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QUANTUM SUPERGROUPS I. FOUNDATIONS

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Abstract. In this part one of a series of papers, we introduce a new version of quantum covering and super groups with no isotropic odd simple root, which is suitable for the study of integrable modules, integral forms, and the bar involution. A quantum covering group involves parameters q and π with $\pi^2 = 1$, and it specializes at $\pi = -1$ to a quantum supergroup. Following Lusztig, we formulate and establish various structural results of the quantum covering groups, including a bilinear form, quasi-$\mathcal{R}$-matrix, Casimir element, character formulas for integrable modules, and higher Serre relations.

Introduction

Quantum groups have been ubiquitous in Lie theory, mathematical physics, algebraic combinatorics, and low-dimensional topology since their introduction by Drinfeld and Jimbo [Dr], [Jim]. We refer to the books of Lusztig and Jantzen [Lu], [Jan] for a systematic development of the structure and representation theory of quantum groups.

In a recent paper [HW] by two of the authors, the spin nilHecke and quiver Hecke algebras (see Wang [Wa], Kang–Kashiwara–Tsuchioka [KKT], Ellis–Khovanov–Lauda [EKL]) were shown to provide a categorification of quantum covering groups with a quantum parameter q and a second parameter π satisfying $\pi^2 = 1$ (we refer to *loc. cit.* for more references on categorification); a quantum covering group specializes at $\pi = -1$ to half of a quantum supergroup with no isotropic odd simple roots, and to half of the Drinfeld–Jimbo quantum group at $\pi = 1$.

In the rank one case, a version of the full quantum covering and super group for $\mathfrak{osp}(1|2)$ suitable for constructing an integral form, as well as integrable modules over $\mathbb{Q}(q)$ corresponding to each nonnegative integer, was formulated by two of the

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authors [CW]. In particular, the structure and representation theories of quantum $\mathfrak{sl}(2)$ and quantum $\mathfrak{osp}(1|2)$ were shown to be in complete agreement; also see [Zou] (in contrast to the classical fact that there are "fewer" integrable modules for $\mathfrak{osp}(1|2)$ than for $\mathfrak{sl}(2)$).

The goal of this paper is to lay the foundations of quantum covering and super groups with no isotropic odd simple roots, following Lusztig [Lu, Part I] as a blueprint. We define a new version of quantum covering and super groups with no isotropic odd simple root, which is suitable for the study of integrable modules for *all* possible dominant integral weights, exactly as for the Drinfeld–Jimbo quantum groups. We formulate and establish various structural results of the quantum covering and super groups, including a bilinear form, twisted derivations, integral forms, bar-involution, quasi-$\mathfrak{R}$-matrix, Casimir, characters for integrable modules, and quantum (higher) Serre relations.

The results of this paper on quantum covering groups reduce to Lusztig's quantum group setting [Lu] when specializing the parameter π to 1, and on the other hand, reduce to quantum supergroup setting when specializing the parameter π to -1. For this reason, we work almost exclusively with quantum covering groups. Even if one is mainly interested in the super case, writing π systematically for the super sign -1 offers a conceptual explanation for various formulas and constructions. For earlier definitions of quantum supergroups, we refer to Yamane [Ya], Musson–Zou [MZ], Benkart–Kang–Melville [BKM].

Let us describe the main results in detail. As in [Kac], a super Cartan datum is a Cartan datum $(I, \cdot)$ with a partition $I = I_{\bar{0}} \sqcup I_{\bar{1}}$ subject to some natural conditions; also see [HW]. Note the only finite type super Cartan datum is of type $B(0, n)$, for $n \geq 1$. In Section 1, we formulate the definition of half a quantum covering group associated to a super Catan datum. We develop the properties of a bilinear form (and a dual version) and twisted derivations on half the quantum covering group systematically. Then we provide a new proof using twisted derivations of a theorem in [HW] (also cf. Yamane [Ya] and Geer [Gr]) that the existence of a non-degenerate bilinear form implies the quantum Serre relations.

Motivated by the rank one construction in [CW], we formulate in Section 2 a new version of quantum super and covering groups with generators E_i, F_i, K_μ, and additional generators J_μ, for $i \in I$ and $\mu \in Y$ (the co-weight lattice). The new generators J_i play a crucial role in formulating the notion of integrable modules of a quantum supergroup for *all* dominant integral weights. A study of all such representations was not possible before (cf. [Kac], [BKM]).

In Section 3, we formulate the quasi-$\mathfrak{R}$-matrix for quantum covering or super groups and establish its basic properties. This generalizes the construction in the rank one case in [CW]. Then we construct the quantum Casimir and use it to prove the complete reducibility of the integrable modules. We show that the simple integrable modules are parametrized by $\pi = \pm 1$ and the dominant integral weights (in contrast to [BKM], [Kac]), and their character formulas coincide with their counterpart for quantum groups (which was established by Lusztig [Lu1]). This character formula (in case $\pi = -1$) is shown to hold for the irreducible integrable modules under some "evenness" restrictions on highest weights as in [BKM] (where a definition of quantum supergroups without operators J_i was used), deforming the construction in [Kac].

The higher Serre relations for quantum covering groups are then established in Section 4.

This paper lays the foundation for further studies of quantum covering and super groups. In a sequel [CHW], we will construct the canonical basis, à la Lusztig and Kashiwara, of quantum covering groups and of integrable modules. In yet another paper, a braid group action on a quantum covering group and its integrable modules will be studied in depth.

1. The algebra f

In this section, starting with the super Cartan datum and root datum, we formulate half a quantum covering group $\mathbf{f}$ in terms of a bilinear form on a free superalgebra $'\mathbf{f}$, and show that the (q,π)-Serre relations are satisfied in $\mathbf{f}$.

1.1. Super Cartan datum

A *Cartan datum* is a pair $(I,\cdot)$ consisting of a finite set I and a symmetric bilinear form $\nu,\nu'\mapsto\nu\cdot\nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i=i\cdot i/2\in\mathbb{Z}_{>0}$;
(b) $2i\cdot j/i\cdot i\in-\mathbb{N}$ for $i\neq j$ in I, where $\mathbb{N}=\{0,1,2,\dots\}$.

If the datum can be decomposed as $I=I_{\overline{0}}\coprod I_{\overline{1}}$ such that

(c) $I_{\overline{1}}\neq\varnothing$,
(d) $2i\cdot j/i\cdot i\in 2\mathbb{Z}$ if $i\in I_{\overline{1}}$,

then it is called a *super Cartan datum.*

The $i\in I_{\overline{0}}$ are called even, $i\in I_{\overline{1}}$ are called odd. We define a parity function $p:I\to\{0,1\}$ so that $i\in I_{\overline{p(i)}}$. We extend this function to the homomorphism $p:\mathbb{Z}[I]\to\mathbb{Z}$. Then p induces a $\mathbb{Z}_2$-grading on $\mathbb{Z}[I]$ which we shall call the parity grading. We define the *height* of $\nu=\sum_{i\in I}\nu_i i\in\mathbb{Z}[I]$ by $\mathrm{ht}(\nu)=\sum\nu_i$. (Note we use different notation than [Lu], where the same quantity is denoted by $\mathrm{tr}(\nu)$.)

A super Cartan datum $(I,\cdot)$ is said to be of *finite* (resp. *affine*) type exactly when $(I,\cdot)$ is of finite (resp. affine) type as a Cartan datum (cf. [Lu, §2.1.3]). In particular, from (a) and (d) we see that the only super Cartan datum of finite type is the one corresponding to the Lie superalgebras of type $B(0,n)$ for $n\geq 1$.

A super Cartan datum is called *bar-consistent* or simply *consistent* if it satisfies

(e) $d_i\equiv p(i) \mod 2,\quad \forall i\in I.$

We note that (e) is almost always satisfied for super Cartan data of finite or affine type (with one exception). A super Cartan datum is not assumed to be (bar-)consistent unless specified explicitly below. (Roughly speaking, the "bar-consistent" condition is imposed whenever a bar involution is involved later on.)

Note that (d) and (e) imply that

(f) $i\cdot j\in 2\mathbb{Z}$ for all $i,j\in I$.

1.2. Root datum

A *root datum* associated to a super Cartan datum $(I,\cdot)$ consists of

(a) two finitely generated free abelian groups Y, X and a perfect bilinear pairing $\langle\cdot,\cdot\rangle:Y\times X\to\mathbb{Z}$;

(b) an embedding $I \subset X$ ($i \mapsto i'$) and an embedding $I \subset Y$ ($i \mapsto i$) satisfying
(c) $\langle i, j' \rangle = 2i \cdot j / i \cdot i$ for all $i, j \in I$.

We will always assume that the image of the imbedding $I \subset X$ (respectively, the image of the imbedding $I \subset Y$) is linearly independent in X (respectively, in Y).

Let $X^+ = \{\lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N} \text{ for all } i \in I\}$. Note that there are no additional "evenness" assumptions for X^+.

Let π be a parameter such that

$$\pi^2 = 1.$$

For any $i \in I$, we set

$$q_i = q^{i \cdot i/2}, \qquad \pi_i = \pi^{p(i)}.$$

Note that when the datum is consistent, $\pi_i = \pi^{i \cdot i/2}$; by induction, we therefore have $\pi^{p(\nu)} = \pi^{\nu \cdot \nu/2}$ for $\nu \in \mathbb{Z}[I]$. We extend this notation so that if $\nu = \sum \nu_i i \in \mathbb{Z}[I]$, then

$$q_\nu = \prod_i q_i^{\nu_i}, \qquad \pi_\nu = \prod_i \pi_i^{\nu_i}.$$

For any ring R we define a new ring $R^\pi = R[\pi]/(\pi^2 - 1)$ (with π commuting with R). We shall need $\mathbb{Q}(q)^\pi$ below.

1.3. Braid group and Weyl group

Assume a Cartan (super) datum $(I, \cdot)$ is given. For $i \neq j \in I$ such that $\langle i, j' \rangle \langle j, i' \rangle > 0$, we define an integer $m_{ij} \in \mathbb{Z}_{\geq 2}$ by $\cos^2(\pi/m_{ij}) = \langle i, j' \rangle \langle j, i' \rangle / 4$ if it exists, and set $m_{ij} = \infty$ otherwise. We have

$\langle i, j' \rangle \langle j, i' \rangle$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

The braid group (associated to I) is the group generated by s_i ($i \in I$) subject to the relations (whenever $m_{ij} < \infty$):

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}. \tag{1.1}$$

The Weyl group W is defined to be the group generated by s_i ($i \in I$) subject to relations (1.1) and additional relations $s_i^2 = 1$ for all i.

For $i \in I$, we let s_i act on X (resp. Y) as follows: for $\lambda \in X, \lambda^\vee \in Y$,

$$s_i(\lambda) = \lambda - \langle i, \lambda \rangle i', \qquad s_i(\lambda^\vee) = \lambda^\vee - \langle \lambda^\vee, i' \rangle i.$$

This defines actions of the Weyl group W on X and Y.

1.4. The algebras $'\mathbf{f}$ and $\mathbf{f}$

Define $'\mathbf{f}$ to be the free associative $\mathbb{Q}(q)^\pi$-superalgebra with 1 and with even generators θ_i for $i \in I_{\bar{0}}$ and odd generators θ_i for $i \in I_{\bar{1}}$. We abuse notation and define the parity grading on $'\mathbf{f}$ by $p(\theta_i) = p(i)$. We also have a weight grading $|\cdot|$ on $'\mathbf{f}$ defined by setting $|\theta_i| = i$.

The tensor product ${}'\mathbf{f} \otimes {}'\mathbf{f}$ as a $\mathbb{Q}(q)^{\pi}$-superalgebra has the multiplication

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{|x_2|\cdot|x_1'|}\pi^{p(x_2)p(x_1')} x_1x_1' \otimes x_2x_2'.$$

Here and below, in all displayed formulas, we will implicitly assume the elements involved are $\mathbb{N}[I] \times \mathbb{Z}_2$*-homogeneous.*

There is a similar multiplication formula in ${}'\mathbf{f} \otimes {}'\mathbf{f} \otimes {}'\mathbf{f}$:

$$\begin{aligned}&(x_1 \otimes x_2 \otimes x_3)(x_1' \otimes x_2' \otimes x_3')\\ &\quad = q^{|x_2|\cdot|x_1'|+|x_3|\cdot|x_2'|+|x_3|\cdot|x_1'|}\pi^{p(x_2)p(x_1')+p(x_3)p(x_2')+p(x_3)p(x_1')} x_1x_1' \otimes x_2x_2' \otimes x_3x_3'.\end{aligned}$$

We will take $r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$ to be an algebra homomorphism such that $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ for all $i \in I$. One checks that the following co-associativity holds:

$$(r \otimes 1)r = (1 \otimes r)r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f} \otimes {}'\mathbf{f};$$

this is an algebra homomorphism.

Proposition 1.4.1. *There exists a unique bilinear form* $(\cdot\,,\cdot)$ *on* ${}'\mathbf{f}$ *with values in* $\mathbb{Q}$ *such that* $(1, 1) = 1$ *and*

(a) $(\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^{-2})^{-1} \quad (\forall i, j \in I)$;
(b) $(x, y'y'') = (r(x), y' \otimes y'') \quad (\forall x, y', y'' \in {}'\mathbf{f})$;
(c) $(xx', y'') = (x \otimes x', r(y'')) \quad (\forall x, x', y'' \in {}'\mathbf{f})$.

Moreover, this bilinear form is symmetric.

Here, the induced bilinear form $({}'\mathbf{f} \otimes {}'\mathbf{f}) \times ({}'\mathbf{f} \otimes {}'\mathbf{f}) \to \mathbb{Q}(q)$ is given by

$$(x_1 \otimes x_2, x_1' \otimes x_2') := (x_1, x_1')(x_2, x_2'), \tag{1.2}$$

for homogeneous $x_1, x_2, x_1', x_2' \in {}'\mathbf{f}$.

This is basically [HW, Prop. 3.3], where $(\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^2)^{-1}$ was imposed (note a different sign on the exponent for q_i^2). These two cases do not exactly match under the bar-involution (which sends $q \mapsto \pi q^{-1}$), and so we redo a careful proof here.

Proof. We follow [Lu, 1.2.3] to define an associative algebra structure on ${}'\mathbf{f}^* := \oplus_\nu {}'\mathbf{f}_\nu^*$ by transposing the "coproduct" $r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$. In particular, for $g, h \in {}'\mathbf{f}^*$, we define $gh(x) := (g \otimes h)(r(x))$, where $(g \otimes h)(y \otimes z) = g(y)h(z)$.

Let $\xi_i \in {}'\mathbf{f}_i^*$ be defined by $\xi_i(\theta_i) = (1 - \pi_i q_i^{-2})^{-1}$. Let $\phi : {}'\mathbf{f} \to {}'\mathbf{f}^*$ be the unique algebra homomorphism such that $\phi(\theta_i) = \xi_i$ for all i. The map ϕ preserves the $\mathbb{N}[I] \times \mathbb{Z}_2$-grading.

Define $(x, y) = \phi(y)(x)$, for $x, y \in {}'\mathbf{f}$. The properties (a) and (b) follow directly from the definition.

Clearly $(x, y) = 0$ unless (homogeneous) x, y have the same weight in $\mathbb{N}[I]$ and the same parity. All elements involved below will be assumed to be homogeneous.

It remains to prove (c). Assume that (c) is known for y'' replaced by y or y' and for any x, x'. We then prove that (c) holds for $y'' = yy'$. Write

$$r(x) = \sum x_1 \otimes x_2, \quad r(x') = \sum x'_1 \otimes x'_2,$$
$$r(y) = \sum y_1 \otimes y_2, \quad r(y') = \sum y'_1 \otimes y'_2.$$

Then

$$r(xx') = \sum q^{|x_2|\cdot|x'_1|} \pi^{p(x_2)p(x'_1)} x_1 x'_1 \otimes x_2 x'_2,$$
$$r(yy') = \sum q^{|y_2|\cdot|y'_1|} \pi^{p(y_2)p(y'_1)} y_1 y'_1 \otimes y_2 y'_2.$$

We have

$$\begin{aligned}(xx', yy') &= (\phi(y)\phi(y'))(xx') = (\phi(y) \otimes \phi(y'))(r(xx')) \\ &= \sum q^{|x_2|\cdot|x'_1|} \pi^{p(x_2)p(x'_1)} (x_1 x'_1, y)(x_2 x'_2, y') \\ &= \sum q^{|x_2|\cdot|x'_1|} \pi^{p(x_2)p(x'_1)} (x_1 \otimes x'_1, r(y))(x_2 \otimes x'_2, r(y')) \\ &= \sum q^{|x_2|\cdot|x'_1|} \pi^{p(x_2)p(x'_1)} (x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2).\end{aligned} \tag{1.3}$$

On the other hand,

$$\begin{aligned}(x \otimes x', r(yy')) &= \sum q^{|y_2|\cdot|y'_1|} \pi^{p(y_2)p(y'_1)} (x \otimes x', y_1 y'_1 \otimes y_2 y'_2) \\ &= \sum q^{|y_2|\cdot|y'_1|} \pi^{p(y_2)p(y'_1)} (x, y_1 y'_1)(x', y_2 y'_2) \\ &= \sum q^{|y_2|\cdot|y'_1|} \pi^{p(y_2)p(y'_1)} (r(x), y_1 \otimes y'_1)(r(x'), y_2 \otimes y'_2) \\ &= \sum q^{|y_2|\cdot|y'_1|} \pi^{p(y_2)p(y'_1)} (x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2).\end{aligned} \tag{1.4}$$

For a summand to make nonzero contribution, we may assume that each of the four pairs $\{x_1, y_1\}, \{x'_1, y_2\}, \{x_2, y'_1\}, \{x'_2, y'_2\}$ has the same weight in $\mathbb{N}[I]$ and the same parity. One checks that the powers of q and π in (1.3) and (1.4) match perfectly. Hence the two sums in (1.3) and (1.4) are equal, and whence (c). □

We set $\mathcal{I}$ to denote the radical of $(\cdot, \cdot)$. As in [Lu], this radical is a 2-sided ideal of $'\mathbf{f}$.

