

ON CLASSIFICATION OF POISSON VERTEX ALGEBRAS

ALBERTO DE SOLE*

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Dedicated to Vladimir Morozov on the 100th anniversary of his birth

Abstract. We describe a conjectural classification of Poisson vertex algebras of CFT type and of Poisson vertex algebras in one differential variable (= scalar Hamiltonian operators).

0. Introduction

Recall that a *Poisson vertex algebra* (PVA) is a unital commutative associative algebra \mathcal{V} with a derivation D , endowed with a λ -bracket $\{ \cdot, \cdot \}_\lambda : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{V}$, which satisfies the axioms of a Lie conformal algebra, and the λ -bracket is related to the product by the left Leibniz rule:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}. \tag{0.1}$$

In this case one says that the differential algebra \mathcal{V} is endowed with a *Poisson λ -bracket*.

Recall, for completeness, that a *Lie conformal algebra* is a $\mathbb{C}[D]$ -module, endowed with a λ -bracket, which satisfies the following three axioms [K]:

$$\begin{aligned} \text{(sesquilinearity)} \quad & \{Da_\lambda b\} = -\lambda\{a_\lambda b\}, \quad \{a_\lambda Db\} = (D + \lambda)\{a_\lambda b\}, \\ \text{(Skew-commutativity)} \quad & \{b_\lambda a\} = -\leftarrow\{a_{-\lambda-D}b\}, \\ \text{(Jacobi identity)} \quad & \{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_\lambda + \mu c\}. \end{aligned}$$

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The left arrow in the second axiom means that D is moved to the left. Extension of these definitions to the super case is straightforward, using the usual sign rule.

Recall that the left Leibniz rule and skew-commutativity imply the right Leibniz rule [DK]:

$$\{bc_\lambda a\} = \{b_{\lambda+D}a\} \rightarrow c + \{c_{\lambda+D}a\} \rightarrow b, \tag{0.2}$$

where the right arrow means that D is moved to the right.

The awkward name ‘‘Poisson vertex algebra’’ comes from the fact that it arises as a quasiclassical limit of a family of vertex algebras [DK] in the same way as a Poisson algebra arises as a quasiclassical limit of a family of associative algebras.

Note that PVA is a local counterpart of a Coisson (= chiral Poisson) algebra, defined in [BD]. PVA can also be obtained as a formal Fourier transform of a local Poisson bracket [BDK], which plays an important role in the theory of infinite-dimensional integrable Hamiltonian systems. In fact, as demonstrated in [BDK], the language of Poisson vertex algebras is often more convenient and transparent than the equivalent languages of local Poisson brackets, used in the book [FT], or of Hamiltonian operators, used in the book [D].

In the present paper we shall discuss the problem of the classification of Poisson λ -brackets on the algebra of differential polynomials

$$\mathcal{R}_\ell = \mathbb{C}[u_i^{(n)} \mid i = 1, \dots, \ell; n \in \mathbb{Z}_+]$$

in ℓ differential variables u_i , where the derivation D is defined in the usual way: $Du_i^{(n)} = u_i^{(n+1)}$, $n \in \mathbb{Z}_+$. As usual, we shall write f' in place of Df , in particular, u_i, u'_i, u''_i, \dots shall often replace $u_i^{(0)}, u_i^{(1)}, u_i^{(2)}, \dots$. In the super case one considers the algebra $\mathcal{R}_{\ell,m}$ of differential polynomials in ℓ even differential variables and m odd ones.

It is clear that, as in the Poisson algebra case, a λ -bracket on \mathcal{R}_ℓ is uniquely determined by the λ -brackets $\{u_i \lambda u_j\}$, $i, j = 1, \dots, \ell$, due to sesquilinearity and the left and right Leibniz rules. It is explained in [BDK] that, as in the Poisson algebra case, the necessary and sufficient conditions for the validity of PVA axioms is skew-commutativity for each pair u_i, u_j and Jacobi identity for each triple u_i, u_j, u_k . As in the Poisson algebra case, there is an explicit formula for the λ -bracket of any $f, g \in \mathcal{R}_\ell$ in terms of λ -brackets of differential variables [DK]:

$$\{f \lambda g\} = \sum_{\substack{1 \leq i, j \leq \ell \\ m, n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (D + \lambda)^n \{u_i \lambda u_j\} \rightarrow (-D - \lambda)^m \frac{\partial f}{\partial u_i^{(m)}}. \tag{0.3}$$

It turns out to be more natural to consider an *algebra of dif and only if erential functions* extension $\tilde{\mathcal{R}}_\ell$ of $\mathcal{R}_\ell[x]$, that is a domain $\tilde{\mathcal{R}}_\ell$, containing $\mathcal{R}_\ell[x]$, such that all partial derivatives $\partial/\partial u_i^{(n)}$ extend to commuting derivations of $\tilde{\mathcal{R}}_\ell$ and only finitely many functions $\partial f/\partial u_i^{(n)}$ are nonzero for each $f \in \tilde{\mathcal{R}}_\ell$. Then D extends to $\tilde{\mathcal{R}}_\ell$ by the formula $D = (\partial/\partial x) + \sum_{\substack{1 \leq i \leq \ell \\ n \in \mathbb{Z}_+}} u_i^{(n+1)} (\partial/\partial u_i^{(n)})$, and formula (0.3) extends the λ -bracket from \mathcal{R}_ℓ to $\tilde{\mathcal{R}}_\ell$, making the latter a PVA as well [BDK]. An

element f of $\widetilde{\mathcal{R}}_\ell$ is called a *quasicontant* (resp. a *constant*) if $\partial f / \partial u_i^{(n)} = 0$ for all i and n (resp. if, in addition, $\partial f / \partial x = 0$). We denote by \mathcal{C} the subalgebra of all constants.

Recall that, given a Poisson λ -bracket on $\widetilde{\mathcal{R}}_\ell$, the associated Hamiltonian operator is the matrix $H = (H_{ij}(D))$, where $H_{ij}(D) = \{u_{jD}u_i\}_\rightarrow$. Conversely, the λ -bracket can be reconstructed from H via

$$\{u_{i\lambda}u_j\} = H_{ji}(D + \lambda)(1). \tag{0.4}$$

Also, given $\int hdx \in \widetilde{\mathcal{R}}_\ell / D\widetilde{\mathcal{R}}_\ell$, the corresponding system of Hamiltonian equations is $du/dt = \{h_\lambda u\}|_{\lambda=0} (= H(\delta \int hdx / \delta u))$ [BDK].

Recall that the skew-commutativity of the λ -bracket is equivalent to the skew-adjointness of H [BDK]. Hence, unlike the Poisson algebra situation, the λ -bracket of a function $f \in \widetilde{\mathcal{R}}_\ell$ with itself can be nonzero. In fact, the skew-commutativity axiom is equivalent to the relation $\{f_\lambda f\} = \sum_{j \text{ odd}} (D + 2\lambda)^j F_j$, $F_j \in \widetilde{\mathcal{R}}_\ell$. Thus, even the cases of one or two differential variables are already highly nontrivial. Apart from a conjecture, stated at the end of the Introduction, we shall be concerned only with these two cases.

In the case $\ell = 1$ we have $\mathcal{R}_1 = \mathbb{C}[u, u', u'', \dots]$ and, according to the above remarks, a skew-commutative λ -bracket on $\widetilde{\mathcal{R}}_1$ is determined by

$$\{u_\lambda u\} = \sum_{\substack{j=1 \\ j \text{ odd}}}^N (D + 2\lambda)^j f_j, \quad f_j \in \widetilde{\mathcal{R}}_1. \tag{0.5}$$

Here N is a positive odd integer, called the *order* of the λ -bracket, provided that $f_N \neq 0$. Note that the Jacobi identity for the triple u, u, u holds in the case when all the f_j in (0.5) are quasicontants. Such a λ -bracket is called a *quasicontant coefficient Poisson λ -bracket*.

For an arbitrary Poisson λ -bracket on $\widetilde{\mathcal{R}}_1$ the Jacobi identity for the triple u, u, u gives a very complicated system of partial differential equations on the functions f_j . In order to state our first result on the structure of these functions, define the *differential order* of $f \in \widetilde{\mathcal{R}}_1$, denoted by $\text{ord}(f)$, as the maximal $m \in \mathbb{Z}_+$, such that $\partial f / \partial u^{(m)} \neq 0$, if f is not a quasicontant, and as $-\infty$ if f is a quasicontant. Define the *level* m of the λ -bracket (0.5) of order N by $m = \max_j \{j + \text{ord}(f_j)\}$. Note that m is a positive integer if the λ -bracket is not a quasicontant coefficient one.

Theorem 0.1. *The possible values m of the level of a nonquasicontant coefficient Poisson λ -bracket (0.5) of order N are $\frac{1}{2}(N - 1) \leq m \leq 2N + 1$, $m \neq 2N$, $m \neq \frac{1}{2}(N + 1)$ if $N \equiv -1 \pmod{4}$, and $m \neq \frac{1}{2}(N - 1)$, $\frac{1}{2}(N + 3)$ if $N \equiv 1 \pmod{4}$.*

The notion of a level in the equivalent language of Hamiltonian operators was considered by I. Dorfman [D], who obtained our upper estimate of the level by a different method. The classification of Hamiltonian operators of order $N = 1, 3$ and 5 , obtained in [V], [GD], [A], [AV], [O], [D], [M], [C], and some further calculations lead to the following conjecture.

Conjecture 0.2. *With the exception of level $m = 1$ in the case $N = 3$, the only possible values of the level of a nonquasiconstant coefficient Poisson λ -bracket of order N are $m = N, N + 1$, or $N + 2$. (Examples below show that all these values of m do occur, with the exception of $m = 2$ in the case $N = 1$).*

This conjecture holds for $N \leq 11$, but for $N > 11$ we can prove only that $m < 2N - 6$ by a more detailed analysis of the Jacobi identity (1.6) for the triple u, u, u . (The proof of Theorem 0.1 uses only the highest total degree in the λ and μ terms in (1.6)).

From the conformal field theory (CFT) point of view, the most interesting PVAs are those which are obtained as a quasiclassical limit from a family of vertex algebras of CFT type, which we shall call the *PVA of CFT type*. By definition, this is an algebra of differential polynomials \mathcal{R}_ℓ with ℓ differential variables $L, W_1, \dots, W_{\ell-1}$, endowed with a λ -bracket, satisfying the axioms of PVAs, and such that:

- (i) $\{L_\lambda L\} = (D + 2\lambda)L + c\lambda^3$, where c is a constant; and
- (ii) $\{L_\lambda W_j\} = (D + \Delta_j\lambda)W_j, \quad j = 1, \dots, \ell - 1$.

