

HERMITIAN STRUCTURES ON SIX-DIMENSIONAL NILMANIFOLDS

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Abstract. Let (J, g) be a Hermitian structure on a six-dimensional compact nilmanifold M with invariant complex structure J and compatible metric g , which is not required to be invariant. We show that, up to equivalence of the complex structure, the strong Kähler with torsion structures (J, g) on M are parametrized by the points in a subset of the Euclidean space, in particular, the region inside a certain ovaloid corresponds to such structures on the Iwasawa manifold and the region outside to strong Kähler with torsion structures with nonabelian J on the nilmanifold $\Gamma \backslash (H^3 \times H^3)$, where H^3 is the Heisenberg group. A classification of six-dimensional nilmanifolds admitting balanced Hermitian structures (J, g) is given, and as an application we classify the nilmanifolds having invariant complex structures which do not admit Hermitian structure with restricted holonomy of the Bismut connection contained in $SU(3)$. It is also shown that on the nilmanifold $\Gamma \backslash (H^3 \times H^3)$ the balanced condition is not stable under small deformations. Finally, we prove that a compact quotient of $H(2, 1) \times \mathbb{R}$, where $H(2, 1)$ is the five-dimensional generalized Heisenberg group, is the only six-dimensional nilmanifold having locally conformal Kähler metrics, and the complex structures underlying such metrics are all equivalent. Moreover, this nilmanifold is a Vaisman manifold for any invariant locally conformal Kähler metric.

Introduction

Let (J, g) be a Hermitian structure on a manifold M , with fundamental 2-form Ω and Lee form θ . The 3-form $Jd\Omega$ can be identified with the torsion of the Bismut connection, i.e., the unique Hermitian connection with totally skew-symmetric torsion [Bi], [Ga], and when $Jd\Omega$ is closed and nonzero (which excludes the Kähler case) the Hermitian structure is called *strong Kähler with torsion* (SKT for short) [AI], [FPS]. Such structures arise in a natural way in physics in the context of supersymmetric σ -models, and in general metric connections with totally skew-symmetric torsion have also applications in type II string theory and black hole moduli spaces (see [P] and references therein).

When the Lee form θ vanishes identically the Hermitian structure is called *balanced*,

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and such structures constitute the class \mathcal{W}_3 in the well known Gray–Hervella classification [GH]. A recent result by Fino and Grantcharov [FG] states that for any compact complex manifold (M, J) with holomorphically trivial canonical bundle, the existence of a balanced structure (J, g) is a necessary condition for the existence of a J -Hermitian metric on M with vanishing Ricci tensor of its Bismut connection (see also [AI], [FPS] for related results).

A Hermitian structure (J, g) is said to be *locally conformal Kähler* (LCK for short) if g is conformal to some local Kähler metric in a neighborhood of each point of M . LCK structures correspond to the Gray–Hervella class \mathcal{W}_4 , and in dimension ≥ 6 they are characterized by the condition $d\Omega = \theta \wedge \Omega$.

Let M be a compact Hermitian non-Kähler manifold of dimension $2n \geq 6$. Then the SKT, balanced, and LCK conditions are complementary to each other. In fact, it is well known that a Kähler metric can be defined as a Hermitian structure in $\mathcal{W}_3 \cap \mathcal{W}_4$. Moreover, Alexandrov and Ivanov prove in [AI] that $d\Omega \neq \theta \wedge \Omega$ if the Hermitian structure is SKT (the compactness of M is only needed here), and a Hermitian structure can only be SKT if $\theta \neq 0$ (see also [FPS]).

In this paper we study SKT, balanced, and LCK geometries on six-dimensional compact nilmanifolds $\Gamma \backslash G$ whose underlying complex structure is invariant, that is, G is a simply connected nilpotent Lie group having a discrete subgroup Γ such that the quotient $\Gamma \backslash G$ is compact, and the complex structure on $\Gamma \backslash G$ stems from a left-invariant one on the Lie group G .

We first observe that such study can be reduced to the particular case when the metric is also invariant. This is shown in [FG] for balanced structures using the “symmetrization” process, which is based on a previous idea of Belgun [Be], and we prove that it also holds for SKT and LCK structures on nilmanifolds (see Propositions 21 and 34). A second reduction comes from the fact that the study of SKT, balanced, and LCK structures can be carried out up to equivalence of the complex structure. Therefore, we can restrict our attention to Hermitian structures at the level of the Lie algebra of G and consider just one representative in each equivalence class of complex structures. Moreover, in Section 1 we prove that in dimension six any invariant complex structure J is equivalent to a complex structure defined by one of two special types of reduced equations, depending on the “nilpotency” of J in the sense of [CFGU2].

Salamon proves in [S] that, up to isomorphism, there are exactly eighteen nilpotent Lie algebras of dimension 6 admitting complex structure, which we shall denote here by \mathfrak{h}_k ($1 \leq k \leq 16$), \mathfrak{h}_{19}^- , and \mathfrak{h}_{26}^+ (see Theorem 8 for details). For instance, the nilpotent Lie algebra \mathfrak{h}_2 is the Lie algebra of $H^3 \times H^3$, where H^3 is the Heisenberg group, \mathfrak{h}_3 is the Lie algebra of $H(2, 1) \times \mathbb{R}$, $H(2, 1)$ being the five-dimensional generalized Heisenberg group, \mathfrak{h}_5 is the Lie algebra underlying the Iwasawa manifold, and \mathfrak{h}_8 is the Lie algebra of $H^3 \times \mathbb{R}^3$. In Section 1 it is shown that any complex structure on \mathfrak{h}_k is nilpotent for $1 \leq k \leq 16$, whereas any complex structure on \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+ is of nonnilpotent type. Since the structure equations of each one of these Lie algebras are rational, their corresponding simply connected nilpotent Lie groups have a discrete subgroup for which the quotient is compact [Ma].

