UNIPOTENT ELEMENTS IN SMALL CHARACTERISTIC, II

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Dedicated to Bert Kostant on the occasion of his 80th birthday

Abstract. Let G be a special orthogonal group over an algebraically closed field of characteristic exponent p. In this paper we extend certain aspects of the Dynkin–Kostant theory of unipotent elements of G (when $p = 1$) to the general case (including $p = 2$).

Introduction

0.1. Let **k** be an algebraically closed field of characteristic exponent $p \geqslant 1$. This paper is a study of unipotent elements in a special orthogonal group SO_Q (where Q is a nondegenerate quadratic form on a finite-dimensional **k**-vector space V) with emphasis on the case where $p = 2$. We develop some of the proposals in [L2] which try to extend the Dynkin–Kostant theory [K] of unipotent elements in the case $p = 1$ to the case $p > 1$.

Namely we show that to any unipotent element $u \in G$ one can associate canonically a filtration of V whose stabilizer in SO_Q is a parabolic subgroup containing u in the unipotent radical and such that this parabolic is of the same type as a parabolic attached in the Dynkin–Kostant theory to a unipotent element in a special orthogonal group with $p = 1$. This allows us to partition the unipotent variety of SO_Q into pieces which are both smooth varieties and unions of (possibly several) unipotent conjugacy classes. We show that (for a split rational structure over a finite field with q elements) each piece has a number of rational points given by a polynomial in q with integer coefficients independent of q and p. (This kind of property was stated without proof in [L1].) A similar statement was established for groups of type C in [L2]; for an exceptional type this is is easily established using the available tables since in that case each piece contains at most two unipotent conjugacy classes. In some sense the behaviour of the pieces is more complicated for orthogonal groups than for other almost simple groups. For example, this is the only case where the number of unipotent conjugacy classes in a given piece (in bad characteristic) is not necessarily a power of 2.

The main result of this paper is Theorem 1.7 which gives a rather explicit decomposition of the unipotent variety in SO_Q into pieces Ξ_ϕ; these are the same as the unipotent pieces of SO_Q in [L2, 1.1] (although their definition is slightly different). Our results verify the validity of properties $\mathfrak{P}_1$–$\mathfrak{P}_5$ in [L2, 1.1] for SO_Q. In particular property $\mathfrak{P}_5$ is proved in our case in 1.8(b). In Section 4 we discuss an (injective) map from unipotent classes in SO_Q to nilpotent orbits in the Lie algebra.

The proof of Theorem 1.7 is given in Section 2 (when $p = 2$) and in Section 3 (when $p \neq 2$).

Notation. We set $\mathbf{Z}' = 2\mathbf{Z}+1, \mathbf{Z}'' = 2\mathbf{Z}$. For $a \in \mathbf{Z}$, we set $\mathbf{Z}'_{\geqslant a} = \{n \in \mathbf{Z}' \mid n \geqslant a\}$, $\mathbf{Z}''_{\geqslant a} = \{n \in \mathbf{Z}'' \mid n \geqslant a\}$. The cardinal of a finite set X is denoted by $|X|$; if X is an infinite set we write $|X| = \infty$.

1. Unipotent elements in orthogonal groups

1.1. In this section we give some preparatory material which allows us to state Theorem 1.7, the main result of this paper.

Let $\mathbf{k}$ be a finite or algebraically closed field of characteristic exponent $p \geqslant 1$. Let $\mathcal{C}$ be the category whose objects are $\mathbf{k}$-vector spaces of finite dimension; the morphisms are linear maps. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ consisting of all $V \in \mathcal{C}$ such that $\dim V \in \mathbf{Z}'$. Let $\mathcal{C}''$ be the subcategory of $\mathcal{C}$ consisting of all $V \in \mathcal{C}$ such that $\dim V \in \mathbf{Z}''$. Let $\bar{C}$ be the category whose objects are $\mathbf{Z}$-graded $\mathbf{k}$-vector spaces $\bar{V} = \bigoplus_{a \in \mathbf{Z}} V^a$ such that $\dim \bar{V} < \infty$; the morphisms are linear maps respecting the grading.

Let $V \in \mathcal{C}$. For $N \in \mathrm{End}\, V$ nilpotent and $i \geqslant 1$ let

$$E_i^N = \ker N^i/(N(\ker N^{i+1}) + \ker N^{i-1}).$$

Then $c_i^N := \dim E_i^N$ is the number of Jordan blocks of size i of N. Let $e = e_N$ be the smallest integer $\geqslant 0$ such that $N^e = 0$.

For any symmetric or symplectic bilinear form $\langle \, , \, \rangle : V \times V \to \mathbf{k}$ let $\mathrm{Rad}_{\langle \, , \, \rangle} = \{x \in V \mid \langle x, V \rangle = 0\}$ be the radical of $\langle \, , \, \rangle$.

In the remainder of this paper we fix $V \in \mathcal{C}$ and a quadratic form $Q : V \to \mathbf{k}$ with associated symmetric bilinear form $\langle \, , \, \rangle : V \times V \to \mathbf{k}$; we have $\langle x, y \rangle = Q(x + y) - Q(x) - Q(y)$ for $x, y \in V$. Let $D = \dim V$. Let $R = \mathrm{Rad}_{\langle \, , \, \rangle}$. We assume that Q is *nondegenerate*, that is, $Q|_R : R \to \mathbf{k}$ is injective. (In this case we must have $R = 0$ unless $V \in \mathcal{C}', p = 2$ when $\dim R = 1$.)

In the case where $D \in \mathbf{Z}''$ let $\mathcal{J}_Q$ be the set of $D/2$-dimensional subspaces of V on which Q is 0. In the case where $|\mathbf{k}| < \infty, D \in \mathbf{Z}''_{\geqslant 2}$ we set $\eta_Q = 1$ if $\mathcal{J}_Q \neq \varnothing$ and $\eta_Q = -1$ if $\mathcal{J}_Q = \varnothing$.

For any subspace W of V we set $W^\perp = \{x \in V \mid \langle x, W \rangle = 0\}$.

Let $\mathrm{O}_Q = \{T \in \mathrm{GL}(V) \mid Q(Tx) = Q(x)\ \forall x \in V\}$ be the orthogonal group of Q. In the case where $|\mathbf{k}| = \infty$ we denote by SO_Q the identity component of O_Q; in the case where $|\mathbf{k}| < \infty$ we set $\mathrm{SO}_Q = \mathrm{O}_Q \cap \mathrm{SO}_{\tilde{Q}}$ where $\tilde{Q}$ is the quadratic form obtained from Q by extension of scalars to an algebraic closure of $\mathbf{k}$. Let

$$\widetilde{\mathcal{M}}_Q = \{N \in \mathrm{End}(V) \mid 1 + N \text{ unipotent in } \mathrm{O}_Q\}$$
$$= \{N \in \mathrm{End}(V) \mid N \text{ nilpotent}, Q(Nx) = -\langle x, Nx \rangle \text{ for all } x \in V\}.$$

Note that if $N \in \widetilde{\mathcal{M}}_Q$ and $x, y \in V$ then

$$\langle x, Ny \rangle + \langle Nx, y \rangle + \langle Nx, Ny \rangle = 0.$$

In particular, for $N \in \widetilde{\mathcal{M}}_Q$, we have $\langle x, Ny \rangle = \langle N^\dagger x, y \rangle$ for $x, y \in V$, where $N^\dagger := (1 + N)^{-1} - 1 = -N + N^2 - N^3 + \cdots \in \widetilde{\mathcal{M}}_Q$.

Let $N \in \widetilde{\mathcal{M}}_Q$. We have

(a) $NV \cap R = 0$.

If $x \in V$ and $Nx \in R$ then $Q(Nx) = -\langle x, Nx \rangle = 0$; since $Q : R \to \mathbf{k}$ is injective we see that $Nx = 0$. Thus (a) holds.

From the definitions we have, for $i \geqslant 1$,

(b) $(\ker N^i)^\perp = N^i V + R$, $(N^i V)^\perp = \ker N^i$.

In particular, $NR \subset R$. Since $\dim R \leqslant 1$, it follows that $NR = 0$.

1.2. In this subsection we assume that $p > 1$ and that $|\mathbf{k}| = q < \infty$. For any $m \in \mathbf{Z}'_{\geqslant 1}$ we set

$$P_m = q^{1/4(m^2 - 2m + 1)}(q^2 - 1)(q^4 - 1) \cdots (q^{m-1} - 1).$$

For any $m \in \mathbf{Z}''_{\geqslant 2}$ and $\delta \in \{1, -1\}$ we set

$$P_m^\delta = q^{1/4(m^2 - 2m)}(q^2 - 1)(q^4 - 1) \cdots (q^{m-2} - 1)(q^{m/2} - \delta).$$

Let $\mathcal{V}$ be a $\mathbf{k}$-vector space with a nondegenerate quadratic form $u : \mathcal{V} \to \mathbf{k}$. Let $(,) : \mathcal{V} \times \mathcal{V} \to \mathbf{k}$ be the associate symmetric bilinear form. Let $\mathcal{R} = \mathrm{Rad}_{(,)}$. Let $s = \dim \mathcal{V}$. Let $k \in [0, s]$. Let $S_k \mathcal{V}$ be the set of k-dimensional subspaces $\mathcal{W}$ of $\mathcal{V}$ such that $u|_\mathcal{W}$ is nondegenerate. If $k \in \mathbf{Z}''_{\geqslant 2}$, we have a partition $S_k \mathcal{V} = \bigsqcup_{\delta \in \{1, -1\}} S_k^\delta \mathcal{V}$ where $S_k^\delta \mathcal{V} = \{\mathcal{W} \in S_k \mathcal{V} \mid \eta_{u|_\mathcal{W}} = \delta\}$. If $s \in \mathbf{Z}''$, $k \in \mathbf{Z}'$ we set $N_{s,k}^{\epsilon,*} = |S_k \mathcal{V}|$. If $s \in \mathbf{Z}'$, $k \in \mathbf{Z}'$, we set $N_{s,k}^{*,*} = |S_k \mathcal{V}|$. If $s \in \mathbf{Z}''$, $k \in \mathbf{Z}''_{\geqslant 2}$ and $\delta \in \{1, -1\}$, we set $N_{s,k}^{\epsilon,\delta} = |S_k^\delta \mathcal{V}|$. If $s \in \mathbf{Z}'$, $k \in \mathbf{Z}''_{\geqslant 2}$ and $\delta \in \{1, -1\}$, we set $N_{s,k}^{*,\delta} = |S_k^\delta \mathcal{V}|$.

(i) If $s \in \mathbf{Z}''_{\geqslant 2}$, $k \in \mathbf{Z}'$, $\epsilon = \eta_u$, then $N_{s,k}^{\epsilon,*} = P_s^\epsilon P_k^{-1} P_{t-k}^{-1}$.

If $p \neq 2$ this is obvious. Assume now that $p = 2$. Let $\mathcal{R}'$ be the radical of $(,)|_\mathcal{W}$ for $\mathcal{W} \in S_k \mathcal{V}$. Then $\mathcal{R}'$ is one of the $q^{(s-2)/2}(q^{s/2} - \epsilon)$ lines in $\mathcal{V}$ on which u is not identically 0. Also $\mathcal{R}'^\perp$ is a hyperplane in $\mathcal{V}$ and $(,)$ induces a nondegenerate symplectic form on $\mathcal{R}'^\perp/\mathcal{R}'$. For each such $\mathcal{R}'$, the number of $\mathcal{W} \in S_k \mathcal{V}$ such that $\mathcal{R}' \subset \mathcal{W} \subset \mathcal{R}'^\perp$ is the number of $(k-1)$-dimensional subspaces of $\mathcal{R}'^\perp/\mathcal{R}'$ on which $(,)$ is nondegenerate, hence it is $P_{s-1} P_k^{-1} P_{s-k}^{-1}$. Hence $N_{s,k}^{\epsilon,*} = q^{(s-2)/2}(q^{s/2} - \epsilon) P_{s-1} P_k^{-1} P_{s-k}^{-1}$, as required.

(ii) If $s \in \mathbf{Z}''_{\geqslant 2}$, $k \in \mathbf{Z}''_{\geqslant 2}$, $k < s$, $\epsilon = \eta_u$, $\delta \in \{1, -1\}$, then

$$N_{s,k}^{\epsilon,\delta} = 2^{-1} P_s^\epsilon (P_k^\delta)^{-1} (P_{s-k}^{\epsilon\delta})^{-1}.$$

This is clear: the orthogonal group of u acts transitively on $S_k^\delta \mathcal{V}$.

(iii) *If $s \in \mathbf{Z}'$, $k \in \mathbf{Z}'$, $k < s$, then $N_{s,k}^{*,*} = 2^{-1} \sum_{\delta \in \{1,-1\}} P_s P_k^{-1} (P_{s-k}^{\delta})^{-1}$.*

If $p \neq 2$ this is obvious. Assume now that $p = 2$. We set $s = 2t+1, k = 2a+1$. Let $\mathcal{R}' = \mathrm{Rad}_{(\ ,\ )|_{\mathcal{W}}}$ for $\mathcal{W} \in S_k \mathcal{V}$. Then $\mathcal{R}'$ is one of the q^{2t} lines in $\mathcal{V}$ on which u is not identically 0. The number of $\mathcal{W} \in S_k \mathcal{V}$, such that the corresponding $\mathcal{R}'$ is equal to $\mathcal{R}$, is the number of $2a$-dimensional subspaces of $\mathcal{V}/\mathcal{R}$ on which $(\ ,\ )$ is nondegenerate, that is, $P_s P_k^{-1} P_{s-k+1}^{-1}$. Let N' be the number of $\mathcal{W} \in S_k \mathcal{V}$ such that the corresponding $\mathcal{R}'$ is not equal to $\mathcal{R}$. Then $N' = (q^{s-1} - 1) N_0'$ where N_0' is the number of $\mathcal{W}$ with prescribed $\mathcal{R}' \neq \mathcal{R}$. To such $\mathcal{W}$ we associate the subspace $\mathcal{W}' = \mathcal{W} + \mathcal{R}$ (of dimension $k+1$) of $\mathcal{R}'^{\perp}$ (of dimension $s-1$). Note that the number of possible such $\mathcal{W}'$ is the number of $(k-1)$-dimensional subspaces of $\mathcal{R}'^{\perp}/(\mathcal{R} + \mathcal{R}')$ on which $(\ ,\ )$ is nondegenerate, that is, $P_{s-2} P_k^{-1} P_{s-k-1}^{-1}$. For any $\mathcal{W}'$ as above the number of $\mathcal{W}$, such that $\mathcal{R}' \subset \mathcal{W} \subset \mathcal{W}'$ and $\mathcal{W} \oplus \mathcal{R} = \mathcal{W}'$, is the number of subspaces $\mathcal{X} \subset \mathcal{W}'/\mathcal{R}'$ such that $\mathcal{X} \oplus \mathcal{R} = \mathcal{W}'/\mathcal{R}'$, that is, q^{k-1}. Thus,

$$N' = (q^{s-1} - 1) q^{k-1} P_{s-2} P_k^{-1} P_{s-k-1}^{-1}$$
$$= q^{2a} q^{-t^2 + (t-1)^2} q^{(t-a)^2 - (t-a-1)^2} (q^{2t-2a} - 1) P_s P_k^{-1} P_{s-k+1}^{-1}.$$

We see that

$$N_{s,k}^{*,*} = P_s P_k^{-1} P_{s-k+1}^{-1} + q^{2a} q^{-t^2 + (t-1)^2} q^{(t-a)^2 - (t-a-1)^2} (q^{2t-2a} - 1) P_s P_k^{-1} P_{s-k+1}^{-1}$$
$$= q^{2t-2a} P_{2t+1} P_{2a+1}^{-1} P_{2t-2a+1}^{-1}$$

and (iii) follows.

(iv) *If $s \in \mathbf{Z}'$, $k \in \mathbf{Z}''_{\geqslant 2}$, $\delta \in \{1,-1\}$, then $N_{s,k}^{*,\delta} = 2^{-1} P_s (P_k^{\delta})^{-1} P_{s-k}^{-1}$.*

This is clear: the orthogonal group of u acts transitively on $S_k^{\delta} \mathcal{V}$.

