Y}}_{\mathbf{I}} \simeq G_0 \times^{T_0} ((T_0)_{\text{reg}} \times M_{\mathbf{I}}),$$

where $M_{\mathbf{I}}$ is defined as in 1.3. As in 3.3, we define $\mathcal{W}^{(\mathbf{m})} = S_{m_1} \times \cdots \times S_{m_r}$ ($= \mathcal{W}_{\mathbf{m}}$), and its subgroup $\mathcal{W}_{\mathbf{m}'}^{(\mathbf{m})}$. For each $\mathbf{m}' \leq \mathbf{m}$, we define $\mathcal{I}^{(\mathbf{m})}(\mathbf{m}')$ as in 3.3. Then a similar formula as (3.3.1) holds for $\tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger$. Let \mathcal{E} be a tame local system on $T^{\iota\theta}$, and we denote by $\mathcal{W}_{\mathcal{E}}^{(\mathbf{m})}$ (resp. $\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})}$) the stabilizer of \mathcal{E} in $\mathcal{W}^{(\mathbf{m})}$ (resp. in $\mathcal{W}_{\mathbf{m}'}^{(\mathbf{m})}$). As in (1.5.2), we have a similar formula as (3.3.2). Note that in the enhanced case, one can check that $\psi_{\mathbf{I}}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}^{(\mathbf{m})}$ (the stabilizer of \mathbf{I} in $\mathcal{W}^{(\mathbf{m})}$). It follows from (3.3.1) and (3.3.2) (corresponding formulas for the enhanced case) that we have

$$(\psi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger} \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})})^\wedge} (\text{Ind}_{\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})}}^{\mathcal{W}_{\mathcal{E}}^{(\mathbf{m})}} \rho) \otimes \mathcal{L}_\rho. \tag{4.1.1}$$

Here we note that

Lemma 4.2.

- (i) $\mathcal{Y}_{\mathbf{m}}$ is open dense in $\mathcal{X}_{\mathbf{m}}$ and $\tilde{\mathcal{Y}}_{\mathbf{m}}$ is open dense in $\tilde{\mathcal{X}}_{\mathbf{m}}$.
- (ii) $\dim \mathcal{X}_{\mathbf{m}} = \dim \tilde{\mathcal{X}}_{\mathbf{m}} = n^2 + \sum_{i=1}^r (r-i)m_i$.
- (iii) For any $(x, \mathbf{v}) \in \mathcal{Y}_{\mathbf{m}}$, $(\psi^{(\mathbf{m})})^{-1}(x, \mathbf{v})$ is a finite set.

Proof. $\widetilde{\mathcal{Y}}_{\mathbf{m}}$ is an open dense subset of $\widetilde{\mathcal{X}}_{\mathbf{m}}$. Since $\widetilde{\mathcal{Y}}_{\mathbf{m}} = (\pi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}})$ and $\pi^{(\mathbf{m})}$ is proper, $\mathcal{Y}_{\mathbf{m}}$ is an open dense subset of $\mathcal{X}_{\mathbf{m}}$. Hence (i) holds. $\widetilde{\mathcal{Y}}_{\mathbf{m}}^{\dagger}$ is an open dense subset of $\widetilde{\mathcal{Y}}_{\mathbf{m}}$, and $\mathcal{Y}_{\mathbf{m}}^0$ is an open dense subset of $\mathcal{Y}_{\mathbf{m}}$. Since $\psi_{\mathbf{I}}$ is a finite Galois covering for $\mathbf{I} \in \mathcal{I}(\mathbf{m})$, we have $\dim \widetilde{\mathcal{Y}}_{\mathbf{m}}^{\dagger} = \dim \mathcal{Y}_{\mathbf{m}}^0$. Hence $\dim \widetilde{\mathcal{Y}}_{\mathbf{m}} = \dim \mathcal{Y}_{\mathbf{m}}$ and (ii) follows from (1.2.1). (iii) is clear since $\psi_{\mathbf{I}}$ is a finite Galois covering for any $\mathbf{I} \in \mathcal{I}(\mathbf{m})(\mathbf{m}')$. \square

Next we show the following proposition.

Proposition 4.3. *For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, $\psi_{\mathbf{I}}^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{Y}_{\mathbf{m}}$ equipped with $\mathcal{W}_{\mathbf{m},\mathcal{E}}$ -action, and is decomposed as*

$$\psi_{\mathbf{I}}^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m},\mathcal{E}}^{\wedge}} \rho \otimes \mathrm{IC}(\mathcal{Y}_{\mathbf{m}}, \mathcal{L}_{\rho})[d_{\mathbf{m}}].$$

Proof. $\psi^{(\mathbf{m})} : \widetilde{\mathcal{Y}}^{(\mathbf{m})} \rightarrow \mathcal{Y}_{\mathbf{m}}$ is proper. By Lemma 4.2 (iii), $\psi^{(\mathbf{m})}$ is semismall. Hence $\psi_{\mathbf{I}}^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf. The definitions of $\mathcal{Y}_{j,k}^0$, $\mathcal{Y}_{j,k}$, $\psi_{j,k}^{(\mathbf{m})}$, $\overline{\psi}_{j,k}^{(\mathbf{m})}$, etc., in 3.4 make sense also in the enhanced case. As in (3.4.1) and (3.4.2), the following formulas hold; first assume that $j = r - 1, 0 \leq k < m_{r-1}$. Then we have

$$(\overline{\psi}_{j,k}^{(\mathbf{m})})! \alpha_0^* \mathcal{E} \simeq \mathcal{N}_{r-1,k}. \tag{4.3.1}$$

Next assume that $j = r - 1, k = m_{r-1}$ or $0 \leq j < r - 1, 0 \leq k \leq m_j$. Then we have

$$(\overline{\psi}_{j,k}^{(\mathbf{m})})! \alpha_0^* \mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m},\mathcal{E}}^{\wedge}} \rho \otimes \mathrm{IC}(\mathcal{Y}'_{\mathbf{m}}, \mathcal{L}_{\rho}) + \mathcal{N}_{j,k}, \tag{4.3.2}$$

where $\mathcal{N}_{j,k}$ is a sum of various $\mathrm{IC}(\mathcal{Y}'_{\mathbf{m}'}, \mathcal{L}_{\rho})$ for $\mathbf{m}' \in \mathcal{Q}(\mathbf{m}; j, k)$ and $\rho \in (\mathcal{W}_{\mathbf{m}',\mathcal{E}}^{(\mathbf{m})})^{\wedge}$ such that $\mathbf{m}' < \mathbf{m}$. (Recall that $\mathcal{Y}'_{\mathbf{m}'}$ denotes the closure of $\mathcal{Y}_{\mathbf{m}'}^0 \subset \mathcal{Y}_{j,k}$ in $\mathcal{Y}_{j,k}$ for any $\mathbf{m}' \leq \mathbf{m}$.)

We show (4.3.1) by induction on k . By (1.7.1) (or directly from the definition of $M^{(r-1,k)}$), $\mathcal{Y}_{r-1,0}(\mathbf{m})$ coincides with $\mathcal{Y}_{\mathbf{m}(0)}^0$. Hence by (4.1.1) for $\mathbf{m}' = \mathbf{m}(0)$, (4.3.1) holds for $k = 0$. A similar argument as in the proof of (1.7.2) shows, thanks to (1.7.1) and (4.1.1), that (4.3.1) holds for $k < m_{r-1}$. Next consider the case where $j = r - 1, k = m_{r-1}$. By (1.7.1), $\mathcal{Y}_{j,k} \setminus \mathcal{Y}_{j,k-1} = \mathcal{Y}_{j,k}^0$, and $\mathcal{Y}_{j,k}^0$ coincides with $\mathcal{Y}_{\mathbf{m}}^0$. Thus (4.1.1) implies that

$$(\psi_{j,k}^{(\mathbf{m})})! \alpha_0^* \mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m},\mathcal{E}}^{\wedge}} \rho \otimes \mathcal{L}_{\rho}.$$

Hence (4.3.2) holds in this case. Now (4.3.2) can be proved by induction on k and by backwards induction on j , starting from $j = r - 2, k = 0$, which case corresponds to the case where $j = r - 1, k = m_{r-1}$ by (1.7.1). Note that in the enhanced case, we do not need a discussion such as in (1.7.5).

Applying (4.3.2) to the case where $j = 0, k = 0$, we obtain the proposition. In fact, in that case, $\mathcal{N}_{0,0}$ is a sum of $A = \mathrm{IC}(\mathcal{Y}_{\mathbf{m}'}, \mathcal{L}_{\rho})$ such that $\mathbf{m}' < \mathbf{m}$. But then $A[d_{\mathbf{m}}]$ is not a perverse sheaf. Since $\psi_{\mathbf{I}}^{(\mathbf{m})} \alpha_0^* \mathcal{E}[d_{\mathbf{m}}]$ is a semisimple perverse sheaf, this implies that $\mathcal{N}_{0,0} = 0$, and the proposition follows. \square

4.4. As a special case of (4.1.1) for $\mathbf{m}' = \mathbf{m}$, we have

$$(\psi_{\mathbf{m}}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \varepsilon}^\wedge} \rho \otimes \mathcal{L}_\rho.$$

Since $\mathcal{Y}_{\mathbf{m}}^0$ is a smooth open dense subset of $\mathcal{X}_{\mathbf{m}}$, one can define a semisimple perverse sheaf $K_{\mathbf{m}, T, \varepsilon}$ on $\mathcal{X}_{\mathbf{m}}$ as

$$K_{\mathbf{m}, T, \varepsilon} = \bigoplus_{\rho \in \mathcal{W}_{\mathbf{m}, \varepsilon}^\wedge} \rho \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}}, \mathcal{L}_\rho)[d_{\mathbf{m}}]. \tag{4.4.1}$$

We consider a diagram

$$T^{i\theta} \xleftarrow{\alpha} \tilde{\mathcal{X}}_{\mathbf{m}} \xrightarrow{\pi^{(\mathbf{m})}} \mathcal{X}_{\mathbf{m}},$$

where α is as in 2.1, and a complex $\pi_1^{(\mathbf{m})} \alpha^* \mathcal{E}[d_{\mathbf{m}}]$ on $\mathcal{X}_{\mathbf{m}}$. We shall prove the following theorem.

Theorem 4.5. *For each $\mathbf{m} \in \mathcal{Q}_{n, r}$, $\pi_1^{(\mathbf{m})} \alpha^* \mathcal{E}[d_{\mathbf{m}}] \simeq K_{\mathbf{m}, T, \varepsilon}$ as perverse sheaves on $\mathcal{X}_{\mathbf{m}}$.*

4.6. For each $\mathbf{m}' \leq \mathbf{m}$, the set $\mathcal{X}_{\mathbf{m}'}^0$ is defined as in 2.3. We define a subvariety $\mathcal{X}_{\mathbf{m}'}^\dagger$ of $\tilde{\mathcal{X}}_{\mathbf{m}}$ by $\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger = (\pi^{(\mathbf{m})})^{-1}(\mathcal{X}_{\mathbf{m}'}^0)$. Then $\tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger$ is an open dense subset of $\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$. The discussion in 2.3 makes sense also for the enhanced case, and $\mathcal{X}_{\mathbf{m}'}^0$ is characterized by a similar formula as (2.3.3). For each $\mathbf{I} \in \mathcal{I}(\mathbf{m}')$, the set $\tilde{\mathcal{X}}_{\mathbf{I}}$ is defined as in 2.3,

$$\tilde{\mathcal{X}}_{\mathbf{I}} = \{(x, \mathbf{v}, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger \mid (x, \mathbf{v}, gB^\theta) \mapsto \mathbf{I}\}.$$

Let $\mathcal{I}^{(\mathbf{m})}(\mathbf{m}')$ be as in 3.3. Then as in Lemma 2.4 (see also 3.6), we have

$$\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger = \coprod_{\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}')} \tilde{\mathcal{X}}_{\mathbf{I}},$$

where $\tilde{\mathcal{X}}_{\mathbf{I}}$ is an irreducible component of $\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$. Let $\pi_{\mathbf{m}'}^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger \rightarrow \mathcal{X}_{\mathbf{m}'}^0$ be the restriction of $\pi^{(\mathbf{m})}$ on $\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger$. The following result is an analogue of Proposition 2.8 (see also Proposition 3.7) and can be proved in a similar way.

Proposition 4.7. *Assume that $\mathbf{m}' \leq \mathbf{m}$. Then $(\pi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}$ is decomposed as*

$$(\pi_{\mathbf{m}'}^{(\mathbf{m})})_! \alpha_0^* \mathcal{E}|_{\tilde{\mathcal{X}}_{\mathbf{m}'}^\dagger} \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}', \varepsilon}^{(\mathbf{m})})^\wedge} (\mathrm{Ind}_{\mathcal{W}_{\mathbf{m}', \varepsilon}^{(\mathbf{m})}}^{\mathcal{W}_\varepsilon^{(\mathbf{m})}} \rho) \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}'}^0, \mathcal{L}_\rho).$$

4.8. By making use of Proposition 4.3 and Proposition 4.7, the theorem is proved by a similar argument as in 2.8 and 2.10.

5. Unipotent variety of enhanced type

In this section, we study the “unipotent part” of the enhanced space, which we call the unipotent variety of enhanced type. First we prepare some combinatorial notation.

5.1. A composition is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with finitely many nonzero terms. A composition λ satisfying the property that $\lambda_1 \geq \lambda_2 \geq \dots$ is called a partition. For a composition λ , we denote by $|\lambda| = \sum_i \lambda_i$ the size of λ . For a positive integer r , an r -tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is called an r -partition. We denote by $|\boldsymbol{\lambda}| = \sum_i |\lambda^{(i)}|$ the size of $\boldsymbol{\lambda}$. We express an r -partition by $\boldsymbol{\lambda} = (\lambda_j^{(i)})$ with partitions $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_m^{(i)})$ by choosing sufficiently large m so that $\lambda_j^{(i)} = 0$ for $j > m$ and for any i . The set of r -partitions of size n is denoted by $\mathcal{P}_{n,r}$. In the case where $r = 1$, the set $\mathcal{P}_{n,1}$ of partitions of n is simply denoted by \mathcal{P}_n . For a given $\mathbf{m} \in \mathcal{Q}_{n,r}$, we denote by $\mathcal{P}(\mathbf{m})$ the set of $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$ such that $|\lambda^{(i)}| = m_i$ for each i .

Let $\boldsymbol{\lambda} = (\lambda_j^{(i)})$ be an r -partition of n . We define a composition $c(\boldsymbol{\lambda})$ of n associated to $\boldsymbol{\lambda}$ by

$$c(\boldsymbol{\lambda}) = (\lambda_1^{(1)}, \lambda_1^{(2)}, \dots, \lambda_1^{(r)}, \lambda_2^{(1)}, \lambda_2^{(2)}, \dots, \lambda_2^{(r)}, \dots, \lambda_m^{(1)}, \lambda_m^{(2)}, \dots, \lambda_m^{(r)}).$$

For example, if $\boldsymbol{\lambda} = (320; 211; 411) \in \mathcal{P}_{15,3}$, we have $c(\boldsymbol{\lambda}) = (324211011)$.

For a composition $\lambda = (\lambda_1, \lambda_2, \dots), \mu = (\mu_1, \mu_2, \dots)$, we denote by $\lambda \leq \mu$ if

$$\lambda_1 + \dots + \lambda_k \leq \mu_1 + \dots + \mu_k$$

for $k = 1, 2, \dots$. We define a dominance order \leq on $\mathcal{P}_{n,r}$ by the condition that $\boldsymbol{\lambda} \leq \boldsymbol{\mu}$ if $c(\boldsymbol{\lambda}) \leq c(\boldsymbol{\mu})$. In the case where $r = 1$, this is the standard dominance order on the set \mathcal{P}_n . In the case where $r = 2$, this is the partial order given in [SS1, 1.7].

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we put $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$. We define a function $n : \mathcal{P}_{n,r} \rightarrow \mathbf{Z}$ by $n(\boldsymbol{\lambda}) = \sum_{i=1}^r n(\lambda^{(i)})$.

5.2. We consider the enhanced space of higher level introduced in 1.2. However in this section, we redefine them directly, without using the symmetric space setting in 1.2. For an integer $r \geq 1$, we consider a variety $\mathcal{X}_{\text{uni}} = G_{\text{uni}} \times V^{r-1}$, where V is an n -dimensional vector space over \mathbf{k} , $G = \text{GL}(V)$ and G_{uni} is the unipotent variety of G . We fix a basis $\{e_1, \dots, e_n\}$ of V , and define M_i as the subspace of V spanned by e_1, \dots, e_i . Let B be a Borel subgroup of G which is the stabilizer of the total flag (M_i) . Let T be a maximal torus of B such that $\{e_i\}$ are weight vectors for T . Then $B = TU$, where U is the unipotent radical of B . For $\mathbf{m} \in \mathcal{Q}_{n,r}$, let $\mathbf{p}(\mathbf{m}) = (p_1, \dots, p_r)$ be as in 1.2. We define

$$\begin{aligned} \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} &= \left\{ (x, \mathbf{v}, gB) \in G_{\text{uni}} \times V^{r-1} \times G/B \mid g^{-1}xg \in U, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}, \\ \mathcal{X}_{\mathbf{m},\text{uni}} &= \bigcup_{g \in G} g \left(U \times \prod_{i=1}^{r-1} M_{p_i} \right). \end{aligned}$$

We define a map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$ by $(x, \mathbf{v}, gB) \mapsto (x, \mathbf{v})$. Clearly $\mathcal{X}_{\mathbf{m},\text{uni}} = \text{Im } \pi_1^{(\mathbf{m})}$, and $\mathcal{X}_{\mathbf{m},\text{uni}}$ coincides with \mathcal{X}_{uni} in the case where $\mathbf{m} = (n, 0, \dots, 0)$. Since $\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \simeq G \times^B (U \times \prod_i M_{p_i})$, $\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}$ is smooth and irreducible. Since $\pi_1^{(\mathbf{m})}$ is proper, $\mathcal{X}_{\mathbf{m},\text{uni}}$ is a closed irreducible subvariety of \mathcal{X}_{uni} .

5.3. We shall define a partition of \mathcal{X}_{uni} into pieces X_λ indexed by $\lambda \in \mathcal{P}_{n,r}$,

$$\mathcal{X}_{\text{uni}} = \coprod_{\lambda \in \mathcal{P}_{n,r}} X_\lambda \tag{5.3.1}$$

satisfying the property such that X_λ is G -stable, and that if $(x, \mathbf{v}) \in X_\lambda$, then the Jordan type of x is $\lambda^{(1)} + \dots + \lambda^{(r)}$. It is known by [AH] and [T] that the set of G -orbits in $G_{\text{uni}} \times V$ is finite, and they are parametrized by $\mathcal{P}_{n,2}$. The labelling is given as follows; take $(x, v) \in G_{\text{uni}} \times V$. Let $E^x = \{y \in \text{End}(V) \mid xy = yx\}$. E^x is a subalgebra of $\text{End}(V)$ stable by the multiplication of x . If we put $W = E^x v$, W is an x -stable subspace of V . We denote by $\lambda^{(1)}$ the Jordan type of $x|_W$, and by $\lambda^{(2)}$ the Jordan type of $x|_{V/W}$. Then the Jordan type of x is $\lambda^{(1)} + \lambda^{(2)}$ and $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_{n,2}$. We denote by \mathcal{O}_λ the G -orbit containing (x, v) . This gives the required labelling of G -orbits in $G_{\text{uni}} \times V$.

If $r \geq 3$, the number of G -orbits in \mathcal{X}_{uni} is infinite. So X_λ should be a union of possibly infinitely many G -orbits. In general, we define X_λ by induction on r . Take $(x, \mathbf{v}) \in \mathcal{X}_{\text{uni}}$ with $\mathbf{v} = (v_1, \dots, v_{r-1})$. Put $W = E^x v_1$, $\bar{V} = V/W$ and $\bar{G} = \text{GL}(\bar{V})$. We consider the variety $\mathcal{X}'_{\text{uni}} = \bar{G} \times \bar{V}^{r-2}$. Assume that $(x, v_1) \in G_{\text{uni}} \times V$ is of type $(\lambda^{(1)}, \nu')$, where $\nu = \lambda^{(1)} + \nu'$ is the type of x . Let \bar{x} be the restriction of x on \bar{V} . Then the type of $\bar{x} \in \text{GL}(\bar{V})$ is ν' . Put $\bar{\mathbf{v}} = (\bar{v}_2, \dots, \bar{v}_{r-1})$, where \bar{v}_i is the image of v_i on \bar{V} . Thus $(\bar{x}, \bar{\mathbf{v}}) \in \mathcal{X}'_{\text{uni}}$. By induction, we have a partition $\mathcal{X}'_{\text{uni}} = \coprod_{\mu \in \mathcal{P}_{n',r-1}} X'_\mu$, where $\dim \bar{V} = n'$. Thus there exists a unique X'_λ containing $(\bar{x}, \bar{\mathbf{v}})$. If we write $\lambda' = (\lambda^{(2)}, \dots, \lambda^{(r)})$, we have $\lambda^{(2)} + \dots + \lambda^{(r)} = \nu'$. It follows that $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$. We define the type of (x, \mathbf{v}) by λ , and define a subset X_λ of \mathcal{X}_{uni} as the set of all (x, \mathbf{v}) with type λ . Then X_λ is a G -stable subset of \mathcal{X}_{uni} , and we obtain the required partition (5.3.1).

We show the following proposition.

Proposition 5.4. *Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$. Then X_λ is a smooth irreducible variety with*

$$\dim X_\lambda = (n^2 - n - 2n(\lambda)) + \sum_{i=1}^{r-1} (r-i)|\lambda^{(i)}|. \tag{5.4.1}$$

Proof. We may assume that $r \geq 2$. Put $\nu = \lambda^{(1)} + \dots + \lambda^{(r)}$ and $\nu' = \lambda^{(2)} + \dots + \lambda^{(r)}$. Let $\mathcal{O} = \mathcal{O}_{(\lambda^{(1)}, \nu')}$ be the G -orbit in $G_{\text{uni}} \times V$ corresponding to $(\lambda^{(1)}, \nu') \in \mathcal{P}_{n,2}$, and \mathcal{O}_ν be the G -orbit in G_{uni} corresponding to $\nu \in \mathcal{P}_n$. We have surjective maps $f_\lambda : X_\lambda \rightarrow \mathcal{O}$, $(x, \mathbf{v}) \mapsto (x, v_1)$, and $h_\lambda : X_\lambda \rightarrow \mathcal{O}_\nu$, $(x, \mathbf{v}) \mapsto x$. For each $x \in \mathcal{O}_\nu$, put $h_\lambda^{-1}(x) = X_{\lambda,x}$. The proposition certainly holds if $r = 2$ by [AH, Prop. 2.8]. We show the proposition, together with the statement (5.4.2), by induction on r .

(5.4.2) $X_{\lambda,x}$ is a smooth and irreducible variety with

$$\dim X_{\lambda,x} = \sum_{i=1}^{r-1} (r-i)|\lambda^{(i)}|.$$

We assume that (5.4.2) holds for \mathcal{X}_{uni} with smaller r . For a fixed $(x, v_1) \in \mathcal{O}$, we consider the variety $\mathcal{X}'_{\text{uni}}$ as in 5.3. Let $\lambda' = (\lambda^{(2)}, \dots, \lambda^{(r)})$. Then by the discussion in 5.3, we see that $f_{\lambda'}^{-1}(x, v_1) \simeq X'_{\lambda',\bar{x}} \times W^{r-2}$. $Z_G(x, v_1)$ stabilizes the subspace W and so acts on $X'_{\lambda',\bar{x}}$. We have

$$X_{\lambda} \simeq G \times^{Z_G(x, v_1)} (X'_{\lambda',\bar{x}} \times W^{r-2}). \tag{5.4.3}$$

By induction, $X'_{\lambda',\bar{x}}$ is smooth and irreducible. Hence X_{λ} is smooth and irreducible. Since

$$X_{\lambda,x} \simeq Z_G(x) \times^{Z_G(x, v_1)} (X'_{\lambda',\bar{x}} \times W^{r-2}), \tag{5.4.4}$$

and $Z_G(x)$ is connected, $X_{\lambda,x}$ is also smooth and irreducible. This proves the first statement of (5.4.1) and (5.4.2). We shall compute $\dim X_{\lambda}$ and $\dim X_{\lambda,x}$. By (5.4.3), we have

$$\dim X_{\lambda} = \dim \mathcal{O}_{(\lambda^{(1)}, \nu')} + \dim X'_{\lambda',\bar{x}} + (r-2)|\lambda^{(1)}|. \tag{5.4.5}$$

Here $\dim \mathcal{O}_{(\lambda^{(1)}, \nu')} = n^2 - n - 2n(\nu) + |\lambda^{(1)}|$ by [AH, Prop. 2.8]. By applying (5.4.2) to $X'_{\lambda',\bar{x}}$, we have

$$\dim X'_{\lambda',\bar{x}} = \sum_{i=2}^{r-1} (r-i)|\lambda^{(i)}|.$$

By substituting those formulas to (5.4.5), we obtain (5.4.1) (note that $n(\nu) = n(\lambda)$).

By comparing (5.4.3) and (5.4.4), we have

$$\begin{aligned} \dim X_{\lambda,x} &= \dim X_{\lambda} - (\dim G - \dim Z_G(x)) \\ &= \dim X_{\lambda} - \dim \mathcal{O}_{\nu}. \end{aligned}$$

Since $\dim \mathcal{O}_{\nu} = n^2 - n - 2n(\nu)$, (5.4.2) follows from (5.4.1). This proves the proposition. \square

5.5. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}(\mathbf{m})$. We write the dual partition $(\lambda^{(i)})^*$ of $\lambda^{(i)}$ as $(\mu_1^{(i)} \leq \mu_2^{(i)} \leq \dots \leq \mu_{\ell_i}^{(i)})$, in the increasing order, where $\ell_i = \lambda_1^{(i)}$. For each $1 \leq i \leq r, 1 \leq j < \ell_i$, we define an integer $n(i, j)$ by

$$n(i, j) = (|\lambda^{(1)}| + \dots + |\lambda^{(i-1)}|) + \mu_1^{(i)} + \dots + \mu_j^{(i)}.$$

Let $P = P_{\lambda}$ be the stabilizer of a partial flag $(M_{n(i,j)})$ in G , and U_P the unipotent radical of P . In particular, P stabilizes the subspaces M_{p_i} . Let us define a variety \tilde{X}_{λ} by

$$\tilde{X}_{\lambda} = \left\{ (x, \mathbf{v}, gP) \in G_{\text{uni}} \times V^{r-1} \times G/P \mid g^{-1}xg \in U_P, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}.$$

We define a map $\pi_{\lambda} : \tilde{X}_{\lambda} \rightarrow \mathcal{X}_{\text{uni}}$ by $(x, \mathbf{v}, gP) \mapsto (x, \mathbf{v})$. Then π_{λ} is a proper map. Since $\tilde{X}_{\lambda} \simeq G \times^P (U_P \times \prod_i M_{p_i})$, \tilde{X}_{λ} is smooth and irreducible.

We have the following lemma.

Lemma 5.6. *Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$.*

- (i) $\dim \tilde{X}_\lambda = \dim X_\lambda$.
- (ii) $\text{Im } \pi_\lambda = \overline{X}_\lambda$, where \overline{X}_λ is the closure of X_λ in \mathcal{X}_{uni} .