Let $\mathbf{f} = {}'\mathbf{f}/\mathcal{I}$ be the quotient algebra of $'\mathbf{f}$ by its radical. Since the different weight spaces are orthogonal with respect to this inner product, the weight space decomposition descends to a decomposition $\mathbf{f} = \bigoplus_\nu \mathbf{f}_\nu$ where $\mathbf{f}_\nu$ is the image of $'\mathbf{f}_\nu$. Each weight space is finite dimensional. The bilinear form descends to a bilinear form on $\mathbf{f}$ which is non-degenerate on each weight space.

Note that the notation of $'\mathbf{f}$ and $\mathbf{f}$ in this paper corresponds to the notation of $'\mathbf{f}^\pi$ and $\mathbf{f}^\pi$ in [HW].

The map $r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$ satisfies $r(\mathcal{I}) \subset \mathcal{I} \otimes {}'\mathbf{f} + {}'\mathbf{f} \otimes \mathcal{I}$ (the proof being entirely the same as in [Lu, §1.2.6]), whence it descends to a well-defined homomorphism $r : \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$.

Let ${}^t r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$ be the composition of r with the permutation map

$$x \otimes y \mapsto y \otimes x$$

of ${}'\mathbf{f} \otimes {}'\mathbf{f}$ to itself. (To have the signs work out below, the tensor permutation cannot be signed.)

The anti-involution $\sigma : {}'\mathbf{f} \to {}'\mathbf{f}$ satisfies $\sigma(\theta_i) = \theta_i$ for each $i \in I$ and

$$\sigma(xy) = \sigma(y)\sigma(x).$$

Lemma 1.4.2.

(a) *We have* $r(\sigma(x)) = (\sigma \otimes \sigma)^t r(x)$, *for all* $x \in {}'\mathbf{f}$.

(b) *We have* $(\sigma(x), \sigma(x')) = (x, x')$ *for all* $x, x' \in {}'\mathbf{f}$.

Proof. Since (b) will follow immediately from (a), it suffices to prove that $r(\sigma(x)) = (\sigma \otimes \sigma)^t r(x)$, for all $x \in {}'\mathbf{f}$. This is obviously true for $x \in \{1, \theta_i : i \in I\}$.

Suppose that $r(\sigma(x')) = (\sigma\otimes\sigma)^t r(x')$ and $r(\sigma(x'')) = (\sigma\otimes\sigma)^t r(x'')$. Let $r(x') = \sum x'_1 \otimes x'_2$ and $r(x'') = \sum x''_1 \otimes x''_2$. Then $r(x'x'') = \sum q^{|x'_2||x''_1|}\pi^{p(x'_2)p(x''_1)} x'_1 x''_1 \otimes x'_2 x''_2$ and we have

$$\begin{aligned} r(\sigma(x'x'')) &= r(\sigma(x''))r(\sigma(x')) \\ &= \left(\sum \sigma(x''_2) \otimes \sigma(x''_1)\right)\left(\sum \sigma(x'_2) \otimes \sigma(x'_1)\right) \\ &= \sum \pi^{p(x'_2)p(x''_1)} q^{|x'_2||x''_1|} \sigma(x'_2 x''_2) \otimes \sigma(x'_1 x''_1) = \sigma \otimes \sigma({}^t r(x'x'')). \end{aligned}$$

The lemma is proved. □

We note that σ descends to $\mathbf{f}$ and shares the above properties.

Let $^- : \mathbb{Q}(q)^\pi \to \mathbb{Q}(q)^\pi$ be the unique $\mathbb{Q}$-algebra involution (called the bar involution) satisfying $\overline{q} = \pi q^{-1}$ and $\overline{\pi} = \pi$.

Assume the super Cartan datum is consistent. Then

$$\overline{q_i} = \pi_i q_i^{-1}. \tag{1.5}$$

We define a bar involution $^- : {}'\mathbf{f} \to {}'\mathbf{f}$ such that $\overline{\theta_i} = \theta_i$ for all $i \in I$ and $\overline{fx} = \overline{f}\,\overline{x}$ for $f \in \mathbb{Q}(q)^\pi$ and $x \in {}'\mathbf{f}$.

Let ${}'\mathbf{f}\overline{\otimes}{}'\mathbf{f}$ be the $\mathbb{Q}(q)^\pi$-vector space ${}'\mathbf{f} \otimes {}'\mathbf{f}$ with multiplication given by

$$(x_1 \otimes x_2)(x'_1 \otimes x'_2) = (\pi q^{-1})^{|x_2|\cdot|x'_1|}\pi^{p(x_2)p(x'_1)} x_1 x'_1 \otimes x_2 x'_2.$$

Define $\overline{r}$ still by $\overline{r}(x) = \overline{r(\overline{x})}$. Then $\overline{r} : {}'\mathbf{f} \to {}'\mathbf{f}\overline{\otimes}{}'\mathbf{f}$ is an algebra homomorphism, being a composition of homomorphisms.

The co-associativity holds for $\overline{r}$:

$$(\overline{r} \otimes 1)(\overline{r}(x)) = \overline{(r \otimes 1)r(\overline{x})} = \overline{(1 \otimes r)r(\overline{x})} = (1 \otimes \overline{r})(\overline{r}(x)).$$

By checking on the algebra generators θ_i, it is an easy computation to see that this is an algebra homomorphism.

Let $\{\cdot,\cdot\} : {}'\mathbf{f} \times {}'\mathbf{f} \to \mathbb{Q}(q)$ be the symmetric bilinear form defined by

$$\{x, y\} = \overline{(\overline{x}, \overline{y})}.$$

It satisfies: $\{1, 1\} = 1$, and

$$\begin{aligned} \{\theta_i, \theta_j\} &= \delta_{i,j}(1 - \pi_i q_i^2)^{-1}; \\ \{x, y'y''\} &= \{\overline{r}(x), y' \otimes y''\}, \text{ for all } x, y', y'' \in {}'\mathbf{f}. \end{aligned}$$

Lemma 1.4.3. *Assume the super Cartan datum is consistent.*

(a) *Let* $r(x) = \sum x_1 \otimes x_2$. *We have*

$$\overline{r}(x) = \sum (\pi q)^{-|x_1|\cdot|x_2|}\pi^{p(x_1)p(x_2)} x_2 \otimes x_1.$$

(b) $\{x, y\} = (-1)^{\mathrm{ht}|x|}\pi^{(p(x)p(y)+p(x))/2} q^{-|x|\cdot|y|/2} q_{-|x|}(x, \sigma(y))$.

Proof. It is straightforward to check both claims are true when $x = \theta_i$ and $y = \theta_j$ for some $i, j \in I$.

Assume (a) holds for x replaced by x' and by x''. We shall prove the claim for $x = x'x''$.

Recall $\overline{q} = \pi q^{-1}$, and $r(\overline{x}) = \overline{\overline{r}(x)}$. Write

$$\begin{aligned} r(x') &= \sum x'_1 \otimes x'_2, \quad r(x'') = \sum x''_1 \otimes x''_2, \\ r(x'x'') &= \sum q^{|x''_1|\cdot|x'_2|}\pi^{p(x''_1)p(x'_2)} x'_1 x''_1 \otimes x'_2 x''_2. \end{aligned} \tag{1.6}$$

By assumption, we have

$$\begin{aligned} r(\overline{x'}) &= \sum q^{|x'_1|\cdot|x'_2|}\pi^{p(x'_1)p(x'_2)} \overline{x'_2} \otimes \overline{x'_1}, \\ r(\overline{x''}) &= \sum q^{|x''_1|\cdot|x''_2|}\pi^{p(x''_1)p(x''_2)} \overline{x''_2} \otimes \overline{x''_1}. \end{aligned}$$

Hence,

$$\begin{aligned} r(\overline{x'})r(\overline{x''}) &= \sum q^{|x'_1|\cdot|x'_2|}\pi^{p(x'_1)p(x'_2)} q^{|x''_1|\cdot|x''_2|}\pi^{p(x''_1)p(x''_2)} (\overline{x'_2} \otimes \overline{x'_1})(\overline{x''_2} \otimes \overline{x''_1}) \\ &= \sum q^{|x'_1|\cdot|x'_2| + |x''_1|\cdot|x''_2|}\pi^{s} q^{|x'_1|\cdot|x''_2|} \overline{x'_2 x''_2} \otimes \overline{x'_1 x''_1}, \end{aligned}$$

where $s = p(x'_1)p(x'_2) + p(x''_1)p(x''_2) + p(x'_1)p(x''_2)$. Then,

$$\begin{aligned} \overline{r}(x'x'') &= \overline{r(\overline{x'})r(\overline{x''})} \\ &= \sum (\pi q)^{-(|x'_1|\cdot|x'_2| + |x''_1|\cdot|x''_2| + |x'_1|\cdot|x''_2|)}\pi^{s} x'_2 x''_2 \otimes x'_1 x''_1 \\ &= \sum (\pi q)^{-|x'_1 x''_1|\cdot|x'_2 x''_2|}\pi^{p(x'_1 x''_1)p(x'_2 x''_2)} q^{|x''_1|\cdot|x'_2|}\pi^{t} x'_2 x''_2 \otimes x'_1 x''_1, \end{aligned}$$

where $t = p(x''_1)p(x'_2) + |x''_1|\cdot|x'_2|$. Now, since the datum is consistent, $|x''_1|\cdot|x'_2| \in 2\mathbb{Z}$, and hence we have

$$\overline{r}(x'x'') = \sum (\pi q)^{-|x'_1 x''_1|\cdot|x'_2 x''_2|}\pi^{p(x'_1 x''_1)p(x'_2 x''_2)} q^{|x''_1|\cdot|x'_2|}\pi^{p(x''_1)p(x'_2)} x'_2 x''_2 \otimes x'_1 x''_1. \tag{1.7}$$

Comparing (1.6) and (1.7), we see that (a) holds.

Let $\mathcal{S}$ be the set of $y \in {'\mathbf{f}}$ such that (b) holds for all $x \in {'\mathbf{f}}$. Let $y', y'' \in \mathcal{S}$; we will show $y = y'y'' \in \mathcal{S}$. Let $x \in {'\mathbf{f}}$ and write $r(x) = \sum x' \otimes x''$ with x, x''

homogeneous. Then

$$\begin{aligned}\{x, y'y''\} &= \{\overline{r}(x), y'\otimes y''\} = \left\{\sum (\pi q)^{-|x'|\cdot|x''|}\pi^{p(x')p(x'')}x''\otimes x', y'\otimes y''\right\}\\ &= \sum q^{-|x'|\cdot|x''|}\pi^{p(x')p(x'')}\{x'', y'\}\{x', y''\}\\ &= \sum (-1)^{\mathrm{ht}|x'|+\mathrm{ht}|x''|}q^{(-|x''|\cdot|y'|-|x'|\cdot|y''|-2|x'|\cdot|x''|)/2}q_{-|x'|-|x''|}\\ &\quad * \pi^{p(x')p(x'')+(p(x')p(y'')+p(x'))/2+(p(x'')p(y')-p(x''))/2}(x'', \sigma(y'))(x', \sigma(y''))\\ &\overset{(\dagger)}{=} \sum (-1)^{\mathrm{ht}|x|}q^{-|x|\cdot|y|/2}q_{-|x|}\pi^{(p(x)p(y)+p(x))/2}(x'\otimes x'', \sigma(y'')\otimes\sigma(y'))\\ &= (-1)^{\mathrm{ht}|x|}q^{-|x|\cdot|y|/2}q_{-|x|}\pi^{(p(x)p(y)+p(x))/2}(x, \sigma(y'y''))\end{aligned}$$

where the equality (†) follows from the observation that the nonzero terms in the sum occur only when the each of the pairs $\{x', y''\}$ and $\{x'', y'\}$ are of the same weight and parity. Therefore we see $y \in \mathcal{S}$. Since the algebra generators lie in $\mathcal{S}$, the claim is proved. □

In particular, we observe the following corollary.

Corollary 1.4.4. *Assume the super Cartan datum is consistent. Then* $^{-}$ *descends to an involution on* $\mathbf{f}$*.*

1.5. The maps r_i and ${}_ir$

Let $i \in I$. Clearly there are unique $\mathbb{Q}(q)^{\pi}$-linear maps $r_i, {}_ir : {}'\mathbf{f} \to {}'\mathbf{f}$ such that $r_i(1) = {}_ir(1) = 0$ and $r_i(\theta_j) = {}_ir(\theta_j) = \delta_{ij}$ satisfying

$$\begin{aligned}{}_ir(xy) &= {}_ir(x)y + \pi^{p(x)p(i)}q^{|x|\cdot i}x\,{}_ir(y),\\ r_i(xy) &= \pi^{p(y)p(i)}q^{|y|\cdot i}r_i(x)y + xr_i(y)\end{aligned}$$

for homogeneous $x, y \in {}'\mathbf{f}$; see [K]. We see that if $x \in {}'\mathbf{f}_\nu$, then ${}_ir(x), r_i(x) \in {}'\mathbf{f}_{\nu-i}$ and moreover that

$$r(x) = r_i(x)\otimes\theta_i + \theta_i\otimes{}_ir(x) + (\dots) \tag{1.8}$$

where $(\dots)$ stands in for other bi-homogeneous terms $x'\otimes x''$ with $|x'| \neq i$ and $|x''| \neq i$. Therefore, we have

$$(\theta_i y, x) = (\theta_i, \theta_i)(y, {}_ir(x)), \quad (y\theta_i, x) = (\theta_i, \theta_i)(y, r_i(x)) \tag{1.9}$$

for all $x, y \in {}'\mathbf{f}$, so ${}_ir(\mathcal{I}) \cup r_i(\mathcal{I}) \subseteq \mathcal{I}$. Hence, both maps descend to maps on $\mathbf{f}$. It is also easy to check that

$$r_i\,\sigma = \sigma\,{}_ir.$$

Indeed, this is trivially true for the generators, and if this holds for $x, y \in \mathbf{f}$, then

$$\begin{aligned}r_i\,\sigma(xy) &= r_i(\sigma(y)\sigma(x)) = \pi^{p(i)p(x)}q^{i\cdot|x|}r_i(\sigma(y))\sigma(x) + \sigma(y)r_i(\sigma(x))\\ &= \sigma(\pi^{p(i)p(x)}q^{i\cdot|x|}x\,{}_ir(y) + {}_ir(x)y) = \sigma\,{}_ir(xy).\end{aligned}$$

Lemma 1.5.1. *Assume $(I, \cdot)$ is consistent. For any homogeneous $x \in \mathbf{f}$, we have*

$$r_i(x) = \pi^{p(x)p(i) - p(i)p(i)} q^{|x|\cdot i - i\cdot i} \overline{{}_i r(\overline{x})}.$$

Proof. This is trivial when $x = \theta_i$. Now assume this is true for $x, y \in {}'\mathbf{f}$. Then

$$\begin{aligned}\overline{{}_i r(\overline{xy})} &= \overline{{}_i r(\overline{x})} y + \pi^{p(x)p(i)} (\pi q)^{-|x|\cdot i} \overline{{}_i r(\overline{y})} \\ &= \pi^{-p(x)p(i)+p(i)p(i)} q^{-|x|\cdot i + i\cdot i} r_i(x) y \\ &\quad + \pi^{-p(y)p(i)+p(i)p(i)} q^{-|y|\cdot i + i\cdot i} \pi^{p(x)p(i)} (\pi q)^{-|x|\cdot i} x r_i(y) \\ &= \pi^{-p(x+y)p(i)+p(i)p(i)} q^{-|x+y|\cdot i + i\cdot i} \left(\pi^{p(y)p(i)} q^{|y|\cdot i} r_i(x) y + x r_i(y) \right) \\ &= \pi^{-p(x+y)p(i)+p(i)p(i)} q^{-|x+y|\cdot i + i\cdot i} r_i(xy).\end{aligned}$$

