ALGEBRAIC DENSITY PROPERTY OF HOMOGENEOUS SPACES

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Abstract. Let X be an affine algebraic variety with a transitive action of the algebraic automorphism group. Suppose that X is equipped with several fixed point free nondegenerate SL_2-actions satisfying some mild additional assumption. Then we prove that the Lie algebra generated by completely integrable algebraic vector fields on X coincides with the space of all algebraic vector fields. In particular, we show that apart from a few exceptions this fact is true for any homogeneous space of form G/R where G is a linear algebraic group and R is a closed proper reductive subgroup of G.

1. Introduction

In this paper we develop further methods introduced by Kutzschebauch and the third author in [KK2] which they used to obtain new results in the Andersén–Lempert theory [A], [AL]. The following notion crucial for this theory was introduced first by Varolin [V1] though its importance for Euclidean spaces was emphasized already in the earlier paper of Rosay [Ro].

Definition 1. A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold X has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly, the algebraic density property for such an X implies the density property for it).

For any complex manifold with the density property the Andersén–Lempert theory is applicable and its effectiveness in complex analysis was demonstrated in several papers (e.g., see [FR], [Ro], [V1], [V2]). However, until recently, the class of manifolds for which this property was established was quite narrow (mostly Euclidean spaces and semisimple Lie groups, and homogeneous spaces of semisimple groups with trivial centers [TV1], [TV2]). In [KK1] and [KK2] this class was shown to contain hypersurfaces of the form $uv = p(\bar{x})$ and connected complex algebraic groups except for $\mathbb{C}_+$ and tori $(\mathbb{C}^*)^n$. Furthermore, it was proven in [KK1], [KK2] that these varieties have the algebraic density property. For $\mathbb{C}_+$ and $(\mathbb{C}^*)^n$ the algebraic density property is not true (for tori of dimension at least 2 this fact is nontrivial and it follows from the result of Andersén [A2] who established that for each completely integrable algebraic vector field on any torus its flow preserves the invariant volume form).

In this paper we study a smooth complex affine algebraic variety X with a transitive action of the algebraic automorphism group $\operatorname{Aut} X$. This is a mild restriction because for an affine algebraic (resp. Stein) manifold with the algebraic density property (resp. density property) the group of holomorphic automorphism generated by elements of flows induced by completely integrable algebraic (resp. holomorphic) vector fields is transitive and even m-transitive for any natural m (this is a consequence of, say, Theorem 0.2 from [V2]). Though the facts we prove about such objects are rather logical extension of [KK2], in combination with Lie group theory they lead to a much wider class of homogeneous spaces with the algebraic density property. Our new technique yields, in particular, to the following.

Theorem A. *Let G be a linear algebraic group and let R be a closed proper reductive subgroup of G such that the homogeneous space G/R has connected components different from $\mathbb{C}_+$ or a torus $(\mathbb{C}^*)^n$. Then G/R has the algebraic density property.*

Besides the criteria developed in [KK2] the main new ingredient of the proof is the Luna slice theorem. For the convenience of the readers we recall it in Section 2 together with basic facts about algebraic quotients and some crucial results from [KK2]. In Section 3 we prove our main theorem. As an application we prove Theorem A in Section 4 using some technical facts from the Lie group theory presented in the Appendix.

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2. Preliminaries

Let us fix some notation first. In this paper X will always be a complex affine algebraic variety and G will be an algebraic group acting on X, i.e., X is a G-variety. The ring of regular functions on X will be denoted by $\mathbb{C}[X]$ and its subring of G-invariant functions by $\mathbb{C}[X]^G$.

2.1. Algebraic (categorical) quotients

Recall that the algebraic quotient $X/\!\!/G$ of X with respect to the G-action is $\operatorname{Spec}(\mathbb{C}[X]^G)$. By $\pi : X \to X/\!\!/G$ we denote the natural quotient morphism gener-

ated by the embedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. The main (universal) property of algebraic quotients is that any morphism from X constant on orbits of G factors through π. In the case of a reductive G several important facts (e.g., see [Sch], [PV], [D], [G]) are collected in the following.

Proposition 1. *Let G be a reductive group.*

(1) *The quotient $X/\!/G$ is an affine algebraic variety which is normal in the case of a normal X and the quotient morphism $\pi : X \to X/\!/G$ is surjective.*
(2) *The closure of every G-orbit contains a unique closed orbit and each fiber $\pi^{-1}(y)$ (where $y \in X/\!/G$) also contains a unique closed orbit O. Furthermore, $\pi^{-1}(y)$ is the union of all those orbits whose closures contain O.*
(3) *In particular, if every orbit of the G-action on X is closed then $X/\!/G$ is isomorphic to the orbit space X/G.*
(4) *The image of a closed G-invariant subset under π is closed.*

If X is a complex algebraic group, and G is a closed subgroup acting on X by multiplication, then all orbits of the action are obviously closed. If G is reductive, the previous proposition implies that the quotient X/G is affine. The next proposition (Matsushima's criterion) shows that the converse is also true for quotients of reductive groups.

Proposition 2. *Let G be a complex reductive group, and let H be a closed subgroup of G. Then the quotient space G/H is affine if and only if H is reductive.*

Besides reductive group actions in this paper, a crucial role will be played by $\mathbb{C}_+$-actions. In general, algebraic quotients in this case are not affine but only quasi-affine [W]. However, we shall later use the fact that for the natural action of any $\mathbb{C}_+$-subgroup of SL_2 generated by multiplication one has $\mathrm{SL}_2/\!/\mathbb{C}_+ \cong \mathbb{C}^2$.

2.2. Luna's slice theorem (e.g., see [D], [PV])

Let us recall some terminology first. Suppose that $f : X \to Y$ is a G-equivariant morphism of affine algebraic G-varieties X and Y. Then the induced morphism $f_G : X/\!/G \to Y/\!/G$ is well defined and the following diagram is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow \\
X/\!/G & \xrightarrow{f_G} & Y/\!/G.
\end{array}
\tag{1}
$$

Definition 2. A G-equivariant morphism f is called strongly étale if

(1) the induced morphism $f_G : X/\!/G \to Y/\!/G$ is étale; and
(2) the quotient morphism $\pi_G : X \to X/\!/G$ induces a G-isomorphism between X and the fibered product $Y \times_{Y/\!/G} (X/\!/G)$.

From the properties of étale maps [D] it follows that f is étale (in particular, quasi-finite).

Let H be an algebraic subgroup of G, and let Z be an affine H-variety. We denote $G \times_H Z$ the quotient of $G \times Z$ with respect to the action of H given

by $h(g, z) = (gh^{-1}, hz)$. The left multiplication on G generates a left action on $G \times_H Z$. The next lemma is an obvious consequence of Proposition 1.

Lemma 3. *Let X be an affine G-variety and G be reductive. Then the H-orbits of $G \times X$ are all isomorphic to H. Therefore, the fibers of the quotient morphism $G \times X \to G \times_H X$ coincide with the H-orbits.*

The isotropy group of a point $x \in X$ will be denoted by G_x. Recall also that an open set U of X is called saturated if $\pi_G^{-1}(\pi_G(U)) = U$. We are ready to state the Luna slice theorem.

Theorem 4. *Let G be a reductive group acting on an affine algebraic variety X, and let $x \in X$ be a point in a closed G-orbit. Then there exists a locally closed affine algebraic subvariety V (called a slice) of X containing x such that*:

  (1) *V is G_x-invariant;*
  (2) *the image of the G-morphism $\varphi : G \times_{G_x} V \to X$ induced by the action is a saturated open set U of X; and*
  (3) *the restriction $\varphi : G \times_{G_x} V \to U$ is strongly étale.*

Given a saturated open set U, we will denote $\pi_G(U)$ by $U /\!\!/ G$. It follows from Proposition 1 that $U /\!\!/ G$ is open. Theorem 4 implies that the following diagram is commutative:

$$
\begin{array}{ccc}
G \times_{G_x} V & \longrightarrow & U \\
\downarrow & & \downarrow \\
V /\!\!/ G_x & \longrightarrow & U /\!\!/ G
\end{array}
\tag{2}
$$

and $G \times_{G_x} V \simeq U \times_{U /\!\!/ G} V /\!\!/ G_x$.

2.3. The compatibility criterion

This section presents the criteria for the algebraic density property, introduced in [KK2], that will be used to prove the main results of this paper.

Definition 3. Let X be an affine algebraic manifold. An algebraic vector field σ on X is semisimple if its flow is an algebraic $\mathbb{C}^*$-action on X. A vector field δ is locally nilpotent if its flow is an algebraic $\mathbb{C}_+$-action on X. In the last case, δ can be viewed as a locally nilpotent derivation on $\mathbb{C}[X]$. That is, for every nonzero $f \in \mathbb{C}[X]$, there is the smallest $n = n(f)$ for which $\delta^n(f) = 0$. We set $\deg_\delta(f) = n - 1$. In particular, elements from the kernel $\operatorname{Ker} \delta$ have degree 0 with respect to δ.

Definition 4. Let δ_1 and δ_2 be nontrivial algebraic vector fields on an affine algebraic manifold X such that δ_1 is a locally nilpotent derivation on $\mathbb{C}[X]$, and δ_2 is either also locally nilpotent or semisimple. That is, δ_i generates an algebraic action of H_i on X where $H_1 \simeq \mathbb{C}_+$ and H_2 is either $\mathbb{C}_+$ or $\mathbb{C}^*$. We say that δ_1 and δ_2 are semicompatible if the vector space $\operatorname{Span}(\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2)$, generated by elements from $\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2$, contains a nonzero ideal in $\mathbb{C}[X]$.

A semicompatible pair is called compatible if in addition one of the following condition holds:

(1) when $H_2 \simeq \mathbb{C}^*$ there is an element $a \in \operatorname{Ker} \delta_2$ such that $\deg_{\delta_1}(a) = 1$, i.e., $\delta_1(a) \in \operatorname{Ker} \delta_1 \setminus \{0\}$; and

(2) when $H_2 \simeq \mathbb{C}_+$ (i.e. both δ_1 and δ_2 are locally nilpotent) there is an element a such that $\deg_{\delta_1}(a) = 1$ and $\deg_{\delta_2}(a) \leq 1$.

Remark 1. If $[\delta_1, \delta_2] = 0$, then conditions (1) and (2) with $a \in \operatorname{Ker} \delta_2$ hold automatically, i.e., any semicompatible pair of commutative algebraic vector fields is always compatible.

Example 1. Consider SL_2 (or even PSL_2) with two natural $\mathbb{C}_+$-subgroups: namely, the subgroup H_1 (resp. H_2) of the lower (resp. upper) triangular unipotent matrices. Denote by

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

an element of SL_2. Then the left multiplication generates actions of H_1 and H_2 on SL_2 with the following associated locally nilpotent derivations on $\mathbb{C}[\mathrm{SL}_2]$,

$$\delta_1 = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2},$$
$$\delta_2 = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}.$$

Clearly, $\operatorname{Ker} \delta_1$ is generated by a_1 and a_2 while $\operatorname{Ker} \delta_2$ is generated by b_1 and b_2. Hence δ_1 and δ_2 are semicompatible. Furthermore, taking $a = a_1 b_2$, we see that condition (2) of Definition 4 holds, i.e., they are compatible.

