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HIGHEST WEIGHT VECTORS OF MIXED TENSOR PRODUCTS OF GENERAL LINEAR LIE SUPERALGEBRAS

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Abstract. In this paper, a notion of cyclotomic (or level k) walled Brauer algebras $\mathscr{B}_{k,r,t}$ is introduced for arbitrary positive integer k. It is proven that $\mathscr{B}_{k,r,t}$ is free over a commutative ring with rank $k^{r+t}(r+t)!$ if and only if it is admissible. Using super Schur–Weyl duality between general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ and $\mathscr{B}_{2,r,t}$, we give a classification of highest weight vectors of $\mathfrak{gl}_{m|n}$-modules M_{pq}^{rt}, the tensor products of Kac-modules with mixed tensor products of the natural module and its dual. This enables us to establish an explicit relationship between $\mathfrak{gl}_{m|n}$-Kac-modules and right cell (or standard) $\mathscr{B}_{2,r,t}$-modules over $\mathbb{C}$. Further, we find an explicit relationship between indecomposable tilting $\mathfrak{gl}_{m|n}$-modules appearing in M_{pq}^{rt}, and principal indecomposable right $\mathscr{B}_{2,r,t}$-modules via the notion of Kleshchev bipartitions. As an application, decomposition numbers of $\mathscr{B}_{2,r,t}$ arising from super Schur–Weyl duality are determined.

Introduction

Motivated by Brundan–Stroppel's work on higher super Schur–Weyl duality in [6], we introduced affine walled Brauer algebras $\mathscr{B}_{r,t}^{\mathrm{aff}}$ in [23] so as to establish higher super Schur–Weyl duality on the tensor product M_{pq}^{rt} of a Kac-module with a mixed tensor product of the natural module and its dual for general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$ under the assumption $r+t \le \min\{m,n\}$ (after we finished [23], Professor Stroppel informed us that Sartori defined affine walled algebras via affine walled Brauer category, independently in [24]). One of purposes of this paper is to generalize super Schur–Weyl duality to the case $r+t > \min\{m,n\}$. For this aim, we need to establish a bijective map from a level two walled Brauer algebra $\mathscr{B}_{2,r,t}$ appearing in [23] to a level two degenerate Hecke algebra $\mathscr{H}_{2,r+t}$.

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This can be done by showing that the dimension of $\mathscr{B}_{2,r,t}$ is $2^{r+t}(r+t)!$ over $\mathbb{C}$. We consider this problem in a general setting by introducing a cyclotomic (or level k) walled Brauer algebra $\mathscr{B}_{k,r,t}$ for arbitrary $k \in \mathbb{Z}^{>0}$. By employing a totally new method, which is independent of seminormal forms of $\mathscr{B}_{k,r,t}$, we prove that $\mathscr{B}_{k,r,t}$ is free over a commutative ring R with rank $k^{r+t}(r+t)!$ if and only if it is admissible in the sense of Definition 2. It is expected that $\mathscr{B}_{k,r,t}$ can be used to study the problem on a classification of finite dimensional simple $\mathscr{B}^{\rm aff}_{r,t}$-modules over an algebraically closed field. Details will be given elsewhere.

The establishment of the higher super Schur–Weyl duality [23] enables us to use the representation theory of $\mathscr{B}_{2,r,t}$ to classify highest weight vectors of M^{rt}_{pq} (at this point, we would like to mention that purely on the Lie superalgebra side, it seems to be hard to construct highest weight vectors of a given module, which is an interesting problem in its own right). On the other hand, a classification of highest weight vectors of M^{rt}_{pq} also enables us to relate the category of finite dimensional $\mathfrak{gl}_{m|n}$-modules with that of $\mathscr{B}_{2,r,t}$, which in turn gives us an efficient way to calculate decomposition numbers of $\mathscr{B}_{2,r,t}$ (cf. [22] for quantum walled Brauer algebras). This is the main motivation of this paper. We explain some details below.

It is proven in [23] that $\mathrm{End}_{U(\mathfrak{gl}_{m|n})}(M^{rt}_{pq})^{\rm op} \cong \mathscr{B}_{2,r,t}$ if $r+t \le \min\{m,n\}$. Since there is a bijection between the dominant weights of M^{rt}_{pq} and the poset $\Lambda_{2,r,t}$ in (33), and since $\mathscr{B}_{2,r,t}$ is a weakly cellular algebra over $\Lambda_{2,r,t}$ in the sense of [12], it is very natural to ask the following problem: whether a $\mathbb{C}$-space of $\mathfrak{gl}_{m|n}$-highest weight vectors of M^{rt}_{pq} with a fixed highest weight is isomorphic to a cell (or standard) module of $\mathscr{B}_{2,r,t}$.

We give an affirmative answer to the problem. In sharp contrast to the Lie algebra case, due to the existence of the parity of $\mathfrak{gl}_{m|n}$ (see, e.g., [4], [25]), the known weakly cellular basis of $\mathscr{B}_{2,r,t}$ in [23] cannot be directly used to establish a relationship between $\mathfrak{gl}_{m|n}$-highest weight vectors of M^{rt}_{pq} and right cell modules of $\mathscr{B}_{2,r,t}$. One has to find new cellular bases of level two Hecke algebra $\mathscr{H}_{2,r}$ which are different from those in [3]. These new cellular bases of $\mathscr{H}_{2,r}$, which relate both trivial and signed representations of symmetric groups, are used to construct a new weakly cellular basis of $\mathscr{B}_{2,r,t}$. Motivated by explicit descriptions of bases of right cell modules for $\mathscr{B}_{2,r,t}$, we construct and classify $\mathfrak{gl}_{m|n}$-highest weight vectors of M^{rt}_{pq}. This leads to a $\mathscr{B}_{2,r,t}$-module isomorphism between each $\mathbb{C}$-space of $\mathfrak{gl}_{m|n}$-highest weight vectors of M^{rt}_{pq} with a fixed highest weight and the corresponding cell module of $\mathscr{B}_{2,r,t}$. Based on the above, we are able to construct a suitable exact functor sending $\mathfrak{gl}_{m|n}$-Kac-modules to right cell modules of $\mathscr{B}_{2,r,t}$. This functor also sends an indecomposable tilting module appearing in M^{rt}_{pq} to a principal indecomposable right $\mathscr{B}_{2,r,t}$-module indexed by a pair of so-called Kleshchev bipartitions in the sense of (36). It gives us an efficient way to calculate decomposition numbers of $\mathscr{B}_{2,r,t}$ via Brundan–Stroppel's result [6] on the multiplicity of a Kac-module in an indecomposable tilting module appearing in M^{rt}_{pq}.

Finally, we would like to say that our method can be used to deal with level k walled Brauer algebras with $k > 2$. In this case, if we consider parabolic subalgebras $\oplus_{i=1}^k \mathfrak{gl}_{m_i}$ of $\mathfrak{gl}_n$ with $\sum_{i=1}^k m_i = n$ and $k > 2$ (for $k = 2$, see [24]), the level

k-walled Brauer algebras with some special parameters will appear. This gives rise to certain relationships between the category of modules for level k-walled Brauer algebras and the parabolic category $\mathcal{O}(\mathfrak{gl}_n)$. We will use the representation theory of level k-walled Brauer algebras (see Remark 2) to classify highest weight vectors of certain tensor modules and hence to use the value at $q = 1$ of certain parabolic inverse Kazhdan–Lusztig polynomials, namely, the multiplicities of simple modules in parabolic Verma modules, to compute the decomposition matrices of such level k walled Brauer algebras. Details will appear in the sequel.

We organize the paper as follows. In section 1, after recalling the definition of $\mathscr{B}^{\mathrm{aff}}_{r,t}$ over a commutative ring R, we introduce cyclotomic walled Brauer algebras $\mathscr{B}_{k,r,t} := \mathscr{B}^{\mathrm{aff}}_{r,t}/I$ for arbitrary $k \in \mathbb{Z}^{>0}$, where I is the two-sided ideal of $\mathscr{B}^{\mathrm{aff}}_{r,t}$ generated by two cyclotomic polynomials $\boldsymbol{f}(x_1)$ and $\boldsymbol{g}(\overline{x}_1)$ of degree k, which satisfy (6)–(8). When $\mathscr{B}^{\mathrm{aff}}_{r,t}$ is admissible in the sense of Definition 2, we describe explicitly an R-basis of I. This enables us to prove that $\mathscr{B}_{k,r,t}$ is free over R with rank $k^{r+t}(r+t)!$ if and only if it is admissible. In section 2, we construct cellular bases of $\mathscr{H}_{2,r}$ and use them to construct a weakly cellular basis of $\mathscr{B}_{2,r,t}$. In section 3, higher super Schur–Weyl dualities in [23] are generalized to the case $r+t > \min\{m, n\}$. In sections 4–5, we classify highest weight vectors of M^{r0}_{pq} and M^{rt}_{pq}. Based on this, we establish an explicit relationship between indecomposable tilting (respectively Kac) modules for $\mathfrak{gl}_{m|n}$ and principal indecomposable (respectively cell) right $\mathscr{B}_{2,r,t}$-modules via a suitable exact functor. This gives us an efficient way to calculate decomposition numbers of $\mathscr{B}_{2,r,t}$ arising from the super Schur–Weyl duality in [23].

1. Affine walled Brauer algebras and their cyclotomic quotients

Throughout, we assume that R is a commutative ring containing $\boldsymbol{\Omega} = \{\omega_a \mid a \in \mathbb{N}\}$ and identity 1. In this section, we introduce a level k walled Brauer algebra $\mathscr{B}_{k,r,t}$ and prove that $\mathscr{B}_{k,r,t}$ is free over R with rank $k^{r+t}(r+t)!$ if and only if $\mathscr{B}_{k,r,t}$ is admissible in the sense of Definition 2. First, we briefly recall the definition of walled Brauer algebras.

Fix $r, t \in \mathbb{Z}^{>0}$. A *walled (r,t)-Brauer diagram* (or simply, a *walled Brauer diagram*) is a diagram with $(r+t)$ vertices on top and bottom rows, and vertices on both rows are labeled from left to right by $r, \ldots, 2, 1, \overline{1}, \overline{2}, \ldots, \overline{t}$, such that every $i \in \{r, \ldots, 2, 1\}$ (respectively, $\overline{i} \in \{\overline{1}, \overline{2}, \ldots, \overline{t}\}$) on each row is connected to a unique $\overline{j}$ (respectively, j) on the same row or a unique j (respectively, $\overline{j}$) on the other row. Thus there are four types of pairs $[i, j]$, $[i, \overline{j}]$, $[\overline{i}, j]$ and $[\overline{i}, \overline{j}]$. The pairs $[i, j]$ and $[\overline{i}, \overline{j}]$ are *vertical edges*, and $[\overline{i}, j]$ and $[i, \overline{j}]$ are *horizontal edges.*

The product of two walled Brauer diagrams D_1 and D_2 can be defined via concatenation. Putting D_1 above D_2 and connecting each vertex on the bottom row of D_1 to the corresponding vertex on the top row of D_2 yields a diagram $D_1 \circ D_2$, called the *concatenation* of D_1 and D_2. Removing all circles of $D_1 \circ D_2$ yields a unique walled Brauer diagram, denoted D_3. Let n be the number of circles appearing in $D_1 \circ D_2$. Then the *product* $D_1 D_2$ is defined to be $\omega_0^n D_3$, where ω_0 is a fixed element in R. The *walled Brauer algebra* [19], [28], [21] $\mathscr{B}_{r,t} := \mathscr{B}_{r,t}(\omega_0)$ with defining parameter ω_0 is the associative R-algebra spanned by all walled Brauer diagrams with product defined in this way.

Let $\mathfrak{S}_r$ (respectively $\overline{\mathfrak{S}}_t$) be the symmetric group in r (respectively t) letters $r, \ldots, 2, 1$ (respectively $\overline{1}, \overline{2}, \ldots, \overline{t}$). It is known that $\mathscr{B}_{r,t}$ contains two subalgebras which are isomorphic to the group algebras of $\mathfrak{S}_r$ and $\overline{\mathfrak{S}}_t$, respectively. More explicitly, the walled Brauer diagram s_i whose edges are of forms $[k,k]$ and $[\overline{k},\overline{k}]$ except two vertical edges $[i,i+1]$ and $[i+1,i]$ can be identified with the basic transposition $(i,i+1)\in\mathfrak{S}_r$, which switches i and $i+1$ and fixes others. Similarly, there is a walled Brauer diagram $\overline{s}_j$ corresponding to $(\overline{j},\overline{j+1})\in\overline{\mathfrak{S}}_t$. Let e_1 be the walled Brauer diagram whose edges are of forms $[k,k]$ and $[\overline{k},\overline{k}]$ except two horizontal edges $[1,\overline{1}]$ on the top and bottom rows. Then $\mathscr{B}_{r,t}$ is the R-algebra [21] generated by e_1, s_i, $\overline{s}_j$ for $1\le i\le r-1$, $1\le j\le t-1$ such that s_i's, $\overline{s}_j$'s are distinguished generators of $\mathfrak{S}_r\times\overline{\mathfrak{S}}_t$ and

$$\begin{gathered}e_1^2=\omega_0e_1,\quad e_1s_1e_1=e_1=e_1\overline{s}_1e_1,\quad s_ie_1=e_1s_i,\quad \overline{s}_je_1=e_1\overline{s}_j\ \ (i,j\neq 1),\\ e_1s_1\overline{s}_1e_1s_1=e_1s_1\overline{s}_1e_1\overline{s}_1,\quad s_1e_1s_1\overline{s}_1e_1=\overline{s}_1e_1s_1\overline{s}_1e_1.\end{gathered}\tag{1}$$

Let $\mathscr{H}_n^{\rm aff}$ be the *degenerate affine Hecke algebra* [11]. As a free R-module, it is the tensor product $R[y_1,y_2,\dots,y_n]\otimes R\mathfrak{S}_n$ of a polynomial algebra with the group algebra of $\mathfrak{S}_n$. The multiplication is defined so that $R[y_1,y_2,\dots,y_n]\equiv R[y_1,y_2,\dots,y_n]\otimes 1$ and $R\mathfrak{S}_n\equiv 1\otimes R\mathfrak{S}_n$ are subalgebras and $s_iy_j=y_js_i$ if $j\neq i,i+1$ and $s_iy_i=y_{i+1}s_i-1$, $1\le i\le n-1$.

Recall that R contains 1 and $\mathbf{\Omega}=\{\omega_a\in R\mid a\in\mathbb{N}\}$. The *affine walled Brauer algebra* $\mathscr{B}_{r,t}^{\rm aff}(\mathbf{\Omega})$ (which is $\widehat{\mathscr{B}_{r,t}}$ in [23, §4]) with respect to the defining parameters ω_a's have been defined via generators and 26 defining relations [23, Def. 2.7]. It follows from [23, Thm. 4.15] that $\mathscr{B}_{r,t}^{\rm aff}(\mathbf{\Omega})$ can be also defined in a simpler way as follows: it is an associative R-algebra generated by e_1, x_1, $\overline{x}_1$, s_i, $\overline{s}_j$ for $1\le i\le r-1$, $1\le j\le t-1$, such that e_1, s_i's, $\overline{s}_j$'s are generators of $\mathscr{B}_{r,t}$ with defining parameter ω_0, and as a free R-module,

$$\mathscr{B}_{r,t}^{\rm aff}(\mathbf{\Omega})=R[\mathbf{x}_r]\otimes\mathscr{B}_{r,t}\otimes R[\overline{\mathbf{x}}_t],\tag{2}$$

the tensor product of the walled Brauer algebra $\mathscr{B}_{r,t}$ with two polynomial algebras

$$R[\mathbf{x}_r]:=R[x_1,x_2,\dots,x_r],\ \text{ and }\ R[\overline{\mathbf{x}}_t]:=R[\overline{x}_1,\overline{x}_2,\dots,\overline{x}_t].$$

The multiplication of $\mathscr{B}_{r,t}^{\rm aff}(\mathbf{\Omega})$ is defined such that $R[\mathbf{x}_r]\otimes 1\otimes 1$, $1\otimes 1\otimes R[\overline{\mathbf{x}}_r]$, $1\otimes\mathscr{B}_{r,t}\otimes 1$, $R[\mathbf{x}_r]\otimes R\mathfrak{S}_r\otimes 1$ and $1\otimes R\overline{\mathfrak{S}}_t\otimes R[\overline{\mathbf{x}}_r]$ are subalgebras isomorphic to $R[\mathbf{x}_r]$, $R[\overline{\mathbf{x}}_r]$, $\mathscr{B}_{r,t}$, $\mathscr{H}_r^{\rm aff}$, and $\mathscr{H}_t^{\rm aff}$ respectively, and (for simplicity, without confusion we identify elements $x_i\otimes 1\otimes 1$, $1\otimes s_i\otimes 1$, $1\otimes e_i\otimes 1$, $1\otimes\overline{s}_i\otimes 1$, $1\otimes 1\otimes\overline{x}_i$ in (2) with $x_i,s_i,e_i,\overline{s}_i,\overline{x}_i$ respectively)

$$e_1(x_1+\overline{x}_1)=(x_1+\overline{x}_1)e_1=0,\quad s_1e_1s_1x_1=x_1s_1e_1s_1,\quad \overline{s}_1e_1\overline{s}_1\overline{x}_1=\overline{x}_1\overline{s}_1e_1\overline{s}_1,\tag{3}$$

$$s_i\overline{x}_1=\overline{x}_1s_i,\quad \overline{s}_ix_1=x_1\overline{s}_i,\quad x_1(e_1+\overline{x}_1)=(e_1+\overline{x}_1)x_1,\tag{4}$$

$$e_1x_1^ke_1=\omega_ke_1,\quad e_1\overline{x}_1^ke_1=\overline{\omega}_ke_1\ \ \forall k\in\mathbb{Z}^{\ge 0},\tag{5}$$

where $\overline{\omega}_a$'s are determined by [23, Cor. 4.3]. If $\overline{\omega}_a$'s do not satisfy [23, Cor. 4.3], and if R is a field, then $e_1=0$ and $\mathscr{B}_{r,t}^{\rm aff}(\mathbf{\Omega})$ turns out to be $\mathscr{H}_r^{\rm aff}\otimes\mathscr{H}_t^{\rm aff}$.

We remark that the isomorphism $R[\mathbf{x}_r]\otimes R\mathfrak{S}_r\otimes 1\cong\mathscr{H}_r^{\rm aff}$ sends $1\otimes s_i\otimes 1$ (respectively $x_1\otimes 1\otimes 1$) to s_i (respectively $-y_1$), and the isomorphism $1\otimes R\overline{\mathfrak{S}}_t\otimes$

$R[\overline{\mathbf{x}}_r] \cong \mathscr{H}_t^{\mathrm{aff}}$ sends $1 \otimes \overline{s}_j \otimes 1$ (respectively $1 \otimes 1 \otimes \overline{x}_1$) to s_j (respectively $-y_1$). So, $x_{i+1} = s_i x_i s_i - s_i$ and $\overline{x}_{j+1} = \overline{s}_j \overline{x}_j \overline{s}_j - \overline{s}_j$ and $y_{i+1} = s_i y_i s_i + s_i$ if all of them make sense.

For the simplification of notation, we denote $\mathscr{B}_{r,t}^{\mathrm{aff}}(\mathbf{\Omega})$ by $\mathscr{B}_{r,t}^{\mathrm{aff}}$. Fix $u_1, u_2, \dots, u_k \in R$ for some $k \in \mathbb{Z}^{>0}$. Let $\boldsymbol{f}(x_1) \in \mathscr{B}_{r,t}^{\mathrm{aff}}$ be such that

$$\boldsymbol{f}(x_1) = \prod_{i=1}^{k} (x_1 - u_i). \tag{6}$$

By [23, Lem. 4.2] (or using (3)–(4)), there is a monic polynomial $\boldsymbol{g}(\overline{x}_1) \in R[\overline{x}_1]$ with degree k such that

$$e_1 \boldsymbol{f}(x_1) = (-1)^k e_1 \boldsymbol{g}(\overline{x}_1). \tag{7}$$

If R is an algebraically closed field, then there are $\overline{u}_1, \overline{u}_2, \dots, \overline{u}_k \in R$ such that

$$\boldsymbol{g}(\overline{x}_1) = \prod_{i=1}^{k} (\overline{x}_1 - \overline{u}_i). \tag{8}$$

Definition 1. Let R be a commutative ring containing 1, $\mathbf{\Omega} = \{\omega_a \in R \mid a \in \mathbb{N}\}$, and $u_i, \overline{u}_i$, $1 \le i \le k$. The cyclotomic (or level k) walled Brauer algebra $\mathscr{B}_{k,r,t}$ is the quotient algebra $\mathscr{B}_{r,t}^{\mathrm{aff}}/I$, where I is the two-sided ideal of $\mathscr{B}_{r,t}^{\mathrm{aff}}$ generated by $\boldsymbol{f}(x_1)$ and $\boldsymbol{g}(\overline{x}_1)$ satisfying (6)–(8).

If $k = 1$, then $\mathscr{B}_{k,r,t}$ is $\mathscr{B}_{r,t}$ with defining parameter ω_0. For some special $u_i, \overline{u}_i$, $i = 1, 2$, $\mathscr{B}_{2,r,t}$ is the level two walled Brauer algebras arising from super Schur–Weyl duality in [23].

Lemma 1. *Let $\boldsymbol{f}(x_1)$ be given in* (6). *Write $\boldsymbol{f}(x_1) = x_1^k + \sum_{i=1}^{k} a_i x_1^{k-i}$. Then e_1 is an R-torsion element of $\mathscr{B}_{k,r,t}$ unless*

$$\omega_\ell = -(a_1 \omega_{\ell-1} + \cdots a_k \omega_{\ell-k}) \ \text{ for all } \ \ell \ge k. \tag{9}$$

Proof. Let $b_\ell = \omega_\ell + a_1 \omega_{\ell-1} + \cdots + a_k \omega_{\ell-k} \in R$. By (5), $b_\ell e_1 = e_1 \boldsymbol{f}(x_1) x_1^{\ell-k} e_1$ in $\mathscr{B}_{r,t}^{\mathrm{aff}}$ and $b_\ell e_1 = 0$ in $\mathscr{B}_{k,r,t}$. Thus, e_1 is an R-torsion element if $b_\ell \ne 0$ for some $\ell \ge k$. □

Definition 2. The algebras $\mathscr{B}_{r,t}^{\mathrm{aff}}$ and $\mathscr{B}_{k,r,t}$ are called *admissible* if (9) holds.