The lemma is proved. □

Lemma 1.5.2. *Let $x \in \mathbf{f}_\nu$ where $\nu \in \mathbb{N}[I]$ is nonzero.*

(a) *If $r_i(x) = 0$ for all $i \in I$, then $x = 0$.*
(b) *If ${}_i r(x) = 0$ for all $i \in I$, then $x = 0$.*

Proof. Suppose that $r_i(x) = 0$ for all i. Using (1.9), this means that $(y\theta_i, x) = 0$ for all $y \in \mathbf{f}$ and all $i \in I$. But since $\mathbf{f}$ is spanned by monomials in the θ_i, this implies $x \in \mathcal{I}$, and so $x = 0$ in $\mathbf{f}$. The proof of (b) proceeds similarly. □

1.6. Gaussian (q, π)-binomial coefficients

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, and let $\mathcal{A}^\pi$ be as in §1.1.3. For $a \in \mathbb{Z}$ and $t \in \mathbb{N}$, we define the (q, π)-*binomial coefficients* to be

$$\begin{bmatrix} a \\ t \end{bmatrix}_i = \frac{\prod_{s=0}^{t-1} ((\pi_i q_i)^{a-s} - q_i^{s-a})}{\prod_{s=1}^{t} ((\pi_i q_i)^s - q_i^{-s})}.$$

We have

$$\begin{bmatrix} a \\ t \end{bmatrix}_i = (-1)^t \pi_i^{ta - \binom{t}{2}} \begin{bmatrix} t-a-1 \\ t \end{bmatrix}_i, \tag{1.10}$$

$$\begin{bmatrix} a \\ t \end{bmatrix}_i = 0 \qquad \text{if } 0 \le a < t, \tag{1.11}$$

$$\prod_{j=0}^{a-1} \left(1 + (\pi_i q_i^2)^j z\right) = \sum_{t=0}^{a} \pi_i^{\binom{t}{2}} q_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_i z^t \quad \text{if } a \ge 0. \tag{1.12}$$

Here z is another indeterminate. From (1.10) and (1.12) we deduce that

$$\begin{bmatrix} a \\ t \end{bmatrix}_i \in \mathcal{A}. \tag{1.13}$$

If a', a'' are integers and $t \in \mathbb{N}$, then

$$\begin{bmatrix} a'+a'' \\ t \end{bmatrix}_i = \sum_{t'+t''=t} \pi_i^{t't''+a't''} q_i^{a't''-a''t'} \begin{bmatrix} a' \\ t' \end{bmatrix}_i \begin{bmatrix} a'' \\ t'' \end{bmatrix}_i . \tag{1.14}$$

We have $\begin{bmatrix} -1 \\ t \end{bmatrix}_i = (-1)^t \pi_i^{\binom{t+1}{2}}$ for any $t \geq 0$, $i \in I$.

For (q,π)-*integers* we shall denote

$$[n]_i = \begin{bmatrix} n \\ 1 \end{bmatrix}_i = \frac{(\pi_i q_i)^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}} \quad \text{for } n \in \mathbb{Z},$$

$$[n]_i^! = \prod_{s=1}^{n} [s]_i \quad \text{for } n \in \mathbb{N},$$

and with this notation we have

$$\begin{bmatrix} a \\ t \end{bmatrix}_i = \frac{[a]_i^!}{[t]_i^! [a-t]_i^!} \quad \text{for } 0 \leq t \leq a.$$

Note that the (q,π)-integers $[n]_i$ and the (q,π)-binomial coefficients in general are not necessarily bar-invariant unless the super Cartan datum is consistent; see (1.5).

If $a \geq 1$, then we have

$$\sum_{t=0}^{a} (-1)^t \pi_i^{\binom{t}{2}} q_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_i = 0 \tag{1.15}$$

which follows from (1.12) by setting $z = -1$.

If x, y are two elements in a $\mathbb{Q}(q)^{\pi}$-algebra such that $xy = \pi_i q_i^2 yx$, then for any $a \geq 0$, we have the quantum binomial formula:

$$(x+y)^a = \sum_{t=0}^{a} q_i^{t(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_i y^t x^{a-t}. \tag{1.16}$$

1.7. Quantum Serre relations

For any $n \in \mathbb{Z}$, let the divided powers $\theta_i^{(n)}$ (in $\mathbf{f}$ or $'\mathbf{f}$) be defined as $\theta_i/[n]_i^!$ if $n \geq 0$ and 0 otherwise.

Lemma 1.7.1. *For any $n \in \mathbb{Z}$ we have*

(a) $r(\theta_i^{(n)}) = \sum_{t+t'=n} q_i^{tt'} \theta_i^{(t)} \otimes \theta_i^{(t')}$,

(b) $\overline{r}(\theta_i^{(n)}) = \sum_{t+t'=n} (\pi_i q_i)^{-tt'} \theta_i^{(t)} \otimes \theta_i^{(t')}$.

Proof. By the quantum binomial formula (1.16) applied to $x = 1 \otimes \theta_i$ and $y = \theta_i \otimes 1$, the formula follows. □

Lemma 1.7.2. *For any $n \geq 0$, we have*

$$(\theta_i^{(n)}, \theta_i^{(n)}) = \prod_{s=1}^{n} \frac{\pi_i^{s-1}}{1 - (\pi_i q_i^{-2})^s} = \pi_i^n q_i^{\binom{n+1}{2}} (\pi_i q_i - q_i^{-1})^{-n} ([n]_i^!)^{-1}.$$

Proof. We prove by induction on n. The lemma is true by definition for $n = 0, 1$. For general n, it follows by Lemma 1.7.1(a) that

$$\begin{aligned}
\big(\theta_i^{(n)}, \theta_i^{(n)}\big) &= [n]_i^{-1}\big(\theta_i^{(n-1)} \otimes \theta_i, r\big(\theta_i^{(n)}\big)\big) \\
&= [n]_i^{-1}\Big(\theta_i^{(n-1)} \otimes \theta_i, \sum_{t+t'=n} q_i^{tt'} \theta_i^{(t)} \otimes \theta_i^{(t')}\Big) \\
&= [n]_i^{-1}\big(\theta_i^{(n-1)} \otimes \theta_i, q_i^{n-1}\theta_i^{(n-1)} \otimes \theta_i\big) \\
&= q_i^{n-1}[n]_i^{-1}(\theta_i, \theta_i)\big(\theta_i^{(n-1)}, \theta_i^{(n-1)}\big).
\end{aligned}$$

Hence by the induction hypothesis, we have

$$\begin{aligned}
\big(\theta_i^{(n)}, \theta_i^{(n)}\big) &= q_i^{n-1}[n]_i^{-1}(1 - \pi_i q_i^{-2})^{-1}\pi_i^{n-1} q_i^{\binom{n}{2}}(\pi_i q_i - q_i^{-1})^{-n+1}([n-1]_i^!)^{-1} \\
&= \pi_i^n q_i^{\binom{n+1}{2}}(\pi_i q_i - q_i^{-1})^{-n}([n]_i^!)^{-1}.
\end{aligned}$$

The lemma is proved. □

Proposition 1.7.3 (Quantum Serre relation). *The generators θ_i of* $\mathbf{f}$ *satisfy the relations*

$$\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0$$

for any $i \neq j$ in I.

Proposition 1.7.3 appeared as [HW, Thm. 3.8]. We shall give a new and simpler proof of Proposition 1.7.3 below after some preparation.

Lemma 1.7.4. *Let $N \in \mathbb{N}$ and $a, a' \in \mathbb{N}$ with $N = a + a'$. Let $i, j, k \in I$ be pairwise distinct. Then*

(a) $r_k(\theta_i^{(a)}\theta_j\theta_i^{(a')}) = 0,$

(b) $r_j(\theta_i^{(a)}\theta_j\theta_i^{(a')}) = q_i^{a'\langle i,j\rangle}\pi_i^{a'p(j)} \begin{bmatrix} N \\ a' \end{bmatrix}_i \theta_i^{(N)},$

(c) $r_i(\theta_i^{(a)}\theta_j\theta_i^{(a')}) = q_i^{a'+(N+\langle i,j\rangle-1)}\pi_i^{a'+p(j)}\theta_i^{(a-1)}\theta_j\theta_i^{(a')} + q_i^{a'-1}\theta_i^{(a)}\theta_j\theta_i^{(a'-1)}.$

Proof. Part (a) is clear from definitions. By (1.8) and Lemma 1.7.1(a) we have

$$r_{i'}(\theta_{j'}^{(a)}) = \delta_{i',j'} q_{i'}^{a-1}\theta_{i'}^{(a-1)}.$$

Parts (b) and (c) follow from this and noting

$$r_i(cba) = cbr_i(a) + \pi^{p(i)p(a)}q^{i\cdot|a|}cr_i(b)a + \pi^{p(i)p(a)+p(i)p(b)}q^{i\cdot|a|+i\cdot|b|}r_i(c)ba.$$

The lemma is proved. □

Proof of Proposition 1.7.3. Let $N = 1 - \langle i, j'\rangle$. By the previous lemma, we have

$$r_k\Big(\sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}\theta_i^{(n)}\theta_j\theta_i^{(n')}\Big) = 0, \quad \text{for } k \neq i, j.$$

In addition, we have

$$\begin{aligned} r_j\Big(&\sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}\theta_i^{(n)}\theta_j\theta_i^{(n')}\Big) \\ &= \sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}q_i^{n'\langle i,j\rangle}\pi_i^{n'p(j)}\begin{bmatrix} N \\ n'\end{bmatrix}_i\theta_i^{(N)} \\ &= \theta_i^{(N)}\sum_{t=0}^{N}(-1)^t\pi_i^{\binom{t}{2}}(q_i)^{t(1-N)}\begin{bmatrix} N \\ t\end{bmatrix}_i. \end{aligned}$$

By Condition 1.1(e), $1 - N \in 2\mathbb{Z}$ if i is odd, so in any case, the right-hand side of the last equation is

$$\theta_i^{(N)}\sum_{t=0}^{N}(-1)^t\pi_i^{\binom{t}{2}}(\pi_i q_i^{-1})^{t(N-1)}\begin{bmatrix} N \\ t\end{bmatrix}_i = 0,$$

where the last equality follows from (1.15). Finally,

$$\begin{aligned} r_i\Big(&\sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}\theta_i^{(n)}\theta_j\theta_i^{(n')}\Big) \\ &= \sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}q_i^{n'}\pi_i^{n'+p(j)}\theta_i^{(n-1)}\theta_j\theta_i^{(n')} \\ &\quad + \sum_{n+n'=N}(-1)^{n'}\pi_i^{n'p(j)+\binom{n'}{2}}q_i^{n'-1}\theta_i^{(n)}\theta_j\theta_i^{(n'-1)} \\ &= \sum_{t=0}^{N-1}(-1)^t\pi_i^{tp(j)+p(j)+\binom{t+1}{2}}q_i^t\theta_i^{(N-1-t)}\theta_j\theta_i^{(t)} \\ &\quad - \sum_{t=0}^{N-1}(-1)^t\pi_i^{(t+1)p(j)+\binom{t+1}{2}}q_i^t\theta_i^{(N-1-t)}\theta_j\theta_i^{(t)} \\ &= 0. \end{aligned}$$

Now Proposition 1.7.3 follows by Lemma 1.5.2. □

Note that the bar map $\bar{\ }$ on $\mathbf{f}$ may not be well-defined when the datum is not consistent. For example, consider the case $(I, \cdot)$ has $i, j \in I_{\bar{0}}$ with $i \cdot j = -1$, hence $d_i = d_j = 1$. Then the calculations above hold; that is, $s(\theta_i, \theta_j) := \theta_i^{(2)}\theta_j - \theta_i\theta_j\theta_i + \theta_j\theta_i^{(2)} = 0$; however, since $\overline{[2]_i} = \pi[2]_i$, it is easy to see that $\overline{s(\theta_i, \theta_j)} \notin \mathfrak{I}$.

Let ${}_{\mathcal{A}}\mathbf{f}$ be $\mathcal{A}^\pi$-subalgebra of $\mathbf{f}$ generated by the elements $\theta_i^{(s)}$ for various $i \in I$ and $s \in \mathbb{Z}$. Since the generators $\theta_i^{(s)}$ are homogeneous, we have ${}_{\mathcal{A}}\mathbf{f} = \bigoplus_\nu {}_{\mathcal{A}}\mathbf{f}_\nu$ where ν runs over $\mathbb{N}[I]$ and ${}_{\mathcal{A}}\mathbf{f}_\nu = {}_{\mathcal{A}}\mathbf{f} \cap \mathbf{f}_\nu$.

2. The quantum covering and super groups

In this section we give the definition of the quantum covering group $\mathbf{U}$ as a Hopf superalgebra, which specializes at $\pi = -1$ to a new variant of a quantum supergroup. We show that $\mathbf{U}$ admits a triangular decomposition $\mathbf{U} = \mathbf{U}^-\mathbf{U}^0\mathbf{U}^+$ with positive/negative parts isomorphic to the algebra $\mathbf{f}$. The novelty here is that $\mathbf{U}^0$ contains some new generators $J_i (i \in I)$ which allow us to construct integrable modules in full generality.

2.1. The algebras $'\mathbf{U}$ and $\mathbf{U}$

Assume that a root datum $(Y, X, \langle\, ,\, \rangle)$ of type $(I, \cdot)$ is given. Consider the associative $\mathbb{Q}(q)^\pi$-superalgebra $'\mathbf{U}$ (with 1) defined by the generators

$$E_i \ \ (i \in I), \quad F_i \ \ (i \in I), \quad J_\mu \ \ (\mu \in Y), \quad K_\mu \ \ (\mu \in Y),$$

where the parity is given by $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$, subject to the relations (a)-(f) below for all $i, j \in I, \mu, \mu' \in Y$:

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{a}$$

$$J_{2\mu} = 1, \quad J_\mu J_{\mu'} = J_{\mu+\mu'}, \tag{b}$$

$$J_\mu K_{\mu'} = K_{\mu'} J_\mu, \tag{c}$$

$$K_\mu E_i = q^{\langle \mu, i' \rangle} E_i K_\mu, \quad J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu, \tag{d}$$

$$K_\mu F_i = q^{-\langle \mu, i' \rangle} F_i K_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{e}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{i,j} \frac{\widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i}}{\pi_i q_i - q_i^{-1}}, \tag{f}$$

where for any element $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$ we have set $\widetilde{K}_\nu = \prod_i K_{d_i \nu_i i}$, $\widetilde{J}_\nu = \prod_i J_{d_i \nu_i i}$. In particular, $\widetilde{K}_i = K_{d_i i}$, $\widetilde{J}_i = J_{d_i i}$. (Under Condition 1.1(e), $\widetilde{J}_i = 1$ for $i \in I_{\overline{0}}$ while $\widetilde{J}_i = J_i$ for $i \in I_{\overline{1}}$.)

We also consider the associative $\mathbb{Q}(q)^\pi$-algebra $\mathbf{U}$ (with 1) defined by the generators

$$E_i \ \ (i \in I), \quad F_i \ \ (i \in I), \quad J_\mu \ \ (\mu \in Y), \quad K_\mu \ \ (\mu \in Y)$$

and the relations (a)-(f) above, together with the additional relations

$$\text{for any } \ f(\theta_i : i \in I) \in \mathfrak{I},\ f(E_i : i \in I) = f(F_i : i \in I) = 0. \tag{g}$$

The algebra $\mathbf{U}$ will be called the *quantum covering group* of type $(I, \cdot)$.

From (g), we see that there are well-defined algebra homomorphisms $\mathbf{f} \to \mathbf{U}$, $x \mapsto x^+$ (with image denoted by $\mathbf{U}^+$) and $\mathbf{f} \to \mathbf{U}$, $x \mapsto x^-$ (with image denoted by $\mathbf{U}^-$) such that $E_i = \theta_i^+$ and $F_i = \theta_i^-$ for all $i \in I$. Clearly, there are well defined algebra homomorphisms $'\mathbf{f} \to {}'\mathbf{U}$ with the aforementioned properties.

(In terms of standard notations used in some other quantum group literature, it is understood that $K_\mu = q^\mu$ and $K_i = q^{h_i}$. It is instructive to see our new generators J's can be understood in the same vein as $J_\mu = \pi^\mu$ and $J_i = \pi^{h_i}$.)

For any $p \geq 0$, we set $E_i^{(p)} = (\theta_i^{(p)})^+$ and $F_i^{(p)} = (\theta_i^{(p)})^-$.