Property (i) says that the differential variable L generates the Virasoro PVA (with central charge $12c$), while property (ii) says that W_j is a *primary element of conformal weight Δ_j* . Important examples of PVA of CFT type are provided by classical W -algebras $\mathcal{W}^k(\mathfrak{g}, f)$ associated to a simple Lie algebra \mathfrak{g} and its nilpotent element f (see, e.g., [DK]). We prove the following theorem.

Theorem 0.3.

- (a) *Let \mathcal{R}_2 be endowed with a PVA structure, generated by a Virasoro differential variable L with $c \neq 0$ and a primary differential variable W of integer conformal weight $\Delta > 2$, such that $\{W_\lambda W\} \neq 0$. Then this PVA is isomorphic to one of the classical W -algebras $\mathcal{W}^k(\mathfrak{g}, f)$, where \mathfrak{g} is a simple Lie algebra of rank 2 and f is a principal nilpotent element of \mathfrak{g} .*
- (b) *Let $\mathcal{R}_{1,1}$ be endowed with a super PVA structure, generated by an even Virasoro differential variable L with $c \neq 0$ and an odd primary differential variable W of conformal weight $\Delta \in \frac{1}{2} + \mathbb{N}$, such that $\{W_\lambda W\} \neq 0$. Then this PVA is isomorphic to the Neveu-Schwarz super PVA, namely, $\Delta = \frac{3}{2}$ and*

$$\{W_\lambda W\} = L + 2\lambda^2 c.$$

This theorem supports the following conjecture.

Conjecture 0.4. *Let $A = \mathbb{C}[D]L \oplus \mathbb{C}[D]W_1 \oplus \dots \oplus \mathbb{C}[D]W_{\ell-1} \oplus \mathbb{C}\mathbb{C}$ be a $\mathbb{C}[D]$ -module with $DC = 0$. Endow $S(A)$ with a Poisson λ -bracket, for which $\{L_\lambda L\} = (D + 2\lambda)L + \lambda^3 C$ and the W_j are primary of conformal weight $\Delta_j \in \mathbb{N}$ (so that $S(A)/(C - c)$ is a PVA of CFT type). Assume that A does not contain $\mathbb{C}[D]$ -submodules I , such that $IS(A)$ is a proper noncentral PVA ideal of $S(A)$. Then $S(A)/(C - c)$ is isomorphic to a classical W -algebra $\mathcal{W}^k(\mathfrak{g}, f)$, where \mathfrak{g} is a simple Lie algebra (including the 1-dimensional one), f is its nilpotent element and $c = -k(x|x)$ (here f, x are elements of an sl_2 triple, such that $[x, f] = -f$).*

One can state a similar conjecture in the super case.

We will prove Theorem 0.1 in Section 1 and Theorem 0.3 in Section 2.

In Section 3 we discuss the problem of the classification of scalar Hamiltonian operators of arbitrary (odd) order N (i.e., the case $\ell = 1$) up to contact transformations. Recall that such a classification for $N \leq 5$ was obtained in a series of papers by Vinogradov [V], Gelfand and Dorfman [GD], Astashov [A], [AV], Mokhov [M], Olver [O], and Cooke [C]. We introduce the following new family of compatible Hamiltonian operators of order $N = 2n + 3 \geq 3$, $H^{(N,0)} = D^2 \circ ((1/u) \circ D)^{2n} \circ D$. We prove in Section 4 that these operators are Hamiltonian and compatible (i.e., any of their linear combinations with constant coefficients is Hamiltonian). Furthermore, in Section 3 we introduce a sequence of Hamiltonian operators $H^{[N,c(x)]}$ of order $N \geq 7$, depending on a linear quasiconstant $c(x)$.

Our main observation is that any Hamiltonian operator of order $N \geq 7$ can be taken by a contact transformation to one of the following four types:

- (1) a skew-adjoint differential operator with quasiconstant coefficients;
- (2) a linear combination with constant coefficients of the operators $H^{(n,0)}$, $3 \leq n \leq N$;
- (3) the operators $H^{[N,c(x)]}$, where $N \geq 9$ and $c''(x) = 0$; and
- (4) a “small” family of exceptional Hamiltonian operators.

We checked that this is indeed true for $7 \leq N \leq 13$, and in Section 3 we exhibit in each of these cases the operators of type (4). The strategy of the proof is the same as in the above-mentioned papers, but the use of the machinery of PVAs considerably simplifies calculations. First, using Conjectures 0.2 and 3.4 on the level and the leading coefficient, one shows that by a contact transformation the leading coefficient can be made equal to 1. After that, using contact transformations that keep the leading coefficient as 1, one reduces the Hamiltonian operator to a canonical form.

Remarkably, it turns out that for $N = 13$ the set of operators of “exceptional” type (4) is empty, i.e. any Hamiltonian operator of order $N = 13$ can be taken by a contact transformation to an operator of type (1), (2) or (3)! We conjecture that the same holds for all $N > 13$.

In Section 3 we also analyze the hierarchies of integrable Hamiltonian equations, obtained by the Lenard–Magri scheme [Ma], [BDK] from a compatible pair of Hamiltonian operators, which we call the bi-Hamiltonian integrable equations, for one of our compatible pairs. On the basis of this analysis we state Conjecture 3.18 on the classification of all scalar bi-Hamiltonian integrable equations.

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1. Proof of Theorem 0.1

The Jacobi identity for the λ -bracket (0.5) reads

$$\{u_\lambda\{u_\mu u\}\} - \{u_\mu\{u_\lambda u\}\} = \{\{u_\lambda u\}_{\lambda+\mu} u\}. \tag{1.6}$$

Substituting (0.5) in (1.6) and using (0.3), we obtain a polynomial equation in λ, μ and $\partial f_j / \partial u^{(i)}$. The highest total degree in λ and μ in this equation is $2N + m$.

Equating this term to 0, we obtain, after dividing by $2^N f_N$,

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{\min\{N,m\}} F_{N+m,j}(\lambda, \mu) \frac{\partial f_j}{\partial u^{(m-j)}} = 0.$$

Here the polynomials $F_{n,j}(u, v)$ for $1 \leq j \leq n, j \text{ odd}$, are as follows:

$$F_{n,j}(u, v) = u^{n-j}(v - w)^j + v^{n-j}(w - u)^j + w^{n-j}(u - v)^j,$$

where we let $w = -(u + v)$. Hence Theorem 0.1 follows immediately from the following proposition.

Proposition 1.1. *Let N be a positive odd integer and let $m = 2N$ or $m \geq 2N + 2$. Then the collection of polynomials $S_{N,m} := \{F_{N+m,j}(u, v)\}_{1 \leq j \leq N, j \text{ odd}}$ is linearly independent.*

The proof of the proposition is based on the following lemma.

Lemma 1.2.

- (a) $(\partial^2/\partial u^2 + \partial^2/\partial v^2 - \partial^2/\partial u\partial v) F_{n,j} = (n - j)(n - j - 1)F_{n-2,j} + 3j(j - 1)F_{n-2,j-2}$ (the right-hand side is zero if $n \leq 2$).
- (b) $4(u^2 + uv + v^2)F_{n,j} = 3F_{n+2,j} + F_{n+2,j+2}$.
- (c) If n is divisible by 3, then the polynomial $u^2 + uv + v^2$ does not divide $F_{n,1}$.

Proof. Part (a) is straightforward. Part (b) is proved, using that $4(u^2 + uv + v^2)$ can be written in the following three forms:

$$3u^2 + (u + 2v)^2 = 3v^2 + (2u + v)^2 = 3(u + v)^2 + (u - v)^2.$$

Then we rewrite the left-hand side of (b), using consistently for each of the three summands these three forms.

Part (c) is proved by noting that $u^2 + uv + v^2$ vanishes at $u = 1, v = \omega := e^{2\pi i/3}$, while $F_{n,1}(1, \omega) = \omega - \omega^2 \neq 0$. \square

Corollary 1.3.

- (a) If the collection of polynomials $S_{N,m}$ is linearly independent, then the collection of polynomials $S_{N,m+2}$ is linearly independent.
- (b) If $N + m + 2$ is divisible by 3 and the collection $S_{N,m}$ is linearly independent, then the collection $S_{N+2,m}$ is linearly independent.

Proof. Applying to a linear dependence of elements from $S_{N,m+2}$, the operator $\partial^2/\partial u^2 + \partial^2/\partial v^2 - \partial^2/\partial u\partial v$ and using Lemma 1.2(a), we obtain a linear dependence between the polynomials $\{(n - j)(n - j - 1)F_{N+m,j} + 3j(j - 1)F_{N+m,j-2}, \text{ where } n = N + m + 2 \text{ and } 1 \leq j \leq N, j \text{ odd}\}$. But this set is linearly independent if the set $S_{N,m}$ is, proving (a).

In order to prove (b), multiplying all elements from $S_{N,m}$ by $u^2 + uv + v^2$, we obtain, due to Lemma 1.2(b), a linearly independent set of polynomials $S = \{3F_{n+2,j} + F_{n+2,j+2}, \text{ where } n = N + m, 1 \leq j \leq N, j \text{ odd}\}$. Since by Lemma 1.2(c),

the polynomial $F_{n+2,1}$ is not divisible by $u^2 + uv + v^2$, we conclude that the set $S \cup \{F_{n+2,1}\}$ is linearly independent, which implies that the set $S_{N+2,m}$ is linearly independent.

Now we can complete the proof of Proposition 1.1 by induction on N . If $N = 1$, then $S_{N,m}$ consists of one nonzero polynomial. Due to Corollary 1.3(a) it suffices to prove the proposition when $m = 2N$ and $m = 2N + 3$. Note that, by the inductive assumption, $S_{N-2,2N}$ is linearly independent, hence by Corollary 1.3(b), $S_{N,2N}$ is linearly independent. Similarly, by the inductive assumption, $S_{N-2,2N+3}$ is linearly independent, hence by Corollary 1.3(b), $S_{N,2N+3}$ is linearly independent. \square

Remark 1.4. If $m < 2N$ or $m = 2N + 1$, then the set of polynomials $S_{N,m}$ is linearly dependent. Indeed, we can view any of the polynomials $F_{N+m,j}$ as an element of the set S_{N+m} of polynomials of the form $(u - v)(v - w)(w - u)f(u, v, w)$ where f is a symmetric polynomial in u, v, w , considered mod $(u + v + w)$, of degree $N + m - 3$. Hence $\dim S_{N+m}$ equals the number of partitions of $N + m - 3$ in a sum of 2's and 3's. But the latter number is smaller than $\dim S_{N,m} = (N + 1)/2$ for the considered values of m .