Lemma 2. *Assume $\boldsymbol{f}(x_1), \boldsymbol{g}(\overline{x}_1) \in \mathscr{B}_{r,t}^{\mathrm{aff}}$ satisfying* (6)–(8) *. If $\mathscr{B}_{r,t}^{\mathrm{aff}}$ is admissible, then*

(i) $e_1 \boldsymbol{f}(x_1) x_1^a e_1 = 0$ *for all* $a \in \mathbb{N}$*;*
(ii) $e_1 \boldsymbol{g}(\overline{x}_1) \overline{x}_1^a e_1 = 0$ *for all* $a \in \mathbb{N}$.

Proof. (i) is trivial since $\mathscr{B}_{r,t}^{\mathrm{aff}}$ is admissible. It is proven in [23] that there is an R-linear anti-involution σ on $\mathscr{B}_{r,t}^{\mathrm{aff}}$, which fixes all generators of $\mathscr{B}_{r,t}^{\mathrm{aff}}$. Applying σ on [23, Lem. 4.2] yields

$$\overline{x}_1^k e_1 = \sum_{i=0}^{k} a_{k,i} x_1^i e_1, \text{ for some } a_{k,i} \in R.$$

So, (ii) follows from (7) and (i), immediately. □

Denote $s_{i,j} = s_i s_{i+1} \cdots s_{j-1}$ if $i < j$, and 1 if $i = j$, and $s_{i-1} s_{i-2} \cdots s_j$ if $i > j$. Denote $\overline{s}_{i,j} \in \overline{\mathfrak{S}}_t$ similarly. Let $e_{i,j}$ be the walled Brauer diagram such that each vertical edge of $e_{i,j}$ is of form $[k, k]$ or $[\overline{k}, \overline{k}]$ and the horizontal edges on the top and bottom rows of $e_{i,j}$ are $[i, \overline{j}]$. Then

$$e_{i,j} = \overline{s}_{j,1} s_{i,1} e_1 s_{1,i} \overline{s}_{1,j} \text{ for } i, j \text{ with } 1 \le i \le r \text{ and } 1 \le j \le t. \tag{10}$$

For each nonnegative integer $f \le \min\{r, t\}$, let

$$e^f = e_1 e_2 \cdots e_f \text{ for } f > 0 \text{ and } e^0 = 1, \text{ where } e_i = e_{i,i}. \tag{11}$$

Set

$$\mathscr{D}^f_{r,t} = \{ s_{f,i_f} \overline{s}_{f,j_f} \cdots s_{1,i_1} \overline{s}_{1,j_1} \,|\, 1 \le i_1 < \cdots < i_f \le r,\ k \le j_k \}. \tag{12}$$

Definition 3. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ and $\beta = (\beta_1, \ldots, \beta_t) \in \mathbb{N}^t$, let $x^\alpha = \prod_{i=1}^r x_i^{\alpha_i}$, $\overline{x}^\beta = \prod_{j=1}^t \overline{x}_j^{\beta_j}$. Let $\mathcal{M}$ be a subset of $\mathscr{B}^{\text{aff}}_{r,t}$ given by

$$\mathcal{M} = \bigcup_{f=0}^{\min\{m,n\}} \{ x^\alpha c^{-1} e^f w d \overline{x}^\beta \,|\, (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t, c, d \in \mathscr{D}^f_{r,t}, w \in \mathfrak{S}_{r-f} \times \overline{\mathfrak{S}}_{t-f} \}. \tag{13}$$

Elements of $\mathcal{M}$ are called *regular monomials* of $\mathscr{B}^{\text{aff}}_{r,t}$.

Theorem 3 ([23, Thm. 4.15]). *The affine walled Brauer algebra $\mathscr{B}^{\text{aff}}_{r,t}$ is free over R with $\mathcal{M}$ as its R-basis.*

We consider $\mathscr{B}^{\text{aff}}_{r,t}$ as a filtrated R-algebra as follows. Let

$$\deg s_i = \deg \overline{s}_j = \deg e_1 = 0 \quad \text{and} \quad \deg x_k = \deg \overline{x}_\ell = 1$$

for all possible i, j, k, ℓ's. Let $(\mathscr{B}^{\text{aff}}_{r,t})^{(k)}$ be the R-submodule spanned by regular monomials with degrees less than or equal to k for $k \in \mathbb{Z}^{\ge 0}$. Then we have the following filtration

$$\mathscr{B}^{\text{aff}}_{r,t} \supset \cdots \supset (\mathscr{B}^{\text{aff}}_{r,t})^{(1)} \supset (\mathscr{B}^{\text{aff}}_{r,t})^{(0)} \supset (\mathscr{B}^{\text{aff}}_{r,t})^{(-1)} = 0. \tag{14}$$

Let $\text{gr}(\mathscr{B}^{\text{aff}}_{r,t}) = \bigoplus_{i \ge 0} (\mathscr{B}^{\text{aff}}_{r,t})^{[i]}$, where $(\mathscr{B}^{\text{aff}}_{r,t})^{[i]} = (\mathscr{B}^{\text{aff}}_{r,t})^{(i)} / (\mathscr{B}^{\text{aff}}_{r,t})^{(i-1)}$. Then $\text{gr}(\mathscr{B}^{\text{aff}}_{r,t})$ is an associated $\mathbb{Z}$-graded algebra. We will use the same symbols to denote elements in $\text{gr}(\mathscr{B}^{\text{aff}}_{r,t})$.

Lemma 4. *Let $x'_i = s_{i-1} x'_{i-1} s_{i-1}$, and $\overline{x}'_j = \overline{s}_{j-1} \overline{x}_{j-1} \overline{s}_{j-1}$ for $i, j \in \mathbb{Z}^{\ge 2}$ with $i \le r$ and $j \le t$, where $x'_1 = x_1$, and $\overline{x}'_1 = \overline{x}_1$.*

(i) *$x_i = x'_i - L_i$, where $L_i = \sum_{1 \le j < i} (j, i)$ and (j, i) is the transposition in $\mathfrak{S}_r$ which switches j, i and fixes others.*

(ii) *$\overline{x}_i = \overline{x}'_i - \overline{L}_i$, where $\overline{L}_i = \sum_{\overline{1} \le \overline{j} < \overline{i}} (\overline{j}, \overline{i})$ and $(\overline{j}, \overline{i})$ is the transposition in $\overline{\mathfrak{S}}_t$ which switches $\overline{j}, \overline{i}$ and fixes others.*

(iii) *Any symmetric polynomial of $L_1, L_2, \ldots, L_r$ (respectively $\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_t$) is a central element of $R\mathfrak{S}_r$ (respectively $R\overline{\mathfrak{S}}_t$).*

Proof. (i)–(ii) are trivial and (iii) is a well-known result. □

The elements L_i's (respectively $\overline{L}_j$'s) are known as Jucys–Murphy elements of $R\mathfrak{S}_r$ (respectively $R\overline{\mathfrak{S}}_t$). Note that $x_ix_j = x_jx_i$ and $\overline{x}_i\overline{x}_j = \overline{x}_j\overline{x}_i$ for all possible i, j. However, x'_i and x'_j (respectively $\overline{x}'_i$ and $\overline{x}'_j$) do not commute each other.

Suppose $0 < f \le \min\{m, n\}$. Denote

$$\vec{i} = (i_1, \ldots, i_f), \quad \vec{j} = (j_1, \ldots, j_f), \qquad e_{\vec{i},\vec{j}} = e_{i_1,j_1}e_{i_2,j_2}\cdots e_{i_f,j_f}, \tag{15}$$

where $i_1, i_2, \ldots, i_f$ are distinct numbers in $\{1, 2, \ldots, r\}$, and $j_1, j_2, \ldots, j_f$ are distinct numbers in $\{\overline{1}, \overline{2}, \ldots, \overline{t}\}$. Then e_{i_k,j_k}'s commute each other. If $f = 0$, we set $\vec{i} = \vec{j} = \varnothing$ and $e_{\vec{i},\vec{j}} = 1$.

We always assume that $\mathfrak{S}_r$ (respectively $\overline{\mathfrak{S}}_t$) acts on the right of $\{r, \ldots, 2, 1\}$ (respectively $\{\overline{1}, \overline{2}, \ldots, \overline{t}\}$).

Lemma 5. *Suppose $a \in \mathbb{Z}^{>0}$, $1 \le i, \ell \le r$ and $1 \le j \le t$.*

(i) *If $w \in \mathfrak{S}_r$, then $w\boldsymbol{f}(x'_i)w^{-1} = \boldsymbol{f}(x'_{(i)w^{-1}})$.*
(ii) *If $w \in \overline{\mathfrak{S}}_t$, then $w\boldsymbol{g}(\overline{x}'_{\overline{j}})w^{-1} = \boldsymbol{g}(\overline{x}'_{(\overline{j})w^{-1}})$.*
(iii) *$x'^a_i\boldsymbol{f}(x'_\ell) = \boldsymbol{f}(x'_\ell)x'^a_i + v$, where $v \in \sum_{b<a}\sum_{h,h_1=1}^{\max\{i,\ell\}} \boldsymbol{f}(x'_h)x'^b_{h_1}R\mathfrak{S}_r$.*
(iv) *$\overline{x}'^a_j\boldsymbol{f}(x'_i) = \boldsymbol{f}(x'_i)\overline{x}'^a_j + v$, where $v \in \sum_{b_1+b_2<a,\, c_1+c_2\le 1} \epsilon\overline{x}'^{b_1}_j e^{c_1}_{ij}\boldsymbol{f}(x'_i)e^{c_2}_{ij}\overline{x}'^{b_2}_j$ for some non-negative integers b_1, b_2, c_1, c_2 and $\epsilon = \pm 1$.*

Proof. (i)–(ii) are trivial. Since $x_2 = x'_2 - s_1$ and $x_2x_1 = x_1x_2$,

$$x'_2\boldsymbol{f}(x_1) = \boldsymbol{f}(x_1)(x'_2 - s_1) + \boldsymbol{f}(x'_2)s_1. \tag{16}$$

Applying the conjugate of $s_{i,2}$ on (16) yields (iii) for $a = 1$ and $\ell = 1$. If $\ell > 1$, then $x'_i\boldsymbol{f}(x'_\ell) = x'_is_{\ell-1}\boldsymbol{f}(x'_{\ell-1})s_{\ell-1} = s_{\ell-1}x'_{(i)s_{\ell-1}}\boldsymbol{f}(x'_{\ell-1})s_{\ell-1}$. Thus, (iii) follows from inductive assumption on $\ell - 1$ and (i) under the assumption $a = 1$. The case $a > 1$ follows by using the previous result on $a = 1$, repeatedly. Finally, (iv) can be checked similarly by induction. We leave the details to the readers. □

Proposition 6. *Let $J_L = \sum_{i=1}^t \mathscr{B}^{\mathrm{aff}}_{r,t}\,\boldsymbol{g}(\overline{x}'_j)$ and $J_R = \sum_{i=1}^r \boldsymbol{f}(x'_i)\mathscr{B}^{\mathrm{aff}}_{r,t}$. We have*

(i) *J_L is a right $R\mathfrak{S}_r \otimes \mathscr{H}^{\mathrm{aff}}_t$-module;*
(ii) *J_R is a left $\mathscr{H}^{\mathrm{aff}}_r \otimes R\overline{\mathfrak{S}}_t$-module;*
(iii) *$I = J_L + J_R$ if $\mathscr{B}^{\mathrm{aff}}_{r,t}$ is admissible, where I is the two-sided ideal of $\mathscr{B}^{\mathrm{aff}}_{r,t}$ generated by $\boldsymbol{f}(x_1)$ and $\boldsymbol{g}(\overline{x}_1)$ satisfying* (6)*–*(8)*.*

Proof. Obviously, both J_L and J_R are $\mathfrak{S}_r \times \mathfrak{S}_{\overline{t}}$-bimodules. By Lemma 5 (iii), $x_1J_R \subseteq J_R$. Similarly, $J_L\overline{x}_1 \subseteq J_L$. This proves (i)–(ii). In order to prove (iii), it suffices to verify that $J_L + J_R$ is a two-sided ideal of $\mathscr{B}^{\mathrm{aff}}_{r,t}$. If so, since $\{\boldsymbol{f}(x_1), \boldsymbol{g}(\overline{x}_1)\} \subset J_L + J_R$, $I = J_L + J_R$, proving the result.

We claim that $e_1(J_L + J_R) \subseteq J_L + J_R$ and $(J_L + J_R)e_1 \subseteq J_L + J_R$. If so, by (4), $(\overline{x}_1 + e_1)\boldsymbol{f}(x_1) = \boldsymbol{f}(x_1)(\overline{x}_1 + e_1)$ and hence $\overline{x}_1\boldsymbol{f}(x_1) \in J_L + J_R$. By (i)–(ii), $\overline{x}_1\boldsymbol{f}(x'_i) = s_{i,1}\overline{x}_1\boldsymbol{f}(x_1)s_{1,i} \in J_L + J_R$, and hence $\overline{x}_1(J_L + J_R) \subseteq J_L + J_R$. Similarly, $(J_L + J_R)x_1 \subseteq J_L + J_R$. Thus the claim implies that $J_L + J_R$ is a two-sided ideal of $\mathscr{B}^{\mathrm{aff}}_{r,t}$.

By symmetry, it remains to prove $e_1(J_L+J_R)\subseteq J_L+J_R$. Obviously, it suffices to verify

$$e_1J_R\subset J_L+J_R. \tag{17}$$

By (3), $e_1\boldsymbol{f}(x_i')=\boldsymbol{f}(x_i')e_1$ for $i\geq 2$. Let $\mathbf{m}$ be a regular monomial of $\mathscr{B}_{r,t}^{\rm aff}$ defined in (13). Then $\mathbf{m}=x^\alpha e_{\vec{i},\vec{j}}w\overline{x}^\beta$ for some $w\in\mathfrak{S}_r\times\overline{\mathfrak{S}}_t$, $(\alpha,\beta)\in\mathbb{N}^r\times\mathbb{N}^t$ and some $\vec{i},\vec{j}$. Using induction on $|\alpha|$, we want to prove

$$e_1\boldsymbol{f}(x_1)\mathbf{m}\in J_L+J_R. \tag{18}$$

If so, then $e_1\boldsymbol{f}(x_1)\mathscr{B}_{r,t}^{\rm aff}\subset J_L+J_R$ and hence (17) follows.

Case $1:|\alpha|=0$.

If $f=0$, then (18) follows from (i) and (7). Suppose $1\leq f\leq\min\{r,t\}$. Since $\mathscr{B}_{r,t}^{\rm aff}$ is admissible, $e_1\boldsymbol{f}(x_1)\mathbf{m}=0$ if e_i is a factor of $e_{\vec{i},\vec{j}}$. Assume that e_1 is not a factor of $e_{\vec{i},\vec{j}}$. If there is an l such that $i_l=p\neq 1$ and $j_l=1$, by (ii),

$$e_1\boldsymbol{f}(x_1)e_{p,1}=s_{p,2}e_1\boldsymbol{f}(x_1)s_1e_1s_{1,p}=s_{p,2}e_1s_1\boldsymbol{f}(x_2')e_1s_{1,p}=s_{p,2}\boldsymbol{f}(x_2')e_1s_{1,p}\in J_R.$$

Suppose $j_l\neq 1$ for all possible l. If there is an l such that $e_{i_l,j_l}=e_{1,p}$ for some $p\neq 1$, then we assume $i_1=1$ and $j_1=p$ without loss of any generality. In this case,

$$\begin{aligned}e_1\boldsymbol{f}(x_1)e_{1,p}&=(-1)^k\overline{s}_{p,2}e_1\boldsymbol{g}(\overline{x}_1)\overline{s}_1e_1\overline{s}_{1,p}\\&=(-1)^k\overline{s}_{p,2}e_1\boldsymbol{g}(\overline{x}_2')\overline{s}_{1,p}=(-1)^k\overline{s}_{p,2}e_1\overline{s}_{1,p}\boldsymbol{g}(\overline{x}_1).\end{aligned}$$

Since $j_l\neq 1$ for $1\leq l\leq f$, by [23, Lem. 4.7 (2)], $\overline{x}_1e_{i_l,j_l}=e_{i_l,j_l}\overline{x}_1$ and hence

$$\boldsymbol{g}(\overline{x}_1)\prod_{l=2}^{f}e_{i_l,j_l}=\prod_{l=2}^{f}e_{i_l,j_l}\boldsymbol{g}(\overline{x}_1)\in J_L. \tag{19}$$

Now, (18) follows from (i). Finally, if $\{i_l,j_l\}\cap\{1\}=\varnothing$ for all possible l, then (18) follows from (i) and the following fact

$$e_1\boldsymbol{f}(x_1)\prod_{l=1}^{f}e_{i_f,j_f}=\prod_{l=1}^{f}e_{i_f,j_f}e_1\boldsymbol{f}(x_1)=(-1)^k\prod_{l=1}^{f}e_{i_f,j_f}e_1\boldsymbol{g}(\overline{x}_1)\in J_L.$$

Case $2:|\alpha|>0$.

If $\alpha_i\neq 0$ for some $2\leq i\leq r$, then $e_1x_i=x_i'e_1-e_1\sum_{j=1}^{i}(j,i)$ and $x_i\boldsymbol{f}(x_1)=\boldsymbol{f}(x_1)x_i$. Let $\mathbf{m}'$ be obtained from $\mathbf{m}$ by removing x_i. Then

$$e_1(1,i)\boldsymbol{f}(x_1)\mathbf{m}'=e_1\boldsymbol{f}(x_i')(1,i)\mathbf{m}'=\boldsymbol{f}(x_i')e_1(1,i)\mathbf{m}'\in J_R.$$

Now, (18) follows from inductive assumption on $|\alpha|$. If $\alpha_i=0$, $2\leq i\leq r$, then $x^\alpha=x_1^{\alpha_1}$ with $\alpha_1>0$. Let $v=e_1\boldsymbol{f}(x_1)\mathbf{m}$. If $j_\ell\neq 1$, $1\leq\ell\leq f$, then by (19), Lemma 5 and inductive assumption,

$$\begin{aligned}v&=e_1\boldsymbol{f}(x_1)x_1^{\alpha_1}e_{\vec{i},\vec{j}}w\overline{x}^\beta=(-1)^ke_1\boldsymbol{g}(\overline{x}_1)x_1^{\alpha_1}e_{\vec{i},\vec{j}}w\overline{x}^\beta\equiv(-1)^ke_1x_1^{\alpha_1}\boldsymbol{g}(\overline{x}_1)e_{\vec{i},\vec{j}}w\overline{x}^\beta\\&=(-1)^ke_1x_1^{\alpha_1}e_{\vec{i},\vec{j}}\boldsymbol{g}(\overline{x}_1)w\overline{x}^\beta\in J_Lw\overline{x}^\beta\subset J_L+J_R,\end{aligned}$$

where the "$\equiv$" is modulo $J_L + J_R$. Finally, if $j_\ell = 1$ for some ℓ, without loss of any generality, we assume $j_1 = 1$. If $i_1 = 1$, by Lemma 2, $v = e_1 \boldsymbol{f}(x_1) x_1^{\alpha_1} e_1 e_{\vec{i}',\vec{j}'} w \overline{x}^\beta = 0$, where $\vec{i}' = (i_2, \ldots, i_f)$ and $\vec{j}' = (j_2, \ldots, j_f)$. Now, we assume $i_1 \neq 1$. Then

$$\begin{aligned} v &= e_1 \boldsymbol{f}(x_1) x_1^{\alpha_1} e_{i_1,1} e_{\vec{i}',\vec{j}'} w \overline{x}^\beta = e_1 e_{i_1,1} \boldsymbol{f}(x_1) x_1^{\alpha_1} e_{\vec{i}',\vec{j}'} w \overline{x}^\beta \\ &= e_1 (1, i_1) \boldsymbol{f}(x_1) x_1^{\alpha_1} e_{\vec{i}',\vec{j}'} w \overline{x}^\beta = e_1 \boldsymbol{f}(x_i') (1, i) x_1^{\alpha_1} e_{\vec{i}',\vec{j}'} w \overline{x}^\beta, \\ &= \boldsymbol{f}(x_i') e_1 (1, i) x_1^{\alpha_1} e_{\vec{i}',\vec{j}'} w \overline{x}^\beta \in J_R. \end{aligned}$$

This completes the proof of (18). $\square$

For $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$, denote $\boldsymbol{f}(x')^\alpha = \boldsymbol{f}(x_1)^{\alpha_1} \cdots \boldsymbol{f}(x_r')^{\alpha_r}$ and $\boldsymbol{g}(\overline{x}')^\beta = \boldsymbol{g}(\overline{x}_1)^{\beta_1} \cdots \boldsymbol{g}(\overline{x}_t')^{\beta_t}$. Let $\mathbb{N}_k^r = \{\alpha \in \mathbb{N}^r \mid \alpha_i \leq k-1, 1 \leq i \leq r\}$ and $\mathbb{N}_k^t = \{\alpha \in \mathbb{N}^t \mid \alpha_i \leq k-1, 1 \leq i \leq r\}$.

Lemma 7. *The affine walled Brauer algebra $\mathscr{B}_{r,t}^{\mathrm{aff}}$ is a free R-module with $\mathcal{N}$ as its R-basis, where*

$$\begin{aligned} \mathcal{N} = \bigcup_{f=0}^{\min\{m,n\}} & \big\{\boldsymbol{f}(x')^\alpha x^\gamma c^{-1} e^f w d \overline{x}^\delta \boldsymbol{g}(\overline{x}')^\beta \mid (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t, \\ & (\gamma, \delta) \in \mathbb{N}_k^r \times \mathbb{N}_k^t,\ \ c, d \in \mathscr{D}_{r,t}^f, w \in \mathfrak{S}_{r-f} \times \overline{\mathfrak{S}}_{t-f}\big\}. \end{aligned} \tag{20}$$

Proof. The result follows from Theorem 3 since the transition matrix between $\mathcal{N}$ and $\mathcal{M}$ in (13) is invertible. $\square$

Lemma 8. *Let I be the two-sided ideal of $\mathscr{B}_{r,t}^{\mathrm{aff}}$ generated by $\boldsymbol{f}(x_1)$ and $\boldsymbol{g}(\overline{x}_1)$ satisfying* (6)–(8)*. If $\mathscr{B}_{r,t}^{\mathrm{aff}}$ is admissible, then S is an R-basis of I, where*

$$S = \{\boldsymbol{f}(x')^\alpha x^\gamma c^{-1} e^f w d \overline{x}^\delta \boldsymbol{g}(\overline{x}')^\beta \in \mathcal{N} \mid \alpha_i + \beta_j \neq 0 \textit{ for some } i, j\}. \tag{21}$$

Proof. Let $M = \mathrm{span}_R S$. By Lemma 7, $\boldsymbol{f}(x_1) \mathscr{B}_{r,t}^{\mathrm{aff}} \subseteq M$. For any positive integer l with $1 \leq l < i$, by Lemma 5 (ii),

$$\boldsymbol{f}(x_i') \boldsymbol{f}(x_l') \in \sum_{j=1}^{i-1} \boldsymbol{f}(x_j') \mathscr{B}_{r,t}^{\mathrm{aff}} + \boldsymbol{f}(x_i') D,$$

such that $D \in \mathscr{B}_{r,t}^{\mathrm{aff}}$ and the degree of D is strictly less then k. Thus, $\boldsymbol{f}(x_i') \mathscr{B}_{r,t}^{\mathrm{aff}} \subseteq M$ which follows from inductive assumption on j with $1 \leq j \leq i-1$ and inductive assumption on degrees. This proves $J_R \subseteq M$. One can check $J_L \subseteq M$ similarly. By Proposition 6 (iii), $I = M$. $\square$

By abuse of notations, a regular monomial $\mathbf{m}$ in Definition 3 is also called a *regular monomial* of $\mathscr{B}_{k,r,t}$ if $0 \leq \alpha_i, \beta_j \leq k-1$ for all i, j with $1 \leq i \leq r$ and $1 \leq j \leq t$. Obviously, the number of all such regular monomials is $k^{r+t}(r+t)!$.