Example 2.1.1. In the case $I = I_{\bar{1}} = \{I\}$, we can identify $Y = X = \mathbb{Z}$ with $i = 1 \in Y$, $i' = 2 \in X$, and $\langle \mu, \lambda \rangle = \mu\lambda$. Then $\mathbf{U}$ is the $\mathbb{Q}(q)^{\pi}$-algebra generated by E,F,K,J such that

$$JK = KJ, \quad JE = EJ, \quad JF = FJ, \quad J^2 = 1,$$
$$KEK = q^2 E, \quad KFK = q^{-2} F,$$
$$EF - \pi FE = \frac{JK - K^{-1}}{\pi q - q^{-1}}.$$

Note that the quotient algebras $\mathbf{U}/((J \pm 1)\mathbf{U})$ are isomorphic to the two variants of the quantum group $\mathbf{U}_q(\mathfrak{osp}(1|2))$ defined in [CW].

2.2. Properties of U

By inspection, there is a unique algebra automorphism (of order 4) $\omega : {}'\mathbf{U} \to {}'\mathbf{U}$ such that

$$\omega(E_i) = \pi_i \widetilde{J}_i F_i, \quad \omega(F_i) = E_i, \quad \omega(K_\mu) = K_{-\mu}, \quad \omega(J_\mu) = J_\mu$$

for $i \in I$, $\mu \in Y$. We have $\omega(x^+) = \pi_{|x|} \tilde{J}_{|x|} x^-$ and $\omega(x^-) = x^+$ for all $x \in \mathbf{f}$, and thus the same formula defines a unique algebra automorphism $\omega : \mathbf{U} \to \mathbf{U}$.

Similarly, there is a unique isomorphism of $\mathbb{Q}(q)^{\pi}$-vector spaces $\sigma : {}'\mathbf{U} \to {}'\mathbf{U}$ such that

$$\sigma(E_i) = E_i, \quad \sigma(F_i) = \pi_i \tilde{J}_i F_i, \quad \sigma(K_\mu) = K_{-\mu}, \quad \sigma(J_\mu) = J_\mu$$

for $i \in I$, $\mu \in Y$ such that $\sigma(uu') = \sigma(u')\sigma(u)$ for $u, u' \in \mathbf{U}$. We have

$$\sigma(x^+) = \sigma(x)^+, \qquad \sigma(x^-) = \pi_{|x|} \tilde{J}_{|x|} \sigma(x)^-, \quad \forall x \in \mathbf{f}. \tag{2.1}$$

Again, this implies that the same formula defines a unique algebra automorphism $\sigma : \mathbf{U} \to \mathbf{U}$. Note that σ on $\mathbf{U}^+$ matches exactly σ on $\mathbf{f}$, but σ on $\mathbf{U}^-$ looks quite different from σ on $\mathbf{f}$ (in contrast to the quantum group setting [Lu]).

Lemma 2.2.1 (Comultiplication**).** *There is a unique algebra homomorphism* $\Delta : {}'\mathbf{U} \to {}'\mathbf{U} \otimes {}'\mathbf{U}$ (*resp.* $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$) *where* ${}'\mathbf{U} \otimes {}'\mathbf{U}$ (*resp.* $\mathbf{U} \otimes \mathbf{U}$) *is regarded as a superalgebra in the standard way, defined by*

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + \widetilde{J}_i \widetilde{K}_i \otimes E_i \quad (i \in I), \\
\Delta(F_i) &= F_i \otimes \widetilde{K}_{-i} + 1 \otimes F_i \quad (i \in I), \\
\Delta(K_\mu) &= K_\mu \otimes K_\mu \quad (\mu \in Y), \\
\Delta(J_\mu) &= J_\mu \otimes J_\mu \quad (\mu \in Y).
\end{aligned}$$

Proof. The relations 2.1 (a)-(c) are trivial to verify. For the relation (d), we have

$$\Delta(E_i)\Delta(F_j) = E_i F_j \otimes \widetilde{K}_{-j} + \widetilde{J}_i \widetilde{K}_i \otimes E_i F_i + E_i \otimes F_j + \pi^{p(i)p(j)} \widetilde{J}_i \widetilde{K}_i F_j \otimes E_i \widetilde{K}_{-j},$$
$$\Delta(F_j)\Delta(E_i) = F_i E_j \otimes \widetilde{K}_{-j} + \widetilde{J}_i \widetilde{K}_i \otimes F_j E_i + \pi^{p(i)p(j)} E_i \otimes F_j + F_j \widetilde{J}_i \widetilde{K}_i \otimes \widetilde{K}_{-j} E_i.$$

So using the fact that $F_j\widetilde{K}_i \otimes \widetilde{K}_{-j}E_i = \widetilde{K}_iF_j \otimes E_i\widetilde{K}_{-j}$, we have

$$\begin{aligned}\Delta(E_i)\Delta(F_j) &- \pi^{p(i)p(j)}\Delta(F_j)\Delta(E_i)\\ &= (E_iF_j - \pi^{p(i)p(j)}F_jE_i) \otimes \widetilde{K}_{-j} + \widetilde{J}_i\widetilde{K}_i \otimes (E_iF_j - \pi^{p(i)p(j)}F_jE_i)\\ &= \delta_{i,j}\left(\frac{\widetilde{J}_i\widetilde{K}_i - \widetilde{K}_{-i}}{\pi_iq_i - q_i^{-1}}\right) \otimes \widetilde{K}_{-j} + \widetilde{J}_i\widetilde{K}_i \otimes \left(\delta_{i,j}\frac{\widetilde{J}_i\widetilde{K}_i - \widetilde{K}_{-i}}{\pi_iq_i - q_i^{-1}}\right)\\ &= \delta_{i,j}\frac{\Delta(\widetilde{J}_i)\Delta(\widetilde{K}_i) - \Delta(\widetilde{K}_{-i})}{\pi_iq_i - q_i^{-1}}.\end{aligned}$$

Finally, define maps $j^\pm : {}'\mathbf{f} \otimes {}'\mathbf{f} \to {}'\mathbf{U} \otimes {}'\mathbf{U}$ given by

$$j^+(x \otimes y) = x^+\widetilde{J}_{|y|}\widetilde{K}_{|y|} \otimes y^+, \qquad j^-(x \otimes y) = x^- \otimes \widetilde{K}_{-|x|}y^-.$$

Then by construction, these maps are algebra homomorphisms, and satisfy

$$j^+r(x) = \Delta(x^+), \quad j^-\overline{r}(x) = \Delta(x^-).$$

Since $r, \overline{r}$ factor through $\mathbf{f}$, so do j^+r and $j^-\overline{r}$, implying that

$$f(\Delta(E_i)) = f(\Delta(F_i)) = 0$$

for all $f(\theta_i : i \in I) \in \mathcal{I}$. $\square$

The previous proof shows that $j^+r(x) = \Delta(x^+)$ and $j^-\overline{r}(x) = \Delta(x^-)$, so in particular, we have

$$\begin{aligned}\Delta(x^+) &= \sum x_1^+\widetilde{J}_{|x_2|}\widetilde{K}_{|x_2|} \otimes x_2^+,\\ \Delta(x^-) &= \sum \pi^{p(x_1)p(x_2)}(\pi q)^{-|x_1|\cdot|x_2|}x_2^- \otimes \widetilde{K}_{-|x_2|}x_1^-,\end{aligned}$$

for $r(x) = \sum x_1 \otimes x_2$. In particular, this yields the formulas

$$\begin{aligned}\Delta(E_i^{(p)}) &= \sum_{p'+p''=p} q_i^{p'p''}\widetilde{J}_i^{p''}E_i^{(p')}\widetilde{K}_i^{p''} \otimes E_i^{(p'')},\\ \Delta(F_i^{(p)}) &= \sum_{p'+p''=p} (\pi_iq_i)^{-p'p''}F_i^{(p')} \otimes \widetilde{K}_i^{-p'}F_i^{(p'')}.\end{aligned}$$

Proposition 2.2.2. *For $x \in {}'\mathbf{f}$ and $i \in I$, we have (in ${}'\mathbf{U}$)*

(a) $x^+F_i - \pi_i^{p(x)}F_ix^+ = \dfrac{r_i(x)^+\widetilde{J}_i\widetilde{K}_i - \widetilde{K}_{-i}\,\pi_i^{p(x)-p(i)}\,{}_ir(x)^+}{\pi_iq_i - q_i^{-1}}$,

(b) $E_ix^- - \pi_i^{p(x)}x^-E_i = \dfrac{\widetilde{J}_i\widetilde{K}_i\,{}_ir(x)^- - \pi_i^{p(x)-p(i)}r_i(x)^-\widetilde{K}_{-i}}{\pi_iq_i - q_i^{-1}}$.

Proof. Assume that (a) is known for x' and x''; we shall show it holds for $x = x'x''$. Let $y' = (x')^+$, ${}_iy' = {}_ir(x')^+$ and similarly for r_i, x'', and x.

$$\begin{aligned}
yF_i &= \pi_i^{p(x'')} y' F_i y'' + \frac{y' y_i'' \widetilde{J}_i \widetilde{K}_i - y' \widetilde{K}_{-i}\ \pi_i^{p(x'')-p(i)}\ {}_iy''}{\pi_i q_i - q_i^{-1}} \\
&= \pi_i^{p(x'x'')} F_i y + \pi_i^{p(x'')} \frac{y_i' \widetilde{J}_i \widetilde{K}_i\, y'' - \widetilde{K}_{-i} \pi_i^{p(x')-p(i)}\ {}_iy' y''}{\pi_i q_i - q_i^{-1}} \\
&\quad + \frac{y'\, y_i'' \widetilde{J}_i \widetilde{K}_i - y' \widetilde{K}_{-i} \pi_i^{p(x'')-p(i)}\ {}_iy''}{\pi_i q_i - q_i^{-1}} \\
&= \pi^{p(x'x'')p(i)} F_i\, y + \frac{y_i \widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i} \pi_i^{p(x)-p(i)}\ {}_iy}{\pi_i q_i - q_i^{-1}}.
\end{aligned}$$

Since (a) holds for the generators, it holds for all $x \in {}'\mathbf{f}$.

If we apply ω^{-1}, we obtain

$$\pi_i \widetilde{J}_i x^- E_i - \pi_i^{p(x)-p(i)} \widetilde{J}_i E_i x^- = \frac{r_i(x)^- \widetilde{J}_i \widetilde{K}_{-i} - \widetilde{K}_i\ \pi_i^{p(x)-p(i)}\ {}_ir(x)^-}{\pi_i q_i - q_i^{-1}},$$

and multiplying both sides by $\pi_i^{p(x)-p(i)} \widetilde{J}_i$ establishes (b). $\square$

We record the following formulas for further use.

Lemma 2.2.3 ([CW, Lemma 2.8]**).** *For any $N, M \geq 0$ we have in* $\mathbf{U}$ *or* ${}'\mathbf{U}$

$$\begin{aligned}
E_i^{(N)} F_i^{(M)} &= \sum_t \pi_i^{MN - \binom{t+1}{2}} F_i^{(M-t)} \begin{bmatrix} \widetilde{K}_i; 2t - M - N \\ t \end{bmatrix}_i E_i^{(N-t)}, \\
F_i^{(N)} E_i^{(M)} &= \sum_t (-1)^t \pi_i^{(M-t)(N-t)-t^2} E_i^{(M-t)} \begin{bmatrix} \widetilde{K}_i; M + N - (t+1) \\ t \end{bmatrix}_i F_i^{(N-t)}, \\
E_i^{(N)} F_j^{(M)} &= \pi^{MNp(i)p(j)} F_j^{(M)} E_i^{(N)} \quad \textit{if } i \neq j,
\end{aligned}$$

where

$$\begin{bmatrix} \widetilde{K}_i; a \\ t \end{bmatrix}_i = \prod_{s=1}^{t} \frac{(\pi_i q_i)^{a-s+1} \widetilde{J}_i \widetilde{K}_i - q_i^{s-a-1} \widetilde{K}_{-i}}{(\pi_i q_i)^s - q_i^{-s}}.$$

The coproduct Δ is coassociative; the verification is the same as in the non-super case. There is a unique algebra homomorphism $e : \mathbf{U} \to \mathbb{Q}(q)^\pi$ satisfying $e(E_i) = e(F_i) = 0$ and $e(J_\mu) = e(K_\mu) = 1$ for all i, μ.

Recall the bar involution $^-$ on $\mathbb{Q}(q)^\pi$ from (1.5). This extends to a unique homomorphism of $\mathbb{Q}$-algebras $^- : \mathbf{U} \to \mathbf{U}$ such that

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{J_\mu} = J_\mu, \quad \overline{K_\mu} = J_\mu K_{-\mu},$$

and $\overline{fx} = \overline{f}\overline{x}$ for all $f \in \mathbb{Q}(q)^\pi$ and $x \in \mathbf{U}$.

Let ${}_\mathcal{A}\mathbf{U}^\pm$ be the images of ${}_\mathcal{A}\mathbf{f}$ defined at the end of §1.7. We define ${}_\mathcal{A}\mathbf{U}$ to be the $\mathcal{A}^\pi$-subalgebra of $\mathbf{U}$ generated by $E_i^{(t)}$, $F_i^{(t)}$, $\begin{bmatrix} K_i; a \\ t \end{bmatrix}_i$, J_μ and K_μ, for all $i \in I$, $\mu \in Y$, and positive integers $a \geq t$.

2.3. Triangular decompositions for $'\mathbf{U}$ and $\mathbf{U}$

If M', M are two $'\mathbf{U}$-modules, then $M' \otimes M$ is naturally a $'\mathbf{U} \otimes {}'\mathbf{U}$-module; hence by restriction to $'\mathbf{U}$ under Δ, it is a $'\mathbf{U}$-module.

Lemma 2.3.1. *Let $\lambda \in X$. There is a unique $'\mathbf{U}$-module structure on the $\mathbb{Q}(q)^\pi$-module $'\mathbf{f}$ such that for any homogeneous $z \in {}'\mathbf{f}$, and $\mu \in Y$ and any $i \in I$, we have*

$$K_\mu \cdot z = q^{\langle \mu, \lambda - |z| \rangle} z, \quad J_\mu \cdot z = \pi^{\langle \mu, \lambda - |z| \rangle} z, \quad F_i \cdot z = \theta_i z, \quad E_i \cdot 1 = 0.$$

Proof. The uniqueness is immediate. To prove the existence, define

$$E_i \cdot z = \frac{-q_i^{\langle i, \lambda \rangle} r_i(z) + \pi_i^{p(z)-p(i)} (\pi_i q_i)^{\langle i, \lambda - |z| + i' \rangle} {}_i r(z)}{\pi_i q_i - q_i^{-1}}.$$

Note that this is essentially the formula prescribed by Proposition 2.2.2. A straightforward computation shows that this, along with the desired formulas for the F and K actions define a $'\mathbf{U}$-module structure on $'\mathbf{f}$. □

We denote this $'\mathbf{U}$-module by $\mathbf{M}_\lambda$ (which is a free $\mathbb{Q}(q)^\pi$-module). Similarly, to an element $\lambda \in X$, we associate a unique $'\mathbf{U}$-module structure on $'\mathbf{f}$ such that for any homogeneous $z \in {}'\mathbf{f}$, any $\mu \in Y$ and any $i \in I$ we have

$$K_\mu \cdot z = q^{\langle \mu, -\lambda + |z| \rangle} z, \quad J_\mu \cdot z = \pi^{\langle \mu, -\lambda + |z| \rangle} z, \quad E_i \cdot z = \theta_i z, \quad F_i \cdot 1 = 0.$$

We denote this $'\mathbf{U}$-module by $\mathbf{M}'_\lambda$ (which is again a free $\mathbb{Q}(q)^\pi$-module). We form the $'\mathbf{U}$-module $\mathbf{M}'_\lambda \otimes \mathbf{M}_\lambda$; we denote the unit element of $'\mathbf{f} = \mathbf{M}_\lambda$ by 1 and that of $'\mathbf{f} = \mathbf{M}'_\lambda$ by $1'$. Thus, we have the canonical element $1' \otimes 1 \in \mathbf{M}'_\lambda \otimes \mathbf{M}_\lambda$. We emphasize that $\mathbf{M}'_\lambda \otimes \mathbf{M}_\lambda$ is again free as a $\mathbb{Q}(q)^\pi$-module.

