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JORDAN GROUPS AND ALGEBRAIC SURFACES

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Abstract. We prove that an analogue of Jordan's theorem on finite subgroups of general linear groups holds for the groups of biregular automorphisms of algebraic surfaces. This gives a positive answer to a question of Vladimir L. Popov.

1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic zero and $\mathbb{P}^1$ is the projective line over k. Let U be an *algebraic variety* over k [14, Vol. 2, Chap. VI, Sect. 1]. Then $U(k)$ and $\mathrm{Aut}(U)$ stand for its set of k-points and the group of biregular k-automorphisms respectively. Unless otherwise stated, by a point of U we mean a k-point. If U is *irreducible* then we write $k(U)$ and $\mathrm{Bir}(U)$ for its field of rational functions and the group of birational k-automorphisms respectively; $\mathrm{Aut}(U)$ is a subgroup of $\mathrm{Bir}(U)$. By an elliptic curve we mean an irreducible smooth projective curve of genus 1 over k. If X is an elliptic curve and $\mathcal{T} \subset X(k)$ is a *nonempty* finite set of points on X then the (sub)group

$$\mathrm{Aut}(X, \mathcal{T}) = \{u \in \mathrm{Aut}(X) \mid u(\mathcal{T}) = \mathcal{T}\} \subset \mathrm{Aut}(X)$$

is *finite*, since $X \setminus \mathcal{T}$ is a *hyperbolic* curve. If $\mathcal{S}$ is a smooth irreducible projective surface over k then an irreducible closed curve C in $\mathcal{S}$ is called a (-1)-curve if it is smooth rational and its self-intersection index is -1.

The following definition was inspired by the classical theorem of Jordan [2, Sect. 36] about finite subgroups of general linear groups over fields of characteristic zero.

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Definition 1.1 (Definition 2.1 of [9])**.** A group B is called a *Jordan group* if there exists a positive integer J_B such that every finite subgroup B_1 of B contains a normal commutative subgroup, whose index in B_1 is at most J_B.

Remark 1.2. Clearly, a subgroup of a Jordan group is also Jordan. If a Jordan group G_1 is a subgroup of *finite* index in a group G then G is also Jordan.

V. L. Popov ([9, Sect. 2], see also [10]) posed a question whether $\mathrm{Aut}(S)$ is a Jordan group when S is an algebraic surface over k. He obtained a positive answer to his question for almost all surfaces. (The case of rational surfaces was treated earlier by J.-P. Serre [12, Sect. 5.4]). The only remaining case is when S is birationally (but not biregularly) isomorphic to a product $X \times \mathbb{P}^1$ of an elliptic curve X and the projective line. In [16] the second named author proved that $\mathrm{Aut}(S)$ is a Jordan group if S is a *projective* surface. The aim of this paper is to extend this result to the case of arbitrary algebraic surfaces. Our main result is the following statement, which gives a positive answer to Popov's question.

Theorem 1.3. *If X is an elliptic curve over k and S is an irreducible normal algebraic surface that is birationally isomorphic to $X \times \mathbb{P}^1$ then* $\mathrm{Aut}(S)$ *is a Jordan group.*

Remark 1.4. The group $\mathrm{Bir}(X \times \mathbb{P}^1)$ is *not* Jordan [15].

Remark 1.5. Suppose that S is a *non*-smooth irreducible normal surface. Since it is normal, there are only finitely many singular points on S. Then, by [10, Sect. 2, Cor. 8], $\mathrm{Aut}(S)$ is Jordan. This implies that in the course of the proof of Theorem 1.3 we may assume that S is smooth. On the other hand, by a theorem of Zariski [17, Cor. II.2.6 on p. 53], every irreducible smooth surface is quasi-projective. This implies that in the course of the proof of Theorem 1.3 we may assume that S is *smooth quasi-projective.*

Corollary 1.6. *Suppose that V is an irreducible normal algebraic variety over k. If* $\dim(V) \le 2$ *then* $\mathrm{Aut}(V)$ *is Jordan.*

Proof of Corollary 1.6. We have $\mathrm{Aut}(V) \subset \mathrm{Bir}(V)$. If V is *not* birationally isomorphic to a product of the projective line and an elliptic curve then $\mathrm{Bir}(V)$ is Jordan ([9, Thm. 2.32]) and therefore its subgroup $\mathrm{Aut}(V)$ is also Jordan. If V is birationally isomorphic to a product of the projective line and an elliptic curve then $\dim(V) = 2$ and Theorem 1.3 implies that $\mathrm{Aut}(V)$ is Jordan. □