Theorem 9. *The cyclotomic walled Brauer algebra $\mathscr{B}_{k,r,t}$ is free over R with rank $k^{r+t}(r+t)!$ if and only if $\mathscr{B}_{k,r,t}$ is admissible.*

Proof. Let M be the R-submodule of $\mathscr{B}_{k,r,t}$ spanned by all regular monomials of $\mathscr{B}_{k,r,t}$. By induction on degrees, it is routine to check that M is a left $\mathscr{B}_{k,r,t}$-module (cf. [23, Prop. 4.12] for $\mathscr{B}^{\mathrm{aff}}_{r,t}$). Since $1 \in M$, we have $M = \mathscr{B}_{k,r,t}$. If $\mathscr{B}_{k,r,t}$ is not admissible, by Lemma 1, e_1 is an R-torsion element. Since $e_1 \in M$, either $\mathscr{B}_{k,r,t}$ is not free or the rank of $\mathscr{B}_{k,r,t}$ is strictly less than $k^{r+t}(r+t)!$. If $\mathscr{B}_{k,r,t}$ is admissible, by Lemmas 7–8, the set of all regular monomials of $\mathscr{B}_{k,r,t}$ is R-linear independent. Thus, $\mathscr{B}_{k,r,t}$ is free over R with rank $k^{r+t}(r+t)!$. □

2. A weakly cellular basis of $\mathscr{B}_{2,r,t}$

The aim of this section is to construct a weakly cellular basis of $\mathscr{B}_{2,r,t}$ in the sense of [12]. This basis will be used to set up a relationship between $\mathfrak{gl}_{m|n}$-Kac-modules and right cell modules of $\mathscr{B}_{2,r,t}$ in section 5.

Recall that a *composition* of r is a sequence of non–negative integers $\tau = (\tau_1, \tau_2, \dots)$ such that $|\tau| := \sum_i \tau_i = r$. If $\tau_i \geq \tau_{i+1}$ for all possible i's, then τ is called a *partition*. Similarly, a *k-partition* of r, or simply a *multipartition* of r, is an ordered k-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ of partitions with $|\lambda| := \sum_{i=1}^{k} |\lambda^{(i)}| = r$. Let $\Lambda_k^+(r)$ be the set of all k-partitions of r. Let $\trianglelefteq$ be the dominant order defined on $\Lambda_k^+(n)$ in the sense that $\lambda \trianglelefteq \mu$ if and only if

$$\sum_{h=1}^{\ell-1} |\lambda^{(h)}| + \sum_{j=1}^{i} \lambda_j^{(\ell)} \leq \sum_{k=1}^{\ell-1} |\mu^{(h)}| + \sum_{j=1}^{i} \mu_j^{(\ell)} \quad \text{for } \ell \leq k \text{ and all possible } i, \tag{22}$$

where $|\lambda^{(0)}| = 0$. Then $\Lambda_k^+(r)$ is a poset with $\trianglelefteq$ as a partial order on it. In this paper, we always assume $k \in \{1, 2\}$.

For each $\lambda \in \Lambda_1^+(r)$, the *Young diagram* $[\lambda]$ is a collection of boxes arranged in left-justified rows with λ_i boxes in the ith row of $[\lambda]$. A *λ-tableau* $\mathfrak{s}$ is obtained by inserting elements i, $1 \leq i \leq r$ into $[\lambda]$ without repetition. A λ-tableau $\mathfrak{s}$ is said to be *standard* if the entries in $\mathfrak{s}$ increase both from left to right in each row and from top to bottom in each column. Let $\mathscr{T}^s(\lambda)$ be the set of all standard λ-tableaux. Let $\mathfrak{t}^\lambda \in \mathscr{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \dots, r$ from left to right along the rows of $[\lambda]$. Let $\mathfrak{t}_\lambda \in \mathscr{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \dots, r$ from top to bottom along the columns of $[\lambda]$. For example, if $\lambda = (3, 2)$, then

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \qquad \text{and} \qquad \mathfrak{t}_\lambda = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}. \tag{23}$$

If $\lambda \in \Lambda_2^+(r)$, then the corresponding Young diagram $[\lambda]$ is $([\lambda^{(1)}], [\lambda^{(2)}])$. In this case, a *$\lambda$-tableau* $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$ is obtained by inserting elements i, $1 \leq i \leq r$ into $[\lambda]$ without repetition. A λ-tableau $\mathfrak{s}$ is said to be *standard* if the entries in $\mathfrak{s}_i$, $1 \leq i \leq 2$ increase both from left to right in each row and from top to bottom in each column. Let $\mathscr{T}^s(\lambda)$ be the set of all standard λ-tableaux. Let $\mathfrak{t}^\lambda \in \mathscr{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \dots, r$ from left to right along the rows of $[\lambda^{(1)}]$ and then $[\lambda^{(2)}]$. Let $\mathfrak{t}_\lambda \in \mathscr{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \dots, r$ from top to bottom along the columns of $[\lambda^{(2)}]$ and then $[\lambda^{(1)}]$. For example, if $\lambda = ((3, 2), (3, 1)) \in \Lambda_2^+(9)$, then

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 6 & 7 & 8 \\ \hline 9 \\ \cline{1-1}\end{array} \right) \qquad \text{and} \qquad \mathfrak{t}_\lambda = \left(\begin{array}{|c|c|c|}\hline 5 & 7 & 9 \\ \hline 6 & 8 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \right). \tag{24}$$

Recall that $\mathfrak{S}_r$ acts on the right of the set $\{1,2,\dots,r\}$ (i.e., the right action). Then $\mathfrak{S}_r$ acts on the right of a λ-tableau $\mathfrak{s}$ by permuting its entries. For example, if $\lambda=((3,2),(3,1))\in\Lambda_2^+(9)$, and $w=s_1s_2$, then

$$\mathfrak{t}^\lambda w=\left(\begin{array}{|c|c|c|}\hline 3&1&2\\\hline 4&5\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 6&7&8\\\hline 9\\\cline{1-1}\end{array}\right). \tag{25}$$

Write $d(\mathfrak{s})=w$ for $w\in\mathfrak{S}_r$ if $\mathfrak{t}^\lambda w=\mathfrak{s}$. Then $d(\mathfrak{s})$ is uniquely determined by $\mathfrak{s}$. Let $w_\lambda=d(\mathfrak{t}_\lambda)$. The row stabilizer $\mathfrak{S}_\lambda$ of $\mathfrak{t}^\lambda$ for $\lambda\in\Lambda_k^+(r)$ is known as the Young subgroup of $\mathfrak{S}_r$ with respect to λ. It is the same as the Young subgroup $\mathfrak{S}_{\lambda_{\mathrm{comp}}}$ with respect to the composition λ_{comp}, which is obtained from λ by concatenation. For example, if $\lambda=((3,2),(3,1))$ then $\lambda_{\mathrm{comp}}=(3,2,3,1)$ with

$$\mathfrak{t}^{\lambda_{\mathrm{comp}}}=\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\hline 6&7&8\\\hline 9\\\cline{1-1}\end{array}.$$

In this case, it is easy to see that the row stabilizer $\mathfrak{S}_{\lambda_{\mathrm{comp}}}$ of $\mathfrak{t}^{\lambda_{\mathrm{comp}}}$ is the subgroup of $\mathfrak{S}_9$ generated by $\{s_1,s_2,s_4,s_6,s_7\}$.

The level two degenerate Hecke $\mathscr{H}_{2,r}$ with defining parameters u_1 and u_2 is $\mathscr{H}_r^{\mathrm{aff}}/I$, where I is the two-sided ideal of $\mathscr{H}_r^{\mathrm{aff}}$ generated by $(y_1-u_1)(y_1-u_2)$, $u_1,u_2\in R$. By definition, $\mathscr{H}_{2,r}$ is an R-algebra generated by s_i, $1\le i\le r-1$ and y_j, $1\le j\le r$ such that

(i) $s_is_j=s_js_i$, $1<|i-j|$,
(ii) $y_iy_\ell=y_\ell y_i$, $1\le i,\ell\le r$,
(iii) $s_iy_i-y_{i+1}s_i=-1$, $y_is_i-s_iy_{i+1}=-1$, $1\le i\le r-1$,
(iv) $s_js_{j+1}s_j=s_{j+1}s_js_{j+1}$, $1\le j\le r-2$,
(v) $s_i^2=1$, $1\le i\le r-1$,
(vi) $(y_1-u_1)(y_1-u_2)=0$.

Following [3], we define $\pi_\lambda=\pi_a(u_2)$ and $\widetilde{\pi}_\lambda=\pi_a(u_1)$ for $\lambda\in\Lambda_2^+(r)$ with $|\lambda^{(1)}|=a$, where for any $u\in R$, $\pi_0(u)=1$ and $\pi_a(u)=\prod_{i=1}^a(y_i-u)$ if $a>0$. Let

$$w_a=\begin{pmatrix}1&2&\cdots&a&a{+}1&a{+}2&\cdots&r\\ r{-}a{+}1&r{-}a{+}3&\cdots&r&1&2&\cdots&r{-}a\end{pmatrix}. \tag{26}$$

It is well known that

$$w_as_j=s_{(j)w_a^{-1}}w_a\quad\text{if } j\neq r-a. \tag{27}$$

Let $\mathfrak{S}_{a,r-a}$ be the Young subgroup with respect to the composition $(a,r-a)$. Then

$$R\mathfrak{S}_{a,r-a}w_a=w_aR\mathfrak{S}_{r-a,a}. \tag{28}$$

For each composition λ of r, we denote

$$x_\lambda=\sum_{w\in\mathfrak{S}_\lambda}w,\quad y_\lambda=\sum_{w\in\mathfrak{S}_\lambda}(-1)^{\ell(w)}w, \tag{29}$$

where $\ell(\cdot)$ is the length function on $\mathfrak{S}_r$. Assume $\lambda\in\Lambda_2^+(r)$ with $|\lambda^{(1)}|=a$. If we denote $\mu^{(i)}=(\lambda^{(i)})'$, the conjugate of $\lambda^{(i)}$ for $i=1,2$, then

$$w_ax_{\mu^{(2)}}y_{\mu^{(1)}}=y_{\mu^{(1)}}x_{\mu^{(2)}}w_a. \tag{30}$$

Remark 1. When we write $x_{\mu^{(2)}}y_{\mu^{(1)}}$, then $x_{\mu^{(2)}}$ (respectively, $y_{\mu^{(1)}}$) is defined via symmetric group on $r-a$ letters $\{1, 2, \dots, r-a\}$ (respectively, on a letters $\{r-a+1, \dots, r\}$). Similarly, when we write $y_{\mu^{(1)}}x_{\mu^{(2)}}$, then $y_{\mu^{(1)}}$ (respectively, $x_{\mu^{(2)}}$) is defined via symmetric group on a letters $\{1, 2, \dots, a\}$ (respectively, on $r-a$ letters $\{a+1, a+2, \dots, r\}$).

Definition 4. For any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)$ with $\lambda \in \Lambda_2^+(r)$, define

(i) $\mathfrak{x}_{\mathfrak{st}} = d(\mathfrak{s})^{-1}\mathfrak{x}_\lambda d(\mathfrak{t})$, where $\mathfrak{x}_\lambda = \pi_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}$,
(ii) $\mathfrak{y}_{\mathfrak{st}} = d(\mathfrak{s})^{-1}\mathfrak{y}_\lambda d(\mathfrak{t})$, where $\mathfrak{y}_\lambda = \widetilde{\pi}_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}$,
(iii) $\overline{\mathfrak{x}}_{\mathfrak{st}} = d(\mathfrak{s})^{-1}\overline{\mathfrak{x}}_\lambda d(\mathfrak{t})$, where $\overline{\mathfrak{x}}_\lambda = \pi_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}$,
(iv) $\overline{\mathfrak{y}}_{\mathfrak{st}} = d(\mathfrak{s})^{-1}\overline{\mathfrak{y}}_\lambda d(\mathfrak{t})$, where $\overline{\mathfrak{y}}_\lambda = \widetilde{\pi}_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}$.

It is proven in [3] that $\mathscr{H}_{2,r}$ is a cellular algebra over R in the sense of [13]. In this paper, we need the following cellular basis of $\mathscr{H}_{2,r}$ so as to construct a new weakly cellular basis of $\mathscr{B}_{2,r,t}$.

Lemma 10. *The set S_i, $i \in \{1, 2, 3, 4\}$, are cellular bases of $\mathscr{H}_{2,r}$ in the sense of* [13]*, where*

(i) $S_1 = \{\mathfrak{x}_{\mathfrak{st}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\}$,
(ii) $S_2 = \{\mathfrak{y}_{\mathfrak{st}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\}$,
(iii) $S_3 = \{\overline{\mathfrak{x}}_{\mathfrak{st}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\}$,
(iv) $S_4 = \{\overline{\mathfrak{y}}_{\mathfrak{st}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\}$.

Proof. Let $S = \{x_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda), \lambda \in \Lambda_2^+(r)\}$ and $x_{\mathfrak{st}} = d(\mathfrak{s})^{-1}\pi_\lambda x_{\lambda^{(1)}} x_{\lambda^{(2)}} d(\mathfrak{t})$. It is proven in [3] that S is a cellular basis of $\mathscr{H}_{2,r}$. If we use $y_{\lambda^{(2)}}$ instead of $x_{\lambda^{(2)}}$ in $x_{\mathfrak{st}}$, we will get $\mathfrak{x}_{\mathfrak{st}}$. However, for any $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{T}^s(\lambda)$, $d(\mathfrak{s})$ can be written uniquely as $d(\mathfrak{s}_1)d(\mathfrak{s}_2)d$ such that d is a distinguished right coset representative of $\mathfrak{S}_a \times \mathfrak{S}_{r-a}$ in $\mathfrak{S}_r$ and $\mathfrak{s}_i \in \mathcal{T}^s(\lambda^{(i)})$, where $a = |\lambda^{(1)}|$. So, the transition matrix between S_1 and S is determined by the transition matrix between the cellular basis

$$\{d(\mathfrak{s}_2)^{-1}x_{\lambda^{(2)}}d(\mathfrak{t}_2) \mid \lambda^{(2)} \in \Lambda^+(r-a), \mathfrak{s}_2, \mathfrak{t}_2 \in \mathcal{T}^s(\lambda^{(2)})\} \text{ and}$$
$$\{d(\mathfrak{s}_2)^{-1}y_{\lambda^{(2)}}d(\mathfrak{t}_2) \mid \lambda^{(2)} \in \Lambda^+(r-a), \mathfrak{s}_2, \mathfrak{t}_2 \in \mathcal{T}^s(\lambda^{(2)})\}$$

of $R\mathfrak{S}_{r-a}$. Thus, S_1 is a basis of $\mathscr{H}_{2,r}$. One can check that S_1 is a cellular basis of $\mathscr{H}_{2,r}$ in the sense of [13] by mimicking Dipper–James–Murphy's arguments in the proof of Murphy basis for Hecke algebras of type B in [9]. We leave the details to the readers. Finally, (ii)–(iv) can be verified similarly. □

By Graham–Lehrer's results on the representation theory of cellular algebras in [13], one can define right cell modules of $\mathscr{H}_{2,r}$ via the cellular bases S_i, $i \in \{1, 2, 3, 4\}$ in Lemma 10. The corresponding right cell modules of $\mathscr{H}_{2,r}$ with respect to S_2 and S_4 are denoted by $\widetilde{\Delta}(\lambda)$, and $\overline{\Delta}(\lambda)$.

For the simplification of discussion, we assume $\mathscr{H}_{2,r}$ is defined over $\mathbb{C}$ in Lemma 11.

Lemma 11. *Suppose $a, b \in \mathbb{N}$. Then*

(i) $\pi_a(u_2)\mathscr{H}_{2,r}\pi_b(u_1) = 0$ *whenever* $a + b > r$ *and* $a, b \in \mathbb{Z}^{>0}$.
(ii) $\pi_a(u_2)\mathscr{H}_{2,r}\pi_{r-a}(u_1) = \pi_a(u_2)w_a\pi_{r-a}(u_1)\mathbb{C}\mathfrak{S}_{r-a,a}$, *where* $\mathfrak{S}_{r-a,a}$ *is as in* (28).
(iii) $\mathfrak{x}_\lambda\mathscr{H}_{2,r}\mathfrak{y}_{\mu'} = 0$ *if* $\lambda, \mu \in \Lambda_2^+(r)$ *with* $\lambda \rhd \mu$.
(iv) $\mathfrak{x}_\lambda\mathscr{H}_{2,r}\mathfrak{y}_{\lambda'} = Span_{\mathbb{C}}\{\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\}$ *if* $\lambda \in \Lambda_2^+(r)$.
(v) $\widetilde{\Delta}(\lambda') \cong \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$.

Proof. (i), (ii) and (iv) can be proven by arguments similar to those for Hecke algebras of type B in [8]. We only give details for (iii) and (v).

If $\lambda \rhd \mu$, then $|\lambda^{(1)}| \geq |\mu^{(1)}|$. If $|\lambda^{(1)}| > |\mu^{(1)}|$, then $|\mu^{(1)}| \neq r$ and the result follows from (i). When $|\lambda^{(1)}| = |\mu^{(1)}|$, by (ii) together with corresponding results for the group algebras of symmetric groups, we have $\lambda^{(i)} \unlhd \mu^{(i)}$ for $i = 1, 2$ if $\mathfrak{x}_\lambda\mathscr{H}_{2,r}\mathfrak{y}_{\mu'} \neq 0$. This proves (iii).

There is a surjective $\mathscr{H}_{2,r}$-homomorphism from $\phi : \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} \to \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$. Let $\mathscr{H}_{2,r}^{\rhd\lambda'}$ be the $\mathbb{C}$-submodule spanned by $\{\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\mu), \mu \rhd \lambda'\}$. It follows from standard results on cellular algebras that $\mathscr{H}_{2,r}^{\rhd\lambda'}$ is a two-sided ideal of $\mathscr{H}_{2,r}$. So, $\mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} + \mathscr{H}_{2,r}^{\rhd\lambda'}/\mathscr{H}_{2,r}^{\rhd\lambda'}$ is isomorphic to a submodule of $\widetilde{\Delta}(\lambda')$. If $\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} \in \mathscr{H}_{2,r}^{\rhd\lambda'}$, we have $\mu \rhd \lambda'$ which is equivalent to $\lambda \rhd \mu'$. By (iii), $x_\lambda w_\lambda \mathfrak{y}_{\mathfrak{s}\mathfrak{t}} = 0$ and $\mathscr{H}_{2,r}^{\rhd\lambda'} \subset \ker\phi$. So, there is an epimorphism from $\mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} + \mathscr{H}_{2,r}^{\rhd\lambda'}/\mathscr{H}_{2,r}^{\rhd\lambda'}$ to $\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$. Mimicking arguments on classical Specht modules for Hecke algebra of type B in [8], we know that $\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$ has a basis $\{\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'} d(\mathfrak{t}) \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$. So,

$$\dim_{\mathbb{C}} \widetilde{\Delta}(\lambda') = \dim_{\mathbb{C}} \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} = \#\mathcal{T}^s(\lambda'),$$

forcing $\mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} + \mathscr{H}_{2,r}^{\rhd\lambda'}/\mathscr{H}_{2,r}^{\rhd\lambda'} \cong \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathscr{H}_{2,r} \cong \widetilde{\Delta}(\lambda')$. □

Now, we use cellular bases S_i of $\mathscr{H}_{2,r}$ in Lemma 10 to construct a weakly cellular basis of $\mathscr{B}_{2,r,t}$ over an arbitrary field in the sense of [12]. We remark that when we use results on level two degenerate Hecke algebra for $\mathscr{B}_{2,r,t}$, we should keep in mind that $\mathscr{B}_{2,r,t}$ contains two subalgebras generated by $\{x_1, s_1, s_2, \ldots, s_{r-1}\}$ and $\{\overline{x}_1, \overline{s}_1, \overline{s}_2, \ldots, \overline{s}_{t-1}\}$, respectively. The first subalgebra is isomorphic to $\mathscr{H}_{2,r}$ with x_1 being sent to $-y_1$ and the second is isomorphic to $\mathscr{H}_{2,t}$ with $\overline{x}_1$ being sent to $-y_1$. Therefore, we have to use $-u_i$ and $-\overline{u}_i$ instead of u_i and $\overline{u}_i$, respectively.