Proposition 2.3.2. *Let $\mathbf{U}^0$ be the associative $\mathbb{Q}(q)^\pi$-algebra with 1 defined by the generators K_μ, J_μ ($\mu \in Y$) and the relations in* §2.1(a),(b). *Then $\mathbf{U}^0$ is isomorphic to the group algebra of $Y \times (Y/2Y)$ over $\mathbb{Q}(q)^\pi$. Moreover,*

(a) *The $\mathbb{Q}(q)^\pi$-linear map $'\mathbf{f} \otimes \mathbf{U}^0 \otimes {}'\mathbf{f} \to {}'\mathbf{U}$ given by $u \otimes J_\nu K_\mu \otimes w \mapsto u^- J_\nu K_\mu w^+$ is an isomorphism.*

(b) *The $\mathbb{Q}(q)^\pi$-linear map $'\mathbf{f} \otimes \mathbf{U}^0 \otimes {}'\mathbf{f} \to {}'\mathbf{U}$ given by $u \otimes J_\nu K_\mu \otimes w \mapsto u^+ J_\nu K_\mu w^-$ is an isomorphism.*

Proof. Note that (b) follows from (a) by applying ω. As a $\mathbb{Q}(q)^\pi$-module, $'\mathbf{U}$ is spanned by words in the E_i, F_i, K_μ, and J_μ. By using the defining relations, we can rewrite any word as a linear combination of words where the F_i come before the J_μ and K_μ, which come before the E_i, thus the given map is surjective.

To prove the map is injective, let $\lambda, \lambda' \in X$, and consider the module $\mathbf{M}'_{\lambda'} \otimes \mathbf{M}_\lambda$ described before. There is a $\mathbb{Q}(q)^\pi$-linear map $\phi : {}'\mathbf{U} \to \mathbf{M}'_{\lambda'} \otimes \mathbf{M}_\lambda$ given by $\phi(u) = u \cdot 1' \otimes 1$. Pick a $\mathbb{Q}(q)^\pi$-basis of $'\mathbf{f}$ consisting of homogeneous elements containing 1. Assume that in $'\mathbf{U}$ there is some relation of the form $\sum_{b',\mu,b} c_{b',\mu,b} b'^- J_\nu K_\mu b^+ = 0$ and let N be the largest integer such that $\mathrm{ht}|b'| = N$ and $c_{b',\mu,b} \neq 0$ for some μ, b.

Then

$$0=\phi\Big(\sum_{b',\mu,\nu,b} c_{b',\mu,\nu,b}b'^{-}J_\nu K_\mu b^+\Big)=\sum_{b',\mu,\nu,b} c_{b',\mu,\nu,b}\Delta(b'^{-}J_\nu K_\mu b^+)\cdot 1\otimes 1.$$

Now

$$\Delta(b'^{-})=\sum_{b'_1,b'_2} g'(b',b'_1,b'_2)b_1'^{-}\otimes \widetilde{K}_{-|b'_1|}b_2'^{-},$$

$$\Delta(b^{+})=\sum_{b_1,b_2} g(b,b_1,b_2)b_1^{+}\widetilde{J}_{|b_2|}\widetilde{K}_{|b_2|}\otimes b_2^{+},$$

so we have

$$0=\sum \pi^{p(b'_2)p(b_1)}c_{b',\mu,\nu,b}g(b,b_1,b_2)g'(b',b'_1,b'_2)b_1'^{-} \\ \times J_\nu K_\mu b_1^+\widetilde{J}_{|b_2|}\widetilde{K}_{|b_2|}\cdot 1'\otimes \widetilde{K}_{-|b'_1|}b_2'^{-}J_\nu K_\mu b_2^+\cdot 1.$$

If $b_2\neq 1$, then $b_2^+\cdot 1=0$ so we must have $b_2=1$ and thus $b_1=b$. Therefore the expression reduces to

$$0=\sum \pi^{p(b'_2)p(b)}c_{b',\mu,\nu,b}g'(b',b'_1,b'_2)b_1'^{-}J_\nu K_\mu b^+\cdot 1'\otimes \widetilde{K}_{-|b'_1|}b_2'^{-}J_\nu K_\mu\cdot 1.$$

By the definition of the module structure, this becomes

$$0=\sum \pi^{p(b'_2)p(b)}c_{b',\mu,\nu,b}g'(b',b'_1,b'_2)\pi^{\langle \nu,\lambda-\lambda'+|b|\rangle}q^{\langle \mu,\lambda-\lambda'+|b|\rangle}b_1'^{-}\cdot b\otimes \widetilde{K}_{-|b'_1|}b'_2.$$

We can now project this equality onto the summand $\mathbf{M}'_{\lambda'}\otimes {}'\mathbf{f}_\nu$ where $\mathrm{ht}\,\nu=N$. Then by construction, $|b'_2|\leq |b|$ and $\mathrm{ht}|b'_2|=N$. Since $c_{b',\mu,b}=0$ if $\mathrm{ht}|b'|>N$, we must have $|b|=|b'_2|$ and thus $b'=b'_2$, $b'_1=1$, so

$$\sum \pi^{p(b')p(b)}c_{b',\mu,\nu,b}\pi^{\langle \nu,\lambda-\lambda'+|b|\rangle}q^{\langle \mu,\lambda-\lambda'+|b|\rangle}b\otimes b'=0.$$

It follows that

$$\sum_{\nu,\mu} c_{b',\mu,\nu,b}\pi^{\langle \nu,\lambda-\lambda'+|b|\rangle}q^{\langle \mu,\lambda-\lambda'+|b|\rangle}=0$$

for all choices of λ,λ',μ,b and b' with $\mathrm{ht}|b'|=N$. Therefore $c_{b',\mu,\nu,b}=0$ for any b' with $\mathrm{ht}|b'|=N$, contradicting the choice of N. □

Corollary 2.3.3.

(a) *The $\mathbb{Q}(q)^\pi$-linear map $\mathbf{f}\otimes\mathbf{U}^0\otimes\mathbf{f}\to\mathbf{U}$ given by $u\otimes J_\nu K_\mu\otimes w\mapsto u^-J_\nu K_\mu w^+$ is an isomorphism.*

(b) *The $\mathbb{Q}(q)^\pi$-linear map $\mathbf{f}\otimes\mathbf{U}^0\otimes\mathbf{f}\to\mathbf{U}$ given by $u\otimes K_\mu\otimes w\mapsto u^+J_\nu K_\mu w^-$ is an isomorphism.*

Proof. Once again (b) follows from (a) by applying the involution ω. Let $J_\pm$ be the two-sided ideal of $'\mathbf{U}$ generated by $\mathcal{I}^\pm = \{x^\pm : x \in \mathcal{I}\}$. Then $\mathbf{U} = {'\mathbf{U}}/(J_+ + J_-)$. Now from Proposition 2.2.2 iterated, we see that

$$('\mathbf{U}^+)\mathcal{I}^- \subseteq \mathcal{I}^-\mathbf{U}^0('\mathbf{U}^+); \qquad \mathcal{I}^+('\mathbf{U}^-) \subseteq ('\mathbf{U}^-)\mathbf{U}^0\mathcal{I}^+.$$

Using the triangular decomposition of $'\mathbf{U}$, we have $J_- = {'\mathbf{U}}\mathcal{I}^-{'\mathbf{U}} \subseteq \mathcal{I}^-\mathbf{U}^0('\mathbf{U}^+) \subseteq J_-$, hence $J_- = \mathcal{I}^-\mathbf{U}^0('\mathbf{U}^+)$. Similarly, $J_+ = ('\mathbf{U}^-)\mathbf{U}^0\mathcal{I}^+$. Therefore,

$$\mathbf{U} = \frac{'\mathbf{U}^- \otimes \mathbf{U}^0 \otimes {'\mathbf{U}^+}}{'\mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathcal{I}^+ + \mathcal{I}^- \otimes \mathbf{U}^0 \otimes {'\mathbf{U}^+}} = \frac{'\mathbf{U}^-}{\mathcal{I}^-} \otimes \mathbf{U}^0 \otimes \frac{'\mathbf{U}^+}{\mathcal{I}^+},$$

from which (a) follows. □

Corollary 2.3.4. *The maps* $^\pm : \mathbf{f} \to \mathbf{U}^\pm$, $x \mapsto x^\pm$, *are* $\mathbb{Q}(q)^\pi$*-algebra isomorphisms, and* $\mathbf{U}^0 \to \mathbf{U}$ *is a* $\mathbb{Q}(q)^\pi$*-algebra embedding.*

For $\nu \in \mathbb{N}[I]$, we shall denote the image $\mathbf{f}_\nu^\pm$ by $\mathbf{U}_\nu^\pm$.

Proposition 2.3.5. *Let* $x \in \mathbf{f}_\nu$ *where* $\nu \in \mathbb{N}[I]$ *is nonzero.*

(a) *If* $x^+F_i = \pi_i^{p(x)}F_ix^+$ *for all* $i \in I$ *then* $x = 0$.
(b) *If* $x^-E_i = \pi_i^{p(x)}E_ix^-$ *for all* $i \in I$ *then* $x = 0$.

Proof. It follows from Proposition 2.2.2 and the linear independence of $r_i(x)^+\widetilde{J}_i\widetilde{K}_i$ (respectively, the linear independence of $\widetilde{J}_i\widetilde{K}_{-i}\,{}_ir(x)^+$) that $r_i(x)^+ = {}_ir(x)^+ = 0$ for all i. Hence $x = 0$ by Lemma 1.5.2. □

2.4. Antipode

For $\nu \in \mathbb{N}[I]$, write $\nu = \sum_i \nu_i i$ and $\nu = \sum_{a=1}^{\mathrm{ht}\nu} i_a$ for $i_a \in I$. Then we set

$$c(\nu) = \nu \cdot \nu/2 - \sum_i \nu_i i \cdot i/2 \in \mathbb{Z},$$

$$e(\nu) = \sum_{a<b} p(i_a)p(i_b) \in \mathbb{Z}.$$

Lemma 2.4.1. *Let* $\nu \in \mathbb{N}[I]$.

(a) *There is a unique* $\mathbb{Q}(q)^\pi$*-linear map* $S : \mathbf{U} \to \mathbf{U}$ *such that*

$$S(E_i) = -\widetilde{J}_{-i}\widetilde{K}_{-i}E_i, \quad S(F_i) = -F_i\widetilde{K}_i, \quad S(K_\mu) = K_{-\mu}, \quad S(J_\nu) = J_{-\nu},$$

and $S(xy) = \pi^{p(x)p(y)}S(y)S(x)$ *for all* $x, y \in \mathbf{U}$.

(b) *For any* $x \in \mathbf{f}_\nu$, *we have*

$$S(x^+) = (-1)^{\mathrm{ht}\nu}\pi^{e(\nu)}(\pi q)^{c(\nu)}\widetilde{J}_{-\nu}\widetilde{K}_{-\nu}\sigma(x)^+,$$
$$S(x^-) = (-1)^{\mathrm{ht}\nu}\pi^{e(\nu)}q^{-c(\nu)}\sigma(x)^-\widetilde{K}_\nu.$$

(c) *There is a unique* $\mathbb{Q}(q)^\pi$*-linear map* $S' : \mathbf{U} \to \mathbf{U}$ *such that*

$$S'(E_i) = -E_i\widetilde{J}_{-i}\widetilde{K}_{-i}, \quad S'(F_i) = -\widetilde{K}_iF_i, \quad S'(K_\mu) = K_{-\mu}, \quad S(J_\nu) = J_{-\nu},$$

and $S'(xy) = \pi^{p(x)p(y)} S'(y)S'(x)$ *for all* $x, y \in \mathbf{U}$.

(d) *For any* $x \in \mathbf{f}_\nu$, *we have*

$$S'(x^+) = (-1)^{\mathrm{ht}\nu} \pi^{e(\nu)} (\pi q)^{-c(\nu)} \sigma(x)^+ \widetilde{J}_{-\nu} \widetilde{K}_{-\nu},$$
$$S'(x^-) = (-1)^{\mathrm{ht}\nu} \pi^{e(\nu)} q^{c(\nu)} \widetilde{K}_\nu \sigma(x)^-.$$

(e) *We have* $SS' = S'S = 1$.

(f) *If* $x \in \mathbf{f}_\nu$, *then* $S(x^+) = (\pi q)^{-f(\nu)} S'(x^+)$ *and* $S(x^-) = q^{f(\nu)} S'(x^-)$ *where* $f(\nu) = \sum_i \nu_i i \cdot i$.

The map S (resp. S') is called the antipode (resp. the skew-antipode) of $\mathbf{U}$. Note that

$$S(E_i^{(n)}) = (-1)^n (\pi_i q_i^2)^{\binom{n}{2}} \widetilde{J}_{-ni} \widetilde{K}_{-ni} E_i^{(n)},$$
$$S'(E_i^{(n)}) = (-1)^n (\pi_i q_i^2)^{-\binom{n}{2}} E_i^{(n)} \widetilde{J}_{-ni} \widetilde{K}_{-ni},$$
$$S(F_i^{(n)}) = (-1)^n (\pi_i q_i^2)^{-\binom{n}{2}} F_i^{(n)} \widetilde{K}_{ni},$$
$$S'(F_i^{(n)}) = (-1)^n (\pi_i q_i^2)^{\binom{n}{2}} \widetilde{K}_{ni} F_i^{(n)}.$$

2.5. Specializations of U at $\pi = \pm 1$

The *specialization at* $\pi = 1$ (respectively, at $\pi = -1$) of a $\mathbb{Q}(q)^\pi$-algebra R is understood as $\mathbb{Q}(q) \otimes_{\mathbb{Q}(q)^\pi} R$, where $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)^\pi$-module with π acting as 1 (respectively, as -1).

Let $\mathfrak{J}$ be the (2-sided) ideal of $\mathbf{U}$ generated by $\{J_\mu - 1 | \mu \in Y\}$.

The specialization at $\pi = -1$ of the algebra $\mathbf{U}/\mathfrak{J}$ is naturally identified with a quantum group associated to the Cartan datum $(I, \cdot)$ (cf. [Lu]). The specialization at $\pi = 1$ of the algebra $\mathbf{U}$, denoted by $\mathbf{U}|_{\pi=1}$, is a variant of this quantum group, with some extra (harmless) central elements J_μ. Specialization at $\pi = 1$ for the rest of the paper essentially reduces our results to those of Lusztig [Lu].

The specialization at $\pi = 1$ of the superalgebra $\mathbf{U}/\mathfrak{J}$ is identified with a quantum supergroup associated to the super Cartan datum $(I, \cdot)$ considered in the literature; cf. [Ya], [BKM]. The specialization at $\pi = -1$ of $\mathbf{U}$, denoted by $\mathbf{U}|_{\pi=-1}$, will also be referred to as a *quantum supergroup* of type $(I, \cdot)$, and the extra generators J_i allow us to formulate integrable modules $V(\lambda)$ for *all* $\lambda \in X^+$, which was not possible before.

All constructions and results in the remainder of this paper clearly afford specializations at $\pi = -1$, *which provide new constructions and new results for quantum supergroups and their representations.*

2.6. The categories $\mathcal{C}$ and $\mathcal{O}$

In the remainder of this paper, by a representation of the algebra $\mathbf{U}$ we mean a $\mathbb{Q}(q)^\pi$-module on which $\mathbf{U}$ acts. Note we have a direct sum decomposition of the $\mathbb{Q}(q)^\pi$-module $\mathbb{Q}(q)^\pi = (\pi+1)\mathbb{Q}(q) \oplus (\pi-1)\mathbb{Q}(q)$, where π acts as 1 on $(\pi+1)\mathbb{Q}(q)$ and as -1 on $(\pi - 1)\mathbb{Q}(q)$.

We define the category $\mathcal{C}$ (of weight $\mathbf{U}$-modules) as follows. An object of $\mathcal{C}$ is a $\mathbb{Z}_2$-graded $\mathbf{U}$-module $M = M_{\overline{0}} \oplus M_{\overline{1}}$, compatible with the $\mathbb{Z}_2$-grading on $\mathbf{U}$, with a given weight space decomposition

$$M = \bigoplus_{\lambda \in X} M^\lambda, \quad M^\lambda = \big\{m \in M \mid K_\mu m = q^{\langle \mu, \lambda \rangle} m, J_\mu m = \pi^{\langle \mu, \lambda \rangle} m, \forall \mu \in Y\big\},$$

such that $M^\lambda = M^\lambda_{\overline{0}} \oplus M^\lambda_{\overline{1}}$ where $M^\lambda_{\overline{0}} = M^\lambda \cap M_{\overline{0}}$ and $M^\lambda_{\overline{1}} = M^\lambda \cap M_{\overline{1}}$. The $\mathbb{Z}_2$-graded structure is only particularly relevant to tensor products, and will generally be suppressed when irrelevant. We have the following $\mathbb{Q}(q)^\pi$-module decomposition for each weight space: $M^\lambda = (\pi+1)M^\lambda \oplus (\pi-1)M^\lambda$; accordingly, we have $M = M_+ \oplus M_-$ as $\mathbf{U}$-modules, where $M_\pm := \oplus_{\lambda \in X}(\pi \pm 1)M^\lambda$ is a $\mathbf{U}$-module on which π acts as ± 1, i.e., a $\mathbf{U}|_{\pi=\pm 1}$-module. Hence the category $\mathcal{C}$ decomposes into a direct sum $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$, where $\mathcal{C}_\pm$ can be identified with categories of weight modules over the specializations $\mathbf{U}|_{\pi=\pm 1}$.