2. Proof of Theorem 0.3

Lemma 2.1. *Let \mathcal{V} be a PVA of CFT type with all conformal weights positive. Then an element P of \mathcal{V} is primary if and only if it is a polynomial in W_1, \dots, W_ℓ .*

Proof. The “if” part is clear by the Leibniz rule (0.1). In order to prove the “only if” part, note that we have, by induction on m ,

$$\{L_\lambda L^{(m)}\} = \lambda^{m+3}c + (\text{lower powers of } \lambda). \tag{2.7}$$

Writing, as usual, $L_\lambda = \sum_{n \in \mathbb{Z}_+} (\lambda^n/n!)L_{(n)}$, we obtain a sequence of derivations $L_{(m)}$ of \mathcal{V} (due to (0.1)). Due to (2.7) we have, in particular,

$$L_{(n)}L^{(m)} = \delta_{n,m+3}\alpha c \quad \text{for } n \geq m + 3, \tag{2.8}$$

where α is a nonzero constant.

Write the element P as a polynomial in $W_j^{(n)}$, $j = 1, \dots, \ell$, $n \in \mathbb{Z}_+$, with coefficients in $\mathbb{C}[L, L', L'', \dots]$. If one of these coefficients, say f , is not constant, let m be the maximal integer for which $L^{(m)}$ occurs in f . Then, due to (2.8), $L_{(m+3)}f = \alpha c(\partial f/\partial L^{(m)}) \neq 0$. Hence f is not primary.

Likewise, we have

$$L_{(n)}W_j^{(m)} = \delta_{n,m+1}\Delta_j \quad \text{if } n \geq m + 1.$$

Hence, if m is maximal such that $W_j^{(m)}$ occurs for some j in a polynomial $P \in \mathbb{C}[W_j^{(n)} \mid j = 1, \dots, \ell, n \in \mathbb{Z}_+]$, we have $L_{(m+1)}P = \sum_j \Delta_j(\partial P/\partial W_j^{(m)})$. Hence P can be primary only if $P \in \mathbb{C}[W_1, \dots, W_\ell]$. \square

Recall that $L_{(0)} = D$ and that $L_{(1)}$ is a diagonalizable operator on a PVA \mathcal{V} of CFT type. If $L_{(1)}P = \Delta P$, $\Delta \in \mathbb{C}$, one says that P has *conformal weight* Δ and puts $\Delta = \Delta_P$. One has

$$\Delta_{PQ} = \Delta_P + \Delta_Q, \quad \Delta_{P'} = \Delta_P + 1,$$

if P and Q are eigenvectors of $L_{(1)}$. Moreover, the coefficient of λ^j in $\{P_\lambda Q\}$ has conformal weight $\Delta_P + \Delta_Q - j - 1$. In other words, all the summands in $\{P_\lambda Q\}$ have conformal weight $\Delta_P + \Delta_Q - 1$ if we put $\Delta_\lambda = 1$.

Now let \mathcal{V} be a PVA as in Theorem 0.3 and assume that $\Delta \in \frac{1}{2}\mathbb{Z}_+$, $\Delta > 1$. It follows from the skew-commutativity of the λ -bracket and the properties of the conformal weight that

$$\{W_\lambda W\} = \sum_{\substack{j=1 \\ j \text{ odd}}}^N (D + 2\lambda)^j P_j, \quad P_N \neq 0, \tag{2.9}$$

where N is a positive odd integer, $1 \leq N \leq 2\Delta - 1$ and P_j has conformal weight $2\Delta - j - 1$. Hence we may write

$$\{L_\lambda P_j\} = (D + (2\Delta - j - 1)\lambda)P_j + \sum_{k=2}^{2\Delta-j} \lambda^k Q_{j,k}, \tag{2.10}$$

where $\Delta_{Q_{j,k}} = 2\Delta - j - k$.

The following lemma is straightforward.

Lemma 2.2. *The Jacobi identity for the triple L, W, W is equivalent to the following equations*

$$Q_{j,2k} = \binom{j+2k}{j} P'_{j+2k}, \quad Q_{j,2k+1} = \left(2\Delta \binom{j+2k}{j} - \binom{j+2k+1}{j} \right) P_{j+2k}. \tag{2.11}$$

First, we assume that Δ is an integer > 2 , and we shall prove claim (a) of the theorem. Denote by a_j the coefficient of $L^{(2\Delta-3-j)}$ in P_j and by b_j the coefficient of $W^{(\Delta-1-j)}$ in P_j and let $g = P_{2\Delta-1}$. From the conformal weight considerations it follows that a_j , b_j and g are constants (depending on c).

Lemma 2.3. *If $g = 0$ and $b_{\Delta-1} = 0$, then $\{W_\lambda W\} = 0$.*

Proof. By Lemma 2.1 the only primary elements in \mathcal{V} are polynomials in W . Since the conformal weight of $\{W_\lambda W\}$ is $2\Delta - 1$, we conclude that it may contain only W^ε where $\varepsilon = 0$ or 1 . Thus, by the conditions of the lemma, all P_j are not primary elements, unless they are 0. Now by downward induction, beginning with $g = P_{2\Delta-1} = 0$, we show, using (2.11), that all P_j are primary, hence zero, and that all $Q_{j,k} = 0$. \square

Introduce the following two polynomials in x and y :

$$G_1(x, y) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{2\Delta-3} a_j x^j y^{2\Delta-3-j}, \quad F_1(x, y) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{\Delta-1} b_j x^j y^{\Delta-1-j},$$

and put

$$G(x, y) = G_1\left(\frac{x-y}{2}, \frac{x+y}{2}\right), \quad F(x, y) = F_1\left(\frac{x+y}{2}, \frac{x-y}{2}\right).$$

We obviously have

$$G(y, x) = -G(x, y), \quad F(y, x) = (-1)^\Delta F(x, y) = -F(-x, -y). \tag{2.12}$$

Then the $W^{(r)}$ terms for $r \in \mathbb{Z}_+$ of the WWW Jacobi identity $\{W_\lambda\{W_\mu W\}\} - \{W_\mu\{W_\lambda W\}\} = \{\{W_\lambda W\}_{\lambda+\mu} W\}$ give the following identity if we let $x = \lambda, y = \mu, z = D + \lambda + \mu,$

$$(x - (\Delta - 1)(y + z))G(y, z) + \text{cycl} = 2F(x, -y)F(x + y, -z) + \text{cycl}, \tag{2.13}$$

where ‘‘cycl’’ means that we add two terms obtained from the first one by cyclically permuting $x, y, z.$

The constant term of the WWW Jacobi identity gives the following identity, multiplied by the constant $g,$

$$x^{2\Delta-1}F(x + y, y) - y^{2\Delta-1}F(x + y, x) = (-1)^\Delta(x + y)^{2\Delta-1}F(x, -y). \tag{2.14}$$

Finally, the constant term of the LWW identity gives

$$cG(x, y) = 48g\left(\Delta \frac{x^{2\Delta-1} - y^{2\Delta-1}}{(x + y)^2} - \frac{x^{2\Delta} - y^{2\Delta}}{(x + y)^3}\right). \tag{2.15}$$

In particular,

$$cG(x, 0) = 48g(\Delta - 1)x^{2\Delta-3}. \tag{2.16}$$

We also have

$$F(0, x) = Ax^{\Delta-1} \text{ for some constant } A. \tag{2.17}$$

Letting $z = 0$ in (2.13), and plugging (2.15), (2.16) and (2.17) in it, we obtain

$$\begin{aligned} &\frac{24gc(\Delta - 1)}{(x + y)^2}((\Delta - 1)(x^{2\Delta-1} - y^{2\Delta-2})(x + y)^2 \\ &\quad + (xy^{2\Delta-3} - x^{2\Delta-3}y)(x + y)^2 \\ &\quad - \Delta(x + y)(x^{2\Delta-1} - y^{2\Delta-1}) + x^{2\Delta} - y^{2\Delta}) \\ &= A(x^{\Delta-1} + y^{\Delta-1} - (x + y)^{\Delta-1})F(-x, y). \end{aligned} \tag{2.18}$$

First, we prove that $g = 0$ if Δ is an odd integer > 3 . In the contrary case, letting $y = 0$ in (2.14), we get $F(x, 0) = F(0, x) = 0$. Hence, letting $z = 0$ in (2.13), we get

$$(x - (\Delta - 1)y)G(y, 0) - (y - (\Delta - 1)x)G(x, 0) = (\Delta - 1)(x + y)G(x + y).$$

Substituting (2.16) in this formula, we obtain (after cancelling $48gc(\Delta - 1)$)

$$\begin{aligned} (\Delta - 1)(x^{2\Delta-2} - y^{2\Delta-2}) - x^{2\Delta-3}y + xy^{2\Delta-3} \\ = (\Delta - 1)x^{2\Delta-2} - (\Delta - 2)x^{2\Delta-3}y + (\Delta - 3)x^{2\Delta-4}y^2 + \dots, \end{aligned}$$

which is impossible if $\Delta > 3$.

Next, we consider the case $g = 0$. Since $\Delta \neq 2$, it follows from (2.18) that $A = 0$, hence, by (2.17),

$$F(0, x) = 0. \tag{2.19}$$

Also, it follows from (2.12) that

$$F(-x, x) = 0. \tag{2.20}$$

Since $g = 0$, by (2.15), $G(x, y) = 0$, hence (2.13) becomes

$$F(x, -y)F(x + y, -z) + F(y, -z)F(y + z, -x) + F(z, -x)F(x + z, -y) = 0. \tag{2.21}$$

We will show, using the above three equations, that $F(x, y) = 0$, which implies, in particular, that $b_{\Delta-1} = 0$, and we can apply Lemma 2.3.

For this, consider the function $\varphi(t) = F(x, tx)/x^{\Delta-1}$. This is a polynomial in t .

Lemma 2.4. *The polynomial $\varphi(t)$ has the following properties:*

- (i) $\varphi(a) = 0, a \neq 0 \Rightarrow \varphi(a^{-1}) = 0$;
- (ii) $\varphi(-2)\varphi(3) = 0$; and
- (iii) $\varphi(a) = 0, a \neq 0 \Rightarrow \varphi(a^{-2})\varphi(a + a^{-1}) = 0$.

Properties (ii) and (iii) imply that $\varphi(t) = 0$.

Proof. Property (i) follows from (2.12) since $F(x, y)$ is a homogeneous polynomial. In order to prove (ii), let $y = -2x, z = x$ in (2.20) and use (2.18) and (2.19). Similarly, (iii) is proved by putting $x = az, y = a^{-1}z$ in (2.20) and using (i).