Fix $r, t, f \in \mathbb{Z}^{>0}$ with $f \leq \min\{r, t\}$. In contrast to (12), we define

$$\mathcal{D}_{r,t}^f = \{s_{r-f+1,i_{r-f+1}}\overline{s}_{t-f+1,j_{t-f+1}} \cdots s_{r,i_r}\overline{s}_{t,j_t} \mid r \geq i_r > \cdots > i_{r-f+1}, j_k \geq k+f-t\}. \quad (31)$$

For each $c \in \mathcal{D}_{r,t}^f$ as in (31), let κ_c be the r-tuple

$$\kappa_c = (k_1, \ldots, k_r) \in \{0,1\}^r \text{ such that } k_i = 0 \text{ unless } i = i_r, i_{r-1}, \ldots, i_{r-f+1}. \quad (32)$$

Note that κ_c may have more than one choice for a fixed c, and it may be equal to κ_d although $c \neq d$ for $c, d \in \mathcal{D}_{r,t}^f$. Let $\mathbf{N}_f = \{\kappa_c \mid c \in \mathcal{D}_{r,t}^f\}$. If $\kappa_c \in \mathbf{N}_f$, define $x^{\kappa_c} = \prod_{i=1}^r x_i^{k_i}$. In [23], we consider poset $(\Lambda_{2,r,t}, \unrhd)$, where

$$\Lambda_{2,r,t} = \left\{(f, \lambda, \mu) \mid (\lambda, \mu) \in \Lambda_2^+(r-f) \times \Lambda_2^+(t-f),\ 0 \leq f \leq \min\{r,t\}\right\}, \quad (33)$$

such that $(f,\lambda,\mu) \trianglerighteq (\ell,\alpha,\beta)$ for $(f,\lambda,\mu),(\ell,\alpha,\beta) \in \Lambda_{2,r,t}$ if either $f > \ell$ or $f = \ell$ and $\lambda \trianglerighteq_1 \alpha$, and $\mu \trianglerighteq_2 \beta$, and in case $f = \ell$, the orders $\trianglerighteq_1$ and $\trianglerighteq_2$ are dominant orders on $\Lambda_2^+(r-f)$ and $\Lambda_2^+(t-f)$ respectively. For each $(f,\mu,\nu) \in \Lambda_{2,r,t}$, let

$$\delta(f,\mu,\nu)=\{(\mathfrak{t},c,\kappa_c)\,|\,\mathfrak{t}=(\mathfrak{t}^{(1)},\mathfrak{t}^{(2)})\in\mathcal{T}^s(\mu)\times\mathcal{T}^s(\nu), c\in\mathcal{D}^f_{r,t} \text{ and } \kappa_c\in\mathbf{N}_f\}. \quad (34)$$

Definition 5. For any $(\mathfrak{s},d,\kappa_d),(\mathfrak{t},c,\kappa_c) \in \delta(f,\mu,\nu)$ with $(f,\mu,\nu) \in \Lambda_{2,r,t}$, define

$$C_{(\mathfrak{s},d,\kappa_d)(\mathfrak{t},c,\kappa_c)} = x^{\kappa_d} d^{-1} \mathfrak{e}^f \mathfrak{n}_{\mathfrak{s}\mathfrak{t}} c x^{\kappa_c}, \quad (35)$$

where, in contrast to notation e^f in (11), we define $\mathfrak{e}^f = e_{r,t}e_{r-1,t-1}\cdots e_{r-f+1,t-f+1}$ if $f \geq 1$ and $\mathfrak{e}^0 = 1$, and $\mathfrak{n}_{\mathfrak{s}\mathfrak{t}} = \mathfrak{y}_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}}\overline{\mathfrak{y}}_{\mathfrak{s}^{(2)}\mathfrak{t}^{(2)}}$ if $\mathfrak{s} = (\mathfrak{s}^{(1)},\mathfrak{s}^{(2)})$ and $\mathfrak{t} = (\mathfrak{t}^{(1)},\mathfrak{t}^{(2)})$ are in $\mathcal{T}^s(\mu) \times \mathcal{T}^s(\nu)$.

Note that $\mathfrak{n}_{\mathfrak{s}\mathfrak{t}}$ in Definition 5 are defined via cellular basis elements of $\mathscr{H}_{2,r-f}$ and $\mathscr{H}_{2,t-f}$ in Lemma 10 (ii) (iv). Since x_i and $\overline{x}_j$ do not commute each other, a cellular basis element of $\mathscr{H}_{2,r-f}$ is always put on the left. Further, we need to use $x_i, -u_1, -u_2$ (respectively $\overline{x}_i, -\overline{u}_1, -\overline{u}_2$) instead of $-y_i$, u_1, u_2 in Lemma 10.

Theorem 12. *If $\mathscr{B}_{2,r,t}$ is admissible, then the set*

$$\mathscr{C} = \{C_{(\mathfrak{s},\kappa_c,c)(\mathfrak{t},\kappa_d,d)} \,|\, (\mathfrak{s},\kappa_c,c),(\mathfrak{t},\kappa_d,d) \in \delta(f,\lambda), \forall (f,\lambda) \in \Lambda_{2,r,t}\}$$

is a weakly cellular basis $\mathscr{B}_{2,r,t}$ over R in the sense of [12].

Proof. Let S be the cellular basis of $\mathscr{H}_{2,r-f}$ (respectively $\mathscr{H}_{2,t-f}$) for $0 \leq f \leq \min\{r,t\}$ defined in the proof of Lemma 10. If we use S instead of the cellular basis S_2 of $\mathscr{H}_{2,r-f}$ and S_4 of $\mathscr{H}_{2,t-f}$ in Lemma 10, we will obtain the weakly cellular basis of $\mathscr{B}_{2,r,t}$ over R in [23, Thm. 6.12] provided that $R = \mathbb{C}$ and $u_1 = -p$, $u_2 = m - q$, $\overline{u}_1 = q$ and $\overline{u}_2 = p - n$ with $r + t \leq \min\{m,n\}$. Since $\mathscr{B}_{2,r,t}$ is admissible, by Theorem 9, the rank of $\mathscr{B}_{2,r,t}$ is $2^{r+t}(k+t)!$. As pointed out in [23, Remark 6.13], [23, Thm. 6.12] holds over R with arbitrary parameters $u_1, u_2, \overline{u}_1, \overline{u}_2$ if the rank of $\mathscr{B}_{2,r,t}$ is $2^{r+t}(r+t)!$. Thus, $\mathscr{C}$ is an R-basis of $\mathscr{B}_{2,r,t}$. Further, the weakly cellularity of $\mathscr{B}_{2,r,t}$ depends only on cellular bases of $\mathscr{H}_{2,r-f}$ and $\mathscr{H}_{2,t-f}$ and does not depend on the explicit descriptions of cellular bases of $\mathscr{H}_{2,r-f}$ and $\mathscr{H}_{2,t-f}$. (cf. the proof of [23, Thm. 6.12]). So, all arguments for the proof of [23, Thm. 6.12] can be used smoothly to prove that $\mathscr{C}$ is a weakly cellular basis $\mathscr{B}_{2,r,t}$ over R. □

Suppose $\mathscr{B}_{2,r,t}$ is defined over a field F. By Theorem 12, one can define right cell modules $C(f,\mu,\nu)$ with respect to $(f,\mu,\nu) \in \Lambda_{2,r,t}$ for $\mathscr{B}_{2,r,t}$. Let $\phi_{f,\mu,\nu}$ be the corresponding invariant form on $C(f,\mu,\nu)$ and let $D^{f,\mu,\nu} = C(f,\mu,\nu)/\operatorname{Rad}\phi_{f,\mu,\nu}$, where $\operatorname{Rad}\phi_{f,\mu,\nu}$ is the radical of $\phi_{f,\mu,\nu}$. By Graham–Lehrer's results in [13] (a weakly cellular algebra has similar representation theory of a cellular algebra in [13]), $D^{f,\mu,\nu}$ is either 0 or irreducible and all non-zero $D^{f,\mu,\nu}$ consist of a complete set of pair-wise non-isomorphic irreducible $\mathscr{B}_{2,r,t}$-modules. Let $\widetilde{\Delta}(\mu)$ (respectively $\overline{\Delta}(\nu)$) be the cell module of $\mathscr{H}_{2,r-f}$ (respectively $\mathscr{H}_{2,t-f}$) defined via S_2 and S_4 in Lemma 10. Similarly, one has the notations D^μ and $\overline{D}^\nu$, respectively.

Proposition 13. *Suppose that $\mathscr{B}_{2,r,t}$ is admissible over F. For any $(f,\mu,\nu) \in \Lambda_{2,r,t}$, $D^{f,\mu,\nu} \neq 0$ if and only if*

(i) $D^{\mu} \neq 0$ *and* $\overline{D}^{\nu} \neq 0$,

(ii) $f \neq r$ *provided* $r = t$ *and* $\omega_0 = \omega_1 = 0$.

Proof. The result can be proven by arguments similar to those for Lemmas 7.3–7.4 in [23]. □

Remark 2. By arguments similar to those for Theorem 12, one can lift cellular bases of $\mathscr{H}_{k,r}$ and $\mathscr{H}_{k,t}$ in [3] to obtain a weakly cellular basis of $\mathscr{B}_{k,r,t}$ over R, provided that $\mathscr{B}_{k,r,t}$ is admissible. Further, it is not difficult to prove the result, which is similar to Proposition 13 for $\mathscr{B}_{k,r,t}$ over an arbitrary field F with characteristic $\operatorname{char} F$ either zero or positive. Let $\mathbf{u} = (u_1, \ldots, u_k) \in F^k$ such that $u_i = d_i \cdot 1_F$ and $0 \le d_i < \operatorname{char} F$ for $1 \le i \le k$. Kleshchev [18] has shown that the simple $\mathscr{H}_{k,n}(\mathbf{u})$-modules are *labeled* by a set of multipartitions which gives the same Kashiwara crystal as the set of $\mathbf{u}$-Kleshchev multipartitions of n in [1, 2]. Thus, the simple $\mathscr{B}_{k,r,t}$-modules are labeled by the set $\{(f,\mu,\nu)\}$, where (i) $0 \le f \le \min\{r,t\}$, (ii) μ's are Kleshchev multipartitions of $r-f$ with respect to $\mathbf{u}$, (iii) ν's are Kleshchev multipartitions of $t-f$ with respect to $\overline{\mathbf{u}} := (\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k)$, (iv) $f \neq r$ if $r = t$ and $\omega_i = 0$ for $0 \le i \le k-1$. By Proposition 13 and [10, Thm. 1.1] or [1, Thm. 1.3], when $\mathscr{B}_{k,r,t}$ is admissible, the simple $\mathscr{B}_{k,r,t}$-modules are always labeled by the $(f,\mu,\nu) \in \Lambda_{k,r,t}$ with $0 \le f \le \min\{r,t\}$ and μ (respectively ν) are Kleshchev multipartitons with respect to $\mathbf{u}$ (respectively $\overline{\mathbf{u}}$) and $f \neq r$ if $r = t$ and $\omega_i = 0$ for $1 \le i \le r$. However, we are not claiming that $D^{f,\mu,\nu} \neq 0$ for the multipartitions μ, ν which Kleshchev [18] uses to label the simple $\mathscr{H}_{k,r-f}(\mathbf{u})$-modules (respectively $\mathscr{H}_{k,r-f}(\overline{\mathbf{u}})$-modules).

We recall the definition of Kleshchev bipartitions over $\mathbb{C}$ as follows (see, e.g., [32]), which will be used in sections 4–5. Fix $u_1, u_2 \in \mathbb{C}$ with $u_1 - u_2 \in \mathbb{N}$. Then $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$ is called a *Kleshchev bipartition* [32] with respect to u_1, u_2 if

$$\lambda^{(1)}_{u_1-u_2+i} \le \lambda^{(2)}_i \text{ for all possible } i. \tag{36}$$

If $u_1 - u_2 \notin \mathbb{Z}$, all bipartitions of r are Kleshchev bipartitions. A pair of bipartitions (μ,ν) is *Kleshchev* if both μ and ν are Kleshchev bipartitions in the sense of (36) with respect to the parameters u_1, u_2 and $\overline{u}_1$, $\overline{u}_2$. The following result will be used in section 5.

Proposition 14. *Suppose $\mathscr{B}_{2,r,t}$ is admissible over $\mathbb{C}$. For each $(f,\mu,\nu) \in \Lambda_{2,r,t}$, let*

$$\widetilde{C}(f,\mu,\nu) := \mathfrak{e}^f \mathfrak{x}_{\mu'} \overline{\mathfrak{x}}_{\nu'} w_{\mu'} w_{\nu'} \mathfrak{y}_{\mu} \overline{\mathfrak{y}}_{\nu} \mathscr{B}_{2,r,t} \pmod{\mathscr{B}^{f+1}_{2,r,t}},$$

where $\mathscr{B}^{f+1}_{2,r,t}$ is the two-sided ideal of $\mathscr{B}_{2,r,t}$ generated by $\mathfrak{e}^{f+1}$. Then $C(f,\mu,\nu) \cong \widetilde{C}(f,\mu,\nu)$.

Proof. Let M_f be the left $\mathscr{B}_{2,r-f,t-f}$-module generated by

$$V^f_{r,t} = \{\mathfrak{e}^f dx^{\kappa_d} \mid (d,\kappa_d) \in \mathcal{D}^f_{r,t} \times \mathbf{N}_f\}. \tag{37}$$

By [23, Prop. 6.10], $M_f = \mathfrak{e}^f \mathscr{B}_{2,r,t}$. By [23, Lem. 6.9], one can use $\mathscr{H}_{2,r-f} \otimes \mathscr{H}_{2,t-f}$ instead of $\mathscr{B}_{2,r-f,t-f}$ in $\mathfrak{x}_{\mu'} \overline{\mathfrak{x}}_{\nu'} w_{\mu'} w_{\nu'} \mathfrak{y}_{\mu} \overline{\mathfrak{y}}_{\nu} M_f \pmod{\mathscr{B}^{f+1}_{2,r,t}}$. Now, the required isomorphism follows from Lemma 11 (v). □

3. Super Schur–Weyl duality

The aim of this section is to generalize super Schur–Weyl duality between general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ and $\mathscr{B}_{2,r,t}$ to the case $r+t > \min\{m,n\}$. Throughout, let $I_0 = \{1,\dots,m\}$, $I_1 = \{m+1,\dots,m+n\}$ and $I = I_0 \cup I_1$.

For any pairs $(i,j) \in I\times I$, let E_{ij} be the matrix unit with parity $[E_{ij}] = [i]+[j]$, where $[i] = a$ if $i \in I_a$, $a = 0,1$. The *general linear Lie superalgebra* $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$, denoted by $\mathfrak{g}$, is $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_{-1} = \mathrm{span}_{\mathbb{C}}\{E_{i,j} \,|\, i \in I_1,\, j \in I_0\}, \quad \mathfrak{g}_1 = \mathrm{span}_{\mathbb{C}}\{E_{i,j} \,|\, i \in I_0,\, j \in I_1\},$$
$$\mathfrak{g}_0 = \mathrm{span}_{\mathbb{C}}\{E_{i,j} \,|\, i,j \in I_0 \ \text{ or } \ i,j \in I_1\}. \tag{38}$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the $\mathbb{C}$-space with basis $\{E_{ii} \,|\, i \in I\}$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$ with dual basis $\{\varepsilon_i \,|\, i \in I\}$. Then any $\xi \in \mathfrak{h}^*$, called a *weight* of $\mathfrak{g}$, can be written as

$$\xi = \sum_{i\in I_0} \xi_i^L \varepsilon_i + \sum_{i\in I_1} \xi_{i-m}^R \varepsilon_i \text{ with } \xi_i^L, \xi_j^R \in \mathbb{C}. \tag{39}$$

Denote ξ by $(\xi_1^L,\dots,\xi_m^L \,|\, \xi_1^R,\dots,\xi_n^R)$. If both $\xi_i^L - \xi_{i+1}^L \in \mathbb{N}$ and $\xi_j^R - \xi_{j+1}^R \in \mathbb{N}$ for all possible i,j, then ξ is called *integral dominant*. Let P^+ be the set of integral dominant weights. For any $\xi \in P^+$, let

$$\xi^\rho := \xi + \rho = (\xi_1^{L,\rho},\dots,\xi_m^{L,\rho} \,|\, \xi_1^{R,\rho},\dots,\xi_n^{R,\rho}), \tag{40}$$

where $\rho = (0,-1,\dots,1-m \,|\, m-1, m-2,\dots,m-n)$. Following [29], [30] (cf. [15], [17]), let

$$\ell = \#\{(i,j) \mid \xi_i^{L,\rho} + \xi_j^{R,\rho} = 0, 1 \le i \le m, 1 \le j \le n\}.$$

Then ξ is called an *ℓ-fold atypical weight* if $\ell > 0$. Otherwise, ξ is called *a typical weight.*

Example 1. For any $p,q \in \mathbb{C}$, let $\lambda_{pq} = (p,\dots,p \,|\, -q,\dots,-q)$. Then λ_{pq} is a typical weight if and only if

$$p-q \notin \mathbb{Z} \ \text{ or } \ p-q \le -m \ \text{ or } \ p-q \ge n. \tag{41}$$

The current q should be regarded as $q+m$ in [6, IV]. In the remaining part of this paper, λ_{pq} is always a typical weight in the sense of (41).

Let $V = \mathbb{C}^{m|n}$ be the natural $\mathfrak{g}$-module with natural basis $\{v_i \,|\, i \in I\}$ such that v_i has parity $[v_i] = [i]$. Then the dual space V^*, which has the dual basis $\{\overline{v}_i \,|\, i \in I\}$, is a left $\mathfrak{g}$-module such that

$$E_{ab}\overline{v}_i = -(-1)^{[a]([a]+[b])}\delta_{ia}\overline{v}_b \text{ for any } (a,b) \in I \times I. \tag{42}$$

In particular, the weight of $\overline{v}_i$ is $-\epsilon_i$. For the simplicity of notation, we set $W = V^*$.

Definition 6. Fix $r,t\in\mathbb{Z}^{>0}$. Let $V^{rt}=V^{\otimes r}\otimes W^{\otimes t}$ and $M_{pq}^{rt}=V^{\otimes r}\otimes K_{\lambda_{pq}}\otimes W^{\otimes t}$, where $K_{\lambda_{pq}}$ is the Kac-module [16] with respect to the highest weight λ_{pq} in Example 1.

Let $\pi : M_{pq}^{rt} \to V^{rt}$ be the projection such that, for any $v \in M_{pq}^{rt}$, $\pi(v)$ is the vector obtained from v by deleting the tensor factor in $K_{\lambda_{pq}}$. Let v_{pq} be the highest weight vector of $K_{\lambda_{pq}}$ with highest weight λ_{pq}. Then v_{pq} is unique up to a scalar. It is well known (see [16]) that $K_{\lambda_{pq}}$ is 2^{mn}-dimensional with a basis

$$B = \Big\{ b^\sigma := \prod_{i=1}^{n}\prod_{j=1}^{m} E_{m+i,j}^{\sigma_{ij}} v_{pq} \,\Big|\, \sigma = (\sigma_{ij})_{i,j=1}^{n,m} \in \{0,1\}^{n\times m}\Big\}, \tag{43}$$

where the products are taken in any fixed order. Define

$$\begin{aligned} I(m|n,r) &= \{\mathbf{i} \mid \mathbf{i} = (i_r, i_{r-1}, \cdots, i_1),\, i_j \in I,\, 1 \le j \le r\},\\ \overline{I}(m|n,t) &= \{\mathbf{j} \mid \mathbf{j} = (j_1, j_2, \cdots, j_t),\, j_i \in I,\, 1 \le i \le t\}. \end{aligned} \tag{44}$$

If $(\mathbf{i}, b, \mathbf{j}) \in I(m|n,r) \times B \times \overline{I}(m|n,t)$, we define

$$v_{\mathbf{i},b,\mathbf{j}} = v_{i_r} \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1} \otimes b \otimes \overline{v}_{j_1} \otimes \overline{v}_{j_2} \otimes \cdots \otimes \overline{v}_{j_t} \in M_{pq}^{rt}. \tag{45}$$

Lemma 15. *Let* $B_M = \{v_{\mathbf{i}} \otimes b \otimes \overline{v}_{\mathbf{j}} \mid (\mathbf{i}, b, \mathbf{j}) \in I(m|n,r) \times B \times \overline{I}(m|n,t)\}$. *Then* B_M *is a basis of* M_{pq}^{rt}.

Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Then M_{pq}^{rt} is a left $U(\mathfrak{g})$-module. Let $J = J_1 \cup \{0\} \cup J_2$ with $J_1 = \{r, \ldots, 2, 1\}$ and $J_2 = \{\overline{1}, \overline{2}, \ldots, \overline{t}\}$. Then $(J, \prec)$ is a total ordered set with

$$r \prec r-1 \prec \cdots \prec 1 \prec 0 \prec \overline{1} \prec \cdots \prec \overline{t}.$$

For any $a, b \in J$ with $a \prec b$, define $\pi_{ab} : U(\mathfrak{g})^{\otimes 2} \to U(\mathfrak{g})^{\otimes(r+t+1)}$ by

$$\pi_{ab}(x\otimes y) = 1\otimes \cdots \otimes 1 \otimes \overset{a\text{th}}{x} \otimes 1 \otimes \cdots \otimes 1 \otimes \overset{b\text{th}}{y} \otimes 1 \otimes \cdots \otimes 1. \tag{46}$$

Let Ω be a *Casimir element* in $\mathfrak{g}^{\otimes 2}$ given by

$$\Omega = \sum_{i,j\in I} (-1)^{[j]} E_{ij} \otimes E_{ji}. \tag{47}$$

In [23], we define operators s_i, $\overline{s}_j$, x_1, $\overline{x}_1$ and e_1 acting on the right of M_{pq}^{rt} via the following formulae:

$$\begin{aligned} s_i &= \pi_{i+1,i}(\Omega)|_{M_{pq}^{rt}}\ (1 \le i < r), \quad \overline{s}_j = \pi_{\overline{j},\overline{j+1}}(\Omega)|_{M_{pq}^{rt}}\ (1 \le j < t),\\ x_1 &= -\pi_{10}(\Omega)|_{M_{pq}^{rt}}, \quad \overline{x}_1 = -\pi_{0\overline{1}}(\Omega)|_{M_{pq}^{rt}}, \quad e_1 = -\pi_{1\overline{1}}(\Omega)|_{M_{pq}^{rt}}. \end{aligned} \tag{48}$$

Then there is an algebra homomorphism $\phi : \mathscr{B}_{2,r,t} \to \mathrm{End}_{U(\mathfrak{g})}(M_{pq}^{rt})^{\mathrm{op}}$ sending the generators $s_i, \overline{s}_j$, x_1, $\overline{x}_1$ and e_1 to the operators s_i, $\overline{s}_j$, x_1, $\overline{x}_1$ and e_1 as above [23]. In this case, we need to use $-p, m-q$, and $q, p-n$ instead of u_1, u_2, $\overline{u}_1$ and $\overline{u}_2$, respectively in Definition 1 for $k = 2$. Further, $\omega_0 = m - n$, $\omega_1 = nq - mp$ and $\omega_a = (m-p-q)\omega_{a-1} - p(q-m)\omega_{a-2}$ for $a \ge 2$ and $\overline{\omega}_a$'s are determined by [23, Cor. 4.3]. Thus, $\mathscr{B}_{2,r,t}$ is admissible in the sense of Definition 2. By Theorem 9, $\dim_{\mathbb{C}} \mathscr{B}_{2,r,t} = 2^{r+t}(r+t)!$. We will always consider $\mathscr{B}_{2,r,t}$ as above in the remaining part of this paper.