Lemma 2.6.1. *A simple $\mathbf{U}$-module is a simple module of either $\mathbf{U}|_{\pi=1}$ or $\mathbf{U}|_{\pi=-1}$.*

Let $M \in \mathcal{C}$ and let $m \in M^\lambda$. The formulas below follow from Lemma 2.2.3.

(a) $E_i^{(N)} F_i^{(M)} m = \sum_t \pi_i^{MN - \binom{t+1}{2}} \begin{bmatrix} N - M + \langle i, \lambda \rangle \\ t \end{bmatrix}_i F_i^{(M-t)} E_i^{(N-t)} m;$

(b) $F_i^{(M)} E_i^{(N)} m = \sum_t \pi_i^{(M-t)(N-t)-t^2} \begin{bmatrix} M - N - \langle i, \lambda \rangle \\ t \end{bmatrix}_i E_i^{(N-t)} F_i^{(M-t)} m;$

(c) $F_i^{(M)} E_j^{(N)} m = E_j^{(N)} F_i^{(M)} m$, for $i \neq j$;

(d) $\begin{bmatrix} \widetilde{K}_i; a \\ t \end{bmatrix}_i m = \begin{bmatrix} \langle i, \lambda \rangle + a \\ t \end{bmatrix}_i m.$

Note that $\mathbf{U} \otimes \mathbf{U}$ is a superalgebra with multiplication $(a \otimes b)(c \otimes d) = \pi^{p(b)p(c)} ac \otimes bd$. A tensor product of $\mathbf{U}$-modules $M \otimes N$ is naturally a $\mathbf{U} \otimes \mathbf{U}$-module with the obvious diagonal grading under the action $(x \otimes y)(m \otimes n) = \pi^{p(y)p(m)} xm \otimes yn$.

The tensor product of modules is naturally a $\mathbf{U}$-module under the coproduct action. Moreover, $\mathcal{C}$ is closed under tensor products. Note that for $a \in \mathbb{Z}_{>0}$, $M', M'' \in \mathcal{C}$, $m' \in M'^{\lambda'}$ and $m'' \in M''^{\lambda''}$, we have

$$E_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} \pi_i^{a'' p(m') + a'' \langle i, \lambda' \rangle} q_i^{a'a'' + a'' \langle i, \lambda' \rangle} E_i^{(a')} m' \otimes E_i^{(a'')} m'',$$

$$F_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} \pi_i^{a'' p(m') + a'a''} q_i^{-a'a'' - a' \langle i, \lambda'' \rangle} F_i^{(a')} m' \otimes F_i^{(a'')} m''.$$

To any $M \in \mathcal{C}$, we can define a new $\mathbf{U}$-module structure via $u \cdot m = \omega(u) m$; we denote this module by ${}^\omega M$. By definition, note that ${}^\omega M^\lambda = M^{-\lambda}$.

Let $\lambda \in X$. Then there is a unique $\mathbf{U}$-module structure on $\mathbf{f}$ such that for any $y \in \mathbf{f}$, $\mu \in Y$ and $i \in I$ we have $K_\mu y = q^{\langle \mu, \lambda - |y| \rangle} y$, $J_\nu y = \pi^{\langle \nu, \lambda - |y| \rangle} y$, $F_i y = \theta_i y$, and $E_i 1 = 0$. As in the non-super case, this follows readily from the triangular decomposition. This module will be called a Verma module and denoted by $M(\lambda)$. The parity grading on $\mathbf{f}$ induces a parity grading on $M(\lambda)$ where $p(1) = 0$. As before, we have a $\mathbf{U}$-module decomposition $M(\lambda) = M(\lambda)_+ \oplus M(\lambda)_-$, where

$M(\lambda)_\pm$ can be identified as the Verma module of $\mathbf{U}|_{\pi=\pm 1}$ (which is a $\mathbb{Q}(q)$-vector space).

For any $M \in \mathcal{C}$ and an element $m \in M^\lambda$ such that $E_i m = 0$ for all i, there is a unique $\mathbf{U}$-homomorphism $M(\lambda) \to M$ via $1 \mapsto m$. This can be proved as in [Lu, 3.4.6], using now Lemma 2.2.3.

Let $\mathcal{O}$ be the full subcategory of $\mathcal{C}$ such that for any M in $\mathcal{O}$ and $m \in M$, there exists an $n \geq 0$ such that $x^+ m = 0$ for all $x \in \mathbf{f}_\nu$ with $\mathrm{ht}\nu \geq n$. Note that $M(\lambda)$ and its quotient $\mathbf{U}$-modules belong to $\mathcal{O}$.

2.7. Category $\mathcal{C}_{\mathbf{int}}$ of integrable modules

An object $M \in \mathcal{C}$ is said to be *integrable* if for any $m \in M$ and any $i \in I$, there exists $n_0 \geq 1$ such that $E_i^{(n)} m = F_i^{(n)} m = 0$ for all $n \geq n_0$. Let $\mathcal{C}_{\mathrm{int}}$ be the full subcategory of $\mathcal{C}$ whose objects are the integrable $\mathbf{U}$-modules.

For $M, M', M'' \in \mathcal{C}_{\mathrm{int}}$, we have ${}^\omega M, M' \otimes M'' \in \mathcal{C}_{\mathrm{int}}$. The proof of the following lemma proceeds as in the non-super case; see [Lu, Lemma 3.5.3].

Lemma 2.7.1. *For $(a_i), (b_i) \in \mathbb{N}^I$ and $\lambda \in X$, let M be the quotient of $\mathbf{U}$ by the left ideal generated by the elements $F_i^{a_i+1}$, $E_i^{b_i+1}$, $K_\mu - q^{\langle \mu, \lambda \rangle}$ with $\mu \in Y$, and $J_\nu - \pi^{\langle \nu, \lambda \rangle}$ with $\nu \in Y$. Then M is an integrable $\mathbf{U}$-module.*

The proof of the following proposition proceeds as in the non-super case; see [Lu, Prop. 3.5.4 and 23.3.11].

Proposition 2.7.2. *If $u \in \mathbf{U}$ such that u acts as zero on every integrable module, then $u = 0$.*

Proposition 2.7.3. *Let $\lambda \in X^+$.*

(a) *Let $\mathcal{T}$ be the left ideal of $\mathbf{f}$ generated by the elements $\theta_i^{\langle i, \lambda \rangle + 1}$ for all $i \in I$. Then $\mathcal{T}$ is a $\mathbf{U}$-submodule of the Verma module $M(\lambda)$.*
(b) *The quotient $\mathbf{U}$-module $V(\lambda) := M(\lambda)/\mathcal{T}$ is integrable.*

The proof is as in the non-super case [Lu, Prop. 3.5.6]. As usual $V(\lambda) = V(\lambda)_+ \oplus V(\lambda)_-$, and $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$; moreover we have the identification $V(\lambda)_\pm = M(\lambda)_\pm / \mathcal{T}_\pm$.

We denote the image of 1 in $V(\lambda)$ by v_λ^+ when convenient. This module has an induced parity grading from the associated Verma module by setting $p(v_\lambda^+) = 0$. When considering the image of 1 in the module ${}^\omega V(\lambda)$, we will denote this vector by v_λ^-.

Proposition 2.7.4. *Let M be an object of $\mathcal{C}_{\mathrm{int}}$ and let $m \in M^\lambda$ be a non-zero vector such that $E_i m = 0$ for all i. Then $\lambda \in X^+$ and there is a unique morphism (in $\mathcal{C}_{\mathrm{int}}$) $t' : V(\lambda) \to M$ sending v_λ^+ to m.*

The proof is as in the non-super case [Lu, Prop. 3.5.8].

3. The quasi-$\mathcal{R}$-matrix and the quantum Casimir

In this section, we introduce the quasi-$\mathcal{R}$-matrix as well as the quantum Casimir for $\mathbf{U}$ and establish their basic properties. Using the Casimir element, we show that the category $\mathcal{O}_{\mathrm{int}}$ is semisimple and classify its simple object by dominant integral weights.

3.1. The quasi-$\mathcal{R}$-matrix Θ

Consider the vector spaces

$$\mathcal{H}_N = \mathbf{U}^+\mathbf{U}^0\Big(\sum_{\mathrm{ht}\nu\geq N}\mathbf{U}^-_\nu\Big)\otimes\mathbf{U} + \mathbf{U}\otimes\mathbf{U}^-\mathbf{U}^0\Big(\sum_{\mathrm{ht}\nu\geq N}\mathbf{U}^+_\nu\Big)$$

for $N\in\mathbb{Z}_{>0}$. Note that $\mathcal{H}_N$ is a left ideal in $\mathbf{U}\otimes\mathbf{U}$; moreover, for any $u\in\mathbf{U}\otimes\mathbf{U}$, we can find an $r\geq 0$ such that $\mathcal{H}_{N+r}u\subset\mathcal{H}_N$.

Let $(\mathbf{U}\otimes\mathbf{U})^\wedge$ be the inverse limit of the vector spaces $(\mathbf{U}\otimes\mathbf{U})/\mathcal{H}_n$. Then the $\mathbb{Q}(q)^\pi$-algebra structure extends by continuity to a $\mathbb{Q}(q)^\pi$-algebra structure on $(\mathbf{U}\otimes\mathbf{U})^\wedge$, and we have the obvious algebra embedding $\mathbf{U}\otimes\mathbf{U}\to(\mathbf{U}\otimes\mathbf{U})^\wedge$.

Let $\bar{\ }:\mathbf{U}\otimes\mathbf{U}\to\mathbf{U}\otimes\mathbf{U}$ be the $\mathbb{Q}$-algebra homomorphism given by $\bar{\ }\otimes\bar{\ }$. This extends to a $\mathbb{Q}$-algebra homomorphism on the completion. Let $\overline{\Delta}:\mathbf{U}\to\mathbf{U}\otimes\mathbf{U}$ be the $\mathbb{Q}(q)^\pi$-algebra homomorphism given by $\overline{\Delta}(x)=\overline{\Delta(\overline{x})}$.

Theorem 3.1.1.

(a) *There is a unique family of elements* $\Theta_\nu\in\mathbf{U}^-_\nu\otimes\mathbf{U}^+_\nu$ (*with* $\nu\in\mathbb{N}[I]$) *such that* $\Theta_0=1\otimes 1$ *and* $\Theta=\sum_\nu\Theta_\nu\in(\mathbf{U}\otimes\mathbf{U})^\wedge$ *satisfies* $\Delta(u)\Theta=\Theta\overline{\Delta}(u)$ *for all* $u\in\mathbf{U}$ (*where this identity is in* $(\mathbf{U}\otimes\mathbf{U})^\wedge$).

(b) *Let* B *be a* $\mathbb{Q}(q)^\pi$*-basis of* $\mathbf{f}$ *such that* $B_\nu=B\cap\mathbf{f}_\nu$ *is a basis of* $\mathbf{f}_\nu$ *for any* ν. *Let* $\{b^*\mid b\in B_\nu\}$ *be the basis of* $\mathbf{f}_\nu$ *dual to* B_ν *under* $(\ ,\ )$. *We have*

$$\Theta_\nu=(-1)^{\mathrm{ht}\nu}\pi^{e(\nu)}\pi_\nu q_\nu\sum_{b\in B_\nu}b^-\otimes b^{*+}\in\mathbf{U}^-_\nu\otimes\mathbf{U}^+_\nu,$$

where $e(\nu)$ *is defined as in §2.4.*

The element Θ will be called the *quasi-$\mathcal{R}$-matrix* for $\mathbf{U}$.

Proof. Consider an element $\Theta\in(\mathbf{U}\otimes\mathbf{U})^\wedge$ of the form $\Theta=\sum_\nu\Theta_\nu$ with $\Theta_\nu=\sum_{b,b'\in B_\nu}c_{b',b}b'^-\otimes b^{*+}$, $c_{b',b}\in\mathbb{Q}(q)^\pi$. The set of $u\in\mathbf{U}$ such that $\Delta(u)\Theta=\Theta\overline{\Delta}(u)$ is clearly a subalgebra of $\mathbf{U}$ containing $\mathbf{U}^0$. Therefore, it is necessary and sufficient that it contains the E_i and F_i. This amounts to showing that

$$\sum_{b_1,b_2\in B_\nu}c_{b_1,b_2}E_ib_1^-\otimes b_2^{*+}+\sum_{b_3,b_4\in B_{\nu-i}}\pi_i^{p(b_3)}c_{b_3,b_4}\widetilde{J}_i\widetilde{K}_ib_3^-\otimes E_ib_4^{*+}$$
$$=\sum_{b_1,b_2\in B_\nu}\pi_i^{p(b_2)}c_{b_1,b_2}b_1^-E_i\otimes b_2^{*+}+\sum_{b_3,b_4\in B_{\nu-i}}c_{b_3,b_4}b_3^-\widetilde{K}_{-i}\otimes b_4^{*+}E_i,$$

and

$$\sum_{b_1,b_2\in B_\nu}\pi_i^{p(b_1)}c_{b_1,b_2}b_1^-\otimes F_ib_2^{*+}+\sum_{b_3,b_4\in B_{\nu-i}}c_{b_3,b_4}F_ib_3^-\otimes\widetilde{K}_{-i}b_4^{*+}$$
$$=\sum_{b_1,b_2\in B_\nu}c_{b_1,b_2}b_1^-\otimes b_2^{*+}F_i+\sum_{b_3,b_4\in B_{\nu-i}}\pi_i^{p(b_4)}c_{b_3,b_4}b_3^-F_i\otimes b_4^{*+}\widetilde{J}_i\widetilde{K}_i.$$

Let $z \in \mathbf{f}$. Then since the inner product is nondegenerate, this equality is equivalent to the equality

$$\sum_{b_1,b_2\in B_\nu} c_{b_1,b_2}(b_2^*,z)(E_ib_1^- - \pi_i^{p(b_2^*)}b_1^-E_i) + \sum_{b_3,b_4\in B_{\nu-i}} c_{b_3,b_4}\big(\pi_i^{p(b_3)}(\theta_ib_4^*,z)\widetilde{J}_i\widetilde{K}_ib_3^- - (b_4^*\theta_i,z)b_3^-\widetilde{K}_{-i}\big) = 0,$$

and

$$\sum_{b_1,b_2\in B_\nu} c_{b_1,b_2}(b_1,z)(\pi_i^{p(b_1)}F_ib_2^{*+} - b_2^{*+}F_i) + \sum_{b_3,b_4\in B_{\nu-i}} c_{b_3,b_4}\big((\theta_ib_3,z)\widetilde{K}_{-i}b_4^{*+} - \pi_i^{p(b_4)}(b_3\theta_i,z)b_4^{*+}\widetilde{J}_i\widetilde{K}_i\big) = 0.$$

Note that $p(b_1) = p(b_2) = p(b_3) + p(i) = p(b_4) + p(i)$. Using Proposition 2.2.2 and the derivations, we have

$$\sum_{b_1,b_2\in B_\nu} (\pi_iq_i - q_i^{-1})^{-1}c_{b_1,b_2}(b_2^*,z)(\widetilde{J}_i\widetilde{K}_i\,{}_ir(b_1)^- - \pi_i^{p(b_1)-p(i)}r_i(b_1)^-\widetilde{K}_{-i}) + \sum_{b_3,b_4\in B_{\nu-i}} c_{b_3,b_4}(\theta_i,\theta_i)\big(\pi_i^{p(b_3)}(b_4^*,{}_ir(z))\widetilde{J}_i\widetilde{K}_ib_3^- - (b_4^*,r_i(z))b_3^-\widetilde{K}_{-i}\big) = 0,$$

and

$$\sum_{b_1,b_2\in B_\nu} -(\pi_iq_i - q_i^{-1})^{-1}c_{b_1,b_2}(b_1,z)(r_i(b_2)^+\widetilde{J}_i\widetilde{K}_i - \pi_i^{p(b_2)-p(i)}\widetilde{K}_{-i}\,{}_ir(b_2)^+) + \sum_{b_3,b_4\in B_{\nu-i}} c_{b_3,b_4}(\theta_i,\theta_i)\big((b_3,{}_ir(z))\widetilde{K}_{-i}b_4^{*+} - \pi_i^{p(b_4)}(b_3,r_i(z))b_4^{*+}\widetilde{J}_i\widetilde{K}_i\big) = 0.$$

Using the triangular decomposition, this is equivalent to the equalities

$$\sum_{b_1,b_2} c_{b_1,b_2}(b_2^*,z)\,{}_ir(b_1) + \sum_{b_3,b_4} \pi_iq_i\pi_i^{p(b_4)}c_{b_3,b_4}(b_4^*,{}_ir(z))b_3 = 0, \tag{3.1}$$

$$\sum_{b_1,b_2} c_{b_1,b_2}\pi_i^{p(b_1)-p(i)}(b_2^*,z)r_i(b_1) + \sum_{b_3,b_4} \pi_iq_ic_{b_3,b_4}(b_4^*,r_i(z))b_3 = 0, \tag{3.2}$$

$$\sum_{b_1,b_2} c_{b_1,b_2}(b_1,z)r_i(b_2) + \sum_{b_3,b_4} \pi_iq_i\pi_i^{p(b_4)}c_{b_3,b_4}(b_3,r_i(z))b_4^* = 0, \tag{3.3}$$

$$\sum_{b_1,b_2} \pi_i^{p(b_2)-p(i)}c_{b_1,b_2}(b_1,z)\,{}_ir(b_2) + \sum_{b_3,b_4} \pi_iq_ic_{b_3,b_4}(b_3,{}_ir(z))b_4^* = 0. \tag{3.4}$$

Now when $c_{b,b'} = (-1)^{\mathrm{ht}(\nu)}\pi^{e(\nu)}\pi_\nu q_\nu \delta_{b,b'}$, we have

$$\sum_b \pi^{e(\nu)} q_\nu(b^*, z)_i r(b) - \sum_{b'} \pi^{e(\nu)}\pi_\nu q_\nu(b'^*, {}_i r(z))b' = 0,$$
$$\sum_b \pi^{e(\nu-i)}\pi_\nu q_\nu(b^*, z) r_i(b) - \sum_{b'} \pi^{e(\nu-i)}\pi_\nu q_\nu(b'^*, r_i(z))b' = 0,$$
$$\sum_b \pi^{e(\nu)}\pi_\nu q_\nu(b, z) r_i(b) - \sum_{b'} \pi^{e(\nu)}\pi_\nu q_\nu(b', r_i(z))b'^* = 0,$$
$$\sum_b \pi^{e(\nu-i)}\pi_\nu q_\nu(b, z)_i r(b) - \sum_{b'} \pi^{e(\nu-i)}\pi_\nu q_\nu(b', {}_i r(z))b'^* = 0.$$

These equalities are easily verified by checking when z is a basis or dual basis element.