Theorem 1.7. *Let V be an irreducible algebraic variety over k. If* $\dim(V) \le 2$ *then* $\mathrm{Aut}(V)$ *is Jordan.*

Proof of Theorem 1.7. Let $\nu : V^\nu \to V$ be the *normalization* of V ([8, Chap. III, Sect. 8], [4, Chap. 2, Sect. 2.14]). Here ν is a birational (surjective) regular map that is called the normalization map for V, and V^ν is an irreducible *normal* variety (of the same dimension as V) over k [8, Thm. 4 on p. 203]. The *universality property* of the normalization map implies that every biregular automorphism of V lifts uniquely to a biregular automorphism of V^ν [4, Chap. 2, Sect. 2.14, Thm. 2.25 on p. 141]. This gives rise to the *embedding* of groups

$$\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(V^\nu).$$

By Corollary 1.6, the group $\mathrm{Aut}(V^{\nu})$ is Jordan. Since $\mathrm{Aut}(V)$ is isomorphic to a subgroup of Jordan group $\mathrm{Aut}(V^{\nu})$, it is also Jordan. □

Corollary 1.8. *Let V be an algebraic variety over k. If* $\dim(V) \leq 2$ *then* $\mathrm{Aut}(V)$ *is Jordan.*

Proof. Let $V_1, \dots, V_r$ be all the *irreducible* components of V. Clearly, all V_i are irreducible algebraic varieties with $\dim(V_i) \leq \dim(V) \leq 2$. By Theorem 1.7, all $\mathrm{Aut}(V_i)$ are Jordan. Now Lemma 1 in Section 2.2 of [10] implies that $\mathrm{Aut}(V)$ is also Jordan. □

Remark 1.9. Suppose that k is the field $\mathbb{C}$ of complex numbers and X is a smooth irreducible quasi-projective non-projective surface. Then $M = X(\mathbb{C})$ carries the natural structure of a connected oriented smooth *real noncompact* fourfold and the group $\mathrm{Aut}(X)$ embeds naturally in the group $\mathrm{Diff}(M)$ of the (real) diffeomorphisms of the fourfold M. While $\mathrm{Aut}(X)$ is always Jordan, there are examples of connected oriented smooth *noncompact* real fourfolds, whose group of diffeomorphisms is *not* Jordan [11].

The paper is organized as follows. In Section 2 we discuss *minimal closures* of surfaces. In Section 3 we prove Theorem 1.3.

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2. Minimal closures

2.1. Let X be an elliptic curve over k and S be a *smooth* irreducible quasi-projective surface over k that is birationally isomorphic to $X \times \mathbb{P}^1$. There exists an irreducible smooth projective surface $\overline{S}$ such that its certain Zariski-open subset is biregularly isomorphic to S (further, we identify S with this open subset). Clearly, the inclusion map $S \subset \overline{S}$ is a birational morphism. This implies that

$$\mathrm{Aut}(S) \subset \mathrm{Bir}(S) = \mathrm{Bir}(\overline{S})$$

and therefore one may view $\mathrm{Aut}(S)$ as a subgroup of $\mathrm{Bir}(\overline{S})$. Since $\overline{S}$ is birationally isomorphic to S, it is also birationally isomorphic to $X \times \mathbb{P}^1$.

Let us fix a birational isomorphism between $\overline{S}$ and $X \times \mathbb{P}^1$. The projection map $X \times \mathbb{P}^1 \to X$ gives rise to a rational map $\overline{\pi} : \overline{S} \to X$ with dense image. Since $\overline{S}$ is smooth and X becomes abelian variety (after a choice of a base point), it follows from a theorem of Weil [1, Sect. 4.4] that $\overline{\pi}$ is regular. Since $\overline{S}$ is projective, $\overline{\pi} : \overline{S} \to X$ is surjective, because its image is closed.

For each $x \in X(k)$ we write $\overline{F}_x$ for the effective divisor $\overline{\pi}^*(x)$ on $\overline{S}$ that is the pullback (under $\overline{\pi}$) of the divisor (x) on X. Clearly, the support of $\overline{F}_x$ coincides with the curve $\overline{\pi}^{-1}(x)$ on $\overline{S}$. One says that the fiber of $\overline{\pi}$ over x is *reduced* if all irreducible components of the divisor $\overline{F}_x$ have multiplicity 1. We say that the fiber of $\overline{\pi}$ over x is *irreducible* if the curve $\overline{\pi}^{-1}(x)$ is irreducible; if this is the case then its multiplicity in $\overline{F}_x$ is 1 [6, Chap. 3, Sect. 1.4, Lemma 1.4.1(1) on p. 195].