Proof. We have

$$\begin{aligned} \dim \tilde{X}_\lambda &= \dim G/P + \dim U_P + \dim \prod_{i=1}^{r-1} M_{p_i} \\ &= 2 \dim U_P + \sum_{i=1}^{r-1} (r-i)|\lambda^{(i)}|. \end{aligned}$$

Here $\dim U_P = (\dim G - \dim L)/2$, where L is a Levi subgroup of P , and

$$\dim L = \sum_{i,j} (\mu_j^{(i)})^2 = n + 2n(\lambda).$$

The second equality follows from the formula $n(\lambda^*) = \sum_i \lambda_i(\lambda_i - 1)/2$ for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and its dual λ^* . By comparing this with Proposition 5.4, we obtain (i).

We show (ii). Take $(x, \mathbf{v}) \in X_\lambda$. By the construction of X_λ in 5.3, one can find a sequence of x -stable subspaces $V_{p_1} \subset V_{p_2} \subset \dots \subset V_{p_r} = V$ of V such that $v_i \in V_{p_i}$ and that the restriction of x on $V_{p_i}/V_{p_{i-1}}$ has type $\lambda^{(i)}$. It is well known that there exists an x -stable flag $(V_{n(i,j)})$ which is a refinement of (V_{p_i}) such that $V_{n(i,j)} = gM_{n(i,j)}$ for some $g \in G$ and that $g^{-1}xg \in U_P$ (see [AH, Prop. 3.3 (4)]). It follows that $(x, \mathbf{v}, gP) \in \tilde{X}_\lambda$, and we see that $(x, \mathbf{v}) \in \text{Im } \pi_\lambda$. This proves that $X_\lambda \subset \text{Im } \pi_\lambda$. Since π_λ is proper, $\text{Im } \pi_\lambda$ is a closed subset of \mathcal{X}_{uni} and we have $\overline{X}_\lambda \subset \text{Im } \pi_\lambda$. Since \overline{X}_λ and $\text{Im } \pi_\lambda$ are both irreducible, (i) implies that $\text{Im } \pi_\lambda = \overline{X}_\lambda$. Hence (ii) holds. The lemma is proved. \square

Remark 5.7. In the case where $r = 2$, Achar-Henderson proved in [AH, Prop. 3.3] that the map $\pi_\lambda : \tilde{X}_\lambda \rightarrow \overline{X}_\lambda$ is a resolution of singularities for \overline{X}_λ . By a similar argument, by using Lemma 5.6 (ii), one can prove that π_λ gives a resolution of singularities for \overline{X}_λ for any $r \geq 2$.

5.8. Recall the map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ in 5.2 for $\mathbf{m} = (m_1, \dots, m_r) \in \mathcal{Q}_{n,r}$. Let us define $\lambda(\mathbf{m}) = (\lambda^{(1)}, \dots, \lambda^{(r)})$ by the condition that $\lambda^{(i)} = (m_i)$ for each i . We consider the varieties \tilde{X}_λ in 5.5 for $\lambda = \lambda(\mathbf{m})$. In this case, $P = B$. \tilde{X}_λ is isomorphic to $\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}$, and π_λ is identified with $\pi_1^{(\mathbf{m})}$. We have the following result.

Proposition 5.9. *For $\mathbf{m} \in \mathcal{Q}_{n,r}$, we have*

- (i) $\mathcal{X}_{\mathbf{m},\text{uni}} = \overline{X}_{\lambda(\mathbf{m})}$.
- (ii) $\dim \mathcal{X}_{\mathbf{m},\text{uni}} = n^2 - n + \sum_{i=1}^{r-1} (r-i)m_i$.
- (iii) For $\mu \in \mathcal{P}(\mathbf{m})$, $X_\mu \subset \mathcal{X}_{\mathbf{m},\text{uni}}$.

Proof. (i) is a direct consequence of Lemma 5.6 (ii) in view of 5.8. (ii) follows from (i) and Proposition 5.4. We show (iii). Take $(x, \mathbf{v}) \in X_\mu$. As in the proof of Lemma 5.6, there exists an x -stable partial flag $(V_{n(i,j)})$ with respect to μ . By our assumption, $v_i \in V_{p_i}$ for each i . Then one can find an x -stable total flag (V_i) as a refinement of this $(V_{n(i,j)})$. This shows that (x, \mathbf{v}) is contained in $\text{Im } \pi_1^{(\mathbf{m})} = \mathcal{X}_{\mathbf{m}, \text{uni}}$. Hence $X_\mu \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$ as asserted. \square

5.10. In the case where $r = 2$, there exists a normal basis for $(x, v) \in G_{\text{uni}} \times V$ (cf. [AH, 2.2], [T]). If $r \geq 3$, one cannot expect such a basis since \mathcal{X}_{uni} has infinitely many G -orbits. However, one can find typical elements in X_λ as follows: Put $\nu = (\nu_1, \dots, \nu_\ell) \in \mathcal{P}_n$ for $\nu = \lambda^{(1)} + \dots + \lambda^{(r)}$. Take $x \in G_{\text{uni}}$ of Jordan type ν , and let $\{u_{j,k} \mid 1 \leq j \leq \ell, 1 \leq k \leq \nu_j\}$ be a Jordan basis of x in V having the property $(x - 1)u_{j,k} = u_{j,k-1}$ with the convention that $u_{j,0} = 0$. We define $v_i \in V$ for $i = 1, \dots, r - 1$ by the condition that

$$v_i = \sum_{1 \leq j \leq \ell} u_{j, \lambda_j^{(1)} + \dots + \lambda_j^{(i)}} \tag{5.10.1}$$

and put $\mathbf{v} = (v_1, \dots, v_{r-1})$. Let W_i be the subspace of V spanned by the basis

$$\{u_{j,k} \mid 1 \leq j \leq \ell, 1 \leq k \leq \lambda_j^{(1)} + \dots + \lambda_j^{(i)}\}.$$

Then $W_1 \subset W_2 \subset \dots \subset W_{r-1} \subset V$, and W_i is an x -stable subspace of V such that $v_i \in W_i$. Let \bar{x} be the restriction of x on $\bar{V} = V/W_{i-1}$, and \bar{v}_i be the image of v_i on \bar{V} . Put $\bar{G} = \text{GL}(\bar{V})$. Then $E^{\bar{x}}\bar{v}_i$ coincides with W_i/W_{i-1} , and the restriction of x on W_i/W_{i-1} has type $\lambda^{(i)}$. It follows that $(x, \mathbf{v}) \in X_\lambda$. Such an element $(x, \mathbf{v}) \in X_\lambda$ is called a standard element. More generally, we consider an element $\mathbf{w} = (w_1, \dots, w_{r-1})$ of the form

$$w_i = \sum_{j=1}^{\ell} \lambda_j^{(1) + \dots + \lambda_j^{(i)}} \sum_{k=1}^{\nu_j} a_{j,k}^i u_{j,k} \tag{5.10.2}$$

with $a_{j,k}^i \in \mathbf{k}$ such that $a_{j, \lambda_j^{(1)} + \dots + \lambda_j^{(i)}}^i \neq 0$. Then $w_i \in W_i$, and $E^{\bar{x}}\bar{w}_i$ coincides with W_i/W_{i-1} . Hence $(x, \mathbf{w}) \in X_\lambda$. Here (x, w_i) is conjugate to an element (x, v_i) as above under the group $Z_G(x)$ for each i . We call (x, \mathbf{w}) a semi-standard element. We define a set X_λ^0 as the set of all G -conjugates of (x, \mathbf{w}) . Hence X_λ^0 is a G -stable subset of X_λ . We note that

(5.10.3) X_λ^0 is a G -stable open dense subset of X_λ .

In fact, take $(x, \mathbf{v}) \in X_\lambda^0$. Under the notation in the proof of Proposition 5.4, put $X_{\lambda,x}^0 = X_\lambda^0 \cap X_{\lambda,x}$. We show simultaneously that $X_{\lambda,x}^0$ is open dense in $X_{\lambda,x}$. Under the isomorphism in (5.4.4), we have

$$X_{\lambda,x}^0 \simeq Z_G(x) \times^{Z_G(x, v_1)} (X_{\lambda', \bar{x}}'^0 \times (W^0)^{r-2}),$$

where $X_{\lambda', \bar{x}}^0$ is a similar variety as $X_{\lambda, x}^0$ defined for $X_{\lambda'}$, and W^0 is an open dense subset of W . Hence by induction on r , $X_{\lambda, x}^0$ is an open dense subset of $X_{\lambda, x}$. Since, under the isomorphism in (5.4.3), X_{λ}^0 can be written as

$$X_{\lambda}^0 \simeq G \times^{Z_G(x, v_1)} (X_{\lambda', \bar{x}}^0 \times (W^0)^{r-2}),$$

X_{λ}^0 is open dense in X_{λ} . Hence (5.10.3) holds.

Concerning the closure \overline{X}_{λ} , we have the following result.

Proposition 5.11. *For each $\lambda \in \mathcal{P}_{n,r}$, we have*

$$\overline{X}_{\lambda} \subset \bigcup_{\mu \leq \lambda} X_{\mu}.$$

For the proof of the proposition, we need a lemma.

Lemma 5.12. *Assume that $(x, \mathbf{v}) \in X_{\lambda}$ is a semi-standard element with $\mathbf{v} = (v_1, \dots, v_{r-1})$. Let U_i be the $\mathbf{k}[x]$ -submodule of V generated by v_i . Then $\dim U_i = \lambda_1^{(1)} + \dots + \lambda_1^{(i)}$. The Jordan type of the restriction of x on V/U_i is $\xi = (\xi_1, \dots, \xi_{\ell})$, where*

$$\xi_j = \lambda_j^{(i+1)} + \dots + \lambda_j^{(r)} + \lambda_{j+1}^{(1)} + \dots + \lambda_{j+1}^{(i)} \quad \text{for } j = 1, \dots, \ell.$$

Proof. In the case where $r = 2$, this result was proved in Lemma 2.5 in [AH]. The general case is reduced to the case where $r = 2$, by considering the double partition $(\mu; \mu')$ with $\mu = \lambda^{(1)} + \dots + \lambda^{(i)}$, $\mu' = \lambda^{(i+1)} + \dots + \lambda^{(r)}$. \square

5.13. We prove the proposition following the strategy in the proof of (a part of) [AH, Thm. 3.9]. We show that $\mu \leq \lambda$ if $(x', \mathbf{v}') \in \overline{X}_{\lambda}$ is of type μ . For this, it is enough to show that

$$\begin{aligned} \sum_{i=1}^k (\mu_i^{(1)} + \dots + \mu_i^{(r)}) + (\mu_{k+1}^{(1)} + \dots + \mu_{k+1}^{(a)}) \\ \leq \sum_{i=1}^k (\lambda_i^{(1)} + \dots + \lambda_i^{(r)}) + (\lambda_{k+1}^{(1)} + \dots + \lambda_{k+1}^{(a)}) \end{aligned} \tag{5.13.1}$$

for $k = 0, \dots, \ell$ and $a = 0, \dots, r - 1$ (we put $\lambda_j^{(0)} = \mu_j^{(0)} = 0$ by convention). Put $\nu = \lambda^{(1)} + \dots + \lambda^{(r)}$ and $\nu' = \mu^{(1)} + \dots + \mu^{(r)}$. Then the Jordan type of x' is ν' . Let \mathcal{O}_{ν} be the G -orbit in G_{uni} corresponding to $\nu \in \mathcal{P}_n$. Since $\overline{X}_{\lambda} \subset \mathcal{O}_{\nu} \times V^{r-1}$, we have $\overline{X}_{\lambda} \subseteq \overline{\mathcal{O}}_{\nu} \times V^{r-1}$. Since $(x', \mathbf{v}') \in \overline{X}_{\lambda}$, we have $x' \in \overline{\mathcal{O}}_{\nu}$ and so $\nu' \leq \nu$. This proves (5.13.1) in the case where $a = 0$. By a similar argument as in the proof of [AH, Thm. 3.9], it follows from Lemma 5.12 that

$$\sum_{i=1}^k (\lambda_i^{(1)} + \dots + \lambda_i^{(r)}) + (\lambda_{k+1}^{(1)} + \dots + \lambda_{k+1}^{(a)}) \tag{5.13.2}$$

is the maximal possible dimension of the x -stable subspace $\mathbf{k}[x]\langle v_a, u_1, \dots, u_k \rangle$ of V , a $\mathbf{k}[x]$ -module generated by v_a, u_1, \dots, u_k , where (x, \mathbf{v}) is a semi-standard element in X_λ with $\mathbf{v} = (v_1, \dots, v_{r-1})$, and u_1, \dots, u_k run over the elements in V . We note that the condition

$$\dim \mathbf{k}[x]\langle v_a, u_1, \dots, u_k \rangle \leq N_a, \quad (u_1, \dots, u_k \in V, 1 \leq a \leq r-1) \tag{5.13.3}$$

for given $\{N_a \mid 1 \leq a \leq r-1\}$ is a closed condition on $(x, \mathbf{v}) \in \mathcal{X}_{\text{uni}}$ (i.e., the set of (x, \mathbf{v}) satisfying the above condition is a closed subset of \mathcal{X}_{uni}) since the condition for a fixed a is a closed condition on \mathcal{X}_{uni} by [loc. cit.]. Since $\overline{X}_\lambda^0 = \overline{X}_\lambda$ by (5.10.3), for any element $(x', \mathbf{v}') \in \overline{X}_\lambda$, the dimension of $\mathbf{k}[x']\langle v'_a, u_1, \dots, u_k \rangle$ is dominated by the number in (5.13.2). The assertion (5.13.1) then follows from this. Hence Proposition 5.11 holds.

Combining with the previous results, we have the following.

Proposition 5.14. *X_λ is open dense in \overline{X}_λ . Hence X_λ is a G -stable locally closed, smooth, irreducible subvariety of \mathcal{X}_{uni} .*

Proof. Since $X_\lambda = \overline{X}_\lambda \setminus \bigcup_{\mu < \lambda} \overline{X}_\mu$ by Proposition 5.11, X_λ is open in \overline{X}_λ . X_λ is smooth and irreducible by Proposition 5.4. \square

5.15. We give here some examples on the closure relation of X_λ . First assume that $n = 1$ and r is arbitrary. Then $G = \text{GL}_1 \simeq \mathbf{k}^*$, the multiplicative group of \mathbf{k} . We have $G_{\text{uni}} = \{1\}$, and $\mathcal{X}_{\text{uni}} \simeq \mathbf{k}^{r-1}$. $\mathcal{P}_{1,r}$ consists of $\{\lambda_1, \dots, \lambda_r\}$, where $\lambda_i = (\lambda^{(1)}, \dots, \lambda^{(r)})$ with $\lambda^{(r+1-i)} = (1)$ and $\lambda^{(j)} = \emptyset$ for $j \neq r+1-i$. Thus the dominance order in $\mathcal{P}_{1,r}$ is given as $\lambda_1 < \lambda_2 < \dots < \lambda_r$. Under the identification $\mathcal{X}_{\text{uni}} \simeq \mathbf{k}^{r-1}$, we have

$$\begin{aligned} X_{\lambda_i} &= \{(v_1, \dots, v_{r-1}) \in \mathbf{k}^{r-1} \mid v_1 = \dots = v_{r-i} = 0, v_{r-i+1} \in \mathbf{k}^*\} \\ &\simeq \begin{cases} \mathbf{k}^{i-2} \times \mathbf{k}^* & \text{if } i \geq 2, \\ \{0\} & \text{if } i = 1. \end{cases} \end{aligned}$$

Thus X_{λ_i} is a locally closed, smooth irreducible subvariety of \mathbf{k}^{r-1} . Since $G \simeq \mathbf{k}^*$, $X_{\lambda_1} = \{0\}$ and $X_{\lambda_2} \simeq \mathbf{k}^*$ are single G -orbits, but other X_{λ_i} are a union of infinitely many G -orbits. Moreover $\overline{X}_{\lambda_i} \simeq \mathbf{k}^{i-1}$, and the closure relation is given as

$$\overline{X}_{\lambda_i} = \bigcup_{j \leq i} X_{\lambda_j}. \tag{5.15.1}$$

This is an analogue of the result in the case where $r = 2$, which asserts that \overline{X}_λ is a union of X_μ for $\mu \leq \lambda$ ([AH, Thm. 3.9]).

However, such a relation does not hold in general for $r \geq 3$. In fact, there exists $\mu < \lambda$ such that $X_\mu \cap \overline{X}_\lambda \neq \emptyset$ and that $X_\mu \not\subset \overline{X}_\lambda$, as the following example shows. We consider the case where $n = 2, r = 3$, hence $\dim V = 2$. Take $\lambda = (1; -, 1)$ and $\mu = (1^2; -, -)$. Then we have $\mu < \lambda$. It is easy to check that

$$\begin{aligned} X_\lambda &= \{(x, (v_1, v_2)) \in G_{\text{uni}} \times V^2 \mid x \neq 1, v_1 \in \text{Ker}(x-1) \setminus \{0\}, v_2 \in \text{Ker}(x-1)\}, \\ X_\mu &= \{(x, (v_1, v_2)) \in G_{\text{uni}} \times V^2 \mid x = 1, v_1 \in V \setminus \{0\}, v_2 \in V\}. \end{aligned}$$

Then for any element $(x, \mathbf{v}) \in X_\lambda$, we have $v_2 \in \mathbf{k}v_1$. It follows that $v_2 \in \mathbf{k}v_1$ for $(x, \mathbf{v}) \in \overline{X}_\lambda$ whenever $v_1 \neq 0$, and we have $X_\mu \cap \overline{X}_\lambda = \{(x, v_1, v_2) \mid x = 1, v_1 \neq 0, v_2 \in \mathbf{k}v_1\}$. In particular, $X_\mu \cap \overline{X}_\lambda \neq \emptyset$ and $X_\mu \not\subset \overline{X}_\lambda$.

The following result is an analogue of [AH, Cor. 3.4].

Proposition 5.16. *Let $(x, \mathbf{v}) \in \mathcal{X}_{\text{uni}}$, and $\lambda \in \mathcal{P}(\mathbf{m})$ for some \mathbf{m} . Then $(x, \mathbf{v}) \in \overline{X}_\lambda$ if and only if there exist $\xi \in \mathcal{P}(\mathbf{m})$ and a flag $(W_i)_{1 \leq i \leq r}$ in V such that $\dim W_i = p_i$ satisfying the following conditions:*

- (i) (W_i) is x -stable,
- (ii) $x|_{W_i/W_{i-1}}$ has type $\xi^{(i)}$ with $\xi^{(i)} \leq \lambda^{(i)}$ for $1 \leq i \leq r$,
- (iii) $v_i \in W_i$ for $1 \leq i \leq r - 1$.

Proof. Let \mathcal{U} be the set of (x, \mathbf{v}) satisfying the condition in the proposition. We show that $\mathcal{U} = \overline{X}_\lambda$. Take $(x, \mathbf{v}) \in \mathcal{U}$. In order to show that $(x, \mathbf{v}) \in \overline{X}_\lambda$, by Lemma 5.6, it is enough to construct $(x, \mathbf{v}, gP_\lambda) \in \widetilde{X}_\lambda$. Under the notation in 5.5, put $V_{n(i,0)} = W_i$ for $i = 1, \dots, r - 1$. By assumption, the restriction of x on W_i/W_{i-1} has type $\xi^{(i)}$. Since $\xi^{(i)} \leq \lambda^{(i)}$, there exists an x -stable flag

$$W_{i-1} = V_{n(i-1,0)} \subset V_{n(i-1,1)} \subset \dots \subset V_{n(i,0)} = W_i.$$

such that $(x - 1)V_{n(i,j)} \subset V_{n(i,j-1)}$. In fact this is an application of Lemma 5.6 for $r = 1$, which is a well-known result for GL_n . There exists $g \in G$ such that $g(M_{n(i,j)}) = (V_{n(i,j)})$, and $(x, \mathbf{v}, gP_\lambda) \in \widetilde{X}_\lambda$. Hence $(x, \mathbf{v}) \in \overline{X}_\lambda$, and we have $\mathcal{U} \subset \overline{X}_\lambda$.

Let P be a parabolic subgroup of G which is the stabilizer of a partial flag $(M_{p_i})_{1 \leq i \leq r}$, and L the Levi subgroup of P containing T . Thus $L \simeq \prod_i \text{GL}_{m_i}$. Let \mathcal{O}'_λ be the L -orbit in L_{uni} corresponding to $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$, and $\overline{\mathcal{O}'_\lambda}$ the closure of \mathcal{O}'_λ in L_{uni} . Let $p : P \rightarrow L$ be the natural projection. We consider a variety

$$\widetilde{\mathcal{P}}_{\mathcal{O}'_\lambda} = \{(x, \mathbf{v}, gP) \in G_{\text{uni}} \times V^{r-1} \times G/P \mid g^{-1}xg \in p^{-1}(\overline{\mathcal{O}'_\lambda}), g^{-1}v_i \in M_{p_i}\},$$

and let $f : \widetilde{\mathcal{P}}_{\mathcal{O}'_\lambda} \rightarrow G_{\text{uni}} \times V^{r-1}$ be the projection on the first two factors. Then $\text{Im } f = \mathcal{U}$. Since f is proper, \mathcal{U} is a closed subvariety of $G_{\text{uni}} \times V^{r-1}$. The construction of X_λ implies that $X_\lambda \subset \mathcal{U}$. It follows that $\overline{X}_\lambda \subset \mathcal{U}$, and so $\mathcal{U} = \overline{X}_\lambda$. The proposition is proved. \square

5.17. Take $\mathbf{m} \in \mathcal{Q}_{n,r}$. For each $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m},\text{uni}}$, put

$$\mathcal{B}_z^{(\mathbf{m})} = \left\{ gB \in G/B \mid g^{-1}xg \in U, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}.$$

$\mathcal{B}_z^{(\mathbf{m})}$ is a closed subvariety of $\mathcal{B} = G/B$, which is isomorphic to the fibre $(\pi_1^{(\mathbf{m})})^{-1}(z)$ and is called the Springer fibre of z . In the case where $r = 2$, $\mathcal{B}_z^{(\mathbf{m})}$ is isomorphic to each other for any $z \in X_\lambda$ since X_λ is a single G -orbit. However, this property does not hold in general for $r \geq 3$ as the next examples show. First assume that

$r = 3$ and $n = 3$. Let $\mathbf{m} = (2, 0, 1)$ and $\boldsymbol{\lambda} = ((1^3), -, -)$. Let v_1, v_2 be linearly independent vectors in V , and take $z = (x, (v_1, v_2)), z' = (x, (v_1, 0))$ with $x = 1$. Then $z, z' \in X_\lambda$. Let $\mathcal{F}(V)$ be the set of total flags (V_i) in V . We have

$$\begin{aligned} \mathcal{B}_z^{(\mathbf{m})} &\simeq \{(V_i) \in \mathcal{F}(V) \mid V_2 = \langle v_1, v_2 \rangle\}, \\ \mathcal{B}_{z'}^{(\mathbf{m})} &\simeq \{(V_i) \in \mathcal{F}(V) \mid v_1 \in V_2\}. \end{aligned}$$

Hence $\dim \mathcal{B}_z^{(\mathbf{m})} = 1, \dim \mathcal{B}_{z'}^{(\mathbf{m})} = 2$, and in particular, $z, z' \in X_\lambda \cap \mathcal{X}_{\mathbf{m}, \text{uni}}$. We have $\dim \mathcal{B}_z^{(\mathbf{m})} \neq \dim \mathcal{B}_{z'}^{(\mathbf{m})}$.

Next assume that $r = 3$ and $n = 4$. Let $\mathbf{m} = (2, 0, 2)$ and $\boldsymbol{\lambda} = ((1^2), -, (1^2))$. We have $X_\lambda \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$. Let x be an element in G_{uni} of type (2^2) , and $\{u_{i,j} \mid 1 \leq i \leq 2, 1 \leq j \leq 2\}$ be a Jordan basis of x in V . Put $v_1 = u_{1,1}, v_2 = u_{2,1}$ so that v_1, v_2 is a basis of $W = \text{Ker}(x - 1)$. Put $z = (x, (v_1, v_2))$ and $z' = (x, (v_1, 0))$. Then $z, z' \in X_\lambda$. We have

$$\begin{aligned} \mathcal{B}_z^{(\mathbf{m})} &\simeq \{(V_i) \in \mathcal{F}_x(V) \mid V_2 = W\}, \\ \mathcal{B}_{z'}^{(\mathbf{m})} &\simeq \{(V_i) \in \mathcal{F}_x(V) \mid v_1 \in V_2\}, \end{aligned}$$

where $\mathcal{F}_x(V)$ is the set of x -stable flags in V . Since $x|_W = 1, x_{V/W} = 1$, we see that $\mathcal{B}_z^{(\mathbf{m})} \simeq \mathcal{F}(W) \times \mathcal{F}(V/W)$. Thus $\mathcal{B}_z^{(\mathbf{m})}$ is irreducible with $\dim \mathcal{B}_z^{(\mathbf{m})} = 2$. On the other hand, if $(V_i) \in \mathcal{B}_{z'}^{(\mathbf{m})}$, either $V_2 = W$ or V_2 is of the form $W_\alpha = \langle u_{1,2} + \alpha v_2, v_1 \rangle$ for $\alpha \in \mathbf{k}$. Since $x|_{W_\alpha}$ and $x|_{V/W_\alpha}$ have both type (2) , W_α determines a unique $(V_i) \in \mathcal{B}_{z'}^{(\mathbf{m})}$. It follows that $\mathcal{B}_{z'}^{(\mathbf{m})} = \mathcal{B}_z^{(\mathbf{m})} \amalg Y$ with $Y = \{(V_i) \mid V_2 = W_\alpha (\alpha \in \mathbf{k})\}$, where Y is irreducible with $\dim Y = 1$. Hence in this case, $\dim \mathcal{B}_z^{(\mathbf{m})} = \dim \mathcal{B}_{z'}^{(\mathbf{m})}$, but $\mathcal{B}_z^{(\mathbf{m})} \not\simeq \mathcal{B}_{z'}^{(\mathbf{m})}$.

6. Unipotent variety of exotic type

The ‘‘unipotent part’’ of the exotic space is called the unipotent variety of exotic type. In this section, first we study the unipotent variety of exotic type in 6.1–6.13. While in 6.14–6.19, we discuss the case of enhanced type. After 6.20, we discuss both cases simultaneously.