It is worth mentioning the following geometrical reformulation of semicompatibility which will be needed further.

Proposition 5. *Suppose that H_1 and H_2 are as in Definition 4, X is a normal affine algebraic variety equipped with nontrivial algebraic H_i-actions where $i = 1, 2$ (in particular, each H_i generates an algebraic vector field δ_i on X). Let $X_i = X /\!\!/ H_i$ and let $\rho_i : X \to X_i$ be the quotient morphisms. Set $\rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2$ and set Z equal to the closure of $\rho(X)$ in Y. Then δ_1 and δ_2 are semicompatible iff $\rho : X \to Z$ is a finite birational morphism.*

Definition 5. A finite subset M of the tangent space $T_x X$ at a point x of a complex algebraic manifold X is called a generating set if the image of M under the action of the isotropy group (of algebraic automorphisms) of x generates $T_x X$.

It was shown in [KK2] that the existence of a pair of compatible derivations δ_1 and δ_2 from Definition 4 implies that $\operatorname{Lie}_{\mathrm{alg}}(X)$ contains a $\mathbb{C}[X]$-submodule $I\delta_2$ where I is a nontrivial ideal in $\mathbb{C}[X]^1$. This yields the central criterion for the algebraic density property [KK2].

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[1] In the case of condition (2) in Definition 4 this fact was proven in [KK2] only for $\deg_{\delta_2}(a) = 0$ but the proof works for $\deg_{\delta_2}(a) = 1$ as well without any change.

Theorem 6. *Let X be a smooth homogeneous (with respect to $\mathrm{Aut}\,X$) affine algebraic manifold with finitely many pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}_{k=1}^m$ such that for some point $x_0 \in X$ vectors $\{\delta_2^k(x_0)\}_{k=1}^m$ form a generating set. Then $\mathrm{Lie}_{\mathrm{alg}}(X)$ contains a nontrivial $\mathbb{C}[X]$-module and X has the algebraic density property.*

As an application of this theorem we have the following.

Proposition 7. *Let X_1 and X_2 be smooth homogeneous (with respect to algebraic automorphism groups) affine algebraic varieties such that each X_i admits a finite number of completely integrable algebraic vector fields $\{\delta_i^k\}_{k=1}^{m_i}$ whose values at some point $x_i \in X_i$ form a generating set and, furthermore, in the case of X_1 these vector fields are locally nilpotent. Then $X_1 \times X_2$ has the algebraic density property.*

We shall also need two technical results (Lemmas 3.6 and 3.7 in [KK2]) that describe conditions under which quasi-finite morphisms preserve semicompatibility.

Lemma 8. *Let $G = \mathrm{SL}_2$ and let X, X' be normal affine algebraic varieties equipped with nondegenerate G-actions (i.e., general G-orbits are of dimension 3). Suppose that subgroups H_1 and H_2 of G are as in Example 1, i.e., they act naturally on X and X'. Let $\rho_i : X \to X_i := X /\!\!/ H_i$ and $\rho_i' : X' \to X_i' := X' /\!\!/ H_i$ be the quotient morphisms and let $p : X \to X'$ be a finite G-equivariant morphism, i.e., we have commutative diagrams:*

$$
\begin{array}{ccc}
X & \xrightarrow{\rho_i} & X_i \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q_i} \\
X' & \xrightarrow{\rho_i'} & X_i'
\end{array}
$$

for $i = 1, 2$. Treat $\mathbb{C}[X_i]$ (resp. $\mathbb{C}[X_i']$) as a subalgebra of $\mathbb{C}[X]$ (resp. $\mathbb{C}[X']$). Let $\mathrm{Span}(\mathbb{C}[X_1] \cdot \mathbb{C}[X_2])$ contain a nonzero ideal of $\mathbb{C}[X]$. Then $\mathrm{Span}(\mathbb{C}[X_1'] \cdot \mathbb{C}[X_2'])$ contains a nonzero ideal of $\mathbb{C}[X']$.

The second result is presented here in a slightly different form but with a much simpler proof.

Lemma 9. *Let the assumption of Lemma 8 hold with two exceptions: we do not assume that G-actions are nondegenerate and instead of the finiteness of p we suppose that there are a surjective étale morphism $r : M \to M'$ of normal affine algebraic varieties equipped with trivial G-actions and a surjective G-equivariant morphism $\tau' : X' \to M'$ such that X is isomorphic to the fibered product $X' \times_{M'} M$ with $p : X \to X'$ being the natural projection (i.e., p is surjective étale). Then the conclusion of Lemma 8 remains valid.*

Proof. By construction, $X_i = X_i' \times_{M'} M$. Thus we have the following commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\rho} & (X_1' \times X_2') \times_{(M' \times M')} (M \times M) & \longrightarrow & M \times M \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (r,r)} \\
X' & \xrightarrow{\rho'} & X_1' \times X_2' & \xrightarrow{(\tau', \tau')} & M' \times M'.
\end{array}
$$

Set Z (resp. Z') equal to the closure of $\rho(X)$ in $X_1 \times X_2$ (resp. $\rho'(X')$ in $X_1' \times X_2'$) and let $D \simeq M$ (resp. $D' \simeq M'$) be the diagonal subset in $M \times M$ (resp. $M' \times M'$). Since $X = X' \times_{M'} M$ we see that $Z = Z' \times_{D'} D$. For any affine algebraic variety Y denote by Y_{norm} its normalization, i.e., $Z_{\mathrm{norm}} = Z'_{\mathrm{norm}} \times_{D'} D$. By Lemma 5, $\rho : X \to Z_{\mathrm{norm}}$ is an isomorphism. Since r is surjective it can happen only when $\rho' : X' \to Z'$ is an isomorphism. Hence the desired conclusion follows from Proposition 5.

The last result from [KK2] that we need allows us to switch from local to global compatibility.

Proposition 10. *Let X be an* SL_2-*variety with associated locally nilpotent derivations δ_1 and δ_2, let Y be a normal affine algebraic variety equipped with a trivial* SL_2-*action, and let $r : X \to Y$ be a surjective* SL_2-*equivariant morphism. Suppose that for any $y \in Y$ there exists an étale neighborhood $g : W \to Y$ such that the vector fields induced by δ_1 and δ_2 on the fibered product $X \times_Y W$ are semicompatible. Then δ_1 and δ_2 are semicompatible.*

3. Algebraic density property and SL_2-actions

Notation 1. We suppose that H_1, H_2, δ_1, and δ_2 are as in Example 1. Note that if SL_2 acts algebraically on an affine algebraic variety X then we have automatically the $\mathbb{C}_+$-actions of H_1 and H_2 on X that generate locally nilpotent vector fields on X which, by abuse of notation, will be denoted by the same symbols δ_1 and δ_2. If X admits several (say, N) SL_2-actions, we denote by $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ the corresponding collection of pairs of locally nilpotent derivations on $\mathbb{C}[X]$.

Recall also that an action of a Lie group of a manifold is nondegenerate if the dimension of general orbits is the same as the dimension of the group. Here is the first main result of this paper.

Theorem 11. *Let X be a smooth complex affine algebraic variety whose group of algebraic automorphisms is transitive. Suppose that X is equipped with N fixed point free nondegenerate actions of* SL_2-*groups $\Gamma_1, \ldots, \Gamma_N$. Let $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ be the corresponding pairs of locally nilpotent vector fields. If $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$ is a generating set at some point $x_0 \in X$ then X has the algebraic density property.*

Remark 2. Note that we can choose any nilpotent element of the Lie algebra of SL_2 as δ_2. Since the space of nilpotent elements generates the whole Lie algebra we can reformulate Theorem 11 as follows: a smooth complex affine algebraic variety X with a transitive group of algebraic automorphisms has the algebraic density property provided it admits "sufficiently many" fixed point free nondegenerate SL_2-actions, where "sufficiently many" means that at some point $x_0 \in X$ the tangent spaces of the corresponding SL_2-orbits through x_0 generate the whole space $T_{x_0}X$.

By virtue of Theorem 6 the main result will be a consequence of the following.

Theorem 12. *Let X be a smooth complex affine algebraic variety equipped with a fixed point free nondegenerate* SL_2-*action that induces a pair of locally nilpotent vector fields $\{\delta_1, \delta_2\}$. Then these vector fields are compatible.*

In preparation for the proof of Theorem 12 we must establish a number of results. From now on, we assume that all SL_2-actions under consideration are nondegenerate.

Lemma 13. *Let the assumption of Theorem* 12 *hold and let* $x \in X$ *be a point contained in a closed* SL_2*-orbit. Then the isotropy group of* x *is either finite, or isomorphic to the diagonal* $\mathbb{C}^*$*-subgroup of* SL_2*, or to the normalizer of this* $\mathbb{C}^*$*-subgroup (which is the extension of* $\mathbb{C}^*$ *by* $\mathbb{Z}_2$*).*

Proof. By Matsushima's criterion (Proposition 2) the isotropy group must be reductive and it cannot be SL_2 itself since the action has no fixed points. The only two-dimensional reductive group is $\mathbb{C}^* \times \mathbb{C}^*$ [FuHa] which is not contained in SL_2. Thus besides finite subgroups we are left to consider the one-dimensional reductive subgroups that include $\mathbb{C}^*$ (which can be considered to be the diagonal subgroup since all tori are conjugated) and its finite extensions. The normalizer of $\mathbb{C}^*$ which is its extension by $\mathbb{Z}_2$ generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is reductive. If we try to find an extension of $\mathbb{C}^*$ by another finite subgroup that contains an element B not from the normalizer then $\mathbb{C}^*$ and $B\mathbb{C}^*B^{-1}$ meet at the identity matrix. In particular, the reductive subgroup must be at least two-dimensional, and we have to disregard this case. $\square$

Proposition 14. *Let* X, δ_1, δ_2 *be as in Theorem* 12. *Then there exists a regular function* $g \in \mathbb{C}[X]$ *such that* $\deg_{\delta_1}(g) = \deg_{\delta_2}(g) = 1$.