It is known [13, Chap. IV] that for all but finitely many $x \in X(k)$ the fiber of $\overline{\pi}$ over x is irreducible and reduced, and the curve $\overline{\pi}^{-1}(x)$ is smooth (and irreducible). We call such fibers nonsingular and other fibers *singular*.

If C is a rational curve on $\overline{S}$ then the restriction of $\overline{\pi}$ to C must be a constant map, because every map from a rational curve to an elliptic curve is constant. This implies that C lies in a fiber of $\overline{\pi}$. (In particular, every (-1)-curve on $\overline{S}$ lies in a fiber of $\overline{\pi}$.) This implies that every birational automorphism of $\overline{S}$ is fiberwise [5, Sect. 13, Thm. 2]; see Section 2.2 below.

However, if $x \in X(k)$ and the fiber $\overline{\pi}^{-1}(x)$ is singular then the corresponding divisor $\overline{F}_x$ enjoys the following properties [6, Chap. I, Sect. 2.12; Chap. 3, Sect. 1.4, Lemma 1.4.1 on p. 195] (see also [3]).

(i) Each irreducible component of $\overline{F}_x$ is a smooth rational curve (and the corresponding graph is a tree) [3, Sect. 3].
(ii) At least one of the irreducible components of $\overline{F}_x$ is a (-1)-curve [3, Sect. 4.2].
(iii) If one of the irreducible components of $\overline{F}_x$ is a (-1)-curve of multiplicity 1 then there is another irreducible (-1)-component of $\overline{F}_x$ [3, Sect. 4.2].

2.2. If $\sigma \in \mathrm{Bir}(\overline{S})$ then there is a unique *biregular* automorphism $\mathfrak{f}(\sigma) : X \to X$ such that the composition $\overline{\pi}\sigma$ is a *regular* map that coincides with the composition

$$\mathfrak{f}(\sigma)\circ\overline{\pi} : \overline{S} \xrightarrow{\overline{\pi}} X \xrightarrow{\mathfrak{f}(\sigma)} X$$

(see, e.g., [7, Lect. V, Sect 1.4, p. 99]). Clearly, σ sends the fiber $\overline{\pi}^{-1}(x)$ to the fiber $\overline{\pi}^{-1}(\mathfrak{f}(\sigma)(x))$ for all $x \in X(k)$. We get a surjective group homomorphism

$$\mathfrak{f} : \mathrm{Bir}(\overline{S}) \to \mathrm{Aut}(X), \quad \sigma \mapsto \mathfrak{f}(\sigma)$$

that fits into a short exact sequence

$$\{1\} \to \mathrm{Bir}_X(\overline{S}) \subset \mathrm{Bir}(\overline{S}) \xrightarrow{\mathfrak{f}} \mathrm{Aut}(X) \to \{1\}$$

where the subgroup $\mathrm{Bir}_X(\overline{S})$ consists of all birational automorphisms $\sigma \in \mathrm{Bir}(\overline{S})$ such that $\overline{\pi}\sigma = \overline{\pi}$ (i.e., σ leaves invariant every fiber of $\overline{\pi}$). In addition, $\mathrm{Bir}_X(\overline{S})$ is isomorphic to the projective linear group $\mathrm{PGL}(2, k(X))$ over the field $k(X)$ of rational functions on X [7, Lect. V, Sect. 1.4, p. 99].

2.3. We write π for the composition

$$S \subset \overline{S} \xrightarrow{\overline{\pi}} X,$$

i.e., for the restriction of $\overline{\pi}$ to S. Recall that $\mathrm{Aut}(S) \subset \mathrm{Bir}(\overline{S})$. Since S is a surface, it is not contained in a union of finitely many fibers of π in $\overline{S}$. This implies that $\pi(S)$ is infinite and therefore is everywhere dense in X. It follows from [14, Vol. 1, Chap. 1, Sect. 5, Thm. 6] that either $\pi(S) = X$ or the complement $T_0 := X(k) \setminus \pi(S(k))$ is a finite set and

$$S \subset \pi^{-1}(X \setminus T_0) \subset \overline{S}.$$

If we write $\mathrm{Aut}_X(S)$ for the intersection (in $\mathrm{Bir}(\overline{S})$) of $\mathrm{Aut}(S)$ and $\mathrm{Bir}_X(\overline{S})$ then we get a short exact sequence

$$\{1\} \to \mathrm{Aut}_X(S) \subset \mathrm{Aut}(S) \xrightarrow{\mathfrak{f}} \mathfrak{f}(\mathrm{Aut}(S)) \to \{1\}$$

where

$$\mathrm{Aut}_X(S) \subset \mathrm{Bir}_X(\overline{S}), \ \mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X).$$