Thus the existence of such a Θ is verified. Suppose Θ'_ν and Θ' also satisfy the conditions in (a). Then $\Theta - \Theta' = \sum c_{b,b'} b^- \otimes b'^+$ must satisfy (3.1)-(3.4) and has $c_{b,b} = 0$ for $b \in B_0$. Suppose $c_{b,b'} = 0$ for $b, b' \in B'_\nu$ for $\mathrm{ht}(\nu') < n$ and assume $\mathrm{ht}(\nu) = n$. Then the second sum in (3.1) is zero, so ${}_i r(\sum_{b_1,b_2} c_{b_1,b_2}(b_2^*, z)b_1) = 0$. But then $\sum_{b_1,b_2} c_{b_1,b_2}(b_2^*, z)b_1 = 0$, whence $(\sum_{b_2} c_{b_1,b_2} b_2^*, z) = 0$ for all $z \in \mathbf{f}$. Therefore $c_{b_1,b_2} = 0$ for all $b_1, b_2 \in B_\nu$. By induction $\Theta - \Theta' = 0$, proving uniqueness. □

Example 3.1.2. Let $I = I_{\bar{1}} = i$ as in Example 2.1.1, and let us determine Θ in this case using Theorem 3.1.1(b). The obvious basis to choose is $B = \{\theta^{(n)} : n \in \mathbb{N}\}$, and then we see from Lemma 1.7.1 that $\Theta = \sum_n a_n F^{(n)} \otimes E^{(n)}$, where $a_n = (-1)^n(\pi q)^{-\binom{n+1}{2}}[n]^!(\pi q - q^{-1})^n$ (compare with [CW, §5]).

Recall that the bar involution on $\mathbf{U}$ makes sense under the assumption that the super Cartan datum is consistent.

Corollary 3.1.3. *Assume that the super Cartan datum is consistent. We have* $\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1 \otimes 1$ *with equality in the completion.*

Proof. First note that by construction Θ is invertible. We have $\Delta(u)\Theta = \Theta\overline{\Delta}(u)$, so $\Theta\Delta(\overline{u}) = \Theta\overline{\Delta}(\overline{u}) = \Theta\overline{\Delta(u)}$. Now applying the bar involution to both sides and rearranging, we get

$$\overline{\Theta}^{-1}\overline{\Delta}(u) = \Delta(u)\overline{\Theta}^{-1}.$$

By uniqueness, $\overline{\Theta}^{-1} = \Theta$. □

We can specialize the identity $\Delta(u)\Theta = \Theta\overline{\Delta}(u)$ to deduce

$$(E_i \otimes 1)\Theta_\nu + (\widetilde{J}_i\widetilde{K}_i \otimes E_i)\Theta_{\nu-i} = \Theta_\nu(E_i \otimes 1) + \Theta_{\nu-i}(\widetilde{K}_{-i} \otimes E_i),$$
$$(1 \otimes F_i)\Theta_\nu + (F_i \otimes \widetilde{K}_{-i})\Theta_{\nu-i} = \Theta_\nu(1 \otimes F_i) + \Theta_{\nu-i}(F_i \otimes \widetilde{J}_i\widetilde{K}_i).$$

Setting $\Theta_{\le p} = \sum_{\mathrm{ht}\nu \le p} \Theta_\nu$, we obtain that

$$\begin{aligned}(E_i \otimes 1 + \widetilde{J}_i\widetilde{K}_i \otimes E_i)\Theta_{\le p} - \Theta_{\le p}(E_i \otimes 1 + \widetilde{K}_{-i} \otimes E_i)\\ = \sum_{\mathrm{ht}\nu=p} (\widetilde{J}_i\widetilde{K}_i \otimes E_i)\Theta_\nu - \sum_{\mathrm{ht}\nu=p} \Theta_\nu(\widetilde{K}_{-i} \otimes E_i),\end{aligned} \tag{3.5}$$

$$\begin{aligned}(F_i \otimes \widetilde{K}_{-i} + 1 \otimes F_i)\Theta_{\le p} - \Theta_{\le p}(F_i \otimes \widetilde{K}_i + 1 \otimes F_i)\\ = \sum_{\mathrm{ht}\nu=p} (F_i \otimes \widetilde{K}_{-i})\Theta_\nu - \sum_{\mathrm{ht}\nu=p} \Theta_\nu(F_i \otimes \widetilde{J}_i\widetilde{K}_i).\end{aligned} \tag{3.6}$$

3.2. The quantum Casimir

Let B, B_ν be as in Theorem 3.1.1. Let S be the antipode and $\mathbf{m} : \mathbf{U} \otimes \mathbf{U} \to \mathbf{U}$ be the multiplication map $u \otimes u' \mapsto uu'$. Applying $\mathbf{m}(S \otimes 1)$ to the identities (3.5) and (3.6), we obtain that, for any $p \ge 0$,

$$\begin{aligned}\sum_{\mathrm{ht}\nu\le p}\sum_{b\in B_\nu} (-1)^{\mathrm{ht}\nu}\pi_\nu q_\nu \big(S(E_i b^-)b^{*+} + \pi_i^{p(\nu)} S(\widetilde{J}_i\widetilde{K}_i b^-)E_i b^{*+}\\ - \pi_i^{p(\nu)} S(b^- E_i)b^{*+} - S(b^-\widetilde{K}_{-i})b^{*+}E_i\big)\\ = \sum_{\mathrm{ht}\nu=p}\sum_{b\in B_\nu} (-1)^p \pi_\nu q_\nu \big(\pi_i^{p(\nu)} S(\widetilde{J}_i\widetilde{K}_i b^-)E_i b^{*+} - S(b^-\widetilde{K}_{-i})b^{*+}E_i\big),\end{aligned}$$

and

$$\begin{aligned}\sum_{\mathrm{ht}\nu\le p}\sum_{b\in B_\nu} (-1)^{\mathrm{ht}\nu}\pi_\nu q_\nu \big(\pi_i^{p(\nu)} S(b^-)F_i b^{*+} + S(F_i b^-)\widetilde{K}_{-i} b^{*+}\\ - S(b^-)b^{*+}F_i - \pi_i^{p(\nu)} S(b^- F_i)b^{*+}\widetilde{J}_i\widetilde{K}_i\big)\\ = \sum_{\mathrm{ht}\nu=p}\sum_{b\in B_\nu} (-1)^p \pi_\nu q_\nu \big(S(F_i b^-)\widetilde{K}_{-i} b^{*+} - \pi_i^{p(\nu)} S(b^- F_i)b^{*+}\widetilde{J}_i\widetilde{K}_i\big).\end{aligned}$$

Now set $\Omega_{\le p} = \sum_{\mathrm{ht}\nu\le p}\sum_{b\in B_\nu} (-1)^{\mathrm{ht}\nu}\pi_\nu q_\nu S(b^-)b^{*+}$. Then observing that

$$\begin{aligned}S(E_i b^-)b^{*+} + \pi_i^{p(\nu)} S(\widetilde{J}_i\widetilde{K}_i b^-)E_i b^{*+}\\ = \pi_i^{p(\nu)} S(b^-)(-\widetilde{J}_{-i}\widetilde{K}_{-i}E_i)b^{*+} + \pi_i^{p(\nu)} S(b^-)\widetilde{J}_{-i}\widetilde{K}_{-i}E_i b^{*+} = 0,\end{aligned}$$

we have

$$\begin{aligned}\widetilde{J}_{-i}\widetilde{K}_{-i}E_i\Omega_{\le p} - \widetilde{K}_i\Omega_{\le p}E_i\\ = \sum_{\mathrm{ht}\nu=p}\sum_{b\in B_\nu} (-1)^p \pi_\nu q_\nu \big(\pi_i^{p(\nu)} S(\widetilde{K}_i b^-)E_i b^{*+} - S(b^-\widetilde{K}_{-i})b^{*+}E_i\big),\\ \Omega_{\le p}F_i - F_i\widetilde{K}_i\Omega_{\le p}\widetilde{J}_i\widetilde{K}_i\\ = -\sum_{\mathrm{ht}\nu=p}\sum_{b\in B_\nu} (-1)^p \pi_\nu q_\nu \big(S(F_i b^-)\widetilde{K}_{-i} b^{*+} - \pi_i^{p(\nu)} S(b^- F_i)b^{*+}\widetilde{J}_i\widetilde{K}_i\big).\end{aligned}$$

Example 3.2.1. Let $I = I_1 = i$ as in Examples 2.1.1 and let Θ be as defined in Example 3.1.2. Then using §2.4,

$$\Omega_{\leq p} = \sum_{1\leq n\leq p} a_n(\pi q^2)^{-\binom{n}{2}} F^{(n)} K^n E^{(n)}.$$

We note that though this is a rather different construction than the Casimir-type element in [CW], it will nevertheless be used toward a similar purpose.

Let $M \in \mathcal{O}$. Then for any $m \in M$ we have that $\Omega(m) = \Omega_{\leq p} m$ is independent of p when p is large enough. We can write

$$\Omega(m) = \sum_b (-1)^{\mathrm{ht}|b|} \pi_{|b|} q_{|b|} S(b^-) b^{*+} m.$$

Then we have

$$\widetilde{J}_{-i}\widetilde{K}_{-i}E_i\Omega = \widetilde{K}_i\Omega E_i, \qquad \Omega F_i = F_i\widetilde{K}_i\Omega\widetilde{J}_i\widetilde{K}_i, \qquad \Omega K_\mu = K_\mu\Omega,$$

as operators on M. Therefore for $m \in M^\lambda$, we have

$$\Omega E_i m = (\pi_i q_i^2)^{-\langle i,\lambda+i'\rangle} E_i\Omega m, \qquad \Omega F_i = (\pi_i q_i^2)^{\langle i,\lambda+i'\rangle} F_i\Omega m.$$

This can be rephrased in terms of the antipode. Define the $\mathbb{Q}(q)^\pi$-linear map $\overline{S} : \mathbf{U} \to \mathbf{U}$ by $\overline{S}(u) = \overline{S(\overline{u})}$. Then $\Omega\overline{S}(u) = S(u)\Omega : M \to M$ for $u \in \mathbf{U}$.

Let C be a fixed coset of X with respect to $\mathbb{Z}[I] \leq X$. Let $G : C \to \mathbb{Z}$ be a function such that

$$G(\lambda) - G(\lambda - i') = \frac{i\cdot i}{2}\langle i,\lambda\rangle \qquad \text{for all } \lambda \in C, i \in I. \tag{3.7}$$

Clearly such a function exists and is unique up to addition of a constant function.

Lemma 3.2.2. *Let $\lambda, \lambda' \in C \cap X^+$. If $\lambda \geq \lambda'$ and $G(\lambda) = G(\lambda')$, then $\lambda = \lambda'$.*

Let $M \in \mathcal{C}$. For each $\mathbb{Z}[I]$-coset C in X, define $M_C = \bigoplus_{\lambda\in C} M^\lambda$. It is clear that

$$M = \bigoplus_{C\in X/\mathbb{Z}[I]} M_C. \tag{3.8}$$

Proposition 3.2.3. *Let $M \in \mathcal{O}$, and let $\Omega : M \to M$ be as above.*

(a) *Assume there exists C as above such that $M = M_C$. Let $G : C \to \mathbb{Z}$ be a function satisfying* (3.7)*. We define a linear map $\Xi : M \to M$ by $\Xi(m) = (\pi q^2)^{G(\lambda)} m$ for all $\lambda \in C$ and $m \in M^\lambda$. Then $\Omega\Xi$ is a locally finite U-module homomorphism.*
(b) *Assume that M is a quotient of $M(\lambda')$. Then $\Omega\Xi$ acts as $(\pi q^2)^{G(\lambda')}$ on M.*
(c) *For M as in* (a)*, the eigenvalues of $\Omega\Xi$ are of the form $(\pi q^2)^c$ for $c \in \mathbb{Z}$.*

The operator $\Omega\Xi$ is called the *Casimir element* of $\mathbf{U}$ (though note that the Casimir element formally lives in a completion of $\mathbf{U}$).

Proof. We compute that for $m \in M^\lambda$,

$$\Omega\Xi E_i m = \Omega(\pi q^2)^{G(\lambda+i')} E_i m = (\pi q^2)^{G(\lambda+i')-G(\lambda)-s_i\langle i,\lambda+i'\rangle} E_i \Omega\Xi m = E_i \Omega\Xi m.$$

A similar argument applies to the F_i, and clearly $\Omega\Xi$ commutes with K_μ, J_μ, proving the first assertion of (a). The local finiteness claim is a standard category $\mathcal{O}$ type argument. Parts (b) and (c) follow now easily. □

3.3. The complete reducibility in $\mathcal{O}_{\mathrm{int}}$

Recall the categories $\mathcal{O}$ and $\mathcal{C}_{\mathrm{int}}$ from §2.6–2.7. Form another category $\mathcal{O}_{\mathrm{int}} := \mathcal{O} \cap \mathcal{C}_{\mathrm{int}}$.

Lemma 3.3.1. *Let $M \in \mathcal{C}$. Assume that M is a nonzero quotient of the Verma module $M(\lambda)$ and that M is integrable. Then*

(a) $\lambda \in X^+$*;*

(b) *M_+ and M_- are either simple or zero.*

Proof. It is clear that (a) holds by some rank one consideration. An argument similar to that for [Lu, Lemma 6.2.1] shows that if $\dim_{\mathbb{Q}(q)} M^\lambda = 1$ then M is simple; in this case, M must be equal to either M_+ or M_-. Otherwise, $\dim_{\mathbb{Q}(q)} M^\lambda = 2$, then $\dim_{\mathbb{Q}(q)} M_+^\lambda = \dim_{\mathbb{Q}(q)} M_1^\lambda = 1$, and we repeat the argument above for the integrable $\mathbf{U}$-module $M_\pm$. □

Theorem 3.3.2. *Let M be a $\mathbf{U}$-module in $\mathcal{O}_{\mathrm{int}}$. Then M is a sum of simple $\mathbf{U}$-submodules.*

Proof. Note that as discussed in §2.6 we may assume that $M = M_+$ or $M = M_-$. Since the case for M_+ follows from [Lu, Thm. 6.2.2], it is enough to prove the theorem for $M = M_-$. Virtually the same argument as in loc. cit. holds, which we will now sketch.

Using (3.8), we may further assume there is a coset C of $\mathbb{Z}[I]$ in X such that $M = M_C$. Then we may pick a function G satisfying (3.7) and avail ourselves of Proposition 3.2.3. Since the Casimir element commutes with the $\mathbf{U}$-action, we may further assume that M lies in a generalized eigenspace of the Casimir element.