Theorem 16 ([23, Thm. 5.16]). *Fix $r, t \in \mathbb{Z}^{>0}$ with $r + t \le \min\{m, n\}$. Then* $\mathrm{End}_{\mathfrak{g}}(M_{pq}^{rt})^{\mathrm{op}} \cong \mathscr{B}_{2,r,t}$.

Theorem 17 ([6, IV, Thm. 3.13]). *If $0 < r \le \min\{m, n\}$, then $End_{U(\mathfrak{g})}(M_{pq}^{r0})^{\mathrm{op}} \cong \mathscr{H}_{2,r}$, the level two Hecke algebra with defining parameters $u_1 = -p$ and $u_2 = m - q$.*

Theorem 18 (Super Schur–Weyl duality). *Keep the condition* (41). *The algebra homomorphism $\phi_1 : \mathscr{B}_{2,r,t} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(M_{pq}^{rt})^{\mathrm{op}}$ is surjective. It is injective if and only if $r + t \le \min\{m, n\}$.*

Proof. By Theorem 16, it suffices to prove that ϕ_1 is surjective, and is not injective if $r + t > \min\{m, n\}$. Note that in diagram (49), θ_1, θ_2 are canonical vector space isomorphisms. Thus as in [7, (7.16)], we can define the map

$$\mathrm{flip}_{r,t} := \theta_2^{-1}\psi\theta_1,$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t}) & \xrightarrow{\mathrm{flip}_{r,t}} & \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t}) \\ \theta_1 \big\downarrow & & \theta_2 \big\downarrow \\ \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}}) \otimes \mathrm{End}_{\mathbb{C}}((V^*)^{\otimes t}) & \xrightarrow{\psi:\, f \otimes g^* \mapsto f \otimes g} & \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}}) \otimes \mathrm{End}_{\mathbb{C}}(V^{\otimes t}). \end{array} \tag{49}$$

It is proven in [7, Lem. 7.6] that $\mathrm{flip}_{r,t}$ is in fact a $\mathfrak{g}$-module isomorphism. Note that $\mathscr{H}_{2,r+t}$ (denoted as $H_{r+t}^{p,q}$ in [6, IV]) is a subspace of $\mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})$, thus (49) induces the following commutative diagram

$$\begin{array}{ccc} \mathscr{B}_{2,r,t} & \xrightarrow{\mathrm{flip}_{r,t}} & \mathscr{H}_{2,r+t} \\ \phi_1 \big\downarrow & & \big\downarrow \pi_1 \\ \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t}) & \xrightarrow{\mathrm{flip}_{r,t}} & \mathrm{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t}). \end{array} \tag{50}$$

By Theorem 9 for $k = 2$, $\dim_{\mathbb{C}} \mathscr{B}_{2,r,t} = 2^{r+t}(r + t)!$. This implies that the top map is a bijection, and the bottom map is a $\mathfrak{g}$-module isomorphism, which induces an isomorphism between two subspaces $\mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t})^{\mathrm{op}}$ and $\mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})^{\mathrm{op}}$. Since, by [6, IV, Thm. 3.21], π_1 is surjectively mapped to $\mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})^{\mathrm{op}}$, we see that $\phi_1 : \mathscr{B}_{2,r,t} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(M_{pq}^{rt})^{\mathrm{op}}$ is surjective. Finally, the second assertion follows from the corresponding result for $t = 0$ in [6, IV, Thm. 3.21]. □

4. Highest weight vectors in $V^{\otimes r} \otimes K_{\lambda_{pq}}$

The aim of this section is to give a classification of highest weight vectors of $M_{pq}^{r0} := V^{\otimes r} \otimes K_{\lambda_{pq}}$ when $r \le \min\{m, n\}$, where V is the natural representation of $\mathfrak{g} := \mathfrak{gl}_{m|n}$ and $K_{\lambda_{pq}}$ is the Kac-module with a highest weight vector $v_{\lambda_{pq}}$ of weight λ_{pq} in Example 1. This will be done in a few steps. First, by noting that $\mathfrak{g}$-highest weight vectors of M_{pq}^{r0} is in one-to-one correspondence with the $\mathfrak{g}_0$-highest weight vectors of $V^{\otimes r}$ (see Remark 3), we are able to reduce the problem to the Lie algebra case. Secondly, since $\mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$, and $V^{\otimes r}$ can be decomposed as a direct sum of tensor products of natural representations of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$, we are able to further simplify the problem to the $\mathfrak{gl}_m$ case.

Remark 3. Any $\mathfrak{g}$-highest weight vector $v_\mu \in M_{pq}^{r0}$ with weight μ corresponds to a unique $\mathfrak{g}_0$-highest weight vector $v'_\eta \in V^{\otimes r}$ of weight $\eta = \mu - \lambda_{pq}$ such that $v_\mu - v'_\eta \otimes v_{\lambda_{pq}} \in V^{\otimes r} \otimes K_+$, where K_+ is the subspace of $K_{\lambda_{pq}}$ spanned by basis elements b^σ's in (43) with $\sigma \neq 0$ (cf. [26, Lems. 5.1–5.2]).

To begin with, we briefly recall the results on a classification of $\mathfrak{gl}_m$-highest weight vectors of $V^{\otimes r}$, where V temporarily denotes the natural representation of $\mathfrak{gl}_m$ over $\mathbb{C}$. Let $\{v_i \mid 1 \leq i \leq m\}$ be a basis of V. Obviously, $V^{\otimes r}$ has a basis $\{v_{\mathbf{i}} \mid \mathbf{i} \in I(m|0,r)\}$, where

$$v_{\mathbf{i}} = v_{i_r} \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1}.$$

We consider a Cashmir element Ω in $\mathfrak{gl}_m^{\otimes 2}$ with

$$\Omega = \sum_{1\leq i,j\leq m} E_{ij} \otimes E_{ji} \in \mathfrak{gl}_m^{\otimes 2}, \tag{51}$$

which is a special case of (47). Define $\mathbf{s}_i = \pi_{i,i+1}(\Omega)$, $1 \leq i \leq r-1$. Then $(i, i+1) \in \mathfrak{S}_r$ acts on $V^{\otimes r}$ via $\mathbf{s}_i$. Thus, $V^{\otimes r}$ is a $(\mathfrak{gl}_m, \mathbb{C}\mathfrak{S}_r)$-bimodule such that

$$v_{\mathbf{i}} w = v_{i_{(r)w^{-1}}} \otimes v_{i_{(r-1)w^{-1}}} \otimes \cdots \otimes v_{i_{(1)w^{-1}}} \text{ for any } w \in \mathfrak{S}_r. \tag{52}$$

For example, $v_{i_3} \otimes v_{i_2} \otimes v_{i_1} s_1 s_2 = v_{i_1} \otimes v_{i_3} \otimes v_{i_2}$. If $r \leq m$, it is well known that

$$\mathrm{End}_{U(\mathfrak{gl}_m)}(V^{\otimes r})^{\mathrm{op}} \cong \mathbb{C}\mathfrak{S}_r.$$

Definition 7. If $\lambda \in \Lambda^+(r, m)$, the set of partitions of r with at most m parts, we define $v_\lambda = v_{\mathbf{i}_\lambda} \in V^{\otimes r}$, where $\mathbf{i}_\lambda = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, m^{\lambda_m})$ and k^{λ_k} denotes the sequence $k, k, \ldots, k$ with multiplicity λ_k.

The following result is well known, and Lemma 20 follows from Lemma 19.

Lemma 19. *Suppose λ and μ are two compositions of r and μ' is the conjugate of μ, and $x_\lambda, y_{\mu'}$ are defined in* (29). *Then $x_\lambda \mathbb{C}\mathfrak{S}_r y_{\mu'} = 0$ unless $\lambda \trianglelefteq \mu$.*

Lemma 20. *There is a bijection between the set of dominant weights of $V^{\otimes r}$ and $\Lambda^+(r, m)$, the set of partitions of r with at most m parts. Further, the $\mathbb{C}$-space of $\mathfrak{gl}_\mathfrak{m}$-highest weight vectors with highest weight λ has a basis $\{v_\lambda w_\lambda y_{\lambda'} d(\mathfrak{t}) \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$.*

Now, we turn to construct $\mathfrak{g}$-highest weight vectors of M_{pq}^{r0}. Since $r \leq \min\{m, n\}$, there is a bijection between the set of dominant weights of M_{pq}^{r0} and $\Lambda_2^+(r)$. Further, if $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$, the corresponding dominant weight of M_{pq}^{r0} is

$$\overline{\lambda} := \lambda_{pq} + \widetilde{\lambda}, \tag{53}$$

where

$$\widetilde{\lambda} = (\lambda_1^{(1)}, \ldots, \lambda_m^{(1)} \mid \lambda_1^{(2)}, \ldots, \lambda_n^{(2)}). \tag{54}$$

For instance, if $\lambda = ((3,1),(2,1))$, then $\widetilde{\lambda} = (3,1,0,\ldots,0 \mid 2,1,0,\ldots,0)$. Recall that Ω is a Casimir element in $\mathfrak{g}^{\otimes 2}$ given in (47). Define operators s_i, x_1 acting on the right of M_{pq}^{r0} via the following formulae: $s_i = \pi_{i+1,i}(\Omega)$, $1 \leq i \leq r-1$ and $x_1 = -\pi_{10}(\Omega)$. In this case, $u_1 = -p$ and $u_2 = m - q$. We recall that Brundan–Stroppel [6] defined x_1 via $\pi_{10}(\Omega)$. So, the current x_1 is $-x_1$ in [6]. Recall that $v_{\mathbf{i}} \otimes v_{pq} = v_{i_r} \otimes \cdots \otimes v_{i_2} \otimes v_{i_1} \otimes v_{pq}$ for any $\mathbf{i} \in I(m|n,r)$ (cf. (44)), and $x'_k = x_k + L_k$ with $L_k = \sum_{i=1}^{k-1}(i,k)$ (see Lemma 4).

Lemma 21 ([6, Lem. 3.1]). *Suppose* $\mathbf{i} \in I(m|n,r)$*, and* $1 \le k \le r$*.*

(i) $v_{\mathbf{i}} \otimes v_{pq} x'_k = -p v_{\mathbf{i}} \otimes v_{pq}$ *if* $1 \le i_k \le m$*.*

(ii) $v_{\mathbf{i}} \otimes v_{pq} x'_k = -q v_{\mathbf{i}} \otimes v_{pq} + \sum_{j=1}^{m} (-1)^{\sum_{l=1}^{k-1}[i_l]} v_{\mathbf{j}} \otimes (E_{i_k,j} v_{pq})$ *if* $m+1 \le i_k \le m+n$*, where* $\mathbf{j} \in I(m|n,r)$ *which is obtained from* $\mathbf{i}$ *by using* j *instead of* i_k *in* $\mathbf{i}$*. In particular, the weight of* $v_{\mathbf{j}}$ *is strictly bigger than that of* $v_{\mathbf{i}}$*.*

Definition 8. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$, define $v_{\widetilde{\lambda}} = v_{\mathbf{i}}$ with $\mathbf{i} = (\mathbf{i}_{\lambda^{(1)}}, \mathbf{i}_{\lambda^{(2)}}) \in I(m|n,r)$.

For instance, $v_{\widetilde{\lambda}} = v_{\mathbf{i}}$ if $\lambda = ((3,1),(2,1))$, where $\mathbf{i} = (1^3, 2, (m+1)^2, m+2)$.

Definition 9. For any $\mathfrak{t} \in \mathscr{T}^s(\lambda')$, we define $v_{\mathfrak{t}} = v_{\widetilde{\lambda}} \otimes v_{pq} w_\lambda \mathfrak{y}_{\lambda'} d(\mathfrak{t})$, where $\mathfrak{y}_{\lambda'}$ is given in Definition 4 (ii).

Theorem 22. *Suppose* $r \le \min\{m,n\}$*. There is a bijection between the set of dominant weights of* M_{pq}^{r0} *and* $\Lambda_2^+(r)$*. Further, the* $\mathbb{C}$*-space* $V_{\widetilde{\lambda}}$ *of* $\mathfrak{g}$*-highest weight vectors of* M_{pq}^{r0} *with highest weight* $\overline{\lambda}$ *has a basis* $\{v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}^s(\lambda')\}$*.*

Proof. The required bijection between $\Lambda_2^+(r)$ and the set of dominant weights of M_{pq}^{r0} is the map sending λ to $\overline{\lambda}$ defined in (53). We claim that each $v_{\mathfrak{t}}$ is killed by $E_{m,m+1}$ and $E_{i,j}$ with $i < j$ and either $i,j \in I_0$ or $i,j \in I_1$. Since M_{pq}^{r0} is $(\mathfrak{g}, \mathscr{H}_{2,r})$-bimodule, we need only consider the case $d(\mathfrak{t}) = 1$. In this case, $\mathfrak{t} = \mathfrak{t}^{\lambda'}$.

Denote $|\lambda^{(1)}| = a$. Recall that $w_{\lambda^{(1)}} \in \mathfrak{S}_a$ and $w_{\lambda^{(2)}} \in \mathfrak{S}_{r-a}$ such that $\mathfrak{t}^{\lambda^{(i)}} w_{\lambda^{(i)}} = \mathfrak{t}_{\lambda^{(i)}}$ for $i = 1,2$. Then

$$w_\lambda = w_{\lambda^{(1)}} w_{\lambda^{(2)}} w_a = w_a w_{\lambda^{(2)}} w_{\lambda^{(1)}}. \tag{55}$$

By (27) and (55),

$$v_{\mathfrak{t}} = v_{\widetilde{\lambda}} \otimes v_{pq} w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}} w_a \pi_{r-a}(-p),$$

where $\mu^{(i)}$ is the conjugate of $\lambda^{(i)}$ for $i = 1,2$. By Lemmas 19–20, $v_{\mathfrak{t}}$ is killed by $E_{i,j}$ with $i < j$ and either $i,j \in I_0$ or $i,j \in I_1$. Since $E_{m,m+1}$ acts on M_{pq}^{r0} via $\sum_{i=1}^{r+1} 1^{\otimes i-1} \otimes E_{m,m+1} \otimes 1^{\otimes r+1-i}$, we have $E_{m,m+1} v_{\widetilde{\lambda}} \otimes v_{pq} = 0$ if v_{m+1} does not occur in $v_{\widetilde{\lambda}}$. Otherwise, $\lambda^{(2)} \ne \varnothing$ and $r - a \ne 0$. In this case, up to a sign, $E_{m,m+1} v_{\widetilde{\lambda}} \otimes v_{pq}$ is equal to

$$v_{\mathbf{j}} \otimes v_{pq}(1 - s_{a+1} + s_{a+1,a+3} + \cdots + (-1)^{b-a} s_{a+1,b+1}),$$

where $b = a + \lambda_1^{(2)} - 1$ and $v_{\mathbf{j}}$ is obtained from $v_{\widetilde{\lambda}}$ by replacing v_{m+1} by v_m at the $(a+1)$th position. Thus, $j_{a+1} = m$. Let

$$h = (1 - s_{a+1} + s_{a+1,a+3} + \cdots + (-1)^{b-a} s_{a+1,b+1}) w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}}.$$

Then $h \in \mathbb{C}\mathfrak{S}_a \otimes \mathbb{C}\mathfrak{S}_{r-a}$. By (27), $h w_a = w_a h_1$ for some $h_1 \in \mathbb{C}\mathfrak{S}_{r-a} \otimes \mathbb{C}\mathfrak{S}_a$. Since $h_1 \pi_{r-a}(-p) = \pi_{r-a}(-p) h_1$, it is enough to prove $v_{\mathbf{j}} \otimes v_{pq} w_a \pi_{r-a}(-p) = 0$. Up to a sign, $v_{\mathbf{j}} \otimes v_{pq} w_a = v_{\mathbf{k}} \otimes v_{pq}$ for some $\mathbf{k}$ such that $v_{k_1} = v_m \in V_0$. Since $r - a \ne 0$,

x_1+p is a factor of $\pi_{r-a}(-p)$. By Lemma 21 (i), $v_{\mathfrak{j}}\otimes v_{pq}w_a\pi_{r-a}(-p)=0$. Thus, $v_{\mathfrak{t}}$ is a highest weight vector of M_{pq}^{r0} if $v_{\mathfrak{t}}\neq 0$.

Note that any vector of M_{pq}^{r0} can be written as $v=\sum_{b\in B}v_b\otimes b$, where B is a basis of $K_{\lambda_{pq}}$ defined in (43) and $v_b\in V^{\otimes r}$. Following [6], v_b is called the *b-component* of v. By Lemma 21 (ii) (or the arguments in the proof of [7, Cor. 3.3]), the v_{pq}-component of $v_{\widetilde{\lambda}}\otimes v_{pq}w_a\pi_{r-a}(-p)$ is $v_{\widetilde{\lambda}}w_a\prod_{i=1}^{r-a}(p-q-L_i)$. By Lemma 4 (iii), $\prod_{i=1}^{r-a}(p-q-L_i)$ is a central element in $\mathbb{C}\mathfrak{S}_{r-a}$, which acts on $v_{\widetilde{\lambda}}\otimes v_{pq}w_{\lambda^{(2)}}x_{\mu^{(2)}}$ as scalar $\prod_{i=1}^{r-a}(p-q-\mathrm{res}_{\mathfrak{t}^{\mu^{(2)}}}(i))$, where $\mu=\lambda'$ and $\mathrm{res}_{\mathfrak{t}^{\mu^{(2)}}}(i)$ is $j-l$ if i is in the lth row and jth column of $\mathfrak{t}^{\mu^{(2)}}$. Since λ_{pq} is typical (cf. (41)), and $r\leq\min\{m,n\}$, $\prod_{i=1}^{r-a}(p-q-\mathrm{res}_{\mathfrak{t}^{\mu^{(2)}}}(i))\neq 0$. So, up to a non-zero scalar,

$$\text{the } v_{pq}\text{-component of } v_{\mathfrak{t}} \;=\; v_{\widetilde{\lambda}}w_aw_{\lambda^{(2)}}x_{\mu^{(2)}}w_{\lambda^{(1)}}y_{\mu^{(1)}}d(\mathfrak{t}). \tag{56}$$

By Lemma 20, it is a $\mathfrak{g}_0$-highest vector of $V^{\otimes r}$ with highest weight $\widetilde{\lambda}$ (cf. (54)), forcing $v_{\mathfrak{t}}\neq 0$.

Now, we prove that $\{v_{\mathfrak{t}}\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$ is $\mathbb{C}$-linear independent. First, consider $V=V_0\oplus V_1$ as a module for $\mathfrak{g}_0=\mathfrak{gl}_m\oplus\mathfrak{gl}_n$. Then $V^{\otimes r}$ can be decomposed as a direct sum of $V_{i_1}\otimes V_{i_2}\otimes\cdots\otimes V_{i_r}$, where $i_j\in\{0,1\}$. As $\mathfrak{g}_0$-modules, $V_{i_1}\otimes V_{i_2}\otimes\cdots\otimes V_{i_r}\cong V_1^{\otimes r-a}\otimes V_0^a$ for some non-negative integer $a\leq r$ with $a=\#\{i_j\mid i_j=0\}$. The corresponding isomorphism is given by acting a unique element w on the right-hand side of $V_1^{\otimes r-a}\otimes V_0^a$, where w is a distinguished right coset representative of $\mathfrak{S}_a\times\mathfrak{S}_{r-a}$ in $\mathfrak{S}_r$. By Lemma 20, all $\mathfrak{g}_0$-highest weight vectors of $V_{i_1}\otimes V_{i_2}\otimes\cdots\otimes V_{i_r}$ with highest weight $\widetilde{\lambda}$ are $v_{\widetilde{\lambda}}w_{\lambda^{(1)}}y_{\mu^{(1)}}w_{\lambda^{(2)}}x_{\mu^{(2)}}d(\mathfrak{t}_1)d(\mathfrak{t}_2)w$ for all $\mathfrak{t}_1\in\mathcal{T}^s(\mu^{(1)})$ and $\mathfrak{t}_2\in\mathcal{T}^s(\mu^{(2)})$. Therefore, the $\mathbb{C}$-space $V_{\widetilde{\lambda}}$ of all $\mathfrak{g}_0$-highest weight vectors of $V^{\otimes r}$ with highest weight $\widetilde{\lambda}$ has a basis $\{v_{\widetilde{\lambda}}w_{\lambda^{(2)}}x_{\mu^{(2)}}w_{\lambda^{(1)}}y_{\mu^{(1)}}d(\mathfrak{t})\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$, where $\mu=\lambda'$. By (56), $\{v_{\mathfrak{t}}\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$ is $\mathbb{C}$-linear independent. Finally, since there is a one-to-one correspondence between $\mathfrak{g}$-highest weight vectors of M_{pq}^{r0} and $\mathfrak{g}_0$-highest weight vectors of $V^{\otimes r}$ (cf. [26, Lems. 5.1–5.2]), and $\dim V_{\widetilde{\lambda}}=\#\{v_{\mathfrak{t}}\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$, one obtains that $\{v_{\mathfrak{t}}\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$ is a basis of $V_{\widetilde{\lambda}}$. $\square$

In the remaining part of this section, we want to establish the relationship between $V_{\widetilde{\lambda}}$ with a special cell module of $\mathscr{H}_{2,r}$ with respect to $\lambda\in\Lambda_2^+(r)$. This result will be needed in section 5. We go on using $-x_1$ instead of x_1 in [6]. In this case, the current $-p$ and $m-q$ are the same as p and q in [6].

Proposition 23. *For any $\lambda\in\Lambda_2^+(r)$, $V_{\widetilde{\lambda}}\cong\mathfrak{x}_\lambda w_\lambda\mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$ as right $\mathscr{H}_{2,r}$-modules, where $V_{\widetilde{\lambda}}$ is defined in Theorem* 22.