Fino, Parton, and Salamon prove in [FPS] that a six-dimensional compact nilmanifold $\Gamma \backslash G$ admits an invariant SKT structure if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_2 , \mathfrak{h}_4 , \mathfrak{h}_5 , or \mathfrak{h}_8 . In Section 2 we prove that the same classification is valid if we do

not require invariance of the metric. We also obtain a more reduced form of the SKT condition, given in [FPS], which allows us to show that the space of SKT structures on a six-dimensional nilmanifold can be parametrized, up to equivalence of the complex structure, by the points in a region of the Euclidean 3-space. More concretely, when the complex structure is not abelian, there is an ovaloid of revolution in the Euclidean space such that the region inside corresponds to SKT structures on the Iwasawa manifold, the region outside to SKT structures on $\Gamma \backslash (H^3 \times H^3)$, and the points on the ovaloid to SKT structures on the nilmanifold with underlying Lie algebra \mathfrak{h}_4 .

A large class of balanced structures is provided by the compact complex parallelizable manifolds M . In fact, any invariant compatible metric on M is balanced [AG], and this property allows us to show in Section 3 that in dimension ≥ 6 such manifolds possess no SKT metrics. We also prove that a compact nilmanifold $\Gamma \backslash G$ of dimension 6 admits a balanced metric compatible with an invariant complex structure if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_{19}^- or \mathfrak{h}_k for some $1 \leq k \leq 6$. Fino and Grantcharov construct in [FG] a family J_t of invariant complex structures on the Iwasawa manifold not admitting balanced metrics, except for the natural complex structure J_0 . Using their above-mentioned result, this family allows them to conclude that for $t \neq 0$ the complex structure J_t does not admit a Hermitian metric whose Bismut connection has restricted holonomy in $SU(3)$, providing counterexamples to a conjecture in [GIP] as well as the nonstability of this property under small deformations. We show that the general situation for six-dimensional compact nilmanifolds $\Gamma \backslash G$ is the following: there exists an invariant complex structure on $\Gamma \backslash G$ not admitting a Hermitian metric whose Bismut connection has restricted holonomy in $SU(3)$ if and only if the Lie algebra of G is not isomorphic to \mathfrak{h}_1 , \mathfrak{h}_6 , or \mathfrak{h}_{19}^- . It is also shown that on the nilmanifold $\Gamma \backslash (H^3 \times H^3)$ the balanced condition is not stable under small deformations.

Section 4 is devoted to LCK geometry on compact nilmanifolds of dimension 6. We prove that such a nilmanifold $\Gamma \backslash G$ admits an LCK metric compatible with an invariant complex structure if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_1 or \mathfrak{h}_3 , that is, apart from the torus, $\Gamma \backslash (H(2, 1) \times \mathbb{R})$ is the only six-dimensional nilmanifold having LCK structures. It is also shown that the complex structures underlying such LCK metrics are all equivalent. Moreover, the Lee form of any invariant LCK metric is parallel with respect to the Levi-Civita connection, so the nilmanifold becomes a Vaisman manifold. As a consequence, the only nontoral five-dimensional nilmanifold admitting an invariant Sasakian structure is a compact quotient of $H(2, 1)$.

1. Invariant complex structures on six-dimensional nilmanifolds

In this paper we deal with compact complex nilmanifolds $(M = \Gamma \backslash G, J)$ endowed with an *invariant* complex structure J , that is, G is a simply connected nilpotent Lie group and Γ a lattice in G of maximal rank, and J stems from a left invariant integrable almost complex structure on G . Since the structure is invariant, we can restrict our attention to the level of the nilpotent Lie algebra \mathfrak{g} of G .

Let \mathfrak{g} be a Lie algebra. An endomorphism $J: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $J^2 = -\text{Id}$ is said to be *integrable* if it satisfies the ‘‘Nijenhuis condition’’

$$[JX, JY] = J[JX, Y] + J[X, JY] + [X, Y]$$

for any $X, Y \in \mathfrak{g}$. In this case we shall say that J is a *complex structure* on \mathfrak{g} .

Let us denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of \mathfrak{g} and by $\mathfrak{g}_{\mathbb{C}}^*$ its dual, which is canonically identified to $(\mathfrak{g}^*)_{\mathbb{C}}$. Given an endomorphism $J: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $J^2 = -\text{Id}$, there is a natural bigraduation induced on the complexified exterior algebra $\bigwedge^* \mathfrak{g}_{\mathbb{C}}^* = \bigoplus_{p,q} \bigwedge^{p,q}(\mathfrak{g}^*)$, where the spaces $\bigwedge^{1,0}(\mathfrak{g}^*)$ and $\bigwedge^{0,1}(\mathfrak{g}^*)$, which we shall also denote by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, are the eigenspaces of the eigenvalues $\pm i$ of J as an endomorphism of $\mathfrak{g}_{\mathbb{C}}^*$, respectively.

Let $d: \bigwedge^* \mathfrak{g}_{\mathbb{C}}^* \rightarrow \bigwedge^{*+1} \mathfrak{g}_{\mathbb{C}}^*$ be the extension to the complexified exterior algebra of the usual Chevalley–Eilenberg differential. It is well known that J is integrable if and only if $\pi_{0,2} \circ d|_{\mathfrak{g}^{1,0}} \equiv 0$, where $\pi_{p,q}: \bigwedge^{p+q} \mathfrak{g}_{\mathbb{C}}^* \rightarrow \bigwedge^{p,q}(\mathfrak{g}^*)$ denotes the canonical projection onto the subspace of forms of type (p, q) .

Next we shall focus on *nilpotent* Lie algebras (NLA for short), that is, the *descending central series* $\{\mathfrak{g}^k\}_{k \geq 0}$ of \mathfrak{g} , which is defined inductively by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \quad k \geq 1,$$

satisfies that $\mathfrak{g}^k = 0$ for some k . If s is the first positive integer with this property, then the NLA \mathfrak{g} is said to be *s-step* nilpotent.

Salamon proves in [S] the following equivalent condition for the integrability of J on a $2n$ -dimensional NLA: J is a complex structure on \mathfrak{g} if and only if $\mathfrak{g}^{1,0}$ has a basis $\{\omega^j\}_{j=1}^n$ such that $d\omega^1 = 0$ and

$$d\omega^j \in \mathcal{I}(\omega^1, \dots, \omega^{j-1}) \quad \text{for } j = 2, \dots, n,$$

where $\mathcal{I}(\omega^1, \dots, \omega^{j-1})$ is the ideal in $\bigwedge^* \mathfrak{g}_{\mathbb{C}}^*$ generated by $\{\omega^1, \dots, \omega^{j-1}\}$.

