

RESEARCH-EXPOSITORY SURVEY

OPEN PROBLEMS ON
CENTRAL SIMPLE ALGEBRAS

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Abstract. We provide a survey of past research and a list of open problems regarding central simple algebras and the Brauer group over a field, intended both for experts and for beginners.

Introduction

There are many accessible introductions to the theory of central simple algebras and the Brauer group, such as [Al61], [He], [Pi], [Dr], [Ro91, Chap. 7], [Ke], [Ja], and [GiS] — or at a more advanced level [CT], [S99], and [Wad]. But there has not been a survey of open problems for a while — the most prominent recent references are the excellent surveys [Am82], [S84], [S92], and [S01]. Since the last survey, major new threads have appeared related to geometric techniques. As examples, we mention (in chronological order):

- Saltman’s results on division algebras over the function field of a p -adic curve, see [S97a], [S07], [B10], [PS99], [PS10];
- De Jong’s result on the Brauer group of the function field of a complex surface, see [dJ], [Lie], and Section 4;

- Harbater, Hartmann, and Krashen’s patching techniques, see [HHK09] and [HHK]; and
- Merkurjev’s bounding of the essential p -dimension of PGL_n , see [M10], [Ma], and Section 6.

Motivated by these and other developments, we now present an updated list of open problems. Some of these problems are—or have become—special cases of much more general problems for algebraic groups. To keep our task manageable, we (mostly) restrict our attention to central simple algebras and PGL_n . The following is an idiosyncratic list that originated during the Brauer group conference held at Kibbutz Ketura in January 2010.

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0. Background

Basic definitions

Let F be a field. In this paper a *central simple F -algebra* is a finite-dimensional F -algebra whose center is F and whose only two-sided ideals are (0) and A . An *F -division algebra* is a central simple F -algebra that is also a division ring. Every central simple F -algebra B is isomorphic to a matrix algebra over a unique F -division algebra A by Wedderburn’s theorem, and A is then said to be *associated* with B . Two algebras are said to be *Brauer-equivalent* or simply *equivalent* if their respective associated division algebras are F -isomorphic. Thus two algebras A and B are equivalent if there exist numbers n and m such that $M_n(A) \cong M_m(B)$. The resulting set of equivalence classes forms the (classical) Brauer group $\mathrm{Br}(F)$, and this set inherits a group structure from the F -tensor product on algebras. The *index* of a central simple algebra is the degree of its associated division algebra, and the *period* (or *exponent*) is the order of its Brauer class.

An F -algebra is *étale* if it is F -isomorphic to a finite product $\prod_{i=1}^n K_i$, where each K_i is a finite separable field extension of F . Note that the fields K_i are uniquely determined up to F -isomorphism. The algebra is *Galois* if it is isomorphic to K^n for a Galois field extension K/F . The *Galois closure* of an étale F -algebra isomorphic to $\prod_{i=1}^n K_i$ is the F -isomorphism class of K^n , where K is the compositum of the Galois closures of the fields K_i .

A Galois F -algebra T has *Galois group* G if G is a (finite) group that acts on T via F -automorphisms such that the induced action on $\mathrm{Hom}_{F\text{-alg}}(T, F_{\mathrm{sep}})$ is simple and transitive, where F_{sep} is a separable closure of F . More formally, $\mathrm{Spec} T \rightarrow \mathrm{Spec} F$ is a G -torsor (see also [Mi80, §I.5]). In contrast to the uniqueness

of the Galois closure, the Galois group of a Galois F -algebra isomorphic to K^n is in general not uniquely defined, and can be any group of order $n \cdot [K : F]$ that contains $\text{Gal}(K/F)$ as a subgroup. For example, K^n admits a cyclic Galois group if and only if K/F is cyclic. To define the action of a suitable group G on K^n , as well as for a more intrinsic definition of Galois closure, see [S99, p. 40].

Definition of a crossed product

The classical theory shows that each central simple F -algebra B of degree n contains a maximal (commutative) étale subalgebra T of degree n over F . Equivalently, there exists an étale extension T/F of degree n that *splits* B , i.e., such that $B \otimes_F T \cong M_n(T)$. By definition, B is a *crossed product* if B contains a *Galois* maximal étale subalgebra, and B is *cyclic* if B contains a cyclic-Galois maximal étale subalgebra. It is easy to see that every B is Brauer-equivalent to a crossed product. For if L/F is the Galois closure of a maximal étale subalgebra $T \subset B$ and $[L : T] = m$, then L is a Galois maximal étale subalgebra of $C = M_m(B)$.

Problems involving crossed products usually focus on the Galois splitting *fields* of division algebras, and their Galois groups. For B is a G -crossed product Brauer-equivalent to A via some G -Galois maximal étale subalgebra L/F if and only if A is split by an H -Galois field extension K/F , where $H \leq G$ and K is the image of L under an (any) F -algebra homomorphism from L to F_{sep} .

It follows from the above that if A is a crossed product or is cyclic, then so is every $B = M_n(A)$. The converse turns out to be false; there exist noncrossed products, as we shall see below. Moreover, in case F has prime characteristic p , Albert proved that every p -algebra (i.e., central simple algebra of period a power of p) is Brauer-equivalent to a cyclic algebra. But [AmSa] gives an example of a noncyclic p -algebra. One can lift this example to a complete discrete valuation ring of characteristic 0 and so find a central simple algebra A over a field of characteristic 0 such that $M_n(A)$ is cyclic for some n but A is not.

Cohomological interpretation

Much of the interest in crossed products stems from the fact that the Galois structure of a crossed product allows us to describe it simply (see Section 6) and explicitly. For if B is a crossed product via a G -Galois extension K/F , then the algebra structure of B is encoded in a Galois 2-cocycle $f \in Z^2(G, K^\times)$, which is easily extracted from B in principle. Conversely, each $f \in Z^2(G, K^\times)$ defines a G -crossed product B_f with maximal subfield K/F , by letting

$$B_f = \bigoplus_{s \in G} K u_s$$

be a K -vector space on formal basis elements u_s , and defining multiplication by $u_s u_t = f(s, t) u_{st}$ and $u_s x = s(x) u_s$ for all $s, t \in G$ and $x \in K$. The equivalence relation on central simple algebras induces an equivalence on cocycles, and the resulting group of equivalence classes is the Galois cohomology group $H^2(G, K^\times)$. Note that every central simple algebra is a crossed product if and only if the correspondence between classes extends to the level of cocycles and algebras.

The crossed product problem for division algebras

The central organizing problem in the theory of central simple algebras and the Brauer group is to determine, possibly for a given field F , the extent to which F -central simple algebras are crossed products and, in particular, cyclic crossed products. Since every central simple algebra representing a given Brauer class is a crossed product if and only if its associated division algebra is a crossed product, the interesting question is whether every *division algebra* is a crossed product, possibly over a given field F .

Fields over which all division algebras are known to be crossed products include: fields of cohomological dimension at most 1, e.g. algebraically closed fields and quasi-finite fields (over which all division algebras are split); global fields and local fields (over which all division algebras are cyclic); and, at least for algebras of period prime to the residue characteristic, strictly Henselian fields (over which all division algebras are abelian crossed products, see [B01, Theorem 1]).

For much of the twentieth century it was reasonable to conjecture that all division algebras over all fields were crossed products. Then, in an astonishing development in 1972, S. A. Amitsur produced noncrossed product division algebras of any degree n divisible by 2^3 or p^2 for an odd prime p , in characteristic 0. The centers of these (universal) division algebras are given as invariant fields whose precise nature is itself a topic of considerable interest (see Section 5). Soon after Amitsur's discovery, in [ScSm], Schacher and Small extended Amitsur's result to include the case where the characteristic is positive and does not divide the degree. In [S78a], Saltman included the p -algebra case, when n is divisible by p^3 . Subsequent constructions of noncrossed products — none improving on the indexes found by Amitsur — have since appeared in [Ri77], [S78b], [Ro81], [T86], [JW86], [B95], [ReY01, Theorem 1.3], [BMT], and others. The most explicit construction to date is in [Han].

Responding to the sensitivity of Amitsur's degree p^2 examples to the ground field's characteristic and roots of unity, Saltman and Rowen observed that every division algebra of degree p^2 becomes a crossed product on some prime-to- p extension (the degree p case being trivial), whereas for degree p^3 and above, examples exist that are stable under such extensions ([RoS92]); see [McK07, Cor. 2.2.2] for the p -algebra case.

1. Noncyclic algebras in prime degree

Perhaps the most important open problem in the theory of central simple algebras is:

Problem 1.1. Given a prime p , construct a noncyclic division algebra of degree p over some field F .

This was listed as Problem 7 in Amitsur's survey [Am82], and as Problem 1 in [S92]. Of course, division algebras of nonprime degree need not be cyclic. This is true already in the smallest possible case of degree 4 ([Al33]).

What is known

Prime-degree division algebras for the two smallest primes are always cyclic. The $p = 2$ case is elementary, and the $p = 3$ case was solved by Wedderburn in 1921 ([Wed]). A result that should be included with these two is that any division algebra of degree $n = 4$ is a crossed product ([Al61]). In 1938 Brauer proved that any division algebra of degree $p = 5$ has a (solvable) Galois splitting field of degree dividing 60 ([Br38]); this result was improved somewhat in [RoS96]. Despite the promising start, this vexing problem remains completely open for $p > 3$. Specific candidates for prime degree noncrossed products have been proposed in [Ro99, §4] and [V04, p. 487].

It is natural to specialize the problem to fields with known properties. For some fields F , one knows that every division algebra of degree p is cyclic: when F has no separable extensions of degree p (trivial case), when F is a local field or a global field (by Albert, Brauer, Hasse, and Noether), and when F is the function field of a ℓ -adic curve for ℓ prime and different from p [S07].

Generalization

In 1934 Albert proved that an F -division algebra A of prime degree p is a crossed product if and only if A contains a noncentral element x such that x^p is in F , called a *p-central element* (“Albert’s cyclicity criterion”, [Al61, XI, Theorem 4.4]). It is natural to ask whether such elements exist in algebras of degree greater than p . When $p = 2$ and the degree is 4, and the characteristic is not 2, Albert’s crossed product result shows that 2-central elements do exist. On the other hand, it is shown in [AmSa] that there exist algebras of degree p^2 and characteristic p with no p -central elements. Using this, it is shown in [S80] that for n a multiple of p^2 , the universal division algebra of degree n over the rational field \mathbb{Q} has no p -central elements. When F has a primitive p th root of unity, there are no known examples of division F -algebras of degree a multiple of p that do not have p -central elements.

2. Other problems regarding crossed products

There are still some important unresolved problems concerning crossed products. The first is a slight generalization of Problem 1.1.

Problem 2.1. Determine whether F -division algebras of prime period are crossed products.

Both Problems 1.1 and 2.1 are related to the problem of computing the essential dimension of algebraic groups described in Section 6.

What is known

This problem is even less resolved than Problem 1.1, in the sense that no noncrossed products of prime period have ever been discovered, and no one has shown that all division algebras of a fixed prime period must be crossed products. For odd primes p the problem is completely open. (Note that the proof of Theorem 6 in [Ro82] concerning the existence of noncrossed products of prime period is incorrect. In the modified version that appeared later as [Ro88, Theorem 7.3.31], the flaw appears in the last paragraphs of page 255.)

Division algebras of $(\text{period}, \text{index}) = (2, 2^n)$ are known to be crossed products for $n \leq 3$, and the problem for $n > 3$ is open. Albert proved the $(2, 4)$ case (“Albert’s theorem”) in [Al32, Theorem 6], see also [Ja, §5.6.9]. Rowen proved the $(2, 8)$ case for any field in [Ro78] and [Ro84], see also [Ja, §5.6.10].

We present two other open problems concerning the existence of noncrossed products. As mentioned above, the smallest noncrossed product p -algebras known have degree p^3 . Therefore we have:

Problem 2.2. Determine whether p -algebras of degree p^2 are crossed products.

In view of the fact that all known examples of noncrossed product algebras occur over fields of cohomological dimension at least 3, it is natural to ask:

Problem 2.3. If F has cohomological dimension at most 2, is every central simple F -algebra a crossed product?

For algebras of period 2, the answer is “yes”, by Merkurjev [K90]. The answer to this question for other periods is unknown, even when F is the function field of a complex surface.

Smallest Galois splitting fields

One way to study the failure of a division algebra to be a crossed product is to determine, for a given division algebra A , the smallest Brauer-equivalent crossed product $B = M_r(A)$. It is equivalent to determining the smallest degree finite Galois splitting fields for A . As Brauer, Hasse, and Noether knew in the 1920s, “minimal” Galois splitting fields can have arbitrarily large degree (Roquette cites [BrN] and [Has] in [Roq04, Chap. 7.1]), so it is important to specify that the degree (rather than the splitting field) be minimal.

To simplify the exposition, we say a finite group G *splits* a central simple F -algebra A if there exists a G -Galois field extension of F that splits A (equivalently, if some $M_m(A)$ is a G -crossed product). Since a splitting field of A may or may not contain a maximal subfield of A , the problem with respect to division algebras divides into parts.

Problem 2.4. Determine the smallest splitting group(s) of an F -division algebra A arising from the Galois closure of a maximal subfield of A . Determine in particular whether every A of degree n has a separable maximal subfield whose Galois closure has group A_n (the alternating group).

Problem 2.5. Determine whether every division algebra is split by an abelian group, and find degree bounds over specific fields.

What is known

Both problems are related to Problem 5 in [Am82] and to Problem 2.5 is Question 1 in [Am82]. In the situation where A is a crossed product, Problem 2.4 becomes a problem of *group admissibility*. It then has a strong number and group-theoretic character, and we put it in Problem 11.1 below.

If A has degree n , then A contains a maximal separable subfield (of degree n) that is contained in an S_n -Galois maximal étale subalgebra, which bounds the answer to the first part of Problem 2.4 from above. It is clear that these are in

general not hard bounds, since if $n = 3$ then C_3 splits A by Wedderburn's results. It is unknown whether there exists a (noncrossed product) division algebra of degree $n > 4$ that is split by no proper subgroup of S_n , prompting Amitsur to ask Problem 2.4.