We now consider a descending sequence $r_0 \geqslant r_2 \geqslant r_4 \geqslant \cdots$ of integers such that $r_0 = s$ and $r_{2n} = 0$ for large n. We denote by $\nu(r_0, r_2, r_4, \dots)$ (if $s \in \mathbf{Z}'$) or by $\nu^{\epsilon}(r_0, r_2, r_4, \dots)$ (if $s \in \mathbf{Z}''_{\geqslant 2}$, $\eta_u = \epsilon$) the number of sequences $U_0 \supset U_2 \supset U_4 \supset \cdots$ of subspaces of $\mathcal{V}$ such that for any $n \geqslant 0$, $\dim U_{2n} = r_{2n}$ and the quadratic form $u|_{U_{2n}}$ is nondegenerate. We show:

(a) *Let $X = \nu(r_0, r_2, r_4, \dots)$ if $r_0 \in \mathbf{Z}'$ and $X = \nu^{\epsilon}(r_0, r_2, r_4, \dots)$ if $r_0 \in \mathbf{Z}''_{\geqslant 2}$ and $\epsilon \in \{1, -1\}$. Then X is a polynomial in q with coefficients in $\mathbf{Z}$ independent of q or p.*

Let M be the number of nonzero terms in the sequence $(r_0, r_2, r_4, \dots)$. We have $M \geqslant 1$. We argue by induction on M. If $M = 1$ we have $X = 1$. Assume now that $M \geqslant 2$. If $r_{2n} = r_{2n+2} > 0$ for some n we have $X = \nu(r_0, \dots, r_{2n}, r_{2n+4}, \dots)$ if $r_0 \in \mathbf{Z}'$, $X = \nu^{\epsilon}(r_0, \dots, r_{2n}, r_{2n+4}, \dots)$ if $r_0 \in \mathbf{Z}''$; the result follows. Hence we may assume that the nonzero terms of $r_0, r_2, \dots$ are distinct. If $r_{2n} \in \mathbf{Z}'$ for some $n > 0$, we have $X = \nu(r_0, \dots, r_{2n}, 0, 0,) \nu(r_{2n}, r_{2n+2}, \dots)$ if $r_0 \in \mathbf{Z}'$, $X = \nu^{\epsilon}(r_0, \dots, r_{2n}, 0, 0,) \nu(r_{2n}, r_{2n+2}, \dots)$ if $r_0 \in \mathbf{Z}''$; in both cases the induction hypothesis applies to the second factor in the right-hand side. Thus we may assume that any odd number in the sequence $r_0, r_2, \dots$ must appear either as r_0 or it must be followed by 0. Thus we must consider four cases.

Case 1. $r_{2n} \in \mathbf{Z}''$ *for all* n. Let r_{2m} be the last nonzero term of $r_0, r_2, \ldots$. If $m = 0$ we have $X = 1$. If $m > 0$ we have, using (ii),

$$X = \sum_{\delta_1, \delta_2, \ldots, \delta_m \in \{1, -1\}} N_{r_0, r_2}^{\epsilon, \delta_1} N_{r_2, r_4}^{\delta_1, \delta_2} \cdots N_{r_{2m-2}, r_{2m}}^{\delta_{m-1}, \delta_m}$$

$$= 2^{-m} \sum_{\delta_1, \delta_2, \ldots, \delta_m \in \{1, -1\}} P_{r_0}^{\epsilon} (P_{r_0 - r_2}^{\epsilon \delta_1})^{-1} (P_{r_2 - r_4}^{\delta_1 \delta_2})^{-1} \cdots (P_{r_{2m-2} - r_{2m}}^{\delta_{m-1} \delta_m})^{-1} (P_{r_{2m}}^{\delta_m})^{-1}.$$

This is clearly a polynomial in q with coefficients in $\mathbf{Z}[2^{-1}]$. It is also the product of a polynomial in q with coefficients in $\mathbf{Z}$ with

$$2^{-m} \sum_{\delta_1, \delta_2, \ldots, \delta_m \in \{1, -1\}} (q^{(r_0 - r_2)/2} - \epsilon \delta_1)^{-1} (q^{(r_2 - r_4)/2} - \delta_1 \delta_2)^{-1} \cdots$$

$$\times (q^{(r_{2m-2} - r_{2m})/2} - \delta_{m-1} \delta_m)^{-1} (q^{r_{2m}/2} - \delta_m)^{-1}$$

$$= (q^{r_0/2} + \epsilon)(q^{r_0 - r_2} - 1)^{-1} (q^{r_2 - r_4} - 1)^{-1} \cdots (q^{r_{2m-2} - r_{2m}} - 1)^{-1} (q^{r_{2m}} - 1)^{-1},$$

hence is a power series with integer coefficients in q. Hence $X \in \mathbf{Z}[q]$.

Case 2. $r_0 \in \mathbf{Z}''$, $r_{2m} \in \mathbf{Z}'$, $r_{2n} \in \mathbf{Z}''$ *for* $0 < n < m$ *and* $r_{2n} = 0$ *for* $n > m$. If $m = 1$ we have $X = N_{r_0, r_2}^{\epsilon, *}$. If $m > 1$ we have

$$X = \sum_{\delta_1, \delta_2, \ldots, \delta_{m-1} \in \{1, -1\}} N_{r_0, r_2}^{\epsilon, \delta_1} N_{r_2, r_4}^{\delta_1, \delta_2} \cdots N_{r_{2m-4}, r_{2m-2}}^{\delta_{m-2}, \delta_{m-1}} N_{r_{2m-2}, r_{2m}}^{\delta_{m-1}, *}.$$

As in Case 1 we see, using (ii) and (i) that $X \in \mathbf{Z}[q]$.

Case 3. $r_0 \in \mathbf{Z}'$, $r_{2n} \in \mathbf{Z}''$ *for* $n > 0$. Let r_{2m} be the last nonzero term of $r_0, r_2, \ldots$. We have

$$X = \sum_{\delta_1, \delta_2, \ldots, \delta_m \in \{1, -1\}} N_{r_0, r_2}^{*, \delta_1} N_{r_2, r_4}^{\delta_1, \delta_2} \cdots N_{r_{2m-2}, r_{2m}}^{\delta_{m-1}, \delta_m}.$$

As in Case 1 we see, using (ii) and (iv), that $X \in \mathbf{Z}[q]$.

Case 4. $r_0 \in \mathbf{Z}'$ *and for some* $m > 0$, $r_{2m} \in \mathbf{Z}'$, $r_{2n} \in \mathbf{Z}''$ *for* $0 < n < m$ *and* $r_{2n} = 0$ *for* $n > m$. If $m = 1$ we have $X = N_{r_0, r_2}^{*, *}$, see (iii). If $m > 1$ we have

$$X = \sum_{\delta_1, \delta_2, \ldots, \delta_{m-1} \in \{1, -1\}} N_{r_0, r_2}^{*, \delta_1} N_{r_2, r_4}^{\delta_1, \delta_2} \cdots N_{r_{2m-4}, r_{2m-2}}^{\delta_{m-2}, \delta_{m-1}} N_{r_{2m-2}, r_{2m}}^{\delta_{m-1}, *}.$$

As in Case 1 we see, using (i), (ii) and (iv), that $X \in \mathbf{Z}[q]$. This completes the proof of (a). $\square$

For any $m \in \mathbf{Z}_{\geq 0}''$, we set

$$R_m = q^{m^2/4} (q^2 - 1)(q^4 - 1) \cdots (q^m - 1).$$

Let $\mathcal{V}'$ be a **k**-vector space with a nondegenerate symplectic form $(,) : \mathcal{V}' \times \mathcal{V}' \to \mathbf{k}$.

We consider a descending sequence $r_1 \geqslant r_3 \geqslant r_5 \geqslant \cdots$ of even integers such that $r_1 = \dim \mathcal{V}'$ and $r_{2n+1} = 0$ for large n. We denote by $\nu'(r_1, r_3, r_5, \ldots)$ the number of sequences $U_1 \supset U_3 \supset U_5 \supset \cdots$ of subspaces of $\mathcal{V}'$ such that for any $n \geqslant 0$, $\dim U_{2n+1} = r_{2n+1}$ and the symplectic form $(,)|_{U_{2n+1}}$ is nondegenerate. Clearly,

(b) $\nu'(r_1, r_3, r_5, \ldots) = R_{r_1} R_{r_1 - r_3}^{-1} R_{r_3 - r_5}^{-1} \cdots$ *is a polynomial in* q *with coefficients in* $\mathbf{Z}$ *independent of* q *or* p.

1.3. Let $\bar{V} \in \bar{\mathcal{C}}$. Define $(f_a)_{a \in \mathbf{Z}}$ by $f_a = \dim \bar{V}^a$. A quadratic form $\bar{Q} : \bar{V} \to \mathbf{k}$ with associated symmetric bilinear form $\langle \, , \rangle$ is said to be *compatible with the grading* if $\bar{Q}|_{\bar{V}^a} = 0$ for $a \neq 0$ and $\langle \bar{V}^a, \bar{V}^{a'} \rangle = 0$ for $a + a' \neq 0$. In this subsection we fix such a $\bar{Q}$ (compatible with the grading) and we assume that it is nondegenerate. Let $\bar{R} = \mathrm{Rad}_{\langle \, , \rangle}$. Then $\bar{R} \subset \bar{V}^0$, $Q|_{\bar{V}^0} : \bar{V}^0 \to \mathbf{k}$ is nondegenerate and $\langle \, , \rangle$ restricts to a perfect pairing $\bar{V}^{-a} \times \bar{V}^a \to \mathbf{k}$ for any $a \geqslant 1$. Hence $f_a = f_{-a}$ for all a. Let

$$E^2 \bar{V} = \{ T \in \mathrm{Hom}_{\mathcal{C}}(\bar{V}, \bar{V}) \mid T(\bar{V}^a) \subset \bar{V}^{a+2} \text{ for any } a \in \mathbf{Z}; \langle Tx, y \rangle$$
$$+ \langle x, Ty \rangle = 0 \text{ for all } x, y \in \bar{V}; \langle x, Tx \rangle = 0 \text{ for any } x \in \bar{V}^{-1} \}.$$

If $T \in E^2 \bar{V}$ we have

   (a) $\langle x, T^a x \rangle = 0$ *for any* $a \in \mathbf{Z}'_{\geqslant 1}$ *and any* $x \in \bar{V}^{-a}$.

   Indeed, we have $a = 2a' + 1$ with $a' \geqslant 0$ and $x' = T^{a'} x \in \bar{V}^{-1}$ satisfies $\langle x', Tx' \rangle = 0$. Thus

$$0 = \langle T^{a'} x, T^{a'+1} x \rangle = (-1)^{a'} \langle x, T^a \rangle$$

and (a) follows.

   Let $E_*^2 \bar{V}$ be the set of all $T \in E^2 \bar{V}$ such that:

   (i) for any $a \in \mathbf{Z}''_{\geqslant 0}$, the quadratic form $\bar{Q}_a : \bar{V}^{-a} \to \mathbf{k}$, $x \mapsto \bar{Q}(T^{a/2}x)$ is nondegenerate;
   (ii) for any $a \in \mathbf{Z}'_{\geqslant 1}$, the symplectic form $\bar{V}^{-a} \times \bar{V}^{-a} \to \mathbf{k}$, $x, y \mapsto \langle x, T^a y \rangle$ is nondegenerate.
   Note that (i) is automatic for $a = 0$.

   For $\tilde{N} \in E^2 \bar{V}$ and $a \geqslant 0$ we set $K_a^T = \ker(T^a : \bar{V}^{-a} \to \bar{V}^a)$. If $a \geqslant 1$ then $K_a^T = \mathrm{Rad}_{\langle \, , \rangle_a}$ where $\langle \, , \rangle_a : \bar{V}^{-a} \times \bar{V}^{-a} \to \mathbf{k}$ is $x, y \mapsto \langle x, T^a y \rangle$. Hence condition (ii) is equivalent to the condition that $K_a^T = 0$ for $a \in \mathbf{Z}'_{\geqslant 1}$. If $a \in \mathbf{Z}''_{\geqslant 0}$ then the symmetric bilinear form associated to $\bar{Q}_a$ is $(-1)^{a/2} \langle \, , \rangle_a$. Hence condition (i) is equivalent to the condition that $\bar{Q}_a : K_a^T \to \mathbf{k}$ is injective for $a \in \mathbf{Z}''_{\geqslant 2}$. (This implies that for such a we have $K_a^T = 0$ unless $p = 2$ and $f_{-a} \in \mathbf{Z}'$, when $\dim K_a^T = 1$.)

   This discussion shows that if $p \neq 2$ an element $T \in E^2 \bar{V}$ belongs to $E_*^2 \bar{V}$ if and only if $T^a : \bar{V}^{-a} \to \bar{V}^a$ is an isomorphism for any $a \geqslant 0$.

   Returning to the general case we reformulate conditions (i) and (ii) for an element $T \in E^2 \bar{V}$ to be in $E_*^2 \bar{V}$ as follows:

   (i') for any $a \in \mathbf{Z}''_{\geqslant 0}$, the map $T^{a/2} : \bar{V}^{-a} \to \bar{V}^0$ is injective and its image I_a^T is such that $\bar{Q}|_{I_a^T}$ is a nondegenerate quadratic form;
   (ii') the symplectic form $\omega_T(x, y) = \langle x, Ty \rangle$ on V^{-1} is nondegenerate; for any $a \in \mathbf{Z}'_{\geqslant 1}$, the map $T^{(a-1)/2} : \bar{V}^{-a} \to \bar{V}^{-1}$ is injective and its image I_a^T is such that $\omega_T|_{I_a^T}$ is a nondegenerate symplectic form.

   Clearly, if (i') holds then (i) holds. If (i) holds and $x \in \ker(T^{a/2} : \bar{V}^{-a} \to \bar{V}^0)$ (with $a \in \mathbf{Z}''_{\geqslant 2}$) then $T^a x = 0$ hence $x \in K_a^T$. We also have $\bar{Q}_a(x) = 0$. Since $\bar{Q}_a : K_a^T \to \mathbf{k}$ is injective we see that $x = 0$. We see that (i') holds.

For $a \in \mathbf{Z}'_{\geqslant 1}$ and $x, y \in \bar{V}^{-a}$ we have

$$\langle x, y \rangle_a = (-1)^{(a-1)/2} \langle T^{(a-1)/2} x, T^{(a-1)/2} y \rangle_1.$$

Hence if (ii′) holds then (ii) holds. If (ii) holds and $x \in \ker(T^{(a-1)/2} : \bar{V}^{-a} \to \bar{V}^{-1})$ (with $a \in \mathbf{Z}'_{\geqslant 1}$) then $T^a x = 0$ hence $x \in K_a^T$ and $x = 0$. We see that (ii′) holds.

If $T \in E_*^2 \bar{V}$, then, clearly,

$$\bar{V}^0 = I_0^T \supset I_2^T \supset I_4^T \supset \cdots \quad \text{and} \quad \bar{V}^{-1} = I_1^T \supset I_3^T \supset I_5^T \supset \cdots.$$

We see that if $E_*^2 \bar{V} \neq \varnothing$, then:

(b) $f_0 \geqslant f_{-2} \geqslant f_{-4} \geqslant \cdots$ and $f_{-1} \geqslant f_{-3} \geqslant f_{-5} \geqslant \cdots$;
(c) $f_a \in \mathbf{Z}''$ if $a \in \mathbf{Z}'$.

We say that $(f_a)_{a \in \mathbf{Z}}$ is *admissible* if it satisfies (b) and (c). In the remainder of this subsection we assume that $(f_a)_{a \in \mathbf{Z}}$ is admissible. Let R^0 be the set of all sequences $U_0 \supset U_2 \supset U_4 \supset \cdots$ of subspaces of $\bar{V}^0$ such that $\dim U_a = f_{-a}$ and $\bar{Q}|_{U_a}$ is a nondegenerate quadratic form for $a = 0, 2, 4, \ldots$. Let R^{-1} be the set of all pairs $(\omega, (U_1, U_3, U_5, \ldots))$ where ω is a nondegenerate symplectic form on $\bar{V}^{-1}$ and $U_1 \supset U_3 \supset U_5 \supset \cdots$ are subspaces of $\bar{V}^{-1}$ such that $\dim U_a = f_{-a}$ and $\omega|_{U_a}$ is a nondegenerate symplectic form for $a = 1, 3, 5, \ldots$. Clearly $R^0 \neq \varnothing, R^{-1} \neq \varnothing$. Define $\psi : E_*^2 \bar{V} \to R^0 \times R^{-1}$ by

$$T \mapsto ((I_0^T, I_2^T, I_4^T, \ldots), (\omega_T, (I_1^T, I_3^T, I_5^T, \ldots))).$$

Clearly,

(d) *the fibre of ψ at* $((U_0, U_2, U_4, \ldots), (\omega, (U_1, U_3, U_5, \ldots))) \in R^0 \times R^{-1}$ *can be identified with* $\prod_{a \geqslant 2} \mathrm{Iso}(\bar{V}^{-a}, U_a)$;

here $\mathrm{Iso}(\bar{V}^{-a}, U_a)$ is the set of vector space isomorphisms $\bar{V}^{-a} \to U_a$.