6.1. We follow the notation in 1.2. Assume that \mathcal{X} is of exotic type. As in the enhanced case we define varieties, for each $\mathbf{m} \in \mathcal{Q}_{n,r}$,

$$\begin{aligned} \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} &= \left\{ (x, \mathbf{v}, gB^\theta) \in G_{\text{uni}}^{\iota\theta} \times V^{r-1} \times H/B^\theta \mid g^{-1}xg \in U^{\iota\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}, \\ \mathcal{X}_{\mathbf{m}, \text{uni}} &= \bigcup_{g \in H} g \left(U^{\iota\theta} \times \prod_{i=1}^{r-1} M_{p_i} \right), \end{aligned}$$

where B, T are as in 1.2, and U is the unipotent radical of B . We define a map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} \rightarrow G_{\text{uni}}^{\iota\theta} \times V^{r-1}$ by $\pi_1^{(\mathbf{m})}(x, \mathbf{v}, gB^\theta) = (x, \mathbf{v})$. Clearly $\mathcal{X}_{\mathbf{m}, \text{uni}} =$

$\text{Im } \pi_1^{(\mathbf{m})}$. As in 1.2, in the case where $\mathbf{m} = (n, 0, \dots, 0)$, we write $\tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}}$, $\mathcal{X}_{\mathbf{m}, \text{uni}}$, $\pi_1^{(\mathbf{m})}$, etc., by $\tilde{\mathcal{X}}_{\text{uni}}$, \mathcal{X}_{uni} , π_1 , etc. Note that $\mathcal{X}_{\mathbf{m}, \text{uni}} \subset \mathcal{X}_{\text{uni}}$ for any \mathbf{m} , but in contrast to the enhanced case, \mathcal{X}_{uni} does not coincide with $G_{\text{uni}}^{u\theta} \times V^{r-1}$ if $r \geq 3$. In fact, \mathcal{X}_{uni} is described as a subset of $G_{\text{uni}}^{u\theta} \times V^{r-1}$ as follows.

$$\mathcal{X}_{\text{uni}} = \{ (x, \mathbf{v}) \in G_{\text{uni}}^{u\theta} \times V^{r-1} \mid \langle x^s v_i, v_j \rangle = 0 \text{ for all } s, i, j \}. \tag{6.1.1}$$

We show (6.1.1). By definition, \mathcal{X}_{uni} coincides with the subset of $G_{\text{uni}}^{u\theta} \times V^{r-1}$ consisting of (x, \mathbf{v}) such that v_1, \dots, v_{r-1} is contained in an x -stable maximal isotropic subspace of V . It is clear that \mathcal{X}_{uni} is contained in the right-hand side of (6.1.1). Conversely, assume that (x, \mathbf{v}) is contained in the right-hand side of (6.1.1). Then $W_0 = \mathbf{k}[x]v_1 + \dots + \mathbf{k}[x]v_{r-1}$ turns out to be an x -stable isotropic subspace of V . One can find an x -stable maximal isotropic subspace of V containing W_0 . Hence $(x, \mathbf{v}) \in \mathcal{X}_{\text{uni}}$. This proves (6.1.1). The variety \mathcal{X}_{uni} is called a unipotent variety of exotic type.

Since $\tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} \simeq H \times B^\theta (U^{l\theta} \times \prod_i M_{p_i})$, $\tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}}$ is smooth and irreducible. Moreover, we have

$$\begin{aligned} \dim \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} &= \dim H/B^\theta + \left(\dim U^{l\theta} + \dim \prod_{i=1}^{r-1} M_{p_i} \right) \\ &= \dim U^\theta + \dim U^{l\theta} + \sum_{i=1}^{r-1} p_i \\ &= 2n^2 - n + \sum_{i=1}^{r-1} (r-i)m_i \end{aligned} \tag{6.1.2}$$

since $\dim U^\theta + \dim U^{l\theta} = \dim U$. Since $\pi_1^{(\mathbf{m})}$ is proper, $\mathcal{X}_{\mathbf{m}, \text{uni}}$ is a closed irreducible subvariety of $G_{\text{uni}}^{u\theta} \times V^{r-1}$.

6.2. Assume that $r \geq 2$, and take $\mathbf{m} \in \mathcal{Q}_{n,r}$. Let P be the θ -stable parabolic subgroup of G such that P^θ is the stabilizer of the isotropic flag $(M_{p_i})_{1 \leq i \leq r-2}$ in V . Let L be the θ -stable Levi subgroup of P containing T and U_P the unipotent radical of P . Put $V'_L = M_{p_{r-2}}$, and $V_L = V/V'_L$. Then P^θ acts naturally on V_L , and we consider the map $\pi_P : P_{\text{uni}}^{u\theta} \times V \rightarrow L_{\text{uni}}^{u\theta} \times V_L$, $(x, v) \mapsto (x', v')$, where $x' = p(x)$ for the natural projection $p : P^{l\theta} \rightarrow L^{l\theta}$, and $v \mapsto v'$ is the projection $V \rightarrow V_L = V/V'_L$. The map π_P is P^θ -equivariant with respect to the diagonal action of P^θ on both varieties.

Let \mathcal{O}' be an L^θ -orbit in $L_{\text{uni}}^{u\theta} \times V_L$. We assume that \mathcal{O}' is P^θ -stable. Since $\pi_P^{-1}(\mathcal{O}') = \bigcup_{g \in P^\theta} g \cdot \pi_P^{-1}(z')$ for $z' \in \mathcal{O}'$, $\pi_P^{-1}(\mathcal{O}')$ is irreducible (note that $\pi_P^{-1}(z')$ is irreducible). Let \mathcal{O} be an H -orbit in $G_{\text{uni}}^{u\theta} \times V$ such that $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$ is open dense in $\pi_P^{-1}(\mathcal{O}')$. Put

$$\mathcal{U} = \{ (z, gP) \in (G_{\text{uni}}^{u\theta} \times V) \times H/P^\theta \mid g^{-1}z \in \pi_P^{-1}(\mathcal{O}') \}.$$

Then $\mathcal{U} \simeq H \times^{P^\theta} \pi_P^{-1}(\mathcal{O}')$ and so \mathcal{U} is an irreducible variety. Let $f : \mathcal{U} \rightarrow G_{\text{uni}}^{u\theta} \times V$ be the first projection, and put $\mathcal{U}_\mathcal{O} = f^{-1}(\mathcal{O})$. Then $\mathcal{U}_\mathcal{O} \simeq H \times^{P^\theta} (\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}'))$, and so $\mathcal{U}_\mathcal{O}$ is irreducible.

Take $z \in \mathcal{O}$, and consider a variety

$$\mathcal{P}_{z, \mathcal{O}'} = \{gP^\theta \in H/P^\theta \mid g^{-1}z \in \pi_P^{-1}(\mathcal{O}')\}.$$

Note that $\mathcal{P}_{z, \mathcal{O}'} \neq \emptyset$. We show the following proposition.

Proposition 6.3. *Under the setting in 6.2, the following hold.*

- (i) $\mathcal{P}_{z, \mathcal{O}'}$ consists of one point.
- (ii) $\dim Z_H(z) = \dim Z_{L^\theta}(z')$ for $z' \in \mathcal{O}'$.
- (iii) Let $z_1 \in \pi_P^{-1}(\mathcal{O}')$ be such that $\dim Z_H(z_1) = \dim Z_H(z)$. Then $z_1 \in \mathcal{O}$.
- (iv) Take $z \in \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$ and put $z' = \pi_P(z)$. Let $Q = Z_P(z')$ be a θ -stable subgroup of P . Then $\dim Z_{Q^\theta}(z) = \dim Z_H(z)$. In particular,

$$Z_H(z) = Z_{P^\theta}(z) = Z_{Q^\theta}(z).$$

- (v) H acts transitively on $\mathcal{U}_{\mathcal{O}}$.
- (vi) P^θ acts transitively on $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$, and Q^θ acts transitively on $\mathcal{O} \cap \pi_P^{-1}(z')$ under the setting in (iv).

Proof. First we show that

$$\dim \mathcal{P}_{z, \mathcal{O}'} = 0. \tag{6.3.1}$$

Replacing z and $\mathcal{P}_{z, \mathcal{O}'}$ by H -conjugate, if necessary, we may assume that $z \in \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$. Put $z' = (x', v') = \pi_P(z)$. Since $\pi_P^{-1}(z') = (x'U_P)^\theta \times (v' + V_L') \simeq U_P^\theta \times V_L'$, we have

$$\dim \pi_P^{-1}(z') = \dim U_P^\theta + \dim V_L' = \dim U_P^\theta. \tag{6.3.2}$$

Put $c = \dim \mathcal{O}$ and $c' = \dim \mathcal{O}'$. Then by [SS1, Prop. 5.7 (i)] (for a correction, see [SS2, Appendix]), we have $\dim(\mathcal{O} \cap \pi_P^{-1}(z')) \leq (c - c')/2$. Since $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$ is open dense in $\pi_P^{-1}(\mathcal{O}')$, $\mathcal{O} \cap \pi_P^{-1}(z')$ is open dense in $\pi_P^{-1}(z')$. It follows that

$$\dim U_P^\theta \leq (c - c')/2.$$

On the other hand, by [SS1, Prop. 5.7(ii)] together with [SS2, Appendix], we have

$$\begin{aligned} \dim \mathcal{P}_{z, \mathcal{O}'} &\leq (\nu_H - c/2) - (\nu_{L^\theta} - c'/2) \\ &= \dim U^\theta - \dim U_L^\theta - (c - c')/2 \\ &= \dim U_P^\theta - (c - c')/2, \end{aligned} \tag{6.3.3}$$

where $\nu_H = \dim U^\theta$, and similarly for ν_{L^θ} . Hence $\dim \mathcal{P}_{z, \mathcal{O}'} \leq 0$. As $\mathcal{P}_{z, \mathcal{O}'} \neq \emptyset$, we obtain (6.3.1).

Substituting this into (6.3.3), we have $c - c' = 2 \dim U_P^\theta$. This implies that $\dim Z_H(z) = \dim Z_{L^\theta}(z')$. Hence (ii) holds.

Under the setting in (iv), Q^θ stabilizes $\pi_P^{-1}(z')$, and so $\mathcal{O}_Q \subset \mathcal{O} \cap \pi_P^{-1}(z')$, where \mathcal{O}_Q is the Q^θ -orbit of z . Since \mathcal{O}' is an P^θ -orbit, we have $\dim Z_{P^\theta}(z') = \dim Z_{L^\theta}(z') + \dim U_P^\theta$. Hence by (6.3.2) we have

$$\dim U_P^\theta \geq \dim \mathcal{O}_Q = \dim Z_{L^\theta}(z') + \dim U_P^\theta - \dim Z_{Q^\theta}(z). \tag{6.3.4}$$

It follows that $\dim Z_{Q^\theta}(z) \geq \dim Z_{L^\theta}(z')$. Since $\dim Z_{Q^\theta}(z) \leq \dim Z_H(z)$, we have $\dim Z_{Q^\theta}(z) = \dim Z_H(z)$ by (6.3.4). Since $Z_H(z)$ is connected, this implies that $Z_H(z) = Z_{Q^\theta}(z)$. Hence $Z_H(z) = Z_{P^\theta}(z)$. Thus (iv) holds.

For any $z \in \mathcal{O}$, $\dim f^{-1}(z) = 0$ by (6.3.1). Thus $\dim \mathcal{U}_{\mathcal{O}} = \dim \mathcal{O}$. We show that H acts transitively on $\mathcal{U}_{\mathcal{O}}$. Take $\xi \in \mathcal{U}_{\mathcal{O}}$, and consider the H -orbit $H\xi$ of ξ . Since f is H -equivariant, f maps $H\xi$ onto \mathcal{O} . Hence $H\xi$ is irreducible with $\dim H\xi = \dim \mathcal{U}_{\mathcal{O}}$. In particular, $H\xi$ is open dense in $\mathcal{U}_{\mathcal{O}}$. If we take another $\xi' \in \mathcal{U}_{\mathcal{O}}$, $H\xi'$ is also open dense in $\mathcal{U}_{\mathcal{O}}$. Hence $H\xi \cap H\xi' \neq \emptyset$. Thus $\mathcal{U}_{\mathcal{O}} = H\xi$ and (v) holds.

Take $z, z_1 \in \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$. Since $(z, P^\theta), (z_1, P^\theta)$ are both contained in $\mathcal{U}_{\mathcal{O}}$, they are in the same H -orbit by (v). Hence there exists $g \in P^\theta$ such that $gz = z_1$. This proves the first statement of (vi). Then for $z_1, z_2 \in \mathcal{O} \cap \pi_P^{-1}(z')$, there exists $p \in P^\theta$ such that $pz_1 = z_2$. But since π_P is P^θ -equivariant, $p \in Z_{P^\theta}(z') = Q^\theta$. This proves the second statement of (vi).

We show (i). We may assume that $z \in \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$. Then $P^\theta \in \mathcal{P}_{z, \mathcal{O}'}$. Assume that $gP^\theta \in \mathcal{P}_{z, \mathcal{O}'}$. Then $(z, P^\theta), (z, gP^\theta)$ are both contained in $\mathcal{U}_{\mathcal{O}}$. By (v) they are conjugate under H . It follows that there exists $h \in Z_H(z)$ such that $gP^\theta = hP^\theta$. But by (iv), we have $h \in P^\theta$, and so $gP^\theta = P^\theta$. This proves (i).

Finally we show (iii). Let \mathcal{O}_1 be the H -orbit of z_1 . Then $\dim \mathcal{O}_1 = c$. By a similar argument as in the proof of (6.3.1), we have $\dim \mathcal{P}_{z_1, \mathcal{O}'} = 0$. Then a similar argument as in the proof of (iv) implies that $Z_H(z_1) = Z_{Q^\theta}(z_1)$. Hence $\dim Z_{Q^\theta}(z) = \dim Z_{Q^\theta}(z_1)$. This shows that the Q^θ -orbit of z_1 in $\pi_P^{-1}(z')$ has the same dimension as the Q^θ -orbit of z . Since both orbits are open dense in $\pi_P^{-1}(z')$, they have an intersection, and so $\mathcal{O} = \mathcal{O}_1$. This proves (iii). The proposition is proved. \square

6.4. Recall that in the enhanced case, the set of G -orbits in $G_{\text{uni}} \times V$ is parametrized by $\mathcal{P}_{n,2}$ if $\dim V = n$. In the exotic case, it is also known by [K], [AH] that the set of H -orbits in $G_{\text{uni}}^\theta \times V$ is parametrized by $\mathcal{P}_{n,2}$. By [AH, Th. 6.1], it is described as follows; we consider the embedding $G_{\text{uni}}^\theta \times V \hookrightarrow G_{\text{uni}} \times V$. The set of G -orbits in $G_{\text{uni}} \times V$ is parametrized by $\mathcal{P}_{2n,2}$. We denote by \mathbb{O}_ξ the G -orbit in $G \times V$ corresponding to $\xi \in \mathcal{P}_{2n,2}$. Then $\mathbb{O}_\xi \cap (G_{\text{uni}}^\theta \times V) \neq \emptyset$ only if ξ is of the form $\xi = (\lambda' \cup \lambda'', \lambda'' \cup \lambda')$ for some $\lambda = (\lambda', \lambda'') \in \mathcal{P}_{n,2}$, and in that case $\mathbb{O}_{(\lambda' \cup \lambda'', \lambda'' \cup \lambda')} \cap (G_{\text{uni}}^\theta \times V)$ turns out to be a single H -orbit, which we denote by \mathbb{O}_λ . This gives the required parametrization.

Let P be as in 6.2. For $\lambda \in \mathcal{P}(\mathbf{m})$, we define a subset \mathcal{M}_λ of $P_{\text{uni}}^\theta \times (\prod_{i=1}^{r-2} M_{p_i} \times M_{p_{r-2}}^\perp)$ as the set of (x, \mathbf{v}) satisfying the following properties. Take (x, \mathbf{v}) such that $x \in P_{\text{uni}}^\theta$ and that $v_i \in M_{p_i}$ for $i = 1, \dots, r-2$, $v_{r-1} \in M_{p_{r-2}}^\perp$. Put $\overline{M}_{p_i} = M_{p_i}/M_{p_{i-1}}$, and let $\overline{v}_i \in \overline{M}_{p_i}$ be the image of $v_i \in M_{p_i}$ for $i = 1, \dots, r-2$ (here we use the convention that $M_{p_0} = 0$). We also put $\overline{M}'_{p_{r-1}} = M_{p_{r-2}}^\perp/M_{p_{r-2}}$. Thus $\overline{M}'_{p_{r-1}}$ has a structure of a symplectic vector space, and one can consider an exotic symmetric space $\text{GL}(\overline{M}'_{p_{r-1}})^{\iota\theta} \times \overline{M}'_{p_{r-1}}$. We also consider enhanced spaces $\text{GL}(\overline{M}_{p_i}) \times \overline{M}_{p_i}$ for $i = 1, \dots, r-2$. Let \overline{v}_{r-1} be the image of v_{r-1} on $\overline{M}'_{p_{r-1}}$. We assume that the $\text{GL}(\overline{M}_{p_i})$ -orbit of $(x|_{\overline{M}_{p_i}}, \overline{v}_i)$ has type $(\lambda^{(i)}, \emptyset)$ for $i = 1, \dots, r-2$,

and that the $\mathrm{GL}(\overline{M}'_{p_{r-1}})^\theta$ -orbit of $(x|_{\overline{M}'_{p_{r-1}}}, \overline{v}_{r-1})$ has type $(\lambda^{(r-1)}, \lambda^{(r)})$. Let \mathcal{O} be the H -orbit of $z = (x, v_{r-1}) \in G_{\mathrm{uni}}^{\iota\theta} \times V$ and \mathcal{O}' the L^θ -orbit of $z' = \pi_P(z)$. Note that \mathcal{O}' is P^θ -stable. We further assume that

$$(6.4.1) \quad \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}') \text{ is open dense in } \pi_P^{-1}(\mathcal{O}').$$

We define X_λ by $X_\lambda = \bigcup_{g \in H} g\mathcal{M}_\lambda$. Note that in the case where $r = 2$, X_λ coincides with the H -orbit \mathcal{O}_λ in [SS1].

It follows from the construction that \mathcal{M}_λ is a P^θ -stable subset of $P^{\iota\theta} \times V^{r-1}$. The closure $\overline{\mathcal{M}}_\lambda$ of \mathcal{M}_λ in $P^{\iota\theta} \times V^{r-1}$ is also P^θ -stable. We define a variety $\widetilde{\mathcal{F}}_\lambda$ by $\widetilde{\mathcal{F}}_\lambda = H \times^{P^\theta} \overline{\mathcal{M}}_\lambda$. Let $\pi_\lambda : \widetilde{\mathcal{F}}_\lambda \rightarrow \mathcal{X}_{\mathrm{uni}}$ be the map induced from the map $H \times \overline{\mathcal{M}}_\lambda \rightarrow \mathcal{X}_{\mathrm{uni}}, (g, z) \mapsto g \cdot z$. Then π_λ is a proper map. We define a subset $\widetilde{\mathcal{F}}_\lambda^0$ of $\widetilde{\mathcal{F}}_\lambda$ by $\widetilde{\mathcal{F}}_\lambda^0 = H \times^{P^\theta} \mathcal{M}_\lambda$, and let π_λ^0 be the restriction of π_λ on $\widetilde{\mathcal{F}}_\lambda^0$. It is clear that π_λ^0 is a surjective map onto X_λ .

6.5. Take $x_i \in \mathrm{GL}(\overline{M}_{p_i})_{\mathrm{uni}}$ with Jordan type $\lambda^{(i)}$ for $1 \leq i \leq r-2$. Then the set of $\overline{v}_i \in \overline{M}_{p_i}$ such that (x_i, \overline{v}_i) is of type $(\lambda^{(i)}, \emptyset)$ is open dense in \overline{M}_{p_i} . Take $(x_{r-1}, \overline{v}_{r-1}) \in \mathrm{GL}(\overline{M}'_{p_{r-1}})^{\iota\theta} \times \overline{M}'_{p_{r-1}}$ with type $(\lambda^{(r-1)}, \lambda^{(r)})$. Now (x_1, \dots, x_{r-1}) determines a unique element in $L_{\mathrm{uni}}^{\iota\theta}$ which we denote by x' . Put $z' = (x', \overline{v}_{r-1})$. The set of $(x, v_{r-1}) \in G_{\mathrm{uni}}^{\iota\theta} \times V$ such that $(x, v_{r-1}) \in \pi_P^{-1}(z')$, satisfying the condition (6.4.1) is open dense in $\pi_P^{-1}(z')$. In fact, it coincides with $\mathcal{O} \cap \pi_P^{-1}(z')$ for an H -orbit \mathcal{O} . Put $\mathbf{v}' = (\overline{v}_1, \dots, \overline{v}_{r-1})$. For a fixed (x', \mathbf{v}') , let $\mathcal{M}_{(x', \mathbf{v}'')}$ be the set of (x, \mathbf{v}) with $\mathbf{v} = (v_1, \dots, v_{r-1})$ such that $(x, v_{r-1}) \in \mathcal{O} \cap \pi_P^{-1}(z')$ as in (6.4.1) and that the image of v_i on \overline{M}_{p_i} coincides with \overline{v}_i . As $\mathcal{M}_{(x', \mathbf{v}'')}$ is an open dense subset of $\prod_{i=1}^{r-3} M_{p_i} \times \pi_P^{-1}(z')$, we have

$$\dim \mathcal{M}_{(x', \mathbf{v}'')} = \sum_{i=1}^{r-3} \dim M_{p_i} + \dim \pi_P^{-1}(z').$$

It follows that $\mathcal{M}_{(x', \mathbf{v}'')}$ is smooth irreducible, and

$$\dim \mathcal{M}_{(x', \mathbf{v}'')} = \sum_{i=1}^{r-3} (r-i-2)m_i + \dim U_P^\theta. \tag{6.5.1}$$

(Note $\dim \pi_P^{-1}(z') = \dim U_P^\theta$ by (6.3.2).) Since

$$\mathcal{M}_\lambda = \bigcup_{g \in L^\theta} g\mathcal{M}_{(x', \mathbf{v}'')} \simeq L^\theta \times^{Z_{L^\theta}(x', \mathbf{v}'')} \mathcal{M}_{(x', \mathbf{v}'')}$$

\mathcal{M}_λ is smooth, irreducible, and open dense in $\overline{\mathcal{M}}_\lambda$. Moreover,

$$\dim \mathcal{M}_\lambda = \dim \mathcal{O}_{(x', \mathbf{v}'')} + \dim \mathcal{M}_{(x', \mathbf{v}'')}$$
(6.5.2)

where $\mathcal{O}_{(x', \mathbf{v}'')}$ is the L^θ -orbit of $(x', \mathbf{v}'') \in L_{\mathrm{uni}}^{\iota\theta} \times (\prod_{i=1}^{r-2} \overline{M}_{p_i} \times \overline{M}'_{p_{r-1}})$. We have $\dim \mathcal{O}_{(x', \mathbf{v}'')} = \sum_{i=1}^{r-1} \dim \mathcal{O}'_i$, where \mathcal{O}'_i is the $\mathrm{GL}(\overline{M}_{p_i})$ -orbit of (x_i, \overline{v}_i) for $i =$

$1, \dots, r - 2$, and \mathcal{O}'_{r-1} is the $\mathrm{GL}(\overline{M}'_{p_{r-1}})^\theta$ -orbit of $(x_{r-1}, \overline{v}_{r-1})$. Hence by [AH, Prop. 2.8] and [SS1, Lem. 2.3],

$$\dim \mathcal{O}'_i = \begin{cases} m_i^2 - 2n(\lambda^{(i)}) & \text{for } 1 \leq i \leq r - 2, \\ 2m_{r-1}^{\prime 2} - 2m'_{r-1} - 4n(\lambda^{(r-1)} + \lambda^{(r)}) + 2|\lambda^{(r-1)}| & \text{for } i = r - 1, \end{cases}$$

where $m'_{r-1} = m_{r-1} + m_r$. This implies that

$$\dim \mathcal{O}_{(x', v')} = \sum_{i=1}^{r-2} m_i^2 + 2m_{r-1}^{\prime 2} - 2m'_{r-1} - 2n(\boldsymbol{\lambda}) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + 2|\lambda^{(r-1)}|. \tag{6.5.3}$$

We have the following lemma.

Lemma 6.6. *$\widetilde{\mathcal{F}}_\lambda$ is an irreducible variety with*

$$\dim \widetilde{\mathcal{F}}_\lambda = 2n^2 - 2n - 2n(\boldsymbol{\lambda}) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + \sum_{i=1}^{r-1} (r - i + 1)|\lambda^{(i)}|.$$

$\widetilde{\mathcal{F}}_\lambda^0$ is a smooth and open dense subvariety of $\widetilde{\mathcal{F}}_\lambda$.

Proof. Since $\widetilde{\mathcal{F}}_\lambda = H \times^{P^\theta} \overline{\mathcal{M}}_\lambda$, and $\overline{\mathcal{M}}_\lambda$ is irreducible, $\widetilde{\mathcal{F}}_\lambda$ is irreducible. Since $\widetilde{\mathcal{F}}_\lambda^0 = H \times^{P^\theta} \mathcal{M}_\lambda$, and \mathcal{M}_λ is smooth and open dense in $\overline{\mathcal{M}}_\lambda$, $\widetilde{\mathcal{F}}_\lambda^0$ is smooth and open dense in $\widetilde{\mathcal{F}}_\lambda$. We have $\dim \widetilde{\mathcal{F}}_\lambda = \dim H/P^\theta + \dim \mathcal{M}_\lambda = \dim U_P^\theta + \dim \mathcal{M}_\lambda$. Here

$$\begin{aligned} 2 \dim U_P^\theta &= \dim H - \dim L^\theta \\ &= (2n^2 + n) - \left(\sum_{i=1}^{r-2} m_i^2 + 2m_{r-1}^{\prime 2} + m'_{r-1} \right). \end{aligned}$$

Then by (6.5.1)–(6.5.3), we see that

$$\begin{aligned} \dim \widetilde{\mathcal{F}}_\lambda &= 2n^2 + n - 3m'_{r-1} - 2n(\boldsymbol{\lambda}) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) \\ &\quad + \sum_{i=1}^{r-3} (r - i - 2)m_i + 2|\lambda^{(r-1)}|. \end{aligned}$$

The lemma follows from this if we note that $m_i = |\lambda^{(i)}|$ for $i = 1, \dots, r$ and $m'_{r-1} = m_{r-1} + m_r$. \square

Proposition 6.7. *Assume that $\lambda \in \mathcal{P}_{n,r}$. Let \overline{X}_λ be the closure of X_λ in $\mathcal{X}_{\mathrm{uni}}$.*

- (i) $\mathrm{Im} \pi_\lambda = \overline{X}_\lambda$.
- (ii) $\pi_\lambda^0 : \widetilde{\mathcal{F}}_\lambda^0 \rightarrow X_\lambda$ gives an isomorphism $\widetilde{\mathcal{F}}_\lambda^0 \xrightarrow{\sim} X_\lambda$.
- (iii) X_λ is smooth, irreducible, and

$$\dim X_\lambda = 2n^2 - 2n - 2n(\boldsymbol{\lambda}) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + \sum_{i=1}^{r-1} (r - i + 1)|\lambda^{(i)}|.$$

- (iv) X_λ is a locally closed subvariety of $\mathcal{X}_{\mathrm{uni}}$.