With the additional structure provided by an involution of the first kind in the period 2 case, some positive results for Problem 2.4 can be proved for $\text{char } F \neq 2$. In particular, if A has period 2 and 2-power degree $2m$, one can prove the existence of a subfield of degree m in a maximal separable subfield of A , and it follows that the Galois closure has group $H = C_2^m \rtimes S_m$. Using the main result of Parimala, Sridharan, and Suresh in [PSS93], one can improve this to $\bar{H} = C_2^{m-1} \rtimes S_m$; we leave the details to the interested reader. Note that H is not contained in A_{2m} , but \bar{H} is.

Amitsur proved that if the universal division algebra (defined in Section 5 below) $UD(k, n)$ over an infinite field k is split by a group G , then every division algebra over a field F containing k is split by (a subgroup of) G ([Am91, Theorem 2]; also see [Am82, p. 15] and [TA85, §7.1]). Thus Problem 2.5 is intimately connected to the problem of splitting the universal division algebra, which is Problem 8 in [Am82]. In [Am82] Amitsur remarked that if n is the composite of r relatively prime numbers n_i , then $S_{n_1} \times \cdots \times S_{n_r}$ splits $UD(k, n)$. Then in [TA85, Theorem 7.3] Tignol and Amitsur established a remarkable lower bound on the order of a splitting group of $UD(k, n)$, for infinite k , and n not divisible by $\text{char } k$. See [TA86, Cor. 9.4] and [ReY01, Theorem 1.3] for improvements and alternative proofs of this bound in the case where n is a prime power.

If $\text{char } F$ does not divide the period n of an F -division algebra A , the Merkurjev–Suslin theorem shows that A has a meta-abelian splitting field, obtained by adjoining n th roots of unity, if necessary, and then a Kummer extension. This theorem solved Amitsur's Question 3 in [Am82], and it solves the first part of Problem 2.5 if F contains the n th roots of unity. Results bounding the “symbol length” over a particular field provide upper bounds for the degree of abelian (or meta-abelian) Galois splitting fields, and for this we refer to Problem 3.5. Tignol and Amitsur submitted a lower bound for the order of an abelian splitting group in [TA85, §7.5].

When F does not contain n th roots of unity, Problem 2.5 has a similar relationship to the more general open problem of determining whether the Brauer group is generated by cyclic classes, and for this we refer to Section 3. This latter problem was Question 2 in [Am82]. The problem is settled for the 2-, 3-, and 5-torsion subgroups of $\text{Br}(F)$.

Smallest Galois splitting fields for p -algebras

Somewhat more is known about Problem 2.5 for p -algebras. If $\text{char } F = p$ and A is an F -division algebra of p -power degree, then Albert showed that A has a cyclic splitting field in [Al36], see also [Al61, Chap. VII, Theorem 9.31]. Louis Isaac Gordon proved the existence of noncyclic 2-algebras (of period and index 4) in [Go] and Amitsur and Saltman proved the existence of noncyclic division p -algebras (which are generic abelian crossed products) of every degree p^n ($n \geq 2$) in [AmSa]. These results raised questions about bounds of cyclic splitting fields. The following result follows from Albert's methods.

Proposition 2.6. *Suppose F is a field that is finitely generated of transcendence degree r over a perfect field k of characteristic $p > 0$. Then a p -algebra of (p -power) period q is split by a cyclic field extension of degree q^r over F .*

Proof. A p -algebra A of period q is split by $F^{1/q}$, by [Al61, Chap. VII.7, Theorem 2], and since $[F^{1/q} : F] = q^r$ ([BourA, A.V.100, Prop. 4] and [BourA, A.V.135, Cor. 3]), A is similar to a tensor product of r cyclic p -algebras of degree q by [Al61, Chap. VII.9, Theorem 28], hence A is split by a cyclic extension of degree q^r , by [Al61, Chap. VII.9, Lemma 13]. \square

In particular, the center $Z(k, n)$ of generic degree n matrices over a field k is known to be finitely generated ([S99, Cor. 14.9]), so if k is a finite field of characteristic p , and $Z(k, n)$ has transcendence degree m over k , then it follows that the universal division algebra $UD(k, p^n)$ is split by a cyclic field extension of degree p^{mn} , and hence for any field F containing k , any F -division algebra A of degree p^n is split by a cyclic extension of degree p^{mn} over F , by [Am82, p. 15].

Because of this connection between the degree of a cyclic splitting field of a p -algebra and the transcendence degree of the center of generic matrices over the prime field, Problem 2.5 for p -algebras is closely tied to Section 6 on essential dimension. As for known bounds on the transcendence degree m of $Z(k, n)$, we have $m \leq p^{2n} + 1$ from general principles, and if p is odd and $p^n \geq 5$, then $m \leq (p^n - 1)(p^n - 2)/2$ by [LRRS]. (Note the $p^n = 2, 3$ cases are settled by Wedderburn’s cyclicity results.)

Crossed products and descent from prime-to- p extensions

Since every division algebra has a maximal separable subfield, every division algebra of prime degree p is a crossed product after scalar extension to a prime-to- p field extension. This has led to some attempts at “descent”: If A is an F -division algebra of prime degree p , and K/F is a Galois splitting field for A whose group G is a semidirect product $G = C_p \rtimes H$ with C_p normal in G and H of order m prime-to- p , then $A \otimes_F K^{C_p}$ is a C_p -crossed product division algebra, and we can ask whether the Galois structure descends to A . This idea goes back to Albert ([Al38]). In the particularly interesting case where $H = C_m$ and $K^{C_p} = F(\zeta_p)$ for a primitive p th root of unity ζ_p , we call A a *quasi-symbol*, after [V04]. Note in this case $\text{char } F \neq p$.

The problem below is strongly related to Problems 3, 4, and 5 in [S92]. As Saltman noted in [S92], showing the division algebra A above is cyclic is equivalent to showing that the G -crossed product $M_m(A)$ is also a G' -crossed product, where G' is a group that has C_p as an image. As in [TA85], write $G \Rightarrow G'$ if every G -crossed product is necessarily a G' -crossed product, and $G \Rightarrow_F G'$ if $G \Rightarrow G'$ for G -crossed products whose centers contain F .

Problem 2.7. Determine groups G and G' and conditions on F under which $G \Rightarrow_F G'$. In particular:

- (1) Determine, for m prime-to- p , whether $C_p \rtimes C_m \Rightarrow_F C_p \times C_m$.
- (2) Determine whether all quasi-symbols are cyclic.

What is known

Background on the first part of Problem 2.7, as well as a basic symmetry result for abelian groups, may be found in [TA85]. We will say that a finite group G splits an F -algebra A if A is split by a G -Galois extension of F . Note that in Problem 2.7(1), $C_p \rtimes C_m \Rightarrow_F C_p \times C_m$ if and only if all F -algebras A of prime degree p that are split by $C_p \rtimes C_m$ are cyclic.

In 1938 Albert showed that if $\text{char } F = p$, and A is a p -algebra of degree p split by a Galois extension whose group is of the form $C_p \rtimes C_m$ for m prime-to- p , then A is cyclic ([Al38]), hence $C_p \rtimes C_m \Rightarrow_F C_p \times C_m$ in this case. He actually proved more generally that for p dividing n and arbitrary m , any $(C_n \rtimes C_m)$ -crossed product is abelian, again for $\text{char } F = p$ (see [ArJa] for a different proof). In 1999 Rowen proved that if A is a p -algebra of odd degree p that becomes equivalent to a certain type (“Type A”) of p -symbol over a quadratic extension E/F , then A is already cyclic over F ([Ro99, Theorem 3.4]). This generalized Wedderburn’s degree 3 theorem (in characteristic 3) and proved the cyclicity of a certain p -algebra of degree p studied by Albert. Rowen constructed a generic p -algebra of degree p that becomes a symbol over a quadratic extension, and conjectured it to be noncyclic for $p > 3$ ([Ro99, §4]).

In 1982, Rowen and Saltman showed that if n is odd, F contains a primitive n th root of unity (so $\text{char } F$ does not divide n), and A is a division algebra of degree n that is split by the dihedral group $D_n = C_n \rtimes C_2$ of order $2n$, then A is cyclic ([RoS82]). Thus $D_n \Rightarrow_F C_n \times C_2$ in this case. Their techniques generalized those used by Albert to prove cyclicity in degree 3 in [Al61]. Mammone and Tignol proved the same result using different methods in [MamT], and Haile gave another proof in [Hai]. Both [HKRT, Cor. 30] and [V04, Theorem 13.6] state that under certain hypotheses, including n odd and $[F(\mu_n) : F] \leq 2$, $D_n \Rightarrow_F C_n \times C_2$. In the $n = 5$ case, Matzri showed how to adapt Rowen and Saltman’s proof to remove the roots of unity and characteristic hypotheses ([Mat]), before using results of [V04] to prove that ${}_5\text{Br}(F)$ is generated by cyclic classes when $\text{char } F \neq 5$.

In 1983 Merkurjev, in a paper that heavily influenced [V04] and [Mat], proved a criterion for quasi-symbols of prime degree to be cyclic. To state this result, we fix some notation and hypotheses. Let p be a prime, $n = p^r$, ζ_n a primitive n th root of unity, $E = F(\zeta_p)$, $\nu : \text{Gal}(E/F) \rightarrow (\mathbb{Z}/p)^\times$ the canonical map, and $\varphi : \text{Gal}(E/F) \rightarrow (\mathbb{Z}/p)^\times$ a character. Then consider the hypothesis

$$\zeta_n \in E \text{ and } K/F \text{ is a } (C_n \rtimes_{\varphi} C_m)\text{-Galois extension, with } E = K^{C_n}. \tag{2.8}$$

In this notation, a quasi-symbol of degree p split by K is said to be of *type* (φ, φ') , where $\varphi' = \nu\varphi^{-1}$. Merkurjev’s criterion is that a quasi-symbol of degree $n = p$ is cyclic if and only if it is of type $(\nu, 1)$ or $(1, \nu)$ ([M83]). Vishne, expanding on Merkurjev’s framework, proved that Merkurjev’s cyclicity criterion holds for quasi-symbols of degrees $n = p^r$ under (2.8), and constructed a generic quasi-symbol, which he conjectured to be noncyclic. He also proved that the type of a quasi-symbol (of degree $n = p^r$) is symmetric in φ and φ' , so that a quasi-symbol split by $C_n \rtimes_{\varphi} C_m$ is also split by $C_n \rtimes_{\varphi'} C_m$.

In 1996 Rowen and Saltman proved that if F contains a p th root of unity, and A is a division algebra of degree p that is split by the semidirect product $C_p \rtimes C_m$ for

$m = 2, 3, 4$, or 6 and dividing $p - 1$, then A is cyclic. Thus $C_p \rtimes C_m \Rightarrow_F C_p \times C_m$ in this case. Vishne extended this result for quasi-symbols of degree $n = p^r$ under (2.8), for any m dividing $p - 1$ ([V04]).

If m divides n and F contains the m th roots of unity, then it is elementary to show $C_n \Rightarrow_F C_{n/m} \times C_m$ (see [V03] for a proof). This is best possible, since there are division algebras D/F where the absolute Galois group of F is rank 2 abelian. As for p -algebras, Saltman proved in [S77] that if $\text{char } F = p$ and n is a power of p , then $C_n \Rightarrow_F G$ for any group G of order n . Finally, Vishne proved in [V03] that if $\text{char } F$ does not divide n and F contains the n/m th roots of unity, then $D_n \Rightarrow_F D_m \times C_{n/m}$ holds for classes of period 2 ([V03]).

Part of the goal of Problem 2.7 is to construct division algebras that are split by a restricted class of groups, and then to say something about these groups. At the extreme, we call a group G *rigid* (resp. *rigid over F*) if $G \Rightarrow G'$ (resp. $G \Rightarrow_F G'$) implies $G \cong G'$. Thus G is rigid (over F) if there exists a central simple algebra (over F) that is a crossed product with respect to G and G only. In general, a central simple algebra can be a crossed product with respect to many different groups; Schacher showed in [Sc] that for any number n there exists a number field F and a central simple F -algebra A that is a crossed product with respect to every group of order n (see Problem 11.1). This result notwithstanding, Amitsur [Am72] proved that elementary abelian groups are rigid, and used this fact in his proof that the generic division algebra is not a crossed product. Saltman, and then Amitsur and Tignol, proved every noncyclic abelian group is rigid in [S78a] and [TA85]. A rigid group (nonabelian) of type $C_p \rtimes C_m$ with m prime-to- p , as mentioned in the beginning of this section, would settle Problem 1.1, and perhaps for this reason the existence of nonabelian rigid groups appeared as Problem 4 in [S92]. Brussel solved this problem for groups of p -power order in 1995, giving examples of rigid nonabelian groups over the fields $\mathbb{Q}(t)$, of the type

$$G = C_{p^2} \rtimes C_p = \{x, y \mid |x| = p^2, |y| = p, yxy^{-1} = x^{p+1}\}$$

for any prime p ([B95]), and then when p is odd, of the type $G = C_{p^3} \rtimes C_p$ ([B96]). All of Brussel's examples depend on the absence of roots of unity in the ground field, and it is unknown whether the same results hold without this assumption.

Note

In the end, the goal of the problems described in this section is to determine the nature of Galois splitting fields of smallest degree for cohomology classes of various kinds. But from an algebra-theoretic point of view, the importance of the crossed product problem is due at least partly to a long history of attempted solutions, together with a sentiment expressed by A. A. Albert, who spent much of his career studying it.