We now assume that $\mathbf{k}, q$ are as in 1.2. For any $m \geqslant 0$ we set

$$A_m = q^{1/2(n^2-n)}(q-1)(q^2-1) \cdots (q^m-1).$$

If $f_0 \in \mathbf{Z}''_{\geqslant 2}$ let $\epsilon = \eta_{\bar{Q}|_{\bar{V}^0}}$. From (d) we see that:

(e) $|E_*^2 \bar{V}| = \prod_{a \geqslant 1} A_{f_a} \xi \nu'(f_1, f_3, f_5, \ldots)(R_{f_1})^{-1}$, where $\xi = \nu(f_0, f_2, f_4, \ldots)$ if $f_0 \in \mathbf{Z}'$; $\xi = \nu^\epsilon(f_0, f_2, f_4, \ldots)$ if $f_0 \in \mathbf{Z}''_{\geqslant 2}$; $\xi = 1$ if $f_0 = 0$.

1.4. Let $X^* = (X^{\geqslant a})_{a \in \mathbf{Z}}$ be a sequence of subspaces of V such that $X^{\geqslant a+1} \subset X^{\geqslant a}$ for any a, $X^{\geqslant a} = 0$ for some a, $X^{\geqslant a} = V$ for some a, that is, a *filtration* (see [L2, 2.2]) of V. We say that X^* is a Q-filtration of V if for any $a \geqslant 1$ we have:

(a) $Q|_{X^{\geqslant a}} = 0$ and $X^{\geqslant 1-a} = (X^{\geqslant a})^\perp$.

Then for any $a \leqslant 0$ we have:

(b) $(X^{\geqslant a})^{\perp} = X^{\geqslant 1-a} \oplus R$ and $X^{\geqslant 1-a} = (X^{\geqslant a})^{\perp} \cap Q^{-1}(0)$.

The first equality follows by applying $^{\perp}$ to both sides of $X^{\geqslant a} = (X^{\geqslant 1-a})^{\perp}$, see (a). If $x \in V^{\geqslant 1-a} \cap R$ then by (a) we have $Q(x) = 0$. Since $Q : R \to \mathbf{k}$ is injective, we have $x = 0$. Thus, $V^{\geqslant 1-a} \cap R = 0$. The second equality in (b) follows from the first equality and (a). This proves (b).

The proof of the following result is standard:

(c) *We can find a direct sum decomposition $V = \bigoplus_{a \in \mathbf{Z}} X^a$ such that $X^{\geqslant a} = \bigoplus_{a'; a' \geqslant a} X^a$ for all a and $\langle X^a, X^{a'} \rangle = 0$ for $a + a' \neq 0$, $Q|_{X^a} = 0$ for $a \neq 0$.*

For any Q-filtration X^* of V we set

$$E^{\geqslant 2} X^* = \{N \in \widetilde{\mathcal{M}}_Q \mid N X^{\geqslant a} \subset X^{\geqslant a+2} \text{ for all } a \in \mathbf{Z}\}.$$

We show:

(d) *If $x \in X^{\geqslant -1}$ then $\langle x, Nx \rangle = 0$.*

It is enough to show that $Q(Nx) = 0$. Since $Nx \in X^{\geqslant 1}$, this follows from (a).

1.5. Let X^* be a Q-filtration of V. We set $\mathrm{gr}^a X^* = X^{\geqslant a}/X^{\geqslant a+1}$. Then $\mathrm{gr}\, X^* = \bigoplus_{a \in \mathbf{Z}} \mathrm{gr}^a X^* \in \bar{\mathcal{C}}$. Let $f_a = \dim \mathrm{gr}^a X^*$. There is a well defined quadratic form $\bar{Q} : \mathrm{gr}\, X^* \to \mathbf{k}$ such that $\bar{Q}(x) = Q(\dot{x}_0) + \sum_{a \geqslant 1} \langle \dot{x}_{-a}, \dot{x}_a \rangle$ where $x = \sum_a x_a$ (with $x_a \in \mathrm{gr}^a X^*$) and $\dot{x}_b$ is a representative of x_b in $X^{\geqslant b}$. The symmetric bilinear form associated to $\bar{Q}$ is $\sum_a x_a, \sum_b x'_b \mapsto \sum_{a+b=0} \langle \dot{x}_a, \dot{x}'_b \rangle$ where $x_a \in \mathrm{gr}^a X^*$, $x'_b \in \mathrm{gr}^b X^*$ and $\dot{x}_a, \dot{x}'_b$ are representatives of x_a, x'_b in $X^{\geqslant a}$, $X^{\geqslant b}$. This form is denoted again by $\langle\ ,\ \rangle$; its radical is the image of R under $X^{\geqslant 0} \to \mathrm{gr}^0 X^*$. It follows that $\bar{Q}$ is nondegenerate. It is clearly compatible with the grading. Hence $f_a = f_{-a}$ for any a.

Now let $N \in E^{\geqslant 2} X^*$. For any a, N restricts to a linear map $X^{\geqslant a} \to X^{\geqslant a+2}$ and $X^{\geqslant a+1} \to X^{\geqslant a+3}$ hence it induces a linear map $\mathrm{gr}^a X^* \to \mathrm{gr}^{a+2} X^*$. Taking the direct sum over a of these linear maps we obtain a linear map $\bar{N} : \mathrm{gr}\, X^* \to \mathrm{gr}\, X^*$. We have $\bar{N} \in E^2 \mathrm{gr}\, X^*$. (If $x \in \mathrm{gr}^a X^*, y \in \mathrm{gr}^b X^*$ and $\dot{x}, \dot{y}$ are representatives for x, y in $X^{\geqslant a}, X^{\geqslant b}$, the sum $\langle \bar{N}x, y \rangle + \langle x, \bar{N}y \rangle$ is 0 unless $a + b + 2 = 0$ in which case it is $\langle N\dot{x}, \dot{y} \rangle + \langle \dot{x} + N\dot{y} \rangle = -\langle N\dot{x}, N\dot{y} \rangle$ which is again zero since $N\dot{x} \in X^{\geqslant a+2}$, $N\dot{y} \in X^{\geqslant b+2} = X^{\geqslant -a}$ and $\langle X^{\geqslant a+2}, X^{\geqslant -a} \rangle = 0$. If x, dx are as above and $a = -1$, we have $\langle x, \bar{N}x \rangle = \langle \dot{x}, N\dot{x} \rangle = -Q(N\dot{x})$ and this is 0 since $N\dot{x} \in X^{\geqslant 1}$ and $Q|_{X^{\geqslant 1}} = 0$.) Thus we have a well-defined map

$$\Phi : E^{\geqslant 2} X^* \to E^2 \mathrm{gr}\, X^*, \quad N \mapsto \bar{N}.$$

Let

$$d = \sum_{a < a'; -a-a' \geqslant 3} f_a f_{a'} + \sum_{a; -2a \geqslant 4} f_a (f_a - 1)/2.$$

We show:

(a) *Φ is an iterated affine space bundle with fibres of dimension d.*

Let $V = \bigoplus_a X^a$ be as in 1.4(c). For any integer $k \geqslant 2$ let Z_k be the set of all collections $(N_b^a)_{a,b \in \mathbf{Z}; 2 \leqslant b-a \leqslant qslant k}$ where $N_b^a : X^a \to X^b$ are linear maps that satisfy:

(i) $\langle x_a, N^{a'}_{-a} x'_{a'}\rangle + \langle N^a_{-a'} x_a, x'_{a'}\rangle = -\sum_{b;-a'-2\geqslant b\geqslant a+2} \langle N^a_b x_a, N^{a'}_{-b} x'_{a'}\rangle$ for any $a \neq a'$ such that $2 \leqslant -a - a' \leqslant k$ and any $x_a \in X^a, x'_{a'} \in X^{a'}$;

(ii) $\langle x_a, N^a_{-a} x_a \rangle = -Q(N^a_0 x_a) - \sum_{b<0;-a-2\geqslant b\geqslant a+2} \langle N^a_b x_a, N^a_{-b} x_a \rangle$ for any a such that $4 \leqslant -2a \leqslant k$ and any $x_a \in X^a$;

(iii) $\langle x_{-1}, N^{-1}_1 x_{-1}\rangle = 0$ for any $x_{-1} \in X^{-1}$.

For large k we may identify $Z_k = E^{\geqslant 2} X^*$ by $(N^a_b) \mapsto N$, $Nx_a = \sum_{b|b\geqslant a+2} N^a_b x_a$ with $x_a \in X^a$. Moreover, we may identify $Z_2 = E^2 \mathrm{gr}\, X^*$ in an obvious way. We have obvious maps $Z_2 \geqslant tsZ_3 \geqslant tsZ_4 \geqslant ts \cdots$. These maps eventually become the identity map of $E^{\geqslant 2} X^*$; their composition may be identified with Φ. It is enough to show that for any $k \geqslant 3$ the obvious map $Z_k \to Z_{k-1}$ is an affine space bundle with fibres of dimension

$$d_k = \sum_{a<a';-a-a'=k} f_a f_{a'} + \sum_{a;-2a=k} f_a(f_a - 1)/2.$$

We shall prove only that the fibre of this map at any given point of Z_{k-1} is an affine space of dimension d_k. This fibre may be identified with the set of all collections $(N^a_b)_{a,b\in\mathbf{Z};b-a=k}$ where $N^a_b : X^a \to X^b$ are linear maps that satisfy (i) with $-a - a' = k$, and (ii) with $-2a = k$. (In these equations the right hand sides involve only the coordinates of the given point in Z_{k-1}.) In equation (i) with $a \neq a', -a - a' = k$ each of $N^{a'}_{-a}, N^a_{-a'}$ determines the other. So the solutions of this equation form an affine space of dimension $f_a f_{a'}$. If $k \in \mathbf{Z}'$ there are no further equations. If $k \in \mathbf{Z}''$, in equation (ii) with $-2a = k$ the right hand-side is a known quadratic form on X^a and the solutions N^a_{-a} form an affine space of dimension $f_a(f_a - 1)/2$. This completes the proof of (a).   □

We show:

(b) $1 + E^{\geqslant 2} X^* \subset \mathrm{SO}_Q$.

To prove this we may assume that $|\mathbf{k}| = \infty$. Since $1 \in 1 + E^{\geqslant 2} X^* \subset \mathrm{O}_Q$ it is enough to show that $E^{\geqslant 2} X^*$ is irreducible. Using (a) we see that it is enough to show that $E^2 \mathrm{gr}\, X^*$ is irreducible. From the definitions we see that $E^2 \mathrm{gr}\, X^*$ is an affine space of dimension $\sum_{a<a';-a-a'=2} f_a f_{a'} + f_{-1}(f_{-1} - 1)/2$. This proves (b).   □

We set $E^{\geqslant 2}_* X^* = \Phi^{-1}(E^2_* \mathrm{gr}\, X^*)$. From (a) we see that:

(c) $E^{\geqslant 2}_* X^*$ is (via Φ) an iterated affine space bundle over $E^2_* \mathrm{gr}\, X^*$ with fibres of dimension d.

Using this and the results in 1.3 we see that if $E^{\geqslant 2}_* X^* \neq \varnothing$ then $(f_a)_{a\in\mathbf{Z}}$ is admissible.

In the remainder of this subsection we assume that $\mathbf{k}, q$ are as in 1.2 and that $(f_a)_{a\in\mathbf{Z}}$ is admissible. From (c) we see that:

(d) $|E^{\geqslant 2}_* X^*| = q^d |E^2_* \mathrm{gr}\, X^*|$.

We denote $|E^{\geqslant 2}_* X^*|$ by $\mathfrak{B}_{(f_a)}$ if $D \in \mathbf{Z}'$, by $\mathfrak{B}^\epsilon_{(f_a)}$ if $D \in \mathbf{Z}''_{\geqslant 2}$ and $\eta_Q = \epsilon$. From (d), 1.3(e), 1.2(a),(b) we see that:

(e) Let $Y = \mathfrak{B}_{(f_a)}$ if $f_0 \in \mathbf{Z}'$, $Y = \mathfrak{B}^\epsilon_{(f_a)}$ if $f_0 \in \mathbf{Z}''$, $\sum_a f_a > 0$, $\epsilon \in \{1, -1\}$. Then Y is a polynomial in q with coefficients in $\mathbf{Z}$ independent of q or p.

Note that if $D \in \mathbf{Z}''_{\geqslant 2}$ we have $\eta_Q = \eta_{\bar{Q}}$.

1.6. In the remainder of this section we assume that $D \geqslant 2$. Let $\widetilde{\mathcal{F}}^D$ be the set of all collections $(f_a)_{a \in \mathbf{Z}}$ of natural numbers such that $\sum_a f_a = D$, $f_a = f_{-a}$ for all a and the admissibility conditions 1.3(b),(c) hold. When $D \in \mathbf{Z}''$, let J be the set of $SO(V)$-orbits on $\mathcal{J}_Q$ (if $|\mathbf{k}| = \infty$) or on $\mathcal{J}_{\widetilde{Q}}$ for $\widetilde{Q}$ as in 1.1 (if $|\mathbf{k}| < \infty$); note that $|J| = 2$; in this case let

$$\widetilde{\mathcal{F}}_0^D = \{(f_a) \in \widetilde{\mathcal{F}}^D \mid f_0 = 0\}, \quad \widetilde{\mathcal{F}}_1^D = \{(f_a) \in \widetilde{\mathcal{F}}^D \mid f_0 > 0\},$$
$$\mathcal{F}^D = \widetilde{\mathcal{F}}_1^D \sqcup (J \times \widetilde{\mathcal{F}}_0^D).$$

When $D \in \mathbf{Z}'$ we set $\mathcal{F}^D = \widetilde{\mathcal{F}}^D$. For $\phi \in \mathcal{F}^D$ of the form (f_a) or $(j, (f_a))$ (where $j \in J$), let $\bar{\mathcal{Y}}_\phi$ be the set of all Q-filtrations X^* of V such that $\dim \mathrm{gr}^a X^* = f_a$ for all a and (in the case where $\phi = (j, (f_a)))$, $X^0 = X^1 \in j$; let $\mathcal{Y}_\phi$ be the set of all pairs (X^*, N) such that $X^* \in \bar{\mathcal{Y}}_\phi$ and $N \in E_*^{\geqslant 2} X^*$. Note that $\bar{\mathcal{Y}}_\phi$ is a partial flag manifold of SO_Q. Moreover, the obvious map $\mathcal{Y}_\phi \to \bar{\mathcal{Y}}_\phi$ has fibres $E_*^{\geqslant 2} X^*$ which are smooth (if $|\mathbf{k}| = \infty$) since $E_*^{\geqslant 2} X^*$ is open in the affine space $E^{\geqslant 2} X^*$. We see that $\mathcal{Y}_\phi$ is naturally a smooth variety (if $|\mathbf{k}| = \infty$).

Define $\Psi : \bigsqcup_{\phi \in \mathcal{F}^D} \mathcal{Y}_\phi \to \{g \in SO_Q \mid g \text{ unipotent}\}$ by $(X^*, N) \mapsto 1 + N$ for $(X^*, N) \in \mathcal{Y}_\phi$. This is well defined by 1.5(b).

Theorem 1.7. *In the setup of* 1.6, Ψ *is a bijection.*

An equivalent statement is:

(a) *For any unipotent element $g \in SO_Q$ there is a unique Q-filtration X^* of V such that $g - 1 \in E_*^{\geqslant 2} X^*$.*

It is enough to prove this in the case where $|\mathbf{k}| = \infty$. The proof is given in 2.12 and 3.3.

For any $\phi \in \mathcal{F}^D$ let $\Xi_\phi = \Psi(\mathcal{Y}_\phi)$. The theorem shows that the sets Ξ_ϕ form a partition of the variety of unipotent elements in SO_Q and that, for any $\phi \in \mathcal{F}^D$, Φ restricts to a bijection $\mathcal{Y}_\phi \to \Xi_\phi$. One can show that if $|\mathbf{k}| = \infty$ this is an isomorphism of the smooth variety $\mathcal{Y}_\phi$ with a subvariety of the variety of unipotent elements of SO_Q. Note that if $|\mathbf{k}| = \infty$ then each $\mathcal{Y}_\phi$ is nonempty hence each Ξ_ϕ is nonempty. (If $|\mathbf{k}| < \infty$ then Ξ_ϕ is nonempty unless $D \in \mathbf{Z}''$, $\eta_Q = -1$, $\phi \in J \times \widetilde{\mathcal{F}}_0^D$ in which case it is empty.) In the case where $|\mathbf{k}| = \infty$ and $p \neq 2$, the subsets Ξ_ϕ are exactly the unipotent conjugacy classes in SO_Q. If $p = 2$, the subsets Ξ_ϕ are unions of unipotent conjugacy classes in SO_Q but not necessarily single conjugacy classes; see Section 4.