Proof. $\pi_\lambda^0 : \tilde{\mathcal{F}}_\lambda^0 \rightarrow X_\lambda$ is H -equivariant, and surjective. Take $(x, \mathbf{v}) \in \mathcal{M}_\lambda$ and put $z = (x, v_{r-1})$, $z' = \pi_P(z)$. Let \mathcal{O}' be the L^θ -orbit of z' . Then

$$(\pi_\lambda^0)^{-1}(x, \mathbf{v}) \simeq \{gP^\theta \in H/P^\theta \mid g^{-1}(x, \mathbf{v}) \in \mathcal{M}_\lambda\} \subset \mathcal{P}_{z, \mathcal{O}'},$$

where $\mathcal{P}_{z, \mathcal{O}'}$ is as in 6.2. By Proposition 6.3 (i), $\mathcal{P}_{z, \mathcal{O}'} = \{P^\theta\}$. By the definition of $\tilde{\mathcal{F}}_\lambda$, $(\pi_\lambda^0)^{-1}(x, \mathbf{v})$ contains P^θ . Hence we see that $(\pi_\lambda^0)^{-1}(x, \mathbf{v}) = \{P^\theta\}$. It follows that π_λ^0 is a bijective morphism. The correspondence $g(x, \mathbf{v}) \mapsto gP^\theta$ gives a well-defined morphism from the H -orbit of $(x, \mathbf{v}) \in \mathcal{M}_\lambda$ to H/P^θ . This induces a morphism from X_λ to $\tilde{\mathcal{F}}_\lambda^0$. It is easy to check that this gives an inverse of π_λ^0 . Hence $\tilde{\mathcal{F}}_\lambda^0 \simeq X_\lambda$ and (ii) holds. (iii) is immediate from (ii) and Lemma 6.6. By (ii) $\text{Im } \pi_\lambda$ contains X_λ . Since π_λ is proper, $\text{Im } \pi_\lambda$ is closed, hence $\text{Im } \pi_\lambda$ contains \overline{X}_λ . We have $\dim \tilde{\mathcal{F}}_\lambda = \dim X_\lambda$ by (iii), and $\text{Im } \pi_\lambda, X_\lambda$ are irreducible. Thus (i) holds. Here $\tilde{\mathcal{F}}_\lambda \setminus \tilde{\mathcal{F}}_\lambda^0$ is a closed subset of $\tilde{\mathcal{F}}_\lambda$, and its image by π_λ coincides with $\overline{X}_\lambda \setminus X_\lambda$. Hence $\overline{X}_\lambda \setminus X_\lambda$ is closed. This implies that X_λ is open dense in \overline{X}_λ , and so (iv) holds. The proposition is proved. \square

6.8 Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. We define a set $\tilde{\mathcal{P}}(\mathbf{m})$ by

$$\tilde{\mathcal{P}}(\mathbf{m}) = \coprod_{0 \leq k \leq m_{r-1}} \mathcal{P}(\mathbf{m}(k)) \tag{6.8.1}$$

(see 1.6 for the definition of $\mathbf{m}(k)$). For $\mathbf{m} \in \mathcal{Q}_{n,r}$, let $\lambda(\mathbf{m}) \in \mathcal{P}_{n,r}$ as in 5.8. As an analogue of Proposition 5.9, we prove the following.

Proposition 6.9. *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$.*

- (i) $\dim \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$.
- (ii) $\mathcal{X}_{\mathbf{m}, \text{uni}} = \overline{X}_{\lambda(\mathbf{m})}$.
- (iii) *Assume that $\mu \in \tilde{\mathcal{P}}(\mathbf{m})$. Then $X_\mu \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$.*

Proof. We consider the surjective map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} \rightarrow \mathcal{X}_{\mathbf{m}, \text{uni}}$. Take $z = (x, \mathbf{v}) \in \mathcal{M}_\mu$ for $\mu \in \mathcal{P}(\mathbf{m}(k))$. Then it follows from the construction of \mathcal{M}_μ , by replacing z by its H -conjugate if necessary, that $(z, B^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}}$. Hence $z \in \mathcal{X}_{\mathbf{m}, \text{uni}}$. Since $\mathcal{X}_{\mathbf{m}, \text{uni}}$ is H -stable, we see that $X_\mu \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$. This proves (iii). Put $\mu = \lambda(\mathbf{m})$. By (iii) we have $X_\mu \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$. By (6.1.2),

$$\dim \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} = 2n^2 - n + \sum_{i=1}^{r-1} (r-i)m_i.$$

On the other hand, by Proposition 6.7, we have

$$\begin{aligned} \dim X_\mu &= 2n^2 - 2n + \sum_{i=1}^{r-1} (r-i+1)m_i \\ &= 2n^2 - n + \sum_{i=1}^{r-1} (r-i)m_i. \end{aligned}$$

(Note that $\sum_{i=1}^{r-1} m_i = n$ since $m_r = 0$.) Since \overline{X}_μ is a closed, irreducible subset of $\mathcal{X}_{\mathbf{m},\text{uni}}$ with $\dim \overline{X}_\lambda = \dim \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}$, we conclude that $\dim \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} = \dim \mathcal{X}_{\mathbf{m},\text{uni}}$ and $\overline{X}_\mu = \mathcal{X}_{\mathbf{m},\text{uni}}$. This proves (i) and (ii). \square

6.10. For a fixed $\mathbf{m} \in \mathcal{Q}_{n,r}$, let P be as in 6.2. For $\lambda \in \mathcal{P}(\mathbf{m})$, we consider the L^θ -orbit \mathcal{O}' and H -orbit \mathcal{O} as in 6.4. Since \mathcal{O}' is determined by λ , we denote it by $\mathcal{O}' = \mathcal{O}'_\lambda$. We shall determine the type of \mathcal{O} . For a partition $\nu = \underbrace{(a, \dots, a)}_{k\text{-times}} = (a^k) \in \mathcal{P}_{ak}$ of rectangular type, we define a double partition $[\nu] \in \mathcal{P}_{ak,2}$ by $[\nu] = (a^{k'}, a^{k''})$, where $k' = [k + 1/2], k'' = k - k'$. In general, a partition ν can be decomposed uniquely as a sum of partitions of rectangular type

$$\nu = (a_1)^{k_1} + (a_2)^{k_2} + \dots + (a_\ell)^{k_\ell}, \tag{6.10.1}$$

where we write the dual partition ν^* as $\nu^* = ((k_1)^{a_1}, (k_2)^{a_2}, \dots, (k_\ell)^{a_\ell})$ for $k_1 > k_2 > \dots > k_\ell > 0$. We define $[\nu] \in \mathcal{P}_{|\nu|,2}$ by

$$[\nu] = [(a_1)^{k_1}] + [(a_2)^{k_2}] + \dots + [(a_\ell)^{k_\ell}]. \tag{6.10.2}$$

For a given $\lambda \in \mathcal{P}_{n,r}$, we define $[\lambda] \in \mathcal{P}_{n,2}$ by

$$[\lambda] = [\lambda^{(1)}] + [\lambda^{(2)}] + \dots + [\lambda^{(r-2)}] + (\lambda^{(r-1)}, \lambda^{(r)}).$$

We have the following lemma.

Lemma 6.11. *Let \mathcal{O} be a unique H -orbit in $G^\theta \times V$ such that $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}'_\lambda)$ is open dense in $\pi_P^{-1}(\mathcal{O}'_\lambda)$. Then $\mathcal{O} = \mathcal{O}_{[\lambda]}$.*

Proof. We show that $\mathcal{O}_{[\lambda]}$ satisfies the required condition. Take $z \in \mathcal{O}_{[\lambda]}$, and $z' \in \mathcal{O}'_\lambda$. First we show that

$$\dim Z_H(z) = \dim Z_{L^\theta}(z'). \tag{6.11.1}$$

In fact, $\dim \mathcal{O}'_\lambda = \sum_{i=1}^{r-2} \dim \mathcal{O}'_{\lambda^{(i)}} + \dim \mathcal{O}'_0$, where $\mathcal{O}'_{\lambda^{(i)}}$ (resp. \mathcal{O}'_0) is the $\text{GL}(\overline{M}_{p_i})$ -orbit of type $\lambda^{(i)}$ (resp. $\text{GL}(\overline{M}_{p_{r-1}})^\theta$ -orbit of type $(\lambda^{(r-1)}, \lambda^{(r)})$) corresponding to \mathcal{O}'_λ . We have

$$\begin{aligned} \dim \mathcal{O}'_{\lambda^{(i)}} &= m_i^2 - m_i - 2n(\lambda^{(i)}) \text{ for } 1 \leq i \leq r-2, \\ \dim \mathcal{O}'_0 &= 2m_{r-1}^{\prime 2} - 2m'_{r-1} - 4n(\lambda^{(r-1)} + \lambda^{(r)}) + 2|\lambda^{(r-1)}|. \end{aligned}$$

(The first formula is well known; see, e.g., [AH, Prop. 2.8 (4)]. The second one is by [SS1, Lem. 2.3].) Hence

$$\dim Z_{L^\theta}(z') = \sum_{i=1}^{r-2} (m_i + 2n(\lambda^{(i)})) + 3m'_{r-1} + 4n(\lambda^{(r-1)} + \lambda^{(r)}) - 2|\lambda^{(r-1)}|. \tag{6.11.2}$$

On the other hand, if we write $[\boldsymbol{\lambda}] = (\nu', \nu'')$, $\dim \mathcal{O}_{[\boldsymbol{\lambda}]}$ is given by a similar formula as that for $\dim \mathcal{O}'_0$. Hence we have

$$\dim Z_H(z) = 3n + 4n([\boldsymbol{\lambda}]) - 2n(\nu'). \tag{6.11.3}$$

Here, for $i = 1, \dots, r - 2$, we show the formula

$$m_i + 2n(\lambda^{(i)}) = 3m_i + 4n([\lambda^{(i)}]) - 2|\nu_{(i)}'|, \tag{6.11.4}$$

where we write $[\lambda^{(i)}]$ as $(\nu_{(i)}', \nu_{(i)}'')$. By the additivity of the n -function together with (6.10.1) and (6.10.2), it is enough to verify (6.11.4) in the case where $\lambda^{(i)}$ is of rectangular type. Assume that $\lambda^{(i)} = (a^{2k-1})$. In this case, $m_i = a(2k-1)$, $|\nu_{(i)}'| = ak$, $n(\lambda^{(i)}) = a(k-1)(2k-1)$ and $n([\lambda^{(i)}]) = a(k-1)^2$. Thus (6.11.4) holds. The case where $\lambda^{(i)} = (a^{2k})$ is dealt with similarly. Now by substituting (6.11.4) into the formula in (6.11.2), and by comparing it with (6.11.3), we obtain (6.11.1).

In order to show the lemma, it is enough to see that $\mathcal{O}_{[\boldsymbol{\lambda}]} \cap \pi_P^{-1}(\mathcal{O}'_\lambda) \neq \emptyset$ by Proposition 6.3 (iii). First we show:

(6.11.5) Assume that $r = 2$ and $m_1 = n, m_2 = 0$. Then $\mathcal{O}_{[\boldsymbol{\lambda}]} \cap \pi_P^{-1}(\mathcal{O}'_\lambda) \neq \emptyset$.

In this case $\boldsymbol{\lambda} = (\lambda^{(1)}, \emptyset)$, $[\boldsymbol{\lambda}] = [\lambda^{(1)}]$. We consider the variety $G_{\text{uni}} \times V$, and we denote by \mathbb{O}_ξ the G -orbit in $G_{\text{uni}} \times V$ corresponding to $\xi \in \mathcal{P}_{2n,2}$. The map π_P is naturally extended to a map $\tilde{\pi}_P : P_{\text{uni}} \times V \rightarrow L_{\text{uni}} \times V_L$. As explained in 6.4, $\mathbb{O}_\xi \cap (G_{\text{uni}}^\theta \times V) \neq \emptyset$ only when ξ is of the form $\xi = (\xi', \xi'')$ with $\xi' = \nu' \cup \nu''$, $\xi'' = \nu'' \cup \nu''$, and in that case $\mathbb{O}_\xi \cap (G_{\text{uni}}^\theta \times V) = \mathcal{O}_{(\nu', \nu'')}$. Write $[\boldsymbol{\lambda}] = (\nu', \nu'')$, and consider $\xi = (\nu' \cup \nu', \nu'' \cup \nu'')$. In order to show (6.11.5), it is enough to see that $\mathbb{O}_\xi \cap \tilde{\pi}_P^{-1}(\mathcal{O}'_\lambda) \neq \emptyset$. For simplicity, we assume that $\lambda^{(1)} = (a^{2k-1})$, i.e., a rectangular type. In that case, $\nu' = (a^k), \nu'' = (a^{k-1})$. We shall take $(x, v) \in G_{\text{uni}} \times V$ as follows. We fix a Jordan basis $\{u_{i,j} \mid 1 \leq i \leq 2k, 1 \leq j \leq \xi'_i + \xi''_i\}$ of V with respect to x , where x acts as $(x-1)u_{i,j} = u_{i,j-1}$ with the convention $u_{i,0} = 0$. We take $v = u_{1,a} + u_{2k-1,a}$. Then (x, v) has type $\xi = ((a^{2k}), (a^{2k-2}))$. We choose a subspace W of V spanned by $\{u_{i,j} \mid 1 \leq i \leq 2k-1, 1 \leq j \leq a\}$. Then W is an x -stable subspace of V containing v such that $x|_W$ has type (a^{2k-1}) , and $x|_{V/W}$ has type (a^{2k-1}) . This implies, after rewriting $\{u_{i,j}\}$ by a suitable symplectic basis of V so that W is a maximal isotropic subspace, that $(x, v) \in \mathbb{O}_\xi \cap \tilde{\pi}_P^{-1}(\mathcal{O}'_\lambda)$. In the case where $\lambda^{(1)} = (a^{2k})$, we have $\xi = ((a^{2k}), (a^{2k}))$, and this case is dealt with similarly (simpler). The general case is also discussed in a similar way, just by a combination of those two cases. Thus (6.11.5) holds.

Next we consider the general case. Let $V = V_1 \oplus \dots \oplus V_{r-1}$ be the decomposition of V into subspaces, where V_i is the symplectic subspace spanned by $\{e_i, f_i \mid p_{i-1} + 1 \leq i \leq p_i\}$ for $i = 1, \dots, r - 2$, and V_{r-1} is the symplectic subspace spanned by $\{e_i, f_i \mid p_{r-2} + 1 \leq i \leq n\}$. We consider the θ -stable subgroup \tilde{L} of G containing L such that $\tilde{L} \simeq G_1 \times \dots \times G_{r-1}$ with $G_i = \text{GL}(V_i)$. Hence $\tilde{L}^\theta \simeq G_1^\theta \times \dots \times G_{r-1}^\theta$ with $G_i^\theta = \text{Sp}(V_i)$. For $i = 1, \dots, r - 2$, let P_i be a θ -stable parabolic subgroup of G_i such that its θ -stable Levi subgroup L_i is isomorphic to $\text{GL}_{m_i} \times \text{GL}_{m_i}$. Let $\mathcal{O}'_{\lambda^{(i)}}$ be the GL_{m_i} -orbit in $(\text{GL}_{m_i})_{\text{uni}}$ as before. We regard it as an L_i^θ -orbit in $(G_i)_{\text{uni}}^\theta$. Let $\mathcal{O}_{[\lambda^{(i)}]}$ be the G_i^θ -orbit in $(G_i)_{\text{uni}}^\theta \times V_i$ corresponding

to $[\lambda^{(i)}]$. Then by (6.11.5), one can find $z_i \in \mathcal{O}_{[\lambda^{(i)}]} \cap \pi_{P_i}^{-1}(\mathcal{O}'_{\lambda^{(i)}})$. Let \mathcal{O}'_0 be the G_{r-1}^θ -orbit in $(G_{r-1})_{\text{uni}}^{\theta}$ corresponding to $(\lambda^{(r-1)}, \lambda^{(r)})$. We choose $z_{r-1} \in \mathcal{O}'_0$. Since $\prod_{i=1}^{r-2} \pi_{P_i}^{-1}(\mathcal{O}'_{\lambda^{(i)}}) \times \mathcal{O}'_{r-1}$ is regarded as a closed subvariety of $\pi_P^{-1}(\mathcal{O}'_\lambda)$, $z = (z_1, \dots, z_{r-1})$ gives an element in $\pi_P^{-1}(\mathcal{O}'_\lambda)$. z is contained in $\tilde{L}^{\theta} \times V$, and actually $z \in \mathcal{O}_{[\lambda]}$. Thus $\mathcal{O}_{[\lambda]} \cap \pi_P^{-1}(\mathcal{O}'_\lambda) \neq \emptyset$. The lemma is proved. \square

Corollary 6.12. *For each $\lambda \in \mathcal{P}_{n,r}$, the map $(x, \mathbf{v}) \mapsto ((x, v_{r-1}), (v_1, \dots, v_{r-2}))$ gives an embedding $X_\lambda \subset \mathcal{O}_{[\lambda]} \times V^{r-2}$.*

Remark 6.13. Proposition 6.9 shows that $\bigcup_{\lambda \in \mathcal{P}_{n,r}} X_\lambda$ covers a dense subset of \mathcal{X}_{uni} . However, it does not coincide with \mathcal{X}_{uni} in general. Also X_λ 's are not mutually disjoint in general, as the following example shows. Assume that $n = 3$ and $r = 4$. Take $\lambda = (1^2; 1; -; -)$ and $\mu = (1; 1^2; -; -)$. By definition and by Lemma 6.11,

$$\begin{aligned} X_\lambda = \{ & (x, v_1, v_2, v_3) \in G_{\text{uni}}^{\theta} \times V^3 \mid v_1 \in W_1, v_2 \in W_2, v_3 \in W_2^\perp, \\ & (x|_{W_1}, v_1) : \text{type } (1^2; -), (x_{W_2/W_1}, \bar{v}_2) : \text{type } (1; -), \\ & (x_{W_2^\perp/W_2}, \bar{v}_3) : \text{type } (-; -), (x, v_3) : \text{type } (2; 1) \\ & \text{for some } x\text{-stable isotropic subspaces } W_1 \subset W_2 \text{ of } V\}, \end{aligned}$$

where \bar{v}_2 is the image of $v_2 \in W_2$ on W_2/W_1 , and \bar{v}_3 is the image of $v_3 \in W_2$ on W_2^\perp/W_2 . Here W_2 is a maximal isotropic subspace, hence $W_2^\perp = W_2$. Similarly,

$$\begin{aligned} X_\mu = \{ & (x, v_1, v_2, v_3) \in G_{\text{uni}}^{\theta} \times V^3 \mid v_1 \in W'_1, v_2 \in W'_2, v_3 \in W_2'^{\perp}, \\ & (x|_{W'_1}, v_1) : \text{type } (1; -), (x_{W'_2/W'_1}, \bar{v}_2) : \text{type } (1^2; -), \\ & (x|_{W_2'^{\perp}/W'_2}, \bar{v}_3) : \text{type } (-; -), (x, v_3) : \text{type } (2; 1) \\ & \text{for some } x\text{-stable isotropic subspaces } W'_1 \subset W'_2 \text{ of } V\}. \end{aligned}$$

Here again W'_2 is a maximal isotropic subspace, and so $W_2'^{\perp} = W'_2$. We fix a symplectic basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ of V and define an action of x on V by $(x - 1)e_i = e_{i-1}$, $(x - 1)f_i = f_{i+1}$ for each i , under the convention $e_0 = f_4 = 0$. Then $x \in G_{\text{uni}}^{\theta}$ and $W = \langle e_1, e_2, f_3 \rangle$ is an x -stable maximal isotropic subspace of V such that $x|_W$ is of type (21). Take $v_1 = e_1, v_2 = e_2, v_3 = e_2$. Then (x, v_2) is of type (2; 1). Put $W_1 = \langle e_1, f_3 \rangle$ and $W_2 = W$. Then $W_1 \subset W_2$ are x -stable isotropic subspaces such that $x|_{W_1}$ is of type (1^2) , and $x|_{W_2/W_1}$ is of type (1). Moreover, $v_1 \in W_1, v_2 \in W_2, v_3 \in W_2^\perp = W_2$. One sees that $(x, v_1, v_2, v_3) \in X_\lambda$. On the other hand, put $W'_1 = \langle e_1 \rangle$, and $W'_2 = W$. Then $W'_1 \subset W'_2$ are x -stable isotropic subspaces such that $x|_{W'_1}$ is of type (1), and $x_{W'_2/W'_1}$ is of type (1^2) . Moreover, $v_1 \in W'_1, v_2 \in W'_2, v_3 \in W_2'^{\perp} = W'_2$. Also we have $(x, v_1, v_2, v_3) \in X_\mu$. It follows that $X_\lambda \cap X_\mu \neq \emptyset$.

6.14. The unipotent variety of enhanced type considered in Section 5 can be interpreted as a closed subvariety of the variety defined in 1.2. So we follow the setting in 1.2, and consider the varieties $\tilde{\mathcal{X}}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}$ as in Section 4. Then the varieties $\mathcal{X}_{\mathbf{m}, \text{uni}}, \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}}$ defined in 5.2 can be identified with closed subvarieties of

$\widetilde{\mathcal{X}}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}$, defined by similar formulas as in 6.1. We shall reformulate X_{λ} defined in Section 5 so that it fits with the discussion in the exotic case.

First we consider an analogue of Propostion 6.3 in the enhanced case. Assume that $r \geq 2$, and take $\mathbf{m} \in \mathcal{Q}_{n,r}$. Let P be a θ -stable parabolic subgroup of G such that P^θ is the stabilizer of the flag $(M_{p_i})_{1 \leq i \leq r}$ in V . Let L be the θ -stable Levi subgroup of P containing T . In contrast to 6.2, we put $V_L = V$. We consider the map $\pi_P : P_{\text{uni}}^{\iota\theta} \times V \rightarrow L_{\text{uni}}^{\iota\theta} \times V_L, (x, v) \mapsto (x', v)$, where $x' = p(x)$ for the natural projection $p : P_{\text{uni}}^{\iota\theta} \rightarrow L_{\text{uni}}^{\iota\theta}$. We define $\mathcal{O}, \mathcal{O}'$ and $\mathcal{P}_{z, \mathcal{O}'}$ as in 6.2. The following result can be proved word by word following the proof of Proposition 6.3. Note that in the enhanced case, $\dim U_P^\theta = \dim U_{P'}^{\theta}$.

Proposition 6.15. *Assume that $\mathcal{X}_{\text{uni}} = G_{\text{uni}}^{\iota\theta} \times V$ is of enhanced type. Then similar statements (i)–(vi) as in Proposition 6.3 hold for \mathcal{X}_{uni} .*

6.16. Let P be as in 6.14. For $\lambda \in \mathcal{P}(\mathbf{m})$, by imitating the definition of \mathcal{M}_{λ} in 6.4, we define a subset \mathcal{M}_{λ} of $P_{\text{uni}}^{\iota\theta} \times \prod_{i=1}^{r-1} M_{p_i}$ as the set of (x, \mathbf{v}) satisfying the following properties. Take (x, \mathbf{v}) such that $x \in P_{\text{uni}}^{\iota\theta}$, and that $v_i \in M_{p_i}$ for $i = 1, \dots, r-1$. Put $\overline{M}_{p_i} = M_{p_i}/M_{p_{i-1}}$, and let $\overline{v}_i \in \overline{M}_{p_i}$ be the image of v_i for each i . We assume that the $\text{GL}(\overline{M}_{p_i})$ -orbit of $(x|_{\overline{M}_{p_i}}, \overline{v}_i)$ in $\text{GL}(\overline{M}_{p_i}) \times \overline{M}_{p_i}$ has type $(\lambda^{(i)}, \emptyset)$ for each i . We further assume that

$$(6.16.1) \quad \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}') \text{ is open dense in } \pi_P^{-1}(\mathcal{O}'), \text{ where } \mathcal{O} \text{ is the } H\text{-orbit of } (x, v) \text{ in } G_{\text{uni}}^{\iota\theta} \times V \text{ and } \mathcal{O}' \text{ is the } L^\theta\text{-orbit of } (p(x), v_{r-1}) \in L_{\text{uni}}^{\iota\theta} \times V.$$

We define X'_{λ} by $X'_{\lambda} = \bigcup_{g \in H} g\mathcal{M}_{\lambda}$. \mathcal{M}_{λ} is a P^θ -stable subset of $P^{\iota\theta} \times V^{r-1}$. In the following lemma, X_{λ}^0 is as defined in 5.10.

Lemma 6.17. *For each $\lambda \in \mathcal{P}_{n,r}$, we have $X_{\lambda}^0 \subset X'_{\lambda}$, hence $\overline{X}_{\lambda} = \overline{X'_{\lambda}}$.*

Proof. Take $(x, \mathbf{v}) \in X_{\lambda}$. By definition in 5.3 together with induction on r , one can find an isotropic flag (W_i) such that W_i is x -stable and that $v_i \in W_i$. By replacing (x, \mathbf{v}) by an H -conjugate if necessary, we may assume that $W_i = M_{p_i}$ for each i . Then $(x|_{\overline{M}_{p_i}}, \overline{v}_i)$ has type $(\lambda^{(i)}, \emptyset)$ for $i = 1, \dots, r-1$. If we further assume that $(x, \mathbf{v}) \in X_{\lambda}^0$, then (x, v_{r-1}) has type $(\lambda^{(1)} + \dots + \lambda^{(r-1)}, \lambda^{(r)})$, and we see that the condition (6.16.1) is satisfied. Thus $(x, \mathbf{v}) \in X'_{\lambda}$, and we have $X_{\lambda}^0 \subset X'_{\lambda}$. Since X_{λ}^0 is open dense in X_{λ} by (5.10.3), we have $\overline{X}_{\lambda} \subset \overline{X'_{\lambda}}$. Conversely, assume that $(x, \mathbf{v}) \in X'_{\lambda}$. Then there exists an isotropic flag (W_i) satisfying the following properties: W_i is x -stable, $v_i \in W_i$, and $x|_{W_i/W_{i-1}}$ has type $\lambda^{(i)}$ for $i = 1, \dots, r-1$. Moreover, since $\mathcal{O}_{\lambda} \cap \pi_P^{-1}(\mathcal{O}'_{\lambda})$ is open dense in $\pi_P^{-1}(\mathcal{O}'_{\lambda})$, (x, v_{r-1}) has type $(\lambda^{(1)} + \dots + \lambda^{(r-1)}, \lambda^{(r)})$. In particular, $x|_{V/W_{r-1}}$ has type $\lambda^{(r)}$. It follows from Proposition 5.16 that $(x, \mathbf{v}) \in \overline{X}_{\lambda}$. Hence $X'_{\lambda} \subset \overline{X}_{\lambda}$, and so $\overline{X'_{\lambda}} \subset \overline{X}_{\lambda}$. We have $\overline{X}_{\lambda} = \overline{X'_{\lambda}}$ as asserted. \square

Remark 6.18. In general, X'_{λ} 's are not mutually disjoint, hence X'_{λ} does not coincide with X_{λ} . In fact, assume that $n = 3, r = 3$, and put $\lambda = (1^2; 1; -), \mu =$

$(1; 1^2; -) \in \mathcal{P}_{3,3}$. We have:

$$\begin{aligned} X'_\lambda &= \{(x, v_1, v_2) \in G_{\text{uni}} \times V^2 \mid v_1 \in W_1, v_2 \in V, (x, v_2) : \text{type } (21; -), \\ &\quad (x|_{W_1}, v_1) : \text{type } (1^2; -), (x|_{V/W_1}, \bar{v}_2) : \text{type } (1; -), \\ &\quad \text{for some } x\text{-stable subspace } W_1 \text{ of } V\}, \\ X'_\mu &= \{(x, v_1, v_2) \in G_{\text{uni}} \times V^2 \mid v_1 \in W'_1, v_2 \in V, (x, v_2) : \text{type } (21; -), \\ &\quad (x|_{W'_1}, v_1) : \text{type } (1; -), (x|_{V/W'_1}, \bar{v}_2) : \text{type } (1^2; -), \\ &\quad \text{for some } x\text{-stable subspace } W'_1 \text{ of } V\}. \end{aligned}$$

Then by a similar argument as in Remark 6.13, we see that $X'_\lambda \cap X'_\mu \neq \emptyset$.

6.19. As in the exotic case, we define a variety $\tilde{\mathcal{F}}_\lambda = H \times^{P^\theta} \overline{\mathcal{M}}_\lambda$, where $\overline{\mathcal{M}}_\lambda$ is the closure of \mathcal{M}_λ in $P^{\iota\theta} \times V^{r-1}$. Then the map $\pi'_\lambda : \tilde{\mathcal{F}}_\lambda \rightarrow \overline{\mathcal{X}}'_\lambda$ is defined as in 6.4. (Here we use the notation π'_λ to distinguish it from the map $\pi_\lambda : \tilde{\mathcal{X}}_\lambda \rightarrow \overline{\mathcal{X}}_\lambda$ in 5.5.) We also define a subset $\tilde{\mathcal{F}}_\lambda^0$ of $\tilde{\mathcal{F}}_\lambda$ by $\tilde{\mathcal{F}}_\lambda^0 = H \times^{P^\theta} \mathcal{M}_\lambda$, and let π_λ^0 be the restriction of π'_λ on $\tilde{\mathcal{F}}_\lambda^0$. The following result can be proved by a similar way as with Proposition 6.7.