Next, by (ii), $\varphi(a_1) = 0$ for some a_1 , such that $|a_1| > 1$. Since $|-a_1^2| > |a_1|$ and $|a_1 + a_1^{-1}| > |a_1|$, it follows from (iii) that $\varphi(a_2) = 0$ for some a_2 , such that $|a_2| > |a_1|$, etc. Thus the polynomial $\varphi(t)$ has infinitely many zeros, hence equals zero. \square

Lemma 2.4 completes the proof of (a) in the case $g = 0$. Now consider the case $g \neq 0$. Then identity (2.14) holds and we also obtain from (2.18), putting there $y = 1$, that

the polynomial

$$H(x) = (\Delta - 3)x^{2\Delta-2}(x + 1) - x^{2\Delta-3} + x^3 - (\Delta - 3)x(x + 1) \tag{2.22}$$

is divisible by the polynomial $f(x) = (x + 1)^{\Delta-1} - x^{\Delta-1} - 1$.

By the above discussion we may assume that Δ is an even integer. Now we will show that (2.22) is impossible for all positive even integers $\Delta > 6$.

In order to show that (2.22) does not hold for these Δ , note that $f(x) = f(-x-1)$, hence the divisibility of $H(x)$ by $f(x)$ implies the divisibility of $H_1(x) := -(x+1)H(-x-1)$ by $f(x)$. Next, we have $H_1(x) \equiv H_0(x) \pmod{f(x)}$, where $H_0(x) = (\Delta - 3)x(x+1)x^{2\Delta-2} - x^{2\Delta-2} + 2(\Delta - 3)(x^{\Delta+1} + x^\Delta) - 2x^{\Delta-1} + (x+1)^4 + (\Delta - 3)x(x+1)^2 + (\Delta - 3)x(x+1) - 1$. Then $P(x) := H_0(x) - xH(x) = 2(\Delta - 3)x^{\Delta+1} + 2(\Delta - 3)x^\Delta - 2x^{\Delta-1} + 2(\Delta - 1)x^3 + 2(2\Delta - 3)x^2 + 2(\Delta - 1)x$ is divisible by $f(x)$ if and only if $H_0(x)$ is. Now assume that $\Delta > 6$. Then, dividing $P(x)$ by $f(x)$, we obtain the remainder $Ax^{\Delta-3}$ + lower degree terms, where $A \neq 0$. Thus, $H(x)$ is not divisible by $f(x)$ for Δ even > 6 .

We conclude the proof of (a) by a direct computation in cases $\Delta = 3, 4$ and 6 . For example, in the case of $\Delta = 3$, from the conformal weight considerations, we have $\{W_\lambda W\} = (D + 2\lambda)P_1 + (D + 2\lambda)^3P_3 + (D + 2\lambda)^5P_5$, where $P_1 = \alpha L^2 + \beta L'' + \gamma W'$, $P_3 = \delta L$, $P_5 = \varepsilon$, and $\alpha, \beta, \gamma, \delta, \varepsilon$ are constants, not all equal to zero. Then the LWW Jacobi identity determines these five constants up to a nonzero common factor, and after rescaling, we get:

$$P_1 = 2^8L^2 + 2^3 \cdot 3cL'', \quad P_3 = 2^3 \cdot 5cL, \quad P_5 = c^2.$$

The WWW Jacobi identity then automatically holds, and we obtain the classical W -algebra $\mathcal{W}^k(\mathfrak{sl}_3, f)$.

Similar, but more complicated computations give, for $\Delta = 4$,

$$\{W_\lambda W\} = \sum_{\substack{j=1 \\ j \text{ odd}}}^7 (D + 2\lambda)^j P_j,$$

where

$$P_1 = 2^{11} \cdot 3^2L^3 + 2^6cL'^2 + 2^7 \cdot 29cLL'' + 2^4 \cdot 3c^2L^{(4)} + 2^4 \cdot 7\sqrt{2}LW + 2\sqrt{2}cW'',$$

$$P_3 = 2^6 \cdot 7^2cL^2 + 2^5 \cdot 7c^2L'' + 2 \cdot 3 \cdot cW, \quad P_5 = 2^4 \cdot 7c^2L, \quad P_7 = c^3.$$

This is the classical W -algebra $\mathcal{W}^k(\mathfrak{sp}_4, f)$.

Finally, for $\Delta = 6$ we get

$$\{W_\lambda W\} = \sum_{\substack{j=1 \\ j \text{ odd}}}^{11} (D + 2\lambda)^j P_j,$$

where $\deg_1 P_j = 11 - j$ with $\deg_1 L = 2, \deg_1 W = 6, \deg_1 D = 1, \deg_1 c = 0$ and $\deg_2 P_j = 5$ with $\deg_2 L = \deg_2 c = 1, \deg_2 W = 3, \deg_2 D = 0$. The explicit formulas for P_j are too long to be reproduced here. This is the classical W -algebra $\mathcal{W}^k(G_2, f)$.

The proof of (b) is similar, but simpler. We again use Lemmas 3.1 and 3.2. In this case the polynomials P_j do not contain the terms, linear in $W^{(m)}$, $m \in \mathbb{Z}_+$, since $\Delta \in \frac{1}{2} + \mathbb{Z}$, hence the linear terms of P_j are linear combinations of $L^{(m)}$,

$m \in \mathbb{Z}_+$, and 1. Denote the coefficient of $L^{(2\Delta-3-j)}$ in P_j by a_j , and let $g = P_{2\Delta-1}$. Then $a_j = 0$ if j is odd. As before, let

$$G_1(x, y) = \sum_{\substack{j=0 \\ j \text{ even}}}^{2\Delta-3} a_j x^j y^{2\Delta-3-j}, \quad G(x, y) = G_1\left(\frac{x-y}{2}, \frac{x+y}{2}\right).$$

As before, computing the $W^{(r)}$ -terms of the WWW Jacobi identity, we get

$$((\Delta - 1)(z - y) + x) G(z, -y) + \text{cycl} = 0. \tag{2.23}$$

The constant term of the LWV identity gives

$$G(x, y) = 48g \sum_{\substack{j=0 \\ j \text{ odd}}}^{2\Delta-3} (-1)^j (j + 1) \left(\Delta - \frac{j + 2}{2}\right) x^{2\Delta-3-j} y. \tag{2.24}$$

Plugging (2.18) in (2.17) and computing the coefficient of z^2 we get

$$48g((3\Delta - 6)(xy^{2\Delta-5} + x^{2\Delta-5}y)) - (\Delta - 1)(\Delta - 3)(x^{2\Delta-4} - y^{2\Delta-4}) = 0.$$

But this is impossible if $\Delta \geq \frac{5}{2}$ and $g \neq 0$. Hence $\{W_\lambda W\} = 0$ if $\Delta \geq \frac{5}{2}$, proving (b).

Example 2.5. The differential algebra \mathcal{R}_2 with W of conformal weight 4 and the λ -bracket

$$\{W_\lambda W\} = (D + 2\lambda)(\alpha L^3 + \beta LW)$$

is a PVA with central charge $c = 0$ for any values of $\alpha, \beta \in \mathbb{C}$, however, only for the value $\alpha/\beta = 2^6 \cdot 3^2 \cdot \sqrt{2}/7$ this PVA is a member of a family of PVAs, depending on an arbitrary central charge c .

3. On classification of Poisson λ -brackets of arbitrary order N in one differential variable

In this section we study Poisson λ -brackets on the algebra of differential functions $\mathcal{V} = \widetilde{\mathcal{R}}_1$. The inclusion of x allows us to consider contact transformations, see [A], [AV], [M], which preserve the order of the λ -bracket (but do not preserve translation invariance, i.e. independence of the coefficients of x).

A *contact transformation* of the differential algebra \mathcal{V} is a transformation of the form

$$x = \varphi(y, v, v_y), \quad u = \psi(y, v, v_y), \quad D_x = \frac{1}{\varphi'} D_y, \tag{3.25}$$

such that the following conditions hold:

$$\frac{\partial \varphi}{\partial v_y} \psi' = \frac{\partial \psi}{\partial v_y} \varphi', \quad \varphi' \text{ and } \rho \varphi' := \frac{\partial \psi}{\partial v} \varphi' - \frac{\partial \varphi}{\partial v} \psi' \text{ are invertible elements of } \mathcal{V}.$$

Note that the Jacobian of the transformation $(y, v, v_y) \mapsto (x, u, u_x)$ equals ρ^2 . An example of a contact transformation is the Legendre transformation: $\varphi = v'$, $\psi = yv' - v$ (for which $\rho = -1$). The contact transformations are precisely all automorphisms of the algebra \mathcal{V} , which leave invariant the contact form $\omega = du - u'dx$ up to multiplication by a function (the factor being ρ), and also precisely those transformations which preserve the order of any Hamiltonian operator [M].

Example 3.1. The Legendre transformation takes the operator D^N to the differential operator $T_N := (1/u'')(D \circ (1/u''))^N$ [M]. It follows that the operators T_N for positive odd N are Hamiltonian and compatible, which answers the question raised in [C1].

Example 3.2. The contact transformation $\varphi = v, \psi = -y$ takes the translation invariant Hamiltonian operator $D \circ ((1/u')D)^{N-1}$ to a quasiconstant coefficient (but not translation invariant) Hamiltonian operator $D^N + 2xD + 1$.

We will say that two λ -brackets on \mathcal{V} or two Hamiltonian operators are *equivalent* if, in an algebra of the differential functions extension of \mathcal{V} , one of them can be transformed to another by a contact transformation. The following result is well known.

Theorem 3.3 ([V], [GD], [A], [AV], [M], [O]). *Any Hamiltonian operator in one function u of order $N = 1$ (resp. $N = 3$) is equivalent to the following (unique) one:*

$$D \quad (\text{resp. } D^3 + a(2uD + u'), \quad a \in \mathcal{C}).$$

Using Conjecture 0.2 for order N Poisson λ -brackets, the first step in their classification is the following conjecture, which we checked for $N \leq 11$.

Conjecture 3.4. *Let f_N be the leading coefficient of a Poisson λ -bracket on \mathcal{V} of order N (recall that N is odd) and level $m = N + \varepsilon$, where $\varepsilon = 1$ or 2 and let $N_\varepsilon = N + 2\varepsilon - 3$. Then*

$$N_\varepsilon f_N \frac{\partial^2 f_N}{\partial u^{(\varepsilon)2}} = (N_\varepsilon + 1) \left(\frac{\partial f_N}{\partial u^{(\varepsilon)}} \right)^2.$$

Equivalently, $f_N = a/((u^{(\varepsilon)} + b)^{N_\varepsilon})$, where $a, b \in \mathcal{V}$ have differential order at most $\varepsilon - 1$.

The following remark shows that Hamiltonian operators remain Hamiltonian under contact transformations.