1.8. In the setup of 1.6 we assume that $\mathbf{k}, q$ are as in 1.2. We assume that Theorem 1.7 holds. Let $\phi \in \mathcal{F}^D$. Assume that if $D \in \mathbf{Z}''$, $\eta_Q = -1$, then $\phi \in \widetilde{\mathcal{F}}_1^D$. We have:

(a) $|\mathcal{Y}_\phi| = |\bar{\mathcal{Y}}_\phi||E_*^{\geqslant 2} X^*|$

for any $X^* \in \bar{\mathcal{Y}}_\phi$. We denote $|\mathcal{Y}_\phi|$ by $\widetilde{\mathfrak{B}}_\phi$ if $D \in \mathbf{Z}'$; by $\widetilde{\mathfrak{B}}_\phi^\epsilon$ if $D \in \mathbf{Z}''$ and $\eta_Q = \epsilon$. From (a) and 1.5(e) we see that:

(b) *Let $\widetilde{Y} = \widetilde{\mathfrak{B}}_\phi$ if $D \in \mathbf{Z}'$; $\widetilde{Y} = \widetilde{\mathfrak{B}}_\phi^\epsilon$ if $D \in \mathbf{Z}''$ and $\eta_Q = \epsilon$. Then $\widetilde{Y}$ is a polynomial in q with coefficients in $\mathbf{Z}$ independent of q or p.*

1.9. Let $N \in \widetilde{\mathcal{M}}_Q - \{0\}, e = e_N$. We have $e \geqslant 2$. Let E be a complement to $\ker N^{e-1}$ in V. We show:

(a) *The* $(-1)^{e-1}$-*symmetric bilinear form* $(\, , \,) : E \times E \to \mathbf{k}$ *given by* $(x,y) = \langle x, N^{e-1}y \rangle$ *is nondegenerate.*

Assume that $v_0 \in E, \langle v_0, N^{e-1}E \rangle = 0$. Then $\langle N^{e-1}v_0, E \rangle = 0$. Also,

$$\langle N^{e-1}v_0, \ker N^{e-1} \rangle = \pm \langle v_0, N^{e-1} \ker N^{e-1} \rangle = 0.$$

Thus $\langle N^{e-1}v_0, E + \ker N^{e-1} \rangle = 0$, that is, $\langle N^{e-1}v_0, V \rangle = 0$ hence $N^{e-1}v_0 \in R$. Using 1.1(a) we deduce $N^{e-1}v_0 = 0$. We see that $v_0 \in \ker N^{e-1} \cap E = 0$. This proves (a).

We show:

(b) *The linear map* $E^{\oplus e} \to V$ *given by* $(v_0, v_1, \ldots, v_{e-1}) \mapsto v_0 + Nv_1 + \cdots + N^{e-1}v_{e-1}$ *is injective and* $\langle \, , \, \rangle$ *is nondegenerate on its image* $W := E + NE + \cdots + N^{e-1}E$.

Assume that $v = v_0 + Nv_1 + \cdots + N^{e-1}v_{e-1}$ with $v_i \in E$ satisfies $0 = \langle v, E \rangle = \langle v, NE \rangle = \cdots = \langle v, N^{e-1}E \rangle$. We must show that $v_k = 0$ for $k \in [0, e-1]$. We argue by induction on k. Assume that $k \in [0, e-1]$ and that $v_{k'} = 0$ for $k' < k$. From $0 = \langle v, N^{e-1-k}E \rangle$ we get $\langle N^k v_k, N^{e-1-k}E \rangle = 0$ hence $(v_k, E) = 0$ hence $v_k = 0$ (using (a)). This proves (b).

We show:

(c) *We have* $V = W \oplus Y$ *(W as in* (b)*,* $Y = W^\perp$*),* $R \subset Y$ *and* Y *is* N-*stable; moreover,* $N^{e-1}Y = 0$.

The first assertion follows from (b). Now let $x \in Y$. We can write $x = x' + x''$ where $x' \in E$, $x'' \in \ker N^{e-1}$. We have $N^{e-1}x = N^{e-1}x' \in W \cap Y$ hence $N^{e-1}x = 0$. This proves (c).

2. The case $p = 2$

2.1. In this section we prove Theorem 1.7 assuming that $p = 2$. In this case the symmetric bilinear form $\langle \, , \, \rangle$ associated to Q is symplectic. Let $N \in \widetilde{\mathcal{M}}_Q$.

For $i \geqslant 1$ we define $\lambda_i^N : \ker N^i \to \mathbf{k}$ by $x \mapsto \langle x, N^{i-1}x \rangle^{1/2}$. Note that $\lambda_i^N = 0$ if $i \in \mathbf{Z}'$ and λ_i^N is linear for any i.

For $i \geqslant 1$ we define $\epsilon_i^N \in \{0, 1\}$ by $\epsilon_i^N = 0$ if $\lambda_i^N = 0$, $\epsilon_i^N = 1$ if $\lambda_i^N \neq 0$. Note that $\epsilon_i^N = 0$ if $i \in \mathbf{Z}'$.

2.2. Let $N \in \widetilde{\mathcal{M}}_Q - \{0\}, e = e_N$. We have $e \geqslant 2$. Let $\lambda = \lambda_e^N$. Define a subspace $L = L_N$ of V by

$$L = N^{e-1}V \quad \text{if } \lambda = 0,$$
$$L = (\ker \lambda)^\perp \quad \text{if } \lambda \neq 0, R = 0,$$
$$L = \{x \in (\ker \lambda)^\perp \mid q(x) = 0\} \quad \text{if } \lambda \neq 0, R \neq 0.$$

If $\lambda \neq 0$ then $\ker \lambda$ is a hyperplane in V containing R; thus L is a line if $R = 0$. If $\lambda \neq 0, R \neq 0$ then $K := (\ker \lambda)^\perp$ is a two dimensional subspace containing R so

that $\langle\ ,\ \rangle$ is zero on K. Hence $q : K \to \mathbf{k}$ is a group homomorphism which restricts to a group isomorphism $R \to \mathbf{k}$ hence induces a group isomorphism $K/L \to \mathbf{k}$. We see that $K = L \oplus R$. Thus L is a line complementary to R in K and $\ker \lambda = L^\perp$.

We show:

(a) $L \subset L^\perp$.

Assume first that $\lambda = 0$. By 1.1(b) it is enough to show that $N^{e-1}V \subset \ker N^{e-1}$ which is clear since $2e - 2 \geqslant e$. If $\lambda \neq 0$ then (a) is clear since L is a line.

We show:

(b) If $\lambda \neq 0$, then $L \subset N^{e-1}(L^\perp)$ if and only if $N^{e-1}V \in \mathcal{C}''$.

Define a symmetric bilinear form $(\ ,\ )$ on V by $(y, y') = \langle y, N^{e-1}y'\rangle$. We have $\mathrm{Rad}_{(\ ,\ )} = \{y' \in V \mid N^{e-1}y' \in R\} = \ker N^{e-1}$ (since $N^{e-1}V \cap R = 0$). We have $\lambda(y) = (y, y)^{1/2}$. Since $\lambda|_{\mathrm{Rad}_{(\ ,\ )}} = 0$, we can find $v \in V$ such that $(v, y) = \lambda(y)$, that is, $\langle N^{e-1}v, y\rangle = \lambda(y)$ for any $y \in V$. Since $\lambda \neq 0$ we have $N^{e-1}v \notin 0$. Clearly, if $y \in \ker \lambda$ then $\langle N^{e-1}v, y\rangle = 0$. Thus $N^{e-1}v \in (\ker \lambda)^\perp$.

The symmetric bilinear form $(\ ,\ )_0$ on V given by $(y, y')_0 = (y, y') + \lambda(y)\lambda(y')$ is symplectic: for $y \in V$ we have $(y, y)_0 = (y, y) + \lambda(y)^2 = 0$. Since $\lambda|_{\ker N^{e-1}} = 0$ we see that $R_0 := \mathrm{Rad}_{(\ ,\ )_0}$ contains $\ker N^{e-1}$. Since $(\ ,\ )_0$ is symplectic we have $\dim R_0 = D \bmod 2$.

Assume now that $N^{e-1}V \in \mathcal{C}''$. Then $\dim \ker N^{e-1} = D \bmod 2$. Hence $\dim \ker N^{e-1} = \dim R_0 \bmod 2$, that is, $R_0/\ker N^{e-1} \in \mathcal{C}''$. If $R_0 \neq \ker N^{e-1}$ then $\dim(R_0/\ker N^{e-1}) \geqslant 2$. Hence the kernel of the linear map $R_0/\ker N^{e-1} \to \mathbf{k}$ induced by $\lambda : R_0 \to \mathbf{k}$ is nonzero. Thus there exists $u \in R_0$ such that $u \notin \ker N^{e-1}$ and $\lambda(u) = 0$. For any $y \in V$ we have $0 = (y, u)_0 = (y, u) + \lambda(u)\lambda(y) = (y, u)$ so that $u \in \mathrm{Rad}_{(\ ,\ )} = \ker N^{e-1}$. This contradiction shows that $R_0 = \ker N^{e-1}$. Since $N^{e-1}v \neq 0$ we have $v \notin \ker N^{e-1}$ hence $v \notin R_0$. Thus there exists $v' \in V$ such that $(v, v')_0 \neq 0$, that is, $(v, v') + (v, v)(v, v') \neq 0$. It follows that $1 + (v, v) \neq 0$. Moreover we have $0 = (v, v)_0 = (v, v) + (v, v)(v, v)$ so that $(v, v)(1 + (v, v)) = 0$. Since $1 + (v, v) \neq 0$ we deduce that $(v, v) = 0$, that is, $\langle v, N^{e-1}v\rangle = 0$. We have $Q(N^{e-1}v) = \langle N^{e-2}v, N^{e-1}v\rangle = \langle v, N^{2e-3}v\rangle$. If $e \geqslant 3$ this is 0 since $2e - 3 \geqslant e$; if $e = 2$ this equals $\langle v, N^{e-1}v\rangle$ which is again 0. We see that $N^{e-1}v$ is a nonzero vector in $\ker(Q : (\ker \lambda)^\perp \to \mathbf{k})$. Hence $N^{e-1}v \in L - 0$. Since L is a line we see that L is spanned by $N^{e-1}v$. To show that $L \subset N^{e-1}(L^\perp)$ it is enough to show that $v \in L^\perp$ or that $\langle v, L\rangle = 0$ or that $\langle v, N^{e-1}v\rangle = 0$. But this equality is already known. We see that $L \subset N^{e-1}(L^\perp)$.

Conversely, assume that $L \subset N^{e-1}(L^\perp)$. Let $x \in L - 0$. We have $x = N^{e-1}t$ with $\langle t, L\rangle = 0$. In particular, $\langle t, x\rangle = 0$, that is, $\lambda(t) = 0$. Since $x \in L$ we have $z \in \ker \lambda \implies \langle x, z\rangle = 0$. Hence there exists $c \in \mathbf{k}$ such that $\langle x, z\rangle = c\lambda(z)$ for all $z \in V$. Since $x \notin R$ we have $c \neq 0$. Replacing x by a nonzero scalar multiple we may assume that $\langle x, z\rangle = \lambda(z)$, that is, $(t, z) = \lambda(z)$ for all $z \in V$. Let $R_1 = \mathrm{Rad}_{(\ ,\ )|_{\ker \lambda}}$. If $u \in R_1$ then we have $w \in \ker \lambda \implies (u, w) = 0$. Hence there exists $c' \in \mathbf{k}$ such that $(u, w) = c'\lambda(w)$, that is, $(u, w) = c'(t, w)$ for any $w \in V$. Hence $u - c't \in \mathrm{Rad}_{(\ ,\ )} = \ker N^{e-1}$. We see that $R_1 \subset \mathbf{k}t + \ker N^{e-1}$. The reverse inclusion is obvious. Thus, $R_1 = \mathbf{k}t + \ker N^{e-1}$. We see that $(\ ,\ )$ induces a nondegenerate symmetric bilinear form on $\ker \lambda/(\mathbf{k}t + \ker N^{e-1})$. This induced

form is symplectic: the equality $(y, y')_0 = (y, y') + \lambda(y)\lambda(y')$ with $(,)_0$ symplectic shows that $(,)|_{\ker \lambda} = (,)_0|_{\ker \lambda}$. We deduce that $\ker \lambda/(\mathbf{k}t + \ker N^{e-1}) \in \mathcal{C}''$. Since $x \neq 0$ we have $t \notin \ker N^{e-1}$ hence $\dim \ker \lambda - 1 - \dim \ker N^{e-1} \in \mathbf{Z}''$, that is, $D - 2 - (D - \dim N^{e-1}V) \in \mathbf{Z}''$. We see that $N^{e-1}V \in \mathcal{C}''$. This proves (b). $\square$

We show:

(c) *We have* $L \subset N^{e-1}V + R$. *More precisely, we have* $L \subset N^{e-1}V$ *except possibly in the case where* $V \in \mathcal{C}', \lambda \neq 0, e = 2, c_e^N \in \mathbf{Z}'$.

If $\lambda = 0$ this is obvious. Assume now that $\lambda \neq 0$. Since $\ker N^{e-1} \subset \ker \lambda$ we have $(\ker \lambda)^\perp \subset (\ker N^{e-1})^\perp$ and using 1.1(b), $(\ker \lambda)^\perp \subset N^{e-1}V + R$. Since $L \subset (\ker \lambda)^\perp$ we have $L \subset N^{e-1}V + R$. If $R = 0$ then clearly $L \subset N^{e-1}V$. If $e \geqslant 3, R \neq 0$ we have $Q|_{N^{e-1}V} = 0$; indeed for $v \in V$ we have $Q(N^{e-1}v) = \langle N^{e-2}v, N^{e-1}v \rangle = \langle v, N^{2e-3}v \rangle = 0$. Hence if $x \in L$ is written as $x = N^{e-1}v + v'$ with $v \in V, v' \in R$ then $0 = Q(x) = Q(v')$. But $Q(v') = 0, v' \in R$ implies $v' = 0$ hence $x = N^{e-1}v$ and $L \subset N^{e-1}V$. If $c_e^N \in \mathbf{Z}''$ then we have $L \subset N^{e-1}V$ by (b). This proves (c). $\square$

We show:

(d) $NL \subset L$, $N(L^\perp) \subset L^\perp$.

From (c) we see that $NL = 0$ (since $N^e = 0$ and $NR = 0$). Thus the first inclusion holds. The second inclusion follows from the first. $\square$

We define a subset $\mathcal{M}_Q$ of $\widetilde{\mathcal{M}}_Q$ as follows:

$$\mathcal{M}_Q = \widetilde{\mathcal{M}}_Q \quad \text{if } V \in \mathcal{C}';$$
$$\mathcal{M}_Q = \{\widetilde{N} \in \widetilde{\mathcal{M}}_Q \mid \ker \widetilde{N} \in \mathcal{C}''\} \quad \text{if } V \in \mathcal{C}''.$$

We show:

(e) *If* $N \in \mathcal{M}_Q - \{0\}$, *then* $Q|_L = 0$.

If $R \neq 0$ the result is obvious. Now assume that $R = 0$. If $e \geqslant 3$ we have $Q|_{N^{e-1}V} = 0$ as in the proof of (c). The same argument shows that $Q|_{N^{e-1}V} = 0$ if $\lambda = 0$. If $\lambda = 0$ or $e \geqslant 3$ we have $L \subset N^{e-1}V$, see (c), hence $Q|_L = 0$. Thus we may assume that $\lambda \neq 0, e = 2$. Since $\ker N \in \mathcal{C}''$ and $V \in \mathcal{C}''$ we have $NV \in \mathcal{C}''$, that is, $N^{e-1}V \in \mathcal{C}''$. As in the proof of (b) we see that L is spanned by $N^{e-1}v$ (v as in that proof) which is contained in $\ker(Q : (\ker \lambda)^\perp \to \mathbf{k})$; the result follows. $\square$

2.3. Let $N \in \mathcal{M}_Q - \{0\}, e = e_N, \lambda = \lambda_e^N, L = L_N$. We have $e \geqslant 2$. We set $V' = L^\perp/L$, see 2.2(a). From 2.2(d) we see that N induces a (nilpotent) endomorphism of V', denoted by N'. Let $e' = e_{N'}$. If $\lambda = 0$ we have $e' \leqslant e - 1$; if $\lambda \neq 0$ we have $e' \leqslant e$. Define a quadratic form $Q' : V' \to \mathbf{k}$ by $Q'(x') = Q(x)$ where x is a representative of $x' \in V'$ in $L^\perp$. (To see that Q' is well defined we use 2.2(e).) The symplectic form associated to Q' is $\langle x', y' \rangle' = \langle x, y \rangle$ where $x', y' \in V'$ and x, y are representatives of x', y' in $L^\perp$. Its radical is

$$\{x \in L^\perp \mid \langle x, L^\perp \rangle = 0\}/L = (L + R)/L.$$

It follows that Q' is nondegenerate. We have $N' \in \widetilde{\mathcal{M}}_{Q'}$. We show:

(a) $N' \in \mathcal{M}_{Q'}$.