The importance of crossed products is due not merely to the fact that up to the present they are the only [central] simple algebras which have actually been constructed but also to Theorem 1: Every [central] simple algebra is similar to a crossed product. — Albert, 1939 [Al61, §V.3]

3. Generation by cyclic algebras

By far the most elementary central simple algebras known are the cyclic algebras, which were the first to be studied, and which to this day have provided an indispensable tool for investigating central simple algebras and the Brauer group. For these reasons, one of the most important practical problems in the study of the Brauer group is to determine whether or not the Brauer group is generated by the classes of cyclic algebras. This appears as Question 2 in [Am82]. Authors who studied this problem in the 1930s in one form or another include Albert ([Al36]), Teichmüller ([Te36]), Nakayama ([N]), and Witt ([Wi37]). For local and global fields, the question has an affirmative answer by the well-known theorems of Hasse and Albert, Brauer, Hasse, and Noether, respectively. There are no known counterexamples.

After Amitsur's work on noncrossed products, an affirmative answer for general F might well have seemed unlikely. But in a seminal breakthrough, Merkurjev and Suslin proved in 1983 that if F contains the n th roots of unity, then ${}_n\text{Br}(F)$ is generated by the classes of (cyclic) symbol algebras of degree n , via the norm residue homomorphism in K -theory ([MS83]). The essence of their result was that when F contains the n th roots of unity, the composition

$$H^1(F, \mu_n) \otimes H^1(F, \mu_n) \rightarrow H^2(F, \mu_n^{\otimes 2}) \rightarrow H^2(F, \mu_n) = {}_n\text{Br}(F)$$

is surjective, where the first map is the cup product, and the second map is the cup product with a chosen generator of $H^0(F, \mu_n^{\otimes(-1)}) = \text{Hom}(\mu_n, \mathbb{Z}/n)$. Thus ${}_n\text{Br}(F)$ is “generated in degree 1”. This work strongly suggests the following natural question, which was posed in the 1930s by Albert:

Problem 3.1 ([Al36, p. 126]). Is the n -torsion subgroup ${}_n\text{Br}(F)$ generated by the classes of cyclic algebras of degree (dividing) n ?

Problem 3.1 has an affirmative answer in the following important cases:

- (1) n is a power of the characteristic of F . This follows from [Al61, Chap. VII.7, Theorem 2] and [Al61, Chap. VII.9, Theorem 28].
- (2) n divides 30. It suffices to note that the answer is “yes” for $n = 2$ by [M81], $n = 3$ by [M83], and $n = 5$ by [Mat].
- (3) n is prime and adjoining a primitive n th root unity yields an extension of degree ≤ 3 by [M83]. This leads to the results summarized in the previous item.
- (4) F contains a primitive n th root of unity. By the Merkurjev–Suslin theorem, as discussed above.

The smallest open case for n prime is therefore:

Problem 3.2. Is ${}_7\text{Br}(F)$ generated by cyclic algebras of degree 7 when $\text{char } F \neq 7$ and F does not contain a primitive seventh root of unity?

(These problems are related to the following very special question stated by M. Mahdavi-Hezavehi: *If the multiplicative group F^\times is divisible, does it follow that $\text{Br}(F)$ is zero?* The answer is “yes” if $\text{Br}(F)$ is generated by cyclic algebras, because cyclic algebras are split over such a field. Therefore the answer is also

“yes” if F has characteristic 0, because in that case F has all the roots of unity so the Merkurjev–Suslin theorem applies.)

For composite n , even the following is unclear:

Problem 3.3. Is ${}_4\text{Br}(F)$ generated by cyclic algebras of degree 4 when $\text{char } F \neq 2$ and $\sqrt{-1} \notin F$?

And one can ask the weaker question:

Problem 3.4. Is ${}_n\text{Br}(F)$ generated by algebras of degree n ?

When n is prime, the answer to Problem 3.4 is “yes” by [M83]. More generally, Merkurjev proved in [M86] that if n is a power of an odd prime p not equal to $\text{char } F$, or if n is a power of 2 $=: p$ and $\sqrt{-1} \in F$, then ${}_n\text{Br}(F)$ is generated by quasi-symbols (for the definition of quasi-symbols, see the discussion preceding Problem 2.7) whose index is bounded by $n^{n/p}$. This result reduces Problem 3.1 to the determination of whether classes of quasi-symbols are generated by cyclic classes. Problem 3.4 is already open in the case $n = 4$.

Symbol length

Once it is known that ${}_n\text{Br}(F)$ is generated by classes of cyclic algebras, the focus turns to the minimal number of cyclic algebras (of fixed degree) needed to represent it in the Brauer group.

Suppose m and n have the same prime factors, $m \mid n$, and ${}_m\text{Br}(F)$ is generated by cyclic algebras. Let $\ell_F(m, n)$ denote the minimal number such that every central simple F -algebra of degree n and period (dividing) m is similar to a tensor product of $\ell_F(m, n)$ cyclic algebras of degree m , unless no such bound exists, in which case set $\ell_F(m, n) = \infty$. If F contains the m th roots of unity, the generic division algebra can be used to show that $\ell_F(m, n)$ is finite, by the Merkurjev–Suslin theorem. In this case, we may assume m is a power of a prime p by [T84, Theorem 2.2], and $\ell_F(m, n)$ is known as the *symbol length*.

Problem 3.5. Suppose p is a prime, $r \leq s$, and ${}_p\text{Br}(F)$ is generated by cyclic algebras. Compute $\ell_F(p^r, p^s)$.

The basic background on this problem can be found in [T84]. Almost all of the known results for $\ell_F(m, n)$ are either for p -algebras, or fields F containing the m th roots of unity. Of course, $\ell_F(p, p) = 1$ for $p = 2, 3$ by the classical theory, for any field F . Albert’s theorem shows that $\ell_F(2, 4) = 2$, and $\ell_F(2, 8) = 4$ when $\text{char } F \neq 2$ by [T84, Theorem 2.6]. For proofs of these and other results that follow more or less immediately from earlier work, see [T84]. In 1937 Teichmüller proved that for p -algebras, $\ell_F(p^r, p^s) \leq p^s! (p^s! - 1)$ ([Te37]).

Lower bounds can be obtained via results on indecomposable algebras, which we treat in Section 9. In [T84, Theorem 2.3], Tignol proved that if p is a prime and F is a field containing the p^r th roots of unity, then $\ell_F(p^r, p^s) \geq s$, by showing that the generic division algebra itself cannot be represented as a sum of fewer than s classes of cyclic algebras of degree p^r . Tignol’s lower bounds were improved by Jacob in [J] in the prime period case, via the construction of indecomposable algebras of period p and index p^n for all $n \geq 1$ except, of course, for $p = 2$ and

$n = 2$. His bounds, valid for F containing a primitive p th root of unity, are $\ell_F(p, p^s) \geq 2s - 1$ for p odd, and $\ell_F(2, 2^s) \geq 2s - 2$ ([J, Remark 3.7]).

For upper bounds, we then have $\ell_F(p, p) \leq \frac{1}{2}(p - 1)!$ for any odd prime p , provided F contains the p th roots of unity, by [Ro88, Theorem 7.2.43]. Kahn established some upper bounds for $\ell_F(2, 2^s)$ in [K00, Theorem 1] for fields of characteristic not 2, and conjectured the bound $\ell_F(2, 2^s) \leq 2^{s-1}$, which holds in the known cases listed above. Finally, Becher and Hoffmann proved that if F contains a p th root of unity and satisfies the (admittedly strong) hypothesis $[F^\times : F^{\times p}] = p^m$, then $\ell_F(p, p^s) = m/2$ for p odd or $p = 2$ and F nonreal. If $p = 2$ and F is real, the result is $(m + 1)/2$ ([BecH]).

In a related result, Mammone and Merkurjev showed that if A is a p -algebra of period p^r and degree p^s , and A becomes cyclic over a finite separable field extension, then A can be represented by at most p^{s-r} classes of cyclic algebras ([MamM, Prop. 5]).

In [S97a] Saltman proved that if F is the function field of an l -adic curve, then the degree of any F -division algebra divides the square of its period, and it follows by Albert’s theorem that if $l \neq 2$, then $\ell_F(2, 4) = 2$. If $l \neq p$ and F contains the p th roots of unity, then Suresh proved that $\ell_F(p, p^2) = 2$ for odd p in [Sur]. Brussel and Tengan removed the roots of unity requirement in [BT], using a different method. The bounds in the $p = 2$ case were crucial to the determination of the u -invariant of function fields of l -adic curves by Parimala and Suresh [PS10], see also [PS99] and [PS05].

The first open case of Problem 3.5 (with or without roots of unity) is:

Problem 3.6. Find an upper bound on $\ell_F(p, p^2)$ and $\ell_F(p^2, p^2)$ for $p \geq 3$.

4. Period-index problem

By the basic theory, the period of a central simple algebra divides its index (i.e., $\text{per}(A) \mid \text{ind}(A)$ for all A), and the two numbers have the same prime factors. For a field F , define the *Brauer dimension* $\text{Br.dim}(F)$ to be the smallest number n such that $\text{ind}(A)$ divides $\text{per}(A)^n$ for every central simple F -algebra A ; if no such n exists, set $\text{Br.dim}(F) = \infty$.

Example 4.1. If $\text{Br}(F) = 0$, then trivially $\text{Br.dim}(F) = 0$. For F a local field or a global field $\text{Br.dim}(F) = 1$ by Albert, Brauer, Hasse, and Noether. For F a field finitely generated and of transcendence degree 2 over an algebraically closed field, $\text{Br.dim}(F) = 1$ by de Jong [dJ] and [Lie, Theorem 4.2.2.3]. In case $\text{Br.dim}(F) = 1$, one says that “ F has period = index”.

One can focus this notion on a particular prime p . Define $\text{Br.dim}_p(F)$ to be the smallest number n such that $\text{ind}(A)$ divides $\text{per}(A)^n$ for every central simple F -algebra A whose index is a power of p . If no such n exists, we put $\text{Br.dim}_p(F) = \infty$.

Examples 4.2.

- (1) M. Artin conjectured in [Ar82] that $\text{Br.dim}(F) = 1$ for every C_2 field F . He proved that $\text{Br.dim}_2(F) = \text{Br.dim}_3(F) = 1$ for such fields, but no more is known.

- (2) For F finitely generated and of transcendence degree 1 over an ℓ -adic field $\text{Br.dim}_p(F) = 2$ for every prime $p \neq \ell$ by [S97a]. Is $\text{Br.dim}_\ell(F)$ finite?
- (3) If F is a complete discretely valued field with residue field k such that $\text{Br.dim}_p(k) \leq d$ for all primes $p \neq \text{char}(k)$, then $\text{Br.dim}_p(F) \leq d + 1$ for all $p \neq \text{char } k$ by [HHK09, Theorem 5.5].

Example 4.3. If F has characteristic p and transcendence degree r over a perfect field k , then $\text{Br.dim}_p(F) \leq r$ by Proposition 2.6.

Problem 4.4. If F is finitely generated over a field F_0 with $\text{Br.dim}(F_0)$ finite, is $\text{Br.dim}(F)$ necessarily finite?

It is natural to start by considering only certain fields F_0 . For example, suppose that F has prime characteristic p and we take F_0 to be its prime field, so $\text{Br.dim}(F_0) = 0$. If F has transcendence degree 1 over F_0 , then F is global and $\text{Br.dim}(F) = 1$. If F has transcendence degree 2 over F_0 , then $\text{Br.dim}(F) \leq 3$ by Lieblich [Lie].

Problem 4.5. Define a notion of “dimension” for some class of fields F such that

$$\text{Br.dim}(F) \leq \dim F - 1 \quad \text{and} \quad \dim(F(t)) = \dim F + 1.$$

One possibility would be to set a C_i field to have dimension i . The notion of cohomological dimension is obviously not the right one in view of Merkurjev’s example of a field F with cohomological dimension 2 and $\text{Br.dim}(F) = \infty$ from [M92].

5. Center of generic matrices

Amitsur’s universal division algebra has already appeared several times in this paper. In this section we discuss some interesting questions regarding its center.

We begin with the definition. Let F be a field, let $V = M_n(F) \oplus M_n(F)$, and let $F(V)$ be the field of rational functions on V . For $k = 1, 2$, let x_{ijk} be the coordinate function defined by the standard elementary matrix $E_{ij} \oplus 0$ or $0 \oplus E_{ij}$, depending on k , and set $X_k = (x_{ijk}) \in M_n(F(V))$. We call the F -algebra $R(F, n)$ generated by X_1 and X_2 the *ring of (two) generic n -by- n matrices over F* ([S99, Chap. 14]). It is a noncommutative domain that can be specialized to give any central simple L -algebra of degree n , for every extension L of F [S99, §14.1]. The field of fractions of its center $C(F, n) \subset R(F, n)$ is denoted by $Z(F, n)$, and called the *center of generic n -by- n matrices over F* . The (central) localization of $R(F, n)$ by the nonzero elements of $C(F, n)$ is denoted by $UD(F, n)$, and is called the *generic division algebra of degree n over F* . In the language of [GMS, p. 11], the class of $UD(F, n)$ in $H^1(Z(F, n), \text{PGL}_n)$ is a versal torsor.

There is another way to view the above construction, due to Procesi. Let $\text{PGL}_n(F) = \text{GL}_n(F)/F^\times$ be the projective linear group. This group has a representation on V via the action $A \cdot (B_1, B_2) = (AB_1A^{-1}, AB_2A^{-1})$. It follows that $\text{PGL}_n(F)$ acts on $F(V)$, and one can show the invariant field $F(V)^{\text{PGL}_n(F)}$ is $Z(F, n)$. Furthermore, $\text{PGL}_n(F)$ acts naturally on $M_n(F(V)) = M_n(F) \otimes_F F(V)$ via the action on each tensor factor, and one can show the invariant ring

$M_n(F(V))^{\text{PGL}_n(F)}$ is $UD(F, n)$ (see [S99, Theorem 14.16] for proofs of both results). This invariant field point of view puts the problems of this section in the bigger context of birational invariant fields of reductive groups and particularly the birational invariant fields of almost free representations of reductive groups.