In particular, Salamon’s condition in six dimensions is equivalent to the existence of a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = A_{12} \omega^{12} + A_{13} \omega^{13} + A_{1\bar{1}} \omega^{1\bar{1}} + A_{1\bar{2}} \omega^{1\bar{2}} + A_{1\bar{3}} \omega^{1\bar{3}}, \\ d\omega^3 = B_{12} \omega^{12} + B_{13} \omega^{13} + B_{1\bar{1}} \omega^{1\bar{1}} + B_{1\bar{2}} \omega^{1\bar{2}} + B_{1\bar{3}} \omega^{1\bar{3}} \\ \quad + B_{23} \omega^{23} + B_{2\bar{1}} \omega^{2\bar{1}} + B_{2\bar{2}} \omega^{2\bar{2}} + B_{2\bar{3}} \omega^{2\bar{3}}, \end{cases} \tag{1}$$

for some complex coefficients A ’s and B ’s. Here ω^{jk} (resp. $\omega^{j\bar{k}}$) means the wedge product $\omega^j \wedge \omega^k$ (resp. $\omega^j \wedge \omega^{\bar{k}}$), where $\omega^{\bar{k}}$ indicates the complex conjugation of ω^k . From now on, we shall use a similar abbreviated notation for “basic” forms of arbitrary bidegree.

1.1. Reduced form of complex structure equations

Next we show that there are two special and disjoint types of complex equations, and that the generic structure equations (1) can always be reduced to one of them, depending on the “nilpotency” of the complex structure.

A complex structure J on a $2n$ -dimensional NLA \mathfrak{g} is called *nilpotent* if there is a basis $\{\omega^j\}_{j=1}^n$ for $\mathfrak{g}^{1,0}$ satisfying $d\omega^1 = 0$ and

$$d\omega^j \in \bigwedge^2 \langle \omega^1, \dots, \omega^{j-1}, \omega^{\bar{1}}, \dots, \omega^{\bar{j-1}} \rangle \quad \text{for } j = 2, \dots, n. \tag{2}$$

Equivalently [CFGU2], the *ascending series* $\{\mathfrak{g}_l^J\}_{l \geq 0}$ for \mathfrak{g} adapted to J , which is defined inductively by $\mathfrak{g}_0^J = 0$ and

$$\mathfrak{g}_l^J = \{X \in \mathfrak{g} \mid [J^k(X), \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J, k = 1, 2\} \text{ for } l \geq 1,$$

satisfies that $\mathfrak{g}_l^J = \mathfrak{g}$ for some positive integer l .

Equations (1) encode information not only about the complex structure J , but also about the structure of the nilpotent Lie algebra \mathfrak{g} itself. Therefore, the coefficients A 's and B 's in (1) must satisfy those compatibility conditions imposed by the Jacobi identity of the Lie bracket $[\cdot, \cdot]$ on \mathfrak{g} (which is equivalent to $d \circ d \equiv 0$), as well as those conditions ensuring the nilpotency of \mathfrak{g} . For instance, if $\{Z_j\}_{j=1}^3$ denotes the dual basis of $\{\omega^j\}_{j=1}^3$, then iterating the bracket $[Z_2, Z_3]$ by Z_2 it is clear that B_{23} must vanish in order for the Lie algebra \mathfrak{g} be nilpotent. The following result is derived by imposing these necessary compatibility conditions and it establishes a first reduction of the generic equations.

Lemma 1. *Let J be a complex structure on an NLA \mathfrak{g} of dimension 6.*

- (a) *If J is nonnilpotent, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying (1) with $A_{1\bar{2}} = B_{1\bar{3}} = B_{23} = B_{2\bar{2}} = B_{2\bar{3}} = 0$, and $A_{1\bar{3}} \neq 0$.*
- (b) *If J is nilpotent, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying (1), where the only nonvanishing coefficients are among $A_{1\bar{1}}, B_{12}, B_{1\bar{1}}, B_{1\bar{2}}, B_{2\bar{1}},$ and $B_{2\bar{2}}$.*

The detailed proof of (a) is given in [CFU2, Lemma 2.1, Props. 2.2 and 2.4]. Part (b) is a direct consequence of (2).

In the following result we give a more reduced form of the equations for nonnilpotent as well as for nilpotent complex structures.

Proposition 2. *Let J be a complex structure on an NLA \mathfrak{g} of dimension 6.*

- (a) *If J is nonnilpotent, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ such that*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = E\omega^{1\bar{3}} + \omega^{1\bar{3}}, \\ d\omega^3 = A\omega^{1\bar{1}} + ib\omega^{1\bar{2}} - ib\bar{E}\omega^{2\bar{1}}, \end{cases} \quad (3)$$

where $A, E \in \mathbb{C}$ with $|E| = 1$, and $b \in \mathbb{R} - \{0\}$.

- (b) *If J is nilpotent, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \epsilon\omega^{1\bar{1}}, \\ d\omega^3 = \rho\omega^{1\bar{2}} + (1 - \epsilon)A\omega^{1\bar{1}} + B\omega^{1\bar{2}} + C\omega^{2\bar{1}} + (1 - \epsilon)D\omega^{2\bar{2}}, \end{cases} \quad (4)$$

where $A, B, C, D \in \mathbb{C}$, and $\epsilon, \rho \in \{0, 1\}$.

Proof. Let us suppose first that J is nonnilpotent. From Lemma 1(a) we have that $A_{1\bar{2}} = B_{1\bar{3}} = B_{23} = B_{2\bar{2}} = B_{2\bar{3}} = 0$ and $A_{1\bar{3}} \neq 0$ in Equations (1) for some $(1, 0)$ -basis $\{\omega^j\}$. The remaining coefficients must guarantee the nilpotency of \mathfrak{g} and the Jacobi identity $d(d\omega^j) = 0$.