Proof. Let $x \in X$ be a point of a closed SL_2-orbit. Luna's slice theorem yields diagram (2) with $G = \mathrm{SL}_2$ and G_x being one of the subgroups described in Lemma 13. That is, we have the natural morphism $\varphi : \mathrm{SL}_2 \times V \to U$ that factors through the étale morphism $\mathrm{SL}_2 \times_{G_x} V \to U$ where V is the slice at x. First, consider the case when G_x is finite. Then φ itself is étale. Furthermore, replacing V by its Zariski open subset and U by the corresponding Zariski open SL_2-invariant subset one can suppose that φ is also finite. Set $f = a_1 b_2$ where a_i, b_i are as in Example 1. Note that each δ_i generates a natural locally nilpotent vector field $\tilde{\delta}_i$ on $\mathrm{SL}_2 \times V$ such that $\mathbb{C}[V] \subset \operatorname{Ker} \tilde{\delta}_i$ and $\varphi_*(\tilde{\delta}_i)$ coincides with the vector field induced by δ_i on X. Treating f as an element of $\mathbb{C}[\mathrm{SL}_2 \times V]$ we have $\deg_{\tilde{\delta}_i}(f) = 1$, $i = 1, 2$. For every $h \in \mathbb{C}[\mathrm{SL}_2 \times V]$ we define a function $\hat{h} \in \mathbb{C}[U]$ by $\hat{h}(u) = \sum_{y \in \varphi^{-1}(u)} h(y)$. One can check that if $h \in \operatorname{Ker} \tilde{\delta}_i$ then $\delta_i(\hat{h}) = 0$. Hence $\delta_i^2(\hat{f}) = 0$ but we also need $\delta_i(\hat{f}) \neq 0$ which is not necessarily true. Thus, multiply f by $\beta \in \mathbb{C}[V]$. Since $\beta \in \operatorname{Ker} \tilde{\delta}_i$ we have $\delta_i(\widehat{\beta f})(u) = \sum_{y \in \varphi^{-1}(u)} \beta(\pi_V(y))\tilde{\delta}_i(f)(y)$. Note that $\tilde{\delta}_i(f)(y_0)$ is not zero at a general $y_0 \in \mathrm{SL}_2 \times V$ since $\tilde{\delta}_i(f) \neq 0$. By a standard application of the Nullstellensatz we can choose β with prescribed values at the finite set $\varphi^{-1}(u_0)$ where $u_0 = \varphi(y_0)$. Hence we can assume that $\delta_i(\widehat{\beta f})(u_0) \neq 0$, i.e., $\deg_{\delta_i}(\widehat{\beta f}) = 1$. There is still one problem: $\widehat{\beta f}$ is regular on U but necessarily not on X. In order to fix it we set $g = \alpha\widehat{\beta f}$ where α is a lift of a nonzero function on $X /\!\!/ G$ that

vanishes with high multiplicity on $(X /\!/ G) \setminus (U /\!/ G)$. Since $\alpha \in \operatorname{Ker} \delta_i$ we still have $\deg_{\delta_i}(g) = 1$ which concludes the proof in the case of a finite isotropy group.

For a one-dimensional isotropy group note that f is $\mathbb{C}^*$-invariant with respect to the action of the diagonal subgroup of SL_2. That is, f can be viewed as a function on $\mathrm{SL}_2 \times_{\mathbb{C}^*} V$. Then we can replace morphism φ with morphism $\psi : \mathrm{SL}_2 \times_{\mathbb{C}^*} V \to U$ that factors through the étale morphism $\mathrm{SL}_2 \times_{G_x} V \to U$. Now ψ is also étale and the rest of the argument remains the same. $\square$

In order to finish the proof of Theorem 12 we need to show semicompatibility of the vector fields δ_1 and δ_2 on X. Let U be a saturated set as in diagram (2) with $G = \mathrm{SL}_2$. Since U is SL_2-invariant it is H_i-invariant (where H_i is from Notation 1) and the restriction of δ_i to U is a locally nilpotent vector field which we denote again by the same letter. Furthermore, the closure of any SL_2-orbit O contains a closed orbit, i.e., O is contained in an open set like U and, therefore, X can be covered by a finite collections of such open sets. Thus Proposition 10 implies the following.

Lemma 15. *If for every U as before the locally nilpotent vector fields δ_1 and δ_2 are semicompatible on U then they are semicompatible on X.*

Notation 2. Suppose further that H_1 and H_2 act on $\mathrm{SL}_2 \times V$ by left multiplication on the first factor. The locally nilpotent vector fields associated with these actions of H_1 and H_2 are, obviously, semicompatible since they are compatible on SL_2 (see Example 1). Consider the SL_2-equivariant morphism $G \times V \to G \times_{G_x} V$ where V, $G = \mathrm{SL}_2$, and G_x are as in diagram (2). By definition $G \times_{G_x} V$ is the quotient of $G \times V$ with respect to the G_x-action whose restriction to the first factor is the multiplication from the right. Hence H_i-action commutes with G_x-action and, therefore, one has the induced H_i-action on $G \times_{G_x} V$. Following the convention of Notation 1 we denote the associated locally nilpotent derivations on $G \times_{G_x} V$ again by δ_1 and δ_2. That is, the SL_2-equivariant étale morphism $\varphi : G \times_{G_x} V \to U$ transforms the vector field δ_i on $G \times_{G_x} V$ into the vector field δ_i on U.

From Lemma 9 and Luna's slice theorem we immediately have the following.

Lemma 16.
   (1) *If the locally nilpotent vector fields δ_1 and δ_2 are semicompatible on $G \times_{G_x} V$ then they are semicompatible on U.*
   (2) *If the isotropy group G_x is finite, δ_1 and δ_2 are, indeed, semicompatible on $G \times_{G_x} V$.*

Now we have to tackle semicompatibility in the case of a one-dimensional isotropy subgroup G_x using Proposition 5 as a main tool. We start with the case of $G_x = \mathbb{C}^*$.

Notation 3. Consider the diagonal $\mathbb{C}^*$-subgroup of SL_2, i.e., elements of form

$$s_\lambda = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

The action of s_λ on $v \in V$ will be denoted by $\lambda.v$. When we speak later about the $\mathbb{C}^*$-action on V we shall mean exactly this action. Set $Y = \mathrm{SL}_2 \times V$, $Y' = \mathrm{SL}_2 \times_{\mathbb{C}^*}$

V, $Y_i = Y /\!\!/ H_i$, $Y_i' = Y' /\!\!/ H_i$. Denote by $\rho_i : Y \to Y_i$ the quotient morphism of the H_i-action and use the similar notation for Y', Y_i'. Set $\rho = (\rho_1, \rho_2) : Y \to Y_1 \times Y_2$ and $\rho' = (\rho_1', \rho_2') : Y' \to Y_1' \times Y_2'$.

Note that $Y_i \simeq \mathbb{C}^2 \times V$ since $\mathrm{SL}_2 /\!\!/ \mathbb{C}_+ \simeq \mathbb{C}^2$. Furthermore, looking at the kernels of δ_1 and δ_2 from Example 1, we see for

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathrm{SL}_2$$

the quotient maps $\mathrm{SL}_2 \to \mathrm{SL}_2 /\!\!/ H_1 \simeq \mathbb{C}^2$ and $\mathrm{SL}_2 \to \mathrm{SL}_2 /\!\!/ H_2 \simeq \mathbb{C}^2$ are given by $A \mapsto (a_1, a_2)$ and $A \mapsto (b_1, b_2)$, respectively. Hence, the morphism $\rho : \mathrm{SL}_2 \times V = Y \to Y_1 \times Y_2 \simeq \mathbb{C}^4 \times V \times V$ is given by

$$\rho(a_1, a_2, b_1, b_2, v) = (a_1, a_2, b_1, b_2, v, v). \tag{3}$$

As we mentioned before, to define $Y' = \mathrm{SL}_2 \times_{\mathbb{C}^*} V$ we let $\mathbb{C}^*$ act on SL_2 via right multiplication. Since H_1 and H_2 act on SL_2 from the left, there are well-defined $\mathbb{C}^*$-actions on Y_1 and Y_2 and a torus $\mathbb{T}$-action on $Y_1 \times Y_2$, where $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$. Namely,

$$(\lambda, \mu).(a_1, a_2, b_1, b_2, v, w) = (\lambda a_1, \lambda^{-1} a_2, \mu b_1, \mu^{-1} b_2, \lambda.v, \mu.w) \tag{4}$$

for $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ and $(\lambda, \mu) \in \mathbb{T}$.

Since the $\mathbb{C}^*$-action on Y and the action of H_i, $i = 1, 2$, are commutative, the following diagram is also commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\ \rho\ } & Y_1 \times Y_2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow{\ \rho'\ } & Y_1' \times Y_2', \end{array} \tag{5}$$

where q (resp. p) is the quotient map with respect to the $\mathbb{T}$-action (resp. $\mathbb{C}^*$-action). It is also worth mentioning that the $\mathbb{C}^*$-action on Y induces the action of the diagonal of $\mathbb{T}$ on $\rho(Y)$, i.e., for every $y \in Y$ we have $\rho(\lambda.y) = (\lambda, \lambda).\rho(y)$.

Lemma 17. *Let $Z = \rho(Y)$ in diagram (5) and let Z' be the closure of $\rho'(Y')$.*

(i) *The map $\rho : Y \to Z$ is an isomorphism and Z is the closed subvariety of $Y_1 \times Y_2 = \mathbb{C}^4 \times V \times V$ that consists of points $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ satisfying the equations $a_1 b_2 - a_2 b_1 = 1$ and $v = w$.*

(ii) *Let T be the $\mathbb{T}$-orbit of Z in $Y_1 \times Y_2$ and let $\bar{T}$ be its closure. Then T coincides with the $(\mathbb{C}^* \times 1)$-orbit (resp. $(1 \times \mathbb{C}^*)$-orbit) of Z. Furthermore, for each $(a_1, a_2, b_1, b_2, v, w) \in \bar{T}$, one has $\pi(v) = \pi(w)$ where $\pi : V \to V /\!\!/ \mathbb{C}^*$ is the quotient morphism.*

(iii) *The restriction of diagram (5) yields the following:*

$$\begin{array}{ccc} Y & \xrightarrow{\ \rho\ } & Z \subset \bar{T} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow{\ \rho'\ } & q(Z) \subset Z', \end{array} \tag{6}$$

where $Y' = Y/\!/\mathbb{C}^* = Y/\mathbb{C}^*$, q is the quotient morphism of the $\mathbb{T}$-action (i.e. $Z' = \bar{T}/\!/\mathbb{T}$), and $q(Z) = \rho'(Y')$.

Proof. The first statement is an immediate consequence of formula (3). The beginning of the second statement follows from the fact that the action of the diagonal $\mathbb{C}^*$-subgroup of $\mathbb{T}$ preserves Z. This implies that for every $t = (a_1, a_2, b_1, b_2, v, w) \in T$ the points $v, w \in V$ belong to the same $\mathbb{C}^*$-orbit and, in particular, $\pi(v) = \pi(w)$. This equality holds for each point in $\bar{T}$ by continuity.

In diagram (5), $Y' = Y/\!/\mathbb{C}^* = Y/\mathbb{C}^*$ because of Proposition 1(3) and Lemma 3, and the equality $q(Z) = \rho'(Y')$ is the consequence of the commutativity of that diagram. Note that $\bar{T}$ is $\mathbb{T}$-invariant. Hence $q(\bar{T})$ coincides with Z' by Proposition 1(4). Being the restriction of the quotient morphism, $q|_{\bar{T}} : \bar{T} \to Z'$ is a quotient morphism itself (e.g., see [D]) which concludes the proof. $\square$

Lemma 18. *There is a rational $\mathbb{T}$-quasi-invariant function f on $\bar{T}$ such that for $t = (a_1, a_2, b_1, b_2, w, v) \in T$ one has:*
(1) $(1/f(t))a_1 b_2 - f(t)a_2 b_1 = 1$ *and* $w = f(t).v$;
(2) *the set $\bar{T} \setminus T$ is contained in $(f)_0 \cup (f)_\infty$; and*
(3) *f generates a regular function on a normalization T_N of T*

Proof. By Lemma 17(ii) any point $t = (a_1, a_2, b_1, b_2, w, v) \in T$ is of the form $t = (\lambda, 1).z_0$ where $z_0 \in Z$ and $\lambda \in \mathbb{C}^*$. Hence formula (4) implies that $w = \lambda.v$ and $\lambda^{-1}a_1 b_2 - \lambda a_2 b_1 = 1$. The last equality yields two possible values (one of which can be ∞ or 0 if any of the numbers a_1, a_2, b_1, or b_2 vanishes)

$$\lambda_\pm = \frac{-1 \pm \sqrt{1 + 4a_1 a_2 b_1 b_2}}{2a_2 b_1}$$

and we assume that

$$\lambda = \lambda_- = \frac{-1 - \sqrt{1 + 4a_1 a_2 b_1 b_2}}{2a_2 b_1},$$

i.e., $w = \lambda_-.v$. Note that $\lambda_+.v = w$ as well only when

$$\tau = \frac{\lambda_+}{\lambda_-} = \frac{-1 + \sqrt{1 + 4a_1 a_2 b_1 b_2}}{-1 - \sqrt{1 + 4a_1 a_2 b_1 b_2}}$$

is in the isotropy group of v.