One can begin with the above construction and modify it to make other generic constructions. For example, put $D := UD(F, n)$ and $Z := Z(F, n)$ and assume m divides n . Let $Z_m(F, n)$ be the generic splitting field of the division algebra equivalent to $D^{\otimes m}$ over Z , and set $UD_m(F, n) = D \otimes_Z Z_m(F, n)$. Then $UD_m(F, n)$ is a generic division algebra of degree n and period m with center $Z_m(F, n)$. In the language of [GMS], the class of $UD_m(F, n)$ in $H^1(Z_m(F, n), \text{GL}_n/\mu_m)$ is a versal torsor. Another example uses Procesi’s result that $Z(F, n) = F(M)^{S_n}$ where S_n is the symmetric group and M is a specific S_n lattice (see [S99, Theorem 14.17]). The group S_n appears because it is the Galois group of the Galois closure of a “generic” maximal subfield of $UD(F, n)$. It follows that if $H \subset S_n$ is a subgroup and we set $Z_H(F, n) = F(M)^H$, then $UD_H(F, n) = D \otimes_Z Z_H(F, n)$ is a generic central simple algebra having H as the Galois group of the Galois closure of a maximal subfield. In particular, if H has order n and acts transitively on $\{1, \dots, n\}$ (we say H is a transitive subgroup of S_n) we have formed the generic H -crossed product algebra and its center. In this context, see also [Ros]. One can combine these constructions and form generic crossed products of period m , but we will have nothing to say about these.

The general problem addressed here concerns the properties of the fields $Z(F, n)$ and their subsidiary fields $Z_m(F, n)$ and $Z_H(F, n)$. To state the questions, define L/F to be *rational* if L is purely transcendental over F . Define L/F to be *stably rational* if there is a rational L'/L such that L'/F is rational. More generally, say that L_1/F and L_2/F are *stably isomorphic* if there are rational L'_i/L_i such that $L'_1 \cong L'_2$ over F . Finally, define L/F to be *retract rational* if the following holds. There is a localized polynomial ring $R = F[x_1, \dots, x_r](1/s)$ and an F subalgebra $S \subset R$ with an F algebra retraction $R \rightarrow S$ (meaning $S \rightarrow R \rightarrow S$ is the identity) such that L is the field of fractions of S . It is pretty clear that rational implies stably rational implies retract rational, and in fact (much harder) these implications cannot be reversed.

To simplify the discussion note the result of Katsylo [Kat] and Schofield [Sch] that if $n = ab$ for a and b relatively prime, then $Z(F, n)$ is stably isomorphic to the field compositum $Z(F, a)Z(F, b)$. Similar statements are possible for the $Z_m(F, n)$ and $Z_H(F, n)$. This often (but not always) allows reduction to the case where n is a prime power.

Problem 5.1 ([Pro, p. 240]). Is $Z(F, n)/F$ rational, stably rational, or retract rational? The same question for $Z_m(F, n)$ and $Z_H(F, n)$.

This appears as Problem 8 in [S92] (see also Saltman’s Problems 9, 10, and 11), and is a major topic of the surveys [Le] and [Form]. We note that $Z(F, n)/F$ is retract rational if and only if division algebras of degree n have the so-called lifting property, see [S99, p. 77]. Similar statements can be made for $Z_m(F, n)$ and $Z_H(F, n)$.

When n is 2, 3, or 4 then $Z(F, n)$ is rational, as proved (respectively) by Sylvester, Procesi, and Formanek. When n is 5 or 7, Bessenrodt and Le Bruyn

showed in [BesL] that $Z(F, n)$ is stably rational, and a second, more elementary proof of this was given by Beneish [Be98].

There are a few results for the $Z_m(F, n)$. The field $Z_2(F, 4)$ is stably rational by Saltman [S02]. Saltman and Tignol [ST] showed that $Z_2(F, 8)$ is retract rational. Beneish [Be05] showed that $Z_2(F, 8)$ is stably rational. Finally, we leave to the well-read reader to prove that when $H \subset S_n$ is cyclic and transitive and F contains a primitive n th root of unity, then $Z_H(F, n)$ is rational (and results *are* available for general F , n not of the form $8m$, and retract rationality). Thus the first interesting specific question along these lines is:

Problem 5.2. Determine if $Z_H(F, 9)$ is rational, stably rational, or retract rational, where $H = C_3 \times C_3 \subset S_9$ is transitive.

6. Essential dimension

Essential dimension counts the number of parameters needed to define an algebraic structure. This notion was introduced in the late 1990s by Buhler and Reichstein [BuRe] and placed in a general functorial context by Merkurjev [BerF]. Given a field F , a functor $\mathcal{F} : \text{Fields}_F \rightarrow \text{Sets}$ from the category of field extensions of F (together with F -embeddings) to the category of sets, a field extension $F \rightarrow K$, and an element $a \in \mathcal{F}(K)$, a *field of definition* of a is a field extension $F \rightarrow L$ and an F -embedding $L \rightarrow K$ such that a is in the image of the map $\mathcal{F}(L) \rightarrow \mathcal{F}(K)$. The *essential dimension of a* (over F), denoted by $\text{ed}_F(a)$, is the infimum of the transcendence degrees $\text{tr deg}_F(L)$ over all fields of definition L of a . Finally, the *essential dimension of \mathcal{F}* is the number

$$\text{ed}_F(\mathcal{F}) = \sup\{\text{ed}_F(a)\},$$

where the supremum is taken over all field extensions $F \rightarrow K$ and all elements $a \in \mathcal{F}(K)$. We will suppress the dependence on the base field F when no confusion may arise.

The *essential p -dimension* of \mathcal{F} , denoted by $\text{ed}_{F,p}(\mathcal{F})$ or simply $\text{ed}_p(\mathcal{F})$, is defined similarly. One replaces $\text{ed}_F(a)$ with $\text{ed}_{F,p}(a)$, which is the infimum of $\text{ed}_F(a_{K'})$ as K' varies over all finite embeddings $K \rightarrow K'$ of prime-to- p degree. Obviously, $\text{ed}_{F,p}(a) \leq \text{ed}_F(a)$ for all a , hence $\text{ed}_{F,p}(\mathcal{F}) \leq \text{ed}_F(\mathcal{F})$.

Definition 6.1. The *essential dimension* $\text{ed}(G) = \text{ed}_F(G)$ of an algebraic group G over F is the essential dimension of the functor $H^1(-, G)$. Similarly, the *essential p -dimension* $\text{ed}_p(G) = \text{ed}_{F,p}(G)$ of G is the essential p -dimension of the functor $H^1(-, G)$.

The essential dimension of an algebraic group G equals the essential dimension of any versal G -torsor, see [BerF, §6]. For a broad survey of essential dimension results, see Reichstein’s sectional address [Re10] to the 2010 ICM in Hyderabad, India.

The functor $H^1(-, \text{PGL}_n)$ is identified via Galois descent with the functor assigning to a field extension $F \rightarrow K$ the isomorphism classes of central simple K -algebras of degree n . As we shall discuss below, computation of the essential dimension of PGL_n is relevant to the theory of central simple algebras, in the

sense that knowing the “number of parameters” needed to define central simple algebras of degree n can lead to conclusions about their structural properties (e.g., concerning decomposability and crossed product structure).

Problem 6.2. Compute the essential dimension and the essential p -dimension of PGL_n . In particular:

- (1) Compute $\mathrm{ed}_F(\mathrm{PGL}_4)$ when $\mathrm{char} F = 2$.
- (2) Compute $\mathrm{ed}_F(\mathrm{PGL}_5)$.

What is known

The generic division algebra $UD(F, n)$ of degree n corresponds to a versal PGL_n -torsor, see Section 5. It is immediate from Procesi’s description of $Z(F, n)$ by means of invariants that $\mathrm{ed}(\mathrm{PGL}_n) \leq n^2 + 1$, see [Pro, Theorem 1.8]. With some additional work, Procesi [Pro, Theorem 2.1] proved that $\mathrm{ed}(\mathrm{PGL}_n) \leq n^2$. The current best upper bounds are

$$\mathrm{ed}(\mathrm{PGL}_n) \leq \begin{cases} \frac{1}{2}(n-1)(n-2) & \text{if } n \geq 5 \text{ and } n \text{ is odd,} \\ n^2 - 3n + 1 & \text{if } n \geq 4, F = F_{\mathrm{sep}}, \text{ and } \mathrm{char}(F) = 0, \end{cases}$$

see [LRRS, Theorem 1.1], [Lem, Prop. 1.6], and [FaF].

Tsen’s theorem can be used to show that $\mathrm{ed}(\mathrm{PGL}_n) \geq 2$ for any $n \geq 2$, see [Re00, Lemma 9.4a]. The current best lower bounds are

$$\mathrm{ed}(\mathrm{PGL}_n) \geq \mathrm{ed}_p(\mathrm{PGL}_n) \geq (r-1)p^r + 1,$$

where p^r is the highest power of p dividing n and we assume $\mathrm{char} F \neq p$, see Merkurjev [Ma] and the discussion below. This improves on the long-standing lower bound $\mathrm{ed}(\mathrm{PGL}_{p^r}) \geq \mathrm{ed}_p(\mathrm{PGL}_{p^r}) \geq 2r$ in [Re99, Theorem 16.1], [ReY00, Theorem 8.6]. Finally, because of the decomposition of central simple algebras into prime powers, $\mathrm{ed}(\mathrm{PGL}_{nm}) \leq \mathrm{ed}(\mathrm{PGL}_n) + \mathrm{ed}(\mathrm{PGL}_m)$ if $(n, m) = 1$.

Table 1 lists current bounds for $\mathrm{ed}(\mathrm{PGL}_n)$ for small values of n , obtained by combining the bounds in this discussion with those after the discussion of Problem 6.4. (A slightly different table on the same topic can be found at the end of Baek’s thesis [Ba].) Of course, the bounds may improve upon further specification of the field. The cases for $n = 4$ in characteristic 2 and for $n = 5$ are the smallest unknown cases, and each has important implications for the crossed product problems of Sections 2 and 1, respectively (see discussion below). The feeble bounds on the second line of the table — when the characteristic divides n — illustrate our lack of knowledge in this case. One of the few positive results is that the upper bound $\mathrm{ed}(\mathrm{PGL}_4) \leq 5$ from [LRRS, Cor. 3.10(a)] holds without hypotheses on the base field.

TABLE 1. Bounds for $\mathrm{ed}(\mathrm{PGL}_n)$, references found in the text.

n	2	3	4	5	6	7	8	9	10
$\mathrm{ed}(\mathrm{PGL}_n)$ $\mathrm{char} F \nmid n$	2	2	5	$2^\dagger - 6$ 3 – 6	2^\dagger 3	$2^\dagger - 15$ 3 – 15	17 – 64	10 – 28	$2^\dagger - 8$ 3 – 8
$\mathrm{ed}(\mathrm{PGL}_n)$ $\mathrm{char} F \mid n$	2	2	2 – 5	2 – 6	2 – 3	2 – 15	2 – 64	2 – 28	2 – 8

\dagger means “when F contains $\zeta_{n/2}$ if n is even or $\zeta_n + \zeta_n^{-1}$ if n is odd”.

Much more is known about essential p -dimension. For example, $\text{ed}_p(\text{PGL}_n) = \text{ed}_p(\text{PGL}_{p^r})$ if p^r is the highest power of p dividing n [ReY00, 8.5.5]; in particular, $\text{ed}_p(\text{PGL}_n) = 0$ if p does not divide n . Every central simple algebra of degree p becomes cyclic over a prime-to- p extension, which gives $\text{ed}_p(\text{PGL}_p) = 2$ [ReY00, 8.5.7]. Using an extension of Karpenko’s incompressibility theorem to products of p -primary Severi–Brauer varieties, Karpenko and Merkurjev [KaM] provide a formula for the essential dimension of any finite p -group, considered over a field containing the p th roots of unity. This general formula was extended to twisted p -groups and algebraic tori in [LMMR], and ultimately used with great success by Merkurjev [M10], [Ma] to establish the formula $\text{ed}_p(\text{PGL}_{p^2}) = p^2 + 1$ for any field F with $\text{char } F \neq p$, and the current best lower bound

$$(r - 1)p^r + 1 \leq \text{ed}_p(\text{PGL}_{p^r}) \leq p^{2r-2} + 1. \tag{6.3}$$

The above upper bound is in a recent preprint by Ruozzi [Ru], improving on [MeRe, Theorem 1.1]. For $p = 2$ and $r = 3$, note that the upper and lower bounds coincide, yielding $\text{ed}_2(\text{PGL}_8) = 17$ when $\text{char } F \neq 2$.

Asymptotic bounds

It might be illustrative to view these bounds on $\text{ed}(\text{PGL}_n)$ asymptotically, in terms of big- O notation. In that language, we have the naive upper bound that $\text{ed}(\text{PGL}_n)$ is $O(n^2)$ and the naive lower bound that it is $\Omega(1)$, because it is between $n^2 + 1$ and 2. The furious profusion of bounds listed in this section are invisible in this context, except for Merkurjev’s lower bound (6.3), which shows that $\text{ed}(\text{PGL}_n)$ is *not* $O(n)$. (This settled in the negative Bruno Kahn’s “barbecue problem” posed in 1992, see [MeRe, §1].) The gap between the upper and lower bounds for PGL_n stands in interesting contrast with the situation for Spin_n : $\text{ed}(\text{Spin}_n)$ is both $O(\sqrt{2}^n)$ and $\Omega(\sqrt{2}^n)$, i.e., is asymptotically bounded both above and below by constants times $\sqrt{2}^n$, by [BRV].