Since $0 = d(d\omega^2) \wedge \omega^{2\bar{3}} = -A_{1\bar{3}}\bar{B}_{12}\omega^{1\bar{2}\bar{3}}$, the coefficient B_{12} must be zero. Moreover, from $0 = d(d\omega^2) \wedge \omega^{2\bar{3}\bar{2}} = A_{1\bar{3}}\bar{B}_{13}\omega^{1\bar{2}\bar{3}\bar{2}}$ it follows that B_{13} also vanishes. Now,

the nilpotency of \mathfrak{g} implies that $A_{12} = 0$, because otherwise $[Z_1, \dots, [Z_1, [Z_1, Z_2]] \dots] = (-A_{12})^k Z_2$ would be a nonzero element in \mathfrak{g}^k for any k . In addition, if we consider the new $(1, 0)$ -basis given by $\tau^1 = \omega^1$, $\tau^2 = \omega^2$, $\tau^3 = \bar{A}_{1\bar{1}} \omega^1 + \bar{A}_{1\bar{3}} \omega^3$, then we can suppose $A_{1\bar{3}} = 1$ and $A_{1\bar{1}} = 0$.

Therefore, there is a basis $\{\omega^j\}$ of $\mathfrak{g}^{1,0}$ satisfying (1), where $A_{1\bar{3}} = 1$ and the remaining nonvanishing coefficients are among A_{13} , $B_{1\bar{1}}$, $B_{1\bar{2}}$, and $B_{2\bar{1}}$. Now, since

$$d(d\omega^2) = (\bar{B}_{1\bar{2}} - A_{13}B_{2\bar{1}}) \omega^{12\bar{1}} \quad \text{and} \quad d(d\omega^3) = (A_{13}B_{2\bar{1}} + B_{1\bar{2}}) \omega^{13\bar{1}} - (\bar{A}_{13}B_{1\bar{2}} + B_{2\bar{1}}) \omega^{1\bar{1}3},$$

the Jacobi identity implies that the following conditions hold:

$$A_{13}B_{2\bar{1}} = \bar{B}_{1\bar{2}} = -B_{1\bar{2}} \quad \text{and} \quad A_{13}\bar{B}_{1\bar{2}} + \bar{B}_{2\bar{1}} = 0.$$

In particular, $B_{1\bar{2}} = ib$ for some $b \in \mathbb{R}$. Notice that $b \neq 0$ because otherwise $B_{1\bar{2}}$ and $B_{2\bar{1}}$ would be zero and the complex structure J should be nilpotent (it suffices to interchange ω^2 with ω^3). Finally, these conditions also imply that $|A_{13}| = 1$, so part (a) of the proposition is proved.

In order to prove part (b), if J is a nilpotent complex structure then Lemma 1(b) implies the existence of a $(1, 0)$ -basis $\{\omega^j\}$ satisfying (1), where all the coefficients A 's vanish except possibly $A_{1\bar{1}}$, and $B_{13} = B_{1\bar{3}} = B_{23} = B_{2\bar{3}} = 0$. Notice that in this case $(d \circ d)\omega^j = 0$ for $j = 1, 2$. Since $(d \circ d)\omega^3 = B_{2\bar{2}}(-\bar{A}_{1\bar{1}}\omega^{12\bar{1}} + A_{1\bar{1}}\omega^{1\bar{1}2})$, the Jacobi identity of the Lie bracket implies that $A_{1\bar{1}}B_{2\bar{2}} = 0$.

Now, if $A_{1\bar{1}} \neq 0$, then $B_{2\bar{2}} = 0$, and we can suppose $A_{1\bar{1}} = 1$ and $B_{1\bar{1}} = 0$ after considering the change of basis $\tau^1 = \omega^1$, $\tau^2 = (1/A_{1\bar{1}})\omega^2$, and $\tau^3 = A_{1\bar{1}}\omega^3 - B_{1\bar{1}}\omega^2$. Finally, notice that if the coefficient of τ^{12} in $d\tau^3$ is nonzero, then we can normalize it. \square

For any election of coefficients on the right-hand side of equations (3) (resp., (4)), it is natural to ask whether the resulting equations are ‘‘admissible’’ in the sense that there exists a nonnilpotent (resp., nilpotent) complex structure J on some six-dimensional NLA \mathfrak{g} having these equations with respect to some $(1, 0)$ -basis. Next we give an affirmative answer to this question, but first we reformulate it in more precise terms.

Let V be a real vector space of dimension $2n$, and denote by $V_{\mathbb{C}}^*$ the dual of the complexification of V . Let us fix a basis $\{\omega^j, \bar{\omega}^j\}_{j=1}^n$ for $V_{\mathbb{C}}^*$, where $\bar{\omega}^j$ denotes the complex conjugate of ω^j . This is equivalent to give an endomorphism $J: V \rightarrow V$ such that $J^2 = -\text{Id}_V$, with respect to which the space $V_{\mathbb{C}}^*$ decomposes as $V_{\mathbb{C}}^* = V^{1,0} \oplus V^{0,1}$, where $V^{1,0} = \langle \omega^j \rangle$ and $V^{0,1} = \langle \bar{\omega}^j \rangle$ are the eigenspaces of the eigenvalues $\pm i$ of the extended endomorphism $J: V_{\mathbb{C}}^* \rightarrow V_{\mathbb{C}}^*$, respectively. Notice that if $\{X_j, Y_j\}$ is the basis of V dual to the basis $\{\alpha^j = \frac{1}{2}\Re\omega^j, \beta^j = \frac{1}{2}\Im\omega^j\}$ of V^* , then the endomorphism J is given by $JX_j = Y_j$ for $j = 1, \dots, n$.

Fixed an n -tuple $\mu = (\mu^1, \dots, \mu^n) \in \bigwedge^2 V_{\mathbb{C}}^* \times \dots \times \bigwedge^2 V_{\mathbb{C}}^*$, we consider the linear mapping $d_{\mu}: V_{\mathbb{C}}^* \rightarrow \bigwedge^2 V_{\mathbb{C}}^*$ defined by $d_{\mu}\omega^j = \mu^j$ and $d_{\mu}\bar{\omega}^j = \bar{\mu}^j$ for $j = 1, \dots, n$, and we extend it to the complexified exterior algebra using the formula $d_{\mu}(\alpha \wedge \beta) = d_{\mu}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_{\mu}\beta$ for $\alpha, \beta \in \bigwedge^* V_{\mathbb{C}}^*$. Let $[\cdot, \cdot]_{\mu}: V \times V \rightarrow V$ be the bracket on V defined by

$$[X, Y]_{\mu} = - \sum_{j=1}^n (\mu^j(X, Y)Z_j + \bar{\mu}^j(X, Y)\bar{Z}_j)$$

for $X, Y \in V$, where $\{Z_j, \bar{Z}_j\}$ is the dual basis of $\{\omega^j, \bar{\omega}^j\}$.