Remark 3.5. Given an element $P = P(x, u, u', \dots) \in \mathcal{V}$ (resp. a differential operator H), denote by \tilde{P} (resp. \tilde{H}) the element (resp. differential operator) obtained from P (resp. H) by the substitution (3.25). Then under the contact transformation (3.25), an evolution partial differential equation $du/dt = P$ gets transformed to $dv/dt = (1/\rho)\tilde{P}$ and, for $\int h dx \in \mathcal{V}/D\mathcal{V}$, the variational derivative $\delta \int h dx / \delta u$ gets transformed to $(1/\rho\varphi')(\delta \int \tilde{h} dy / \delta v)$. A Hamiltonian (evolution) partial differential equation $du/dt = H(\delta \int h dx / \delta u)$, where H is a Hamiltonian operator, gets transformed to $dv/dt = H_{\text{new}}(\delta \int \tilde{h} dy / \delta v)$, where $H_{\text{new}} = (1/\rho)\tilde{H} \circ (1/\rho\varphi')$ is again a Hamiltonian operator [M].

The following remark shows how the first two coefficients of a λ -bracket change under contact transformations.

Remark 3.6. A contact transformation takes the λ -bracket (0.5) to a λ -bracket of the form (0.5) with some coefficients $g_j \in \mathcal{V}$, where $g_N = \tilde{f}_N / \rho^2 \varphi'^{N+1}$. Furthermore, $f_N = g_N = 1$ if and only if the contact transformation has the form $\varphi = \varphi(y)$,

$\psi = \varphi'(y)^{-(N+1)/2}v + f(y)$, and then $g_{N-2} = \varphi'^2 \tilde{f}_{N-2} + \frac{1}{3}n(n^2 - 1)S(\varphi)$, where $S(\varphi) = \varphi'''/\varphi' - 3\varphi''^2/2\varphi'^2$ is the Schwarz derivative.

A classification of Hamiltonian operators of order $N = 5$ was obtained by Cooke [C], who showed that such an operator can be transformed either to a quasiconstant coefficient skew-adjoint operator, or to a certain canonical form, depending on one parameter. This canonical form is not translation invariant, but it can be slightly simplified, using the contact transformation with $\varphi = e^y, \psi = e^{-3y}v$, to make it translation invariant. It turned out that, by the further contact transformation $\varphi = y, \psi = v^2/2$, the canonical form can be recast in a beautiful form, which can be easily generalized to arbitrary order N .

Theorem 3.7.

- (a) Any Poisson Hamiltonian operator of order $N = 5$ on \mathcal{V} is equivalent to either a quasiconstant coefficient skew-adjoint operator or to one of the following translation invariant Hamiltonian operators ($b, c \in \mathbb{C}$):

$$(D^2 - c^2) \circ \frac{1}{u} \circ D \circ \frac{1}{u} (D^2 - c^2) + bD(D^2 - c^2). \tag{3.26}$$

These two types of Hamiltonian operators are not equivalent. The Hamiltonian operators (3.26), corresponding to parameters (b, c) and (b_1, c_1) , are equivalent if and only if either $b \neq 0$ and $bc = \pm b_1c_1$ or $b = b_1 = 0$.

- (b) The (compatible for all $c \in \mathbb{C}$) Hamiltonian operators

$$H^{(5,c)} = (D^2 - c^2) \circ \frac{1}{u} \circ D \circ \frac{1}{u} (D^2 - c^2) \quad \text{and} \quad K_c = D(D^2 - c^2)$$

give rise to the Lenard–Magri scheme

$$K_c \xi_{j+1,c} = H^{(5,c)} \xi_{j,c}, \quad j = 0, 1, 2, \dots, \tag{3.27}$$

with $\xi_{0,c} = -\frac{2}{c^2}$ if $c \neq 0, \quad \xi_{0,0} = x^2, \quad \xi_{1,c} = \frac{1}{u^2},$

where, for $j \geq 1, \xi_{j,c}$ lie in $\mathcal{V}_0 := \mathcal{C}[u, u^{-1}, u^{(n)}; n \geq 1]$ and depend polynomially on c . This scheme produces an integrable hierarchy of Hamiltonian evolutionary equations $du/dt_j = K_c \xi_{j+1,c}, j = 0, 1, 2, \dots,$ the first one being (after rescaling):

$$\frac{du}{dt_0} = \left(\frac{u''}{u^3} - 3\frac{u'^2}{u^4} - \frac{c^2}{2} \frac{1}{u^2} \right). \tag{3.28}$$

Proof. The proof of (a) consists of three steps (cf. [AV], [M], [C]). First, one proves that the level $m \leq 7$ (cf. Conjecture 0.2, which is proved for $N = 5$). Second, one shows, as in [C], that, by a contact transformation one can make the leading coefficient equal to 1. (Here one uses Conjecture 3.4, which is proved for $N = 5$.) Third, one shows that any Poisson λ -bracket $\{u_\lambda u\} = \sum_{j=1,3,5} (D + 2\lambda)^j f_j$ with $f_5 = 1$, either has f_1 and f_3 quasiconstant, or has the following f_1 and f_3 , where

one can add to u an arbitrary quasiconstant (this is Proposition 4.10 from [C] recast in terms of λ -brackets):

$$\begin{aligned}
 f_1 &= \frac{1}{4}c_1^2 + 3c_1'' + 2c_1 \frac{u''}{u} - c_1 \frac{u'^2}{u^2} - 4c_1' \frac{u'}{u} \\
 &\quad + \frac{1}{2}c_1c_2u + 3c_2u'' - 3c_2 \frac{u'^2}{u} - 3c_2'u + 4 \frac{u^{(4)}}{u} \\
 &\quad - 12 \frac{u'u'''}{u^2} + 12 \frac{u''^2}{u^2} - 24 \frac{u'^2u''}{u^3} + 21 \frac{u'^4}{u^4}, \\
 f_3 &= c_1 + c_2u + 12 \frac{u''}{u} - 14 \frac{u'^2}{u^2},
 \end{aligned}
 \tag{3.29}$$

where the $c_j = c_j(x)$ are quasiconstants satisfying the following equation:

$$2c_1c_2' + c_2c_1' + 4c_2''' = 0. \tag{3.30}$$

By Remark 3.5 a contact transformation keeps the leading coefficient to be 1 if and only if $\varphi = \varphi(y)$ and $\psi = \varphi'(y)^{-3}v + f(y)$, and this transformation takes the pair of quasiconstants c_1, c_2 to the pair of quasiconstants \bar{c}_1, \bar{c}_2 , where $\bar{c}_1 = \varphi'^2c_1(\varphi) + 4S(\varphi), \bar{c}_2 = c_2(\varphi)/\varphi'$. By such a transformation one can make $c_1 = 0$, so that $c_2(x) = Ax^2 + Bx + C$ where $A, B, C \in \mathbb{C}$. We can make $C = 0$, replacing x by $x + \text{const}$. Then, applying the transformation with $\varphi = y^{-1}$, we make $A = 0$ and keep $c_1 = 0$ since $S(\varphi) = 0$ and, if $B \neq 0$, we can make $C = 0$ replacing x by $x + \text{const}$. Taking further $\varphi = e^y$, we make both c_1, c_2 constant and, finally, by the transformation $\varphi = y, \psi = v^2/2$, the corresponding Hamiltonian operator is reduced to the form, described in (a), where $c = c_1, b = c_2$. The equivalence, stated in (a), is immediate by the above remarks. All these computations use Mokhov’s transformation formula (see Remark 3.5). For $N > 5$ the strategy is the same, but the computational difficulties increase exponentially.

In order to prove (b), let $\bar{H}_c = (1/u) \circ D \circ (1/u) \circ (D^2 - c^2), \bar{K} = D$, so that $H^{(5,c)} = (D^2 - c^2)\bar{H}_c, K_c = (D^2 - c^2)\bar{K}$. Obviously, any solution to the Lenard–Magri scheme

$$D\xi_{j+1,c} = \bar{H}_c\xi_{j,c}, \quad j = 0, 1, 2, \dots, \tag{3.31}$$

with $\xi_{0,c} = -2/c^2$ if $c \neq 0, \xi_{0,0} = x^2, \xi_{1,c} = 1/u^2$, is a solution to the Lenard–Magri scheme (3.27). Note that $\bar{H}_c = D \circ H_c - \frac{1}{2}K_c\xi_{1,c}$, where $H_c = (1/u^2) \circ (D^2 - c^2) + (u'/u^3)(D + c) - (D + c) \circ (u'/u^3)$. Hence (3.31) becomes

$$D\xi_{j+1,c} = D(H_c\xi_{j,c}) - \frac{1}{2}\xi_{j,c}(K_c\xi_{1,c}). \tag{3.32}$$

We construct a solution $\xi_{0,c}, \dots, \xi_{n,c}$ to (3.31), such that $\xi_{j,c} \in \mathcal{V}_0$ for $j = 1, \dots, n$, by induction on n . We already have it for $n = 1$. If we have it for n , then it is also a solution to the Lenard–Magri scheme (3.25) with skew-adjoint operators, hence, by [BDK, Lemma 2.6], $\xi_{j,c}(K_c\xi_{i,c}) \in D\mathcal{V}_0$ for all $i, j = 1, \dots, n$. In particular, this holds for $j = n$ and $i = 1$, which implies that equation (3.32)

has a solution $\xi_{j,c} \in \mathcal{V}_0$ for $j = 1, \dots, n + 1$. Using the recurrent formula (3.32), it is easy to prove by induction that the $\xi_{j,c}$ are linearly independent and, for $j \geq 1$, depend polynomially on c .

Letting $h_0 = -2u/c^2$ if $c \neq 0$ and $h_0 = x^2u$ if $c = 0$ and $h_1 = -1/u$, so that $\xi_{j,c} = \delta \int h_j dx / \delta u$ for $j = 0, 1$ by [BDK, Theorem 2.7 and Prop. 1.9], there exist $\int h_2 dx, \int h_3 dx, \dots \in \mathcal{V}_0 / D\mathcal{V}_0$, such that $\xi_{j,c} = \delta \int h_j dx / \delta u$ for $j \geq 2$, hence the hierarchy, defined in (b), is integrable. \square

Equation (3.28) appears in [MSS] in a classification of integrable evolution equations (see equation (4.1.26) there).