Essential dimension and crossed products

Let G be a finite group of order $n \geq 2$. Let CPAlg_n (resp. Alg_G) be the functor $\text{Fields}_F \rightarrow \text{Sets}$ assigning to $F \rightarrow K$ the set of isomorphism classes of crossed product (resp. G -crossed product) K -algebras of degree n . Of course, $\text{ed}(\text{CPAlg}_n)$ is the maximum of $\text{ed}(\text{Alg}_G)$ over all groups G of order n . These functors are relevant to the crossed product problem discussed in Section 2. For example, $\text{ed}(\text{Alg}_G) \leq \text{ed}(\text{CPAlg}_n) \leq \text{ed}(\text{PGL}_n)$, with equality if every central simple algebra of degree n over every field extension of F is a G -crossed product. An inequality $\text{ed}(\text{CPAlg}_n) < \text{ed}(\text{PGL}_n)$ would imply the existence of a noncrossed product algebra of degree n over some field extension of F .

Problem 6.4. Determine structural conclusions about central simple algebras from essential (p -)dimension bounds.

- (1) Calculate bounds for $\text{ed}(\text{CPAlg}_n)$ and $\text{ed}(\text{Alg}_G)$.
- (2) Compute the difference $\text{ed}(\text{Alg}_G) - \text{ed}(G)$.
- (3) If $\text{ed}(\text{CPAlg}_n) = \text{ed}(\text{PGL}_n)$, determine whether every algebra of degree n over every extension of F is a crossed product.

What is known

By [Al61, Chap. IX.6, Theorem 9], $H^1(-, \text{PGL}_n)$ is isomorphic to the functor Alg_G for $n = 2, 3, 6$ and $G = C_n$, and for $n = 4$ and $G = C_2 \times C_2$. A result communicated to us by Merkurjev states that $\text{ed}(\text{Alg}_{C_n}) = \text{ed}(C_n) + 1$ when $\text{char } F$ does not divide n (see Proposition 6.6). Since $\text{ed}(C_2) = \text{ed}(C_3) = 1$ (over any field, see also [Se08, §1.1], [JLY, §2.1]), and $\text{ed}(C_6)$ equals 1 or 2 (depending on whether F contains the sixth roots of unity or not), we obtain the values listed in Table 1 for $n = 2, 3, 6$ and $\text{char } F$ not dividing n . When $n = 6$ and $\text{char } F$ divides n , one can show that $\text{ed}(C_6) = 2$ using [Led, Theorem 1], and therefore $2 \leq \text{ed}(\text{PGL}_6) \leq 3$. Note that Proposition 6.6 resolves Problem 6.4(2) for $G = C_n$, when $\text{char } F$ does not divide n .

In [Ma, Theorem 7.1], Merkurjev proved that $\text{ed}(\text{Alg}_{C_2 \times C_2}) = 5$ when $\text{char } F \neq 2$, and thus $\text{ed}(\text{PGL}_4) = 5$ (see also Rost [R]). Note that since $\text{ed}(\text{Alg}_{C_4}) = \text{ed}(C_4) + 1 \leq 3$ when $\text{char } F \neq 2$, this implies the existence of noncyclic algebras of degree 4. These algebras necessarily have period 4 because otherwise they would be biquaternion. Examples of such algebras were exhibited in [Al33].

One can bound $\text{ed}(\text{Alg}_G)$ by bounding the number of generators of G . If a finite group G of order n can be generated by $r \geq 2$ elements, then $\text{ed}(\text{Alg}_G) \leq (r - 1)n + 1$, see [LRRS, Cor. 3.10(a)]. This shows $\text{ed}(\text{PGL}_4) \leq 5$ when $\text{char } F = 2$. The bound is sharp for $\text{char } F \neq p$: if $G = C_p^r$ for $r \geq 2$ and $\text{char } F \neq p$, then $\text{ed}(\text{Alg}_G) = \text{ed}_p(\text{Alg}_G) = (r - 1)p^r + 1$ by [Ma, Theorem 7.1]. Note that this resolves Problem 6.4(1) for elementary abelian G . One can bound the number of generators by $r \leq \log_2(n)$ for $n \geq 4$, see [LRRS, Cor. 3.10(b)], and this bound is realized on elementary abelian 2-groups. Thus $\text{ed}(\text{CPAlg}_n) \leq (\log_2(n) - 1)n + 1$ for $n \geq 4$. Now we have

$$\text{ed}(\text{CPAlg}_8) \leq 17 = \text{ed}_2(\text{PGL}_8) \leq \text{ed}(\text{PGL}_8).$$

As there exist noncrossed products of degree 8, it is reasonable to guess that at least one of the two inequalities is strict.

Algebras with small exponent

The functor $H^1(-, \text{GL}_n/\mu_m)$ assigns to a field K the isomorphism classes of central simple K -algebras of degree n and period dividing m . Of course, $H^1(-, \text{GL}_n/\mu_n) = H^1(-, \text{PGL}_n)$. By Albert’s theorem, every algebra of degree 4 and period 2 is biquaternion, ultimately yielding $\text{ed}(\text{GL}_4/\mu_2) = \text{ed}_2(\text{GL}_4/\mu_2) = 4$ when $\text{char } F \neq 2$, see [BaM10, Remark 8.2]. When $\text{char } F = 2$, all algebras of degree 4 and period 2 are cyclic by [Al34], hence $\text{ed}(\text{GL}_4/\mu_2) = \text{ed}(\text{Alg}_{C_4}) \leq \text{ed}(C_4) + 1 = 3$ (one can bound $\text{ed}(C_4)$ using Witt vectors of length 2, see [JLY, Theorem 8.4.1]). Similarly, for GL_8/μ_2 , Rowen [Ro78] produced triquadratic splitting fields, ultimately yielding $\text{ed}(\text{GL}_8/\mu_2) = \text{ed}_2(\text{GL}_8/\mu_2) = 8$ when $\text{char } F \neq 2$, see [BaM10, Cor. 8.3]. In particular, this implies the existence of algebras of degree 8 and period 2 (when $\text{char } F \neq 2$) that are not decomposable as a product of three quaternion algebras; such examples were exhibited in [AmRT].

When $\text{char } F \neq p$, the current best bounds for $\text{ed}_p(\text{GL}_{p^r}/\mu_{p^s})$ are

$$\left. \begin{matrix} (r - 1)2^{r-1} \\ (r - 1)p^r + p^{r-s} \end{matrix} \right\} \leq \text{ed}_p(\text{GL}_{p^r}/\mu_{p^s}) \leq \begin{cases} 2^{2r-2} & \text{for } p = 2 \text{ and } s = 1, \\ p^{2r-2} + p^{r-s} & \text{otherwise,} \end{cases}$$

for $r \geq 2$ and $1 \leq s \leq r$, see [BaM10, Theorem 6.1, 7.2], [Ru, Theorem 1.1], and [Ba, Theorem 4.1.30]. For p odd and $r = 2$, the upper and lower bounds coincide, yielding

$$\text{ed}_p(\text{GL}_{p^2}/\mu_{p^2}) = p^2 + 1 \quad \text{and} \quad \text{ed}_p(\text{GL}_{p^2}/\mu_p) = p^2 + p.$$

This implies the existence of indecomposable algebras of degree p^2 and period p , as exhibited previously in [T87], and is an example of an essential dimension result having implications on the existence of algebras with certain structural properties.

The smallest open cases seem to be:

Problem 6.5. Compute $\text{ed}_F(\text{GL}_8/\mu_2)$ when $\text{char } F = 2$; $\text{ed}_F(\text{GL}_{16}/\mu_2)$ when $\text{char } F \neq 2$; and $\text{ed}_F(\text{GL}_{p^2}/\mu_p)$ for odd primes p and all F .

The last part overlaps with Problem 2.1.

We prove the essential dimension result stated above:

Proposition 6.6. *If $\text{char } F$ does not divide $n > 1$, then $\text{ed}_F(\text{Alg}_{C_n}) = \text{ed}_F(C_n) + 1$.*

Proof. Since clearly $\text{ed}_F(\text{Alg}_{C_n}) \leq \text{ed}_F(C_n) + 1$, it suffices to produce an algebra with essential dimension at least $\text{ed}_F(C_n) + 1$. Suppose K/F is a field extension, L/K is a cyclic field extension of degree n , and $A = (L/K, \sigma, t)$ is a cyclic crossed product over the rational function field $K(t)$. The parameter t defines a discrete valuation v on K , and we have a residue $\partial_v(A) = \chi$, for an element χ of order n in the character group $X(K)$. If A is defined as A_0 over an extension E/F , then t defines a discrete valuation v_0 on E , and $\partial_{v_0}(A_0) = \chi_0 \in X(\kappa(v_0))$. Since the residue map commutes with scalar extension—scaled by the ramification index if necessary— χ_0 has order at least n , hence $\text{ed}_F(C_n) \leq \text{tr deg}_F(\kappa(v_0))$. Since $\text{tr deg}_F(\kappa(v_0)) < \text{tr deg}_F(E)$, we conclude $\text{ed}_F(C_n) < \text{ed}_F(A)$. \square

7. Springer problem for involutions

Involutions on central simple algebras are ring-antiautomorphisms of period 2. They come in different types, depending on their action on the center and on the type of bilinear forms they are adjoint to after scalar extension to an algebraic closure of the center: an involution is of *orthogonal* (resp. *symplectic*) *type* if it is adjoint to a symmetric, nonalternating bilinear form (resp. to an alternating bilinear form) over an algebraic closure; it is of *unitary type* if its restriction to the center is not the identity. The correspondence between central simple algebras with involution and linear algebraic groups of classical type was first pointed out by Weil [Wei]; it is a deep source of inspiration for the development of the theory of central simple algebras with involution, see [KMRT]. For semisimple linear algebraic groups, Tits defined in [Ti66] a notion of index, which generalizes the Schur index of central simple algebras and the Witt index of quadratic forms. Anisotropic general linear groups arise from division algebras, and anisotropic quadratic forms yield anisotropic orthogonal groups. More generally, every adjoint group of type ${}^1\text{D}_n$ or ${}^2\text{D}_n$ over a field of characteristic different from 2 can be represented as the group of automorphisms of a central simple algebra A of degree $2n$ with orthogonal involution σ ; the group is anisotropic if and only if σ is anisotropic in the following

sense: for $x \in A$, the equation $\sigma(x)x = 0$ implies $x = 0$. The behavior of the index under scalar extension is a major subject of study, to which the following problem, first formulated in [BayST, p. 475], pertains:

Problem 7.1. Suppose σ is an anisotropic orthogonal or symplectic involution on a central simple algebra A over a field F . Does σ remain anisotropic after scalar extension to any odd-degree field extension of F ?

(If $\text{char } F = 2$, the question makes sense, but to preserve the relation with linear algebraic groups one should replace the orthogonal involution σ with a quadratic pair [KMRT, §5B].)

Representing A as $\text{End}_D V$ for some vector space V over a division algebra D , we may rephrase the problem in terms of Hermitian forms: if a Hermitian or skew-Hermitian form over a division algebra with orthogonal or symplectic involution is anisotropic, does it remain anisotropic after any odd-degree extension? When the central simple algebra is split (which is always the case if its degree $\deg A$ is odd), orthogonal involutions are adjoint to symmetric bilinear forms, and the problem has an affirmative solution by a well-known theorem of Springer [EKM, Cor. 18.5] (actually first proved—but not published—by E. Artin in 1937, see [K08, Remark 1.5.3]). On the other hand, every symplectic involution on a split algebra is hyperbolic, hence isotropic, so the problem does not arise for symplectic involutions on split algebras. At the other extreme, if A is a division algebra, then the solution is obviously affirmative, since A remains a division algebra after any odd-degree extension and every involution on a division algebra is anisotropic.

Variants of Problem 7.1 take into account the “size” of the isotropy: call a right ideal $I \subset A$ *isotropic* for an involution σ if $\sigma(I) \cdot I = \{0\}$, and define its *reduced dimension* by $\text{rdim } I = \dim I / \deg A$. The reduced dimension of a right ideal can be any multiple of the Schur index $\text{ind } A$ between $\text{ind } A$ and $\deg A$, but for isotropic ideals we have $\text{rdim } I \leq \frac{1}{2} \deg A$, see [KMRT, §6A]. The involution σ is called *metabolic* if A contains an isotropic ideal of reduced dimension $\frac{1}{2} \deg A$. A weak version of Springer’s theorem holds for arbitrary involutions: Bayer and Lenstra proved in [BayL, Prop. 1.2] that an involution that is not metabolic cannot become metabolic over an odd-degree field extension. As a result, Problem 7.1 also has an affirmative solution when $\text{ind } A = \frac{1}{2} \deg A$, for in this case isotropy implies metabolicity. (See [BerFT, Theorem 1.14] for the analogue for quadratic pairs.)

If $\text{char } F \neq 2$ and $\text{ind } A = 2$, Problem 7.1 was solved in the affirmative by Parimala, Sridharan, and Suresh [PSS01]. The symplectic case is easily reduced to the case of quadratic forms by an observation of Jacobson relating Hermitian forms over quaternion algebras to quadratic forms. The orthogonal case also is reduced to the case of quadratic forms, using scalar extension to the function field of the conic that splits A . That approach relates Problem 7.1 to another important question, to which Parimala, Sridharan, and Suresh gave a positive solution when $\text{ind } A = 2$ and $\text{char } F \neq 2$:

Problem 7.2. Suppose σ is an anisotropic orthogonal involution on a central simple algebra A over a field F . Does σ remain anisotropic after scalar extension to the function field F_A of the Severi–Brauer variety of A ?

(This is a special case of a general problem concerning semisimple algebraic

groups, namely whether a projective homogeneous variety under one group has a rational point over the function field of a projective homogeneous variety under another group. From this perspective, Problem 7.2 is asking for an analogue of the index reduction results from [SchB], [MPW96], [MPW98], etc.)

By Springer’s theorem, an affirmative answer to Problem 7.2 readily implies that Problem 7.1 has an affirmative solution for orthogonal involutions. Surprisingly, Karpenko [Ka09a] recently proved that the converse also holds when $\text{char } F \neq 2$: an orthogonal involution cannot become isotropic over F_A unless it also becomes isotropic over some odd-degree extension of F .