We introduce the following notation: $d_\mu(\mu) \equiv (d_\mu\mu^1, \dots, d_\mu\mu^n)$ and $\mu_{0,2} \equiv (\pi_{0,2}(\mu^1), \dots, \pi_{0,2}(\mu^n))$, where $\pi_{0,2}: \bigwedge^2 V_{\mathbb{C}}^* \rightarrow \bigwedge^{0,2}(V^*)$ is the canonical projection onto the subspace of elements of type $(0, 2)$.

Lemma 3. *Let V be a real vector space of dimension $2n$, and fix a basis $\{\omega^j, \bar{\omega}^j\}_{j=1}^n$ of $V_{\mathbb{C}}^*$. Given an n -tuple $\mu \in \bigwedge^2 V_{\mathbb{C}}^* \times \dots \times \bigwedge^2 V_{\mathbb{C}}^*$, we define J , d_μ , and $[\cdot, \cdot]_\mu$ as above.*

- (a) *If $d_\mu(\mu) = 0$, then $\mathfrak{g}_\mu = (V, [\cdot, \cdot]_\mu)$ is a Lie algebra.*
- (b) *If, in addition, $\mu_{0,2} = 0$, then J is a complex structure on \mathfrak{g}_μ .*

Proof. From the definitions we have $\omega^j([X, Y]_\mu) = -d_\mu\omega^j(X, Y)$. Now (a) is clear because the bracket $[\cdot, \cdot]_\mu$ satisfies the Jacobi identity if and only if $d_\mu(d_\mu\omega^j) = 0$ for $j = 1, \dots, n$, that is, $d_\mu(\mu) = 0$. To see (b), just notice that the Nijenhuis condition is equivalent to the vanishing of the $(0, 2)$ -type component in $d_\mu\omega^j = \mu^j$ for $j = 1, \dots, n$. \square

In general, the Lie algebra \mathfrak{g}_μ may not be nilpotent. For example, if we consider a 3-tuple $\mu = (d\omega^1, d\omega^2, d\omega^3)$ given by (1) and satisfying $d_\mu(\mu) = 0$, then it determines a Lie algebra \mathfrak{g}_μ for which the endomorphism J above is a complex structure, however, \mathfrak{g}_μ cannot be nilpotent if $B_{23} \neq 0$.

Next we show that for any μ given by (3) or (4), we always obtain a nilpotent Lie algebra \mathfrak{g}_μ . Thus, the following result can be considered as the converse to Proposition 2.

Proposition 4. *In the conditions of Lemma 3 we have:*

- (a) *If $\mu = (0, E\omega^{13} + \omega^{1\bar{3}}, A\omega^{1\bar{1}} + ib\omega^{12} - ib\bar{E}\omega^{2\bar{1}})$ with $A, E \in \mathbb{C}$, $|E| = 1$, and $b \in \mathbb{R} - \{0\}$, then \mathfrak{g}_μ is an NLA and J is a nonnilpotent complex structure on \mathfrak{g}_μ .*
- (b) *If $\mu = (0, \epsilon\omega^{1\bar{1}}, \rho\omega^{12} + (1 - \epsilon)A\omega^{1\bar{1}} + B\omega^{1\bar{2}} + C\omega^{2\bar{1}} + (1 - \epsilon)D\omega^{2\bar{2}})$ with $A, B, C, D \in \mathbb{C}$ and $\epsilon, \rho \in \{0, 1\}$, then \mathfrak{g}_μ is an NLA and J is a nilpotent complex structure on \mathfrak{g}_μ .*

Proof. First, let μ be given as in (a). It is easy to check that $d_\mu(\mu) = 0$, so the Jacobi identity holds for the bracket $[\cdot, \cdot]_\mu$. In terms of the complex basis $\{Z_j, \bar{Z}_j\}$ dual to $\{\omega^j, \bar{\omega}^j\}$, this bracket is given by

$$\begin{aligned} [Z_1, Z_3]_\mu &= -E Z_2, \\ [Z_1, \bar{Z}_3]_\mu &= -Z_2, \\ [Z_1, \bar{Z}_2]_\mu &= -ib(Z_3 - E\bar{Z}_3), \\ [Z_1, \bar{Z}_1]_\mu &= -A Z_3 + \bar{A}\bar{Z}_3, \end{aligned}$$

and their complex conjugates. Therefore, if $E \neq 1$, then the derived algebra $(\mathfrak{g}_\mu)^1 = [V, V]_\mu$ is contained in the space $\langle \Re(Z_2), \Im(Z_2), (1 - \bar{E})(Z_3 - E\bar{Z}_3), i(A Z_3 - \bar{A}\bar{Z}_3) \rangle$. Notice that the element $(1 - \bar{E})(Z_3 - E\bar{Z}_3)$ is in the center of \mathfrak{g}_μ and that it is a multiple of $i(A Z_3 - \bar{A}\bar{Z}_3)$ if and only if $\bar{A} = AE$. Thus,

$$\begin{aligned} (\mathfrak{g}_\mu)^2 &= [[V, V]_\mu, V]_\mu \subseteq \langle \Re(Z_2), \Im(Z_2), (1 - \bar{E})(Z_3 - E\bar{Z}_3) \rangle, \\ (\mathfrak{g}_\mu)^3 &= [[[V, V]_\mu, V]_\mu, V]_\mu \subseteq \langle (1 - \bar{E})(Z_3 - E\bar{Z}_3) \rangle, \end{aligned}$$

and $(\mathfrak{g}_\mu)^4 = 0$, that is, the Lie algebra \mathfrak{g}_μ is nilpotent in step $s \leq 4$.