Consider the set of points $t \in T$ such that v is not a fixed point of the $\mathbb{C}^*$-action on V and $\tau.v = v$. Denote its closure by S. Since S is a proper subvariety of T, one has a well-defined branch λ_- of the two-valued function $\lambda_\pm$ on the complement to S. Its extension to $\bar{T}$, which is denoted by f, satisfies (1).

Let $t_n \in T$ and $t_n \to t \in \bar{T}$ as $n \to \infty$. By Lemma 17(ii), t_n is of the form $t_n = (f(t_n)a_1^n, (1/f(t_n))a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n)$ where

$$\begin{pmatrix} a_1^n & a_2^n \\ b_1^n & b_2^n \end{pmatrix} \in \mathrm{SL}_2 \quad \text{and} \quad v = \lim_{n \to \infty} v_n.$$

If sequences $\{f(t_n)\}$ and $\{1/f(t_n)\}$ are bounded, then switching to a subsequence one can suppose that $f(t_n) \to f(t) \in \mathbb{C}^*$, $w = f(t)v$, and $t = (f(t)a_1', (1/f(t))a_2', b_1', b_2', f(t).v, v)$ where

$$\begin{pmatrix} a_1' & a_2' \\ b_1' & b_2' \end{pmatrix} \in \mathrm{SL}_2 \,,$$

i.e., $t \in T$. Hence $\bar{T} \setminus T$ is contained in $((f)_0 \cup (f)_\infty)$ which is (2).

The function f is regular on $T \setminus S$ by construction. Consider $t \in S$ with w and v in the same nonconstant $\mathbb{C}^*$-orbit, i.e., $w = \lambda.v$ for some $\lambda \in \mathbb{C}^*$. Then $w = \lambda'.v$ if and only if only λ' belongs to the coset Γ of the isotropy subgroup of v in $\mathbb{C}^*$. For any sequence of points t_n convergent to t one can check that $f(t_n) \to \lambda \in \Gamma$ by continuity, i.e., f is bounded in a neighborhood of t. Let $\nu : T_N \to T$ be a normalization morphism. Then the function $f \circ \nu$ extends regularly to $\nu^{-1}(t)$ by the Riemann extension theorem. The set of points of S for which v is a fixed point of the $\mathbb{C}^*$-action is of codimension at least 2 in T. By the Hartogs theorem, $f \circ \nu$ extends regularly to T_N which concludes (3).

Remark 3. Consider the rational map $\kappa : T \to Z$ given by $t \mapsto (1/f(t), 1).t$. It is regular on $T \setminus S$ and if $t \in T \setminus S$ and $z \in Z$ are such that $t = (\lambda, 1).z$ then $\kappa(t) = z$. In particular, κ sends $\mathbb{T}$-orbits from T into $\mathbb{C}^*$-orbits of Z. Furthermore, the morphism $\kappa_N = \kappa \circ \nu : T_N \to Z$ is regular because this is the case for the function $f \circ \nu$.

Lemma 19. *Let $E_i = \{t = (a_1, a_2, b_1, b_2, w, v) \in \bar{T} \mid b_i = 0\}$ and let $\bar{T}^b$ coincide with $T \cup ((f)_0 \setminus E_2) \cup ((f)_\infty \setminus E_1)$. Suppose that $\bar{T}_N^b$ is a normalization of $\bar{T}^b$. Then there is a regular extension of $\kappa_N : T_N \to Z$ to a morphism $\bar{\kappa}_N^b : \bar{T}_N^b \to Z$.*

Proof. Since the set $(f)_0 \cap (f)_\infty$ is of codimension 2 in $\bar{T}$, the Hartogs theorem implies that it suffices to prove the regularity of $\bar{\kappa}_N^b$ on the normalization of $\bar{T}^b \setminus ((f)_0 \cap (f)_\infty)$. Furthermore, by the Riemann extension theorem it is enough to construct a continuous extension of κ from $T \setminus S$ to $\bar{T}^b \setminus (S \cup ((f)_0 \cap (f)_\infty))$.

By Lemma 18(2) we need to consider this extension, say, at $t = (a_1, a_2, b_1, b_2, w, v) \in (f)_0 \setminus (f)_\infty$. Let $t_n \to t$ as $n \to \infty$ where

$$t_n = \left(f(t_n)a_1^n, \frac{1}{f(t_n)}a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n \right) \in T$$

with $a_1^n b_2^n - a_2^n b_1^n = 1$ and $f(t_n) \to 0$. Perturbing, if necessary, this sequence $\{t_n\}$ we can suppose every $t_n \notin S$, i.e., $\kappa(t_n) = (a_1^n, a_2^n, b_1^n, b_2^n, v_n, v_n)$. Note that $\lim v_n = v$, $b_k = \lim b_k^n$, $k = 1, 2$, and $a_2^n \to 0$ since a_2 is finite. Hence, $1 = a_1^n b_2^n - a_2^n b_1^n \approx a_1^n b_2$ and $a_1^n \to 1/b_2$ as $n \to \infty$. Now we get a continuous extension of κ by putting $\kappa(t) = (1/b_2, 0, b_1, b_2, v, v)$. This yields the desired conclusion. $\square$

Remark 4. If we use the group $(1 \times \mathbb{C}^*)$ instead of the group $(\mathbb{C}^* \times 1)$ from Lemma 17(ii) in our construction this would lead to the replacement of f by f^{-1}. Furthermore, for the variety $\bar{T}^a = T \cup ((f)_0 \setminus \{a_1 = 0\}) \cup ((f)_\infty \setminus \{a_2 = 0\})$, we obtain a morphism $\bar{\kappa}_N^a : \bar{T}_N^a \to Z$ similar to $\bar{\kappa}_N^b$.

The next fact is intuitively obvious but the proof requires some work.

Lemma 20. *The complement $\bar{T}^0$ of $\bar{T}^a \cup \bar{T}^b$ in $\bar{T}$ (which is $\bar{T}^0 = (\bar{T} \setminus T) \cap \bigcup_{i \neq j}\{a_i = b_j = 0\}$) has codimension at least 2.*

Proof. Let $t_n \to t = (a_1, a_2, b_1, b_2, w, v)$ be as in the proof of Lemma 19. Since for a general point of the slice V the isotropy group is finite after perturbation we can suppose that each v_n is contained in a nonconstant $\mathbb{C}^*$-orbit $O_n \subset V$. Treat v_n and $f(t_n).v_n$ as numbers in $\mathbb{C}^* \simeq O_n$ such that $f(t_n).v_n = f(t_n)v_n$. Let $|v_n|$ and $|f(t_n).v_n|$ be their absolute values. Then one has the annulus $A_n = \{|f(t_n).v_n| < \zeta < |v_n|\} \subset O_n$, i.e., $\zeta = \eta v_n$ where $|f(t_n)| < |\eta| < 1$ for each $\zeta \in A_n$. By Lemma 17(iii) $\pi(v) = \pi(w)$ but by Lemma 18(3) the $\mathbb{C}^*$-orbit $O(v)$ and $O(w)$ are different unless $w = v$ is a fixed point of the $\mathbb{C}^*$-action. In any case, by Proposition 1(2) the closures of these orbits meet at a fixed point $\bar{v}$ of the $\mathbb{C}^*$-action.

Consider a compact neighborhood $W = \{u \in V \mid \varphi(u) \leq 1\}$ of $\bar{v}$ in V where φ is a plurisubharmonic function on V that vanishes at $\bar{v}$ only. Note that the sequence $\{(\lambda, \mu).t_n\}$ is convergent to $(\lambda a_1, a_2/\lambda, \mu b_1, b_2/\mu, \lambda.w, \mu.v)$. In particular, replacing $\{t_n\}$ by $\{(\lambda, \mu).t_n\}$ with appropriate λ and μ we can suppose that the boundary ∂A_n of any annulus A_n is contained in W for sufficiently large n. By the maximum principle $\bar{A}_n \subset W$. The limit $A = \lim_{n \to \infty} \bar{A}_n$ is a compact subset of W that contains both v and w, and also all points $\eta.v$ with $0 < |\eta| < 1$ (since $|f(t_n)| \to 0$). Unless $O(v) = \bar{v}$ only one of the closures of the subsets $\{\eta.v \mid 0 < |\eta| < 1\}$ or $\{\eta.v \mid |\eta| > 1\}$ in V is compact and contains the fixed point $\bar{v}$ (indeed, otherwise the closure of $O(v)$ is a complete curve in the affine variety V). The argument above shows that it is the first one.

That is, $\mu.v \to \bar{v}$ when $\mu \to 0$. Similarly, $\lambda.w \to \bar{v}$ when $\lambda \to \infty$. It is not difficult to check now that the dimension of the set of such pairs (w, v) is at most $\dim V$.

Consider the set $(\bar{T} \setminus T) \cap \{a_1 = b_2 = 0\}$. It consists of points $t = (0, a_2, b_1, 0, w, v)$ and, therefore, its dimension is at most $\dim V + 2$. Thus it has codimension at least 2 in $\bar{T}$ whose dimension is $\dim V + 4$. This yields the desired conclusion. $\square$

The next technical fact may be somewhere in the literature, but, unfortunately, we did not find a reference.

Proposition 21. *Let a reductive group G act on an affine algebraic variety X and let $\pi : X \to Q := X /\!/ G$ be the quotient morphism such that one of closed G-orbits O is contained in the smooth part of X [2]. Suppose that $\nu : X_N \to X$ and $\mu : Q_N \to Q$ are normalization morphisms, i.e., $\pi \circ \nu = \pi_N \circ \mu$ for some morphism $\pi_N : X_N \to Q_N$. Then $Q_N \simeq X_N /\!/ G$ for the induced G-action on X_N and π_N is the quotient morphism.*

Proof. Let $\psi : X_N \to R$ be the quotient morphism. By the universal property of quotient morphims, $\pi_N = \varphi \circ \psi$ where $\varphi : R \to Q_N$ is a morphism. It suffices to show that φ is an isomorphism. The points of Q (resp. R) are nothing but the closed G-orbits in X (resp. X_N) by Proposition 1, and above each closed orbit in X we have only a finite number of closed orbits in X_N because ν is finite. Hence, $\mu \circ \varphi : R \to Q$ and, therefore, $\varphi : R \to Q_N$ are at least quasi-finite. There is only one closed orbit O_N in X_N above orbit $O \subset \operatorname{reg} X$. Thus φ is injective in a

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[2]Without this assumption on smoothness the proposition does not hold.

neighborhood of $\psi(O_N)$. That is, φ is birational and by Zariski's Main Theorem it is an embedding.