Remark 3.8. The contact transformation $\varphi = y, \psi = v^2/2$ takes the Poisson λ -brackets with $f_5 = 1$ and f_1, f_3 as in (3.29), to the Poisson λ -brackets $\{u_\lambda u\}_{(5,c_1,c_2)} = \sum_{j=1,3,5} (D + 2\lambda)^j g_j$, where

$$\begin{aligned} g_1 &= \frac{1}{u^6} (\frac{1}{4}c_1^2 u^4 + c_1 c_2 u^6 - 2c_1 u^3 u'' + 6c_1 u^2 u'^2 \\ &\quad + 3c_1'' u^4 - 6c_2'' u^6 - 8c_1' u^3 u' - 2u^3 u^{(4)} \\ &\quad + 24u^2 u' u''' + 18u^2 u''^2 - 144uu' u'' + 120u^{(4)}), \\ g_3 &= \frac{1}{u^4} (c_1 u^2 + 2c_2 u^4 + 4uu'' - 12u'^2), \quad g_5 = \frac{1}{u^2}, \end{aligned}$$

and $c_i = c_i(x)$ satisfy (3.30). The corresponding Hamiltonian operator is, provided that $c_1(x) \neq 0$,

$$H_{(5,c_1,c_2)} = B_{(2,c_1,c_2)}^* \circ D \circ B_{(2,c_1,c_2)},$$

where $B_{(2,c_1,c_2)} = (1/u)D^2 + a(x)D - a'(x) - a''(x)/(a(x)u)$, $c_1(x) = -8a''(x)/a(x)$, $c_2(x) = -2a(x)^2$, $a(x)$ is a nonzero quasiconstant, and

$$H_{(5,0,c(x))} = H^{(5,0)} + \frac{1}{2}c(x)D^3 + \frac{3}{4}c'(x)D^2,$$

where $c'''(x) = 0$. If $c_2 = 0$, then $H_{(5,c_1,c_2)}$ is equivalent to $H^{(5,0)}$; otherwise, it is equivalent to $H^{(5,c)} + \frac{1}{2}K_c$, where $8c^2 = 2c_2'^2 - 4c_2c_2'' - c_1c_2^2$ (which is a constant due to (3.30)).

Motivated by Theorem 3.7, introduce the following skew-adjoint differential operator of order $N = 2n + 3 \geq 3$ and the leading coefficient u^{3-N} ,

$$H^{(N,0)} = D^2 \circ \left(\frac{1}{u}D\right)^{2n} \circ D. \tag{3.33}$$

We show in the next section that all the operators $H^{(N,0)}$ are Hamiltonian and compatible. Note that two linear combinations of the form $H^{(N,0)} + a_1 H^{(N-2,0)} + a_2 H^{(N-4,0)} + \dots + a_n H^{(3,0)}$, where all $a_j \in \mathcal{C}$, are equivalent if and only if there exists a nonzero constant s , such that a_j is replaced by $s^j a_j$ for all j .

It is straightforward to show that the contact transformation $\varphi = (e^{ay} - 1)/a, \psi = e^{-by}v$, where $a = (n + 1)c, b = (n + 2)c, c \in \mathcal{C}$, takes the Hamiltonian operator $H^{(N,0)}$ to the Hamiltonian operator

$$H^{(N,c)} = (-1)^n (D - c) \circ (B^{(n,c)})^* \circ D \circ B^{(n,c)} \circ (D + c),$$

where

$$B^{(n,c)} = \frac{1}{u}(D - c) \circ \frac{1}{u} \circ (D - 2c) \circ \frac{1}{u} \circ \dots \circ \frac{1}{u}(D - nc).$$

Here $*$ stands for taking the adjoint differential operator, and $B^{(0,c)} = 1$.

The same arguments as in the proof of Theorem 3.7(a) lead to a classification of Hamiltonian operators of orders 7, 9 and 11. We state here the results, omitting the detailed proofs.

Theorem 3.9. *Any Hamiltonian operator of order 7 is equivalent either to a quasiconstant coefficient skew-adjoint differential operator or to the operator $H_{(7,c(x))} + b^2D^3$, where*

$$H_{(7,c(x))} = -B_{(3,c(x))}^* \circ D \circ B_{(3,c(x))}, \quad B_{(3,c(x))} = \frac{1}{u}D \circ \frac{1}{u}D^2 + c(x)D - \frac{1}{2}c'(x),$$

and $c'''(x) = 0, b \in \mathcal{C}$. These two types of Hamiltonian operators are not equivalent. The Hamiltonian operators $H_{(7,c(x))} + b^2D^3$ and $H_{(7,c_1(x))} + b_1^2D^3$ are equivalent if and only if $\alpha^2c_1(x) = c(\alpha^3x + \beta)$ and $\alpha^2b_1 = \pm b$ for some constants $\alpha \neq 0$ and β . Such a Hamiltonian operator is equivalent to a linear combination of the operators $H^{(j,0)}$ if and only if $c(x) = c \in \mathcal{C}$, and one has $H_{(7,c)} = H^{(7,0)} + 2cH^{(5,0)} + c^2H^{(3,0)}$.

Remark 3.10. The compatible pair of Hamiltonian operators $H_{(7,c(x))}$, where $c'''(x) = 0$, and D^3 give rise to a Lenard–Magri scheme with $\xi_0 = 1$ if $c'''(x) \neq 0$ (resp. $\xi_0 = c(x)$ if $c'''(x) = 0, c'(x) \neq 0$) and $\bar{\xi}_0 = -\frac{1}{2}x^2$. The first Hamiltonian equations of the resulting hierarchy are (up to a constant factor)

$$\begin{aligned} \frac{du}{dt_0} &= H_{(7,c(x))}\xi_0 = \left(\frac{u''}{u^3} - \frac{3u'^2}{u^4} - \frac{1}{4} \frac{(c(x)^2)''}{c''(x)} \right)' \left(\text{resp.} = \left(\frac{u''}{u^3} - \frac{3u'^2}{u^4} - \frac{3}{2}c(x) \right)' \right), \\ \frac{du}{d\bar{t}_0} &= H_{(7,c(x))}\bar{\xi}_0 \\ &= \left(\frac{u^{(4)}}{u^5} - 15 \frac{u'u'''}{u^6} - 10 \frac{u''^2}{u^6} + 105 \frac{u''u'^2}{u^7} - 105 \frac{u'^4}{u^8} \right. \\ &\quad + \left(2c(x) - \frac{1}{4}c''(x)x^2 + \frac{1}{2}c'(x)x \right) \frac{u''}{u^3} - \left(\frac{3}{2}c'(x)x + 6c(x) - \frac{3}{4}c''(x)x^2 \right) \frac{u'^2}{u^4} \\ &\quad + 5c'(x) \frac{u'}{u^3} - \frac{5}{4} \frac{c''(x)}{u^2} - \frac{15}{16}c(x)^2 + \frac{9}{16}c(x)c''(x)x^2 - \frac{9}{8}c(x)c'(x)x \\ &\quad \left. + \frac{3}{64}c''(x)^2x^4 - \frac{3}{16}c'(x)c''(x)x^3 + \frac{3}{16}c'(x)^2x^2 \right)' = D^3 \frac{\delta h_1}{\delta u}, \end{aligned}$$

where

$$h_1 = a(x)u + \frac{1}{u} \left(c(x) - \frac{1}{8}c''(x)x^2 + \frac{1}{4}c'(x)x \right) - \frac{u'^2}{2u^5}$$

and

$$\begin{aligned} a(x) &= \frac{x^2}{32} \left(-18c(x)^2 - 6c(x)c''(x)x^2 + 20c(x)c'(x)x \right. \\ &\quad \left. - c''(x)^2x^4 + 5c'(x)c''(x)x^3 - 7c'(x)^2x^2 \right). \end{aligned}$$

We can show that this Lenard–Magri scheme produces an integrable hierarchy of Hamiltonian equations. The first of these equations includes equations (4.1.23) and (4.1.24) from [MSS], while the second seems to be new.

For $N = 2n + 5 \geq 5$ and a quasiconstant $c(x)$, such that $c''(x) = 0$, we introduce the following skew-adjoint differential operator of order N with the leading coefficient u^{3-N} ,

$$H^{[N,c(x)]} = (-1)^n (B^{[n+2,c(x)]})^* \circ D \circ B^{[n+2,c(x)]},$$

where

$$B^{[n+2,c(x)]} = \left(\frac{1}{u}D - c(x)\right) \dots \left(\frac{1}{u}D - nc(x)\right) \left(\frac{1}{u}D^2 + c(x)D - c'(x)\right)$$

is a differential operator of order $n + 2$. By the same method, as in Section 4, one shows that $H^{[N,c(x)]}$ is a Hamiltonian operator.

For $N = 7$ we have $H_{(7,-c(x)^2)} = H^{[7,c(x)]}$ if $c''(x) = 0$.

Theorem 3.11. *Any Hamiltonian operator of order 9 is equivalent either to a quasiconstant coefficient skew-adjoint differential operator, or to the Hamiltonian operator $H^{[9,c(x)]} + aD^3$, where $c''(x) = 0$, $a \in \mathcal{C}$, or to the Hamiltonian operator $H_{(9,c(x))} + a^2H_{(5,0,c(x)/2)} + b^3D^3$, where*

$$H_{(9,c(x))} = B^* \left(D \circ \frac{1}{u} \circ D \circ \frac{1}{u} D + c(x)D + \frac{1}{2}c'(x) \right) B,$$

$$B = \frac{1}{u} \circ D \circ \frac{1}{u} D^2 + c(x)D - \frac{1}{2}c'(x),$$

$H_{(5,0,c(x))}$ is defined in Remark 3.8, and $a, b \in \mathcal{C}$, $c''(x) = 0$. These three types of Hamiltonian operators of order 9 are not equivalent. The Hamiltonian operators, corresponding to the triples $(c(x), a, b)$ and $(c_1(x), a_1, b_1)$, are equivalent if and only if $\alpha c_1(x) = c(\alpha^2 x + \beta)$, $\alpha a_1 = \pm a$, and $\alpha b_1 = \varepsilon b$ for some constants $\alpha \neq 0$, β and a cube root of unity ε . An operator $H_{(9,c(x))}$ is equivalent to a linear combination of the operators $H^{(j,0)}$ with constant coefficients if and only if $c(x) = c \in \mathcal{C}$, and one has $H_{(9,c)} = H^{(9,0)} + 3cH^{(7,0)} + 3c^2H^{(5,0)} + c^3H^{(3,0)}$.

Remark 3.12. The compatible, for $N \leq 9$, pair of Hamiltonian operators $H^{[N,c(x)]}$ and D^3 gives rise to a Lenard–Magri scheme with $\xi_0 = 1$ and $\bar{\xi}_0 = x^2$. We can show that this Lenard–Magri scheme produces an integrable hierarchy of Hamiltonian equations if $N = 9$. The simplest equation of this hierarchy is, up to a constant factor, the following equation of order 5, which seems to be new (here $c''(x) = 0$, $c'(x) \neq 0$),

$$\begin{aligned} \frac{du}{dt_0} = H^{[9,c(x)]} \mathbf{1} = & \left(\frac{u^{(4)}}{u^5} - 15 \frac{u'u'''}{u^6} - 10 \frac{u''^2}{u^6} + 105 \frac{u''u'^2}{u^7} - 105 \frac{u'^4}{u^8} \right. \\ & + 20c'(x) \frac{u'^2}{u^5} - 5c'(x) \frac{u''}{u^4} + 5c'(x)^2 \frac{1}{u^2} - 20c(x)c'(x) \frac{u'}{u^3} \\ & \left. + 15c(x)^2 \frac{u'^2}{u^4} - 5c(x)^2 \frac{u''}{u^3} - 5c(x)^4 \right)'. \end{aligned}$$

(In this way for $N = 5$ we get the integrable hierarchy in Theorem 3.7 and for $N = 7$ a special case of the integrable hierarchy of Remark 3.10.)