Similarly to the case of projective surfaces, if $x \in X(k)$ then we write F_x for the effective divisor $\pi^*(x)$ on S that is the pullback (under π) of the divisor (x) on S. Clearly, the support of F_x coincides with the curve $\pi^{-1}(x)$ on S. It is also clear that the divisor F_x on S is the pullback of the divisor $\overline{F}_x$ on $\overline{S}$ under the (open) inclusion map $S \subset \overline{S}$. One says that the fiber of π over x is *reduced* if all irreducible components of the divisor F_x have multiplicity 1. We say that the fiber of π over x is irreducible if it is a multiple of a *simple* divisor, i.e., the curve $\overline{\pi}^{-1}(x)$ is irreducible. Clearly, if the fiber of $\overline{\pi}$ over x is irreducible (resp. reduced, resp. smooth) then the fiber of π over x is irreducible (resp. reduced, resp. smooth). On the other hand, if an irreducible component $\overline{C}$ of $\overline{F}_x$ has multiplicity $m > 1$ and meets S, then $\overline{C} \cap S$ is an irreducible component of F_x with the same multiplicity m. In particular, the fiber of π over x is *not* reduced. Notice also that if $\overline{C}_1$ and $\overline{C}_2$ are distinct irreducible components of $\overline{F}_x$ that meet F_x then $C_1 := \overline{C}_1 \bigcap S$ and $C_2 := \overline{C}_2 \bigcap S$ are *distinct* irreducible components of F_x; in particular, the fiber of π over x is *not* irreducible.

It follows from the results about the fibers of $\overline{\pi}$ mentioned in Section 2.1 (see also theorems of Bertini [14, Vol. 1, Chap. 2, Sects. 6.1 and 6.2]) that either all the fibers of π are smooth irreducible reduced or the set T_1 of points $x \in \pi(S(k)) \subset X(k)$ such that, at least, one of these properties does not hold, is finite. Clearly,

$$\mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_0), \ \ \mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_1).$$

This implies that if either T_0 or T_1 is *non*-empty then $\mathfrak{f}(\mathrm{Aut}(S))$ is a *finite* group and $\mathrm{Aut}_X(\overline{S})$ is a subgroup of *finite index* in $\mathrm{Aut}(S)$.

2.4. It follows from the theorem of Jordan that the projective linear group $\mathrm{PGL}(2, k(X))$ is Jordan [9], [16]. Since $\mathrm{Bir}_X(\overline{S})$ is isomorphic to $\mathrm{PGL}(2, k(X))$ (see Section 2.2), it is also a Jordan group. This implies in turn that its subgroup $\mathrm{Aut}_X(S)$ is also Jordan. It follows that if either T_0 or T_1 is *non*-empty then $\mathrm{Aut}(S)$ contains the Jordan subgroup $\mathrm{Aut}_X(S)$ of finite index and therefore is Jordan itself, thanks to Remark 1.2.

In order to handle the case of empty T_0 and T_1, we need additional ideas.

Definition 2.5. The projective surface $\overline{S}$ is called a (relative) *minimal closure* of S if every (-1)-curve on $\overline{S}$ meets S. See [3, Sect. 4.9]. A minimal closure of S always exists [3, Prop. 4.10]. (Warning: if $\overline{S}$ is a minimal closure then the complement of S in $\overline{S}$ does *not* have to be a divisor!)

Lemma 2.6 (Lemma 4.12 of [3])**.** *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced. If $\overline{S}$ is a minimal closure of S then all the fibers of $\overline{\pi} : \overline{S} \to X$ are irreducible.*

Proof. Suppose that there exists $x \in X(k)$ such that the fiber of $\overline{\pi}$ over x is not irreducible and therefore is singular. Then $\overline{F}_x$ contains as an irreducible component a (-1)-curve, say $\overline{C}_1$ with multiplicity $m \geq 1$ (Section 2.1). The minimality of $\overline{S}$ implies that $C_1 = \overline{C}_1 \bigcap S$ is non-empty and therefore is an irreducible component of F_x with the same multiplicity m (Section 2.3). Since the fiber of π over x is reduced, $m = 1$. This implies that $\overline{F}_x$ contains another irreducible component $\overline{C}_2$ that is also a (-1)-curve. Again $C_2 = \overline{C}_2 \bigcap S$ is an irreducible component of F_x that does not coincide with C_1. This implies that the fiber of π over x is *not* irreducible, which is not the case. □

Theorem 2.7. *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced. Let $\overline{S}$ be a minimal closure of S. Then every biregular automorphism of S extends uniquely to a biregular automorphism of $\overline{S}$. In other words,*

$$\mathrm{Aut}(S) \subset \mathrm{Aut}(\overline{S}) \subset \mathrm{Bir}(\overline{S}).$$

Proof. By Lemma 2.6, every fiber $\overline{F}_x$ is an irreducible curve isomorphic to $\mathbb{P}^1$.