Problem 7.2 is addressed in the following papers (besides [PSS01] and [Ka09a]): [Ka00], [Ka09b], [G09], [Ka10b]. In [Ka00] and [Ka10a, §2.5], Karpenko gives an affirmative solution when A is a division algebra. This result is superseded in [Ka09b], where he proves that for any quadratic pair on a central simple algebra A of arbitrary characteristic, the Witt index of the quadratic form to which the quadratic pair is adjoint over F_A is a multiple of $\text{ind } A$. In particular, it follows that if the quadratic pair becomes hyperbolic over F_A , then $\text{ind } A$ divides $\frac{1}{2} \text{deg } A$. An alternative proof of the latter result is given by Zainouline in [G09, App. A]. A much stronger statement was soon proved by Karpenko [Ka10b]: if $\text{char } F \neq 2$, an orthogonal involution that is not hyperbolic cannot become hyperbolic over F_A . (The special case where $\text{deg } A = 8$ and $\text{ind } A = 4$ was obtained earlier by Sivatski in [Si, Prop. 3], using Laghribi’s work on eight-dimensional quadratic forms [Lag, Theorem 4].)

Note that the orthogonal and symplectic cases of Problem 7.1 are related by the following observation: tensoring a given central simple F -algebra with involution (A, σ) with the “generic” quaternion algebra $(x, y)_{F(x, y)}$ with its conjugation involution yields a central simple $F(x, y)$ -algebra with involution (A', σ') , which is anisotropic if and only if σ is anisotropic. If $\text{char } F \neq 2$, the involution σ' is orthogonal (resp. symplectic) if and only if σ is symplectic (resp. orthogonal). Therefore, a negative solution to Problem 7.1 for a given degree d and index i in the orthogonal (resp., symplectic) case readily yields a negative solution in the symplectic (resp. orthogonal) case in degree $2d$ and index $2i$. The same idea can be used to obtain symplectic versions of the results on Problem 7.2, where F_A is replaced by a field over which the index of A is generically reduced to 2, see [Ka10b, App. A].

If $\text{char } F \neq 2$, the smallest index for which Problem 7.1 is open is 4, and the smallest degree is 12. Since Parimala, Sridharan, and Suresh do not address the characteristic 2 case in [PSS01], Problem 7.1 — restated for quadratic pairs — seems to be open in this case already for index 2 and degree 8. (The answer to Problem 7.1 is “yes” in the degree 6 case. This can be seen via the exceptional isomorphism $D_3 = A_3$.)

Unitary involutions

The analogue of Problem 7.1 for unitary involutions was solved in the negative in [PSS01]: for any odd prime p , there is a central simple algebra A of degree $2p$ and index p with anisotropic unitary involution over a field $F = \ell((t))$, where ℓ is a ramified quadratic extension of a p -adic field k , which becomes isotropic over any odd-degree extension of F that splits A . If $p \geq 5$, there are unitary involutions on division algebras of degree p that become isotropic after scalar extension to

any maximal subfield. As Parimala, Sridharan, and Suresh suggest at the end of their paper, the correct analogue of Problem 7.1 for unitary involutions should probably ask: Does the involution remain anisotropic after scalar extension to a field extension of degree coprime to $2 \operatorname{ind} A$?

Motivated by this observation, we can generalize Problem 7.1 by asking:

Problem 7.3. Let G be an absolutely almost simple linear algebraic group that is anisotropic over F . Does G remain anisotropic over every finite extension K/F of dimension not divisible by any prime in $S(G)$?

Here $S(G)$ denotes the set of “homological torsion primes” from [Se95, §2.2.3], exhibited in Table 2. Roughly speaking, the results on Problem 7.1 for orthogonal and symplectic involutions concern the types B, C, and D cases of Problem 7.3 and the results for unitary involutions concern the type A case. The answer to Problem 7.3 is “yes” for G of type G_2 or F_4 ; this can be seen by inspecting the cohomological invariants of these groups described in [KMRT].

TABLE 2. The set $S(G)$ of homological torsion primes for an absolutely almost simple algebraic group G .

type of G	elements of $S(G)$
A_n	2 and prime divisors of $n + 1$
B_n, C_n, D_n ($n > 4$), G_2	2
D_4, F_4, E_6, E_7	2 and 3
E_8	2, 3, and 5

8. Artin–Tate conjecture

Motivated by an analogue of the Birch and Swinnerton-Dyer conjecture for abelian varieties over global fields of characteristic p , Tate (reporting on joint work with Artin) [Ta66b] introduces a host of conjectures for surfaces over finite fields. As most of the progress on these conjectures has been made from their connection with the Tate conjecture on algebraic cycles and the Birch and Swinnerton-Dyer conjecture, a natural challenge arises: *Can progress be made on the conjectures in this section using “central simple algebra techniques”?*

Let $k = \mathbb{F}_q$ be a finite field of characteristic p and X a geometrically connected, smooth, and projective k -surface. Let $\operatorname{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$, which equals the Azumaya Brauer group since X is regular over a field.

Conjecture (Artin–Tate conjecture A). *The Brauer group of X is finite.*

Since $\operatorname{Br}(X) = \operatorname{Br}_{\text{ur}}(k(X))$, where $\operatorname{Br}_{\text{ur}}(k(X))$ denotes the unramified (at all codimension 1 points) Brauer group of the function field $k(X)$ of X , Conjecture A is equivalent to:

Conjecture (Artin–Tate conjecture A). *If $k(X)$ is a function field of transcendence degree 2 over a finite field, then there are finitely many unramified classes in $\operatorname{Br}(k(X))$.*

Relationship with the Tate conjecture on algebraic cycles

Let $\ell \neq p$ be a prime. Let $P_2(X, t)$ be the characteristic polynomial of the action of the (geometric) Frobenius morphism on the ℓ -adic cohomology $H^2(\overline{X}, \mathbb{Q}_\ell(1))$, where $\overline{X} = X \times_k k_{\text{sep}}$. Let $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ denote the Néron–Severi group of X and $\rho(X) = \text{rank}_{\mathbb{Z}} \text{NS}(X)$ its Picard number. Tate [Ta65, §3] has earlier conjectured a relationship between the Picard number and this characteristic polynomial for surfaces.

Conjecture (Tate’s conjecture T). *The Picard number of X is equal to the multiplicity of the root $t = q^{-1}$ in the polynomial $P_2(X, t)$.*

Tate [Ta66b, §4, Conj. C] refines this by providing a conjectural leading term for the polynomial $P_2(X, t)$ at $t = q^{-1}$.

Conjecture (Artin–Tate conjecture C).

$$\lim_{s \rightarrow 1} \frac{P_2(X, q^{-s})}{(1 - q^{1-s})^{\rho(X)}} = \frac{|\text{Br}(X)| |\det(h_{\text{NS}(X)})|}{q^{\alpha(X)} |\text{NS}(X)_{\text{tors}}|^2},$$

where $h_{\text{NS}(X)}$ is the intersection form on the Néron–Severi group modulo torsion, $\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim \mathcal{P}ic_X$, and $\mathcal{P}ic_X$ is the Picard variety of X .

In fact, it turns out that showing the finiteness of $\text{Br}(X)$ is the “hard part,” and all of the above conjectures are equivalent.

Theorem 8.1. *The following statements are equivalent:*

- (1) *Conjecture A holds for X , i.e. $\text{Br}(X)$ is finite.*
- (2) *${}_p\text{Br}(X)$ is finite for some prime ℓ (with $\ell = p$ allowed).*
- (3) *Conjecture T holds for X .*
- (4) *Conjecture C holds for X .*

In [Ta66b, Theorem 5.2] the prime-to- p part of this theorem is proved, i.e., the finiteness of the prime-to- p part of the Brauer group ${}_p\text{Br}(X)$ is equivalent to Conjecture C up to a power of p . Milne [Mi75, Theorem 4.1] proves the general case using comparisons between étale and crystalline cohomology. Note that in [Mi75], Milne assumes that $p \neq 2$, though on his website addendum page, he points out that by appealing to Illusie [Ill] in place of a preprint of Bloch, this hypothesis may be removed.

In turn, Conjecture T is a special case of the Tate conjecture on algebraic cycles [Ta65, Conj. 1], made at the AMS Summer Institute at Woods Hole, 1964. Denote by $Z^i(X)$ the group of algebraic cycles of codimension i on a variety X .

Conjecture (Tate conjecture). *Let k be a field finitely generated over its prime field, $\Gamma = \text{Gal}(k_{\text{sep}}/k)$, and X a geometrically connected, smooth, and projective k -variety. Then the image of the cycle map*

$$\text{cl} : Z^i(\overline{X}) \rightarrow H^{2i}(\overline{X}, \mathbb{Q}_\ell(i))$$

spans the \mathbb{Q}_ℓ -vector subspace $H^{2i}(\overline{X}, \mathbb{Q}_\ell(i))^\Gamma$ of Galois invariant classes.

In the case of interest to us, of (co)dimension 1 cycles on a surface over a finite field, the cycle map factors through the Néron–Severi group and Milne [Mi75, Theorem 4.1] (following Tate [Ta65, §3]) proves that the Tate conjecture (in this case, equivalent to $\rho(X) = \text{rank}_{\mathbb{Z}} H^2(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma$) is equivalent to Conjecture T.

What is known

Conjecture T is known for many special classes of surfaces: rational surfaces (Milne [Mi70]), Fermat hypersurfaces $X_0^n + X_1^n + X_2^n + X_3^n = 0$, with $p \equiv 1 \pmod n$ or $p^a \equiv -1 \pmod n$ for some a (Tate [Ta65, §3]), abelian surfaces and products of two curves (Tate [Ta66a, Theorem 4]), $K3$ surfaces with a pencil of elliptic curves (Artin and Swinnerton-Dyer [ArSD, Theorem 5.2]), $K3$ surface of finite height (where $p \geq 5$) (Nygaard [Ny, Cor. 3.4] and Nygaard and Ogus [NyO, Theorem 0.2]), and certain classes of Hilbert modular surfaces (Harder, Langlands, and Rapoport [HLR], Murty and Ramakrishnan [MuRa], Langer [Lan00], [Lan04]). Conjecture T respects birational equivalence. Furthermore, if $X \rightarrow X'$ is a finite morphism and Conjecture T holds for X , then it holds for X' . In particular, Conjecture T holds for unirational surfaces and Kummer surfaces. Also, Conjecture T holds for any surface that lifts to a smooth surface in characteristic 0 with geometric genus $p_g = 0$.

Relationship with the Birch and Swinnerton-Dyer conjecture

The refined statement of the Birch and Swinnerton-Dyer (BSD) conjecture over global fields of characteristic p appears in [Ta66b, §1].

Conjecture (BSD conjecture). *Let K be a global field of characteristic p , A an abelian K -variety, and $L^*(A, s)$ the L -function of A with respect to a suitably normalized Tamagawa measure. Then $L^*(A, s)$ has an analytic continuation to the complex plane and*

$$\lim_{s \rightarrow 1} \frac{L^*(A, s)}{(s - 1)^r} = \frac{|\text{III}(A)| |\det(h_A)|}{|A(K)_{\text{tors}}| |\hat{A}(K)_{\text{tors}}|},$$

where h_A is the Néron–Tate height pairing on $A(K)$ modulo torsion, \hat{A} is the dual abelian variety, and $r = \text{rank}_{\mathbb{Z}} A(K)$ is the rank of the Mordell–Weil group.

Milne [Mi68] proves the BSD conjecture for constant abelian varieties. Otherwise, there are a host of results on the BSD conjecture for particular classes of abelian varieties over global fields of characteristic p . For a modern treatment of the BSD conjecture for abelian varieties over number fields, see Hindry and Silverman [HS, App. F.4].

Now let C be a geometrically connected, smooth, and projective k -curve with function field $K = k(C)$ and suppose that there is a proper flat morphism $f : X \rightarrow C$ with generic fiber X_K a geometrically connected, smooth, and projective K -curve. Let A be the Jacobian of X_K .

Theorem 8.2 (Artin–Tate conjecture d). *Conjecture C for the k -surface X is equivalent to the BSD conjecture for the Jacobian A of the K -curve X_K .*

Tate explains that “this conjecture gets only a small letter d as label, because it is of a much more elementary nature” than Conjecture C or the BSD conjecture, see [Ta66b, §4]. The Artin–Tate conjecture d is proved by Liu, Lorenzini, and Raynaud [LLR05], following a suggestion of Leslie Saper. In fact, as shown in [LLR05, Proof of Theorem 1], any geometrically connected, smooth, and projective k -surface X is birational to another such surface X' admitting a proper flat morphism $f' :$

$X' \rightarrow \mathbb{P}^1$ with geometrically connected, smooth, and projective generic fiber. In particular, up to birational transformations, we may replace any surface by a surface fibered over a curve to which Conjecture d applies.

Square order of the Brauer group

If either group is assumed to be finite, then a precise relationship between $|\mathrm{Br}(X)|$ and $|\mathrm{III}(A)|$ is given in Liu, Lorenzini, and Raynaud [LLR04, Theorem 4.3], [LLR05, Cor. 3] in terms of the local and global periods and indices of the curve X_K .

Theorem 8.3 (Liu, Lorenzini, and Raynaud [LLR05, Cor. 3]). *Let $f : X \rightarrow C$ be as above. For each closed point v of C , denote by K_v the completion of $K = k(C)$ at v , and by δ_v and δ'_v the index and period, respectively, of the K_v -curve X_{K_v} . Denote by δ the index of the K -curve X_K . Assume that either $\mathrm{Br}(X)$ or $\mathrm{III}(A)$ is finite, where A is the Jacobian of X_K . Then*

$$|\mathrm{III}(A)| \prod_{v \in C} \delta_v \delta'_v = |\mathrm{Br}(X)| \delta^2$$

and $|\mathrm{Br}(X)|$ is a square.