It remains to show that φ is proper. Recall that G is a complexification of a compact subgroup $G^{\mathbb{R}}$ and there is a so-called Kempf–Ness real algebraic subvariety $X^{\mathbb{R}}$ of X such that the restriction $\pi|_{X^{\mathbb{R}}}$ is nothing but the standard quotient map $X^{\mathbb{R}} \to X^{\mathbb{R}}/G^{\mathbb{R}} = Q$ which is automatically proper (e.g., see [Sch]). Set $X_N^{\mathbb{R}} = \nu^{-1}(X^{\mathbb{R}})$. Then the restriction of $\pi \circ \nu$ to $X_N^{\mathbb{R}}$ is proper being the composition of two proper maps. On the other hand, the restriction of $\mu \circ \pi_N = \pi \circ \nu$ to $X_N^{\mathbb{R}}$ is proper only when morphism φ, through which it factors, is proper which concludes the proof.

Proposition 22. *The morphism $\rho' : Y' \to Z'$ from diagram (6) is finite birational.*

Proof. The morphism ρ' factors through $\rho'_N : Y' \to Z'_N$ where $\mu : Z'_N \to Z'$ is a normalization of Z' and the statement of the proposition is equivalent to the fact that ρ'_N is an isomorphism. Set $Z'(b) = q(\bar{T}^b)$ and $Z'(a) = q(\bar{T}^a)$. Note that $Z' \setminus (Z'(a) \cup Z'(b))$ is in the q-image of the $\mathbb{T}$-invariant set $\bar{T}^0$ from Lemma 20. Hence $Z' \setminus (Z'(b) \cup Z'(a))$ is of codimension 2 in Z' and by the Hartogs theorem it suffices to prove that ρ'_N is invertible over $Z(b)'$ (resp. $Z'(a)$).

By Remark 3, $\bar{\kappa}_N^b$ sends each orbit of the induced $\mathbb{T}$-action on $\bar{T}_N^b$ onto a $\mathbb{C}^*$-orbit in Z. Thus the composition of $\bar{\kappa}_N^b$ with $p : Z \simeq Y \to Y'$ is constant on $\mathbb{T}$-orbits and by the universal property of quotient spaces it must factor through the quotient morphism $q_N^b : \bar{T}_N^b \to Q$. By Proposition 21, $Q = Z'_N(b)$ where $Z'_N(b) = \mu^{-1}(Z'(b))$. That is, $p \circ \bar{\kappa}_N^b = \tau^b \circ q_N^b$ where $\tau^b : Z'_N(b) \to Y'$. Our construction implies that τ^b is the inverse of ρ'_N over $Z'_N(b)$. Hence, ρ'_N is invertible over $Z'_N(b)$ which concludes the proof. $\square$

3.1. Proof of Theorems 12 and 11

Let $G = \mathrm{SL}_2$ act algebraically on X as in Theorem 12 and let V be the slice of this action at a point $x \in X$ so that there is an étale morphism $G \times_{G_x} V \to U$ as in Theorem 4. By Lemmas 15 and 16 for the validity of Theorem 12 it suffices to prove semicompatibility of the vector fields δ_1 and δ_2 on $\mathcal{Y} = G \times_{G_x} V$, which was already done in the case of a finite isotropy group G_x (see Lemma 16(2)). Consider the quotient morphisms $\varrho_i : \mathcal{Y} \to \mathcal{Y}_i := \mathcal{Y}/\!\!/ H_i$ where H_i, $i = 1, 2$, are as in Example 1. Set $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \to \overline{\varrho(\mathcal{Y})} \subset \mathcal{Y}_1 \times \mathcal{Y}_2$. By Proposition 5, Theorem 12 is true if ϱ is finite birational. If $G_x = \mathbb{C}^*$ then $\varrho : \mathcal{Y} \to \mathcal{Y}_1 \times \mathcal{Y}_2$ is nothing but morphism $\rho' : Y' \to Y_1' \times Y_2'$ from Proposition 22, i.e., we are done in this case as well. By Lemma 13 the only remaining case is when G_x is an extension of $\mathbb{C}^*$ by $\mathbb{Z}_2$. Then one has a $\mathbb{Z}_2$-action on Y' such that it is commutative with H_i-actions on Y' and $\mathcal{Y} = Y'/\!\!/\mathbb{Z}_2$. Since the vector fields δ_1 and δ_2 are semicompatible on Y' by Propositions 22 and 5, they also generate semicompatible vector fields on $\mathcal{Y}$ by Lemma 8. This concludes Theorem 12 and, therefore, Theorem 11. $\square$

Remark 5. (1) Consider $\mathcal{Y} = G \times_{G_x} V$ in the case when $G = G_x = \mathrm{SL}_2$, i.e., the SL_2-action has a fixed point. It is not difficult to show that morphism $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \to \overline{\varrho(\mathcal{Y})} \subset \mathcal{Y}_1 \times \mathcal{Y}_2$ as in the proof above is not quasi-finite. In particular, δ_1 and

δ_2 are not compatible. However, we do not know if the condition about the absence of fixed points is essential for Theorem 11. In examples we know the presence of fixed points is not an obstacle for the algebraic density property. Say, for $\mathbb{C}^n$ with $2 \le n \le 4$ any algebraic SL_2-action is a representation in a suitable polynomial coordinate system (e.g., see [Po]) and, therefore, it has a fixed point; but the algebraic density property is a consequence of the Andersén–Lempert work.

(2) Similarly, we do not know whether the nondegeneration assumption is required for Theorem 11. The simplest case of a degenerate SL_2-action is presented by the homogeneous space $SL_2/\mathbb{C}^*$ where $\mathbb{C}^*$ is the diagonal subgroup. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

be a general element of SL_2. Then the ring of invariants of the $\mathbb{C}^*$-action is generated by $u = a_1 a_2$, $v = b_1 b_2$, and $z = a_2 b_1 + \frac{1}{2}$ (since $a_1 b_2 = 1 + a_2 b_1 = \frac{1}{2} + z$). Hence $SL_2/\mathbb{C}^*$ is isomorphic to a hypersurface S in $\mathbb{C}^3_{u,v,z}$ given by the equation $uv = z^2 - \frac{1}{4}$. This hypersurface has the algebraic density property by [KK1].

(3) The situation is a bit more complicated if we consider the normalizer T of the diagonal $\mathbb{C}^*$-subgroup of SL_2 (i.e., T is an extension of $\mathbb{C}^*$ by $\mathbb{Z}_2$). Then $\mathcal{P} = SL_2/T$ is isomorphic to $S/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action is given by $(u, v, z) \to (-u, -v, -z)$. This interesting surface[3] has also the algebraic density property. To see this, note that the diagonal $\mathbb{C}^*$-subgroup of SL_2 contains the center of SL_2 which implies that both S and $\mathcal{P}$ are quotients of PSL_2. Since PSL_2 has a trivial center it has the density property by the Toth–Varolin theorem in [TV2]. Furthermore, the analysis of their paper shows that they used only completely integrable algebraic vector fields in the proof, i.e., they established, in fact, the algebraic density property. In brief, we have the algebraic density property for any homogeneous space that is a quotient of SL_2.

4. Applications

Theorem 11 is applicable to a wide class of homogeneous spaces. Let us start with the following observation: given a reductive subgroup R of a linear algebraic group G, any SL_2-subgroup $\Gamma < G$ yields a natural Γ-action on G/R. Furthermore, for each point $aR \in G/R$, its isotropy subgroup under this action is isomorphic to $\Gamma \cap aRa^{-1}$. In particular, the action has no fixed points if $a^{-1}\Gamma a$ is not contained in R for any $a \in G$ and it is nondegenerate if $\Gamma^a := a^{-1}\Gamma a \cap R \simeq \Gamma \cap aRa^{-1}$ is finite for some $a \in R$. Thus Theorem 11 implies the following.

Proposition 23. *Let G be an algebraic group and let $\Gamma_1, \dots, \Gamma_k$ be SL_2-subgroups of G such that at some $x \in G$ the set $\{\delta_2^i(x)\}$ is a generating one (where (δ_1^i, δ_2^i) is the corresponding pair of locally nilpotent vector fields on G generated by the natural Γ_i-action). Suppose that for each $i = 1, \dots, k$ and any $a \in G$, the group*

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[3]It can be shown that $\mathcal{P}$ is the only $\mathbb{Q}$-homology plane which is simultaneously a Danilov–Gizatullin surface (i.e., it has a trivial Makar-Limanov invariant (see [FKZ])), and its fundamental group is $\mathbb{Z}_2$.

$\Gamma_i^a := a^{-1}\Gamma_i a \cap R$ *is not isomorphic to* Γ_i *and, furthermore,* Γ_i^a *is finite for some* a. *Then* G/R *has the algebraic density property.*

Note that for a simple Lie group G, a generating set at any $x \in G$ consists of one nonzero vector since the adjoint representation is irreducible. Therefore, in this case the algebraic density property is a consequence of the following.

Theorem 24. *Let* G *be a simple Lie group with Lie algebra different from* $\mathfrak{sl}_2$, *and let* R *be a proper closed reductive subgroup of* G. *Then there exists an* SL_2-*subgroup* Γ *in* G *such that* Γ^a *is not isomorphic to* Γ *for any* $a \in G$ *and, furthermore,* Γ^a *is finite for some* $a \in G$.

Surprisingly enough, the proof of Theorem 24 (at least in our presentation) requires some serious facts from Lie group theory and we shall postpone it untill the Appendix.

Corollary 25. *Let* $X = G/R$ *be an affine homogeneous space of a semisimple Lie group* G *different from* SL_2. *Then* X *is equipped with* N *pairs* $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ *of compatible derivations such that the collection* $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$ *is a generating set at some point* $x_0 \in X$. *In particular,* X *has the algebraic density property by Theorem* 6.

Proof. Note that R is reductive by Proposition 2 (Matsushima's theorem). Then X is isomorphic to a quotient of form G/R where $G = G_1 \oplus \cdots \oplus G_N$, each G_i is a simple Lie group, and R is not necessarily connected. However, we can suppose that R is connected by virtue of Proposition 8. Consider the projection homomorphism $\pi_k : G \to G_k$ and $R_k = \pi_k(R)$ which is reductive being the image of a reductive group. If N is the minimal possible then $R_k \neq G_k$ for every k. Indeed, if say $R_N = G_N$, then $X = (G_1 \oplus \cdots \oplus G_{N-1})/\tilde{R}$ where $\tilde{R} = \mathrm{Ker}\,\pi_N$ which contradicts minimality.

Assume first that none of the G_is is isomorphic to SL_2. By Theorem 24 one can choose an SL_2-subgroup $\Gamma_k < G_k$ such that the natural Γ_k-action on $G_k/\pi_k(R)$ and, therefore, on G/R is fixed point free. We can also assume that each Γ_k-action is nondegenerate. Denote by δ_1^k and δ_2^k the corresponding pair of locally nilpotent derivations for the Γ_k-action. Since the adjoint representation is irreducible for a simple Lie group, $\{\delta_2^k(e)\}_{k=1}^N$ is a generating set of the tangent space T_eG at $e = e_1 \oplus \cdots \oplus e_N \in G$, where e_k is the unity of G_k. Consider $X = G/R$ as the set of left cosets, i.e., X is the quotient of G with respect to the action generated by multiplication by elements of R from the right. Hence, this action commutes with multiplication by elements of Γ_k from the left and, therefore, it commutes with any field δ_k^i. Pushing the actions of Γ_ks to X we get fixed point free nondegenerate SL_2-actions on X and the desired conclusion in this case follows from by Theorem 11.