Proposition 6.20. *Assume that \mathcal{X} is of enhanced type. For each $\lambda \in \mathcal{P}_{n,r}$,*

- (i) $\text{Im } \pi'_\lambda = \overline{\mathcal{X}}'_\lambda = \overline{\mathcal{X}}_\lambda$.
- (ii) $\pi_\lambda^0 : \tilde{\mathcal{F}}_\lambda^0 \rightarrow X'_\lambda$ gives an isomorphism $\tilde{\mathcal{F}}_\lambda^0 \xrightarrow{\sim} X'_\lambda$.

6.21. In the remainder of this section, we assume that \mathcal{X} is of exotic type or of enhanced type. We follow the formulation in Section 1. Put $\mathcal{B} = H/B^\theta$. For each $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m},\text{uni}}$, put

$$\begin{aligned} \mathcal{B}_z &= \{gB^\theta \in \mathcal{B} \mid g^{-1}xg \in U^{\iota\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\}, \\ \mathcal{B}_z^{(\mathbf{m})} &= \left\{ gB^\theta \in \mathcal{B} \mid g^{-1}xg \in U^{\iota\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i} \right\}. \end{aligned}$$

In the exotic case, \mathcal{B}_z is a closed subvariety of \mathcal{B} , which is isomorphic to the fibre $\pi_1^{-1}(z)$, and is called the Springer fibre of z . $\mathcal{B}_z^{(\mathbf{m})}$ is a closed subvariety of \mathcal{B}_z isomorphic to $(\pi_1^{(\mathbf{m})})^{-1}(z)$, which we call the small Springer fibre. In the enhanced case, we only consider the Springer fibre $\mathcal{B}_z^{(\mathbf{m})}$ as already defined in 5.18.

We fix $\mathbf{m} \in \mathcal{Q}_{n,r}$. For an integer $d \geq 0$, we define a subset $X(d)$ of $\mathcal{X}_{\mathbf{m},\text{uni}}$ by

$$X(d) = \{z \in \mathcal{X}_{\mathbf{m},\text{uni}} \mid \dim \mathcal{B}_z^{(\mathbf{m})} = d\}. \tag{6.21.1}$$

Then $X(d)$ is a locally closed subvariety of $\mathcal{X}_{\mathbf{m},\text{uni}}$, and $\mathcal{X}_{\mathbf{m},\text{uni}} = \coprod_{d \geq 0} X(d)$. We consider the Steinberg varieties $\mathcal{Z}^{(\mathbf{m})}$ and $\mathcal{Z}_1^{(\mathbf{m})}$ as follows:

$$\begin{aligned} \mathcal{Z}^{(\mathbf{m})} &= \{(z, gB^\theta, g'B^\theta) \in \mathcal{X} \times \mathcal{B} \times \mathcal{B} \mid (z, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}, (z, g'B^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}\}, \\ \mathcal{Z}_1^{(\mathbf{m})} &= \{(z, gB^\theta, g'B^\theta) \in \mathcal{X}_{\text{uni}} \times \mathcal{B} \times \mathcal{B} \mid (z, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}, (z, g'B^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}\}. \end{aligned}$$

Recall that $W = N_H(T^\theta)/T^\theta$ is the Weyl group W_n of type C_n (respectively S_n) in the exotic case (respectively in the enhanced case). In the exotic case, we assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, while in the enhanced case, we consider an arbitrary $\mathbf{m} \in \mathcal{Q}_{n,r}$. We define a subgroup $\mathcal{W}_{\mathbf{m}}^{\natural}$ of $N_H(T^\theta)/T^\theta$ by

$$\mathcal{W}_{\mathbf{m}}^{\natural} = \begin{cases} S_{m_1} \times \cdots \times S_{m_{r-2}} \times W_{m_{r-1}} & \text{(exotic case),} \\ S_{m_1} \times \cdots \times S_{m_r} & \text{(enhanced case).} \end{cases} \tag{6.21.2}$$

We show the following lemma.

Lemma 6.22. *Under the notation of 6.21,*

- (i) $\dim \mathcal{Z}_1^{(\mathbf{m})} = \dim \mathcal{X}_{\mathbf{m},\text{uni}}$. *The set of irreducible components of $\mathcal{Z}_1^{(\mathbf{m})}$ with maximal dimension is parametrized by $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$.*
- (ii) $\dim \mathcal{Z}^{(\mathbf{m})} = \dim \mathcal{Z}_1^{(\mathbf{m})} + n$. *The set of irreducible components of $\mathcal{Z}^{(\mathbf{m})}$ with maximal dimension is parametrized by $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$.*
- (iii) *Assume that $X(d) \neq \emptyset$. For any $z \in X(d)$ we have*

$$\dim \mathcal{B}_z^{(\mathbf{m})} \leq \frac{1}{2}(\dim \mathcal{X}_{\mathbf{m},\text{uni}} - \dim X(d)).$$

In particular, $\pi^{(\mathbf{m})}$ is semismall with respect to the stratification $\mathcal{X}_{\mathbf{m},\text{uni}} = \coprod_d X(d)$.

Proof. Let $p_1 : \mathcal{Z}_1^{(\mathbf{m})} \rightarrow \mathcal{B} \times \mathcal{B}$ be the projection on the second and third factors. For each $w \in W$, let \mathcal{O}_w be the H -orbit of (B^θ, wB^θ) in $\mathcal{B} \times \mathcal{B}$. We have $\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} \mathcal{O}_w$. Put $Z_w = p_1^{-1}(\mathcal{O}_w)$. Then Z_w is a vector bundle over $\mathcal{O}_w \simeq H/(B^\theta \cap wB^\theta w^{-1})$ with fibre isomorphic to

$$(U^{\theta} \cap wU^{\theta} w^{-1}) \times \prod_{i=1}^{r-1} (M_{p_i} \cap w(M_{p_i})).$$

First we consider the exotic case. We identify W with a subgroup of S_{2n} which is the stabilizer of the element $(1, n+1)(2, n+2) \cdots (n, 2n)$. Let b_w be the number of i such that $w^{-1}(e_i) \in M_n$, i.e., $w^{-1}(i) \in [1, n]$. Then we have $\dim(U^{\theta} \cap wU^{\theta} w^{-1}) = \dim(U^{\theta} \cap wU^{\theta} w^{-1}) - b_w$. Also we have $\dim(M_n \cap w(M_n)) = b_w$. By our assumption $M_{p_{r-1}} = M_n$, we have

$$\dim Z_w = \dim H - \dim T^\theta + \sum_{i=1}^{r-2} \dim(M_{p_i} \cap w(M_{p_i})).$$

Here $\dim(M_{p_i} \cap w(M_{p_i})) = \#\{j \in [1, p_i] \mid w^{-1}(j) \in [1, p_i]\} \leq p_i$, and the equality holds if and only if w leaves the set $[1, p_i]$ invariant. It follows that $\dim Z_w$ takes the maximal value $2n^2 + \sum_{i=1}^{r-2} p_i$ if and only if $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$. Since

$$\sum_{i=1}^{r-2} p_i = -n + \sum_{i=1}^{r-1} (r-i)m_i$$

(note that $m_r = 0$), in that case $\dim Z_w = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$ by Proposition 6.9 together with (6.1.1). Since $\{\overline{Z}_w \mid w \in \mathcal{W}_{\mathbf{m}}^{\natural}\}$ gives the set of irreducible components of $\mathcal{Z}_1^{(\mathbf{m})}$ with maximal dimension, (i) follows. In the enhanced case, a similar computation shows that

$$\dim Z_w = \dim H - \dim T^\theta + \sum_{i=1}^{r-1} \dim(M_{p_i} \cap w(M_{p_i}))$$

for $w \in S_n$ (note that in this case $\dim(U^{\iota\theta} \cap wU^{\iota\theta}w^{-1}) = \dim(U^\theta \cap wU^\theta w^{-1})$), and $\dim Z_w$ takes the maximal value $n^2 - n + \sum_{i=1}^{r-1} p_i$ if and only if $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$. Since $\sum_{i=1}^{r-1} p_i = \sum_{i=1}^{r-1} (r-i)m_i$, in this case $\dim Z_w = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$ by Proposition 5.9 (ii). Hence (i) holds also in the enhanced case.

For (ii), we consider $\tilde{Z}_w = p^{-1}(\mathcal{O}_w)$, where $p : \mathcal{Z}^{(\mathbf{m})} \rightarrow \mathcal{B} \times \mathcal{B}$ is the projection on the second and third factors. Then \tilde{Z}_w is a locally trivial fibration over \mathcal{O}_w with fibre isomorphic to

$$T^{\iota\theta} \times (U^{\iota\theta} \cap wU^{\iota\theta}w^{-1}) \times \prod_{i=1}^{r-1} (M_{p_i} \cap w(M_{p_i})). \tag{6.22.1}$$

Hence (ii) is proved by a similar argument as (i).

We show (iii). Let $q_1 : \mathcal{Z}_1^{(\mathbf{m})} \rightarrow \mathcal{X}_{\mathbf{m}, \text{uni}}$ be the projection on the first factor. For each $z \in \mathcal{X}_{\mathbf{m}, \text{uni}}$, $q_1^{-1}(z) \simeq \mathcal{B}_z^{(\mathbf{m})} \times \mathcal{B}_z^{(\mathbf{m})}$. By (6.21.1), we have

$$\dim q_1^{-1}(X(d)) = \dim X(d) + 2d.$$

Since $\dim q_1^{-1}(X(d)) \leq \dim \mathcal{Z}_1^{(\mathbf{m})} = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$, we see that $2d \leq \dim \mathcal{X}_{\mathbf{m}, \text{uni}} - \dim X(d)$. This proves (iii). The lemma is proved. \square

7. Springer correspondence

7.1. In this section, we assume that \mathcal{X} is of exotic type or of enhanced type. We shall prove the Springer correspondence for \mathcal{X}_{uni} . However, the method used in [SS1], which is based on an evaluation of the number of irreducible components of Springer fibres (see [SS1, Lem. 2.5]), does not work well for the case where $r \geq 3$. Instead, we apply the method used by Lusztig [Lu] to prove the generalized Springer correspondence for reductive groups, which makes use of the Steinberg map. We follow the notation in Section 1. In the exotic case, let $\omega' : G \rightarrow T/S_{2n}$ be the Steinberg map with respect to G . Then we have $\Xi = T^{\iota\theta}/S_n \hookrightarrow T/S_{2n}$, where S_n is embedded in S_{2n} as a subgroup of the centralizer of $(1, n+1)(2, n+2) \cdots (n, 2n)$. In the enhanced case, let $\omega' : G \rightarrow T/(S_n \times S_n)$ be the Steinberg map and consider $\Xi = T^{\iota\theta}/S_n \hookrightarrow T/(S_n \times S_n)$. In either case, we denote by ω the map $G^{\iota\theta} \rightarrow \Xi$ induced from ω' . Take $\mathbf{m} \in \mathcal{Q}_{n,r}$, and consider the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$ as in 1.1. Let $\mathcal{Z}^{(\mathbf{m})}$ be the Steinberg variety with respect to $\pi^{(\mathbf{m})}$ defined in 6.21.

We denote by φ the natural map $\mathcal{Z}^{(\mathbf{m})} \rightarrow \mathcal{X}_{\mathbf{m}}$. We define a map $\tilde{\alpha} : \mathcal{Z}^{(\mathbf{m})} \rightarrow T^{\iota\theta}$ by $(x, \mathbf{v}, gB^\theta, g'B^\theta) \mapsto p_T(g^{-1}xg)$. Then we have a commutative diagram

$$\begin{CD} \mathcal{Z}^{(\mathbf{m})} @>\tilde{\alpha}>> T^{\iota\theta} \\ @V\varphi VV @VV\omega_T V \\ \mathcal{X}_{\mathbf{m}} @>\tilde{\omega}>> \Xi, \end{CD}$$

where $\tilde{\omega}$ is the composite of the projection $\mathcal{X}_{\mathbf{m}} \rightarrow G^{\iota\theta}$ with ω , and ω_T is the restriction of ω on $T^{\iota\theta}$. Note that ω_T is a finite morphism.

As in 6.21, we assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$ in the exotic case, while we consider arbitrary $\mathbf{m} \in \mathcal{Q}_{n,r}$ in the enhanced case. Recall that $d'_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m},\text{uni}}$, and put $\sigma = \omega_T \circ \tilde{\alpha}$. We define a constructible sheaf \mathcal{T} on Ξ by

$$\mathcal{T} = \mathcal{H}^{2d'_{\mathbf{m}}}(\sigma_! \bar{\mathbf{Q}}_l) = R^{2d'_{\mathbf{m}}}\sigma_! \bar{\mathbf{Q}}_l. \tag{7.1.1}$$

Recall the definition of perfect sheaves in [Lu, 5.4]. A constructible sheaf \mathcal{E} on an irreducible variety X is said to be perfect if \mathcal{E} coincides with an intersection cohomology complex (reduced to a single sheaf on degeree zero) on X , and the support of any nonzero constructible subsheaf of \mathcal{E} is X .

The following gives examples of perfect sheaves.

(7.1.2) If $\pi : Y \rightarrow X$ is a finite morphism with Y smooth and if \mathcal{E}' is a locally constant sheaf on Y , then $\mathcal{E} = \pi_* \mathcal{E}'$ is a perfect sheaf on X .

(7.1.3) If $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$ is an exact sequence of constructible sheaves on X , with $\mathcal{E}_1, \mathcal{E}_3$ perfect, then \mathcal{E}_2 is perfect.

As in Section 6, let $W = N_H(T^\theta)/T^\theta$ be the Weyl group of H . We show the following lemma.

Lemma 7.2. *The sheaf \mathcal{T} is a perfect sheaf on Ξ .*

Proof. Under the notation in the proof of Lemma 6.22, we consider $\tilde{Z}_w = p^{-1}(\mathcal{O}_w)$ for each $w \in W$. Let σ_w be the restriction of σ on \tilde{Z}_w , and put $\mathcal{T}_w = \mathcal{H}^{2d'_{\mathbf{m}}}((\sigma_w)_! \bar{\mathbf{Q}}_l)$. We also put α_w as the restriction of $\tilde{\alpha}$ on \tilde{Z}_w . Let $\mathcal{W}_{\mathbf{m}}^h$ be the subgroup of W given in (6.21.2). We show that

$$\mathcal{T}_w \simeq \begin{cases} (\omega_T)_! \bar{\mathbf{Q}}_l & \text{if } w \in \mathcal{W}_{\mathbf{m}}^h, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.1}$$

In fact, since $\tilde{Z}_w \rightarrow \mathcal{O}_w$ is a locally trivial fibration with fibre isomorphic to (6.22.1), we see that α_w is a locally trivial fibration with fibre isomorphic to

$$H \times^{(B^\theta \cap wB^\theta w^{-1})} \left((U^{\iota\theta} \cap wU^{\iota\theta} w^{-1}) \times \prod_{i=1}^{r-1} (M_{p_i} \cap w(M_{p_i})) \right). \tag{7.2.2}$$

By the computation in the proof of Lemma 6.22, we see that $\dim \alpha_w^{-1}(s) \leq d'_m$ for any $s \in T^{u\theta}$, and that the equality holds if and only if $w \in \mathcal{W}_m^h$. Moreover, in that case, each fibre is an irreducible variety. It follows that

$$R^{2d'_m}(\alpha_w)_! \bar{\mathbf{Q}}_l \simeq \begin{cases} \bar{\mathbf{Q}}_l & \text{if } w \in \mathcal{W}_m^h, \\ 0 & \text{otherwise.} \end{cases} \quad (7.2.3)$$

Since ω_T is a finite morphism, we have $R^{2d'_m}(\sigma_w)_! \bar{\mathbf{Q}}_l \simeq R^0(\omega_T)_! R^{2d'_m}(\alpha_w)_! \bar{\mathbf{Q}}_l$. Thus (7.2.1) follows from (7.2.3).

Since ω_T is a finite morphism, (7.2.1) and (7.1.2) imply that \mathcal{T}_w is a perfect sheaf if $w \in \mathcal{W}_m^h$. By (7.2.2), each fibre $\alpha_w^{-1}(s)$ is a vector bundle over \mathcal{O}_w . In turn, \mathcal{O}_w is a vector bundle over H/B^θ with fibre isomorphic to $U^\theta \cap wU^\theta w^{-1}$. It follows that $H_c^i(\alpha_w^{-1}(s), \bar{\mathbf{Q}}_l) = 0$ for odd i . This implies that $R^{2d'_m-1}(\sigma_w)_! \bar{\mathbf{Q}}_l = 0$ for $w \in \mathcal{W}_m^h$. Now \mathcal{W}_m^h is a parabolic subgroup of W , and the closure relations for $\mathcal{O}_w (w \in \mathcal{W}_m^h)$ are described by the Bruhat order on \mathcal{W}_m^h . It follows, by a similar argument as in [Lu, 5.4] by using the property (7.1.3), we see that \mathcal{T} is a perfect sheaf on Ξ . The lemma is proved. \square

Proposition 7.3. $\mathcal{T} \simeq \bigoplus_{w \in \mathcal{W}_m^h} \mathcal{T}_w$ as sheaves on Ξ .

Proof. Put $\Xi_{\text{reg}} = \omega_T(T^{u\theta}_{\text{reg}})$. Then Ξ_{reg} is an open dense subset of Ξ . Since \mathcal{T} and $\bigoplus_w \mathcal{T}_w$ are perfect sheaves on Ξ , it is enough to show that their restrictions on Ξ_{reg} are isomorphic. Put $\mathcal{Z}_0^{(\mathbf{m})} = \sigma^{-1}(\Xi_{\text{reg}})$. Then $\mathcal{Z}_0^{(\mathbf{m})} \simeq \tilde{\mathcal{Y}}_m \times_{\mathcal{Y}_m} \tilde{\mathcal{Y}}_m$. Let σ_0 be the restriction of σ on $\mathcal{Z}_0^{(\mathbf{m})}$, which is a composite of the natural map $\mathcal{Z}_0^{(\mathbf{m})} \rightarrow \mathcal{Y}_m$ with the map $\mathcal{Y}_m \rightarrow \Xi$. The restriction of \mathcal{T} on Ξ_{reg} is isomorphic to $R^{2d'_m}(\sigma_0)_! \bar{\mathbf{Q}}_l$. Put

$$\mathbf{M}^b = \prod_{i=1}^{r-1} (M_{[p_{i-1}+1, p_i]} + M_{p_{i-1}}).$$

Then \mathbf{M}^b coincides with $\mathbf{M}_{\mathbf{I}^\circ}$ for $\mathbf{I}^\circ = (I_1^\circ, \dots, I_r^\circ)$ in 1.3, where $I_i^\circ = [p_{i-1}+1, p_i]$. Put $\tilde{\mathcal{Y}}^b = \tilde{\mathcal{Y}}_{\mathbf{I}^\circ}$ in the notation in 1.3 (see also 4.1). Under the isomorphism (1.2.2), $\tilde{\mathcal{Y}}^b$ is regarded as an open dense subset of $\tilde{\mathcal{Y}}_m$. Put $\mathcal{Z}^b = \tilde{\mathcal{Y}}^b \times_{\mathcal{Y}_m^0} \tilde{\mathcal{Y}}^b$ and let σ_b be the restriction of σ_0 on \mathcal{Z}^b . For each $w \in W_n$, let σ_b^w be the restriction of σ_b on $\mathcal{Z}^b(w) = \mathcal{Z}^b \cap \tilde{\mathcal{Z}}_w$. Note that $\mathcal{Z}^b(w) = \emptyset$ unless $w \in \mathcal{W}_m^h$. Now \mathcal{Z}^b is an open subset of $\mathcal{Z}_0^{(\mathbf{m})}$ and the inclusion map $\mathcal{Z}^b \hookrightarrow \mathcal{Z}_0^{(\mathbf{m})}$ induces a morphism $R^i(\sigma_b)_! \bar{\mathbf{Q}}_l \rightarrow R^i(\sigma_0)_! \bar{\mathbf{Q}}_l$ for each i . Similarly, we have a morphism $R^i(\sigma_b^w)_! \bar{\mathbf{Q}}_l \rightarrow R^i(\sigma_0^w)_! \bar{\mathbf{Q}}_l$ for each w . We have

$$\begin{aligned} R^{2d'_m}(\sigma_b)_! \bar{\mathbf{Q}}_l &\simeq R^{2d'_m}(\sigma_0)_! \bar{\mathbf{Q}}_l, \\ R^{2d'_m}(\sigma_b^w)_! \bar{\mathbf{Q}}_l &\simeq R^{2d'_m}(\sigma_0^w)_! \bar{\mathbf{Q}}_l \quad \text{for } w \in \mathcal{W}_m^h. \end{aligned} \quad (7.3.1)$$

We show (7.3.1). Let $\tilde{\alpha}_0 : \mathcal{Z}_0^{(\mathbf{m})} \rightarrow T^{u\theta}$ be the restriction of $\tilde{\alpha}$ on $\tilde{\mathcal{Z}}_0^{(\mathbf{m})}$, and α_0^w the restriction of $\tilde{\alpha}_0$ on $\tilde{\mathcal{Z}}_w$. We also denote by α_b (resp. α_b^w) the restriction of $\tilde{\alpha}_0$ on \mathcal{Z}^b (resp. on $\mathcal{Z}^b(w)$). From the computation in the proof of Lemma 7.2, we know that the map $R^{2d'_m}(\alpha_b^w)_! \bar{\mathbf{Q}}_l \rightarrow R^{2d'_m}(\alpha_0^w)_! \bar{\mathbf{Q}}_l$ is surjective. For any $\mathbf{I} \subset \mathcal{I}(\mathbf{m}')$ with

$\mathbf{m}' \leq \mathbf{m}$, we consider $\psi_{\mathbf{I}} : \tilde{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}_{\mathbf{m}'}^0$ as in 1.3. For each $\mathbf{I}, \mathbf{J} \in \mathcal{I}(\mathbf{m}')$, put $\mathcal{Z}_{\mathbf{I}, \mathbf{J}} = \tilde{\mathcal{Y}}_{\mathbf{I}} \times_{\mathcal{Y}_{\mathbf{m}}} \tilde{\mathcal{Y}}_{\mathbf{J}}$ under the inclusion $\mathcal{Y}_{\mathbf{m}'}^0 \hookrightarrow \mathcal{Y}_{\mathbf{m}}$. Note that $\mathcal{Z}_{\mathbf{I}, \mathbf{J}} = \emptyset$ unless $\mathbf{J} = w(\mathbf{I})$ for some $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$. We have a partition $\mathcal{Z}_0^{(\mathbf{m})} = \coprod_{\mathbf{I}, \mathbf{J}} \mathcal{Z}_{\mathbf{I}, \mathbf{J}}$ by locally closed subsets $\mathcal{Z}_{\mathbf{I}, \mathbf{J}}$, and \mathcal{Z}^b coincides with $\mathcal{Z}_{\mathbf{I}^{\circ}, \mathbf{I}^{\circ}}$. Let \mathcal{Z}_1 be the complement of \mathcal{Z}^b in $\mathcal{Z}_0^{(\mathbf{m})}$, and put $\mathcal{Z}_1(w) = \mathcal{Z}_1 \cap \tilde{\mathcal{Z}}_w$. Hence $\mathcal{Z}_1 = \coprod_w \mathcal{Z}_1(w)$. Again, by a similar computation as in the proof of Lemma 7.2, we see that $\dim \alpha_w^{-1}(s) \cap \mathcal{Z}_1(w) < d'_{\mathbf{m}}$ for any $s \in T^{\iota\theta}$. This implies that $R^{2d'_{\mathbf{m}}}(\alpha_b^w)_! \bar{\mathbf{Q}}_l \simeq R^{2d'_{\mathbf{m}}}(\alpha_0^w)_! \bar{\mathbf{Q}}_l$. By applying $R^0(\omega_T)_!$ on both sides, we have $R^{2d'_{\mathbf{m}}}(\sigma_b^w)_! \bar{\mathbf{Q}}_l \simeq R^{2d'_{\mathbf{m}}}(\sigma_0^w)_! \bar{\mathbf{Q}}_l$ for any $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$. This proves the second statement of (7.3.1). By considering the long exact sequence arising from the stratification $\mathcal{Z}_0^{(\mathbf{m})} = \coprod_w (\mathcal{Z}_0^{(\mathbf{m})} \cap \tilde{\mathcal{Z}}_w)$, we obtain the first statement. Thus (7.3.1) holds.

By (7.3.1), the proof of the proposition is reduced to showing

$$R^{2d'_{\mathbf{m}}}(\sigma_b)_! \bar{\mathbf{Q}}_l \simeq \bigoplus_{w \in \mathcal{W}_{\mathbf{m}}^{\natural}} R^{2d'_{\mathbf{m}}}(\sigma_b^w)_! \bar{\mathbf{Q}}_l. \tag{7.3.2}$$

Note that in this case the natural projection $\varphi_b : \mathcal{Z}^b \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ is a finite Galois covering. (This is clear in the enhanced case. In the exotic case, since $m_r = 0$, we have $\mathbf{I}_r^{\circ} = \emptyset$. Hence $\tilde{\mathcal{Y}}_b \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ is a finite Galois covering.) Also note that $\mathcal{Z}^b(w) = \emptyset$ unless $w \in \mathcal{W}_{\mathbf{m}}^{\natural}$. In that case, $\mathcal{Z}^b(w)$ is an open and closed subset of \mathcal{Z}^b since φ_b is a finite Galois covering. We have a decomposition $\mathcal{Z}^b = \coprod_{w \in \mathcal{W}_{\mathbf{m}}^{\natural}} \mathcal{Z}^b(w)$ into open and closed subsets. This implies that

$$(\sigma_b)_! \bar{\mathbf{Q}}_l \simeq \bigoplus_{w \in \mathcal{W}_{\mathbf{m}}^{\natural}} (\sigma_b^w)_! \bar{\mathbf{Q}}_l. \tag{7.3.3}$$

Hence (7.3.2) holds. The proposition is proved. \square

7.4. By the Künneth formula, $\varphi_! \bar{\mathbf{Q}}_l \simeq \pi_1^{(\mathbf{m})} \bar{\mathbf{Q}}_l \otimes \pi_1^{(\mathbf{m})} \bar{\mathbf{Q}}_l$. By Theorem 3.2 and Theorem 4.5, $\pi_1^{(\mathbf{m})} \bar{\mathbf{Q}}_l$ has a natural structure of $\mathcal{W}_{\mathbf{m}}^{\natural}$ -module. Hence $\varphi_! \bar{\mathbf{Q}}_l$ has a structure of $\mathcal{W}_{\mathbf{m}}^{\natural} \times \mathcal{W}_{\mathbf{m}}^{\natural}$ -module. It follows that $\mathcal{T} = \mathcal{H}^{2d'_{\mathbf{m}}}(\sigma_! \bar{\mathbf{Q}}_l) \simeq \mathcal{H}^{2d'_{\mathbf{m}}}(\tilde{\omega}_!(\varphi_! \bar{\mathbf{Q}}_l))$ has a natural action of $\mathcal{W}_{\mathbf{m}}^{\natural} \times \mathcal{W}_{\mathbf{m}}^{\natural}$. Under the decomposition of \mathcal{T} in Proposition 7.3, the action of $\mathcal{W}_{\mathbf{m}}^{\natural} \times \mathcal{W}_{\mathbf{m}}^{\natural}$ has the following property. For each $w_1, w_2 \in \mathcal{W}_{\mathbf{m}}^{\natural}$,

$$(w_1, w_2) \cdot \mathcal{T}_w = \mathcal{T}_{w_1 w_2^{-1}}. \tag{7.4.1}$$

In fact, since \mathcal{T} is a perfect sheaf by Lemma 7.2, it is enough to check (7.4.1) for the restriction of \mathcal{T} on Ξ_{reg} . Here $(\varphi_b)_! \bar{\mathbf{Q}}_l$ has already a structure of $\mathcal{W}_{\mathbf{m}}^{\natural} \times \mathcal{W}_{\mathbf{m}}^{\natural}$ -module. Hence by (7.3.1), it is enough to check a similar property for the corresponding decomposition of $(\varphi_b)_! \bar{\mathbf{Q}}_l$ in (7.3.3) (note that $\sigma_b = \tilde{\omega} \circ \varphi_b$). But this can be verified directly from the discussion in 7.3. Hence (7.4.1) holds.