Consider the set of singular vectors of M(that is, the set of vectors $m \in M$ for which $E_i m = 0$ for all $i \in I$) and let M' be the submodule they generate. Then each homogeneous singular vector generates a simple submodule by virtue of Lemma 3.3.1, so M' is a sum of simple modules.

It remains to show that $M = M'$, so take $M'' = M/M'$ and suppose $M'' \neq 0$. Then there is a maximal weight $\lambda \in C$ such that $M''^\lambda \neq 0$. Then the Casimir element acts on the submodule generated by a nonzero $m_1 \in M''^\lambda$ by $(-q^2)^{G(\lambda)}$ by Proposition 3.2.3, and so in particular M must lie in the generalized $(-q^2)^{G(\lambda)}$-eigenspace of the Casimir element.

On the other hand, m is the image of a vector $\tilde{m} \in M \setminus M'$. The $\mathbf{U}^+$-module generated by $\tilde{m}$ contains a singular vector m_2 of weight $\eta \geq \lambda$, and the Casimir element acts on the module generated by m_2 as $(-q^2)^{G(\eta)}$. Then $G(\eta) = G(\lambda)$ and $\eta \geq \lambda$, so by Lemma 3.2.2 $\eta = \lambda$. But the $\tilde{m}$ is a singular vector, contradicting that our choice of m_1 was nonzero. □

Corollary 3.3.3.

(a) *For $\lambda \in X^+$, the* **U***-modules* $V(\lambda)_+$ *and* $V(\lambda)_-$ *are simple objects of* $\mathcal{O}_{\rm int}$.

(b) *For $\lambda, \lambda' \in X^+$, the* **U***-modules* $V(\lambda)_+$ *and* $V(\lambda')_+$*, and respectively* $V(\lambda)_-$ *and* $V(\lambda')_-$*, are isomorphic if and only if* $\lambda = \lambda'$*. (Clearly,* $V(\lambda)_+$ *and* $V(\lambda')_-$ *are non-isomorphic.)*

(c) *Any integrable module in* $\mathcal{O}$ *is a direct sum of simple modules of the form* $V(\lambda)_\pm$ *for various* $\lambda \in X^+$.

Proof. The argument in [Lu, Cor. 6.2.3] holds using our Lemma 3.3.1 above. □

3.4. Character formula

Denote by $\rho \in X$ such that $\langle i, \rho \rangle = 1$ for all $i \in I$. We claim the following character formula of $V(\lambda)$ for *every* $\lambda \in X^+$:

$$\mathrm{ch}\, V(\lambda)_\pm = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)-\rho}}. \tag{3.9}$$

This is equivalent to claiming $V(\lambda)$ is always a $\mathbb{Q}(q)^\pi$-free module for each $\lambda \in X^+$. This character formula holds for $V(\lambda)_+$ with $\lambda \in X^+$ by a theorem of Lusztig [Lu1]. A proof of this formula for $V(\lambda)_-$ is possible, but requires techniques outside the scope of this paper.

Assume now that $\lambda \in X^+$ satisfies an *evenness condition*

$$\langle i, \lambda \rangle \in 2\mathbb{Z}_+, \quad \forall i \in I_{\bar{1}}. \tag{3.10}$$

Then the action of **U** on $V(\lambda)$ factors through an action of the algebra $\mathbf{U}/\mathcal{J}$ (see §2.5), and (3.9) holds by [BKM, Thm. 4.9] on the characters of integrable modules of the usual quantum groups. The irreducible integrable modules of the corresponding Kac–Moody superalgebras were known [Kac] to be parametrized by highest weights $\lambda \in X^+$ satisfying (3.10). Hence, for $\lambda \in X^+$ which does not satisfy (3.10), the usual q-deformation argument cannot be applied directly to $V(\lambda)_-$.

Note there are always weights λ satisfying (3.10) which are large enough relative to every $i \in I$. Therefore, the same type of arguments as in [Lu, Chap. 33] show that the algebra **f** and hence **U** admit the following equivalent formulations.

Proposition 3.4.1. *The algebra* **f** *is isomorphic to the algebra generated by* $\theta_i, i \in I$*, subject to the quantum Serre relation as in Proposition* 1.7.3.

Proposition 3.4.2. *The algebra* **U** *is isomorphic to the algebra generated by* E_i, F_i $(i \in I)$ *and* J_μ, K_μ $(\mu \in Y)$*, subject to the relations* 2.1(a)–(f) *and the quantum Serre relations for* E_i*'s as well as for* F_i*'s (in place of* θ_i*'s in Proposition* 1.7.3*).*

As a consequence of (3.9) and Proposition 3.4.1, the character of $\mathbf{U}^-$ is given by

$$\mathrm{ch}\, \mathbf{U}^- = \frac{1}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)-\rho}} = \prod_{\alpha > 0} (1 - e^{-\alpha})^{(-1)^{1+p(\alpha)} \dim \mathfrak{g}_\alpha}, \tag{3.11}$$

where $\mathfrak{g}$ denotes the Kac–Moody superalgebra of type $(I, \cdot)$ (cf. [Kac]), "$\alpha > 0$" denotes positive roots of $\mathfrak{g}$, $p(\cdot)$ denotes the parity function, and $\mathfrak{g}_\alpha$ denotes the α-root space.

4. Higher Serre relations

In this section we formulate and establish the higher Serre relations, which will be instrumental in determining the action of a braid group on a quantum covering group and integrable modules in a future work.

4.1. Higher Serre elements

For $i, j \in I$, and $n, m \geq 0$, set

$$p(n, m; i, j) = mnp(i)p(j) + \binom{m}{2} p(i).$$

For $i \neq j$, define the elements

$$e_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} E_i^{(r)} E_j^{(n)} E_i^{(s)}, \tag{4.1}$$

$$e'_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} E_i^{(s)} E_j^{(n)} E_i^{(r)}, \tag{4.2}$$

$$f_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)} q_i^{r(n\langle i,j'\rangle+m-1)} F_i^{(s)} F_j^{(n)} F_i^{(r)}, \tag{4.3}$$

$$f'_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)} q_i^{r(n\langle i,j'\rangle+m-1)} F_i^{(r)} F_j^{(n)} F_i^{(s)}. \tag{4.4}$$

When there is no confusion by fixing i and j, we will abbreviate $e_{i,j;n,m} = e_{n,m}$, $e'_{i,j;n,m} = e'_{n,m}$, $f_{i,j;n,m} = f_{n,m}$, $f'_{i,j;n,m} = f'_{n,m}$. Note that we have the equalities

$$e'_{n,m} = \sigma(e_{n,m}), \quad f'_{n,m} = \sigma\omega^2(f_{n,m}), \quad e_{n,m} = \omega(\overline{f'_{n,m}}), \quad e'_{n,m} = \omega(\overline{f_{n,m}}). \tag{4.5}$$

4.2. Commutations with divided powers

Lemma 4.2.1. *The following hold:*

(a) $-q_i^{-n\langle i,j'\rangle-2m} \pi_i^{m+np(j)} E_i e_{n,m} + e_{n,m} E_i = [m+1]_i e_{n,m+1}$.
(b) $-F_i e_{n,m} + \pi_i^{m+np(j)} e_{n,m} F_i = [-n\langle i,j'\rangle - m + 1]_i \pi_i^{np(j)+1} \widetilde{K}_i^{-1} e_{n,m-1}$.

Proof. When $i \in I_{\bar{0}}$ this is [Lu, Lemma 7.1.2]. We therefore assume $i \in I_{\bar{1}}$. Then, $\langle i, j'\rangle \in 2\mathbb{Z}$ by 1.1(d). The left-hand side of (a) is

$$\begin{aligned}
&\sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} [s+1]_i E_i^{(r)} E_j^{(n)} E_i^{(s+1)} \\
&\qquad + \sum_{r+s=m} (-1)^{r+1} \pi_i^{p(n,r;i,j)+np(j)+m} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)-n\langle i,j'\rangle-2m} \\
&\qquad \times [r+1]_i E_i^{(r+1)} E_j^{(n)} E_i^{(s)} \\
&= \sum_{r+s=m+1} (-1)^r \pi_i^{p(n,r;i,j)} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m)} \\
&\qquad \times \big((\pi_i q_i)^{r-(m+1)} \pi_i^{r-1+m} [r]_i + (\pi_i q_i)^r [s]_i\big) E_i^{(r)} E_j^{(n)} E_i^{(s)},
\end{aligned}$$

where we have used

$$p(n,r-1;i,j) \equiv p(n,r;i,j) + np(i)p(j) + (r-1)p(i) \pmod 2 \tag{4.6}$$

in the last line. Part (a) now follows from the computation

$$(\pi_i q_i)^{r-(m+1)}\pi_i^{r-1+m}[r]_i + (\pi_i q_i)^r[s]_i = q_i^{-s}[r]_i + (\pi_i q_i)^r[s]_i = [r+s]_i = [m+1]_i.$$

To prove (b), observe that

$$\begin{aligned} F_i E_i^{(r)} E_j^{(n)} E_i^{(s)} =& \pi_i^{r+np(j)+s} E_i^{(r)} E_j^{(n)} E_i^{(s)} F_i - \pi_i^{r+np(j)+1} E_i^{(r)} E_j^{(n)} E_i^{(s-1)} [\widetilde{K}_i; s-1] \\ & - \pi_i E_i^{(r-1)} [\widetilde{K}_i; r-1] E_j^{(n)} E_i^{(s)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & -F_i e_{n,m} + \pi_i^{m+np(j)} e_{n,m} F_i \\ &= \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)+r+np(j)+1} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} E_i^{(r)} E_j^{(n)} E_i^{(s-1)} [\widetilde{K}_i; s-1] \\ &\quad + \sum_{r+s=m} (-1)^r \pi_i^{p(n,r;i,j)+1} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} E_i^{(r-1)} [\widetilde{K}_i; r-1] E_j^{(n)} E_i^{(s)} \\ &= \sum_{r+s=m-1} (-1)^r \pi_i^{p(n,r;i,j)+r+np(j)+1} (\pi_i q_i)^{-r(n\langle i,j'\rangle+m-1)} E_i^{(r)} E_j^{(n)} E_i^{(s)} [\widetilde{K}_i; s] \\ &\quad + \sum_{r+s=m-1} (-1)^{r-1} \pi_i^{p(n,r+1;i,j)+1} (\pi_i q_i)^{-(r+1)(n\langle i,j'\rangle+m-1)} E_i^{(r)} [\widetilde{K}_i; r] E_j^{(n)} E_i^{(s)} \\ &= \sum_{r+s=m-1} (-1)^r \pi_i^{p(n,r;i,j)} (\pi_i q_i)^{-r(n\langle i,j'\rangle+(m-1)-1)} \pi_i^{np(j)+1} \pi_i^r (\bigstar) E_i^{(r)} E_j^{(n)} E_i^{(s)} \end{aligned}$$

where, using (4.6) we compute

$$\begin{aligned} (\bigstar) =& (\pi_i q_i)^{-r} [\widetilde{K}_i; -s - n\langle i,j'\rangle - 2r] - (\pi_i q_i)^{-n\langle i,j'\rangle-m+1-r} [\widetilde{K}_i; -r] \\ =& \frac{(\pi_i q_i)^{-r}(\pi_i q_i)^{-m+1-n\langle i,j'\rangle-r} - (\pi_i q_i)^{-n\langle i,j'\rangle-m+1-r}(\pi_i q_i)^{-r}}{\pi_i q_i - q_i^{-1}} \widetilde{J}_i \widetilde{K}_i \\ & + \frac{(\pi_i q_i)^{-n\langle i,j'\rangle-m+1-r} q_i^r - (\pi_i q_i)^{-r} q_i^{n\langle i,j'\rangle+m-1+r}}{\pi_i q_i - q_i^{-1}} \widetilde{K}_i^{-1} \\ =& \pi_i^r \frac{(\pi_i q_i)^{-n\langle i,j'\rangle-m+1} - q_i^{n\langle i,j'\rangle+m-1}}{\pi_i q_i - q_i^{-1}} \widetilde{K}_i^{-1}. \end{aligned}$$

This proves (b). □

The next result, which is a π-analogue of [Lu, Lemma 7.1.3], follows by a straightforward induction argument.

Lemma 4.2.2. *The following formulas hold:*

(a)

$$E_i^{(N)} e_{n,m} = \sum_{k=0}^{N} (-1)^k q_i^{N(n\langle i,j'\rangle+2m)+(N-1)k} \pi_i^{N(np(j)+m)+\binom{k}{2}} \times \begin{bmatrix} m+k \\ k \end{bmatrix}_i e_{n,m+k} E_i^{(N-k)};$$

(b)

$$F_i^{(M)} e_{n,m} = \sum_{h=0}^{M} (-1)^h q_i^{-(M-1)h} \pi_i^{M(m+np(j))+(M-m)h} \times \begin{bmatrix} -n\langle i,j'\rangle - m + h \\ h \end{bmatrix}_i \widetilde{K}_i^{-h} e_{n,m-h} F_i^{(M-h)}.$$

Lemma 4.2.3. *Let* $m = 1 - n\langle i, j'\rangle$. *Then*

$$F_j e_{n,m} - \pi_j^{mp(i)+n} e_{n,m} F_j = \pi_j^n \left(\widetilde{K}_j^{-1} \frac{q_j^{n-1}}{\pi_j q_j - q_j^{-1}} e_{n-1,m} - \widetilde{J}_j \widetilde{K}_j \frac{q_j^{1-n}}{\pi_j q_j - q_j^{-1}} \overline{e_{n-1,m}} \right).$$

Proof. To begin, if $r + s = m$, then

$$F_j E_i^{(r)} E_j^{(n)} E_i^{(s)} = \pi_j^{mp(i)+n} E_i^{(r)} E_j^{(n)} E_i^{(s)} F_j - \pi_j^{rp(i)+1} E_i^{(r)} E_j^{(n-1)} [\widetilde{K}_j, n-1] E_i^{(s)}.$$

Since $m = 1 - \langle i, j'\rangle$, the exponent of $\pi_i q_i$ in $e_{n,m}$ is 0; see (4.1). Therefore,

$$\begin{aligned} &F_j e_{n,m} - \pi_j^{mp(i)+n} e_{n,m} F_j \\ &= -\pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)} [\widetilde{K}_j; 1-n-ra_{ji}] E_i^{(r)} E_j^{(n-1)} E_i^{(s)} \\ &= -\pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)} q_i^{-r\langle i,j'\rangle} \frac{(\pi_j q_j)^{1-n}}{\pi_j q_j - q_j^{-1}} \widetilde{J}_j \widetilde{K}_j E_i^{(r)} E_j^{(n-1)} E_i^{(s)} \\ &\quad + \pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)} (\pi_i q_i)^{r\langle i,j'\rangle} \frac{(\pi_j q_j)^{n-1}}{\pi_j q_j - q_j^{-1}} \widetilde{K}_j^{-1} E_i^{(r)} E_j^{(n-1)} E_i^{(s)}. \end{aligned}$$

We have used $p(n, r; i, j) = p(n-1, r; i, j) + rp(i)p(j)$ to simplify the second line, and $q_i^{-r\langle i,j'\rangle} = q_j^{-r\langle j,i'\rangle}$ and $\pi_j^{r\langle j,i'\rangle} = 1 = \pi_i^{r\langle i,j'\rangle}$ in the two subsequent lines. Since $(n-1)\langle i, j'\rangle + m - 1 = -\langle i, j'\rangle$, the result follows. □

As a consequence of the previous lemmas we obtain a generalization of the quantum Serre relations.

Proposition 4.2.4 (Higher Serre Relations)**.** *Let* $i, j \in I$ *be distinct. If* $m > -n\langle i, j'\rangle$, *then* $e_{i,j;n,m} = 0$.

Proof. As before, fix i and j and write $e_{n,m} = e_{i,j;n,m}$. Note that $e'_{1,1-\langle i,j'\rangle} = \sigma(e_{1,1-\langle i,j'\rangle})$ is just the usual quantum Serre relations (see Proposition 1.7.3). Using Lemma 4.2.1(a), it follows by induction on m that $e_{1,m} = 0$ for $m \geq 1-\langle i,j'\rangle$. Now, let $n > 1$ and assume that $e_{n-1,m} = 0$ for all $m > (1-n)\langle i,j'\rangle$. By Lemma 4.2.1(b), $e_{n,1-n\langle i,j'\rangle}$ supercommutes with F_i, and by Lemma 4.2.3 and induction, it supercommutes with F_j (note that $m = 1-n\langle i,j'\rangle > (1-n)\langle i,j'\rangle$). It trivially supercommutes with F_k for $k \neq i,j$. Therefore, by Proposition 2.3.5 we deduce that $e_{n,1-n\langle i,j'\rangle} = 0$. Again, using Lemma 4.2.1(a) and induction m the $e_{n,m} = 0$ for $m \geq 1-n\langle i,j'\rangle$. □

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