In the case when some of the G_ks are isomorphic to SL_2 we cannot assume that each Γ_k-action is nondegenerate, but now $N \geq 2$ and the Γ_k-actions are still fixed point free. Consider an isomorphism $\varphi_k : \Gamma_k \to \Gamma_1$. Then we have an SL_2-group $\Gamma^{\varphi_k} = \{(\varphi_k(\gamma), \gamma) \mid \gamma \in \Gamma_k\} < \Gamma_1 \times \Gamma_k$ acting naturally on $G_1 \times G_k$ and, therefore, on G. This isomorphism φ_k can be chosen so that the Γ^{φ_k}-action is nondegenerate. Indeed, if, say a $\mathbb{C}^*$-subgroup $L < \Gamma_k$ acts on G_k trivially, choose φ_k so that $\varphi_k(L)$ acts nontrivially on G_1 which makes the action nondegenerate. In

particular, by Theorem 12 we get pairs of compatible locally nilpotent derivations $\tilde{\delta}_1^{\varphi_k}$ and $\tilde{\delta}_2^{\varphi_k}$ corresponding to such actions. Set $G' = G_2 \oplus \cdots \oplus G_N$ and $e' = e_2 \oplus \ldots \oplus e_N \in G'$. Since the adjoint representation is irreducible for a simple Lie group the orbit of the set $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ under conjugations generates a subspace of S of T_eG such that the restriction of the natural projection $T_eG \to T_{e'}G'$ to S is surjective. In order to enlarge $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ to a generating subset of T_eG consider an isomorphism $\psi_2 : \Gamma_2 \to \Gamma_1$ different from φ_2 and such that the Γ^{ψ_2}-action is nondegenerate. Denote the corresponding compatible locally nilpotent derivations by $\tilde{\delta}_1^{\psi_2}$ and $\tilde{\delta}_2^{\psi_2}$ on $G_1 \oplus G_2$ (and also by abusing notation on G). Note that the vectors $\tilde{\delta}_2^{\varphi_2}(e_1 \oplus e_2)$ and $\tilde{\delta}_2^{\psi_2}(e_1 \oplus e_2)$ can be assumed different with an appropriate choice of ψ_2. Hence these two vectors form a generating subset of $T_{e_1 \oplus e_2} G_1 \oplus G_2$. Taking into consideration the remark about S we see that $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N \cup \{\tilde{\delta}_2^{\psi_2}(e)\}$ is a generating subset of T_eG. Now pushing these SL_2-actions to X we get the desired conclusion. $\square$

Theorem 26. *Let G be a linear algebraic group and let R be a closed proper reductive subgroup of G such that the homogeneous space G/R has connected components different from $\mathbb{C}_+$ or a torus $(\mathbb{C}^*)^n$. Then G/R has the algebraic density property.*

Proof. Since all components of G/R are isomorphic as varieties we can suppose that G is connected. Furthermore, by Corollary 25 and Remark 5(3), we are done with a semisimple G.

Let us consider first the case of a reductive but not semisimple G. Then the center $Z \simeq (\mathbb{C}^*)^n$ of G is nontrivial. Let S be the semisimple part of G. Assume for the time being that G is isomorphic as a group to the direct product $S \times Z$ and consider the natural projection $\tau : G \to Z$. Set $Z' = \tau(R) = R/R'$ where $R' = R \cap S$. Since we are going to work with compatible vector fields we can suppose that R is connected by virtue of Lemma 8. Then Z' is a subtorus of Z and also R' is reductive by Proposition 2. Hence $G/R = (G/R')/Z'$ and $G/R' = S/R' \times Z$. Note that there is a subtorus Z'' of Z such that $Z'' \simeq Z/Z'$ and $Z' \cdot Z'' = Z$. (Indeed, $Z' \simeq (\mathbb{C}^*)^k$ generates a sublattice $L \simeq \mathbb{Z}^k$ of homomorphisms from $\mathbb{C}^*$ into Z' of the similar lattice $\mathbb{Z}^n$ of $Z \simeq (\mathbb{C}^*)^n$ such that the quotient $\mathbb{Z}^n/L$ has no torsion, i.e., it is isomorphic to $\mathbb{Z}^{n-k}$. Since any short exact sequence of free $\mathbb{Z}$-modules splits we have a $\mathbb{Z}$-submodule $K \simeq \mathbb{Z}^{n-k}$ in $\mathbb{Z}^n$ such that $K + L = \mathbb{Z}^n$. This lattice K yields a desired subtorus $Z'' \simeq (\mathbb{C}^*)^{n-k}$.) Hence G/R is isomorphic to $\varrho^{-1}(Z'') \simeq S/R' \times Z''$ where $\varrho : G/R' \to Z$ is the natural projection. Note that both factors are nontrivial since otherwise G/R is either a torus or we are in the semisimple case again. Thus X has the algebraic density property by Proposition 7 with S/R' playing the role of X_1 and Z'' of X_2. In particular, we have a finite set of pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}$ as in Theorem 6. Furthermore, one can suppose that the fields δ_1^k correspond to one-parameter subgroups of S isomorphic to $\mathbb{C}_+$ and δ_2^k to one-parameter subgroups of Z isomorphic to $\mathbb{C}^*$. In the general case G/R is the factor of X with respect to the natural action of a finite (central) normal subgroup $F < G$. Since F is central, the fields δ_1^k, δ_2^k induce completely integrable vector fields $\tilde{\delta}_1^k$, $\tilde{\delta}_2^k$ on G/R while $\tilde{\delta}_2^k(x_0)$ is a generating set for some $x_0 \in G/R$. By Lemma 8 the pairs $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$ are compatible and the algebraic density

property for G/R follows again from Theorem 6.

In the case of a general linear algebraic group G different from a reductive group, $\mathbb{C}^n$, or a torus $(\mathbb{C}^*)^n$ consider the nontrivial unipotent radical $\mathcal{R}_u$ of G. It is automatically an algebraic subgroup of G, see [Ch, p. 183]. By Mostow's theorem [Mo] (see also [Ch, p. 181)]) G contains a (Levi) maximal closed reductive algebraic subgroup G_0 such that G is the semi-direct product of G_0 and $\mathcal{R}_u$, i.e., G is isomorphic as an affine variety to the product $\mathcal{R}_u \times G_0$. Furthermore, any other maximal reductive subgroup is conjugated to G_0. Hence, replacing G_0 by its conjugate, we can suppose that R is contained in G_0. Therefore, G/R is isomorphic as an affine algebraic variety to the $G_0/R \times \mathcal{R}_u$ and we are done now by Proposition 7 with $\mathcal{R}_u$ playing the role of X_1 and G_0/R of X_2.

Remark 6. (1) The algebraic density property implies, in particular, that the Lie algebra generated by completely integrable algebraic (and, therefore, holomorphic) vector fields is infinite dimensional, i.e., this is true for homogeneous spaces from Theorem 26. For Stein manifolds of dimension at least 2 that are homogeneous spaces of holomorphic actions of a connected complex Lie group the infinite dimensionality of such algebras was also established by Huckleberry and Isaev [HI].

(2) Note that as in [KK2] we actually proved a stronger fact for a homogeneous space $X = G/R$ from Theorem 26. Namely, it follows from the construction that the Lie algebra generated by vector fields of form $f\sigma$, where σ is either locally nilpotent or semisimple and $f \in \operatorname{Ker}\sigma$ for semisimple σ and $\deg_\sigma f \leq 1$ in the locally nilpotent case, coincides with $\mathrm{AVF}(X)$.

5. Appendix: The proof of Theorem 24

Let us start with the following technical fact.

Proposition 27. *Let G be a semisimple group and let R be a semisimple subgroup of G. Suppose that the number of orbits of nilpotent elements in the Lie algebra $\mathfrak{r}$ of R under the adjoint action is less than the number of orbits of nilpotent elements in the Lie algebra of G under the adjoint action. Then G contains an SL_2-subgroup Γ such that $\Gamma^g := g^{-1}\Gamma g \cap R$ is different from $g^{-1}\Gamma g$ for any $g \in G$.*

Proof. By the Jacobson–Morozov theorem (e.g., see Proposition 2 and Corollary in [Bo, Sect. 8.11.2]) for any semisimple group G there is a bijection between the set of G-conjugacy classes of $\mathfrak{sl}_2$-triples and the set of G-conjugacy classes of nonzero nilpotent elements from G which implies the desired conclusion.

In order to exploit Proposition 27 we need to recall some terminology and results from [Bo].

Definition 6. (1) Recall that a semisimple element h of a Lie algebra is regular, if the kernel of its adjoint action is a Cartan subalgebra. An $\mathfrak{sl}_2$-subalgebra of the Lie algebra $\mathfrak{g}$ of a semisimple group G is called principal if in its triple of standard generators the semisimple element h is regular and the adjoint action of h has even eigenvalues (see Definition 3 in [Bo, Sect. 8.11.4]). The subgroup generated by this subalgebra is called a principal SL_2-subgroup of G. As an example of such a principal subgroup one can consider an SL_2-subgroup of SL_n that acts irreducibly

on the natural representation space $\mathbb{C}^n$. In general, principal $\mathfrak{sl}_2$-subalgebras exist in any semisimple Lie algebra $\mathfrak{g}$ (see Proposition 8 in [Bo, Sect. 8.11.4]). Any two principal SL_2-subgroups are conjugated (see Proposition 6 in [Bo, Sect. 8.11.3] and Proposition 9 in [Bo, Sect. 8.11.4]).

(2) A connected closed subgroup P of G is called principal if it contains a principal SL_2-subgroup[4]. The rank of P is the rank of the maximal torus it contains. If this rank is 1 then P coincides with its principal SL_2-subgroup (see Exercise 21 in [Bo, Sect. 9.4]).

Proposition 28. *Let R be a proper reductive subgroup of a simple group G different from* SL_2 *or* PSl_2. *Then there exists an* SL_2*-subgroup* Γ *of* G *such that* $\Gamma^g := g^{-1}\Gamma g \cap R$ *is different from* $g^{-1}\Gamma g$ *for any* $g \in G$.

Proof. If R is not principal it cannot contain a principal SL_2-subgroup and we are done. Thus it suffices to consider the case of the principal subgroup R only.

Suppose first that R is of rank 1. If R contains $g^{-1}\Gamma g$ it must coincide with this subgroup by the dimension argument. Hence it suffices to choose nonprincipal Γ to see the validity of the proposition in this case.

Suppose now that R is of rank at least 2. Then there are the following possibilities (Exercises 20c–e in [Bo, Sect. 9.4]):

(1) R is of type B_2 and G is of type A_3 or A_4;

(2) R is of type G_2 and G is of type $B_3, D_4,$, or A_6;

(3) G is of type A_{2l} with $l \geq 3$ and R is of type B_l;

(4) G is of type A_{2l-1} with $l \geq 3$ and R is of type C_l;

(5) G is of type D_l with $l \geq 4$ and R is of type B_{l-1};

(6) G is of type E_6 and R is of type F_4.