As explained in the Introduction of [LLR05], the fact that the Brauer group of X has (conjecturally) square order is surprising given the history of the subject. Tate [Ta66b] (see also Milne [Mi75, Theorem 2.4]) exhibits a skew-symmetric bilinear pairing,

$$\begin{aligned} \mathrm{Br}(X) \times \mathrm{Br}(X) &\rightarrow \mathbb{Q}/\mathbb{Z}, \\ (x, y) &\mapsto \mathrm{tr}(\tilde{x} \cup \beta \tilde{y}), \end{aligned}$$

on the Brauer group of a surface over a finite field, defined by taking lifts $\tilde{x}, \tilde{y} \in H_{\text{ét}}^2(X, \mu_n)$ of $x, y \in {}_n\mathrm{Br}(X)$, where $\beta : H_{\text{ét}}^2(X, \mu_n) \rightarrow H_{\text{ét}}^3(X, \mu_n)$ is the coboundary map arising from the exact sequence $1 \rightarrow \mu_n \rightarrow \mu_{n^2} \rightarrow \mu_n \rightarrow 1$, and $\mathrm{tr} : H_{\text{ét}}^5(X, \mu_n^{\otimes 2}) \cong \mathbb{Z}/n\mathbb{Z}$ is the arithmetic trace map induced from duality and the Hochschild–Serre spectral sequence, as in [Ta66b, Formula 5.4]. This pairing is nondegenerate on the prime-to- p part of the Brauer group and has kernel consisting of the maximal divisible subgroup of $\mathrm{Br}(X)$. In particular, if $\mathrm{Br}(X)$ is assumed to be finite, it must have order either a square or twice a square.

In the late 1960s Manin published examples of rational surfaces over finite fields with Brauer group of order 2. Finally, in 1996, Urabe found a mistake in Manin’s examples and proved that rational surfaces always have Brauer groups of square order. The proof that ${}_2\mathrm{Br}(X)$ is a square in [LLR05] uses knowledge of ${}_2\mathrm{III}(A)$, via Theorem 8.2.

The (conjectural) square order of $\mathrm{Br}(X)$ would immediately result from the existence of a nondegenerate alternating pairing. Of course, this is only a nontrivial question on the 2-primary torsion $\mathrm{Br}(X)_2$ of the Brauer group.

Problem 8.4. Given a geometrically connected, smooth, projective surface X over a finite field k , is there a nondegenerate alternating pairing on $\mathrm{Br}(X)_2$ with values in \mathbb{Q}/\mathbb{Z} ?

This is addressed in Urabe [U, Theorem 0.3], where it is proved that the restriction of Tate’s skew-symmetric form to $\ker(\mathrm{Br}(X) \rightarrow \mathrm{Br}(\overline{X}))$ is alternating.

As a corollary, this settles the question whenever $H_{\text{ét}}^3(\overline{X}, \mathbb{Z}_2(1))_{\text{tors}}$ is trivial, in particular, this is the case for rational surfaces, ruled surfaces, abelian surfaces, K3 surfaces, or complete intersections in projective space. The question is still open, for example, if X is an Enriques surface. Also see Poonen and Stoll's [PooS, §11] possible strategy for finding a counterexample.

9. The tensor product problem

Both surveys [Am82] and [S92] address problems concerning division algebra decomposability. We say an F -division algebra D is *decomposable* if there exist F -division algebras A and B such that $D \simeq A \otimes_F B$. All division algebras decompose into p -primary tensor factors, and all p -primary factors with equal period and index are indecomposable, so the problem of when a division algebra decomposes is only interesting for division algebras of unequal prime-power period and index. Albert proved that any division algebra of period 2 and index 4 is decomposable in [Al32], and for a long time all known division algebras of unequal period and index were decomposable. However, in 1979 Saltman [S79] and Amitsur, Rowen, and Tignol [AmRT] independently constructed nontrivially indecomposable division algebras. In 1991 Jacob produced indecomposable division algebras of prime period p and index p^n for any $n \geq 1$, except when $p = 2$ and $n = 2$ ([J]). Thus by the time of Saltman's survey [S92], examples had been produced of many period-index combinations, though not in every characteristic.

Shortly after [S92], Karpenko produced indecomposable division algebras (of a generic type) in any characteristic with any odd prime-power period p^m and index p^n , for any $m \leq n$ (see also [McK08]), and period 2 and index 2^n for $n \geq 3$. The odd degree examples used the index reduction formula of Schofield and van den Bergh [SchB] and the theory of cycles on Severi–Brauer varieties ([Ka95]); the 2-power degree examples used Karpenko's theorem [Ka98, Prop. 5.3] that a division algebra A of prime period is indecomposable if the torsion subgroup of the Chow group of codimension 2 cycles on the Severi–Brauer variety of A is nontrivial.

There were two problems on indecomposability in [S92], one (Problem 6) on the existence of an indecomposable division algebra of p -primary degree that becomes decomposable after a prime-to- p extension, and the other (Problem 7) on the existence of indecomposable division algebras of unequal period and index over fields finitely generated over a prime field. Both were solved by Brussel, in [B00, Problem 6] and [B96, Problem 7]. However, Brussel's solution to Problem 6 relied on the absence of a p th root of unity in the center. Karpenko's examples, on the other hand, are manifestly stable under prime-to- p extension.

In spite of the fact that some gaps remain (no one has submitted an indecomposable division algebra of period 4 and index 16, for example), recent work has focused less on producing examples of indecomposable division algebras of given index and period, and more on understanding the significance of indecomposability and its relation to common subfields.

For a fixed primitive n th root of unity ξ , and $a, b \in F^\times$, let $(a, b)_n$ denote the symbol algebra of degree n over F , generated by elements u and v under the relations $u^n = a$, $v^n = b$, $uv = \xi vu$.

Problem 9.1. Suppose $(a, b)_n$ and $(c, d)_n$ are symbol algebras of degree n over F , and $\text{ind}((a, b)_n \otimes_F (c, d)_n) < n^2$. Determine whether there are necessarily symbol algebras $(a', b')_n$ and $(c', d')_n$ that share a common maximal subfield, and such that $(a, b)_n \otimes_F (c, d)_n \cong (a', b')_n \otimes_F (c', d')_n$?

One can also consider splitting fields instead of maximal subfields:

Problem 9.2. Suppose A and B are F -division algebras such that $\text{ind}(A \otimes B) < \text{ind}(A) \cdot \text{ind}(B)$. Determine the common subfields of A and B .

What is known

The first major result on Problem 9.1 was Albert's theorem that in characteristic not 2, the tensor product of two quaternion division algebras is a division algebra if and only if the quaternions do not have a common maximal subfield ([Al72]). The characteristic 2 analog was proved by Sah [Sah], and improved by Draxl: if the tensor product of two quaternion division algebras is not a division algebra, then they have a common separable maximal subfield (also see [L]).

Using a similar proof, Risman [Ri75] showed that for any quaternion algebra A and any division algebra B , if $A \otimes B$ is not a division algebra, then either B contains a field K such that $A \otimes K$ is not a division algebra, or A contains a field K such that $K \otimes B$ is not a division algebra.

In spite of the Albert–Sah theorem for quaternions, Tignol and Wadsworth [TW, Prop. 5.1] produced for any odd integer n an example in which A and B are division algebras of degree n , $A \otimes_F B$ is not a division algebra, and yet A and B have no common maximal subfields. Saltman pointed out that for these examples, $A \otimes_F B^{\text{op}}$ is a division algebra, and noted that a better question to ask is whether one could have $A^{\otimes i} \otimes B$ not a division algebra for all i , with A and B still having no common subfields. Mammone answered the question in [Mam], producing examples $A \otimes_F B$ with A of degree n and B of degree n^2 for any n . Then Jacob and Wadsworth [JW93, Theorem 1] constructed examples of degree p for any odd prime p , using valuation-theoretic methods. The Jacob and Wadsworth examples directly support an affirmative answer to Problem 9.1. In [Ka99, Theorem II.1], Karpenko produced an essentially different family of examples, also of equal odd prime degrees, using the theory of the Chow group of Severi–Brauer varieties.

As for Problem 9.2, it follows from Albert's and Risman's examples that if A is a quaternion, B has degree 2^n , and $A \otimes B$ is not a division algebra, then A and B have a common splitting field of degree less than or equal to 2^n over F . The Mammone, Jacob and Wadsworth, and Karpenko examples showed that the results of Albert and Risman could not be generalized in the obvious way; in particular any common splitting field of the degree p examples is divisible by p^2 . However, Karpenko proceeded to generalize the results of Albert and Risman as follows. Let A be a central simple F -algebra of degree p , let B_1, \dots, B_{p-1} be central simple F -algebras of degrees $p^{n_1}, \dots, p^{n_{p-1}}$, and let $n = n_1 + \dots + n_{p-1}$. Suppose that for every $i = 1, \dots, p-1$, $A \otimes_F B_i$ has zero divisors. Then there exists a field extension E/F of degree less than or equal to p^n that splits all of the algebras A, B_1, \dots, B_{p-1} ([Ka99, Theorem 1.1]). In a similar vein, Krashen proved that if A_1, \dots, A_{p^k} are central simple F -algebras of degrees p^n for a prime p , and $A_1 \otimes_F \dots \otimes_F A_{p^k}$ has index p^k , then the A_i are split by an étale extension

E/F of degree $p^{n(p^k-1)}m$, where m is prime-to- p ([Kr10, Cor. 4.4]). Problems such as these are connected with the notion of canonical dimension, see [Ka10a] for a survey.

As noted above, Karpenko's examples made use of the seminal index reduction formula due to Schofield and Van den Bergh from [SchB]. Such formulas should play a role in helping to resolve Problem 9.2, which is framed quite generally. For example, there has been considerable recent interest in the special case: *If A and B have the same splitting fields (chosen from some class of extension fields), then are A and B isomorphic?* We address this in the next section.

10. Amitsur's conjecture

As a central simple F -algebra A is largely understood via the study of its splitting fields (see Section 0), it is natural to try to determine exactly how much information about A is captured by its splitting fields alone.

In [Am55] Amitsur constructed for any F -division algebra A a *generic splitting field* $F(A)$, whose defining property is that a field extension L/F splits A if and only if $F(A)$ has a place in L (see [BourC, VI.2]). The first field of this type was constructed by Witt in [Wi34], for quaternion algebras. Amitsur proved that his $F(A)$ is the function field of the Severi–Brauer variety $SB(A)$ of A , first studied by Châtelet in [Châ]. Nowadays, the generic splitting field of a central simple algebra is viewed as a special instance of the generic splitting field of a (reductive) algebraic group, as described in [KeR]. Amitsur proved:

Theorem 10.1 ([Am55, §9]). *Let A and B be central simple F -algebras.*

- (1) *A and B have the same splitting fields if and only if A is Brauer-equivalent to $B^{\otimes i}$ for some integer i relatively prime to the period of A .*
- (2) *If $F(A)$ is isomorphic to $F(B)$, then A and B generate the same subgroup of the Brauer group.*

He then posed the following converse of (2):

Conjecture 10.2 (Amitsur's conjecture). *If A and B have the same degree and generate the same subgroup of the Brauer group, then $F(A)$ and $F(B)$ are isomorphic.*

The question was raised in [Am55], and appears as Question 6 in [Am82].

By definition, the generic splitting fields $F(A)$ and $F(B)$ are isomorphic if and only if the Severi–Brauer varieties $SB(A)$ and $SB(B)$ are F -birational. It is not hard to show that if A and B are as in the conjecture, then $F(A)$ and $F(B)$ are stably isomorphic.

What is known

Amitsur was able to prove his conjecture in either the case where A has a maximal subfield that is cyclic Galois, or in the case where A is arbitrary and B is A^{op} [Am55]. Roquette later extended these results to the case where A has a maximal subfield which is solvable Galois [Roq64]. Tregub, using very different techniques, proved the result for A arbitrary of odd degree in the case where B is equivalent to

$A^{\otimes 2}$ [Tr]. Krashen generalized Roquette’s results to the case where A has a maximal subfield whose Galois closure is dihedral (hence quadratic over the maximal subfield), and also showed that the conjecture may be reduced to considering the case where every nontrivial subfield of A is a maximal subfield [Kr03].

Theorem 10.1(1) says that the splitting fields almost determine the algebra. But it may be too much to expect to know all of the splitting fields of a central simple algebra, and so we will pose a finer problem. Note first that Theorem 10.1(2) says that if algebras A and B have degree n , then the fields $F(A)$ and $F(B)$ have transcendence degree $n-1$. It follows that *if A and B have the same splitting fields of transcendence degree $< n$, then they generate the same subgroup of the Brauer group*. It is therefore natural to ask about splitting fields of smaller transcendence degrees, or even finite splitting fields.

Problem 10.3. Determine whether two division algebras that have the same splitting fields in a given class (say finite, or of transcendence degree at most 1), necessarily generate the same subgroup in the Brauer group.

The smallest open case appears to be:

Problem 10.4. Determine whether two division algebras of degree 3 that have the same splitting fields of transcendence degree ≤ 1 generate the same subgroup of the Brauer group.

In the case of finite extensions, one has to be a bit more careful; one may easily construct examples of central simple algebras A and B over global fields which have all the same finite splitting fields, but which do not generate the same cyclic subgroup of the Brauer group [PraR, p. 149]. On the other hand, it is easy to show that if A and B are *quaternion* algebras over a global field that have the same collection of quadratic splitting fields, then A and B must be isomorphic. This motivates the following.

Problem 10.5. Show that if F is a field which is finitely generated over a prime field or over an algebraically closed field, then any two quaternion division algebras having the same finite splitting fields (or even just the same maximal subfields) must be isomorphic.

What is known

The restriction to finitely generated fields is necessary. One can take colimits of function fields in the style of [M92] to construct nonisomorphic quaternion division algebras with the same maximal subfields, see, e.g., [GS, §2.1].