Theorem 3.13. *Any Hamiltonian operator of order 11 is equivalent either to a quasiconstant coefficient skew-adjoint differential operator, or to a linear combination with constant coefficients of the operators $H^{(j,0)}$ with $3 \leq j \leq 11$, j odd, or to the operator*

$$\begin{aligned}
 H^{[11,c(x)]} &+ a \left(\frac{1}{u^4} D^7 - 14 \frac{u'}{u^5} D^6 + \frac{1}{3u^6} \left(-10c(x)^2 u^4 - 8c'(x)u^3 \right. \right. \\
 &- 60uu'' + 285u'^2 \Big) D^5 + \frac{5}{3u^7} \left(10c(x)^2 u^4 u' \right. \\
 &- 10c(x)c'(x)u^5 + 12c'(x)u^3 u' - 9u^2 u''' + 120uu'u'' - 225u'^3 \Big) D^4 \\
 &+ \frac{1}{3u^8} \left(7c(x)^4 u^8 + 40c(x)^2 u^5 u'' - 120c(x)^2 u^4 u'^2 + 174c(x)c'(x)u^5 u' \right. \\
 &- 40c'(x)^2 u^6 + 48c'(x)u^4 u'' - 192c'(x)u^3 u'^2 - 18u^3 u^{(4)} + 300u^2 u' u''' \\
 &+ 210u^2 u''^2 - 2340uu'^2 u'' + 2520u'^4 - \frac{4}{3}au^8 \Big) D^3 + \frac{1}{3u^9} \left(42c(x)^3 c'(x)u^9 \right. \\
 &+ 10c(x)^2 u^6 u''' - 90c(x)^2 u^5 u'u'' + 120c(x)^2 u^4 u'^2 + 81c(x)c'(x)u^6 u'' \\
 &- 243c(x)c'(x)u^5 u'^2 + 81c'(x)^2 u^6 u' + 12c'(x)u^5 u''' - 144c'(x)u^4 u' u'' \\
 &+ 240c'(x)u^3 u'^3 - 3u^4 u^{(5)} + 60u^3 u' u^{(4)} + 105u^3 u'' u''' - 585u^2 u'^2 u''' \\
 &- 810u^2 u' u''^2 + 3465uu'^3 u'' - 2520u'^5 \Big) D^2 + \frac{7}{3u^5} c(x)c'(x) \left(3c(x)c'(x)u^5 \right. \\
 &+ u^2 u''' - 9uu'u'' + 12u'^3 \Big) D + \frac{7}{3u^5} c'(x)^2 \left(-3c(x)c'(x)u^5 - u^2 u''' \right. \\
 &\left. \left. + 9uu'u'' - 12u'^3 \right) \right),
 \end{aligned}$$

where $c''(x) = 0$, $a \in \mathbb{C}$, or to the operator $H_{(11,c(x))} + aH_{(5,0,2c(x))}$, where

$$\begin{aligned}
 H_{(11,c(x))} &= -B_{(11,c(x))}^* \circ D \circ B_{(11,c(x))}, \\
 B_{(11,c(x))} &= \left(\frac{1}{u} D \circ \frac{1}{u} D + \frac{1}{4}c(x) \right) \circ \left(\frac{1}{u} D \circ \frac{1}{u} D^2 + c(x)D - \frac{1}{2}c'(x) \right),
 \end{aligned}$$

and $a \in \mathbb{C}$, $c''(x) = 0$, or to the operator $H_{[11,c(x)]} + aH_{(5,0,14c(x))}$, $a \in \mathbb{C}$, $c''(x) = 0$, where $H_{[11,c(x)]}$ is too long to be reproduced here (its computer printout takes seven pages).

Remark 3.14. The compatible pairs of Hamiltonian operators

$$\left(H_{(11,c(x))}, H_{(5,0,2c(x))} \right) \quad \text{and} \quad \left(H_{[11,c(x)]}, H_{(5,0,14c(x))} \right)$$

give rise to Lenard–Magri schemes with $\xi_0 = 1$ and $\bar{\xi}_0 = c(x)$. We conjecture that these two Lenard–Magri schemes produce integrable hierarches of Hamiltonian equations.

Now we proceed to state our conjectures on the classification of Hamiltonian operators (in one differential variable u) of arbitrary (odd) order N .

Conjecture 3.15.

- (a) Any Hamiltonian operator is equivalent to a Hamiltonian operator with the leading coefficient 1. Note that for $N = 2n + 3 \geq 3$ the latter is equivalent to a Hamiltonian operator H with the leading coefficient u^{3-N} by the contact transformation $x = y, u = v^{n+1}/(n + 1)$.
- (b) If H is a translation invariant nonconstant coefficient Hamiltonian operator of order $N = 2n + 3 \geq 7$ with the leading coefficient 1, then after the contact transformation $x = y, u = v^{n+1}/(n + 1) + \text{const.}$ it becomes either $H^{(N,c)}$ with $c \in \mathcal{C}$, or a linear combination with constant coefficients of the operators $H^{(j,0)}$ with $3 \leq j \leq N, j$ odd.
- (c) Any translation invariant Hamiltonian operator of order $N \geq 7$ is equivalent to either a quasiconstant coefficient skew-adjoint differential operator, or to a linear combination of the operators $H^{(j,0)}$ with $3 \leq j \leq N, j$ odd.
- (d) For $N \geq 7$ the Hamiltonian operator $H^{(N,1)}$ is not compatible with any translation invariant Hamiltonian operator other than a scalar multiple of itself.
- (e) Any Hamiltonian operator of order $N \geq 13$ is equivalent to either a quasiconstant coefficient skew-adjoint differential operator, or to a linear combination with constant coefficients of the operators $H^{(j,0)}$ with $3 \leq j \leq N, j$ odd, or to the Hamiltonian operator $H^{[N,c(x)]}$, where $c''(x) = 0$.
- (f) For $N \geq 11$ the Hamiltonian operator $H^{[N,c(x)]}$ with $c'(x) \neq 0, c''(x) = 0$ is compatible only with a constant multiple of itself.

We verified Conjectures 3.15(a), (b), (d), (e) and (f) for $N \leq 13$, but were unable to prove (c) even for $N = 7$. Note that Conjecture 3.15(a) follows from Conjectures 0.2 and 3.4 and, conversely, it implies these conjectures.

Remark 3.16. It follows from (3.33) that for $\xi_j := \xi_{j,c=0}$, where $\xi_{j,c}$ are the same as in Theorem 3.7(b), we have

$$D^3 \xi_{j+n} = H^{(2n+3,0)} \xi_j, \quad j = 0, 1, 2, \dots, n = 1, 2, \dots$$

In particular, $\overline{\xi_j} := \xi_{nj}$ give a solution to the Lenard–Magri scheme $D^3 \overline{\xi_{j+1}} = H^{(2n+3,0)} \overline{\xi_j}, j = 0, 1, \dots$

Remark 3.17. Let $H_{(5,ax)}$ denote the Hamiltonian operator, corresponding to the λ -bracket $\{u_\lambda u\}_{(5,c_1,c_2)}$ from Remark 3.8 with $c_1 = 0, c_2 = ax$, where a is a nonzero constant, and let $H_{(9,ax)}$ be the Hamiltonian operator from Theorem 3.11 with $c(x) = ax$. Then the triple of Hamiltonian operators $H_{(9,ax)}, H_{(5,ax)}, D^3$ is compatible. Let ξ_n be the sequence produced by the Lenard–Magri scheme

$$H_{(5,ax)} \xi_n = D^3 \xi_{n+1}, \quad \xi_0 = x^2.$$

Then $H_{(9,ax)} \xi_0 = H_{(5,ax)} \xi_2$.

In view of Conjecture 3.15 and Remarks 3.16 and 3.17, the following conjecture, consistent with the known classification results of general integrable equations [MSS], seems natural.

Conjecture 3.18. *Any integrable bi-Hamiltonian equation in u is equivalent by a contact transformation to one, contained in either the linear hierarchy, or the KdV hierarchy, or the HD hierarchy, or the hierarchy defined in Theorem 3.7(b), or the hierarchies discussed in Remarks 3.10, 3.12 and 3.14. (For the definition of integrability of a Hamiltonian equation and the construction via the Lenard–Magri scheme of the KdV and HD hierarchies, see, e.g., [BDK].)*

4. Compatible family of Hamiltonian operators $H^{(N,0)}(D)$

Recall the definition (3.33) of the operator $H^{(N,0)}(D)$, where $N \geq 3$,

$$H^{(N,0)}(D) = D^2 \circ B^{(N-3)}(D) \circ D, \quad \text{where } B^{(n)}(D) := \left(\frac{1}{u}D\right)^n. \tag{4.34}$$

We denote by $\{\cdot, \lambda \cdot\}_N$ the corresponding λ -bracket, which is given by

$$\{u_\lambda u\}_N = (\lambda + D)^2 B^{(N-3)}(\lambda + D)\lambda. \tag{4.35}$$

Theorem 4.1. *The operators $H^{(N,0)}(D)$, $N \geq 3$ odd, are compatible Hamiltonian operators. Namely, any linear combination*

$$\{u_\lambda u\} = \sum_{\substack{N \geq 3, \text{ odd} \\ \text{(finite)}}} \alpha_N \{u_\lambda u\}_N, \tag{4.36}$$

with constant coefficients α_N , is a Poisson λ -bracket.