When $E = 1$, the elements $i(Z_3 - \bar{Z}_3)$ and $i(AZ_3 - \bar{A}\bar{Z}_3)$ of $[V, V]_\mu$ are linearly dependent if and only if the coefficient A is real. In any case, $i(Z_3 - \bar{Z}_3)$ is a central element and, therefore, if $A \in \mathbb{R}$, then $\mathfrak{g}_\mu^1 = \langle \Re(Z_2), \Im(Z_2), i(Z_3 - \bar{Z}_3) \rangle$, $\mathfrak{g}_\mu^2 = \langle i(Z_3 - \bar{Z}_3) \rangle$, and $\mathfrak{g}_\mu^3 = 0$; if A is not real, then $\mathfrak{g}_\mu^1 = \langle \Re(Z_2), \Im(Z_2), \Re(Z_3), \Im(Z_3) \rangle$ and \mathfrak{g}_μ is 4-step nilpotent.

Finally, the bracket relations above imply that any term in the ascending series $\{(\mathfrak{g}_\mu)_l^J\}_{l \geq 0}$ adapted to J is zero, so the complex structure J is nonnilpotent. This completes the proof of (a).

Now, suppose that μ is given as in (b). Since $d_\mu(\mu) = 0$, the bracket $[\cdot, \cdot]_\mu$ satisfies the Jacobi identity. The Lie algebra $\mathfrak{g}_\mu = (V, [\cdot, \cdot]_\mu)$ is nilpotent in step $s \leq 3$, because $(\mathfrak{g}_\mu)^2 = [[V, V]_\mu, V]_\mu \subseteq \langle \Re Z_3, \Im Z_3 \rangle$, and $\Re Z_3, \Im Z_3$ are central elements of \mathfrak{g}_μ .

The terms in the ascending series $\{(\mathfrak{g}_\mu)_l^J\}_{l \geq 0}$ adapted to J satisfy: $(\mathfrak{g}_\mu)_1^J \supseteq \langle \Re Z_3, \Im Z_3 = -J(\Re Z_3) \rangle$, $(\mathfrak{g}_\mu)_2^J \supseteq \langle \Re Z_2, \Im Z_2 = -J(\Re Z_2), \Re Z_3, \Im Z_3 = -J(\Re Z_3) \rangle$, and $(\mathfrak{g}_\mu)_3^J = \mathfrak{g}_\mu$. Therefore, J is a nilpotent complex structure, and part (b) of the proposition is proved. \square

Remark 1. Let us consider a family of μ 's such that $d_\mu(\mu)$ and $\mu_{0,2}$ vanish.

(a) From Lemma 3 we get a family of Lie algebras $\mathfrak{g}_\mu = (V, [\cdot, \cdot]_\mu)$ on which the endomorphism $J: V \rightarrow V$ (which is independent on μ) is integrable. Let us fix an inner product $\langle \cdot, \cdot \rangle$ on V compatible with J which does not depend on μ . Now, in the case that \mathfrak{g}_μ is nilpotent for each μ , our construction is related to [L], where it is investigated the space of all “nilpotent” Lie brackets $[\cdot, \cdot]_\mu$ for which J is integrable and compatible with $\langle \cdot, \cdot \rangle$, i.e., $(J, \langle \cdot, \cdot \rangle)$ is a fixed Hermitian structure on each NLA $\mathfrak{g}_\mu = (V, [\cdot, \cdot]_\mu)$.

(b) Notice that the Lie algebras \mathfrak{g}_μ might be nonisomorphic to each other. When \mathfrak{g}_μ and $\mathfrak{g}_{\mu'}$ are both isomorphic to a Lie algebra \mathfrak{g} , we can interpret this situation as having two complex structures J_μ and $J_{\mu'}$ on the same Lie algebra \mathfrak{g} .

As a consequence of Propositions 2 and 4 we have the following result.

Theorem 5. *A six-dimensional NLA \mathfrak{g} admits a nonnilpotent (resp., nilpotent) complex structure if and only if there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying (3) (resp., (4)).*

1.2. Classification of NLAs admitting complex structure

Next we show that a six-dimensional NLA cannot support nilpotent and nonnilpotent complex structures at the same time, and then we classify the NLAs according to the nilpotency of the complex structures that they admit.

Proposition 6. *Let \mathfrak{g} be an NLA of dimension 6 having a nonnilpotent complex structure. Then, the center of \mathfrak{g} is one-dimensional.*

Proof. From Proposition 2(a), there is a $(1, 0)$ -basis $\{\omega^j\}_{j=1}^3$ with reduced equations (3). Then, in terms of its dual basis $\{Z_j\}$, any central element T of \mathfrak{g} is expressed as $T = \sum_{j=1}^3 (\lambda_j Z_j + \bar{\lambda}_j \bar{Z}_j)$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. A direct calculation shows that $0 = [T, Z_3] = -E\lambda_1 Z_2 - \bar{\lambda}_1 \bar{Z}_2$, which implies $\lambda_1 = 0$. Moreover,

$$0 = [T, Z_1] = (E\lambda_3 + \bar{\lambda}_3)Z_2 + ib\bar{\lambda}_2 Z_3 - ibE\bar{\lambda}_2 \bar{Z}_3.$$

Thus $\lambda_2 = 0$, because $b \neq 0$, and $\bar{\lambda}_3 = -E\lambda_3$. Therefore, $T = \lambda_3 Z_3 - E\lambda_3 \bar{Z}_3 = \lambda_3(Z_3 - E\bar{Z}_3)$. If $E = 1$, then $T = i\lambda(Z_3 - \bar{Z}_3)$, $\lambda \in \mathbb{R}$. If $E \neq 1$, then $T = \lambda(1 - \bar{E})(Z_3 - E\bar{Z}_3)$, $\lambda \in \mathbb{R}$, because $|E| = 1$. Thus, we conclude that in any case the center of \mathfrak{g} is one-dimensional. \square

Corollary 7. *Let \mathfrak{g} be a six-dimensional NLA admitting complex structures. Then all of them are either nilpotent or nonnilpotent.*

Proof. If \mathfrak{g} has a nilpotent complex structure J , then the first term \mathfrak{g}_1^J in the ascending series for \mathfrak{g} adapted to J is nonzero. By definition, \mathfrak{g}_1^J is a J -invariant ideal of \mathfrak{g} contained in the center, so if \mathfrak{g} has a nilpotent J , then its center is at least two-dimensional. From Proposition 6 it follows that \mathfrak{g} has no nonnilpotent complex structures. Thus, any complex structure on \mathfrak{g} must be nilpotent. \square

Remark 2. Proposition 6 and Corollary 7 do not hold in higher dimension. In fact, in [CFGU2] it is given a ten-dimensional NLA with center of dimension 2 having both nilpotent and nonnilpotent complex structures.