To do this we may assume that $V \in \mathcal{C}''$. Then $R = 0$. It is enough to show that $\dim \ker N' = \dim \ker N \mod 2$. We have $\ker N' = \{x \in L^\perp \mid Nx \in L\}/L$. We have an exact sequence

$$0 \to L \xrightarrow{a} \ker N \xrightarrow{b} \{x \in L^\perp \mid Nx \in L\}/L \xrightarrow{c} L \cap N(L^\perp) \to 0,$$

where a is the inclusion, b is induced by the inclusion $\ker N \subset L^\perp$ and c is induced by $x \mapsto Nx$. Thus $\dim L - \dim \ker N + \dim \ker N' - \dim(L \cap N(L^\perp)) = 0$ and it is enough to show that $\dim L = \dim(L \cap N(L^\perp)) \mod 2$.

Assume first that $\lambda \neq 0$. We show that $L \subset N(L^\perp)$ (this implies that $\dim L = \dim(L \cap N(L^\perp))$). If $e = 2$ this follows from 2.2(b). (We have $NV \in \mathcal{C}''$ since $\ker N \in \mathcal{C}''$ and $V \in \mathcal{C}''$.) Assume now that $e \geqslant 3$. Since $R = 0$ we see from 2.2(c) that L is spanned by $N^{e-1}x$ for some $x \in V$. It is enough to show that $N^{e-2}x \in L^\perp$; since $e \geqslant 3$ it is enough to show that $NV \subset L^\perp$; this is clear since $L^\perp$ is N-stable of codimension 1.

Assume next that $\lambda = 0$. We must show that

$$\dim N^{e-1}V = \dim(N^{e-1}V \cap N(\ker N^{e-1})) \mod 2.$$

If $e \geqslant 3$ we have $N^{e-1}V \subset N(\ker N^{e-1})$ hence $\dim N^{e-1}V = \dim(N^{e-1}V \cap N(\ker N^{e-1}))$. If $e = 2$ we have $N(\ker N^{e-1}) = 0$ and it is enough to show that $NV \in \mathcal{C}''$. But this follows from $\ker N \in \mathcal{C}''$ and $V \in \mathcal{C}''$. This completes the proof of (a). $\square$

2.4. Let $N \in \mathcal{M}_Q - \{0\}$. Let $e, \lambda, L, V', N', Q'$ be as in 2.2. Properties (i)–(iii) below describe the invariants $c_i^{N'}, \epsilon_i^{N'}$ of N'.

(i) *Assume that* $\lambda = 0$. *We have* $c_e^{N'} = 0$; $c_{e-2}^{N'} = c_{e-2}^N + c_e^N$ *(if* $e > 2$); $c_i^{N'} = c_i^N$ *if* $i \neq e, i \neq e - 2$; $\epsilon_i^{N'} = \epsilon_i^N$ *for any* i.

(ii) *Assume that* $\lambda \neq 0$ *and* $c_e^N \in \mathbf{Z}''$. *Then* $c_e^{N'} = c_e^N - 2$; $c_{e-1}^{N'} = c_{e-1}^N + 2$; $c_i^{N'} = c_i^N$ *for* $i \neq e, i \neq e - 1$; $\epsilon_e^{N'} = 0$; $\epsilon_i^{N'} = \epsilon_i^N$ *for any* $i \neq e$.

(iii) *Assume that* $\lambda \neq 0$, $c_e^N \in \mathbf{Z}'$. *Then* $c_e^{N'} = c_e^N - 1$; $c_{e-2}^{N'} = c_{e-2}^N + 1$ *(if* $e > 2$); $c_i^{N'} = c_i^N$ *for* $i \neq e, i \neq e - 2$; $\epsilon_e^{N'} = 0$; $\epsilon_{e-2}^{N'} = 1$ *(if* $e > 2$); $\epsilon_i^{N'} = \epsilon_i^N$ *for any* $i \notin \{e, e - 2\}$.

We have $N'^{e-1}V' = (N^{e-1}(L^\perp) + L)/L = N^{e-1}(L^\perp)/(N^{e-1}(L^\perp) \cap L)$.

If $\lambda = 0$ we have $L^\perp = \ker N^{e-1}$ and $N^{e-1}(L^\perp) = 0$. Hence $N'^{e-1}V' = 0$ and $c_i^{N'} = 0$ for $i \geqslant e$.

If $\lambda \neq 0$ and $N^{e-1}V \in \mathcal{C}''$ (that is $c_e^N \in \mathbf{Z}''$) we have $L \subset N^{e-1}(L^\perp)$ (see 2.2(b)) so that $N'^{e-1}V' = N^{e-1}(L^\perp)/L$. If $\lambda \neq 0$ and $N^{e-1}V \in \mathcal{C}'$ (that is, $c_e^N \in \mathbf{Z}'$) we have $L \not\subset N^{e-1}(L^\perp)$ (see 2.2(b)) and, since L is a line, we have $N^{e-1}(L^\perp) \cap L = 0$ and $N'^{e-1}V' = N^{e-1}(L^\perp)$. We have an exact sequence

$$0 \to \ker N^{e-1} \cap L^\perp \to L^\perp \xrightarrow{N^{e-1}} N^{e-1}(L^\perp) \to 0.$$

Now $\ker N^{e-1} \subset \ker \lambda = L^\perp$ hence this exact sequence becomes

$$0 \to \ker N^{e-1} \to \ker \lambda \to N^{e-1}(L^\perp) \to 0.$$

We see that

$$\dim N^{e-1}(L^{\perp}) = \dim \ker \lambda - \dim \ker N^{e-1} = \dim V - \dim \ker N^{e-1} - 1 = c_e^N - 1,$$

so that $c_e^{N'} = \dim N'^{e-1}V'$ equals $c_e^N - 1$ if $c_e^N \in \mathbf{Z}'$ and equals $c_e^N - 2$ if $c_e^N \in \mathbf{Z}''$.

We write $V = W \oplus Y$ as in 1.9(c). Let $N_1 = N|_W, N_2 = N|_Y$. Let us now assume that we are not in the case

$$V \in \mathcal{C}', \quad \lambda \neq 0, \quad e = 2, \quad c_e^N \in \mathbf{Z}'. \tag{$*$}$$

By 2.2(c) we have $L \subset N^{e-1}V$. Since $N^{e-1}Y = 0$ we have $L \subset N^{e-1}W$ hence $L \subset W$. Since $\ker N^{e-1} \subset \ker \lambda = L^{\perp}$ and $Y \subset \ker N^{e-1}$ we have $Y \subset L^{\perp}$. Hence we have, canonically, $V' = W' \oplus Y$ where $W' = L^{\perp'}/L'$ (and $L^{\perp'} = \{x \in W \mid \langle x, L \rangle = 0\}$) with both W', Y, N'-stable. Let $N_1' = N'|_{W'}$. If $\lambda = 0$ then $c_{e-2}^{N_1'} = c_e^{N_1} = c_e^N$ and $c_i^{N_1'} = 0$ for $i \neq e - 2$. If $\lambda \neq 0$ we have either

$$c_e^{N_1'} = c_e^{N_1} - 1 = c_e^N - 1, \quad c_{e-2}^{N_1'} = 1, \quad c_i^{N_1'} = 0 \quad \text{for } i \notin \{e, e-2\}$$

or

$$c_e^{N_1'} = c_e^{N_1} - 2 = c_e^N - 2, \quad c_{e-1}^{N_1'} = 2, \quad c_i^{N_1'} = 0 \quad \text{for } i \notin \{e, e-1\};$$

the formulas for $c_i^{N'}$ in (i)–(iii) follow since

$$c_i^{N'} = c_i^{N_1'} + c_i^{N_2} = c_i^{N_1'} + c_i^N \quad \text{for } i \leqslant e - 1.$$

(These formulas also hold in the excluded case $(*)$. Indeed in that case we have $c_2^{N'} = c_2^N - 1$ as we have seen already. For $i > 2$ we have $c_i^{N'} = 0$ and $c_1^{N'} + 2c_2^{N'} = \dim V' = \dim V - 2 = c_1^N + 2c_2^N - 2$ hence $c_1^{N'} = c_1^N$.)

Assume now that $l \neq 0$. We show that $\epsilon_e^{N'} = 0$. It is enough to show that if $x \in L^{\perp}$ then $\langle x, N^{e-1}x \rangle = 0$. This follows from $L^{\perp} = \ker \lambda$. We have $\epsilon_{e-1}^{N'} = 0$ since $e - 1 \in \mathbf{Z}'$. (This determines completely $\epsilon_i^{N'}$ in the case where $e = 2$.)

Assume that $\lambda \neq 0, c_e^N \in \mathbf{Z}', e > 2$. We show that $\epsilon_{e-2}^{N'} = 1$. It is enough to show that there exists $x \in V$ such that $N^{e-2}x \in L$ and $\langle x, N^{e-3}x \rangle \neq 0$. (For such x we have automatically $x \in L^{\perp}$ since $N^{e-1}x = 0$ and $\ker N^{e-1} \subset \ker \lambda = L^{\perp}$.) By 2.2(c) we can find $y \in V$ such that $N^{e-1}y$ is a basis element of L. We have $\langle y, N^{e-1}y \rangle \neq 0$. (If not we would have $y \in L^{\perp}$ and $L \subset N^{e-1}(L^{\perp})$ contradicting 2.2(b).) Let $x = Ny$. We have $N^{e-2}x = N^{e-1}y \in L$ and $\langle x, N^{e-3}x \rangle = \langle Ny, N^{e-2}y \rangle = \langle y, N^{e-1}y \rangle \neq 0$, as required.

We may now assume that we are not in the case $(*)$. We use the decompositions $V = W \oplus Y, V' = W' \oplus Y$ as above. The proof of the remaining assertions on $\epsilon_i^{N'}$ in (i)–(iii) is standard. $\square$

2.5. Let $N \in \mathcal{M}_Q$. We associate to N a collection of subspaces $(V^{\geqslant a})_{a \in \mathbf{Z}} = (V_N^{\geqslant a})_{a \in \mathbf{Z}}$ of V using induction on D. Let $e = e_N$. If $N = 0$ we set $V^{\geqslant a} = V$ for $a \leqslant 0, V^{\geqslant a} = 0$ for $a \geqslant 1$. Thus $V^{\geqslant a}$ are defined when $D \leqslant 1$. We may assume that $D \geqslant 2, N \neq 0$ and that $V^{\geqslant a}$ are already defined when V is replaced by a vector space of dimension $< D$. We have $e \geqslant 2$. Let λ, L, V', N', Q' be as in 2.2

and 2.3. Let $\rho : L^\perp \to V'$ be the obvious map. Since $L \neq 0$, we have $\dim V' < D$. By the induction hypothesis, $V'^{\geqslant a} = V'^{\geqslant a}_{N'}$ is defined for any $a \in \mathbf{Z}$. If $\lambda = 0$ we set

$$V^{\geqslant a} = V \text{ for } a \leqslant 1-e; \quad V^{\geqslant a} = \rho^{-1}(V'^{\geqslant a}) \text{ for } a \in [2-e, e-1]; \quad V^{\geqslant a} = 0 \text{ for } a \geqslant e.$$

If $\lambda \neq 0$ we set

$$V^{\geqslant a} = V \text{ for } a \leqslant -e; \quad V^{\geqslant a} = \rho^{-1}(V'^{\geqslant a}) \text{ for } a \in [1-e, e]; \quad V^{\geqslant a} = 0 \text{ for } a \geqslant 1+e.$$

This completes the definition of the subspaces $V^{\geqslant a}$.

From the definition it is clear that $V^{\geqslant a+1} \subset V^{\geqslant a}$ for any a. Thus $(V^{\geqslant a})$ is a filtration of V. We show:

(a) *If* $a \geqslant 1$ *we have* $Q|_{V^{\geqslant a}} = 0$ *and* $V^{\geqslant 1-a} = (V^{\geqslant a})^\perp$.

We use induction on D. For $N = 0$ the result is obvious. Thus the result holds when $D \leqslant 1$. Now assume that $D \geqslant 2$, $N \neq 0$. Let λ, L, V', Q', N' be as above. By the induction hypothesis we have $Q'|_{V'^{\geqslant a}_{N'}} = 0$ and $V'^{\geqslant 1-a}_{N'} = (V'^{\geqslant a}_{N'})^{\perp'}$ with $\perp'$ relative to V'. It follows that $Q|_{V^{\geqslant a}} = 0$ and $V^{\geqslant 1-a} = (V^{\geqslant a})^\perp$ for $a \in [1, e-1]$ (if $\lambda = 0$) and for $a \in [1, e]$ (if $\lambda \neq 0$). For $a \geqslant e$ (with $\lambda = 0$) and for $a \geqslant 1+e$ (with $\lambda \neq 0$) the result is again true since $Q|_{\{0\}} = 0$ and $\{0\}^\perp = V$. This proves (a). □

2.6. Let $N \in \mathcal{M}_Q - \{0\}$. Let $\lambda, L, V', N', Q', e'$ be as in 2.2 and 2.3. Let $\rho : L^\perp \to V'$ be the obvious map. Define $\perp'$ in terms of V' in the same way as $\perp$ is defined in terms of V. If $e' \geqslant 1$ we set $\lambda' = \lambda^{N'}_{e'}$. If $e' \geqslant 2$ we define L', V'', N'', Q'', e'' in terms of V', N', Q' in the same way as L, V', N', Q', e' were defined in terms of V, N, Q. If $e' \geqslant 2$ and $e'' \geqslant 1$ we set $\lambda'' = \lambda^{N''}_{e''}$.

Let $V = W \oplus Y$ be as in 1.9(c). For a subspace Z of Y let $Z^\vdash = \{y \in Y \mid \langle y, Z \rangle = 0\}$.

In the case where $\lambda = 0, e' = e - 1 \geqslant 2, \lambda' \neq 0$ we can view L' as a line in Y as follows. We have $N^{e-1}V \subset W$, $Y \subset \ker N^{e-1}$ hence $V' = W' \oplus Y$ where $W' = (\ker N^{e-1} \cap W)/N^{e-1}V$. Since $W' \subset \ker N'^{e-2} \subset \ker \lambda'$ we have $(\ker \lambda')^{\perp'} \subset W'^{\perp'} = Y$. Hence $L' \subset Y$ and $L'^{\perp'} = W' \oplus L'^\vdash$.

In this subsection we describe explicitly $V^{\geqslant a}$ for certain a.

(i) *If* $\lambda \neq 0$ *then* $V^{\geqslant e} = L$.

We have $e' \leqslant e$. If $e' < e$ then $V'^{\geqslant e} = 0$. If $e' = e$ then (using 2.4(ii), (iii)) we have $\epsilon^{N'}_e = 0$ hence $\lambda' = 0$ and $V'^{\geqslant e} = 0$. Hence $V^{\geqslant e} = \rho^{-1}(0) = L$.

(ii) *If* $\lambda \neq 0, e' < e$ *then* $V^{\geqslant e-1} = L$.

If $e' < e - 1$ then $V'^{\geqslant e-1} = 0$. If $e' = e - 1$ and $\lambda' = 0$ then $V'^{\geqslant e-1} = 0$. If $e' = e-1$ and $\lambda' \neq 0$ then e', e are even, contradiction. Hence $V^{\geqslant e-1} = \rho^{-1}(0) = L$.

(iii) *If* $\lambda = 0$ *and either* $e' \leqslant e - 2$ *or* $e' = e - 1, \lambda' = 0$ *then* $V^{\geqslant e-1} = N^{e-1}V$.

If $e' \leqslant e-2$ we have $V'^{\geqslant e-1} = 0$. If $e' = e-1, \lambda' = 0$ we have again $V'^{\geqslant e-1} = 0$. Hence $V^{\geqslant e-1} = \rho^{-1}(0) = L = N^{e-1}V$.