Proof. By Lemma 11 (ii), $S^\lambda:=\mathfrak{x}_\lambda w_\lambda\mathfrak{y}_{\lambda'}\mathscr{H}_{2,r}$ has a basis $M=\{\mathfrak{x}_\lambda w_\lambda\mathfrak{y}_{\lambda'}d(\mathfrak{t})\mid \mathfrak{t}\in\mathcal{T}^s(\lambda')\}$. It follows from Theorem 22 that there is a linear isomorphism $\phi: V_{\widetilde{\lambda}}\to S^\lambda$ sending $v_{\mathfrak{t}}$ to $\mathfrak{x}_\lambda w_\lambda\mathfrak{y}_{\lambda'}d(\mathfrak{t})$. Obviously, ϕ is a right $\mathfrak{S}_r$-homomorphism. In order to show that ϕ is a right $\mathscr{H}_{2,r}$-homomorphism, it suffices to prove that

$$\phi(v_{\mathfrak{t}}x_k)=\phi(v_{\mathfrak{t}})x_k,\ \text{for } 1\leq k\leq r. \tag{57}$$

Denote $a=|\lambda^{(1)}|$. If $1\leq k\leq r-a$, then

$$\widetilde{\pi}_{\lambda'}x_k=\pi_{r-a}(m-q)x_k=\pi_{r-a}(m-q)(-p-L_k).$$

Since ϕ is a right $\mathfrak{S}_r$-homomorphism, (57) holds for $1 \le k \le r-a$. If $r-a+1 \le k \le r$, then $x_k = s_{k,r-a+1}x_{r-a+1}s_{r-a+1,k} - \sum_{j=r-a+1}^{k-1}(j,k)$. By Lemma 11 (i),

$$\pi_\lambda w_a \widetilde{\pi}_{\lambda'} x_k = \pi_\lambda w_a \widetilde{\pi}_{\lambda'}\Big(-p-\sum_{j=r-a+1}^{k-1}(j,k)\Big). \tag{58}$$

On the other hand, $\widetilde{\pi}_{\lambda'}x_k = x_k\widetilde{\pi}_{\lambda'}$ and $v_{\overline{\lambda}} \otimes v_{pq} w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}} w_a$ is a linear combination of elements $v_{\mathbf{i}} \otimes v_{pq}$, for some $\mathbf{i} \in I(m|n,r)$ such that $v_{i_j} \in V_0$ for all $r-a+1 \le j \le r$. By Lemma 21 (i), x_k acts on $v_{\mathbf{i}} \otimes v_{pq}$ as $-p-L_k$. In order to verify (57) for $k \ge r-a+1$, by (58), it remains to show that

$$v_{\mathbf{i}} \otimes v_{pq}(i,k)\widetilde{\pi}_{\lambda'} = 0 \ \text{ for all } i, 1 \le i \le r-a. \tag{59}$$

Write $v_{\mathbf{i}} \otimes v_{pq}(i,k) = v_{\mathbf{j}}$ up to a sign. Then $v_{j_i} \in V_0$ and $v_{\mathbf{j}}(1,i)\widetilde{\pi}_{\lambda'} = 0$ by Lemma 21 (i). Since $(1,i)\widetilde{\pi}_{\lambda'} = \widetilde{\pi}_{\lambda'}(1,i)$, and $(1,i)$ is invertible, $v_{\mathbf{j}}\widetilde{\pi}_{\lambda'} = 0$, proving (59). □

Corollary 24. *Suppose $\lambda \in \Lambda_2^+(r)$. As right $\mathscr{H}_{2,r}$-modules,*

$$\mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{r0}) \cong \widetilde{\Delta}(\lambda') \tag{60}$$

where $\widetilde{\Delta}(\lambda')$ is the right cell module defined via the cellular basis of $\mathscr{H}_{2,r}$ in Lemma 10(ii).

Proof. For any $\mathfrak{g}$-highest weight vector v of M_{pq}^{r0} with highest weight $\overline{\lambda}$, there is a unique $U(\mathfrak{g})$-homomorphism $f_v : K_{\overline{\lambda}} \to \mathbf{U}(\mathfrak{g})v \subset M_{pq}^{r0}$ sending $v_{\overline{\lambda}}$ to v, where $v_{\overline{\lambda}}$ is the highest weight vector of $K_{\overline{\lambda}}$. Further, f_v can be considered as a homomorphism in $\mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{r0})$ by composing the embedding homomorphism.

For any $0 \ne f \in \mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{r0})$, $f(v_{\overline{\lambda}})$ is a highest weight vector of M_{pq}^{r0}. By Theorem 22, $f(v_{\overline{\lambda}})$ is a linear combination of $v_{\mathfrak{t}}$'s, for $\mathfrak{t} \in \mathcal{T}^s(\lambda')$. So, f can be written as a linear combination of $f_{v_{\mathfrak{t}}}$'s. Thus, $\{f_{v_{\mathfrak{t}}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ is a basis of $\mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{r0})$. Let $V_{\overline{\lambda}}$ be defined in Theorem 22. Then the linear isomorphism $\phi : \mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{r0}) \to V_{\overline{\lambda}}$ sending $f_{v_{\mathfrak{t}}}$ to $v_{\mathfrak{t}}$ for any $\mathfrak{t} \in \mathcal{T}^s(\lambda')$ is a right $\mathscr{H}_{2,r}$-homomorphism. By Lemma 11 (v) and Proposition 23, $V_{\overline{\lambda}} \cong \widetilde{\Delta}(\lambda')$, proving (60). □

In the remaining part of this section, we always assume $p-q \le -m$. If $p-q \ge n$, one can switch roles between p and q (or consider the dual module of M_{pq}^{r0}). Without loss of any generality, we assume $p, q \in \mathbb{Z}$.

Let $\lambda \in \Lambda_2^+(r)$ with $r \le \min\{m,n\}$. Then λ corresponds to a dominant weight $\overline{\lambda}$ defined in (53). In particular, $\overline{\varnothing} = \lambda_{pq}$. Following [6, 14, 20, 27], we are going to represent a dominant weight $\overline{\lambda}$ in a unique way by a weight diagram D_λ. First we write (cf. (40))

$$\overline{\lambda}^\rho = \overline{\lambda} + \rho = (\overline{\lambda}_1^{L,\rho}, \dots, \overline{\lambda}_m^{L,\rho} \,|\, \overline{\lambda}_1^{R,\rho}, \dots, \overline{\lambda}_n^{R,\rho}). \tag{61}$$

Denote

$$S(\lambda)_{\mathrm{L}} = \{\overline{\lambda}_i^{L,\rho} \,|\, i = 1, \dots, m\}, \qquad S(\lambda)_{\mathrm{R}} = \{-\overline{\lambda}_j^{R,\rho} \,|\, j = 1, \dots, n\},$$
$$S(\lambda) = S(\lambda)_{\mathrm{L}} \cup S(\lambda)_{\mathrm{R}}, \qquad S(\lambda)_{\mathrm{B}} = S(\lambda)_{\mathrm{L}} \cap S(\lambda)_{\mathrm{R}}.$$

Definition 10. The *weight diagram* D_λ associated with the dominant weight $\overline{\lambda}$ is a line with vertices indexed by $\mathbb{Z}$ such that each vertex i is associated with a symbol $D^i_\lambda = \varnothing, <, >$ or $\times$ according to whether $i \notin S(\lambda)$, $i \in S(\lambda)_{\rm R} \setminus S(\lambda)_{\rm B}$, $i \in S(\lambda)_{\rm L} \setminus S(\lambda)_{\rm B}$ or $i \in S(\lambda)_{\rm B}$.

For example, if $p, q \in \mathbb{Z}$ with $p \le q - m$, then the weight diagram $D_\varnothing$ of $\overline{\varnothing} = \lambda_{pq}$ is given by

$$\cdots \text{——} \underset{p-m+1}{\overset{>}{\circ}} \text{—} \overset{>}{\circ} \text{—} \cdots \text{—} \underset{p}{\overset{>}{\circ}} \text{—} \circ \text{—} \circ \text{—} \cdots \text{—} \underset{q-m+1}{\overset{<}{\circ}} \text{—} \overset{<}{\circ} \text{—} \cdots \text{—} \overset{<}{\circ} \text{—} \underset{q-m+n}{\overset{<}{\circ}} \text{—} \circ \text{—} \cdots, \tag{62}$$

where, for simplicity, we have associated vertex i with nothing if $D^i_\lambda = \varnothing$. Note that $\sharp S(\varnothing)_{\rm B} = 0$, i.e., λ_{pq} is typical.

Definition 11. Let $\overline{\lambda}$ be as in (53), where $\lambda \in \Lambda^+_2(r)$.

(i) Let $\overline{\lambda}^{\rm top}$ be the unique dominant weight such that $L_{\overline{\lambda}}$ is the simple submodule of the Kac-module $K_{\overline{\lambda}^{\rm top}}$. Then $\overline{\lambda}^{\rm top}$ is obtained from $\overline{\lambda}$ via the unique longest right path (cf. [27, Def. 5.2], [31, Conjecture 4.4]) or via a raising operator (cf. [5]). For example, if D_λ is given by

$$\cdots\text{—} 0 \text{—} \overset{\times}{1} \text{—} \overset{\times}{2} \text{—} 3 \text{—} \overset{\times}{4} \text{—} \overset{>}{5} \text{—} 6 \text{—} \overset{\times}{7} \text{—} \overset{<}{8} \text{—} 9 \text{—} \overset{<}{10} \text{—} 11 \text{—} \cdots, \tag{63}$$

then the weight diagram $D_{\lambda^{\rm top}}$ of $\overline{\lambda}^{\rm top}$ is given by

$$\cdots\text{—} 0 \text{—} 1 \text{—} 2 \text{—} \overset{\times}{3} \text{—} 4 \text{—} \overset{>}{5} \text{—} \overset{\times}{6} \text{—} 7 \text{—} \overset{<}{8} \text{—} \overset{\times}{9} \text{—} \overset{<}{10} \text{—} \overset{\times}{11} \text{—} \cdots, \tag{64}$$

where the $\times$'s at vertices $9, 6, 3, 11$ in (64) are respectively obtained from the $\times$'s at vertices $7, 4, 2, 1$ in (63) (thus every symbol "$\times$" is always moved to the unique empty place at its right side which is closest to it, under the rule that the rightmost "$\times$" should be moved first, as indicated in (64)). Alternatively, $\overline{\lambda}$ is obtained from $\overline{\lambda}^{\rm top}$ via the unique longest left path.

(ii) Write $\overline{\lambda}^{\rm top} = \lambda_{pq} + \widetilde{\lambda}^{\rm top}$ (cf. (54) and (53)) with $\widetilde{\lambda}^{\rm top} = (\lambda^{({\rm top},1)} \mid \lambda^{({\rm top},2)})$ and denote $\lambda^{\rm top} = (\lambda^{({\rm top},1)}, \lambda^{({\rm top},2)})$, where $\lambda^{({\rm top},1)} = (\lambda^{\rm top}_1, \dots, \lambda^{\rm top}_m)$, $\lambda^{({\rm top},2)} = (\lambda^{\rm top}_{m+1}, \dots, \lambda^{\rm top}_{m+n})$ for some $\lambda^{\rm top}_i \in \mathbb{Z}$. Then obviously $\sum_i \lambda^{\rm top}_i = r$. Thus $\lambda^{\rm top} \in \Lambda^+_2(r)$ if and only if $\lambda^{\rm top}_i \in \mathbb{Z}^{\ge 0}$ for all possible i.

Write $p = q - m - k$ for some $k \in \mathbb{N}$. If $\mu = ((\mu^L_1, \dots, \mu^L_m), (\mu^R_1, \dots, \mu^R_n)) \in \Lambda^+_2(r)$, then μ' is Kleshchev with respect to $u_1 = -p$, $u_2 = m - q$ (cf. (36)) if and only if

$$\mu^L_i \ge \mu^R_i - k \ \text{ for all possible } i. \tag{65}$$

Following [6, IV], we denote $I^+_{pq} = \{p - m + 1, p - m + 2, \dots, q - m + n\}$. For any $\lambda \in \Lambda^+_2(r)$ and any $j \in I^+_{pq}$, set

$$I^\varnothing_{\ge j}(\lambda) = \mathbb{Z}^{\ge j} \cap (I^+_{pq} \setminus S(\lambda) \cap I^+_{pq}), \tag{66}$$

$$I^\varnothing_{\le j}(\lambda) = \mathbb{Z}^{\le j} \cap (I^+_{pq} \setminus S(\lambda) \cap I^+_{pq}), \tag{67}$$

$$I^\times_{\ge j}(\lambda) = \mathbb{Z}^{\ge j} \cap (I^+_{pq} \cap S(\lambda)_{\rm B}, \tag{68}$$

$$I^\times_{\le j}(\lambda) = \mathbb{Z}^{\le j} \cap (I^+_{pq} \cap S(\lambda)_{\rm B}. \tag{69}$$

In terms of the above notations, Brundan and Stroppel [6, IV, Lemma 2.6] have proved that the indecomposable tilting module $T_{\overline{\lambda}}$ is a direct summand of M_{pq}^{r0} if

$$S(\lambda) \subset I_{pq}^{+} \ \text{ and } \ \#I_{\geq j}^{\varnothing}(\lambda) \geq \#I_{\geq j}^{\times}(\lambda) \text{ for all } j \in I_{pq}^{+}. \tag{70}$$

These two conditions on bipartition λ (or weight $\overline{\lambda}$) are equivalent to the following conditions on λ^{top} (which can be seen from (63)–(64) in case $I_{pq}^{+} = \{1, 2, \ldots, 11\}$):

$$S(\lambda^{\text{top}}) \subset I_{pq}^{+} \ \text{ and } \ \#I_{\leq j}^{\varnothing}(\lambda^{\text{top}}) \geq \#I_{\leq j}^{\times}(\lambda^{\text{top}}) \text{ for all } j \in I_{pq}^{+}. \tag{71}$$

Lemma 25. *Let $\mu \in \Lambda_2^{+}(r)$ such that μ' is Kleshchev with respect to $u_1 = -p, u_2 = m - q$, where $p = q - m - k$ with $k \in \mathbb{N}$. Then*

$$S(\mu) \subset I_{pq}^{+} \ \textit{ and } \ \#I_{\leq j}^{\varnothing}(\mu) \geq \#I_{\leq j}^{\times}(\mu) \textit{ for all } j \in I_{pq}^{+}. \tag{72}$$

Proof. We have (cf. (40))

$$\lambda_{pq} + \rho = (q - m - k, \ldots, q - 2m - k + 1 \,|\, -q + m - 1, \ldots, -q + m - n). \tag{73}$$

Thus for $i = 1, \ldots, m$, we have (cf. (61)) $\overline{\mu}_i^{L,\rho} = \mu_i + q - m - k \geq q - 2m - k + 1$ and $\overline{\mu}_i^{L,\rho} \leq q + n - m$ (as $\mu_i \leq r \leq n$), i.e., $\overline{\mu}_i^{L,\rho} \in I_{pq}^{+}$. Similarly, $-\overline{\mu}_j^{R,\rho} \in I_{pq}^{+}$ for $j = 1, \ldots, n$. Hence, $S(\mu) \subset I_{pq}^{+}$.

To prove the other assertion of (72), note that the weight diagram D_μ of $\overline{\mu}$ is obtained from $D_\varnothing$ (cf. (62)) by moving the "$>$" at vertex $p - i$ for all i with $0 \leq i \leq m - 1$ to its right side to vertex $p - i + \mu_{i+1}^{L}$, and moving the "$<$" at vertex $q - m + j$ for all j with $1 \leq j \leq n$ to its left side to vertex $q - m + j - \mu_j^{R}$ (if "$<$" meets "$>$" at the destination vertex, then two symbols "$<$" and "$>$" are combined to become the symbol "$\times$"). Since μ' is Kleshchev, condition (65) shows that in order to produce a "$\times$" at some vertex i of D_μ, a "$>$" at some vertex j with $j < i$ must be moved to vertex i, i.e., an "$\varnothing$" must appear in some vertex j' with $j' \leq j < i$, i.e., (71) holds. □

Corollary 26. *Suppose $\lambda \in \Lambda_2^{+}(r)$ such that $\lambda^{\text{top}} \in \Lambda_2^{+}(r)$ and $(\lambda^{\text{top}})'$ is Kleshchev, where $(\lambda^{\text{top}})'$ is the conjugate of $\lambda^{\text{top}} \in \Lambda_2^{+}(r)$. Then $T_{\overline{\lambda}}$ is a direct summand of M_{pq}^{r0}. Further, any indecomposable direct summand of M_{pq}^{r0} is of form $T_{\overline{\lambda}}$ for some $\lambda \in \Lambda_2^{+}(r)$ such that $\lambda^{\text{top}} \in \Lambda_2^{+}(r)$ and $(\lambda^{\text{top}})'$ is Kleshchev.*

Proof. The first assertion follows from [6, IV, Lem. 2.6] and Lemma 25. To prove the last assertion, since $r \leq \min\{m, n\}$, by Theorem 17, $\text{End}_{U(\mathfrak{gl}_{m|n})}(M_{pq}^{r0})^{\text{op}} \cong \mathscr{H}_{2,r}$. So, the number of non-isomorphic indecomposable direct summands of $\mathfrak{gl}_{m|n}$-module M_{pq}^{r0} is equal to that of non-isomorphic irreducible $\mathscr{H}_{2,r}$-modules, which is equal to the number of Kleshchev bipartitions in $\Lambda_2^{+}(r)$. Now, everything is
clear. □

Corollary 27. *Suppose $\lambda \in \Lambda_2^{+}(r)$ such that $\lambda^{\text{top}} \in \Lambda_2^{+}(r)$ and $(\lambda^{\text{top}})'$ is Kleshchev. As right $\mathscr{H}_{2,r}$-modules,*

$$\text{Hom}_{U(\mathfrak{g})}(T_{\overline{\lambda}}, M_{pq}^{r0}) \cong P^{(\lambda^{\text{top}})'}, \tag{74}$$

where $P^{(\lambda^{\text{top}})'}$ is the projective cover of $D^{(\lambda^{\text{top}})'}$ which is the simple head of $\widetilde{\Delta}((\lambda^{\text{top}})')$.

Proof. Since $r \le \min\{m,n\}$, $\lambda^{\rm top} \in \Lambda_2^+(r)$ and $(\lambda^{\rm top})'$ is Kleshchev, by Corollary 26, $T_{\overline{\lambda}}$ is a direct summand of M_{pq}^{r0}, forcing $0 \ne {\rm Hom}_{U(\mathfrak{g})}(T_{\overline{\lambda}}, M_{pq}^{r0})$ to be a direct summand of $\mathscr{H}_{2,r}$. We claim that ${\rm Hom}_{U(\mathfrak{g})}(T_{\overline{\lambda}}, M_{pq}^{r0})$ is indecomposable. If not, then the number of indecomposable direct summands of the right $\mathscr{H}_{2,r}$-module $\mathscr{H}_{2,r}$ is strictly bigger than $\sum_{\overline{\lambda}} \ell_{\overline{\lambda}}$ if we write M_{pq}^{r0} as $M_{pq}^{r0} = \bigoplus_{\overline{\lambda}} T_{\overline{\lambda}}^{\oplus \ell_{\overline{\lambda}}}$ with $\ell_{\overline{\lambda}} \ne 0$.

On the other hand, since M_{pq}^{r0} is a right $\mathscr{H}_{2,r}$-module, we can consider the right exact functor $\mathfrak{F} := M_{pq}^{r0} \otimes_{\mathscr{H}_{2,r}} ?$ from the category of left $\mathscr{H}_{2,r}$-modules to the category of left $U(\mathfrak{g})$-modules. We have an epimorphism from $\mathfrak{F}(P^\mu)$ to $\mathfrak{F}(\widetilde{\Delta}(\mu))$, where P^μ is any principal indecomposable left $\mathscr{H}_{2,r}$-module and $\widetilde{\Delta}(\mu)$ temporally denotes the left cell module of $\mathscr{H}_{2,r}$ defined via the cellular basis of $\mathscr{H}_{2,r}$ in Lemma 10 (i) with the simple head D^μ. By Lemma 11 (v) and Theorem 17, $\mathfrak{F}(\widetilde{\Delta}(\mu)) \ne 0$, forcing $\mathfrak{F}(P^\mu) \ne 0$. So, $\mathfrak{F}(P^\mu)$ is a direct sum of indecomposable direct summand of $U(\mathfrak{g})$-module M_{pq}^{r0}. In particular, $\sum_{\overline{\lambda}} \ell_{\overline{\lambda}}$ is no less than the number of indecomposable direct summands of left $\mathscr{H}_{2,r}$-module $\mathscr{H}_{2,r}$. This is a contradiction since the number of indecomposable direct summands of left $\mathscr{H}_{2,r}$-module $\mathscr{H}_{2,r}$ is equal to that of indecomposable direct summands of right $\mathscr{H}_{2,r}$-module $\mathscr{H}_{2,r}$. So, $\mathfrak{F}(T_{\overline{\lambda}})$ is a principal indecomposable right $\mathscr{H}_{2,r}$-module. Since $K_{\lambda^{\rm top}} \hookrightarrow T_{\overline{\lambda}}$, ${\rm Hom}_{U(\mathfrak{g})}(T_{\overline{\lambda}}, M_{pq}^{r0}) \twoheadrightarrow {\rm Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}^{\rm top}}, M_{pq}^{r0})$. By Corollary 24, ${\rm Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}^{\rm top}}, M_{pq}^{r0}) \cong \widetilde{\Delta}((\lambda^{\rm top})')$. Since ${\rm Hom}_{U(\mathfrak{g})}(T_{\overline{\lambda}}, M_{pq}^{r0})$ is a principal indecomposable right $\mathscr{H}_{2,r}$-module, it implies that $\widetilde{\Delta}((\lambda^{\rm top})')$ has the unique simple head, denoted by $D^{(\lambda^{\rm top})'}$. Thus, ${\rm Hom}_{\mathfrak{g}}(T_{\overline{\lambda}}, M_{pq}^{r0}) \cong P^{(\lambda^{\rm top})'}$. □

Brundan–Stroppel have already proved that decomposition numbers of $\mathscr{H}_{2,r}$ arising from super Schur–Weyl duality in [6] can be determined by the multiplicity of Kac-modules in indecomposable tilting modules appearing in M_{pq}^{r0}. This result can also be seen via the exact functor ${\rm Hom}_{U(\mathfrak{g})}(?, M_{pq}^{r0})$.

5. Highest weight vectors in M_{pq}^{rt}

In this section, we classify $\mathfrak{g}$-highest weight vectors of $\mathfrak{gl}_{m|n}$-module M_{pq}^{rt} over $\mathbb{C}$. As an application, we set up an explicit relationship between Kac (respectively indecomposable tilting) modules of $\mathfrak{g}$ and cell (respectively principal indecomposable) modules of $\mathscr{B}_{2,r,t}$. This gives us an efficient way to calculate decomposition numbers of $\mathscr{B}_{2,r,t}$. Throughout, assume $r,t \in \mathbb{Z}^{>0}$ such that $r+t \le \min\{m,n\}$. The case $t=0$ has been dealt with in section 4. By symmetry, one can also classify highest weight vectors of M_{pq}^{0t} via those in section 4. The following result, which is the counterpart of Lemma 21, can be verified directly.