Let a_0 be the element in Ξ corresponding to the S_n -orbit of $1 \in T^{\iota\theta}$, and \mathcal{T}_{a_0} be the stalk of \mathcal{T} at $a_0 \in \Xi$. By Proposition 7.3, we have a decomposition

$$\mathcal{T}_{a_0} = \bigoplus_{w \in \mathcal{W}_{\mathbf{m}}^{\natural}} (\mathcal{T}_w)_{a_0},$$

where $(\mathcal{T}_w)_{a_0}$ is the stalk of \mathcal{T}_w at a_0 . $\mathcal{W}_{\mathbf{m}}^{\natural} \times \mathcal{W}_{\mathbf{m}}^{\natural}$ acts on \mathcal{T}_{a_0} following (7.4.1). By (7.2.1), $\mathcal{T}_w \simeq (\omega_T)_! \bar{\mathbf{Q}}_l$. Since $\omega_T^{-1}(a_0) = \{1\} \subset T^{\iota\theta}$, $(\mathcal{T}_w)_{a_0} \simeq H_c^0(\omega_T^{-1}(a_0), \bar{\mathbf{Q}}_l) \simeq \bar{\mathbf{Q}}_l$. Thus we have proved

Proposition 7.5. \mathcal{T}_{a_0} has a structure of $\mathcal{W}_{\mathbf{m}}^{\mathfrak{h}} \times \mathcal{W}_{\mathbf{m}}^{\mathfrak{h}}$ -module, which coincides with the two-sided regular representation of $\mathcal{W}_{\mathbf{m}}^{\mathfrak{h}}$.

The following lemma is a variant of [Lu, Lem. 6.7].

Lemma 7.6. Let A, A' be simple perverse sheaves on $\mathcal{X}_{\mathbf{m}, \text{uni}}$. Then we have

$$\dim \mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, A \otimes A') = \begin{cases} 1 & \text{if } A' \simeq D(A), \\ 0 & \text{otherwise,} \end{cases}$$

where $D(A)$ is the Verdier dual of A .

Proof. Assume that $A = \text{IC}(\overline{X}, \mathcal{E})[\dim X]$ and $A' = \text{IC}(\overline{X}', \mathcal{E}')[\dim X']$, where X, X' are smooth irreducible subvarieties of $\mathcal{X}_{\mathbf{m}, \text{uni}}$ and \mathcal{E} (respectively \mathcal{E}') is a simple local system on X (respectively on X'). We have $\mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, A \otimes A') \simeq \mathbf{H}_c^0(\overline{X} \cap \overline{X}', A \otimes A')$. First assume that $\overline{X} \neq \overline{X}'$, and put $Y = \overline{X} \cap \overline{X}'$. We show

$$\mathbf{H}_c^0(Y, A \otimes A') = 0. \tag{7.6.1}$$

For this, by using the hypercohomology spectral sequence, it is enough to show the following.

$$(7.6.2) \text{ If } H_c^i(Y, \mathcal{H}^j A \otimes \mathcal{H}^{j'} A') \neq 0, \text{ then } i + j + j' < 0.$$

We show (7.6.2). Suppose that $H_c^i(Y, \mathcal{H}^j A \otimes \mathcal{H}^{j'} A') \neq 0$. Put

$$Y_{j, j'} = \text{supp } \mathcal{H}^j A \cap \text{supp } \mathcal{H}^{j'} A' \subset Y.$$

We have $H_c^i(Y_{j, j'}, \mathcal{H}^j A \otimes \mathcal{H}^{j'} A') \simeq H_c^i(Y, \mathcal{H}^j A \otimes \mathcal{H}^{j'} A') \neq 0$. It follows that $i \leq 2 \dim Y_{j, j'}$. By using the property of intersection cohomology, we have

$$\begin{aligned} \dim Y_{j, j'} &\leq \dim \text{supp } \mathcal{H}^j A \leq -j, \\ \dim Y_{j, j'} &\leq \dim \text{supp } \mathcal{H}^{j'} A' \leq -j' \end{aligned}$$

and so

$$j \leq -\dim Y_{j, j'}, \quad j' \leq -\dim Y_{j, j'}. \tag{7.6.3}$$

Since $\overline{X} \neq \overline{X}'$, we have $\dim Y < \dim X$ or $\dim Y < \dim X'$, and one of the inequalities in (7.6.3) is a strict inequality. It follows that $i + j + j' < 0$. Hence (7.6.2) holds and (7.6.1) follows.

Next assume that $\overline{X} = \overline{X}'$. We may assume that X, X' are open dense in \overline{X} . By replacing X, X' by $X \cap X'$, if necessary, we may assume that $X = X'$. Put $Y = \overline{X} \setminus X$. We show that

$$\mathbf{H}_c^0(Y, A \otimes A') = 0 \text{ and } \mathbf{H}_c^{-1}(Y, A \otimes A') = 0. \tag{7.6.4}$$

As in the previous case, we consider the hypercohomology spectral sequence. Suppose that $H_c^i(Y, \mathcal{H}^j A \otimes \mathcal{H}^{j'} A') \neq 0$. We may replace Y by $Y_{j, j'}$, where $Y_{j, j'} = Y \cap \text{supp } \mathcal{H}^j A \cap \text{supp } \mathcal{H}^{j'} A'$. Then we have $i \leq 2 \dim Y_{j, j'}$, and we obtain

a similar formula as (7.6.3), but in this case, both of them are strict inequalities since $\dim Y < \dim X$. It follows that $i + j + j' < -1$. This proves (7.6.4). By using the cohomology long exact sequence with respect to $Y = \overline{X} \setminus X$, we see that

$$\mathbf{H}_c^0(\overline{X}, A \otimes A') \simeq \mathbf{H}_c^0(X, A \otimes A') \simeq H_c^{2 \dim X}(X, \mathcal{E} \otimes \mathcal{E}').$$

The last space is isomorphic to $\overline{\mathbf{Q}}_l$ if $\mathcal{E}' \simeq \mathcal{E}^\vee$, the dual local system, and is equal to zero otherwise. Since $D(A) = \text{IC}(\overline{X}, \mathcal{E}^\vee)[\dim X]$, the lemma is proved. \square

7.7. We consider the map $\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}, \text{uni}} \rightarrow \mathcal{X}_{\mathbf{m}, \text{uni}}$. Put $K_{\mathbf{m}, 1} = (\pi_1^{(\mathbf{m})})_! \overline{\mathbf{Q}}_l[d'_{\mathbf{m}}]$. By Lemma 6.22 (iii), the map $\pi_1^{(\mathbf{m})}$ is semi-small. Hence $K_{\mathbf{m}, 1}$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}, \text{uni}}$ and is decomposed as

$$K_{\mathbf{m}, 1} \simeq \bigoplus_{A \in \mathcal{C}_{\mathbf{m}}} V_A \otimes A, \tag{7.7.1}$$

where $\mathcal{C}_{\mathbf{m}}$ is the set of (isomorphism classes of) simple perverse sheaves appearing in the decomposition of $K_{\mathbf{m}, 1}$, and $V_A = \text{Hom}(K_{\mathbf{m}, 1}, A)$ is the multiplicity space for A . We have the following.

Proposition 7.8. *Under the notation as above, put $m_A = \dim V_A$ for each $A \in \mathcal{C}_{\mathbf{m}}$. Then we have*

$$\sum_{A \in \mathcal{C}_{\mathbf{m}}} m_A^2 = |\mathcal{W}_{\mathbf{m}}^{\natural}|.$$

Proof. By the computation in 7.4, we have

$$\begin{aligned} \mathcal{T}_{a_0} &\simeq \mathbf{H}_c^{2d'_{\mathbf{m}}}(\mathcal{X}_{\mathbf{m}, \text{uni}}, (\pi_1^{(\mathbf{m})})_! \overline{\mathbf{Q}}_l \otimes (\pi_1^{(\mathbf{m})})_! \overline{\mathbf{Q}}_l) \\ &\simeq \mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, K_{\mathbf{m}, 1} \otimes K_{\mathbf{m}, 1}). \end{aligned}$$

Hence by (7.7.1), we have

$$\dim \mathcal{T}_{a_0} = \sum_{A, A' \in \mathcal{C}_{\mathbf{m}}} (m_A m_{A'}) \dim \mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, A \otimes A').$$

By Lemma 7.6, $\mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, A \otimes A') \neq 0$ only when $D(A) = A'$, in which case, $\dim \mathbf{H}_c^0(\mathcal{X}_{\mathbf{m}, \text{uni}}, A \otimes A') = 1$. But since $K_{\mathbf{m}, 1}$ is self dual, $m_A = m_{D(A)}$ for each A . It follows that $\dim \mathcal{T}_{a_0} = \sum_{A \in \mathcal{C}_{\mathbf{m}}} m_A^2$. On the other hand, by Proposition 7.5, we have $\dim \mathcal{T}_{a_0} = |\mathcal{W}_{\mathbf{m}}^{\natural}|$. This proves the proposition. \square

7.9. Since $\pi_1^{(\mathbf{m})} \overline{\mathbf{Q}}_l$ is equipped with the $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action, for each $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}$, $H^i((\pi^{(\mathbf{m})})^{-1}(z), \overline{\mathbf{Q}}_l)$ turns out to be a $\mathcal{W}_{\mathbf{m}}^{\natural}$ -module. In the case where $z_0 = (1, \mathbf{0})$, $(\pi^{(\mathbf{m})})^{-1}(z_0) \simeq H/B^\theta$ and so $H^i(H/B^\theta, \overline{\mathbf{Q}}_l)$ has a structure of $\mathcal{W}_{\mathbf{m}}^{\natural}$ -module. Note that $\mathcal{W}_{\mathbf{m}}^{\natural}$ is a subgroup of W_n (respectively of S_n) in the exotic case (respectively the enhanced case). It is well known that the Weyl group W_n (respectively S_n) acts naturally on $H^i(H/B^\theta, \overline{\mathbf{Q}}_l)$, which we call the classical action of W_n (respectively S_n) on it. We have the following lemma.

Lemma 7.10. *The action of $\mathcal{W}_{\mathbf{m}}^{\natural}$ on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$ coincides with the restriction of the classical action of W_n (respectively S_n) on it in the exotic case (respectively the enhanced case).*

Proof. First assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type. Put $\mathbf{m}' = (0, \dots, 0, n, 0) \in \mathcal{Q}_{n,r}^0$. Then $\mathcal{Y}_{\mathbf{m}'} \subset \mathcal{Y}_{\mathbf{m}}$, and $\mathcal{X}_{\mathbf{m}'} \subset \mathcal{X}_{\mathbf{m}}$ (in fact, $\mathcal{Y}_{\mathbf{m}'} = \{(x, \mathbf{v}) \in \mathcal{Y}_{\mathbf{m}} \mid v_1 = \dots = v_{r-2} = 0\}$ and similarly for $\mathcal{X}_{\mathbf{m}'}$). We have $\mathcal{W}_{\mathbf{m}'}^{\natural} = W_n$. In this case, $\psi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l|_{\mathcal{Y}_{\mathbf{m}'}} \simeq \psi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l$, and $\pi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l|_{\mathcal{X}_{\mathbf{m}'}} \simeq \pi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l$. It follows from the construction that the action of $\mathcal{W}_{\mathbf{m}}^{\natural}$ on $\psi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l|_{\mathcal{Y}_{\mathbf{m}'}}$ coincides with the restriction of the action of $\mathcal{W}_{\mathbf{m}'}^{\natural} \simeq W_n$ on $\psi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l$. Hence a similar fact holds also for $\pi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l$. In particular, the $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action on $\mathcal{H}_{z_0}^i(\pi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l)$ coincides with the restriction of the $\mathcal{W}_{\mathbf{m}'}^{\natural}$ -action on $\mathcal{H}_{z_0}^i(\pi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l)$. Here $\mathcal{X}_{\mathbf{m}'} \simeq G^{u\theta} \times V$, and $\pi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l$ is isomorphic to $\pi_! \bar{\mathbf{Q}}_l$, where $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X} = G^{u\theta} \times V$ is the map defined in 1.2 for the case $r = 2$. The complex $\pi_! \bar{\mathbf{Q}}_l$ was studied in [SS1], and one sees that the action of $\mathcal{W}_{\mathbf{m}'}^{\natural}$ on $\pi_!^{(\mathbf{m}')} \bar{\mathbf{Q}}_l$ is nothing but the W_n -action on $\pi_! \bar{\mathbf{Q}}_l$ constructed in [SS1]. It induces a W_n -action on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$, which is called the exotic action of W_n . Hence in order to prove the lemma, it is enough to see that the exotic action and the classical action of W_n on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$ coincide with each other. But this is proved in [SS1, Lem. 5.2]. Hence the lemma is proved in the exotic case.

Next assume that $\mathcal{X}_{\mathbf{m}}$ is of enhanced type. In this case, we consider $\mathcal{Y}_{\mathbf{m}'} \subset \mathcal{Y}_{\mathbf{m}}$ and $\mathcal{X}_{\mathbf{m}'} \subset \mathcal{X}_{\mathbf{m}}$ for $\mathbf{m}' = (0, \dots, 0, n) \in \mathcal{Q}_{n,r}$. Then $\mathcal{X}_{\mathbf{m}'} \simeq G^{u\theta} \times V$, and as in the exotic case, the proof is reduced to the case where $r = 2$. So we consider $\mathcal{X} = G \times V$ for $G = \text{GL}(V)$ and let $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be the corresponding map. We have an action of S_n on $H^i(G/B, \bar{\mathbf{Q}}_l)$, called the enhanced action of S_n , obtained from the S_n -action on $\pi_! \bar{\mathbf{Q}}_l$. One can prove that the enhanced action coincides with the classical action, by a similar (but simpler) argument as in the proof of Lemma 5.2 in [SS1]. Hence the lemma holds for the enhanced case. \square

7.11. We keep the assumption as before. By applying Theorem 3.2 and Theorem 4.5 to the case where $\mathcal{E} = \bar{\mathbf{Q}}_l$, one can write

$$\pi_!^{(\mathbf{m})} \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^\wedge} \rho \otimes K_\rho, \tag{7.11.1}$$

where $d_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$, K_ρ is a simple perverse sheaf on $\mathcal{X}_{\mathbf{m}}$ as given in Theorem 3.2 and Theorem 4.5. More precisely, if $\mathcal{X}_{\mathbf{m}}$ is of exotic type then $K_\rho = \text{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_{\rho_1})[d_{\mathbf{m}(k)}]$ for $\rho \simeq V_{\rho_1}^{\natural}$ with $\rho_1 \in \mathcal{W}_{\mathbf{m}(k)}^\wedge$, and if $\mathcal{X}_{\mathbf{m}}$ is of enhanced type then $K_\rho = \text{IC}(\mathcal{X}_{\mathbf{m}}, \mathcal{L}_\rho)[d_{\mathbf{m}}]$. We consider the complex $(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}]$, where $d_{\mathbf{m}'} = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$. The following result gives the Springer correspondence with respect to the action of $\mathcal{W}_{\mathbf{m}}^{\natural}$. In the exotic case, this result is regarded as a weak version of the Springer correspondence.

Theorem 7.12 (Springer correspondence for $\mathcal{W}_{\mathbf{m}}^{\natural}$). *Let $\mathcal{X}_{\mathbf{m}}$ be of exotic type or of enhanced type for $\mathbf{m} \in \mathcal{Q}_{n,r}$. In the exotic case, assume further that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$.*

Then $(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d'_\mathbf{m}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ equipped with $\mathcal{W}_\mathbf{m}^{\natural}$ -action, and is decomposed as

$$(\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d'_\mathbf{m}] \simeq \bigoplus_{\rho \in (\mathcal{W}_\mathbf{m}^{\natural})^\wedge} \rho \otimes L_\rho, \tag{7.12.1}$$

where L_ρ is a simple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ such that

$$K_\rho|_{\mathcal{X}_{\mathbf{m},\text{uni}}} \simeq L_\rho[d_\mathbf{m} - d'_\mathbf{m}]. \tag{7.12.2}$$

Proof. As discussed in 7.7, $K_{\mathbf{m},1} = (\pi_1^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d'_\mathbf{m}]$ is a semisimple perverse sheaf. Since $K_{\mathbf{m},1}$ is the restriction of $(\pi^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l$ on $\mathcal{X}_{\mathbf{m},\text{uni}}$, $K_{\mathbf{m},1}$ has a structure of $\mathcal{W}_\mathbf{m}^{\natural}$ -module. Thus we can define an algebra homomorphism

$$\bar{\mathbf{Q}}_l[\mathcal{W}_\mathbf{m}^{\natural}] \xrightarrow{\alpha} \text{End } K_{\mathbf{m},1} \xrightarrow{\beta} \text{End } H^\bullet(H/B^\theta, \bar{\mathbf{Q}}_l).$$

In order to show (7.12.1), it is enough to see that α gives an isomorphism

$$\alpha : \bar{\mathbf{Q}}_l[\mathcal{W}_\mathbf{m}^{\natural}] \xrightarrow{\sim} \text{End } K_{\mathbf{m},1}. \tag{7.12.3}$$

We show (7.12.3). We assume that $\mathcal{X}_\mathbf{m}$ is of exotic type. The proof for the enhanced case is similar. $\beta \circ \alpha$ is a homomorphism induced from the action of $\mathcal{W}_\mathbf{m}^{\natural}$ on $H^\bullet(H/B^\theta, \bar{\mathbf{Q}}_l)$. By Lemma 7.10, this action is the restriction of the classical action of W_n on $H^\bullet(H/B^\theta, \bar{\mathbf{Q}}_l)$. Since $H^\bullet(H/B^\theta, \bar{\mathbf{Q}}_l) \simeq \bar{\mathbf{Q}}_l[W_n]$ as W_n -module, with respect to the classical action, $\bar{\mathbf{Q}}_l[W_n] \rightarrow H^\bullet(H/B^\theta, \bar{\mathbf{Q}}_l)$ is injective. Hence $\beta \circ \alpha$ is injective. It follows that α is injective. On the other hand, Proposition 7.8 implies that $\dim \text{End } K_{\mathbf{m},1} = |\mathcal{W}_\mathbf{m}^{\natural}|$. This shows that α is surjective, and so (7.12.3) holds.

(7.12.2) now follows easily by comparing (7.11.1) and (7.12.1). The theorem is proved. \square

7.13. We now return to the setting in 1.6, and consider the complex reflection group $W_{n,r}$. For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, we denote by $(W_{n,r}^\wedge)_\mathbf{m}$ the set of irreducible representations \tilde{V}_ρ (up to isomorphism) of $W_{n,r}$ obtained from $\rho \in \mathcal{W}_{\mathbf{m}(k)}^\wedge$ for various $0 \leq k \leq m_{r-1}$ as in 1.6. Then we have

$$W_{n,r}^\wedge = \coprod_{\mathbf{m} \in \mathcal{Q}_{n,r}^0} (W_{n,r}^\wedge)_\mathbf{m}. \tag{7.13.1}$$

It follows from the construction of \tilde{V}_ρ and of V_ρ^\natural , there exists a natural bijection between $(W_{n,r}^\wedge)_\mathbf{m}$ and $(\mathcal{W}_\mathbf{m}^{\natural})^\wedge$. We denote by $V(\rho)$ the irreducible representation of $W_{n,r}$ belonging to $(W_{n,r}^\wedge)_\mathbf{m}$ corresponding to $\rho \in (\mathcal{W}_\mathbf{m}^{\natural})^\wedge$. Now assume that $\mathcal{X}_\mathbf{m}$ is of exotic type. We consider the map $\bar{\pi}_\mathbf{m} : \pi^{-1}(\mathcal{X}_\mathbf{m}) \rightarrow \mathcal{X}_\mathbf{m}$, and the complex $(\bar{\pi}_\mathbf{m})_! \bar{\mathbf{Q}}_l[d_\mathbf{m}]$ as in 2.1. Then by Theorem 2.2, $(\bar{\pi}_\mathbf{m})_! \bar{\mathbf{Q}}_l[d_\mathbf{m}]$ is a semisimple perverse sheaf equipped with $W_{n,r}$ -action, and is decomposed as

$$(\bar{\pi}_\mathbf{m})_! \bar{\mathbf{Q}}_l[d_\mathbf{m}] \simeq \bigoplus_{\rho \in (\mathcal{W}_\mathbf{m}^{\natural})^\wedge} V(\rho) \otimes K_\rho, \tag{7.13.2}$$

where K_ρ is a simple perverse sheaf on $\mathcal{X}_\mathbf{m}$ given in (7.11.1). Let $\bar{\pi}_{\mathbf{m},1} : \pi^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}}) \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ be the restriction of $\bar{\pi}_\mathbf{m}$ on $\mathcal{X}_{\mathbf{m},\text{uni}}$. Since $\bar{\pi}_{\mathbf{m},1}$ is proper, $\bar{\pi}_{\mathbf{m},1} \bar{\mathbf{Q}}_l[d'_\mathbf{m}]$ is a semisimple complex on $\mathcal{X}_{\mathbf{m},\text{uni}}$. By applying (7.12.2), we see that $\bar{\pi}_{\mathbf{m},1} \bar{\mathbf{Q}}_l[d'_\mathbf{m}]$ is a semisimple perverse sheaf. As a corollary to Theorem 7.12, we obtain the Springer correspondence for $W_{n,r}$.

Corollary 7.14 (Springer correspondence for $W_{n,r}$). *Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type with $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $\bar{\pi}_{\mathbf{m},1} \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ with $W_{n,r}$ -action, and is decomposed as*

$$(\bar{\pi}_{\mathbf{m},1})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (W_{\mathbf{m}}^{\mathfrak{h}})^{\wedge}} V(\rho) \otimes L_{\rho},$$

where L_{ρ} is a simple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ as in Theorem 7.12, and $V(\rho)$ is an irreducible representation of $W_{n,r}$ as defined in 7.13.

8. Determination of the Springer correspondence

8.1. Assume that $r \geq 2$. In this section we shall determine L_{ρ} appearing in the Springer correspondence explicitly. For a fixed $\mathbf{m} \in \mathcal{Q}_{n,r}$, we consider $\mathcal{X}_{\mathbf{m}}$ of exotic type or of enhanced type. In the exotic case, we assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. First we consider the case where $\mathcal{X}_{\mathbf{m}}$ is of exotic type. We define a variety $\mathcal{G}_{\mathbf{m}}$ by

$$\mathcal{G}_{\mathbf{m}} = \{(x, \mathbf{v}, (W_i)_{1 \leq i \leq r-2}) \mid (x, \mathbf{v}) \in \mathcal{X}, (W_i) : \text{isotropic flag in } V, \dim W_i = p_i, x(W_i) = W_i, v_i \in W_i (1 \leq i \leq r-2), v_{r-1} \in W_{r-2}^{\perp}\}.$$

Let $\zeta : \mathcal{G}_{\mathbf{m}} \rightarrow \mathcal{X}$ be the projection to the first two factors. We consider the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$. Then $\pi^{(\mathbf{m})}$ is decomposed as

$$\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \xrightarrow{\varphi} \mathcal{G}_{\mathbf{m}} \xrightarrow{\zeta} \mathcal{X}_{\mathbf{m}},$$

where φ is defined by $(x, \mathbf{v}, gB^{\theta}) \mapsto (x, \mathbf{v}, (gM_{p_i})_{1 \leq i \leq r-2})$. The map φ is surjective since $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. It follows that $\zeta(\mathcal{G}_{\mathbf{m}})$ coincides with $\mathcal{X}_{\mathbf{m}}$. Since $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, we have $\dim \tilde{\mathcal{X}}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$. This implies that $\dim \mathcal{G}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$. Note that in the case where $r = 2$, we have $\mathbf{m} = (n, 0)$, and $\mathcal{G}_{\mathbf{m}} = G^{i\theta} \times V = \mathcal{X}_{\mathbf{m}}$. By modifying the definition of $\mathcal{K}_{\mathbf{m}}$ in 2.5, we define a variety $\mathcal{H}_{\mathbf{m}}$ by

$$\mathcal{H}_{\mathbf{m}} = \{(x, \mathbf{v}, (W_i), \phi_1, \phi_2) \mid (x, \mathbf{v}, (W_i)) \in \mathcal{G}_{\mathbf{m}}, \phi_1 : W_1 \xrightarrow{\sim} V_0, \phi_2 : W_1^{\perp}/W_1 \xrightarrow{\sim} \bar{V}_0 \text{ (symplectic isom.)}\},$$

where $V_0 = M_{p_1}$ and $\bar{V}_0 = M_{p_1}^{\perp}/M_{p_1}$. We also define a variety $\tilde{\mathcal{Z}}_{\mathbf{m}}$ by

$$\tilde{\mathcal{Z}}_{\mathbf{m}} = \{(x, \mathbf{v}, gB^{\theta}, \phi_1, \phi_2) \mid (x, \mathbf{v}, gB^{\theta}) \in \tilde{\mathcal{X}}_{\mathbf{m}}, \phi_1 : g(M_{p_1}) \xrightarrow{\sim} V_0, \phi_2 : g(M_{p_1})^{\perp}/g(M_{p_1}) \xrightarrow{\sim} \bar{V}_0\}.$$

Assume that $r \geq 3$, and let $\mathbf{m}' = (m_2, \dots, m_r)$ for $\mathbf{m} = (m_1, \dots, m_r)$, and $G_1, \tilde{G}_1, \mathcal{X}'_{\mathbf{m}'}, G_2$, etc., be as in 2.5. $\tilde{\mathcal{X}}_{\mathbf{m}'}$ is defined for $\mathcal{X}'_{\mathbf{m}'}$ in a similar way as $\tilde{\mathcal{X}}_{\mathbf{m}}$ is defined

for $\mathcal{X}_{\mathbf{m}}$. (Hence in the case where $r = 3$, $\mathcal{X}'_{\mathbf{m}'} = \mathcal{G}'_{\mathbf{m}'} = G_2^{u\theta} \times \bar{V}_0$.) We have the following commutative diagram

$$\begin{array}{ccccc}
 \tilde{G}_1 \times \tilde{\mathcal{X}}'_{\mathbf{m}'} & \xleftarrow{\tilde{\sigma}} & \tilde{\mathcal{Z}}_{\mathbf{m}} & \xrightarrow{\tilde{q}} & \tilde{\mathcal{X}}_{\mathbf{m}} \\
 \pi^1 \times \varphi' \downarrow & & \downarrow \tilde{\varphi} & & \downarrow \varphi \\
 G_1 \times \mathcal{G}'_{\mathbf{m}'} & \xleftarrow{\sigma} & \mathcal{H}_{\mathbf{m}} & \xrightarrow{q} & \mathcal{G}_{\mathbf{m}} \\
 \text{id} \times \zeta' \downarrow & & & & \downarrow \zeta \\
 G_1 \times \mathcal{X}'_{\mathbf{m}'} & & & & \mathcal{X}_{\mathbf{m}},
 \end{array} \tag{8.1.1}$$

where q, σ are defined in a similar way as in 2.5, and

$$\begin{aligned}
 \tilde{q} &: (x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mapsto (x, \mathbf{v}, gB^\theta), \\
 \tilde{\varphi} &: (x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mapsto (x, \mathbf{v}, (gM_{p_i})_{1 \leq i \leq r-2}, \phi_1, \phi_2).
 \end{aligned}$$

The map $\tilde{\sigma}$ is defined as follows. Let $\mathcal{F}^\theta(V)$ be the set of complete isotropic flags in V , and $\tilde{\mathcal{F}}_{\mathbf{m}}^\theta$ the set of $((x, \mathbf{v}), (V_i)) \in \mathcal{X}_{\mathbf{m}} \times \mathcal{F}^\theta(V)$ such that (V_i) is x -stable and that $v_i \in V_{p_i}$ for $i = 1, \dots, r-1$. Then $\tilde{\mathcal{F}}_{\mathbf{m}}^\theta$ is isomorphic to $\tilde{\mathcal{X}}_{\mathbf{m}}$, and

$$\tilde{\mathcal{Z}}_{\mathbf{m}} \simeq \{(x, \mathbf{v}, (V_i), \phi_1, \phi_2) \mid (x, \mathbf{v}, (V_i)) \in \tilde{\mathcal{F}}_{\mathbf{m}}^\theta, \phi_1 : V_{p_1} \xrightarrow{\sim} V_0, \phi_2 : V_{p_1}^\perp / V_{p_1} \xrightarrow{\sim} \bar{V}_0\}.$$

We define $\tilde{\sigma} : \tilde{\mathcal{Z}}_{\mathbf{m}} \rightarrow \tilde{G}_1 \times \tilde{\mathcal{X}}'_{\mathbf{m}'}$ by $(x, \mathbf{v}, (V_i), \phi_1, \phi_2) \mapsto (\xi_1, \xi_2)$ with

$$\begin{aligned}
 \xi_1 &= (\phi_1(x')\phi_1^{-1}, (\phi_1(V_i))_{i \leq p_1}) \in \tilde{G}_1, \\
 \xi_2 &= (\phi_2(x'')\phi_2^{-1}, \phi_2(\bar{\mathbf{v}}), (\phi_2(V_i/V_{p_1}))_{i \geq p_1}) \in \tilde{\mathcal{X}}'_{\mathbf{m}'},
 \end{aligned}$$

where x' (respectively x'') is the restriction of x on V_{p_1} (respectively $V_{p_1}^\perp / V_{p_1}$), $\bar{\mathbf{v}} = (\bar{v}_2, \dots, \bar{v}_{r-1})$ with \bar{v}_i the image of v_i on $V_{p_1}^\perp / V_{p_1}$ for $\mathbf{v} = (v_1, \dots, v_{r-1})$. One can check that the squares in the diagram are both cartesian squares. Put $H_0 = G_1 \times G_2^\theta$. Then as in (2.5.2) and (2.5.3), we have:

(8.1.2) q is a principal bundle with fibre isomorphic to H_0 , and σ is a locally trivial fibration with smooth fibre of dimension $\dim H + (r-2)m_1$.