(iv) *If $\lambda \neq 0, e' = e$ then $V^{\geqslant e-1} = N^{e-1}L^{\perp} + L \subset \ker N$.*
Using 2.4(ii), (iii) we have $\lambda' = 0$. Hence $e'' < e'$. If either $e'' \leqslant e' - 2$ or $e'' = e' - 1, \lambda'' = 0$ then from (iii) applied to V' we get $V'^{\geqslant e-1} = N'^{e-1}V'$ and the result follows. If $e'' = e' - 1, \lambda'' \neq 0$ then e'', e are even, a contradiction. We have $NV^{\geqslant e-1} = 0$ since $N^e = 0$ and $NL = 0$.

(v) *If $\lambda = 0$ and either $e' \leqslant e - 3$ or $e' = e - 2 \geqslant 1, \lambda' = 0$ or $e = 2, e' = 0$ then $V^{\geqslant e-2} = N^{e-1}V$.*

If $e' \leqslant e - 3$ then $V'^{\geqslant e-2} = 0$. If $e' = e - 2 \geqslant 1, \lambda' = 0$ then $V'^{\geqslant e-2} = 0$. If $e = 2, e' = 0$ then $V' = 0$ and $V'^{\geqslant e-2} = 0$. Hence $V^{\geqslant e-2} = \rho^{-1}(0) = L = N^{e-1}V$.

(vi) *If $\lambda = 0, e' = e-1, \lambda' \neq 0$ then $V^{\geqslant e-1} = \{x \in V \mid N^{e-1}x = 0, \langle x, N^{e-2}x \rangle = 0\}^{\perp} \cap q^{-1}(0)$,*
$$V^{\geqslant e-1} = N^{e-1}V \oplus L' \subset \ker N.$$

By (i) for V' we have $V'^{\geqslant e-1} = L' = \{x \in V' \mid \langle x, N^{e-2}x \rangle = 0\}^{\perp'} \cap q'^{-1}(0)$. Hence

$$V^{\geqslant e-1} = \ker N^{e-1} \cap \{x \in \ker N^{e-1} \mid \langle x, N^{e-2}x \rangle = 0\}^{\perp} \cap q^{-1}(0).$$

In the last equality of (vi) we regard $N^{e-1}V$ as a subspace of W and L' as a subspace of Y (as earlier in this subsection).

(vii) *If $\lambda = 0, e' = e - 2 \geqslant 1, \lambda' \neq 0$ we have $V^{\geqslant e-2} = \{x \in V \mid N^{e-1}x = 0, \langle x, N^{e-3}x \rangle = 0\}^{\perp} \subset \ker N$.*

By (i) for V' we have $V'^{\geqslant e-2} = \{x' \in V' \mid \langle x', N'^{e-3}x' \rangle 0\}^{\perp'} \cap Q'^{-1}(0)$. Hence

$$V^{\geqslant e-2} = \ker N^{e-1} \cap \{x \in \ker N^{e-1} \mid \langle x, N^{e-3}x \rangle = 0\}^{\perp} \cap Q^{-1}(0).$$

We have $NV \subset \{x \in \ker N^{e-1} \mid \langle x, N^{e-3}x \rangle = 0\}$ since $\lambda = 0$. Taking $\perp$ we obtain $\{x \in \ker N^{e-1} \mid \langle x, N^{e-3}x \rangle = 0\}^{\perp} \subset \ker N$. Since $\ker N \subset \ker N^{e-1}$ we see that $V^{\geqslant e-2}$ is as required.

(viii) *If $\lambda = 0, e' = e - 1, \lambda' \neq 0, e'' \leqslant e' - 1$, we have $V^{\geqslant e-2} = \{x \in V \mid N^{e-1}x = 0, \langle x, N^{e-2}x \rangle = 0\}^{\perp} \cap Q^{-1}(0) \subset \ker N$.*

By (ii) for V' we have $V'^{\geqslant e-2} = \{x' \in V' \mid \langle x', N'^{e-2}x' \rangle 0\}^{\perp} \cap Q'^{-1}(0)$. Hence

$$V^{\geqslant e-2} = \ker N^{e-1} \cap \{x \in \ker N^{e-1} \mid \langle x, N^{e-2}x \rangle = 0\}^{\perp} \cap Q^{-1}(0).$$

We have $NV \subset \{x \in \ker N^{e-1} \mid \langle x, N^{e-2}x \rangle = 0\}$. Taking $\perp$ we obtain $\{x \in \ker N^{e-1} \mid \langle x, N^{e-2}x \rangle = 0\}^{\perp} \subset \ker N$. Since $\ker N \subset \ker N^{e-1}$ we see that $V^{\geqslant e-2}$ is as required.

(ix) *If $\lambda = 0, e' = e - 1, \lambda' = 0$ and either $e' = 1$ or $e \geqslant 3, e'' \leqslant e' - 2$ or $e \geqslant 3, e'' = e' - 1, \lambda'' = 0$ then $V^{\geqslant e-2} = N^{e-2}(\ker N^{e-1}) + N^{e-1}V \subset \ker N$.*

By (iii) for V' we have $V'^{\geqslant e-2} = N'^{e-2}V'$. Hence $V^{\geqslant e-2}$ is as required.

(x) *If $\lambda = 0, e' = e-1, \lambda' \neq 0, e'' = e'$, then $V^{\geqslant e-2} = N^{e-1}V \oplus (N^{e-2}L'^{\vdash} + L') \subset \ker N$.*

Here we regard $N^{e-1}V$ as a subspace of W and L' as a subspace of Y (as earlier in this subsection). Then $N^{e-2}L'^{\vdash} + L'$ is a subspace of Y. By (iv) for V' we have $V'^{\geqslant e-2} = N'^{e-2}L'^{\perp'} + L'$. Since $N'^{e-2}W' = 0$ we have $V'^{\geqslant e-2} = N'^{e-2}L'^{\vdash} + L' \subset Y$ and $V^{\geqslant e-2} = N^{e-1}V \oplus (N^{e-2}L'^{\vdash} + L')$. By (iv) for V' we have $N(N^{e-2}L'^{\vdash} + L') = 0$. Hence $NV^{\geqslant e-2} = 0$.

(xi) *If* $\lambda = 0, e' = e - 1, e \geqslant 3, \lambda' = 0, e'' = e' - 1, \lambda'' \neq 0$ *then* $V^{\geqslant e-2} = N^{e-1}V \oplus (\{b \in Y \mid N^{e-2}b = 0, \langle b, N^{e-3}b\rangle = 0\}^{\vdash} \cap q^{-1}(0)) \subset \ker N$.

By (vi) for V' we have $V'^{\geqslant e-2} = U'^{\perp'} \cap Q'^{-1}(0)$ where $U' = \{x \in V' \mid N'^{e-2}x = 0, \langle x, N'^{e-3}x' = 0\}$. We write $V' = W' \oplus Y$ as in the proof of (x). We write the condition that $x = a + b$ with $a \in W', b \in Y$ is in U' in terms of a, b. Note that $N'^{e-2}a = 0$ and $\langle a, N'^{e-3}a\rangle = 0$. (The last equality follows from $\tilde{a} \in W, N^{e-1}\tilde{a} = 0 \implies \langle \tilde{a}, N^{e-3}\tilde{a}\rangle = 0$. Indeed we have $\tilde{a} = Nc$ with $c \in W$ and $\langle \tilde{a}, N^{e-3}\tilde{a}\rangle = \langle Nc, N^{e-3}Nc\rangle = \langle c, N^{e-1}c\rangle = 0$ since $\lambda = 0$.) We see that $U' = W' \oplus \{b \in Y \mid N^{e-2}b = 0, \langle b, N^{e-3}b\rangle = 0\}$ and

$$V'^{\geqslant e-2} = \{b \in Y \mid N^{e-2}b = 0, \langle b, N^{e-3}b\rangle = 0\}^{\vdash} \cap Q'^{-1}(0),$$
$$V^{\geqslant e-2} = N^{e-1}V \oplus (\{b \in Y \mid N^{e-2}b = 0, \langle b, N^{e-3}b\rangle = 0\}^{\vdash} \cap Q^{-1}(0)).$$

By (vi) for V' we have $N(\{b \in Y \mid N^{e-2}b = 0, \langle b, N^{e-3}b\rangle = 0\}^{\vdash} \cap Q^{-1}(0)) = 0$. Hence $NV^{\geqslant e-2} = 0$.

2.7. Let $N \in \mathcal{M}_Q, e = e_N$. Let $V^{\geqslant a}$ be as in 2.5. We show:

(a) $NV^{\geqslant a} \subset V^{\geqslant a+2}$ *for any* $a \in \mathbf{Z}$.

When $N = 0$ the result is obvious. Now assume that $N \neq 0$. Then $e \geqslant 2$. Let λ, L, V', N', Q' be as in 2.2 and 2.3. We may assume that (a) holds when V, N are replaced by V', N'. We may assume that (a) holds when V, N are replaced by V', N'.

Assume first that $\lambda = 0$. If $a \geqslant e$ then $V^{\geqslant a} = 0$ and (a) is obvious. If $a \in \{e - 2, e - 1\}$ then $NV^{\geqslant a} = 0$ by 2.6 and (a) holds. Assume now that $a = -e$ or that $a = 1 - e, e \geqslant 3$. To prove (a) in this case it is enough to show that $NV \subset V^{\geqslant a+2}$, that is (using 2.5(a)), $NV \subset (V^{\geqslant -1-a})^{\perp}$ or that $N^{\dagger}(V^{\geqslant -1-a}) \in R$. This follows from $N(V^{\geqslant -1-a}) = 0$ which has been noted earlier. If $a = -1, e = 2$ we have $NV = V^{\geqslant a+2}$ and (a) holds. If $a \leqslant -1 - e$ then $V^{\geqslant a+2} = V$ and (a) is obvious. If $2 - e \leqslant a \leqslant e - 3$ then $V^{\geqslant a} = \rho^{-1}(V'^{\geqslant a})$, $V^{\geqslant a+2} = \rho^{-1}(V'^{\geqslant a+2})$ (notation of 2.5). Since $N'V'^{\geqslant a} \subset V'^{\geqslant a+2}$ we see that (a) holds.

Assume next that $\lambda \neq 0$. If $a \geqslant e+1$ then $V^{\geqslant a} = 0$ and (a) is obvious. If $a \in \{e-1, e\}$ then $NV^{\geqslant a} = 0$ by 2.6 and (a) holds. Assume now that $a \in \{-e, -1-e\}$. To prove (a) in this case it is enough to show that $NV \subset V^{\geqslant a+2}$, that is (using 2.5(a)), $NV \subset (V^{\geqslant -1-a})^{\perp}$ or that $N^{\dagger}(V^{\geqslant -1-a}) \in R$. This follows from $N(V^{\geqslant -1-a}) = 0$ which has been noted earlier. If $a \leqslant -2 - e$ then $V^{\geqslant a+2} = V$ and (a) is obvious. If $1 - e \leqslant a \leqslant qslante - 2$ then $V^{\geqslant a} = \rho^{-1}(V'^{\geqslant a})$, $V^{\geqslant a+2} = \rho^{-1}(V'^{\geqslant a+2})$ (notation of 2.5). Since $N'V'^{\geqslant a} \subset V'^{\geqslant a+2}$ we see that (a) holds. This proves (a).

For any $a \in \mathbf{Z}$ we set $\bar{V}^a = V^{\geqslant a}/V^{\geqslant a+1}$. From (a) we see that N induces a linear map $\bar{N} : \bar{V}^a \to \bar{V}^{a+2}$.

2.8. In the setup of 2.6 taking $\perp$ in 2.6(i),(iii),(vi) we obtain:

If $\lambda \neq 0$ then $V^{\geqslant 1-e} = L^\perp$.
If $\lambda = 0$ and either $e' \leqslant e - 2$ or $e' = e - 1, \lambda' = 0$ then $V^{\geqslant 2-e} = \ker N^{e-1}$.
If $\lambda = 0, e' = e - 1, \lambda' \neq 0$ then $V^{\geqslant 2-e} = (\ker N^{e-1} \cap W) \oplus L'^\vdash$.

2.9. For any $a \geqslant 0$ we set $K_a = \ker \bar{N}^a : \bar{V}^{-a} \to \bar{V}^a$. For any $a \in \mathbf{Z}''_{\geqslant 2}$ we define a quadratic form $Q_a : \bar{V}^{-a} \to \mathbf{k}$ by $Q_a(x) = Q_0(\bar{N}^{a/2}x)$. (Note that $\bar{N}^{a/2}x \in \bar{V}^0$.) If $\dot{x}$ is a representative of x in $V^{\geqslant -a}$ we have

$$Q_a(x) = Q(N^{a/2}\dot{x}) = \langle N^{a/2-1}\dot{x}, N^{a/2}\dot{x}\rangle = \langle \dot{x}, N^{a-1}\dot{x} + cN^a\dot{x} + \cdots \rangle,$$

where $c \in \mathbf{k}$. If $f > 0$ we have $\langle x, N^{a+f}x\rangle = 0$ since $\langle V^{\geqslant -a}, V^{\geqslant a+2f}\rangle = 0$. Moreover,

$$\langle x, N^a x\rangle = \langle N^{a/2}x + c'N^{a/2+1}x + \cdots, N^{a/2}x\rangle$$
$$= \langle c'N^{a/2+1}x + \cdots, N^{a/2}x\rangle = c'\langle x, N^{a+1}x\rangle = 0,$$

where $c' \in \mathbf{k}$. Hence

$$Q_a(x) = \langle \dot{x}, N^{a-1}\dot{x}\rangle.$$

Let $\langle\ ,\ \rangle_a$ be the symplectic form on $\bar{V}^{-a}$ associated to Q_a. Thus

$$\langle x, x'\rangle_a = \langle \bar{N}^{a/2}x, \bar{N}^{a/2}x'\rangle_0$$

for $x, x' \in \bar{V}^{-a}$. If $\dot{x}, \dot{x}'$ are representatives of x, x' in $V^{\geqslant -a}$ we have

$$\langle x, x'\rangle_a = \langle N^{a/2}\dot{x}, N^{a/2}\dot{x}'\rangle = \langle \dot{x}, N^a\dot{x}' + cN^{a+1}\dot{x}' + \cdots\rangle = \langle \dot{x}, N^a\dot{x}'\rangle = \langle x, \bar{N}^a x'\rangle.$$

(We use that $\langle V^{\geqslant -a}, V^{\geqslant a+2}\rangle = 0$.) Let R_a be the radical of $\langle\ ,\ \rangle_a$. If $x' \in R_a$ then $\langle x, \bar{N}^a x'\rangle = 0$ for all $x \in \bar{V}^{-a}$ hence $\bar{N}^a x' = 0$. Thus $R_a = K_a$. We show:

(a) If $a \in \mathbf{Z}'$ we have $K_a = 0$. If $a \in \mathbf{Z}''_{\geqslant 0}$ then Q_a is nondegenerate; hence $\dim K_a \in \{0, 1\}$.

If $N = 0$ we have $V^{\geqslant -a} = V_{\geqslant 1-a} = V$ hence $\bar{V}^{-a} = 0$ so that $K_a = 0$ as required. Now assume that $N \neq 0$ so that $e \geqslant 2$. Let λ, L, V', N', Q' be as in 2.2 and 2.3. Let $\bar{V}'^a, \bar{N}'^a, Q'_a$ be the analogues of $\bar{V}^a, \bar{N}^a, Q_a$ for V', N' instead of V, N. We may assume that the analogue of (a) holds when V, N is replaced by V', N'. If $\lambda \neq 0, a > e$ or if $\lambda = 0, a \geqslant e$ we have $\bar{V}^{-a} = 0$ hence $K_a = 0$ as required. If $\lambda \neq 0, a = e$ we have $\bar{V}^{-e} = V/L^\perp, \bar{V}^e = L, bN^e = 0$ hence $K_a = V/L^\perp$. We must show that Q_a is not identically zero. It is enough to show that $\dot{x} \mapsto \langle \dot{x}, N^{a-1}\dot{x}\rangle$ is not identically zero on V; this holds since $\lambda \neq 0$. If $\lambda = 0, a = e - 1, \epsilon^{N'}_{e-1} = 0$ then $\bar{V}^{1-e} = V/\ker N^{e-1}, \bar{V}^{e-1} = N^{e-1}V$ and $\bar{N}^a : \bar{V}^{-a} \to \bar{V}^a$ is an isomorphism; hence $K_a = 0$, as required. If $\lambda = 0, a = e - 1, \epsilon^{N'}_{e-1} = 1$ then with the notation of 2.6(vi), 2.8 we have $\bar{V}^{1-e} = W/(\ker N^{e-1}|W) \oplus Y/L'^\vdash, \bar{V}^{e-1} = N^{e-1}V \oplus L'$; now $\bar{N}^{e-1}$ restricts to an isomorphism $W/(\ker N^{e-1}|W) \to N^{e-1}V$ and to the zero map $Y/L'^\vdash \to L'$ (since $N^{e-1}Y = 0$). Hence $K_a = Y/L'^\vdash$. We must show that Q_a is not identically 0 on $Y/L'^\vdash$ or that $x \mapsto \langle x, N^{e-2}x\rangle$ is not identically 0 on Y.