Let $g : S \to S$ be a biregular automorphism of S. Let us extend g to a birational map

$$\overline{g} : \overline{S} \to \overline{S}.$$

Assume that $\overline{g}$ is *not* a regular map. Let S' be a *resolution of the indeterminacies* of $\overline{g}$, i.e., a smooth irreducible surface included into the following commutative diagram

$$\begin{array}{ccc} S' & & \\ {\scriptstyle u}\downarrow & \overset{g'}{\searrow} & \\ \overline{S} & \overset{\overline{g}}{\dashrightarrow} & \overline{S} \\ \cup & & \cup \\ S & \xrightarrow{g} & S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{h} & X \end{array},$$

where u is a birational morphism that is a composition of finitely many blow ups and induces a biregular isomorphism between $u^{-1}(S)$ and S (such a u exists, because g is defined on S), g' and $\overline{\pi}' = \overline{\pi} \circ u$ are morphisms, and $h = \mathfrak{f}(g) \in \mathrm{Aut}(X)$ is a biregular automorphism of X. (The group homomorphism $\mathfrak{f}$ is defined in Section 2.2.) Let $D' \subset S'$ be the union of all exceptional curves for g' and let $D = g'(D') \subset \overline{S}$, which is a finite set.

Every point z of $\overline{S}$ that does not lie on D has only one preimage ${g'}^{-1}(z) \in S'$ ([14, Chap. 2, Sect. 4, Thm. 2]).

Let B' be the union of exceptional curves for u. Clearly,

$$B' \subset S' \setminus u^{-1}(S).$$

This implies that

$$u(B') \bigcap S = \varnothing.$$

We want to show that $B' \subset D'$, because then g' contracts all components of B' and $\overline{g}$ appears to be a morphism.

Let C' be an irreducible component of B'. The *point* $u(C')$ lies in $u(B')$ and therefore does *not* belong to S.

Since X is an elliptic curve, and C' is rational, $\overline{\pi}(g'(C'))$ is a point $x \in X(k)$. Thus, since all the fibers of $\overline{\pi}$ are irreducible (thanks to Lemma 2.6), either

Case 1. $g'(C')$ is a point and therefore $C' \subset D'$;

or

Case 2. $g'(C') = \overline{F}_x = \overline{\pi}^{-1}(x) \subset \overline{S}$. Let us put $x_1 := h^{-1}(x) \in X(k)$. Then $x = h(x_1) \in X(k)$. Let $s \in F_x \setminus (F_x \cap D) \subset S$ be a point of the fiber F_x, which is not in the image of D'. Therefore it has only one *preimage* $s_1 := {g'}^{-1}(s)$. Moreover, $s_1 \in u^{-1}(S)$, because $s \in S$. On the other hand, since $g'(C') = \overline{F}_x$, there is a point $c \in C' \subset S' \setminus u^{-1}(S)$ such that $g'(c) = s$. Clearly, $c \neq s_1$ and we get a contradiction that shows that the Case 2 does not occur.

This proves that every $g \in \mathrm{Aut}(S)$ extends to a regular birational map $\overline{g} : \overline{S} \to \overline{S}$. Since the same is true for $g^{-1} \in \mathrm{Aut}(S)$, the map $\overline{g}$ is a biregular automorphism of $\overline{S}$. $\square$

3. Proof of Theorem 1.3

Remark 1.5 tells us that we may assume that S is a smooth quasi-projective surface. In light of the results of Section 2.4, we may also assume that every fiber of π is smooth irreducible and reduced, and $\pi(S) = X$. Let $\overline{S}$ be a minimal closure of S. By Theorem 2.7, $\mathrm{Aut}(S)$ is a subgroup of $\mathrm{Aut}(\overline{S})$. Since $\overline{S}$ is projective, the results of [16] imply that the group $\mathrm{Aut}(\overline{S})$ is Jordan and therefore its every subgroup is Jordan. It follows that $\mathrm{Aut}(S)$ is Jordan.

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