A complex structure J satisfying $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$, is obviously nilpotent and is called *abelian*, because $\mathfrak{g}^{1,0}$ is an abelian complex Lie algebra. It is easily seen that abelian complex structures correspond to the case $\rho = 0$ in the reduced equations (4).

The following result gives a classification of six-dimensional NLAs in terms of the different types of complex structures that they admit.

Theorem 8. *Let \mathfrak{g} be an NLA of dimension 6. Then, \mathfrak{g} has a complex structure if and only if it is isomorphic to one of the following Lie algebras:¹*

$$\begin{array}{ll}
 \mathfrak{h}_1 = (0, 0, 0, 0, 0, 0), & \mathfrak{h}_{10} = (0, 0, 0, 12, 13, 14), \\
 \mathfrak{h}_2 = (0, 0, 0, 0, 12, 34), & \mathfrak{h}_{11} = (0, 0, 0, 12, 13, 14 + 23), \\
 \mathfrak{h}_3 = (0, 0, 0, 0, 0, 12 + 34), & \mathfrak{h}_{12} = (0, 0, 0, 12, 13, 24), \\
 \mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23), & \mathfrak{h}_{13} = (0, 0, 0, 12, 13 + 14, 24), \\
 \mathfrak{h}_5 = (0, 0, 0, 0, 13 + 42, 14 + 23), & \mathfrak{h}_{14} = (0, 0, 0, 12, 14, 13 + 42), \\
 \mathfrak{h}_6 = (0, 0, 0, 0, 12, 13), & \mathfrak{h}_{15} = (0, 0, 0, 12, 13 + 42, 14 + 23), \\
 \mathfrak{h}_7 = (0, 0, 0, 12, 13, 23), & \mathfrak{h}_{16} = (0, 0, 0, 12, 14, 24), \\
 \mathfrak{h}_8 = (0, 0, 0, 0, 0, 12), & \mathfrak{h}_{19}^- = (0, 0, 0, 12, 23, 14 - 35), \\
 \mathfrak{h}_9 = (0, 0, 0, 0, 12, 14 + 25), & \mathfrak{h}_{26}^+ = (0, 0, 12, 13, 23, 14 + 25).
 \end{array}$$

Moreover:

- (a) *Any complex structure on \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+ is nonnilpotent.*
- (b) *For $1 \leq k \leq 16$, any complex structure on \mathfrak{h}_k is nilpotent.*
- (c) *Any complex structure on \mathfrak{h}_1 , \mathfrak{h}_3 , \mathfrak{h}_8 , and \mathfrak{h}_9 is abelian.*
- (d) *There exist both abelian and nonabelian nilpotent complex structures on \mathfrak{h}_2 , \mathfrak{h}_4 , \mathfrak{h}_5 and \mathfrak{h}_{15} .*
- (e) *Any complex structure on \mathfrak{h}_6 , \mathfrak{h}_7 , \mathfrak{h}_{10} , \mathfrak{h}_{11} , \mathfrak{h}_{12} , \mathfrak{h}_{13} , \mathfrak{h}_{14} , and \mathfrak{h}_{16} is not abelian.*

Proof. Salamon proves in [S] that \mathfrak{g} has a complex structure J if and only if it is isomorphic to one of the Lie algebras appearing in the list above. Now, using Proposition 6 we have that a nonnilpotent J can only live on \mathfrak{h}_{19}^- or \mathfrak{h}_{26}^+ , because the center of these NLAs is one-dimensional. Corollary 7 implies that any J on \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+ is nonnilpotent, and (a) is proved.

¹Here we use a mixed notation combining the structure description of the NLAs as it appears in [S] and the notation \mathfrak{h}_k in [CFGU1]. For instance, $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$ means that there is a basis $\{\alpha^j\}_{j=1}^6$ such that the Chevalley–Eilenberg differential is given by $d\alpha^1 = d\alpha^2 = d\alpha^3 = d\alpha^4 = 0$, $d\alpha^5 = \alpha^1 \wedge \alpha^2$, $d\alpha^6 = \alpha^3 \wedge \alpha^4$; equivalently, the Lie bracket is given in terms of its dual basis $\{X_j\}_{j=1}^6$ by $[X_1, X_2] = -X_5$ and $[X_3, X_4] = -X_6$.

In [CFGU1] it is shown that if J is nilpotent, then \mathfrak{g} must be isomorphic to \mathfrak{h}_k for some $1 \leq k \leq 16$. By Corollary 7, any complex structure on \mathfrak{h}_k , $1 \leq k \leq 16$, is nilpotent, so (b) is proved.

In order to prove (c), notice that if $\rho \neq 0$ in equations (4), then the first Betti number $\dim(\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}])$ of \mathfrak{h} is equal to 3 or 4 depending on $\epsilon = 1$ or $\epsilon = 0$, respectively. Since the Lie algebras \mathfrak{h}_3 and \mathfrak{h}_8 have first Betti number equal to 5, any complex structure must be abelian. On the other hand, since the Lie algebra \mathfrak{h}_9 is 3-step and its complex structures are all nilpotent, the coefficient ϵ in (4) must be equal to 1: in fact, $\epsilon = 0$ implies that the Lie algebra is nilpotent in step ≤ 2 . Therefore, $\rho = 0$ because the first Betti number of \mathfrak{h}_9 is equal to 4, so (c) is proved.

In [CFU1] it is proved that a six-dimensional nilpotent Lie algebra admits an abelian J if and only if it is isomorphic to \mathfrak{h}_k for $k = 1, 2, 3, 4, 5, 8, 9$, or 15. This proves (e).