But this follows from $\epsilon_{e-1}^{N'} = 1$. If $\lambda \neq 0, a \in [1, e-1]$ or if $\lambda = 0, a \in [1, e-2]$, we have $\bar{V}^{-a} = \bar{V}'^{-a}$, $\bar{V}^a = \bar{V}'^a$. We can identify $\bar{N}'^a = \bar{N}^a$ and (if $a \in \mathbf{Z}''$) Q_a with Q_a'. Hence the result follows from the induction hypothesis. If $a = 0$ the result is obvious. This completes the proof.

We show:

(b) *Assume that* $a \in \mathbf{Z}''_{\geqslant 2}$. *We set* $\phi_a = |\{b \in \mathbf{Z}'' \mid c_b^N \in \mathbf{Z}', b > a\}|$, $\xi_a = \dim K_a$. *We have* $c_a^N \in \mathbf{Z}' \implies \xi_a = 1$; $c_a^N \in \mathbf{Z}'', \phi_a \in \mathbf{Z}' \implies \xi_a = 1$; $c_a^N \in \mathbf{Z}'', \phi_a \in \mathbf{Z}'' \implies \xi_a = \epsilon_a^N$.

If $N = 0$ the result is obvious. Now assume that $N \neq 0$ so that $e \geqslant 2$. Let λ, V', N', Q' be as in 2.2 and 2.3. Let ξ_a', ϕ_a' be the analogues of ξ_a, ϕ_a for V', N' instead of V, N. We may assume that the analogue of (b) holds when V, N is replaced by V', N'. If $\lambda \neq 0, a > e$ or if $\lambda = 0, a \geqslant e$ we have (by the proof of (a)) $\xi_a = 0$, as required. If $\lambda \neq 0, a = e$ we have (by the proof of (a)) $\xi_a = 1$ as required. If $\lambda = 0, a = e - 1$ then (by the proof of (a)) $\xi_a = \epsilon_{e-1}^{N'} = \epsilon_{e-1}^N$ as required. In the remainder of the proof we assume that either $\lambda \neq 0, a \in [1, e-1]$ or $\lambda = 0, a \in [1, e-2]$. Then (by the proof of (a)), $\xi_a = \xi_a'$. Using the induction hypothesis we see that if $c_a^{N'} \in \mathbf{Z}'', \phi_a' \in \mathbf{Z}''$ then $\xi_a = \epsilon_a'^N$; otherwise, $\xi_a = 1$.

Assume first that $\lambda \neq 0$. Then $e \in \mathbf{Z}''$ hence $a \leqslant e - 2$.

If $a = e - 2$, $c_{e-2}^N \in \mathbf{Z}'$ and $c_e^N \in \mathbf{Z}'$ then $c_{e-2}^{N'} \in \mathbf{Z}''$, $c_e^{N'} \in \mathbf{Z}''$ and $\epsilon_{e-2}^{N'} = 1$ so that $\xi_{e-2} = 1$.

If $a = e - 2$, $c_{e-2}^N \in \mathbf{Z}'$ and $c_e^N \in \mathbf{Z}''$ then $c_{e-2}^{N'} \in \mathbf{Z}'$, $c_e^{N'} \in \mathbf{Z}''$ so that $\xi_{e-2} = 1$.

If $a = e - 2$, $c_{e-2}^N \in \mathbf{Z}''$ and $c_e^N \in \mathbf{Z}'$ then $c_{e-2}^{N'} \in \mathbf{Z}'$, $c_e^{N'} \in \mathbf{Z}''$ and $\epsilon_{e-2}^{N'} = 1$ so that $\xi_{e-2} = 1$. Also, $\phi_{e-2} = 1$.

If $a = e - 2$, $c_{e-2}^N \in \mathbf{Z}''$ and $c_e^N \in \mathbf{Z}''$ then $c_{e-2}^{N'} \in \mathbf{Z}''$, $c_e^{N'} \in \mathbf{Z}''$, $x_{e-2} = \epsilon_{e-2}^{N'} = \epsilon_{e-2}^N$.

If $a \leqslant e - 4$ we have $c_a^{N'} = c_a^N$. If these are odd then $\xi_a = 1$. If these are even then $\phi_a = \phi_a' \mod 2$ (if $c_e^N \in \mathbf{Z}', c_{e-2}^N \in \mathbf{Z}'$ then $\phi_a = \phi_a' + 2$; otherwise, $\phi_a = \phi_a'$).

Assume next that $\lambda = 0$.

If $a = e - 2$ then $c_e^N \in \mathbf{Z}''$ and $c_a^{N'} = c_a^N \mod 2$. If $c_a^{N'}, c_a^N$ are odd we have $\xi_a = 1$. If $c_a^{N'}, c_a^N$ are even we have $\xi_a = \epsilon_a^{N'} = \epsilon_a^N$.

If $a \leqslant e - 3$ we have $c_a^{N'} = c_a^N$. If these are both odd then $\xi_a = 1$; if these are both even then $\phi_a = \phi_a'$.

We see that (b) holds. $\square$

We show:

(c) *If* $a \in \mathbf{Z}'_{\geqslant 1}$ *then the bilinear pairing* $\bar{V}^{-a} \times \bar{V}^{-a} \to \mathbf{k}$, $x, y \mapsto \langle x, \bar{N}^a y \rangle$ *is symplectic; it is nondegenerate by* (a). *Hence* $\bar{V}^{-a} \in \mathcal{C}''$.

If $N = 0$ we have $\bar{V}^{-a} = 0$ if $a \neq 0$ and (c) is obvious. Assume that $N \neq 0$. Then $e \geqslant 2$. Let λ, V', N' be as in 2.2 and 2.3. We may assume that (c) holds when V, N are replaced by V', N'. If $\lambda \neq 0, a > e$ or if $\lambda = 0, a \geqslant e$ we have $\bar{V}^{-a} = 0$ and (c) is obvious. The case $\lambda \neq 0, a = e$ cannot occur since $e \in \mathbf{Z}''$. If $\lambda = 0, a = e - 1$ then $e - 1 \in \mathbf{Z}'$ and $\epsilon_{e-1}^{N'} = 0$; we have $\bar{V}^{1-e} = V/\ker N^{e-1}$ and $\langle x, N^{e-1} x \rangle = 0$ for all x. Hence (c) holds. If $\lambda \neq 0, a \in [1, e-1]$ or if $\lambda = 0, a \in [1, e-2]$, we have

$\bar{V}^{-a} = \bar{V}'^{-a}$. Hence the result follows the induction hypothesis. The case $a = 0$ does not arise. This completes the proof. $\square$

2.10. Let $X^* = (X^{\geqslant a})_{a \in \mathbf{Z}}$ be a Q-filtration of V (see 1.4). Let $N \in \mathcal{M}_Q, e = e_N$. We say that X^* is N-*adapted* if conditions (i)–(iii) below hold:

 (i) $NX^{\geqslant a} \subset X^{\geqslant a+2}$ for any a;
   For any $a \geqslant 0$ let $\mathcal{K}_a$ be the kernel of the map $\nu^a : \mathrm{gr}^{-a} X^* \to \mathrm{gr}^a X^*$ induced by N^a. We state conditions (ii) and (iii).
 (ii) for any $a \in \mathbf{Z}'_{\geqslant 1}$ we have $\mathcal{K}_a = 0$;
 (iii) for any $a \in \mathbf{Z}''_{\geqslant 2}$ we have $\mathcal{K}_a = 0$ or $\dim \mathcal{K}_a = 1$; in the latter case the map $\mathcal{K}_a \to \mathbf{k}$, $x \mapsto \langle \dot{x}, N^{a-1} \dot{x} \rangle$ is a bijection.
   (For $x \in \mathrm{gr}^a X^*$ we denote by $\dot{x}$ a representative of x in $X^{\geqslant a}$.) Note that $(V^{\geqslant a})_{a \in \mathbf{Z}}$ is N-adapted where $V^{\geqslant a} = V_N^{\geqslant a}$. We show:

 (a) *If $(X^{\geqslant a})_{a \in \mathbf{Z}}$ is an N-adapted filtration of V then $X^{\geqslant a} = V^{\geqslant a}$ for any a.*

If $V = 0$ the result is obvious. Now assume that $V \neq 0$. If $N = 0$ and $a \geqslant 1$ then $\nu^a : \mathrm{gr}^{-a} X^* \to \mathrm{gr}^a X^*$ is 0 and $\mathcal{K}_a = 0$. Hence $X^{\geqslant a} = X^{\geqslant a+1}$ and $X^{\geqslant -a} = X^{\geqslant -a+1}$ so that we have $X^{\geqslant 1} = X^{\geqslant 2} = \cdots = 0$ and $X_{\geqslant 0} = X_{\geqslant -1} = \cdots = V$ as required. We now assume that $N \neq 0$ hence $e \geqslant 2$. Let $\lambda, L, V', N', Q', \rho, \epsilon'$ be as in 2.2 and 2.3. If $\epsilon' \geqslant 1$ let λ' be as in 2.6. If $\epsilon' \geqslant 2$ let $Y, L', L'^{\vdash}$ be as in 2.6. We may assume that (a) holds when V, N are replaced by V', N'.

If $a \geqslant e + 1$ or if $a = e, \lambda = 0$, then $\nu^a : \mathrm{gr}^{-a} X^* \to \mathrm{gr}^a X^*$ is 0 and $\mathcal{K}_a = 0$. Hence $X^{\geqslant a} = X^{\geqslant a+1}$ and $X^{\geqslant -a} = X^{\geqslant -a+1}$ so that if $\lambda = 0$ we have

$$X^{\geqslant e} = X^{\geqslant e+1} = \cdots = 0 \quad \text{and} \quad X_{\geqslant 1-e} = X_{\geqslant -e} = \cdots = V;$$

if $\lambda \neq 0$ we have

$$X^{\geqslant e+1} = X^{\geqslant e+2} = \cdots = 0 \quad \text{and} \quad X_{\geqslant -e} = X_{\geqslant -1-e} = \cdots = V.$$

Hence $X^{\geqslant a} = V^{\geqslant a}$ if $a \geqslant e + 1$ or if $a = e, \lambda = 0$ or if $a \leqslant -e$ or if $a = 1 - e, \lambda = 0$.

Assume that $\lambda \neq 0$. Then $\mathcal{K}_e$ is the kernel of $0 = \nu^e : V/X^{\geqslant 1-e} \to X^{\geqslant e}$, that is, $\mathcal{K}_e = V/X^{\geqslant 1-e}$. Also $\mathcal{K}_e \to \mathbf{k}$, $x \mapsto \langle \dot{x}, N^{e-1} \dot{x} \rangle$ is not identically zero hence $\dim V/X^{\geqslant 1-e} = 1$. Since for $x \in \mathrm{gr}^{1-e} X^*$ we have $\dot{x}, N^{e-1} \dot{x} \rangle = 0$ we see that $X^{\geqslant 1-e} \subset \ker \lambda = L^{\perp}$. Since $\dim X^{\geqslant 1-e} = \dim L^{\perp}$ we see that $X^{\geqslant 1-e} = L^{\perp}$. We have $X^{\geqslant e} = (X^{\geqslant 1-e})^{\perp} \cap Q^{-1}(0) = L$ as required.

Next we assume that $\lambda = 0$. Then $\mathcal{K}_{e-1}$, the kernel of $\nu^a : V/X^{\geqslant 2-e} \to X^{\geqslant e-1}$, is

$$\{x \in V/X^{2-e} \mid N^{e-1} \dot{x} \in X^{\geqslant e} = 0\} = \{x \in V/X^{2-e} \mid N^{e-1} \dot{x} = 0\}$$

$$= \ker N^{e-1}/X^{\geqslant 2-e}.$$

If $\epsilon_{e-1}^N = 0$ then $\langle \dot{x}, N^{e-2} \dot{x} \rangle = 0$ for x in this kernel, that is, for $x \in \mathcal{K}_{e-1}$. Hence in this case we have $\mathcal{K}_{e-1} = 0$. Thus N^{e-1} induces an isomorphism $V/X^{\geqslant 2-e} \to X^{\geqslant e-1}$ so that $X^{\geqslant e-1} = N^{e-1}V, X^{\geqslant 2-e} = \ker N^{e-1}$, as required. We have $X^{\geqslant 2-e} = (X^{\geqslant e-1})^{\perp} = (V^{\geqslant e-1})^{\perp} = V^{\geqslant 2-e}$ as required.

Now assume that $\lambda = 0, \epsilon_{e-1}^N = 1$. In this case we have an isomorphism $\ker N^{e-1}/X^{2-e} \to \mathbf{k}$ induced by λ_{e-1}^N, that is, we have

$$X^{\geqslant 2-e} = \{x \in \ker N^{e-1} \mid \lambda_{e-1}^N x = 0\}.$$

This is the same as $(\ker N^{e-1} \cap W) \oplus L'^{\vdash} = V^{\geq 2-e}$, see 2.8. Taking $\perp$ in $X^{\geq 2-e} = V^{\geq 2-e}$ and intersecting with $Q^{-1}(0)$ we obtain $X^{\geq e-1} = V^{\geq e-1}$.

If $\lambda = 0, a \in [2-e, e-1]$ we have $N^{e-1}V \subset X^{\geq a} \subset \ker N^{e-1}$ and we denote by $X'^{\geq a}$ the image of $X^{\geq a}$ under $\rho : \ker N^{e-1} \to V' = \ker N^{e-1}/N^{e-1}V$. For $a \leq 1-e$ we set $X'^{\geq a} = V'$ and for $a \geq e$ we set $X'^{\geq a} = 0$. Now $(X'^{\geq a})_{a \in \mathbf{Z}}$ is an N'-adapted filtration of V'. (We must only show the analogue of (ii) and (iii) for N' with $a = e-1$. We have $X'^{\geq 1-e}/X'^{\geq 2-e} = \ker N^{e-1}/X^{\geq 2-e}$; this is $Y/L'^{\vdash}$ if $\epsilon' = \epsilon - 1, \lambda' \neq 0$ and 0 otherwise. We have $X'^{\geq e-1}/X'^{\geq e} = X^{\geq e-1}/N^{e-1}V$. This is L' if $\epsilon' = \epsilon - 1, \lambda' \neq 0$ and 0 otherwise. Hence (ii) and (iii) are obvious in this case.) By the induction hypothesis we have $X'^{\geq a} = V'^{\geq a}_{N'}$ for all a. Taking the inverse image under ρ we see that for $a \in [2-e, e-1]$ we have $X^{\geq a} = V^{\geq a}$.

If $\lambda \neq 0, a \in [1-e, e]$ we have $L \subset X^{\geq a} \subset L^{\perp}$ and we denote by $X'^{\geq a}$ the image of $X^{\geq a}$ under $\rho : L^{\perp} \to V' = L^{\perp}/L$. For $a \leq -e$ we set $X'^{\geq a} = V'$ and for $a \geq e+1$ we set $X'^{\geq a} = 0$. Now $(X'^{\geq a})_{a \in \mathbf{Z}}$ is an N'-adapted filtration of V'. (We have $X'^{\geq -e}/X'^{\geq 1-e} = 0$, $X'^{\geq e}/X'^{\geq e+1} = 0$.) By the induction hypothesis we have $X'^{\geq a} = V'^{\geq a}_{N'}$ for all a. Taking the inverse image under ρ we see that for $a \in [1-e, e]$ we have $X^{\geq a} = V^{\geq a}$. This completes the proof of (a). $\square$

2.11. Assume that $V \in \mathcal{C}''$. If $g \in \mathrm{O}_Q$ and $S \in \mathcal{J}_Q$, we set $\delta_g = \dim(S/(S \cap g(S)))$ mod 2. It is known that δ_g is independent of the choice of S and that $\delta_g = 0$ if and only if $g \in \mathrm{SO}_Q$.

Let $N \in \widetilde{\mathcal{M}}_Q$. We show that:

(a) $\delta_{1+N} = \dim \ker N$ mod 2.