8.2 For a fixed k , we consider the variety $\tilde{\mathcal{Y}}_{\mathbf{m}(k)}^\dagger = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}(k)}^0)$ as in 3.3. Let $\mathcal{G}_{\mathbf{m}(k), \text{reg}} = \zeta^{-1}(\mathcal{Y}_{\mathbf{m}(k)}^0)$ be a locally closed subvariety of $\mathcal{G}_{\mathbf{m}}$ (not of $\mathcal{G}_{\mathbf{m}(k)}$; note that $\mathcal{G}_{\mathbf{m}(k)}$ is not defined since $\mathbf{m}(k) \notin \mathcal{Q}_{m,r}^0$). We define $\mathbf{m}'(k) = \mathbf{m}'(r-2, k)$ similarly to $\mathbf{m}(k)$, by replacing \mathbf{m} by \mathbf{m}' . Then the varieties $\mathcal{Y}_{\mathbf{m}'(k)}^0, \tilde{\mathcal{Y}}_{\mathbf{m}'(k)}^\dagger$ and $\mathcal{G}'_{\mathbf{m}'(k), \text{reg}}$ are defined similarly with respect to $\mathcal{X}'_{\mathbf{m}'}$. The commutative diagram

(8.1.1) induces a commutative diagram

$$\begin{array}{ccccc}
 \tilde{G}_{1,\text{reg}} \times \tilde{\mathcal{Y}}_{\mathbf{m}'(k)}^\dagger & \longleftarrow & \tilde{\mathcal{Z}}_{\mathbf{m}(k)}^\dagger & \longrightarrow & \tilde{\mathcal{Y}}_{\mathbf{m}(k)}^\dagger \\
 \downarrow & & \downarrow & & \downarrow \varphi_0 \\
 G_{1,\text{reg}} \times \mathcal{G}'_{\mathbf{m}'(k),\text{reg}} & \longleftarrow & \mathcal{H}_{\mathbf{m}(k),\text{reg}} & \longrightarrow & \mathcal{G}_{\mathbf{m}(k),\text{reg}} \\
 \downarrow & & & & \downarrow \zeta_0 \\
 G_{1,\text{reg}} \times \mathcal{Y}_{\mathbf{m}'(k)}^0 & & & & \mathcal{Y}_{\mathbf{m}(k)}^0,
 \end{array} \tag{8.2.1}$$

where $\mathcal{H}_{\mathbf{m}(k),\text{reg}} = q^{-1}(\mathcal{G}_{\mathbf{m}(k),\text{reg}})$, $\tilde{\mathcal{Z}}_{\mathbf{m}(k)}^\dagger = \tilde{q}^{-1}(\tilde{\mathcal{Y}}_{\mathbf{m}(k)}^\dagger)$, and φ_0, ζ_0 are restrictions of φ, ζ , respectively. Again, the squares in the diagrams are both cartesian.

By (3.3.1) we have a decomposition

$$\tilde{\mathcal{Y}}_{\mathbf{m}(k)}^\dagger = \coprod_{\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}(k))} \tilde{\mathcal{Y}}_{\mathbf{I}}, \tag{8.2.2}$$

into irreducible components, where

$$\mathcal{I}^{(\mathbf{m})}(\mathbf{m}(k)) = \{\mathbf{I} \in \mathcal{I}(\mathbf{m}(k)) \mid I_i = I_i^\circ \ (1 \leq i \leq r-2), I_{r-1}, I_r \subset I_{r-1}^\circ\}.$$

Let $\psi_{\mathbf{I}}$ be the restriction of the map $\tilde{\mathcal{Y}}_{\mathbf{m}(k)}^\dagger \rightarrow \mathcal{Y}_{\mathbf{m}(k)}^0$ on $\tilde{\mathcal{Y}}_{\mathbf{I}}$ for each $\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}(k))$ as in 1.3. Under the notation in 1.3, the map $\psi_{\mathbf{I}}$ factors through $\hat{\mathcal{Y}}_{\mathbf{I}}$ as $\psi_{\mathbf{I}} = \eta_{\mathbf{I}} \circ \xi_{\mathbf{I}}$ (see (1.3.1)). We define a map $\eta_{\mathbf{I}}^{(\mathbf{m})} : \hat{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{G}_{\mathbf{m}(k),\text{reg}}$ as the quotient of the map $(g, (t, \mathbf{v})) \mapsto (gtg^{-1}, g\mathbf{v}, (g\overline{M}_{I_i})_{1 \leq i \leq r-2})$. Thus $\eta_{\mathbf{I}} : \hat{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}_{\mathbf{m}(k)}^0$ factors through $\mathcal{G}_{\mathbf{m}(k),\text{reg}}$ as $\eta_{\mathbf{I}} = \zeta_0 \circ \eta_{\mathbf{I}}^{(\mathbf{m})}$.

The variety $\tilde{\mathcal{Y}}_{\mathbf{m}'(k)}^\dagger$ is also decomposed into irreducible components as in (8.2.2), by using the parameter set $\mathcal{I}^{(\mathbf{m}')}(\mathbf{m}'(k))$. For $\mathbf{I}' \in \mathcal{I}^{(\mathbf{m}')}(\mathbf{m}'(k))$, the varieties $\hat{\mathcal{Y}}_{\mathbf{I}'}$ are defined with respect to $\mathcal{X}'_{\mathbf{m}'}$. Note that the set $\mathcal{I}^{(\mathbf{m}')}(\mathbf{m}'(k))$ is in bijection with the set $\mathcal{I}^{(\mathbf{m})}(\mathbf{m}(k))$ under the correspondence $\mathbf{I}' \leftrightarrow \mathbf{I} = (I_1^\circ, \mathbf{I}')$. Now the commutative diagram (8.2.1) implies the following commutative diagram for each $\mathbf{I} \in \mathcal{I}^{(\mathbf{m})}(\mathbf{m}(k))$:

$$\begin{array}{ccccc}
 \tilde{G}_{1,\text{reg}} \times \hat{\mathcal{Y}}_{\mathbf{I}'} & \longleftarrow & \hat{\mathcal{Z}}_{\mathbf{I}} & \longrightarrow & \hat{\mathcal{Y}}_{\mathbf{I}} \\
 \psi_0^1 \times \eta_{\mathbf{I}'}^{(\mathbf{m}')} \downarrow & & \downarrow & & \downarrow \eta_{\mathbf{I}}^{(\mathbf{m})} \\
 G_{1,\text{reg}} \times \mathcal{G}'_{\mathbf{m}'(k),\text{reg}} & \longleftarrow & \mathcal{H}_{\mathbf{m}(k),\text{reg}} & \longrightarrow & \mathcal{G}_{\mathbf{m}(k),\text{reg}},
 \end{array} \tag{8.2.3}$$

where $\hat{\mathcal{Z}}_{\mathbf{I}}$ is the fibre product of $\mathcal{H}_{\mathbf{m}(k),\text{reg}}$ and $\hat{\mathcal{Y}}_{\mathbf{I}}$ over $\mathcal{G}_{\mathbf{m}(k),\text{reg}}$. Both squares in the diagram are cartesian squares. By 1.3, we know that $\eta_{\mathbf{I}} : \hat{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}_{\mathbf{m}(k)}^0$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$. Here we note that

(8.2.4) $\eta_{\mathbf{I}}^{(\mathbf{m})} : \widehat{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{G}_{\mathbf{m}(k),\text{reg}}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$. In particular, the restriction of ζ_0 gives an isomorphism $\mathcal{G}_{\mathbf{m}(k),\text{reg}} \xrightarrow{\sim} \mathcal{Y}_{\mathbf{m}(k)}^0$.

In fact, this is clear in the case where $r = 2$, since $\mathcal{G}_{\mathbf{m}(k),\text{reg}} = \mathcal{Y}_{\mathbf{m}(k)}^0$. Assume that $r \geq 3$, and that (8.2.4) holds for $r - 1$. By induction, $\eta_{\mathbf{I}'}^{(\mathbf{m}')}$: $\widehat{\mathcal{Y}}_{\mathbf{I}'}$ \rightarrow $\widehat{\mathcal{G}}'_{\mathbf{m}'(k),\text{reg}}$ is a finite Galois covering with group $\mathcal{W}'_{\mathbf{I}'}$. Then $\psi_0^1 \times \eta_{\mathbf{I}'}^{(\mathbf{m}')}$ is a finite Galois covering with group $S_{m_1} \times \mathcal{W}'_{\mathbf{I}'} = \mathcal{W}_{\mathbf{I}}$. Since both squares in the diagram (8.2.3) are cartesian, $\eta_{\mathbf{I}}^{(\mathbf{m})}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$. Since $\eta_{\mathbf{I}}$ is also a finite Galois covering with group $\mathcal{W}_{\mathbf{I}}$, we conclude that $\mathcal{G}_{\mathbf{m}(k),\text{reg}} \simeq \mathcal{Y}_{\mathbf{m}(k)}^0$. This proves (8.2.4).

Since $\eta_{\mathbf{I}}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}} \simeq \mathcal{W}_{\mathbf{m}(k)}$, by (1.5.3) we have

$$(\eta_{\mathbf{I}})! \bar{\mathbf{Q}}_l \simeq \bigoplus_{\rho_0 \in \mathcal{W}_{\mathbf{m}(k)}^\wedge} \rho_0 \otimes \mathcal{L}_{\rho_0},$$

where \mathcal{L}_{ρ_0} is a simple local system on $\mathcal{Y}_{\mathbf{m}(k)}^0$. We regard \mathcal{L}_{ρ_0} as a simple local system on $\mathcal{G}_{\mathbf{m}(k),\text{reg}}$ under the isomorphism $\mathcal{G}_{\mathbf{m}(k),\text{reg}} \simeq \mathcal{Y}_{\mathbf{m}(k)}^0$.

Now take $\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^\wedge$. There exist a unique integer k and $\rho_0 \in \mathcal{W}_{\mathbf{m}(k)}^\wedge$ such that $\rho = V_{\rho_0}^{\natural}$. Then we have $K_\rho = \text{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_{\rho_0})[d_{\mathbf{m}(k)}]$. Put $A_\rho = \text{IC}(\bar{\mathcal{G}}_{\mathbf{m}(k)}, \mathcal{L}_{\rho_0})[d_{\mathbf{m}(k)}]$. A_ρ is an H -equivariant simple perverse sheaf on $\bar{\mathcal{G}}_{\mathbf{m}(k)}$, and we regard it as a perverse sheaf on $\mathcal{G}_{\mathbf{m}}$ by extension by zero.

We show the following fact.

Proposition 8.3. *Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type.*

- (i) $\varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{G}_{\mathbf{m}}$ equipped with $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action, and is decomposed as

$$\varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^\wedge} \rho \otimes A_\rho.$$

- (ii) $\zeta_! A_\rho \simeq K_\rho$.

Proof. We prove the proposition by induction on r . In the case where $r = 2$, $\mathbf{m} = (m_1, m_2) = (n, 0)$ since $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Thus $\mathcal{G}_{\mathbf{m}} = \mathcal{X}_{\mathbf{m}} = \mathcal{X}$ and $\varphi = \pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$. Moreover, ζ is the identity map, and $\mathcal{W}_{\mathbf{m}}^{\natural} = W_n$. Hence the proposition follows from [SS1, Thm. 4.2]. Assume that $r \geq 3$, and that the proposition holds for $r - 1$. We know, under the notation of 2.6, that

$$\pi_1^1 \bar{\mathbf{Q}}_l[\dim G_1] \simeq \bigoplus_{\rho_1 \in S_{m_1}^\wedge} \rho_1 \otimes \text{IC}(G_1, \mathcal{L}_{\rho_1})[\dim G_1]. \tag{8.3.1}$$

On the other hand, by applying the induction hypothesis to $\mathcal{X}'_{\mathbf{m}'}$, we have

$$\varphi'_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}'}] \simeq \bigoplus_{\rho' \in (\mathcal{W}_{\mathbf{m}'}^{\natural})^\wedge} \rho' \otimes A_{\rho'}, \tag{8.3.2}$$

where $A_{\rho'}$ is a simple perverse sheaf on $\mathcal{G}'_{\mathbf{m}'}$ defined similarly to A_{ρ} . By applying the argument in 2.6, one can find a unique H -equivariant simple perverse sheaf \tilde{A}_{ρ} on $\mathcal{G}_{\mathbf{m}}$ such that

$$q^* \tilde{A}_{\rho}[\beta_2] \simeq \sigma^*(K_{\rho_1} \boxtimes A_{\rho'})[\beta_1], \quad (8.3.3)$$

where $\beta_1 = \dim H + (r-2)m_1$, $\beta_2 = \dim H_0$, and $K_{\rho_1} = \mathrm{IC}(G_1, \mathcal{L}_{\rho_1})[\dim G_1]$. It follows from the discussion in 8.2 that \tilde{A}_{ρ} actually coincides with A_{ρ} . Put $K = \varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$, $K' = \varphi'_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}'}]$ and $K'' = (\pi^1)_! \bar{\mathbf{Q}}_l[\dim G_1]$. Since both squares in (8.1.1) are cartesian, we have

$$q^* K[\beta_2] \simeq \sigma^*(K'' \boxtimes K')[\beta_1].$$

Combining (8.3.1), (8.3.2), and (8.3.3), we obtain

$$K \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^{\wedge}} \rho \otimes A_{\rho}.$$

By this decomposition, $K = \varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is regarded as a complex with $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action. This proves (i).

Next we show (ii). Since ζ is proper, $\zeta_! A_{\rho}$ is a semisimple complex on $\mathcal{X}_{\mathbf{m}}$. By (i), $K = \varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf. Since $\zeta_! K \simeq (\pi^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf, it follows that $\zeta_! A_{\rho}$ is a semisimple perverse sheaf. By (8.2.4) we have $\zeta_! A_{\rho}|_{\mathcal{Y}_{\mathbf{m}(k)}^0} \simeq K_{\rho}|_{\mathcal{Y}_{\mathbf{m}(k)}^0}$. Hence $\zeta_! A_{\rho}$ contains K_{ρ} as a direct summand. By applying $\zeta_!$ to the formula in (i), we have

$$(\pi^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^{\wedge}} \rho \otimes \zeta_! A_{\rho}.$$

By Theorem 3.2, we have

$$(\pi^{(\mathbf{m})})_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^{\wedge}} \rho \otimes K_{\rho}.$$

By comparing these two formulas, we obtain (ii). The proposition is proved. \square

8.4. For each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, put $\mathcal{G}_{\mathbf{m}, \mathrm{uni}} = \zeta^{-1}(\mathcal{X}_{\mathbf{m}, \mathrm{uni}})$. Then the map $\pi_1^{(\mathbf{m})}$ is decomposed as

$$\pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}, \mathrm{uni}} \xrightarrow{\varphi_1} \mathcal{G}_{\mathbf{m}, \mathrm{uni}} \xrightarrow{\zeta_1} \mathcal{X}_{\mathbf{m}, \mathrm{uni}},$$

where φ_1, ζ_1 are restrictions of φ, ζ , respectively. Note that φ_1 is surjective. Put $\mathcal{H}_{\mathbf{m}, \mathrm{uni}} = q^{-1}(\mathcal{G}_{\mathbf{m}, \mathrm{uni}})$. The inclusion map $\mathcal{G}_{\mathbf{m}, \mathrm{uni}} \hookrightarrow \mathcal{G}_{\mathbf{m}}$ is compatible with the diagram (8.1.1); namely, we have a commutative diagram

$$\begin{array}{ccccc} G_1 \times \mathcal{G}'_{\mathbf{m}'} & \xleftarrow{\sigma} & \mathcal{H}_{\mathbf{m}} & \xrightarrow{q} & \mathcal{G}_{\mathbf{m}} \\ \uparrow & & \uparrow & & \uparrow \\ G_{1, \mathrm{uni}} \times \mathcal{G}'_{\mathbf{m}', \mathrm{uni}} & \xleftarrow{\sigma_1} & \mathcal{H}_{\mathbf{m}, \mathrm{uni}} & \xrightarrow{q_1} & \mathcal{G}_{\mathbf{m}, \mathrm{uni}} \end{array} \quad (8.4.1)$$

where σ_1, q_1 are restrictions of σ, q respectively, and vertical maps are natural inclusions. A similar property as (8.1.2) still holds for q_1, σ_1 , and both squares are cartesian squares.

For each $\lambda \in \mathcal{P}(\mathbf{m}(k))$, we define a subset \mathcal{G}_λ of $\mathcal{G}_{\mathbf{m}, \text{uni}}$ inductively as follows; Write $\lambda = (\lambda^{(1)}, \lambda')$ with $\lambda' \in \mathcal{P}_{n-m_1, r-1}$. Assume that the G_2^θ -stable subset $\mathcal{G}'_{\lambda'}$ of $\mathcal{G}'_{\mathbf{m}', \text{uni}}$ was defined. Let $\mathcal{O}_{\lambda^{(1)}}$ be the G_1 -orbit in $(G_1)_{\text{uni}}$ corresponding to $\lambda^{(1)}$, and put $Z = \mathcal{O}_{\lambda^{(1)}} \times \mathcal{G}'_{\lambda'}$. Then $\sigma_1^{-1}(Z)$ is an H_0 -stable subset of $\mathcal{H}_{\mathbf{m}, \text{uni}}$, and $q_1 \circ \sigma_1^{-1}(Z)$ coincides with the quotient of $\sigma_1^{-1}(Z)$ by H_0 . We define \mathcal{G}_λ by $\mathcal{G}_\lambda = q_1 \circ \sigma_1^{-1}(Z)$. In the case where $r = 3$, $\mathbf{m} = (m_1, m_2, 0)$ and $\mathbf{m}(k) = (m_1, k, k')$ with $k+k' = m_2$. $\mathbf{m}' = (m_2, 0)$ and $\mathcal{G}'_{\mathbf{m}', \text{uni}} = \mathcal{X}'_{\text{uni}}$, where \mathcal{X}' is the exotic symmetric space for $r = 2$ associated to G_2 . In this case, $\lambda' = (\lambda^{(2)}, \lambda^{(3)}) \in \mathcal{P}_{m_2, 2}$, and we take $\mathcal{G}'_{\lambda'}$ as the G_2^θ -orbit in $\mathcal{X}'_{\text{uni}}$ corresponding to λ' . Thus \mathcal{G}_λ is well defined, and \mathcal{G}_λ turns out to be a smooth irreducible H -stable subvariety of $\mathcal{G}_{\mathbf{m}, \text{uni}}$. By induction on r , we show the following formulas:

$$\dim \mathcal{G}_\lambda = 2n^2 - 2n - 2n(\lambda) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + \sum_{i=1}^{r-1} (r-i+1)|\lambda^{(i)}|. \tag{8.4.2}$$

In fact, by (8.1.2), we have

$$\dim \mathcal{G}_\lambda = \dim \mathcal{O}_{\lambda^{(1)}} + \dim \mathcal{G}'_{\lambda'} + (\dim H + (r-2)m_1) - \dim H_0. \tag{8.4.3}$$

By applying the induction hypothesis for λ' , (with $n' = n - m_1$), we have

$$\dim \mathcal{G}'_{\lambda'} = 2n'^2 - 2n' - 2n(\lambda') - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + \sum_{i=2}^{r-1} (r-i+1)|\lambda^{(i)}|,$$

and $\dim \mathcal{O}_{\lambda^{(1)}} = m_1^2 - m_1 - 2n(\lambda^{(1)})$. Moreover, $\dim H = 2n^2 + n$, $\dim H_0 = m_1^2 + 2n'^2 + n'$. Substituting these formulas into (8.4.3), we obtain (8.4.2).

Let $\overline{\mathcal{G}}_\lambda$ be the closure of \mathcal{G}_λ in $\mathcal{G}_{\mathbf{m}, \text{uni}}$. Recall $\tilde{\mathcal{F}}_\lambda, \tilde{\mathcal{F}}_\lambda^0$ in 6.4. It follows from the construction, $\tilde{\mathcal{F}}_\lambda$ is a closed subset of $\mathcal{G}_{\mathbf{m}, \text{uni}}$. We have the following lemma.

Lemma 8.5. *Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type.*

- (i) $\overline{\mathcal{G}}_\lambda$ coincides with $\tilde{\mathcal{F}}_\lambda$. In particular, $\tilde{\mathcal{F}}_\lambda^0$ is an open dense subset of $\overline{\mathcal{G}}_\lambda$.
- (ii) $\zeta_1(\overline{\mathcal{G}}_\lambda) = \overline{X}_\lambda$, and $\zeta_1^{-1}(X_\lambda) = \tilde{\mathcal{F}}_\lambda^0$. Hence the restriction of ζ_1 on $\zeta_1^{-1}(X_\lambda)$ gives an isomorphism $\zeta_1^{-1}(X_\lambda) \xrightarrow{\sim} X_\lambda$.

Proof. By induction on r , one can verify that $\tilde{\mathcal{F}}_\lambda^0 \subset \mathcal{G}_\lambda$. Hence $\tilde{\mathcal{F}}_\lambda \subset \overline{\mathcal{G}}_\lambda$. By Lemma 6.6 and by (8.4.2), $\dim \tilde{\mathcal{F}}_\lambda = \dim \overline{\mathcal{G}}_\lambda$. Since both are irreducible closed subsets of $\mathcal{G}_{\mathbf{m}, \text{uni}}$, we have $\tilde{\mathcal{F}}_\lambda = \overline{\mathcal{G}}_\lambda$. This proves (i). Then the restriction of ζ_1 on $\overline{\mathcal{G}}_\lambda$ coincides with the map $\pi_\lambda : \tilde{\mathcal{F}}_\lambda \rightarrow \overline{X}_\lambda$. Hence (ii) follows from Proposition 6.7. \square

8.6 Recall the set $\tilde{\mathcal{P}}(\mathbf{m})$ in (6.8.1) for each $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. It is well known that $(\mathcal{W}_{\mathbf{m}}^{\mathfrak{h}})^\wedge$ is naturally parametrized by the set $\tilde{\mathcal{P}}(\mathbf{m})$. We denote by ρ_λ the irreducible representation of $\mathcal{W}_{\mathbf{m}}^{\mathfrak{h}}$ corresponding to $\lambda \in \tilde{\mathcal{P}}(\mathbf{m})$.

For each $\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})$, let $V(\lambda)$ be the irreducible representation of $W_{n,r}$ obtained from ρ_λ as in 7.13. Then we have

$$W_{n,r}^\wedge \simeq \coprod_{\mathbf{m} \in \mathcal{Q}_{n,r}^0} \widetilde{\mathcal{P}}(\mathbf{m}). \tag{8.6.1}$$

We have the following refinements of Theorem 7.12 and Corollary 7.14.