In order to apply Proposition 27 to these cases we need the Dynkin classification of nilpotent orbits (with Elkington's corrections) as described in the Bala–Carter paper [BaCa2, pp. 6–7].

By this classification the number a_n of such orbits in a simple Lie algebra of type A_n coincides with the number of partitions λ of $n + 1$, i.e., $\lambda = (\lambda_1, \ldots, \lambda_k)$ with natural λ_i such that $|\lambda| = \lambda_1 + \cdots + \lambda_k = n + 1$.

For a simple Lie algebra of type B_m the number b_m of nilpotent orbits coincides with the number of partitions λ and μ such that $2|\lambda| + |\mu| = 2m + 1$ where μ is a partition with distinct odd parts.

For a simple Lie algebra of type C_m the number c_m of nilpotent orbits coincides with the number of partitions λ and μ such that $|\lambda| + |\mu| = m$ where μ is a partition with distinct parts.

For a simple Lie algebra of type D_m the number d_m of nilpotent orbits coincides with the number of partitions λ and μ such that $2|\lambda| + |\mu| = 2m$ where μ is a partition with distinct odd parts.

The numbers of nilpotent orbits of algebras of type G_2, F_4, E_6, E_7, E_8 are 5, 16, 21, 45, and 70, respectively.

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[4]The definition of a principal subgroup in [Bo] is different (see Exercise 18 in Section 9.4) but it coincides with this one in the case of a complex Lie group (see Exercise 21c in Section 9.4).

Now one has $a_4 > a_3 = 5 > b_2 = 4$ which settles case (1) by Proposition 27. Then $b_3, d_4, a_6 > 5$ which settles case (2). Similarly, $a_{2l} > b_l$ for $l \geq 3$, $a_{2l-1} > c_l$ for $l \geq 3$, $d_l > b_{l-1}$ for $l \geq 4$, and $21 > 16$ which settles cases (3)–(6) and concludes the proof. $\square$

Remark 7. In fact, the statement of Proposition 28 is true for any proper maximal subgroup R of G. This can be deduced from Dynkin's classification of maximal subalgebras in semisimple Lie algebras. We outline the argument below.

Let us consider a maximal subalgebra $\mathfrak{r}$ in $\mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra. If $\mathfrak{r}$ is regular (i.e., if its normalizer contains some Cartan subalgebra in $\mathfrak{g}$), then $\mathfrak{r}$ does not contain any principal $\mathfrak{sl}_2$-triple [OVG, Sect. 6.2.4]. Thus we may assume that $\mathfrak{r}$ is nonregular.

If $\mathfrak{g}$ is exceptional, the list of such $\mathfrak{r}$ is given in Theorems 6.3.4 and 6.3.5 from [OVG]. All of them are semisimple, and we will only consider simple subalgebras (otherwise, $\mathfrak{r}$ once again does not contain any principal $\mathfrak{sl}_2$s). The list of simple maximal nonregular subalgebras of rank ≥ 2 in exceptional Lie algebras is short: B_2 in E_8, A_2 in E_7 and A_2, G_2, C_4, F_4 in E_6. In all these cases, Proposition 27 applies.

It remains to consider nonregular maximal subalgeras $\mathfrak{r}$ of classical Lie algebras. Any such $\mathfrak{r}$ is simple, and an embedding of $\mathfrak{r}$ in $\mathfrak{g}$ is defined by a nontrivial linear irreducible representation $\varphi : \mathfrak{r} \to \mathfrak{sl}(V)$. Let $n = \dim V$ and $m = [n/2]$. If the module V is not self-dual, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = \mathsf{A}_{n-1}$. If V is self-dual and endowed with a skew-symmetric invariant form, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = \mathsf{C}_m$; and if V is self-dual with a symmetric invariant form, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = \mathsf{B}_m$ or D_m. Denote by $o(V)$ the number of nilpotent orbits in $\mathfrak{g}$, then

$$o(V) \geq o_n = \begin{cases} \min(a_{n-1}, b_m) & \text{if } n \text{ odd,} \\ \min(a_{n-1}, d_m, c_m) & \text{if } n \text{ is even.} \end{cases}$$

We want to check that for any irreducible $\mathfrak{r}$-module V (except those corresponding to the trivial embedding $\mathfrak{r} = \mathfrak{g}$), the number $o(\mathfrak{r})$ of the nilpotent orbits in $\mathfrak{r}$ is less than $o(V)$. In what follows, representations φ generating trivial embeddings of $\mathfrak{r}$ in $\mathfrak{g}$ are excluded. For exceptional $\mathfrak{r}$ of types G_2, F_4, E_6, E_7, E_8 the smallest irreducible representation has dimension $n = 7, 26, 27, 56, 248$ respectively. In all cases, the inequality $o(\mathfrak{r}) < o_n$ holds.

If $\mathfrak{r}$ is of type A_k, then either V is not self-dual and $n > k + 1$ (in which case $o(\mathfrak{r}) = a_k < a_{n-1} = o(V)$) or V is self-dual and $n > 2(k + 1)$. Then $a_k < o_n \leq o(V)$.

If $\mathfrak{r}$ is of type B_k $(k \geq 2)$, then all irreducible V are self-dual and $n > 2k + 1$, hence $b_k < o_n$. If $\mathfrak{r}$ is of type C_k $(k \geq 3)$, then all irreducible V are self-dual and $n > 4k$, hence $c_k < o_n$. If $\mathfrak{r}$ is of type D_k $(k \geq 4)$, then for any irreducible V, $n > 3k$ and $d_k < o_n$.

From this we conclude that Proposition 27 applies to any simple nonregular $\mathfrak{r}$ in $\mathfrak{g}$, where $\mathfrak{g}$ is a classical simple Lie algebra.

Lemma 29. *Each orbit O of a fixed point free degenerate SL_2-action on an affine algebraic variety X is two dimensional and closed, and the isotropy group of any point $x \in X$ is either $\mathbb{C}^*$ or a $\mathbb{Z}_2$-extension of $\mathbb{C}^*$.*

Proof. In the case of a fixed point free SL_2-action the isotropy group I_x of a point from a closed orbit is either finite or $\mathbb{C}^*$ or $\mathbb{Z}_2$-extension of $\mathbb{C}^*$ by Lemma 13. Because the action is also degenerate I_x cannot be finite and and, therefore, the closed orbit SL_2/I_x is two dimensional. By Proposition 1 (2) the closure of O must contain a closed orbit. Since O itself is at most two dimensional it must coincide with this closed orbit. $\square$

Next we need two lemmas with the proof of the first one being straightforward.

Lemma 30. *Let G be a simple Lie group of dimension N and rank n, let a be an element of G and let $C(a)$ be its centralizer. Suppose that k is the dimension of $C(a)$. Then the dimension of the orbit O of a under conjugations is $N - k$. In particular, when a is a regular element (i.e,. $\dim C(a) = n$) we have $\dim O = N - n$ coincides with the codimension of the centralizer of a.*

Lemma 31. *Let G be a simple Lie group of dimension N and rank n, let R be a proper reductive subgroup of G whose dimension is M and rank is m and let Γ be an SL_2-subgroup of G such that its natural action on G/R is fixed point free degenerate. Suppose that a is a semisimple nonidentical element of Γ and k is the dimension of $C(a)$. Then $M \geq N - k - 1$. Furthermore, if a is regular, $M = N - n + m - 2$.*

Proof. Since the Γ-action on G/R is fixed point free and degenerate the isotropy group of any element $gR \in G/R$ is either $\mathbb{C}^*$ or a $\mathbb{Z}_2$ extension of $\mathbb{C}^*$ by Lemma 29. Recall that this isotropy group is $\Gamma \cap gRg^{-1}$ and, therefore, R contains a unique subgroup of the form $g^{-1}L'g$ where L' is a $\mathbb{C}^*$-subgroup of Γ. That is, $L' = \gamma_0^{-1}L\gamma_0$ for some $\gamma_0 \in \Gamma$ where L is the $\mathbb{C}^*$-subgroup generated by a. Furthermore, this γ_0 is unique modulo a normalizer of L in Γ because, otherwise, Γ^g contain another $\mathbb{C}^*$-subgroup of $g^{-1}\Gamma g$ and, therefore, it would be at least two dimensional. The two-dimensional variety $W_{a,g} = \{(\gamma g)^{-1}a(\gamma g) \mid \gamma \in \Gamma\}$ meets R exactly at two points $(\gamma_0 g)^{-1}a(\gamma_0 g)$ and $(\gamma_0 g)^{-1}a^{-1}(\gamma_0 g)$ (since the normalizer of L has two components). Varying g we can suppose that $W_{a,g}$ contains a general point of the G-orbit O_a of a under conjugations. Since it meets subvariety $R \cap O_a$ of O_a at two points we see that $\dim R \cap O_a = \dim O_a - 2 = N - k - 2$ by Lemma 30. Thus, with a running over L we have $\dim R \geq N - k - 1$.

For the second statement note that $b = g^{-1}ag \in R$ is a regular element in G. Hence the maximal torus in G (and, therefore, in R) containing b is determined uniquely. Assume that two elements $b_l = g_l^{-1}ag_l \in R$, $l = 1, 2$, are contained in the same maximal torus T' of R and, therefore, the same maximal torus T of G. Then $g_2 g_1^{-1}$ belongs to the normalizer of T, i.e. b_2 is of the form $w^{-1}b_1 w$ where w is an element of the Weyl group of T. Thus $R \cap O_a$ meets each maximal torus T' at a finite number of points. The space of the maximal tori of R is naturally isomorphic to R/T'_{norm} where T'_{norm} is the normalizer of T'. Hence $\dim R \cap O_a = \dim R - \dim T'_{\mathrm{norm}} = M - m$. We showed already that the last dimension is also $N - n - 2$ which implies $M = N - n + m - 2$.

Proposition 32. *Let the assumption of Lemma 31 hold. Then a cannot be a regular element of G.*

Proof. Assume the contrary, i.e., a is regular. Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) be the linear space generated by positive (resp. negative) root spaces. Set $\mathfrak{s} = \mathfrak{r}_+ + \mathfrak{r}_-$ and suppose that $\mathfrak{g}', \mathfrak{h}', \mathfrak{s}'$ are the similar objects for R with $\mathfrak{h}' \subset \mathfrak{h}$. Put $\mathfrak{r}'_\pm = \mathfrak{s}' \cap \mathfrak{r}_\pm$.