If $N = 0$ this is clear. Assume now that $N \neq 0$. Let $e = e_N, L = L_N$. We have $e \geq 2$. Let λ, L be as in 2.2. We may assume that $V \in \mathcal{C}''$. Assume first that $e \geq 3$. As in the proof of 2.2(e) we have $Q|_L = 0$. We set $V' = L^{\perp}/L$. The nondegenerate quadratic form $Q' : V' \to \mathbf{k}$ can be defined as in 2.3. The nilpotent endomorphism $N' : V' \to V'$ induced by N belongs to $\widetilde{\mathcal{M}}_{Q'}$. As in the proof of 2.3(a) we see that $\dim \ker N' = \dim \ker N$ mod 2. We have $\delta_{1+N} = \delta_{1+N'}$. Since the result may be assumed to hold for N' we see that (a) holds. We now assume that $e = 2$, that is, $N^2 = 0, N \neq 0$.

Assume that $\lambda \neq 0$. We can find $x \in V$ such that $\langle x, Nx \rangle \neq 0$. Then x, Nx span a two-dimensional N-stable subspace P of V on which $\langle \, , \, \rangle$ is nondegenerate. Let $V' = P^{\perp}$ and let $Q' = Q|_{V'}$. Then $V = P \oplus V'$, Q' is nondegenerate and N restricts to a nilpotent map $N' : V' \to V'$. Note that $\dim \ker N = \dim \ker N' + 1$ and $\delta_{1+N} = \delta_{1+N'} + 1$ mod 2. Since the result may be assumed to hold for N' we see that (a) holds.

Assume that $\lambda = 0$. We write $V = W \oplus Y$ as in 1.9(c). Then W, Y are N-stable nondegenerate even-dimensional subspaces of V with $\langle W, Y \rangle = 0$; moreover, $NY = 0$. Hence $\dim \ker N = \dim \ker(N|_W) + \dim Y$, $\delta_{1+N} = \delta_{1+N|_W}$. If $Y \neq 0$ we may assume that the result holds for $N|_W$; we see that (a) holds. Thus we may assume that $Y = 0$. We have $V = E \oplus NE$ with E as in 1.9(a). Note that $\dim NE = D/2$, see 1.9(b). Since $\lambda = 0$ we have $Q|_{NE} = 0$. Clearly NE is $1 + N$ stable. We see that $\delta_{1+N} = 0$. The nondegenerate symmetric bilinear form on E described in 1.9(a) is symplectic since $\lambda = 0$. Hence $\dim E \in \mathbf{Z}''$ and $\dim NE \in \mathbf{Z}''$. We see that $\dim \ker N \in \mathbf{Z}''$. Thus (a) holds.

From (a) we deduce that for $N \in \widetilde{\mathcal{M}}_Q$ we have $N \in \mathcal{M}_Q$ if and only if $1 + N \in \mathrm{SO}_Q$.

2.12. We prove Theorem 1.7 (with $p = 2$) in the form 1.7(a). Let $T \in \mathrm{SO}_Q$ be unipotent. Then $N = T - 1 \in \mathrm{O}_Q$ and by 2.11 we have $N = T - 1 \in \mathcal{M}_Q$. In 2.5 we have attached to N a Q-filtration $X^* = (V_N^{\geqslant a})$ of V. In 2.7 and 2.9 we have shown that $N \in E_*^{\geqslant 2} X^*$. In 2.10 we have shown that the last property determines X^* uniquely. Thus 1.7(a) is established. $\square$

3. The case $p \neq 2$

3.1. In this section we prove Theorem 1.7 assuming that $p \neq 2$. In this case we have $R = 0$.

Let $N \in \widetilde{\mathcal{M}}_Q - \{0\}, e = e_N$. We have $e \geqslant 2$. Let $L = L_N = N^{e-1}V$, a subspace of V. By 1.1(b) we have $L^\perp = \ker N^{e-1}$. Since $2e - 2 \geqslant e$ we have $N^{e-1}V \subset \ker N^{e-1}$ hence:

(a) $L \subset L^\perp$.

Clearly,

(b) $NL = 0, NV \subset L^\perp$.

We show:

(c) $Q|_L = 0$.

Let $v \in V$. We have $q(N^{e-1}v) = \langle N^{e-2}v, N^{e-1}v \rangle = \pm \langle v, N^{2e-3}v \rangle$. If $e \geqslant 3$ this is 0 since $2e - 3 \geqslant e$. If $e = 2$ this is 0 since $\langle v, Nv \rangle = -\langle Nv, v \rangle$ so that $2\langle v, Nv \rangle = 0$ and $\langle v, Nv \rangle = 0$.

3.2. Let $N \in \widetilde{\mathcal{M}}_Q - \{0\}, e = e_N, L = L_N$. We have $e \geqslant 2$. We set $V' = L^\perp/L$, see 3.1(a). From 3.1(b) we see that N induces a (nilpotent) endomorphism of V', denoted by N'. Let $e' = e_{N'}$. We have $e' \leqslant e - 1$. Define a quadratic form $Q' : V' \to \mathbf{k}$ by $Q'(x') = Q(x)$ where x is a representative of $x' \in V'$ in $L^\perp$. (To see that Q' is well defined we use 3.1(c).) The symmetric bilinear form associated to Q' is $\langle x', y' \rangle' = \langle x, y \rangle$ where $x', y' \in V'$ and x, y are representatives of x', y' in $L^\perp$. Its radical is $\{x \in L^\perp \mid \langle x, L^\perp \rangle = 0\}/L = 0$. It follows that Q' is nondegenerate. We have $N' \in \widetilde{\mathcal{M}}_{Q'}$.

3.3. Let $N \in \widetilde{\mathcal{M}}_Q, e = e_N$. We associate to N a collection of subspaces

$$(V^{\geqslant a})_{a \in \mathbf{Z}} = (V_N^{\geqslant a})_{a \in \mathbf{Z}}$$

of V using induction on D. If $N = 0$ we set $V^{\geqslant a} = V$ for $a \leqslant 0$, $V^{\geqslant a} = 0$ for $a \geqslant 1$. Thus $V^{\geqslant a}$ are defined when $D \leqslant 1$. We may assume that $D \geqslant 2$, $N \neq 0$ and that $V^{\geqslant a}$ are already defined when V is replaced by a vector space of dimension $< D$. We have $e \geqslant 2$. Let L, V', N', Q' be as in 3.1 and 3.2. Let $\rho : L^\perp \to V'$ be the obvious map. Since $L \neq 0$, we have $\dim V' < D$. By the induction hypothesis, $V'^{\geqslant a} = V_{N'}'^{\geqslant a}$ is defined for any $a \in \mathbf{Z}$. We set

$$V^{\geqslant a} = V \text{ for } a \leqslant 1 - e; \quad V^{\geqslant a} = \rho^{-1}(V'^{\geqslant a}) \text{ for } a \in [2 - e, e - 1]; \quad V^{\geqslant a} = 0 \text{ for } a \geqslant e.$$

This completes the definition of the subspaces $V^{\geqslant a}$.

From the definition it is clear that $(V^{\geqslant a})$ is a filtration of V. We show:

(a) *If $a \geqslant 1$ we have $Q|_{V^{\geqslant a}} = 0$ and $V^{\geqslant 1-a} = (V^{\geqslant a})^{\perp}$.*

We use induction on D. For $N = 0$ the result is obvious. Thus the result holds when $D \leqslant 1$. Now assume that $D \geqslant 2$, $N \neq 0$. Let L, V', Q', N' be as above. By the induction hypothesis we have $Q'|_{V'^{\geqslant a}_{N'}} = 0$ and $V'^{\geqslant 1-a}_{N'} = (V'^{\geqslant a}_{N'})^{\perp'}$ with $\perp'$ relative to V'. It follows that $Q|_{V^{\geqslant a}} = 0$ and $V^{\geqslant 1-a} = (V^{\geqslant a})^{\perp}$ for $a \in [1, e-1]$. For $a \geqslant e$ the result is again true since $Q|_{\{0\}} = 0$ and $\{0\}^{\perp} = V$. This proves (a). $\square$

We see that $V^* = (V^{\geqslant a})$ is a Q-filtration of V. Clearly, $V^{\geqslant a}$ is the same as the subspace $V^N_{\geqslant a}$ defined in terms of the nilpotent endomorphism $N : V \to V$ (without reference to Q) in [L2, 2.3, 2.4]. It follows that $NV^{\geqslant a} \subset V^{\geqslant a+2}$ for any $a \in \mathbf{Z}$. For any $a \in \mathbf{Z}$ we set $\bar{V}^a = V^{\geqslant a}/V^{\geqslant a+1}$. We see that N induces a linear map $\bar{N} : \bar{V}^a \to \bar{V}^{a+2}$. From [L2, 2.3] we see that for any $a \geqslant 0$, $\bar{N}^a : \bar{V}^{-a} \to \bar{V}^{-a}$ is an isomorphism. It follows that $N \in E^{\geqslant 2}_* V^*$. Conversely if X^* is a Q-filtration of V such that $N \in E^{\geqslant 2}_* X^*$ we see that for any $a \geqslant 0$ the kernel $\mathcal{K}_a$ of the map $\mathrm{gr}^{-a} X^* \to \mathrm{gr}^a X^*$ induced by N^a is the radical of the symmetric bilinear form attached to a nondegenerate quadratic form on $\mathrm{gr}^{-a} X^*$; since $p \neq 2$ it follows that $\mathcal{K}_a = 0$. Hence the map $\mathrm{gr}^{-a} X^* \to \mathrm{gr}^a X^*$ induced by N^a is an isomorphism. Using [L2, 2.4] it follows that $X^* = V^*$. Thus 1.7(a) holds. $\square$

4. On unipotent conjugacy classes in SO_Q ($p = 2$)

4.1. In this section we assume that $\mathbf{k}$ is algebraically closed. Moreover, in this and the next subsection, we assume that $p = 2$. Assume that $D \geqslant 2$. Let $\phi \in \widetilde{\mathcal{F}}^D_1$. (See 1.6.) Thus $\phi = (f_a)$ where $f_0 > 0$. Let

$$\mathfrak{X}_\phi = \{i \in \mathbf{Z}'_{\geqslant 1} \mid i = f_a \text{ for some } a \in \mathbf{Z}''_{\geqslant 0}\},$$

this is a finite set. Let $\mathcal{E}_\phi$ be the set of all subsets of $\mathfrak{X}_\phi \times \mathfrak{X}_\phi$ which are equivalence relations on $\mathfrak{X}_\phi$. To any $X^* \in \bar{\mathcal{Y}}_\phi$ (see 1.6) and any $\bar{N} \in E^2_* \mathrm{gr} X^*$ we associate an element $S \in \mathcal{E}_\phi$ as follows. For any $i \in \mathfrak{X}_\phi$ let Z_i be the subspace of $\mathrm{gr}^0 X^*$ given by the image of the embedding $\mathrm{gr}^{-a/2} X^* \to \mathrm{gr}^0 X^*$ induced by $\bar{N}^{a/2}$ for some/any $a \in \mathbf{Z}''_{\geqslant 0}$ such that $i = f_a$; the natural symplectic form $\langle \, , \, \rangle$ on $\mathrm{gr}^0 X^*$ restricts to a symplectic form on Z_i with one-dimensional radical denoted by L_i. By definition, $S = \{(i,j) \in \mathfrak{X}_\phi \times \mathfrak{X}_\phi \mid L_i = L_j\}$. For $X^* \in \bar{\mathcal{Y}}_\phi$ and $S \in \mathcal{E}_\phi$ let $E^2_* \mathrm{gr}^* X^*_S$ be the set of all $\bar{N} \in E^2_* \mathrm{gr}^* X^*$ such that $(X^*, \bar{N})$ give rise to S as above. Let $E^{\geqslant 2} X^*_S = \Phi^{-1}(E^2_* \mathrm{gr}^* X^*_S)$ where Φ is as in 1.5. We thus obtain a partition

$$E^{\geqslant 2}_* X^* = \bigsqcup_{S \in \mathcal{E}_\phi} E^{\geqslant 2}_* X^*_S$$

into finitely many locally closed subvarieties. For $S \in \mathcal{E}_\phi$ let

$$\Xi^S_\phi = \{g \in \mathrm{SO}_Q \mid g \text{ unipotent } g - 1 \in E^{\geqslant 2}_* X^*_S\}.$$

Hence Ξ_ϕ (see 1.7) is partitioned as $\Xi_\phi = \bigsqcup_{S \in \mathcal{E}_\phi} \Xi^S_\phi$. Note that each Ξ^S_ϕ is stable under conjugation by SO_Q.

4.2. Let $\mathcal{N}_Q$ be the set of nilpotent elements $\nabla \in \operatorname{End}(V)$ such that $\langle x, \nabla y \rangle + \langle \nabla x, y \rangle = 0$ for all x, y in V and $\langle x, \nabla x \rangle = 0$ for all $x \in V$. We can view $\mathcal{N}_Q$ as the set of nilpotent elements in the Lie algebra of SO_Q. Note that SO_Q acts by conjugation on $\mathcal{N}_Q$.

Let u be a unipotent element in SO_Q. We associate to u an SO_Q-conjugacy class in $\mathcal{N}_Q$ as follows. Let $N = u - 1 \in \mathcal{M}_Q$. Let $X^* = V_N^*$. Then $N \in E^{\geqslant 2} X^*$ and $\bar{N} \in E^2 \operatorname{gr} X^*$ is defined as in 1.5. Let $\bar{Q}$ be the quadratic form on $\operatorname{gr} X^*$ defined in 1.5. We have $\bar{N} \in \mathcal{N}_{\bar{Q}}$. Note that if $D \in \mathbf{Z}''$ then the set of connected components of $\mathcal{J}_Q$ and that of $\mathcal{J}_{\bar{Q}}$ may be naturally identified. We can find an isomorphism of vector spaces $\operatorname{gr} X^* \xrightarrow{\sim} V$ which carries $\bar{Q}$ to Q and (when $D \in \mathbf{Z}''$) induces the identity map from the set of connected components of $\mathcal{J}_{\bar{Q}}$ to that of $\mathcal{J}_Q$. This isomorphism carries $\bar{N}$ to an element $\nabla \in \mathcal{N}_Q$ whose SO_Q-orbit is independent of the choice of isomorphism. Note that $1 + N \mapsto \nabla$ defines a map

$$\{\operatorname{SO}_Q - \text{conjugacy classes of unipotent elements in } \operatorname{SO}_Q\}$$
$$\rightarrow \{\operatorname{SO}_Q - \text{conjugacy classes of nilpotent elements in Lie } \operatorname{SO}_Q\}.$$

One can show that this map is injective; it is not in general surjective.

4.3. The last map also makes sense in the more general framework of [L2]. Assume that $p > 1$. Let G be as in [L2, 0.1]. We assume that property $\mathfrak{P}_1$ in [L2, 1.1] holds for G. Let u be a unipotent element in G. By $\mathfrak{P}_1$ we can find a unique sequence $\triangle = (G_0^\triangle \supset G_1^\triangle \supset G_2^\triangle \supset \cdots)$ in D_G such that $u \in X^\triangle$ (notation of [L2, 1.1]). In particular we have $u \in G_2^\triangle$. Let $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{g}_n^\triangle = \operatorname{Lie} G_n^\triangle$. Let $\bar{u}$ be the image of u in $G_2^\triangle / G_3^\triangle$. Since $G_2^\triangle / G_3^\triangle$ is a connected commutative unipotent group in which the pth power of any element is 1 we see that it is canonically isomorphic to its Lie algebra $\mathfrak{g}_2^\triangle / \mathfrak{g}_3^\triangle$. Hence $\bar{u}$ can be identified with an element of $\mathfrak{g}_2^\triangle / \mathfrak{g}_3^\triangle$. By definition there exists a homomorphism of algebraic groups $h : \mathbf{k}^* \rightarrow G$ such that if we denote by $\mathfrak{g}^n$ the n-eigenspace of the action $x \mapsto \operatorname{Ad}(h(x))$ of $\mathbf{k}^*$ on $\mathfrak{g}$ ($n \in \mathbf{Z}$) we have $\mathfrak{g}_n^\triangle = \mathfrak{g}^n \oplus \mathfrak{g}^{n+1} \oplus \cdots$ for any $n \geqslant 0$; moreover, h is unique up to $G_0^\triangle$-conjugacy. Using the decomposition $\mathfrak{g}_2^\triangle = \mathfrak{g}^2 \oplus \mathfrak{g}_3^\triangle$ we can identify $\mathfrak{g}_2^\triangle / \mathfrak{g}_3^\triangle$ with $\mathfrak{g}^2$ and we can view $\bar{u}$ as an element of $\mathfrak{g}^2$ hence as a nilpotent element of $\mathfrak{g}$. This element is well defined up to $G_0^\triangle$-conjugacy. This defines a map

$$\{\text{unipotent } G\text{-conjugacy classes in } G\} \rightarrow \{\text{nilpotent } G\text{-conjugacy classes in } \mathfrak{g}\}.$$

We expect that this map is always injective.

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