Theorem 8.7. *Assume that $\mathcal{X}_{\mathbf{m},\text{uni}}$ is of exotic type with $\mathbf{m} \in \mathcal{Q}_{n,r}^0$.*

- (i) *Let L_ρ be as in Theorem 7.12. Assume that $\rho = \rho_\lambda \in (\mathcal{W}_{\mathbf{m}}^{\natural})^\wedge$ for $\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})$. Then we have*

$$L_\rho \simeq \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda].$$

- (ii) *(Springer correspondence for $\mathcal{W}_{\mathbf{m}}^{\natural}$)*

$$(\pi_1^{(\mathbf{m})})_! \overline{\mathbf{Q}}_l[d'_\mathbf{m}] \simeq \bigoplus_{\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})} \rho_\lambda \otimes \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda].$$

- (iii) *(Springer correspondence for $W_{n,r}$)*

$$(\overline{\pi}_{\mathbf{m},1})_! \overline{\mathbf{Q}}_l[d'_\mathbf{m}] \simeq \bigoplus_{\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})} V(\lambda) \otimes \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda].$$

Proof. By Proposition 8.3, we know that $\zeta_1 A_\rho = K_\rho$. Hence, by the base change theorem, $(\zeta_1)_!(A_\rho|_{\mathcal{G}_{\mathbf{m},\text{uni}}}) \simeq K_\rho|_{\mathcal{X}_{\mathbf{m},\text{uni}}}$. For $\lambda \in \mathcal{P}(\mathbf{m}(k))$, we define a simple perverse sheaf B_λ on $\mathcal{G}_{\mathbf{m},\text{uni}}$ inductively as follows: in the case where $r = 2$, put $B_\lambda = \text{IC}(\overline{\mathcal{O}}_\lambda, \overline{\mathbf{Q}}_l)[\dim \mathcal{O}_\lambda]$, where \mathcal{O}_λ is the H -orbit in $G_{\text{uni}}^{\text{orb}} \times V$ corresponding to λ . In general for $\lambda \in \mathcal{P}(\mathbf{m}(k))$, put $\lambda = (\lambda^{(1)}, \lambda')$ with $|\lambda^{(1)}| = m_1$, $\lambda' \in \mathcal{P}(\mathbf{m}'(k))$. We assume that a simple perverse sheaf $B_{\lambda'}$ on $\mathcal{G}'_{\mathbf{m}',\text{uni}}$ is already defined. By a similar construction as \widetilde{A}_ρ in the proof of Proposition 8.3, there exists a unique simple perverse sheaf B_λ on $\mathcal{G}_{\mathbf{m},\text{uni}}$ satisfying the relation

$$q_1^* B_\lambda[\beta_2] \simeq \sigma_1^*(K_{\lambda^{(1)}} \boxtimes B_{\lambda'})[\beta_1],$$

where $K_{\lambda^{(1)}} = \text{IC}(\overline{\mathcal{O}}_{\lambda^{(1)}}, \overline{\mathbf{Q}}_l)[\dim \mathcal{O}_{\lambda^{(1)}}]$ for the GL_{m_1} -orbit $\mathcal{O}_{\lambda^{(1)}}$ in $(\text{GL}_{m_1})_{\text{uni}}$ corresponding to $\lambda^{(1)}$, and β_1, β_2 are as in (8.3.3). Assume that $\rho = \rho_\lambda$. By comparing the construction of \widetilde{A}_ρ and of B_λ , and by using the induction hypothesis, we see that the restriction of A_ρ on $\mathcal{G}_{\mathbf{m},\text{uni}}$ coincides with B_λ , up to shift. Moreover, by induction, one can show that the restriction of B_λ on \mathcal{G}_λ is a constant sheaf $\overline{\mathbf{Q}}_l$. In particular, $\text{supp } B_\lambda = \overline{\mathcal{G}}_\lambda$. Then by Lemma 8.5, the support of $(\zeta_1)_! B_\lambda$ coincides with \overline{X}_λ . By (7.12.2), we know that the restriction of K_ρ on $\mathcal{X}_{\mathbf{m},\text{uni}}$ is a simple perverse sheaf L_ρ , up to shift. We show that

$$L_\rho \simeq \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda]. \tag{8.7.1}$$

For this it is enough to see that $L_\rho|_{X_\lambda}$ is a constant sheaf $\overline{\mathbf{Q}}_l$. But by Lemma 8.5 (ii), $\zeta_1^{-1}(X_\lambda) = \widetilde{\mathcal{F}}_\lambda^0 \subset \mathcal{G}_\lambda$, and the restriction of B_λ on \mathcal{G}_λ is the constant sheaf $\overline{\mathbf{Q}}_l$, up to shift. Since $\zeta_1^{-1}(X_\lambda) \xrightarrow{\sim} X_\lambda$ by Lemma 8.5 (ii), we see that $(\zeta_1)_! B_\lambda|_{X_\lambda}$ coincides with $\overline{\mathbf{Q}}_l$ up to shift. Thus (8.7.1) holds, and (i) follows. (ii) and (iii) then follow from Theorem 7.12 and Corollary 7.14. The theorem is proved. \square

8.8. We now assume that \mathcal{X} is of enhanced type with $\mathbf{m} \in \mathcal{Q}_{n,r}$. We define a variety $\mathcal{G}_{\mathbf{m}}$ by

$$\mathcal{G}_{\mathbf{m}} = \{(x, \mathbf{v}, (W_i)_{1 \leq i \leq r-1}) \mid (x, \mathbf{v}) \in \mathcal{X}, (W_i) : \text{partial flag in } V, \\ \dim W_i = p_i, x(W_i) = W_i, v_i \in W_i \ (1 \leq i \leq r-1)\}.$$

Let $\zeta : \tilde{\mathcal{G}}_{\mathbf{m}} \rightarrow \mathcal{X}$ be the projection to the first and second factors. Then the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$ is decomposed as $\pi^{(\mathbf{m})} = \zeta \circ \varphi$, where $\varphi : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \tilde{\mathcal{G}}_{\mathbf{m}}, \zeta : \tilde{\mathcal{G}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$ are defined similarly to the exotic case. Here φ and ζ are surjective maps. Since $\dim \tilde{\mathcal{X}}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$ by Lemma 4.2, we have $\dim \tilde{\mathcal{G}}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$. Put $V_0 = M_{p_1}, \bar{V}_0 = V/M_{p_1}$. We define a variety $\mathcal{H}_{\mathbf{m}}$ by

$$\mathcal{H}_{\mathbf{m}} = \{(x, \mathbf{v}, (W_i), \phi_1, \phi_2) \mid (x, \mathbf{v}, (W_i)) \in \mathcal{G}_{\mathbf{m}}, \\ \phi_1 : W_1 \xrightarrow{\sim} V_0, \phi_2 : V/W_1 \xrightarrow{\sim} \bar{V}_0\}.$$

We also define a variety $\tilde{\mathcal{Z}}_{\mathbf{m}}$ by

$$\tilde{\mathcal{Z}}_{\mathbf{m}} = \{(x, \mathbf{v}, gB^\theta, \phi_1, \phi_2) \mid (x, \mathbf{v}, gB^\theta) \in \tilde{\mathcal{X}}_{\mathbf{m}}, \\ \phi_1 : g(M_{p_1}) \xrightarrow{\sim} V_0, \phi_2 : V/g(M_{p_1}) \xrightarrow{\sim} \bar{V}_0\}.$$

Assume that $r \geq 3$, and let $\mathbf{m}' = (m_2, \dots, m_r)$ for $\mathbf{m} = (m_1, \dots, m_r)$. Let $G_1 = \text{GL}(V_0)$ and $G_2 = \text{GL}(\bar{V}_0) \times \text{GL}(\bar{V}_0)$ with the permutation action $\theta : G_2 \rightarrow G_2$. Then $\tilde{G}_1, \tilde{\mathcal{X}}'_{\mathbf{m}'}, \mathcal{X}'_{\mathbf{m}'}$, etc., are defined similarly to the exotic case. Note that $\mathcal{X}'_{\mathbf{m}'}$ is a closed subvariety of $G_2^{u_\theta} \times \bar{V}_0^{r-2} \simeq \text{GL}(\bar{V}_0) \times \bar{V}_0^{r-2}$. Then a similar commutative diagram to (8.1.1) holds also for the enhanced case. We use the same notation as in (8.1.1). One can check that the maps q, σ satisfy similar properties as in (8.1.2); namely, we have:

(8.8.1) q is a principal bundle with fibre isomorphic to H_0 , and σ is a locally trivial fibration with smooth fibre of dimension $\dim H + (r-1)m_1$. (It should be noticed that the dimension of the fibre is different from the exotic case. Also note in this case, $H_0 \simeq \text{GL}(V_0) \times \text{GL}(\bar{V}_0)$, and $H \simeq \text{GL}(V)$.)

8.9. For $\mathbf{m} \in \mathcal{Q}_{n,r}$, put $\tilde{\mathcal{Y}}_{\mathbf{m}}^\dagger = (\psi^{(\mathbf{m})})^{-1}(\mathcal{Y}_{\mathbf{m}}^0)$. Then under the notation in 4.1, $\tilde{\mathcal{Y}}_{\mathbf{m}}^\dagger$ coincides with $\tilde{\mathcal{Y}}_{\mathbf{I}}^\dagger$ for $\mathbf{I} = \mathbf{I}^\bullet$ in 3.3. In particular, $\psi_{\mathbf{I}} : \tilde{\mathcal{Y}}_{\mathbf{I}} \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{I}} = \mathcal{W}_{\mathbf{m}}^\natural$. $\tilde{\mathcal{Y}}_{\mathbf{m}}^\dagger$ is an open dense subset of $\tilde{\mathcal{Y}}_{\mathbf{m}}$. Put $\mathcal{G}_{\mathbf{m},\text{reg}} = \zeta^{-1}(\mathcal{Y}_{\mathbf{m}}^0)$. $\mathcal{G}_{\mathbf{m},\text{reg}}$ is an open subset of $\mathcal{G}_{\mathbf{m}}$. The varieties $\mathcal{Y}_{\mathbf{m}'}^0, \tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger$ and $\mathcal{G}'_{\mathbf{m}',\text{reg}}$ are defined similarly. As in the exotic case, we have the following commutative diagram:

$$\begin{array}{ccccc} \tilde{G}_{1,\text{reg}} \times \tilde{\mathcal{Y}}_{\mathbf{m}'}^\dagger & \longleftarrow & \tilde{\mathcal{Z}}_{\mathbf{m}}^\dagger & \longrightarrow & \tilde{\mathcal{Y}}_{\mathbf{m}}^\dagger \\ \downarrow & & \downarrow & & \downarrow \varphi_0 \\ G_{1,\text{reg}} \times \mathcal{G}'_{\mathbf{m}',\text{reg}} & \longleftarrow & \mathcal{H}_{\mathbf{m},\text{reg}} & \longrightarrow & \mathcal{G}_{\mathbf{m},\text{reg}} \\ \downarrow & & & & \downarrow \zeta_0 \\ G_{1,\text{reg}} \times \mathcal{Y}_{\mathbf{m}'}^0 & & & & \mathcal{Y}_{\mathbf{m}}^0 \end{array} \tag{8.9.1}$$

By using (8.9.1) and by induction on r , one can show that $\varphi_0 : \widetilde{\mathcal{Y}}_{\mathbf{m}}^{\dagger} \rightarrow \mathcal{G}_{\mathbf{m},\text{reg}}$ is a finite Galois covering with group $\mathcal{W}_{\mathbf{m}}^{\natural}$. It follows that:

$$(8.9.2) \quad \zeta_0 \text{ gives an isomorphism } \mathcal{G}_{\mathbf{m},\text{reg}} \xrightarrow{\sim} \mathcal{Y}_{\mathbf{m}}^0.$$

For each $\rho \in \mathcal{W}_{\mathbf{m}}^{\natural}$, we consider $K_{\rho} = \text{IC}(\mathcal{X}_{\mathbf{m}}, \mathcal{L}_{\rho})[d_{\mathbf{m}}]$, where \mathcal{L}_{ρ} is a simple local system on $\mathcal{Y}_{\mathbf{m}}^0$ obtained from the Galois covering $\mathcal{W}_{\mathbf{m}}^{\natural}$. We regard \mathcal{L}_{ρ} as a local system on $\mathcal{G}_{\mathbf{m},\text{reg}}$ under the isomorphism $\mathcal{G}_{\mathbf{m},\text{reg}} \simeq \mathcal{Y}_{\mathbf{m}}^0$. Put $A_{\rho} = \text{IC}(\mathcal{G}_{\mathbf{m},\text{reg}}, \mathcal{L}_{\rho})[d_{\mathbf{m}}]$.

The following result can be proved in a similar way as Proposition 8.3, by making use of Theorem 4.5, instead of Theorem 3.2.

Proposition 8.10. *Assume that $\mathcal{X}_{\mathbf{m}}$ is of enhanced type.*

- (i) $\varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{G}_{\mathbf{m}}$ equipped with $\mathcal{W}_{\mathbf{m}}^{\natural}$ -action, and is decomposed as

$$\varphi_! \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in (\mathcal{W}_{\mathbf{m}}^{\natural})^{\wedge}} \rho \otimes A_{\rho}.$$

- (ii) $\zeta_! A_{\rho} \simeq K_{\rho}$.

8.11. For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, put $\mathcal{G}_{\mathbf{m},\text{uni}} = \zeta^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}})$. Then the map $\pi_1^{(\mathbf{m})}$ is decomposed as $\pi_1^{(\mathbf{m})} = \zeta_1 \circ \varphi_1$ as in the exotic case (see 8.4), where $\varphi_1 : \widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{G}_{\mathbf{m},\text{uni}}$ and $\zeta_1 : \mathcal{G}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ are restrictions of φ, ζ . Note that φ_1 is surjective. Put $\mathcal{H}_{\mathbf{m},\text{uni}} = q^{-1}(\mathcal{G}_{\mathbf{m},\text{uni}})$. Then we have a similar commutative diagram as (8.4.1).

For $\lambda \in \mathcal{P}(\mathbf{m})$, we define a subset \mathcal{G}_{λ} of $\mathcal{G}_{\mathbf{m},\text{uni}}$ inductively, by applying the discussion in 8.4 for the exotic case. Note that in the case where $r = 2$, $\mathbf{m} = (m_1, m_2)$ and $\mathbf{m}' = (m_2)$. $\mathcal{G}_{\mathbf{m}'} = \mathcal{X}'_{\text{uni}}$. In this case, we take \mathcal{G}'_{λ} as the G_2^{θ} -orbit in $\mathcal{X}'_{\text{uni}}$ corresponding to $\lambda' = \lambda^{(2)}$. Thus \mathcal{G}_{λ} is defined, and \mathcal{G}_{λ} is a smooth irreducible H -stable subvariety of $\mathcal{G}_{\mathbf{m},\text{uni}}$.

As in the exotic case (see (8.4.2)), one can compute the dimension of \mathcal{G}_{λ} by making use of (8.8.1). We have

$$\dim \mathcal{G}_{\lambda} = n^2 - n - 2n(\lambda) + \sum_{i=1}^{r-1} (r-i)|\lambda^{(i)}|. \tag{8.11.1}$$

Recall the definition of $\widetilde{\mathcal{F}}_{\lambda}$ and $\widetilde{\mathcal{F}}_{\lambda}^0$ (in the enhanced case) in 6.18. The following lemma can be proved in a similar way as Lemma 8.5. Note that $\dim \widetilde{\mathcal{F}}_{\lambda} = \dim \bar{\mathcal{G}}_{\lambda}$ by Proposition 5.4 and Proposition 6.20, together with (8.11.1). We use Proposition 6.20 instead of Proposition 6.7.

Lemma 8.12. *Assume that \mathcal{G}_{λ} is of enhanced type. A similar statement as in Lemma 8.5 holds also for \mathcal{G}_{λ} , by replacing X_{λ} by X'_{λ} .*

We can now obtain a refinement of Theorem 7.12 in the enhanced case. The proof is similar to the proof of Theorem 8.7. Note that Theorem 8.13 (ii) is obtained by Li [Li, Thm. 3.2.6] by a different method.

Theorem 8.13. *Assume that $\mathcal{X}_{\mathbf{m},\text{uni}}$ is of enhanced type.*

- (i) Let L_ρ be as in Theorem 7.12. Assume that $\rho = \rho_\lambda \in (\mathcal{W}_m^h)^\wedge$ for $\lambda \in \mathcal{P}(m)$. Then we have

$$L_\rho \simeq \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda].$$

- (ii) (Springer correspondence for \mathcal{W}_m^h)

$$(\pi_1^{(m)})_! \overline{\mathbf{Q}}_l[d'_m] \simeq \bigoplus_{\lambda \in \mathcal{P}(m)} \rho_\lambda \otimes \text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)[\dim X_\lambda].$$

8.14. Assume that \mathcal{X}_m is of exotic type or of enhanced type, under the setting in 8.1. For each $z \in \mathcal{X}_m$, we consider the (small) Springer fibre $\mathcal{B}_z^{(m)} = (\pi^{(m)})^{-1}(z)$. In the exotic case, we also consider the Springer fibre $\mathcal{B}_z = \pi^{-1}(z)$ (see 6.21). We have $\mathcal{B}_z^{(m)} \subset \mathcal{B}_z$. The cohomology group $H^i(\mathcal{B}_z^{(m)}, \overline{\mathbf{Q}}_l)$ has a structure of \mathcal{W}_m^h -module. In turn, $H^i(\mathcal{B}_z, \overline{\mathbf{Q}}_l)$ has a structure of $W_{n,r}$ -module. Put $d_\lambda = (\dim \mathcal{X}_{m,\text{uni}} - \dim X_\lambda)/2$ for $\lambda \in \tilde{\mathcal{P}}(m)$ in the exotic case, and for $\lambda \in \mathcal{P}(m)$ in the enhanced case. Explicitly, we have

$$d_\lambda = \begin{cases} (m_{r-1} - k) + n(\lambda) + n(\lambda^{(r-1)} + \lambda^{(r)}) & : \text{exotic case, } \lambda \in \mathcal{P}(m(k)), \\ n(\lambda) & : \text{enhanced case, } \lambda \in \mathcal{P}(m). \end{cases} \tag{8.14.1}$$

As shown in the example in 5.17, $\dim \mathcal{B}_z^{(m)}$ is not constant for $z \in X_\lambda$. We show the following lemma.

Lemma 8.15. *Assume that \mathcal{X}_m is of exotic type or of enhanced type.*

- (i) For any $z \in X_\lambda$, $\dim \mathcal{B}_z^{(m)} \geq d_\lambda$. The set $z \in X_\lambda$ such that $\dim \mathcal{B}_z^{(m)} = d_\lambda$ forms an open dense subset of X_λ .
- (ii) For any $z \in X_\lambda$, $H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \overline{\mathbf{Q}}_l)$ contains an irreducible \mathcal{W}_m^h -module ρ_λ .

Proof. First we show (ii). Assume that \mathcal{X}_m is of exotic type. For any $z \in \mathcal{X}_{m,\text{uni}}$, Theorem 8.7(ii) implies that

$$H^i(\mathcal{B}_z^{(m)}, \overline{\mathbf{Q}}_l) \simeq \bigoplus_{\mu \in \tilde{\mathcal{P}}(m)} \rho_\mu \otimes \mathcal{H}_z^{i-d'_m + \dim X_\mu}(\text{IC}(\overline{X}_\mu, \overline{\mathbf{Q}}_l)) \tag{8.15.1}$$

as \mathcal{W}_m^h -modules. Assume that $z \in X_\lambda$, and put $i = 2d_\lambda$. Since $\mathcal{H}_z^0(\text{IC}(\overline{X}_\lambda, \overline{\mathbf{Q}}_l)) = \overline{\mathbf{Q}}_l$, $H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \overline{\mathbf{Q}}_l)$ contains ρ_λ . This proves (ii). The enhanced case is proved in a similar way by using Theorem 8.13 (ii).

(ii) implies, in particular, $\dim \mathcal{B}_z^{(m)} \geq d_\lambda$. Put $d = \dim(\pi^{(m)})^{-1}(X_\lambda) - \dim X_\lambda$. Let $X(d)$ be as in 6.21. Then $X(d) \cap X_\lambda$ is open dense in X_λ . Hence $\dim X_\lambda \leq \dim X(d)$. By Lemma 6.22 (iii), we have

$$\dim \mathcal{B}_z^{(m)} \leq \frac{1}{2}(\dim \mathcal{X}_{m,\text{uni}} - \dim X(d)) \leq \frac{1}{2}(\dim \mathcal{X}_{m,\text{uni}} - \dim X_\lambda) = d_\lambda$$

for any $z \in X_\lambda \cap X(d)$. Hence $\dim \mathcal{B}_z^{(m)} = d_\lambda$ and $d = d_\lambda$. This proves (i). \square

We show the following result. In the enhanced case, a similar result was proved in [Li, Cor. 3.2.9] for the Borel–Moore homology.

Proposition 8.16. *Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type or of enhanced type. Take $z \in X_{\lambda}$ such that $\dim \mathcal{B}_z^{(\mathbf{m})} = d_{\lambda}$.*

- (i) $H^{2d_{\lambda}}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l) \simeq \rho_{\lambda}$ as $\mathcal{W}_{\mathbf{m}}^{\mathfrak{h}}$ -modules.
- (ii) *Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type. Then $\dim \mathcal{B}_z = d_{\lambda}$, and $H^{2d_{\lambda}}(\mathcal{B}_z, \bar{\mathbf{Q}}_l) \simeq V(\lambda)$ as $W_{n,r}$ -modules.*

Proof. We prove (i) by induction on r . Assume that $r = 2$. In the exotic case, (i) holds by [SS1, Rems. 5.5 (ii)]. A similar method also works for the enhanced case, since the number of H -orbits in \mathcal{X}_{uni} is finite. Assume that $r \geq 3$, and that (i) holds for $r - 1$. We consider the diagram as in (8.1.1) restricted to the unipotent varieties as discussed in 8.4. Put $\lambda = (\lambda^{(1)}, \lambda')$ as in 8.4. By (8.14.1), we have $d_{\lambda} = d_{\lambda'} + d_{\lambda^{(1)}}$, where $d_{\lambda'}$ is defined similarly to d_{λ} , and $d_{\lambda^{(1)}} = (\dim G_{1,\text{uni}} - \dim \mathcal{O}_{\lambda^{(1)}})/2$. Take $z \in X_{\lambda}$ such that $\dim \mathcal{B}_z^{(\mathbf{m})} = d_{\lambda}$. By Lemma 8.5 and Lemma 8.12, ζ_1 gives an isomorphism $\zeta_1^{-1}(X_{\lambda}) \rightarrow X_{\lambda}$. Hence there exists a unique $z_* \in \zeta_1^{-1}(X_{\lambda})$ such that $\zeta_1(z_*) = z$. Then by using the diagram (8.1.1), one can find $(x_1, z') \in \mathcal{O}_{\lambda^{(1)}} \times X'_{\lambda'}$ and $z'_* = (\zeta'_1)^{-1}(z')$ such that $\sigma_1^{-1}(x_1, z'_*) = q_1^{-1}(z_*)$. Here $z' \in X'_{\lambda'}$ satisfies the condition that $\dim \mathcal{B}_{z'_*}^{(\mathbf{m}')} = d_{\lambda'}$. By using the isomorphism $(\zeta'_1)^{-1}(X'_{\lambda'}) \simeq X'_{\lambda'}$, we have

$$H^{2d_{\lambda'}}(\mathcal{B}_{z'_*}^{(\mathbf{m}')}, \bar{\mathbf{Q}}_l) \simeq (R^{2d_{\lambda'}}(\pi'_1)_! \bar{\mathbf{Q}}_l)_{z'_*} \simeq (R^{2d_{\lambda'}}(\varphi'_1)_! \bar{\mathbf{Q}}_l)_{z'_*},$$

where φ'_1 is the restriction of φ' . Similarly we have the isomorphism $H^{2d_{\lambda}}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l) \simeq (R^{2d_{\lambda}}(\varphi_1)_! \bar{\mathbf{Q}}_l)_{z_*}$ by using $\zeta^{-1}(X_{\lambda}) \simeq X_{\lambda}$. Let ξ be an element contained in $\sigma_1^{-1}(x_1, z'_*) = q_1^{-1}(z_*)$. By (8.1.1), we have

$$(R^{2d_{\lambda^{(1)}}}(\pi_1^1)_! \bar{\mathbf{Q}}_l)_{x_1} \otimes (R^{2d_{\lambda'}}(\varphi'_1)_! \bar{\mathbf{Q}}_l)_{z'_*} \simeq (R^{2d_{\lambda}}(\tilde{\varphi}_1)_! \bar{\mathbf{Q}}_l)_{\xi} \simeq (R^{2d_{\lambda}}(\varphi_1)_! \bar{\mathbf{Q}}_l)_{z_*},$$

where $\pi_1^1, \tilde{\varphi}_1$ are restrictions of $\pi^1, \tilde{\varphi}$, respectively. By induction, we know that $\dim H^{2d_{\lambda'}}(\mathcal{B}_{z'_*}^{(\mathbf{m}')}, \bar{\mathbf{Q}}_l) = \dim \rho_{\lambda'}$. It is well known that $\dim H^{2d_{\lambda^{(1)}}}(\mathcal{B}_{x_1}, \bar{\mathbf{Q}}_l) = \dim \rho_{\lambda^{(1)}}$. Hence $\dim H^{2d_{\lambda}}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l) = \dim \rho_{\lambda^{(1)}} + \dim \rho_{\lambda'} = \dim \rho_{\lambda}$. On the other hand, by Lemma 8.15, $H^{2d_{\lambda}}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l)$ contains ρ_{λ} . Thus $H^{2d_{\lambda}}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l) \simeq \rho_{\lambda}$. (i) is proved.

Next we show (ii). Assume that $\mathcal{X}_{\mathbf{m}}$ is of exotic type and $z \in X_{\lambda}$ such that $\dim \mathcal{B}_z^{(\mathbf{m})} = d_{\lambda}$. We consider the decomposition (8.15.1) in the case where $z \in X_{\lambda}$. By Theorem 8.7 (iii), we have a similar decomposition:

$$H^i(\mathcal{B}_z, \bar{\mathbf{Q}}_l) \simeq \bigoplus_{\mu \in \tilde{\mathcal{P}}(\mathbf{m})} V(\mu) \otimes \mathcal{H}_z^{i-d'_{\mathbf{m}}+\dim X_{\mu}}(\text{IC}(\bar{X}_{\mu}, \bar{\mathbf{Q}}_l)). \tag{8.16.1}$$

(8.15.1) shows that $\mathcal{H}_z^{i-d'_{\mathbf{m}}+\dim X_{\mu}}(\text{IC}(\bar{X}_{\mu}, \bar{\mathbf{Q}}_l)) = 0$ for any choice of $i > 2d_{\lambda}$ and of $\mu \in \tilde{\mathcal{P}}(\mathbf{m})$. This implies, by (8.16.1), that $H^i(\mathcal{B}_z, \bar{\mathbf{Q}}_l) = 0$ for any $i > 2d_{\lambda}$. Since $\mathcal{B}_z^{(\mathbf{m})} \subset \mathcal{B}_z$, we conclude that $\dim \mathcal{B}_z = \dim \mathcal{B}_z^{(\mathbf{m})} = d_{\lambda}$. Now assume that $i = 2d_{\lambda}$. By (i) and (8.15.1), we see that $\mathcal{H}_z^{2d_{\lambda}-d'_{\mathbf{m}}+\dim X_{\mu}}(\text{IC}(\bar{X}_{\mu}, \bar{\mathbf{Q}}_l)) = 0$ for any $\mu \neq \lambda$. Hence by (8.16.1), $H^{2d_{\lambda}}(\mathcal{B}_z, \bar{\mathbf{Q}}_l) \simeq V(\lambda)$. The proposition is proved. \square

References

- [AH] P. Achar, A. Henderson, *Orbit closures in the enhanced nilpotent cone*, Adv. in Math. **219** (2008), 27–62. Corrigendum, *ibid.* **228** (2011), 2984–2988.
- [FGT] M. Finkelberg, V. Ginzburg, R. Travkin, *Mirabolic affine Grassmannian and character sheaves*, Selecta Math. **14** (2009), 607–628.
- [K] S. Kato, *An exotic Deligne–Langlands correspondence for symplectic groups*, Duke Math. J. **148** (2009), 306–371.
- [Li] Y. Li, *A class of perverse sheaves on framed representation varieties of the Jordan quiver*, J. Algebra **386** (2013), 113–130.
- [Lu] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), 205–272.
- [SS1] T. Shoji, K. Sorlin, *Exotic symmetric space over a finite field*, I, Transform. Groups **18** (2013), 877–929.
- [SS2] T. Shoji, K. Sorlin, *Exotic symmetric space over a finite field*, III, Transform. Groups **19** (2014), 1149–1198.
- [T] R. Travkin, *Mirabolic Robinson–Schensted correspondence*, Selecta Math. **14** (2009), 727–758.