Each element x' of a root space from $\mathfrak{s}'$ is of the form $x' = h_0 + x_+ + x_-$ where $h_0 \in \mathfrak{h}$ and $x_\pm \in \mathfrak{r}_\pm$. Then there exists an element $h' \in \mathfrak{h}'$ such that the Lie bracket $[x', h']$ is a nonzero multiple of x' which implies that $h_0 = 0$, since $[h_0, h'] = 0$. Thus $\mathfrak{s}' \subset \mathfrak{s}$. By assumption $\mathfrak{h}'$ is a linear subspace of $\mathfrak{h}$ of codimension $n - m$. Hence Lemma 31 implies that $\mathfrak{s}'$ is of codimension 2 in $\mathfrak{s}$. We have two possibilities: (1) either, say, $\mathfrak{r}'_+ = \mathfrak{r}_+$ and $\mathfrak{r}'_-$ is of codimension 2 in $\mathfrak{r}_-$; or (2) $\mathfrak{r}'_\pm$ is of codimension 1 in $\mathfrak{r}_\pm$. In the first case, each element of a root space $x \in \mathfrak{r}_+$, being in an eigenspace of $\mathfrak{h}' \subset \mathfrak{h}$, is also an element of a root space of $\mathfrak{g}'$. However, for each root the negative of it is also contained in $\mathfrak{g}'$ which implies that $\mathfrak{r}'_- = \mathfrak{r}_-$. A contradiction.

In case (2) consider the generators $x_1, \ldots, x_l$ (resp. $y_1, \ldots, y_l$) of all root spaces in $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) such that $h_i = [x_i, y_i]$ is a nonzero element of $\mathfrak{h}$. Their linear combination $\sum_{i=1}^l c_i^+ x_i$ is contained in $\mathfrak{r}'_+$ if and only if its coefficients satisfy a nontrivial linear equation $\sum_{i=1}^l d_i^+ c_i^+ = 0$. Similarly, $\sum_{i=1}^l c_i^- y_i$ is contained in $\mathfrak{r}'_-$ if and only if its coefficients satisfy a nontrivial linear equation $\sum_{i=1}^l d_i^- c_i^- = 0$. Note that $d_i^+ = 0$ if and only if $d_i^- = 0$ since, otherwise, one can find a root of $\mathfrak{g}'$ whose negative is not a root. Without loss of generality we suppose that the simple roots are presented by $x_1, \ldots, x_n$, i.e. $h_1, \ldots, h_n$ is a basis of $\mathfrak{h}$. Hence at least one coefficient $d_i^+ \neq 0$ for $i \leq n$. Indeed, otherwise $\mathfrak{r}'_+$ contains $x_1, \ldots, x_n$ which implies that $\mathfrak{r}'_+ = \mathfrak{r}_+$, contrary to our assumption. Note that $[a, x_i] = 2x_i$ for $i \leq n$ (Proposition 8 in [Bo, Sect. 8.11.4]). Furthermore, since any x_j, $j \geq n+1$, is a Lie bracket of simple elements one can check via the Jacobi identity that $[a, x_j] = sx_j$ where s is an even number greater than 2. If we assume that $d_j^+ \neq 0$ then a linear combination $x_i + cx_j$, $c \neq 0$, is contained in $\mathfrak{r}'_+$ for some x_i, $i \leq n$. Taking a Lie bracket with a we see that $2x_i + scx_j \in \mathfrak{r}'_+$. Hence $x_j \in \mathfrak{r}'_+$, i.e., $d_j^+ = 0$ which is absurd. Thus $d_k^+ \neq 0$ only for $k \leq n$. We can suppose that $d_i^+ \neq 0$ for $i \leq l_o \leq n$ and $d_j^+ = 0$ for any $j \geq l_o + 1$. Note that $h_j \in \mathfrak{h}'$. If $l_o \geq 3$ pick any three distinct numbers i, j, and $k \leq l_o$. Then up to nonzero coefficients $x_i + x_j \in \mathfrak{r}'_+$ and $y_i + y_k \in \mathfrak{r}'_-$. Hence $h_i = [x_i + x_j, y_i + y_k] \in \mathfrak{h}'$, i.e., $\mathfrak{h}' = \mathfrak{h}$. In this case we can find $h \in \mathfrak{h}'$ such that $[h, x_i] = s_i x_i$ and $[h, x_j] = s_j x_j$ with $s_i \neq s_j$. As before this implies that $x_i \in \mathfrak{r}'_+$, which is a contradiction. Thus we can suppose that at most d_1^+ and d_2^+ are different from zero.

If $l_0 \leq 2$ and $n \geq 3$ we can suppose that $[x_2, x_3]$ is a nonzero nilpotent element. The direct computation shows that up to nonzero coefficients $[[x_2, x_3], [y_2, y_3]]$ coincides with $h_2 - h_3$. Since $h_3 \in \mathfrak{h}'$, so is h_2. The same argument works for h_1, i.e., $\mathfrak{h}' = \mathfrak{h}$ again which leads to a contradiction as before. If $n = 2$ then the rank m of R is 1 (since we do not want $\mathfrak{h}' = \mathfrak{h}$), i.e., R is either $\mathbb{C}^*$ or SL_2. In both cases $\dim R < \dim G - n + m - 2$ contrary to Lemma 31 which yields the desired conclusion. $\square$

Combining this result with Definition 6 and Proposition 28 we get the following.

Corollary 33. *Let G be a simple Lie group and let R be its reductive nonprincipal subgroup. Then for any principal SL_2-subgroup $\Gamma < G$ we have a finite Γ^{g_0} for some*

$g_0 \in G$ and Γ^g is different from $g^{-1}\Gamma g$ for any $g \in G$.

Lemma 34. *Let R be a principal subgroup of G. Then there exists an SL_2-subgroup $\Gamma < G$ such that Γ^{g_0} is finite for some $g_0 \in G$ and Γ^g is different from $g^{-1}\Gamma g$ for any $g \in G$.*

Proof. Recall that the subregular nilpotent orbit is the unique nilpotent orbit of codimension rank $\mathfrak{g} + 2$ in $\mathfrak{g}$ [CM, Sect. 4.1]. It can be characterized as the unique open orbit in the boundary of the principal nilpotent orbit. The corresponding $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}$ is also called subregular. The dimension of the centralizer of the semisimple subregular element H in this triple is rank $\mathfrak{g} + 2$ [CM]. We denote the subregular SL_2 subgroup of G by Γ_{sr}.

G	R	rank $G + 3$	$\dim G - \dim R$
B_3	G_2	6	7
D_4	G_2	7	14
A_6	G_2	9	34
E_6	F_4	9	26
A_{2l-1}	C_l	$2l + 2$	$l(2l - 1) - 1$
A_{2l}	B_l	$2l + 3$	$2l^2 + 3l$
D_l	B_{l-1}	$l + 3$	$2l - 1$

We will demonstrate that in the cases listed in the above table, no conjugates of Γ_{sr} can belong to R. Then by Lemma 31 the statement of the current lemma follows whenever $\dim G - \dim R >$ rank $G + 3$. From the table above we see that it covers all the principal embeddings from the proof of Proposition 28, with the exceptions of the inclusions $\mathsf{B}_3 \subset \mathsf{D}_4$ and $\mathsf{C}_2 \subset \mathsf{A}_3$.

For $G = \mathsf{A}_r$, the subregular $\mathfrak{sl}_2$ corresponds to the partition $(r, 1)$. If r is odd, this partition is not symplectic (since in symplectic partitions all odd entries occur with even multiplicity), and if r is even, this partition is not orthogonal (since in orthogonal partitions all even entries occur with even multiplicity). In other words, the subregular SL_2-subgroup Γ_{sr} in A_{2l-1} (resp., A_{2l}) does not preserve any nondegenerate symplectic (resp., orthogonal) form on $\mathbb{C}^{2l}$ ($\mathbb{C}^{2l+1}$) and thus does not belong to $R = \mathsf{C}_l$ (resp., $R = \mathsf{B}_l$). The same is true for any conjugate of Γ_{sr} in G.

If $G = \mathsf{D}_l$, the embedding of $R = \mathrm{SO}_{2l-1}$ in SO_{2l} is defined by the choice of the nonisotropic vector $v \in \mathbb{C}^{2l}$ which is fixed by R. The subregular $\mathfrak{sl}_2$ in $\mathfrak{so}_{2l}$ corresponds to the partition $(2l - 3, 3)$. Thus we see that $\Gamma_{\mathrm{sr}} \subset \mathrm{SO}_{2l}$ does not fix any one-dimensional subspace in $\mathbb{C}^{2l}$ (its invariant subspaces have dimensions $2l - 3$ and 3) and hence none of its conjugates can belong to R. Moreover, we can choose v such that $xv \neq v$ for $x \in \Gamma_{\mathrm{sr}}$, $x \neq 1$. Thus $\Gamma_{\mathrm{sr}} \cap \mathrm{SO}_{2l-1} = \{e\}$. This establishes the desired conclusion for the embeddings $\mathrm{SO}_7 \subset \mathrm{SO}_8$ (i.e., the case of $\mathsf{B}_3 \subset \mathsf{D}_4$) and $\mathrm{SO}_5 \subset \mathrm{SO}_6$ (i.e., the case of $\mathsf{B}_2 \simeq \mathsf{C}_2 \subset \mathsf{A}_3 \simeq \mathsf{D}_3$), in which the dimension count of Lemma 31 by itself is not sufficient.

The alignments of $\mathfrak{sl}_2$-triples in the exceptional cases were analyzed in [LMW]. In particular, it was observed there that any conjugacy class of $\mathfrak{sl}_2$-triples in $\mathfrak{f}_4$ lifts uniquely to a conjugacy class of $\mathfrak{sl}_2$-triples in $\mathfrak{e}_6$. Consulting the explicit

correspondence given in [LMW, Sect. 2.2], we observe that the largest nonprincipal nilpotent orbit in $\mathfrak{e}_6$ which has nonempty intersection with $\mathfrak{f}_4$ has codimension 10. This implies that no $\mathfrak{sl}_2$-triple in $\mathfrak{f}_4$ lifts to a subregular $\mathfrak{sl}_2$ in $\mathfrak{e}_6$. In other words, no conjugate of $\Gamma_{\mathrm{sr}} \subset \mathsf{E}_6$ belongs to F_4.

When $R = \mathsf{G}_2$, its embedding in SO_8 is defined by the triality automorphism $\tau : \mathrm{SO}_8 \to \mathrm{SO}_8$, with R being a fixed point group of this automorphism. Equivalently, $R = \mathrm{SO}_7 \cap \tau(\mathrm{SO}_7)$. In particular, only those $\mathfrak{sl}_2$-triples in $\mathfrak{so}_8$ which are fixed under the triality automorphism belong to $\mathfrak{g}_2$. Observe that the subregular $(5,3)$ $\mathfrak{sl}_2$-triple is not fixed by triality (cf. Remark 2.6 in [LMW]). Similarly, the subregular $\mathfrak{sl}_2$-triple in $\mathfrak{so}_7$ corresponds to the partition $(5,1,1)$. Since the $(5,1,1,1)$ $\mathfrak{sl}_2$-triple in $\mathfrak{so}_8$ is not invariant under triality, neither is subregular $\mathfrak{sl}_2$ in $\mathfrak{so}_7$. Thus no conjugates of Γ_{sr} in B_3 or D_4 are fixed by τ, and no conjugates of Γ_{sr} belong to G_2.

Finally, when $G = \mathsf{A}_6$, the subregular triple in A_6 does not belong to B_3 (see above), and thus none of its conjugates lie in $\mathsf{G}_2 \subset \mathsf{B}_3$. $\quad\square$

Now Theorem 24 follows immediately from the combined statements of Corollary 33 and Lemma 34.

Remark 8. Note that we proved slightly more than required. Namely, the SL$_2$-subgroup Γ in Theorem 24 can be chosen either principal or subregular.

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