Garibaldi and Saltman [GS] showed that if F has trivial “unramified Brauer group” (in a sense including the Archimedean places), then any two quaternion division algebras over F having the same maximal subfields must be isomorphic. Rapinchuk and Rapinchuk proved similar results for period 2 division algebras in [RR]. Krashen and McKinnie [KrM] showed that if quaternion algebras over F may be distinguished by their finite splitting fields, then the same is true over $F(x)$.

For algebras of higher degree, the situation for global fields leads one to the following problem:

Problem 10.6. Show that if F is a field which is finitely generated over a prime field or over an algebraically closed field, and D is a division algebra, then there are only finitely many other division algebras which share the same collection of finite splitting fields.

What is known

One may easily verify this for global fields. Krashen and McKinnie [KrM] showed that if this is true for a field F , it remains true for the field $F(x)$.

11. Other problems

11.A. Group admissibility and division algebras

Let F be a field and G a finite group. We say G is F -admissible if there exists a G -crossed product F -division algebra A .

Problem 11.1. Determine which groups are F -admissible, for a field F .

This problem is an extension of the inverse Galois problem, and is especially interesting for $F = \mathbb{Q}$. It appears as part of Problem 5 in [Am82].

The next problem, also known as the \mathbb{Q} -admissibility conjecture, was first recognized in [Sc].

Problem 11.2. Prove every Sylow-metacyclic group is \mathbb{Q} -admissible.

What is known

We discuss only the number field case of Problem 11.1. Schacher initiated the study of admissibility in his thesis, where he proved that for any finite group G there exists a number field F over which G is F -admissible ([Sc, Theorem 9.1]). His key observation was that if F is a global field, a group G is F -admissible if and only if (a) G is the group for a Galois extension L/F ; and (b) for every prime p dividing $|G|$, there are two places v_1 and v_2 such that $\text{Gal}(L_{v_i}/F_{v_i})$ contains a p -Sylow subgroup of G ([Sc, Prop. 2.5]). It follows easily that a \mathbb{Q} -admissible group is *Sylow-metacyclic*: every Sylow subgroup P of G contains a normal subgroup H such that both H and P/H are cyclic [Sc, Theorem 4.1]. This motivated Problem 11.2.

All Sylow-metacyclic abelian groups [Sc, Theorem 6.1] and all metacyclic p -groups [GoS79] are \mathbb{Q} -admissible. Liedahl extended the latter result to give necessary and sufficient conditions on a pair (G, F) for G to be F -admissible, where G is a metacyclic p -group and F is a number field [Li]. In 1983 Sonn settled Problem 11.2 for solvable groups, proving they are all \mathbb{Q} -admissible in [So].

To help organize the work for nonsolvable groups we make the following observations. Schacher showed that S_n is \mathbb{Q} -admissible if and only if $n \leq 5$ in [Sc, Theorem 7.1]. He showed that the alternating group A_n is not \mathbb{Q} -admissible for $n \geq 8$ in [Sc, Theorem 7.4], but did not solve the problem for $n = 4, 5, 6$, and 7 , though Gordon and Schacher proved $A_4 = \text{PSL}(2, 3)$ is \mathbb{Q} -admissible in [GoS77]. Schacher also noted that the groups $\text{PSL}(2, p)$, which are simple for $p \geq 5$, are Sylow-metacyclic for p a prime, making them a natural target for attempts at Problem 11.2. Chillag and Sonn noted that the central extensions $\text{SL}(2, p^n)$ of $\text{PSL}(2, p^n)$ are Sylow-metacyclic for $n \leq 2$ and $p \geq 5$, and that a nonabelian finite simple Sylow-metacyclic group belongs to the set $\{A_7, M_{11}, \text{PSL}(2, p^n) \text{ for } n \leq 2 \text{ and } p^n \neq 2, 3\}$

([CS, Lemma 1.5]). In 1993, Fein, Saltman, and Schacher proved that any finite Sylow-metacyclic group that appears as a Galois group over \mathbb{Q} is $\mathbb{Q}(t)$ -admissible [FSS].

Here is a list of known nonsolvable \mathbb{Q} -admissible groups and the papers in which the results appear: $A_5 = \text{PSL}(2, 4) = \text{PSL}(2, 5)$ ([GoS79]); $A_6 = \text{PSL}(2, 9)$ and A_7 ([FeV], [ScSo]); $\text{SL}(2, 5)$ ([FeFe]); the double covers $\tilde{A}_6 = \text{SL}(2, 9)$ and \tilde{A}_7 ([Fe94]); $\text{PSL}(2, 7)$ and $\text{PSL}(2, 11)$ ([AlSc]); and $\text{SL}(2, 11)$ ([Fe02]). In some cases these groups are known to be F -admissible in more general circumstances. For example, the condition “2 has at least two prime divisors in F or F does not contain $\sqrt{-1}$ ” is necessary and sufficient for the F -admissibility of the groups $\text{SL}(2, 5)$ ([FeFe]); A_6 and A_7 ([ScSo]), and $\text{PSL}(2, 7)$ ([AlSc]). Feit proved $\text{PSL}(2, 11)$ is F -admissible over all number fields in [Fe04]. Problem 11.2 for $\text{PSL}(2, 13)$ remains open.

Interestingly, an analog of the \mathbb{Q} -admissibility conjecture was recently proved by Harbater, Hartmann, and Krashen for the function field of a curve over a discretely valued field with algebraically closed residue field, using patching machinery [HHK].

11.B. SK_1 of central simple algebras

For a central simple F -algebra A , write $\text{SL}_1(A)(F)$ for the elements of A with reduced norm 1 and

$$\text{SK}_1(A) := \text{SL}_1(A)(F)/[A^\times, A^\times].$$

This group depends only on the Brauer class of A [Dr, p. 160]. If A is not division, then $\text{SK}_1(A)$ is the Whitehead group of the linear algebraic group $\text{SL}_1(A)$ appearing in the Kneser–Tits problem, see [Ti78] and [Gi].

The basic question is: What is this group? It is always abelian (obviously) and torsion with period dividing the index of A [Dr, p. 157, Lemma 2]. It was for a long time an open question (called *the Tannaka–Artin problem*) whether it is always zero. For example, Wang proved in [Wang] that it is zero if the index of A is square-free. Then in [Pl] Platonov exhibited a biquaternion algebra with nonzero SK_1 ; reworked versions of this example can be found also in [KMRT] and [Dr, §24]. Moreover, every countable abelian torsion group of finite exponent — e.g., a finite abelian group — is $\text{SK}_1(A)$ for some algebra A over some field.

One can also consider SK_1 simultaneously over all extensions of F , which is connected with the structure of $\text{SL}_r(A)$ as a variety:

Theorem 11.3. *For a central simple F -algebra A , $\text{SK}_1(A \otimes L) = 0$ for every extension L/F if and only if $\text{SL}_r(A)$ is retract rational for every (resp., some) $r \geq 2$.*

The “if” direction can be found in [Vo, p. 186]. The full theorem amounts to combining the results from *ibid.* with [Gi, Theorem 5.9]. Gille’s theorem furthermore shows that the conditions are equivalent to the abstract group $\text{SL}_r(A)(L)$ being “projectively simple” for every extension L/F and every $r \geq 2$.

The main problem from this point of view is to prove the following converse to Wang’s result:

Conjecture 11.4 (Suslin [Sus, p. 75]). *If $\mathrm{SK}_1(A \otimes L) = 0$ for every extension L/F , then the index of A is square-free.*

The main known result on this conjecture is due to Merkurjev in [M93] and [M06]: If 4 divides the index of A , then $\mathrm{SK}_1(A \otimes L) \neq 0$ for some L/F .

11.C. Relative Brauer groups

Let F be a field with nontrivial Brauer group. Which subgroups of the Brauer group of F are *algebraic* relative Brauer groups; i.e., of the form $\mathrm{Br}(K/F)$ for some *algebraic* field extension K of F ? Which subgroups of the Brauer group of F are *abelian* relative Brauer groups; i.e., of the form $\mathrm{Br}(K/F)$ for some *abelian* field extension K of F ?

In particular, if $n > 1$ when is the n -torsion subgroup ${}_n\mathrm{Br}(F)$ of $\mathrm{Br}(F)$ an algebraic (resp. abelian) relative Brauer group?

It is conjectured that if F is finitely generated (and infinite), then ${}_n\mathrm{Br}(F)$ is an abelian relative Brauer group for all n if and only if F is a global field.

What is known

If F is a global field, then ${}_n\mathrm{Br}(F)$ is an abelian relative Brauer group for all n ([KS03]; [Po]; [KS06]). The conjecture above is that the converse holds. So far the following is known [PoSW]: if ℓ is a prime different from the characteristic of F and F contains the ℓ^2 roots of unity, then the conjecture holds for $n = \ell$, i.e., there is no abelian extension K/F such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(K/F)$.

11.D. The Brumer–Rosen conjecture

Amongst the first examples one sees, probably the fields \mathbb{R} or iterated Laurent series $\mathbb{R}((x_1))((x_2)) \cdots ((x_n))$ are the only ones whose Brauer group is nonzero and finite. This leads to the following naive question:

Problem 11.5. For which natural numbers n is there a field F such that $|\mathrm{Br}(F)| = n$?

Fein and Schacher [FS] noted that every power of 2 can occur. The following, finer conjecture would imply that these are the only possibilities. Specifically, for any field F , we can decompose $\mathrm{Br}(F)$ into a direct sum of its p -primary components $\mathrm{Br}(F)_p$ and ask if the following holds:

Conjecture 11.6 (Brumer–Rosen [BR]). *For a given field F and each prime p , one of the following holds:*

- (a) $\mathrm{Br}(F)_p = 0$.
- (b) $\mathrm{Br}(F)_p$ contains a nonzero divisible subgroup.
- (c) $p = 2$ and $\mathrm{Br}(F)_2$ is an elementary abelian 2-group.

Some remarks:

- (1) In contrast with the case for fields covered by this conjecture, every finite abelian group can be obtained as the Brauer group of some *commutative ring* [Ford].
- (2) Every divisible torsion abelian group is the Brauer group of some field, see [M85]. (Interestingly, this paper does not appear in MathSciNet, is

not listed in the journal’s table of contents, and has apparently not been translated into English.)

- (3) $\text{Br}(F)_2$ is an elementary abelian 2-group if and only if the cup product map $? \cdot (-1): H^2(F, \mu_2) \rightarrow H^3(F, \mu_2^{\otimes 2})$ is an injection [LLT, Cor. A4].

“Many” cases of the conjecture are known:

- If $p = \text{char } F$. In this case, $\text{Br}(F)_p$ is p -divisible by Witt’s theorem [Dr, p. 110].
- If $p = 2$ or 3 . These are Theorem 3 and Corollary 2 in [M83].
- If p is odd and there is a division algebra of degree p that is cyclic. This is Theorem 2 in [M83]. This gives a connection between the Brumer–Rosen conjecture and Problems 1.1 and 3.1.
- If $p = 5$ or F contains a primitive p th root of unity. These follow from the previous bullet, because in these cases the p -torsion in $\text{Br}(F)$ is generated by cyclic algebras of degree p .

Clearly, Merkurjev has largely settled the conjecture, in the same sense that the Merkurjev–Suslin theorem settled a large portion of Problem 3.1 on generation by cyclic algebras. The smallest open case is $p = 7$, and this case would be settled by a “yes” answer to Problem 3.2. Does there exist a field with Brauer group $\mathbb{Z}/7$?

We remark that Fein and Schacher’s main result in [FS] was that “many” of the possibilities predicted by Brumer and Rosen actually occur: every countable abelian torsion group of the form $D \oplus V$ for D divisible and V a vector space over \mathbb{F}_2 occurs as the Brauer group of some algebraic extension of \mathbb{Q} . Efrat [Ef] gives general criteria on the field F for $\text{Br}(F)_2$ to be finite and nonzero.

11.E. Invariants of central simple algebras

Once one is interested in central simple F -algebras of degree n —equivalently, elements of $H^1(F, \text{PGL}_n)$ —it is natural to want to know the invariants of PGL_n in the sense of [GMS] with values in, say, mod-2 Galois cohomology $H^i(-, \mathbb{Z}/2\mathbb{Z})$ or the Witt ring $W(-)$. (In this subsection, we assume for simplicity that the characteristic of F does not divide $2n$.)

Note that this problem is related to Problem 5.1: If PGL_n has a nonconstant invariant that is unramified in the sense of [GMS, p. 87], then the center of generic matrices $Z(F, n)$ from Section 5 is not a rational extension of F , see *ibid.*, page 87. There are no nonconstant unramified invariants with values in $H^i(-, \mathbb{G}_m)$ for $i \leq 3$ by [S85] (for $i = 2$) and [S97b] (for $i = 3$).

This problem is also related to essential dimension as in Section 6, because cohomological invariants provide lower bounds on essential dimension. Specifically: If F is algebraically closed and there is a nonzero invariant $H^1(-, \text{PGL}_n) \rightarrow H^i(-, C)$ for some finite Galois module C , then $\text{ed}(\text{PGL}_n) \geq i$ by [GMS, p. 32].

What is known

The case $n = 2$ concerns quaternion algebras and the invariants are determined in [GMS, pp. 43, 66]. For n an odd prime, there are no nonconstant Witt invariants (for trivial reasons) and the mod- n cohomological invariants are linear combinations of constants and the connecting homomorphism $H^1(-, \text{PGL}_n) \rightarrow H^2(-, \mu_n)$, see [G, §6.1].

The case $n = 4$ is substantial, but solved. Rost, Serre, and Tignol [RST] construct invariants of algebras of degree 4, including one that detects whether an algebra is cyclic. (See [T06] for analogous results in characteristic 2.) Rost proved that, roughly speaking, the invariants from [RST] generate all cohomological invariants of algebras of degree 4—see [Ba, §3.4] for details.

The next interesting case is $n = 8$. Here, Baek and Merkurjev [BaM09] solve the subproblem of determining the invariants of algebras of degree 8 and period 2, i.e., of GL_8/μ_2 . Beyond this, not much is known.

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