Lemma 28. *Suppose* $\mathbf{i} \in I(m|n,r)$, $\mathbf{j} \in \overline{I}(m|n,t)$ *(cf. (44)) and* $1 \le k \le t$.

(i) $v_{\mathbf{i}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{j}} \overline{x}'_k = q v_{\mathbf{i}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{j}}$ *if* $1+m \le j_k \le m+n$.

(ii) $v_{\mathbf{i}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{j}} \overline{x}'_k = p v_{\mathbf{i}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{j}} + \sum_{j=m+1}^{m+n} (-1)^{\sum_{l=1}^{k-1}[j_l]} v_{\mathbf{i}} \otimes (E_{jj_k} v_{pq}) \otimes \overline{v}_\ell$ *if* $1 \le j_k \le m$, *where* $\ell \in \overline{I}(m|n,t)$ *which is obtained from* $\mathbf{j}$ *by using* j *instead of* j_k *in* $\mathbf{j}$. *In particular, the weight of* $\overline{v}_\ell$ *is strictly bigger than that of* $\overline{v}_{\mathbf{j}}$.

For any integral weight ξ of $\mathfrak{g}$ written as

$$\xi = (\xi_1, \dots, \xi_m \,|\, \xi_{m+1}, \dots, \xi_{m+n}), \tag{75}$$

let

$$\xi^L = (\xi_1^L, \dots, \xi_m^L) = (\xi_1, \dots, \xi_m), \text{ and } \xi^R = (\xi_1^R, \dots, \xi_m^R) = (\xi_{m+1}, \dots, \xi_{m+n}).$$

We define two bicompositions μ, ν such that all $\mu_i^{(1)}, \mu_j^{(2)}, \nu_i^{(1)}, \nu_j^{(2)}$ are zero except that

(i) for $1 \le i \le m$, $\mu_i^{(1)} = \xi_i^L$ if $\xi_i^L > 0$ or $\nu_{m-i+1}^{(1)} = -\xi_i^L$ if $\xi_i^L < 0$.
(ii) for $1 \le j \le n$, $\mu_j^{(2)} = \xi_j^R$ if $\xi_j^R > 0$ or $\nu_{n-j+1}^{(2)} = -\xi_j^R$ if $\xi_j^R < 0$.

Then both μ and ν correspond to integral weights of $\mathfrak{g}$. In particular, $\xi = \mu - \widehat{\nu}$ with

$$\widehat{\nu} = (\nu_m^{(1)}, \dots, \nu_1^{(1)} \,|\, \nu_n^{(2)}, \dots, \nu_1^{(2)}) \in \mathfrak{h}^*. \tag{76}$$

Conversely, if μ and ν are two bicompositions, then $\xi = \mu - \widehat{\nu}$ is a integral weight of $\mathfrak{g}$. For instance, if $\xi = (r-4, 1, 0, \dots, 0, -1, -(t-5) \,|\, 2, 1, 0, \dots, 0, -1, -3)$, then $\mu = ((r-4, 1), (2, 1))$ and $\nu = ((t-5, 1), (3, 1))$ such that $\xi = \mu - \widehat{\nu}$.

Definition 12. For any $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ with μ, ν written as in (75), let $\overline{\lambda} := \lambda_{pq} + \mu - \widehat{\nu}$ and $\widetilde{\lambda} := \mu - \widehat{\nu}$. Since $r + t \le \min\{m, n\}$, both μ and ν correspond to integral weights of $\mathfrak{g}$ as above such that

$$\mu_i \nu_{m+1-i} = 0 \ \text{ for } \ 1 \le i \le m \ \text{ and } \ \mu_{m+j} \nu_{m+n+1-j} = 0 \ \text{ for } \ 1 \le j \le n\ , \tag{77}$$

Lemma 29. *For any $\mathfrak{g}$-highest weight Λ of M_{pq}^{rt}, there is a unique triple $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ such that $\Lambda = \overline{\lambda}$.*

Proof. By [23, Lem. 5.20], $\Lambda = \lambda_{pq} + \eta - \zeta$ for some bicompositions (or weights) η and ζ (written as in (75)) of sizes r and t, respectively. For $i \in I$, let $\xi_i = \min\{\eta_i, \zeta_i\}$ and $f = \sum_{i \in I} \xi_i$. Then we obtain a weight ξ, and two bicompositions $\mu := \eta - \xi$ and $\gamma := \zeta - \xi$ such that $|\mu| = r - f$, $|\gamma| = t - f$ and $\Lambda = \lambda_{pq} + \mu - \gamma$. Set $\nu = \widehat{\gamma}$, then $\Lambda = \overline{\lambda}$ and (77) is satisfied by definition of ξ. Since Λ is dominant, μ, ν must be bipartitions. Thus Λ corresponds to $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$. Such a λ is unique. □

Definition 13. For each $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$, denote $v_\lambda = v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}}$, where

$$\mathbf{i} = (\mathbf{i}_{\mu^{(1)}}, \mathbf{i}_{\mu^{(2)}}, \underbrace{1, \dots, 1}_{f}) \in I(m|n, r), \text{ and } \mathbf{j} = (\mathbf{j}_{\nu^{(2)}}, \mathbf{j}_{\nu^{(1)}}, \underbrace{1, \dots, 1}_{f}) \in \overline{I}(m|n, t),$$

such that

(i) $\mathbf{j}_{\nu^{(2)}}$ is obtained from $\mathbf{i}_{\nu^{(2)}}$ by using $m+n-i+1$ instead of i for $1 \le i \le n$,
(ii) $\mathbf{j}_{\nu^{(1)}}$ is obtained from $\mathbf{i}_{\nu^{(1)}}$ by using $m-i+1$ instead of i for $1 \le i \le m$.

For instance, if $\lambda = (1, \mu, \nu) \in \Lambda_{2,8,10}$ with $\mu = ((3,1),(2,1))$ and $\nu = ((4,1),(3,1))$, then $\mathbf{i} = (1^3, 2, (m+1)^2, (m+2), 1)$ and $\mathbf{j} = ((m+n)^3, (m+n-1), m^4, (m-1), 1)$. Thus,

$$v_\lambda = v_1 \otimes v_{m+2} \otimes v_{m+1}^{\otimes 2} \otimes v_2 \otimes v_1^{\otimes 3} \otimes v_{pq} \otimes \overline{v}_{m+n}^{\otimes 3} \otimes \overline{v}_{m+n-1} \otimes \overline{v}_m^{\otimes 4} \otimes \overline{v}_{m-1} \otimes \overline{v}_1.$$

Definition 14. For any $(f,\mu,\nu)\in\Lambda_{2,r,t}$, define

(i) $w_{\mu,\nu} = w_\mu w_{\nu^o}$, with $\nu^o = (\nu^{(2)},\nu^{(1)})$, $w_\mu = d(\mathfrak{t}_\mu)\in\mathfrak{S}_{r-f}$ and $w_{\nu^o} = d(\mathfrak{t}_{\nu^o})\in\overline{\mathfrak{S}}_{t-f}$,

(ii) $v_{\lambda,\mathfrak{t},d,\kappa_d} = v_\lambda \mathfrak{e}^f w_{\mu,\nu}\mathfrak{y}_{\mu'}\overline{\mathfrak{y}}_{(\nu^o)'}d(\mathfrak{t})dx^{\kappa_d}$, $\mathfrak{t}\in\mathcal{T}^s(\mu')\times\mathcal{T}^s((\nu^o)')$, $d\in\mathcal{D}^f_{r,t}$ and $\kappa_d\in\mathbf{N}_f$.

Theorem 30. *Suppose $r+t\le\min\{m,n\}$.*

(i) *There is a bijection between the set of dominant weights of M^{rt}_{pq} and $\Lambda_{2,r,t}$.*

(ii) *If $\lambda=(f,\mu,\nu)\in\Lambda_{2,r,t}$, then $V_{\overline{\lambda}}$, the $\mathbb{C}$-space of all $\mathfrak{g}$-highest weight vectors of M^{rt}_{pq} with highest weight $\overline{\lambda}$, has a basis*

$$S := \{v_{\lambda,\mathfrak{t},d,\kappa_d}\mid \mathfrak{t}\in\mathcal{T}^s(\mu')\times\mathcal{T}^s((\nu^o)'), d\in\mathcal{D}^f_{r,t},\kappa_d\in\mathbf{N}_f\}.$$

Proof. Obviously, (i) follows from Lemma 12. To obtain (ii), we prove that for each $\lambda=(f,\mu,\nu)\in\Lambda_{2,r,t}$, $V_{\overline{\lambda}}$ has the required basis in the case either $f=0$ or $f>0$.

Case $1:\ f=0$.

By Definition 14, $v_{\lambda,\mathfrak{t},d,\kappa_d} = v_{\mathbf{i}}\otimes v_{pq}\otimes\overline{v}_{\mathbf{j}}w_\mu\mathfrak{y}_{\mu'}d(\mathfrak{t}_1)w_{\nu^o}\overline{\mathfrak{y}}_{(\nu^o)'}d(\mathfrak{t}_2)$, where $\mathbf{i},\mathbf{j}$ are defined in Definition 13. By Theorem 22, $v_{\mathbf{i}}\otimes v_{pq}\otimes\overline{v}_{\mathbf{j}}w_\mu\mathfrak{y}_{\mu'}d(\mathfrak{t}_1)$ can be regarded as a $\mathfrak{g}$-highest weight vector of M^{r0}_{pq}. Similarly, $v_{\mathbf{i}}\otimes v_{pq}\otimes\overline{v}_{\mathbf{j}}w_{\nu^o}\overline{\mathfrak{y}}_{(\nu^o)'}d(\mathfrak{t}_2)$ can be regarded as a $\mathfrak{g}$-highest weight vector of M^{0t}_{pq}. Thus, $v_{\lambda,\mathfrak{t},d,\kappa_d}$ is a $\mathfrak{g}$-highest weight vector of M^{rt}_{pq}. The last assertion follows from arguments on counting the dimensions of $V_{\overline{\lambda}}$ and that of $\mathfrak{g}_0$-highest weight vectors of $V^{rt}:=V^{\otimes r}\otimes W^{\otimes t}$ with highest weight $\mu-\widehat{\nu}$.

Case $2:\ f>0$.

For any $i\in I$,

$$v_i\otimes\overline{v}_ie_1 = (-1)^{[i]}\sum_{j\in I}v_j\otimes\overline{v}_j.$$

Thus $v_i\otimes\overline{v}_ie_1$ is unique up to a sign for different i's. Since M^{rt}_{pq} is a $(\mathfrak{g},\mathscr{B}_{2,r,t})$-bimodule, we can switch $v_{i_{r-k}}$ and $\overline{v}_{j_{t-k}}$ in v_λ with $i_{r-k}=j_{t-k}$ to v_o and $\overline{v}_o$ for any fixed $o, 1\le o\le m+n$ simultaneously when we consider the action of $E_{j,\ell}$ on i_{r-k}th (respectively j_{t-k}th) tensor factor of $v_{\lambda,\mathfrak{t},d,\kappa_d}$ for $0\le k\le f-1$. Let $v_{\mathfrak{t}}$ be

$$v_{i_{r-f}}\otimes\cdots\otimes v_{i_1}\otimes v_{pq}\otimes\overline{v}_{j_1}\otimes\cdots\otimes\overline{v}_{j_{t-f}}w_{\mu,\nu}x_{\alpha^{(2)}}y_{\alpha^{(1)}}y_{\beta^{(1)}}x_{\beta^{(2)}}\pi_{r-f-a}(-p)\pi_b(q)d(\mathfrak{t}),\quad(78)$$

where $\alpha^{(i)}$ (respectively $\beta^{(i)}$) is the conjugate of $\mu^{(i)}$ (respectively $\nu^{(i)}$), $i=1,2$. Applying Theorem 22 to both $V^{\otimes r-f}\otimes K_{\lambda_{pq}}$ and $K_{\lambda_{pq}}\otimes W^{\otimes t-f}$ yields $E_{j,\ell}v_{\mathfrak{t}}=0$. So, $E_{j,\ell}v_{\lambda,\mathfrak{t},d,\kappa_d}=0$ for any $j<\ell$.

We claim that S is linear independent, where S is given in (ii). If so, each $v_{\lambda,\mathfrak{t},d,\kappa_d}\ne 0$, forcing $v_{\lambda,\mathfrak{t},d,\kappa_d}$ to be a $\mathfrak{g}$-highest weight vector of M^{rt}_{pq} with highest weight $\overline{\lambda}$.

Suppose $\mathbf{i}\in I(m|n,r_1-1)$ and $\mathbf{j}\in\overline{I}(m|n,t_1-1)$ with $r_1\le r$ and $t_1\le t$ such that there are at least some $k_0\in I_0$ and $\ell_0\in I_1$ satisfying $k_0,\ell_0\notin\{i_l,j_o\}$ for all possible i,o's. We consider $\sum_{k\in I}v_k\otimes v_{\mathbf{i}}\otimes v\otimes v_{\mathbf{j}}\otimes\overline{v}_k\in M^{r_1,t_1}_{pq}$, where $v\in B$ is a basis element of $K_{\lambda_{pq}}$ in (43). Since $x'_{r_1} = x_{r_1} + L_{r_1}$ and x'_{r_1} acts on $M_{pq}^{r_1,t_1}$ as $-\pi_{r_1,0}(\Omega)$, where Ω is given in (47), we have

$$\begin{aligned}\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \overline{v}_k(x_{r_1} + L_{r_1})& \\ &= -\pi_{r_1,0}(\Omega)\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \overline{v}_k \\ &= -\sum_{k,i\in I}(-1)^{[k]+([k]+[i])([k]+[\mathbf{i}])} v_i \otimes v_{\mathbf{i}} \otimes E_{k,i}v \otimes \overline{v}_{\mathbf{j}} \otimes \overline{v}_k,\end{aligned}$$

where $[\mathbf{i}] = \sum_{j=1}^{r_1-1}[i_j]$. So, up to some scalar a, $\sum_{k=1}^{m+n} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \overline{v}_k x_{r_1}$ contains the unique term $v_{k_0} \otimes v_{\mathbf{i}} \otimes v \otimes \overline{v}_{\mathbf{j}} \otimes \overline{v}_{k_0}$. In particular, if $v \neq v_{pq}$, $\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \overline{v}_k x_{r_1}$ does not contribute terms with form $v_{k_0} \otimes v_{\mathbf{i}'} \otimes v_{pq} \otimes v_{\mathbf{j}'} \otimes \overline{v}_{k_0}$ for all possible $\mathbf{i}'$ and $\mathbf{j}'$. If $v = v_{pq}$, by Lemma 21, the previous scalar is $-p$. Similarly, the coefficient of $v_{\ell_0} \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_{\ell_0}$ in the expression of $\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \overline{v}_k x_{r_1}$ is $-q$. Assume

$$c\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_k x_{r_1} + d\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_k = 0 \qquad (79)$$

for some $c, d \in \mathbb{C}$. Then $d = cp = cq$ by considering the coefficients of $v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_k$, $k \in \{k_0, \ell_0\}$ in the expression of LHS of (79). If $c \neq 0$, then $p - q = 0$. This is a contradiction since λ_{pq} is typical in the sense of (41). So, $c = d = 0$ and hence $\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_k x_{r_1}$ and $\sum_{k\in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \overline{v}_k$ are linear independent. Now, we assume

$$\sum_{\mathfrak{t},d,\kappa_d} r_{\mathfrak{t},d,\kappa_d} v_{\lambda,\mathfrak{t},d,\kappa_d} = 0 \text{ for some } r_{\mathfrak{t},d,\kappa_d} \in \mathbb{C}. \qquad (80)$$

We claim that $r_{\mathfrak{t},d,\kappa_d} = 0$ for all possible $\mathfrak{t}, d, \kappa_d$. If not, then we pick up a $d \in \mathcal{D}_{r,t}^f$ such that

(i) $r_{\mathfrak{t},d,\kappa_d} \neq 0$,
(ii) $d = s_{r-f+1,i_{r-f+1}}\overline{s}_{t-f+1,j_{t-f+1}} \cdots s_{r,i_r}\overline{s}_{t,j_t}$ and $i_r > i_{r-1} > \cdots > i_{r-f+1}$,
(iii) $(i_r, \dots, i_{r-f+1})$ is maximal with respect to lexicographic order.

Since $r + t \leq \min\{m, n\}$ and $0 < f \leq \min\{r, t\}$, we can pick f pairs (k_i, ℓ_i), $r - f + 1 \leq i \leq r$ such that

(i) $k_i \in I_0$, $\ell_i \in I_1$, $k_i > k_j$ and $\ell_i > \ell_j$ if $i > j$;
(ii) both v_{k_i} and v_{ℓ_i} are not a tensor factor of $v_{\mathbf{i}_\mu}$;
(iii) both $\overline{v}_{k_i}$ and $\overline{v}_{\ell_i}$ are not a tensor factor of $\overline{v}_{\mathbf{j}}$ and $\mathbf{j} = (\mathbf{j}_{\upsilon^{(2)}}, j_{\nu^{(1)}})$.

We consider the terms $v_{\mathbf{a}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{b}}$'s in the expressions of $v_{\lambda,\mathfrak{t},d,\kappa_d}$'s in the left-hand side of (80) with $r_{\mathfrak{t},d,\kappa_d} \neq 0$ such that either $v_{a_{i_h}} = v_{k_h}$ and $\overline{v}_{b_{i_{t-r+h}}} = \overline{v}_{k_h}$ or $v_{a_{i_h}} = v_{\ell_h}$ and $\overline{v}_{b_{i_{t-r+h}}} = \overline{v}_{\ell_h}$ for $r - f + 1 \leq h \leq r$. Such terms occur in the expression of $v_1^{\otimes f} \otimes \widetilde{v}_{\mathfrak{t}} \otimes \overline{v}_1^{\otimes f}\mathfrak{e}^f dx^{\kappa_d}$, where $\widetilde{v}_{\mathfrak{t}}$ is a linear combination of the terms in $v_{\mathfrak{t}}$'s (cf. (78)) with forms $v_{\mathbf{i}'} \otimes v_{pq} \otimes \overline{v}_{\mathbf{j}'}$. If $v_{a_h} = v_{k_h}$ and $\overline{v}_{b_{i_{t-r+h}}} = \overline{v}_{k_h}$, by previous arguments, the coefficient of $v_{\mathbf{a}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{b}}$ in $v_1^{\otimes f} \otimes v_{\mathfrak{t}} \otimes \overline{v}_1^{\otimes f} e^f dx^{\kappa_d}$

is $\prod_{h=r}^{r-f+1}(-p)^{\epsilon_h}$, where $\epsilon_h = 1$ if $\kappa_h = 1$ and 0 if $\kappa_h = 0$. If $v_{a_h} = v_{\ell_h}$ and $\overline{v}_{b_{i_{t-r+h}}} = \overline{v}_{\ell_h}$, then the coefficient of $v_{\mathbf{a}} \otimes v_{pq} \otimes \overline{v}_{\mathbf{b}}$ in $v_1^{\otimes f} \otimes \widetilde{v}_{\mathfrak{t}} \otimes \overline{v}_1^{\otimes f} \mathfrak{e}^f dx^{\kappa_d}$ is $\prod_{h=r}^{r-f+1}(-q)^{\epsilon_h}$, where $\epsilon_h = 1$ if $\kappa_h = 1$ and 0 if $\kappa_h = 0$. By (80), $\sum_{\mathfrak{t}} r_{\mathfrak{t},d,\kappa_d} \widetilde{v}_{\mathfrak{t}} = 0$ for any fixed κ_d. Thus, we can assume that $\kappa_d = (0, \cdots, 0) \in \mathbf{N}_f$. If we identify $\widetilde{v}_{\mathfrak{t}}$ with its v_{pq}-component, then $\widetilde{v}_{\mathfrak{t}}$ can be considered as $\mathfrak{g}_0$-highest weight vectors of $V^{\otimes r-f} \otimes W^{\otimes t-f}$ (cf. arguments in the proof of Theorem 22) of the form

$$\widetilde{v}_{\mathfrak{t}} = v_{i_{r-f}} \otimes \cdots v_{i_1} \otimes \overline{v}_{j_1} \otimes \cdots \overline{v}_{j_{t-f}} w_{\mu,\nu} x_{\alpha^{(2)}} y_{\alpha^{(1)}} \overline{y}_{\beta^{(1)}} \overline{x}_{\beta^{(2)}} d(\mathfrak{t}).$$

So, $r_{\mathfrak{t},d,\kappa_d} = 0$, a contradiction. This proves that S is $\mathbb{C}$-linear independent. Further, S is a basis of $V_{\overline{\lambda}}$ since the cardinality of S is $2^f |\mathcal{D}_{r,t}^f| \cdot |\mathcal{T}^s(\mu')| \cdot |\mathcal{T}^s(\nu')|$, which is the dimension of space consisting of $\mathfrak{g}_0$-highest weight vectors of V^{rt} with highest weight $\mu - \widehat{\nu}$. □

Definition 15. Let $\mathfrak{F} = \mathrm{Hom}_{U(\mathfrak{g})}(?, M_{pq}^{rt})$ be the functor from the category of finite-dimensional left $\mathfrak{g}$-modules to the category of right $\mathscr{B}_{2,r,t}$-modules over $\mathbb{C}$.