Lemma 4.2. *For every $m, n \in \mathbb{Z}_+$ we have*

$$\begin{aligned} & \{B^{(m)}(\lambda + D)1_{\lambda+\mu}u\}_{n+3} \\ &= -\lambda(\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \left(B^{(m)}(\lambda + D) \frac{1}{u} \right. \\ & \quad \left. - (-1)^m B^{(m)}(\mu + D) \frac{1}{u} \right). \end{aligned} \tag{4.37}$$

Proof. We prove equation (4.37) by induction on $m \in \mathbb{Z}_+$. For $m = 0$ both sides of (4.37) are zero. For $m \geq 0$ we have, by the definition (4.34) of the operator $B^{(n)}(D)$,

$$\begin{aligned} \{B^{(m+1)}(\lambda + D)1_{\lambda+\mu}u\}_{n+3} &= \left\{ \frac{1}{u}(\lambda + D)B^{(m)}(\lambda + D)1_{\lambda+\mu}u \right\}_{n+3} \\ &= -\{u_{\lambda+\mu+D}u\}_{n+3} \frac{1}{u^2}(\lambda + D)B^{(m)}(\lambda + D)1 \\ & \quad - \{B^{(m)}(\lambda + D)1_{\lambda+\mu+D}u\}_{n+3}(\mu + D) \frac{1}{u}. \end{aligned} \tag{4.38}$$

In the last identity we used sesquilinearity and the right Leibniz formula. By the definition (4.35) of the λ -bracket $\{\cdot \lambda \cdot\}_{n+3}$, the first term in the right-hand side of (4.38) is equal to

$$\begin{aligned}
 & -(\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D)(\lambda + \mu + D) \frac{1}{u} B^{(m+1)}(\lambda + D) 1 \\
 & = -\lambda(\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \left(B^{(m+1)}(\lambda + D) \frac{1}{u} \right. \\
 & \quad \left. + \left(B^m(\lambda + D) \frac{1}{u} \right) \left((\mu + D) \frac{1}{u} \right) \right). \tag{4.39}
 \end{aligned}$$

By inductive assumption, the second term in the right-hand side of (4.38) is equal to

$$\begin{aligned}
 & -\lambda(\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \left(\left(-B^{(m)}(\lambda + D) \frac{1}{u} \right) \left((\mu + D) \frac{1}{u} \right) \right. \\
 & \quad \left. + (-1)^m B^{(m+1)}(\mu + D) \frac{1}{u} \right). \tag{4.40}
 \end{aligned}$$

Combining (4.39) and (4.40), we conclude that the right-hand side of (4.38) is equal to

$$-\lambda(\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \left(B^{(m+1)}(\lambda + D) \frac{1}{u} + (-1)^m B^{(m+1)}(\mu + D) \frac{1}{u} \right),$$

thus proving the claim. \square

Lemma 4.3. *For every $m, n \in \mathbb{Z}_+$ we have*

$$\begin{aligned}
 & -\frac{1}{u} \left((\lambda + D) B^{(m)}(\lambda + D) \lambda \right) \left((\mu + D) B^{(n)}(\mu + D) \mu \right) + \{u_\lambda B^{(n)}(\mu + D) \mu\}_{m+3} \\
 & = -\lambda^2 \mu^2 B^{(n)}(\lambda + \mu + D) B^{(m)}(\lambda + D) \frac{1}{u}. \tag{4.41}
 \end{aligned}$$

Proof. We prove equation (4.41) by induction on $n \in \mathbb{Z}_+$. For $n = 0$ both sides of (4.41) are equal to $-\lambda^2 \mu^2 B^{(m)}(\lambda + D)(1/u)$. For $n \geq 0$, we have, by sesquilinearity and the left Leibniz formula,

$$\begin{aligned}
 \{u_\lambda B^{(n+1)}(\mu + D) \mu\}_{m+3} & = \{u_\lambda \frac{1}{u} (\mu + D) B^{(n)}(\mu + D) \mu\}_{m+3} \\
 & = -\frac{1}{u^2} \{u_\lambda u\}_{m+3} (\mu + D) B^{(n)}(\mu + D) \mu \\
 & \quad + \frac{1}{u} (\lambda + \mu + D) \{u_\lambda B^{(n)}(\mu + D) \mu\}_{m+3}. \tag{4.42}
 \end{aligned}$$

The first term in the right-hand side of (4.42) is equal to

$$-\frac{1}{u} \left((\lambda + D)^2 B^{(m)}(\lambda + D) \lambda \right) \left(B^{(n+1)}(\mu + D) \mu \right),$$

and summing this to

$$-\frac{1}{u}((\lambda + D)B^{(m)}(\lambda + D)\lambda)((\mu + D)B^{(n+1)}(\mu + D)\mu),$$

we get

$$-\frac{1}{u}(\lambda + \mu + D)\frac{1}{u}((\lambda + D)B^{(m)}(\lambda + D)\lambda)((\mu + D)B^{(n)}(\mu + D)\mu).$$

Combining the above results we get, by the inductive assumption,

$$\begin{aligned} &-\frac{1}{u}((\lambda + D)B^{(m)}(\lambda + D)\lambda)((\mu + D)B^{(n+1)}(\mu + D)\mu) + \{u_\lambda B^{(n+1)}(\mu + D)\mu\}_{m+3} \\ &= \frac{1}{u}(\lambda + \mu + D)\left(-\frac{1}{u}((\lambda + D)B^{(m)}(\lambda + D)\lambda)((\mu + D)B^{(n)}(\mu + D)\mu) \right. \\ &\quad \left. + \{u_\lambda B^{(n)}(\mu + D)\mu\}_{m+3}\right) \\ &= -\lambda^2\mu^2 B^{(n+1)}(\lambda + \mu + D)B^{(m)}(\lambda + D)\frac{1}{u}, \end{aligned}$$

as we wanted. \square

Proof of Theorem 4.1. Skew-commutativity of the λ -bracket (4.36) is clear, since, for odd N , the operators $H^{(N,0)}(D)$ are skew-adjoint. We thus only have prove the Jacobi identity

$$\{u_\lambda\{u_\mu u\}\} - \{u_\mu\{u_\lambda u\}\} = \{\{u_\lambda u\}_\lambda\}_{\lambda + \mu}u. \tag{4.43}$$

By Lemma 4.3, the first term in the left-hand side of (4.43) is, denoting $M = m + 3$, $N = n + 3$,

$$\begin{aligned} &\sum_{M,N} \alpha_M \alpha_N \{u_\lambda\{u_\mu u\}_N\}_M \\ &= \sum_{M,N} \alpha_M \alpha_N (\lambda + \mu + D)^2 \{u_\lambda B^{(n)}(\mu + D)\mu\}_M \\ &= \sum_{M,N} \alpha_M \alpha_N (\lambda + \mu + D)^2 \left(\frac{1}{u}((\lambda + D)B^{(m)}(\lambda + D)\lambda)((\mu + D)B^{(n)}(\mu + D)\mu) \right. \\ &\quad \left. - \lambda^2\mu^2 B^{(n)}(\lambda + \mu + D)B^{(m)}(\lambda + D)\frac{1}{u}\right). \end{aligned}$$

The first term in the right-hand side of (4) is invariant under the exchange of λ and μ , hence it gives no contribution to the Jacobi identity. Hence, the left-hand side of (4.43) is

$$\begin{aligned} &-\lambda^2\mu^2 \sum_{M,N} \alpha_M \alpha_N (\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \\ &\quad \cdot \left(B^{(m)}(\lambda + D)\frac{1}{u} - B^{(m)}(\mu + D)\frac{1}{u}\right). \end{aligned} \tag{4.44}$$

By Lemma 4.2, the right-hand side of (4.43) is

$$\begin{aligned} & \sum_{M,N} \alpha_M \alpha_N \{ \{ u_\lambda u \}_{M_{\lambda+\mu}} u \}_N \\ &= \sum_{M,N} \alpha_M \alpha_N \lambda \mu^2 \{ B^{(m)}(\lambda + D) 1_{\lambda+\mu} u \}_N \\ &= -\lambda^2 \mu^2 \sum_{M,N} \alpha_M \alpha_N (\lambda + \mu + D)^2 B^{(n)}(\lambda + \mu + D) \\ & \quad \cdot \left(B^{(m)}(\lambda + D) \frac{1}{u} - B^{(m)}(\mu + D) \frac{1}{u} \right), \end{aligned}$$

which is equal to (4.44). \square

References

[A] A. M. Асташов, *Нормальные формы гамильтоновых операторов в теории поля*, ДАН СССР **270** (1983), 1033–1037. Engl. transl.: A. M. Astashov, *Normal forms of Hamiltonian operators in field theory*, Sov. Math., Dokl. **27** (1983), 685–689.

[AV] A. M. Astashov, A. M. Vinogradov, *On the structure of Hamiltonian operators in field theory*, J. Geom. Phys. **2** (1986), 263–287.

[BD] A. Beilinson, V. G. Drinfeld, *Chiral Algebras*, AMS Colloquium Publications, Vol 51, Amer. Math. Soc., Providence, RI, 2004.

[BDK] A. Barakat, A. De Sole, V. G. Kac, *Poisson vertex algebras in the theory of Hamiltonian equations*, Jpn. J. Math. **4** (2009), 141–252.

[C] D. Cooke, *Classification results and Darboux’ theorem for low order Hamiltonian operators*, J. Math. Phys. **32** (1991), 109–119.

[C1] D. Cooke, *Compatibility conditions for Hamiltonian pairs*, J. Math. Phys. **32** (1991), 3071–3076.

[D] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Wiley, New York, 1993.

[DK] A. De Sole, V. G. Kac, *Finite vs. affine W-algebras*, Jpn. J. Math. **1** (2006), 137–261.

[GD] И. М. Гельфанд, И. Дорфман, *Гамильтоновы операторы и связанные с ними алгебраические структуры*, Функци. анализ. и его прилож. **13** (1979), no. 4, 13–30. Engl. transl.: I. M. Gelfand, I. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funct. Anal. Appl. **13** (1980), 248–262.

[FT] Л. А. Тахтаджян, Л. Д. Фаддеев, *Гамильтонов подход в теории солитонов*, Наука, М., 1986. Engl. transl.: L. D. Faddeev, L. A. Takhtadzhyan, *The Hamiltonian Methods in the Theory of Solitons*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1987.

[K] V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10, Amer. Math. Soc., Providence, RI, 1996. Second edition, 1998. Russian transl.: В. Г. Кац, *Вертексные алгебры для начинающих*, МЦНМО, М., 2005.

[Ma] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19** (1978), 1156–1162.

- [MSS] A. V. Mikhailov, A. V. Shabat, V. V. Sokolov *The symmetry approach to classification of integrable equations*, in: V. E. Zakharov, ed., *What is Integrability?*, Springer Series in Non-Linear Dynamics, Springer-Verlag, New York, 1991, pp, 115–184.
- [M] О. И. Мохов, *Гамильтоновы дифференциальные операторы и контактная геометрия*, Функц. анализ и его прилож. **21** (1987), no.3, 53–60. Engl. transl.: O. I. Mokhov, *Hamiltonian differential operators and contact geometry*, Funct. Anal. Appl. **21** (1987), 217–223.
- [O] P. J. Olver, *Darboux' theorem for Hamiltonian operators*, J. Differential Equations **71** (1988), 10–33.
- [V] А. М. Виноградов, *О гамильтоновых структурах в теории поля*, ДАН СССР **241** (1978), 18–21. Engl. transl.: A. M. Vinogradov, *On Hamiltonian structures in field theory*, Sov. Math., Dokl. **19** (1978), 790–794.