Lemma 31. *The functor $\mathfrak{F}$ is exact.*

Proof. In the category of finite-dimensional $\mathfrak{gl}_{m|n}$-modules, a module is injective if and only if it is tilting if and only if it is projective (e.g., [6, IV]). Since λ_{pq} is typical, $K_{\lambda_{pq}}$ is injective, and hence M_{pq}^{rt} is injective as a left $\mathfrak{g}$-module. So, $\mathfrak{F}$ is exact. □

Proposition 32. *Suppose $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$. Then $\mathfrak{F}(K_{\overline{\lambda}}) \cong C(f, \mu', (\nu^o)')$, where $\nu^o = (\nu^{(2)}, \nu^{(1)})$.*

Proof. By Proposition 14, there is an explicit linear isomorphism between $C(f, \mu', (\nu^o)')$ and $V_{\overline{\lambda}}$, where $V_{\overline{\lambda}}$ is given in Theorem 30. By Proposition 23 and [23, Prop. 6.10], this linear isomorphism is a $\mathscr{B}_{2,r,t}$-homomorphism. Thus, $C(f, \mu', (\nu^o)') \cong V_{\overline{\lambda}}$ as right $\mathscr{B}_{2,r,t}$-modules. Using the universal property of Kac-modules yields $\mathrm{Hom}_{U(\mathfrak{g})}(K_{\overline{\lambda}}, M_{pq}^{rt}) \cong V_{\overline{\lambda}}$ as $\mathscr{B}_{2,r,t}$–modules (cf. the proof of Corollary 24). Now, everything is clear. □

In the remaining part of this section, we calculate decomposition matrices of $\mathscr{B}_{2,r,t}$. We always assume that $p \in \mathbb{Z}$. Otherwise, one can use $x_1 + p_1$ instead of x_1 for any $p_1 \in \mathbb{C}$ with $p - p_1 \in \mathbb{Z}$. Since λ is typical, we have $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. In the first case, by [23, Thm. 5.21], $\mathscr{B}_{2,r,t}$ is semisimple and hence its decomposition matrix is the identity matrix. We assume that $p - q \leq -m$. If $p - q \geq n$, one can switch the roles between p and q (or by considering the dual module of M_{pq}^{rt}) in the following arguments. Since M_{pq}^{rt} is a tilting module, it can be decomposed into the direct sum of indecomposable tilting modules

$$M_{pq}^{rt} = \bigoplus_{\xi \in P^+} T_\mu^{\oplus \ell_\xi} \quad \text{for some} \quad \ell_\xi \in \mathbb{N}. \tag{81}$$

In the remaining part of this paper, we denote $\mathbb{T}$ to be the following finite subset of P^+:

$$\mathbb{T} := \{\xi \in P^+ \mid \ell_\xi \neq 0\}. \tag{82}$$

Suppose $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$. Let $\overline{\lambda} = \lambda_{pq} + \widetilde{\lambda} = \lambda_{pq} + \mu - \widehat{\nu}$ be as in Definition 12. Denote by $T_{\overline{\lambda}}$ the indecomposable tilting module, which is the projective cover of $L_{\overline{\lambda}}$, where $L_{\overline{\lambda}}$ is the simple $\mathfrak{g}$-module with highest weight $\overline{\lambda}$. It is known that $T_{\overline{\lambda}}$ has filtrations of Kac-modules. Let $K_{\overline{\lambda}^{\rm top}}$ be the unique bottom of $T_{\overline{\lambda}}$. Then $L_{\overline{\lambda}}$ is the simple $\mathfrak{g}$-module of $K_{\overline{\lambda}^{\rm top}}$. Further, $\overline{\lambda}^{\rm top}$ is the dominant weight defined in Definition 11 (i). Note that any dominant weight especially $\overline{\lambda}^{\rm top}$ can be uniquely written as (cf. (76) for notation $\widehat{\tau}$)

$$\begin{aligned}\overline{\lambda}^{\rm top} = \lambda_{pq} + \varepsilon - \widehat{\tau}, \text{ where } \varepsilon &= (\varepsilon_1, ..., \varepsilon_{k_1}, 0, ..., 0 \,|\, \varepsilon_{m+1}, ..., \varepsilon_{m+\ell_1}, 0, ..., 0) \in P^+,\\ \tau &= (\tau_1, ..., \tau_{k_2}, 0, ..., 0 \,|\, \tau_{m+1}, ..., \tau_{m+\ell_2}, 0, ..., 0) \in P^+,\end{aligned} \tag{83}$$

for some $\varepsilon_i, \tau_j \in \mathbb{Z}^{>0}$ and $k_1, k_2, \ell_1, \ell_2 \in \mathbb{Z}^{\geq 0}$ with $k_1 + k_2 \leq m$, $\ell_1 + \ell_2 \leq n$. Denote $|\varepsilon| := \sum_i \varepsilon_i$, $|\tau| := \sum_i \tau_i$. Obviously $r + t = |\varepsilon| + |\tau|$. Denote $g = r - |\varepsilon| = t - |\tau|$, and set $\lambda^{\rm top} = (g, \varepsilon, \tau)$. Thus $\lambda^{\rm top} \in \Lambda_{2,r,t}$ if and only if $g \geq 0$. For any $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$, we define $\lambda' = (f, \mu', (\nu^o)') \in \Lambda_{2,r,t}$, where ν^o is defined as in Definition 14 (i) and $\mu', (\nu^o)'$ are conjugates of μ, ν^o, respectively.

Now parallel to Corollary 26, we have the following.

Lemma 33. *Let $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ such that $\lambda^{\rm top} \in \Lambda_{2,r,t}$ and $(\lambda^{\rm top})'$ is Kleshchev (cf. statements after (36)). Then $T_{\overline{\lambda}}$ is a direct summand of M_{pq}^{rt}.*

Proof. First we clarify some notations: by Definition 12, any $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ corresponds to a unique dominant weight $\overline{\lambda}$, thus corresponds to a unique dominant weight $\overline{\lambda}^{\rm top}$ by Definition 11 (i). We claim that $T_{\overline{\lambda}}$ is a direct summand in $M_{pq}^{r-f,t-f}$. If so, then

$$v_1^f \otimes T_{\overline{\lambda}} \otimes \overline{v}_1^f \mathfrak{e}^f$$

is obviously a tilting submodule in M_{pq}^{rt} which is isomorphic to $T_{\overline{\lambda}}$. Thus the claim implies the result. Therefore, it suffices to consider the case $f = 0$.

Denote $\overline{\nu} = \lambda_{pq} - \widehat{\nu}$. Since we assume $p \leq q - m$, the weight diagram D_ν (cf. Definition 10) of $\overline{\nu}$ is obtained from that of λ_{pq} in (62) by moving the "$>$" at vertex $p - i + 1$ to its left side at vertex $p - i + 1 - \nu_{m-i+1}^{(1)}$ for each i with $1 \leq i \leq m$, and moving the "$<$" at vertex $q - m + j$ to its right side at vertex $q - m + j + \nu_{n-j+1}^{(2)}$ for each j with $1 \leq j \leq n$ (cf. (76)). Thus no "$\times$" can be produced, i.e., $\overline{\nu}$ is typical. Hence $K_{\overline{\nu}}$ is a direct summand in M_{pq}^{0t}. Thus, it suffices to prove that $T_{\overline{\lambda}}$ is a direct summand in $V^{\otimes r} \otimes K_{\overline{\nu}} \ltimes M_{pq}^{rt}$, where $\ltimes$ means direct summand of M_{pq}^{rt}. For this, we can apply [6, IV, Lems. 2.4 and 2.6]. Note from [6, IV, Lem. 2.4] that the action of the functor F_i on $K_{\overline{\nu}}$ defined in [6, IV] only depends on symbols at vertices i and $i + 1$ of the weight diagram D_ν of $\overline{\nu}$ (we remark that symbols $\circ, \wedge, \vee, \times$ in [6, IV] are respectively symbols $<, \times, \varnothing, >$ in this paper). Due to condition (77), for any $i \in I_{pq} := I_{pq}^+ \setminus \{q - m + n\}$ such that i is involved in a path in the crystal graph in [6, IV, Lemma 2.6], the symbols at vertex i and $i + 1$ in the weight diagram D_ν of $\overline{\nu}$ are the same as those in the weight diagram $D_\varnothing$ of λ_{pq}. This shows that $T_{\overline{\lambda}}$ is a direct summand in $V^{\otimes r} \otimes K_{\overline{\nu}}$ if and only if $T_{\lambda_{pq}+\widetilde{\mu}}$ is a direct summand in $V^{\otimes r} \otimes K_{\lambda_{pq}}$; more precisely, [6, IV, Lem. 2.6] implies

$$F_{i_r} \cdots F_{i_1} K_{\overline{\nu}} \cong T_{\overline{\lambda}}^{\otimes 2^\ell} \iff F_{i_r} \cdots F_{i_1} K_{\lambda_{pq}} \cong T_{\lambda_{pq}+\widetilde{\mu}}^{\otimes 2^\ell},$$

where ℓ is the number of edges in the given path of the form $\varnothing\times \to <>$. Thus the result follows from Corollary 26. □

We remark that there is a bijection between $\mathbb{T}$ defined in (82) and the set of pair-wise non-isomorphic simple $\mathscr{B}_{2,r,t}$-modules. See [23, Thm. 7.5]. For any $\xi\in\mathbb{T}$ as above, parallel to Definition 11, we define $\xi^{\rm top}$ to be the unique dominant weight such that L_ξ is the simple submodule of $K_{\xi^{\rm top}}$. To avoid confusion of notations, we emphasis that $\xi\in\mathbb{T}$ is not an element in $\Lambda_{2,r,t}$, but a dominant weight in P^+ (thus in fact, $\xi=\overline{\lambda}$ for some $\lambda\in\Lambda_{2,r,t}$ by Lemma 29).

Proposition 34. *For any $\xi\in\mathbb{T}$, there is a unique $\lambda=(f,\mu,\nu)\in\Lambda_{2,r,t}$ such that $\xi^{\rm top}=\lambda_{pq}+\mu-\widehat{\nu}$ (i.e., $\xi^{\rm top}=\overline{\lambda}$ by Definition 12). Further, $\mathfrak{F}(T_\xi)$ is isomorphic to the projective cover of $D^{f,\mu',(\nu^o)'}$, where $D^{f,\mu',(\nu^o)'}$ is the simple head of $C(f,\mu',(\nu^o)')$.*

Proof. If $\xi\in\mathbb{T}$, then T_ξ is an indecomposable tilting module with $\ell_\xi>0$. By Theorem 16, $\mathfrak{F}(T_\xi)$ is a direct sum of certain principle indecomposable right $\mathscr{B}_{2,r,t}$-modules. We claim that $\mathfrak{F}(T_\xi)$ is indecomposable for any $\xi\in\mathbb{T}$. Otherwise, $\sum_{\xi\in\mathbb{T}}\ell_\xi$ is strictly less than the number of principal indecomposable direct summands of right $\mathscr{B}_{2,r,t}$-module $\mathscr{B}_{2,r,t}$. However, for each principal indecomposable direct summand P of left $\mathscr{B}_{2,r,t}$-module $\mathscr{B}_{2,r,t}$, P has to be a projective cover of irreducible left $\mathscr{B}_{2,r,t}$-module, say D, which is the simple head of a left cell module, say $\Delta(\ell,\alpha,\beta)$ for some $(\ell,\alpha,\beta)\in\Lambda_{2,r,t}$, where $\Delta(\ell,\alpha,\beta)$ is defined via a weakly cellular basis of $\mathscr{B}_{2,r,t}$. So, there is an epimorphism from P to $\Delta(\ell,\alpha,\beta)$. Since $\mathfrak{G}:=M^{rt}_{pq}\otimes_{\mathscr{B}_{2,r,t}}?$ is right exact, there is an epimorphism from $\mathfrak{G}(P)$ to $\mathfrak{G}(\Delta(\ell,\alpha,\beta))$. If $\mathfrak{G}(\Delta(\ell,\alpha,\beta))\neq 0$, then $\mathfrak{G}(P)$ is a non-zero direct summand of M^{rt}_{pq}. This implies that the number of indecomposable direct summands of left $\mathscr{B}_{2,r,t}$-module $\mathscr{B}_{2,r,t}$ is strictly less than $\sum_{\xi\in\mathbb{T}}\ell_\xi$. This is a contradiction since the number of principal indecomposable direct summands of left $\mathscr{B}_{2,r,t}$-module $\mathscr{B}_{2,r,t}$ is equal to that of right $\mathscr{B}_{2,r,t}$-module $\mathscr{B}_{2,r,t}$. So, $\mathfrak{F}(T_\xi)$ is indecomposable. Since $K_{\xi^{\rm top}}\hookrightarrow T_\xi$, we have $\mathfrak{F}(T_\xi)\twoheadrightarrow\mathfrak{F}(K_{\xi^{\rm top}})$. By Proposition 32, $\mathfrak{F}(K_{\xi^{\rm top}})\cong C(f,\mu',(\nu^o)')$. Thus, $C(f,\mu',(\nu^o)')$ has the simple head, denoted by $D^{f,\mu',(\nu^o)'}$, and hence $\mathfrak{F}(T_\xi)=P^{f,\mu',(\nu^o)'}$. Since $\xi\in\mathbb{T}$, by Lemma 33, both μ' and $(\nu^o)'$ are Kleshchev in the sense of (36) with respect to $-p,m-q$ and $q,p-n$.

It remains to prove $\mathfrak{G}(\Delta(\ell,\alpha,\beta^o))\neq 0$ for any $\delta:=(\ell,\alpha,\beta)\in\Lambda_{2,r,t}$. By Theorem 30, $V_{\overline{\delta}}$ contains a non-zero vector $v:=v_1^{\otimes\ell}\otimes v_{\mathbf{i}}\otimes v_{pq}\otimes v_{\mathbf{j}}\otimes\overline{v}_1^{\otimes\ell}\mathfrak{e}^f w_{\alpha,\beta}\mathfrak{y}_{\alpha'}\overline{\mathfrak{y}}_{(\beta^o)'}$, where $\mathbf{i}$ and $\mathbf{j}$ are defined as in Definition 13. So, it is enough to show $v\in\mathfrak{G}(\Delta(\ell,\alpha,\beta^o))$, where $\Delta(\ell,\alpha,\beta^o)$ is defined via a suitable weakly cellular basis of $\mathscr{B}_{2,r,t}$. We use cellular bases of $\mathscr{H}_{2,r-f}$ and $\mathscr{H}_{2,t-f}$ in Lemma 10 (i) (iii) to construct a weakly cellular basis of $\mathscr{B}_{2,r,t}$, which is similar to that in Theorem 12. Let $\Delta(\ell,\alpha,\beta^o)$ be the corresponding left cell module with respect to $(\ell,\alpha,\beta^o)\in\Lambda_{2,r,t}$. By arguments similar to those for the proof of Proposition 14, one can verify

$$\Delta(\ell,\alpha,\beta^o)\cong\mathscr{B}_{2,r,t}\mathfrak{e}^f\mathfrak{x}_\alpha\overline{\mathfrak{x}}_{\beta^o}w_{\alpha,\beta}\mathfrak{y}_{\alpha'}\overline{\mathfrak{y}}_{(\beta^o)'}\pmod{\mathscr{B}_{2,r,t}^{\ell+1}}.$$

Let $M=\widetilde{v}\mathscr{B}_{2,r,t}$ be the cyclic $\mathscr{B}_{2,r,t}$-module generated by $\widetilde{v}:=v_1^{\otimes\ell}\otimes v_{\mathbf{i}}\otimes v_{pq}\otimes v_{\mathbf{j}}\otimes\overline{v}_1^{\otimes\ell}$. Then $M\otimes_{\mathscr{B}_{2,r,t}}\Delta(\ell,\alpha,\beta^o)$ is a subspace of $\mathfrak{G}(\Delta(\ell,\alpha,\beta^o))$. Since $\mathscr{B}_{2,r,t}^{\ell+1}$ acts on M trivially, there is a $\mathbb{C}$-linear map $\phi: M\otimes_{\mathscr{B}_{2,r,t}}\Delta(\ell,\alpha,\beta^o)\to M$ such that

$\phi(m \otimes \overline{h}) = mh$ for any $\overline{h} \in \mathscr{B}_{2,r,t}\mathfrak{e}^f \mathfrak{x}_\alpha \overline{\mathfrak{x}}_{\beta^o} w_{\alpha,\beta} \mathfrak{y}_{\alpha'} \overline{\mathfrak{y}}_{(\beta^o)'}$ (mod $\mathscr{B}^{\ell+1}_{2,r,t}$). Since λ_{pq} is typical and the ground filed is $\mathbb{C}$, up to a non-zero scalar, we have $v = \phi(\widetilde{v} \otimes \overline{h})$, where $h \equiv \mathfrak{e}^f w_{\alpha,\beta} \mathfrak{y}_{\alpha'} \overline{\mathfrak{y}}_{(\beta^o)'}$ (mod $\mathscr{B}^{\ell+1}_{2,r,t}$). Thus, $\mathfrak{G}(\Delta(\ell, \alpha, \beta^o)) \neq 0$. □

Remark 4. Proposition 34 implies that $C(f, \mu, \nu)$ has the simple head if μ and ν are Kleshchev bipartitions with respect to $-p, m - q$ and $q, p - n$ in the sense of (36). Further, all non-isomorphic simple $\mathscr{B}_{2,r,t}$-modules can be realized in this way.

Proposition 35. *Suppose $\xi \in P^+$. Then $\mathfrak{F}(L_\xi) = 0$ if $\xi \notin \mathbb{T}$ (cf. (82)) and $\mathfrak{F}(L_\xi) \cong D^{f,\mu',(\nu^o)'}$ if $\xi \in \mathbb{T}$, where $\xi^{\rm top} = \lambda_{pq} + \mu - \widehat{\nu}$ with $(f, \mu, \nu) \in \Lambda_{2,r,t}$.*

Proof. By (81), $\mathfrak{F}(L_\xi) = \bigoplus_{\zeta \in \mathbb{T}} \mathrm{Hom}_{\mathfrak{g}}(L_\xi, T_\zeta^{\otimes \ell_\zeta})$. Suppose $0 \neq f \in \mathrm{Hom}_{U(\mathfrak{g})}(L_\xi, T_\zeta^{\oplus \ell_\zeta})$. Then $L_\xi \cong f(L_\xi)$ is a simple submodule of $T_\zeta^{\oplus \ell_\zeta}$. Since T_ζ has the unique simple submodule L_ζ, $\mathfrak{F}(L_\xi) = 0$ if $\xi \notin \mathbb{T}$. If $\xi \in \mathbb{T}$, then

$$\mathfrak{F}(L_\xi) = \mathrm{Hom}_{U(\mathfrak{g})}(L_\xi, T_\xi^{\oplus \ell_\xi}), \tag{84}$$

which is obviously ℓ_ξ-dimensional. Let $v_\xi^1, \ldots, v_\xi^{\ell_\xi} \in T_\xi^{\oplus \ell_\xi}$ be the generators of the tilting module $T_\xi^{\oplus \ell_\xi}$ (then $v_\xi^1, \ldots, v_\xi^{\ell_\xi}$ span the generating space, denoted $\boldsymbol{V}$, of $T_\xi^{\oplus \ell_\xi}$), and $v_\xi'^1, \ldots, v_\xi'^{\ell_\xi} \in L_\xi^{\oplus \ell_\xi}$, the corresponding generators of the submodule $L_\xi^{\oplus \ell_\xi}$ of $T_\xi^{\oplus \ell_\xi}$. Thus, there exists a unique $u \in U(\mathfrak{g})$ such that

$$v_\xi'^i = u v_\xi^i \quad \text{for} \quad i = 1, \ldots, \ell_\xi. \tag{85}$$

Let $\widetilde{v}_\xi \in L_\xi$ be the generator of the simple module L_ξ. As in the proof of Corollary 27, we can define $f^i : L_\xi \to T_\xi^{\oplus \ell_\xi}$ to be the $U(\mathfrak{g})$-homomorphism sending $\widetilde{v}_\xi$ to $v_\xi'^i$ for $i = 1, \ldots, \ell_\xi$. Then $(f^1, \ldots, f^{\ell_\xi})$ is obviously a basis of $\mathfrak{F}(L_\xi)$ (cf. (84)).

For any $A \in M_{\ell_\xi}$ (the algebra of $\ell_\xi \times \ell_\xi$ complex matrices), we can define an element $\phi_A \in \mathrm{End}_{U(\mathfrak{g})}(M^{rt}_{pq})^{\rm op} = \mathscr{B}_{2,r,t}$ as follows: $\phi_A\big|_{T_\zeta^{\oplus \ell_\zeta}} = 0$ if $\zeta \neq \xi$ and

$$\phi_A\big|_{T_\xi^{\oplus \ell_\xi}} : (v_\xi^1, \ldots, v_\xi^{\ell_\xi}) \mapsto (v_\xi^1, \ldots, v_\xi^{\ell_\xi})A, \tag{86}$$

where the right-hand side is regarded as vector-matrix multiplication, i.e., the transition matrix of the action of $\phi_A\big|_{T_\xi^{\oplus \ell_\xi}}$ on the generating space $\boldsymbol{V}$ of $T_\xi^{\oplus \ell_\xi}$ under the basis $(v_\xi^1, \ldots, v_\xi^{\ell_\xi})$ is A. By the universal property of projective modules, this uniquely defines an element $\phi_A \in \mathscr{B}_{2,r,t}$. Thus we have the embedding $\phi : M_{\ell_\xi} \to \mathscr{B}_{2,r,t}$ sending A to ϕ_A. Write A as $A = (a_{ij})_{i,j=1}^{\ell_\xi}$. Then by (86) and definition of the right action of $\mathscr{B}_{2,r,t}$ on M^{rt}_{pq}, we have

$$\begin{aligned} f^i(\widetilde{v}_\xi)\phi_A &= v_\xi'^i \phi_A = (u v_\xi^i)\phi_A = u(v_\xi^i \phi_A) \\ &= u \sum_{j=1}^{\ell_\xi} a_{ji} v_\xi^j = \sum_{j=1}^{\ell_\xi} a_{ji} v_\xi'^j = \Big(\sum_{j=1}^{\ell_\xi} a_{ji} f^j\Big)(\widetilde{v}_\xi), \end{aligned} \tag{87}$$

i.e., the transition matrix of the action of ϕ_A on $\mathfrak{F}(L_\xi)$ under the basis $(f^1, \ldots, f^{\ell_\xi})$ is A. Thus $\phi(M_{\ell_\xi})$ acts transitively on the ℓ_ξ-dimensional space $\mathfrak{F}(L_\xi)$ and hence $\mathfrak{F}(L_\xi)$ is a simple $\mathscr{B}_{2,r,t}$-module. Finally, since $L_\xi \hookrightarrow K_{\xi^{\rm top}}$, we have $\mathfrak{F}(K_{\xi^{\rm top}}) \twoheadrightarrow \mathfrak{F}(L_\xi)$. Note that $D^{f,\mu',(\nu^o)'}$ is the simple head of $\mathfrak{F}(K_{\xi^{\rm top}})$. Thus, $\mathfrak{F}(L_\xi) \cong D^{f,\mu',(\nu^o)'}$. □

Theorem 36. *Suppose* $(f,\alpha,\beta)\in\Lambda_{2,r,t}$ *such that there is a* $\lambda\in\mathbb{T}$ (*cf.* (82)) *satisfying* $\lambda^{\rm top}=\lambda_{pq}+\alpha-\widehat{\beta}$. *If* $\mu:=(\ell,\gamma,\delta)\in\Lambda_{2,r,t}$, *then* $[C(\ell,\gamma',(\delta^o)'):D^{f,\alpha',(\beta^o)'}]=(T_\lambda:K_{\overline{\mu}})$.

Proof. The result follows from Lemma 31, Propositions 32 and 35, together with the BGG reciprocity formula for $\mathfrak{g}$. □

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