Finally, to see (d) we observe that the equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{2}} + C\omega^{2\bar{1}},$$

define, in the sense of Proposition 4(b), a nilpotent complex structure on \mathfrak{h}_2 for $C = 1$, and on \mathfrak{h}_4 for $C = 2$. On the other hand, the equations

$$d\omega^1 = 0, \quad d\omega^2 = \epsilon\omega^{1\bar{1}}, \quad d\omega^3 = \omega^{12},$$

define a nilpotent complex structure on \mathfrak{h}_5 for $\epsilon = 0$, and on \mathfrak{h}_{15} for $\epsilon = 1$. Since in each case the coefficient of ω^{12} in $d\omega^3$ is nonzero, the complex structures are not abelian. This, together with the fact that \mathfrak{h}_k has abelian complex structures for $k = 2, 4, 5$, and 15, proves (d) and so the proof of the theorem is complete. \square

Remark 3. If \mathfrak{g} is a complex Lie algebra, then its canonical complex structure J satisfies $[JX, Y] = J[X, Y]$ for all $X, Y \in \mathfrak{g}$. Any complex structure J on an NLA \mathfrak{g} satisfying this condition is obviously nilpotent. Moreover, $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*)$, so in dimension 6 the corresponding equations are of the form (4) with $\rho = 0, 1$ and all the remaining coefficients are equal to zero. Therefore, these complex structures only live on the abelian Lie algebra \mathfrak{h}_1 and on the Lie algebra \mathfrak{h}_5 underlying the Iwasawa manifold. We shall refer to them as *complex parallelizable* structures, because the corresponding complex nilmanifolds possess three holomorphic 1-forms which are linearly independent at each point.

Remark 4. The deformation of abelian complex structures on 2-step nilmanifolds is studied in [MPPS], where it is proved that the Kuranishi process preserves the invariance of the deformed complex structures, at least for small deformations. Conditions under which the deformed structures remain abelian are also investigated there. In this context, it follows from Theorem 8 that in dimension 6 all the complex structures obtained by such small deformations are always of nilpotent type. Recently, the result of [MPPS] has been generalized to any nilmanifold with abelian complex structure in [CFP], where it is also given an example of an abelian complex structure on a ten-dimensional 3-step nilmanifold which deforms into nonnilpotent complex structures.

As a consequence of Theorem 8 we find reduced complex structure equations for the Lie algebras \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+ .

Proposition 9. *For any complex structure on \mathfrak{h}_{19}^- (resp., on \mathfrak{h}_{26}^+), there is a $(1, 0)$ -basis satisfying (3) with $\bar{A} = AE$ (resp., $\bar{A} \neq AE$).*

Proof. Since any complex structure on \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+ is nonnilpotent, there exists a $(1, 0)$ -basis satisfying (3). But an NLA \mathfrak{g} defined by (3) is isomorphic to \mathfrak{h}_{19}^- if and only if its first Betti number $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ is equal to 3, which is equivalent to the closedness of the real 1-form $i(1 - \bar{E})(E\omega^3 + \omega^{\bar{3}})$. This latter condition is satisfied if and only if $\bar{A} = AE$. \square

1.3. Complex structure equations on 2-step NLAs

Here we shall arrive at more reduced equations which describe any complex structure on each 2-step NLA.

Proposition 10. *Let \mathfrak{g} be a six-dimensional NLA endowed with a nilpotent complex structure J . Then, the coefficient ϵ vanishes in the reduced equations (4) corresponding to J if and only if the Lie algebra \mathfrak{g} is nilpotent in step $s \leq 2$ and its first Betti number is ≥ 4 . In this case, \mathfrak{g} must be isomorphic to \mathfrak{h}_8 or \mathfrak{h}_k for some $1 \leq k \leq 6$.*

Proof. Clearly, $\epsilon = 0$ in (4) implies that \mathfrak{g} is nilpotent in step ≤ 2 and $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \geq 4$.

Suppose that the Lie algebra \mathfrak{g} has first Betti number ≥ 4 and is nilpotent in step $s \leq 2$. Let (4) be equations corresponding to J on \mathfrak{g} , and suppose that $\epsilon = 1$. First, the coefficient ρ must vanish, because otherwise the first Betti number would be 3. Moreover, if B and C are not both zero, then $BC \neq 0$ in order to have the first Betti number at least 4. Now, if $\{Z_j\}$ is the dual basis of $\{\omega^j\}$, then the element $\Im Z_2 \in [\mathfrak{g}, \mathfrak{g}]$ satisfies $[\Im Z_2, \mathfrak{g}] \neq 0$, that is, the Lie algebra is not nilpotent in step $s \leq 2$. Therefore, if $\epsilon = 1$, then $B = C = 0$, but in such a case we can choose $\epsilon = 0$ after interchanging ω^2 with ω^3 .

Finally, if \mathfrak{g} has first Betti number ≥ 4 , then Theorem 8 implies that \mathfrak{g} cannot be isomorphic to \mathfrak{h}_7 or \mathfrak{h}_k for any $k \geq 10$. On the other hand, \mathfrak{h}_9 is 3-step nilpotent, so \mathfrak{g} cannot be isomorphic to \mathfrak{h}_9 if $\epsilon = 0$. \square

The following lemma provides a further reduction of the equations on 2-step NLAs.

Lemma 11. *Let J be a complex structure on a 2-step NLA \mathfrak{g} of dimension 6 with first Betti number ≥ 4 . If J is not complex parallelizable, then there is a basis $\{\omega^j\}_{j=1}^3$ of $\mathfrak{g}^{1,0}$ such that*

$$\begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \rho\omega^{12} + \omega^{1\bar{1}} + B\omega^{1\bar{2}} + D\omega^{2\bar{2}}, \end{cases} \quad (5)$$

where $B, D \in \mathbb{C}$, and $\rho = 0, 1$.

Proof. First, by the preceding proposition we can suppose $\epsilon = 0$ in the reduced equations (4) corresponding to J . Next, we distinguish several cases depending on the vanishing of the coefficients A and D .

If $A \neq 0$, then we consider the change of basis given by $\omega^1 = \omega'^1 - C\omega'^2$, $\omega^2 = A\omega'^2$, $\omega^3 = A\omega'^3$. It is easy to check that with respect to the new $(1, 0)$ -basis $\{\omega'^j\}$ the equations become

$$d\omega'^1 = d\omega'^2 = 0, \quad d\omega'^3 = \rho\omega'^{12} + \omega'^{1\bar{1}} + B'\omega'^{1\bar{2}} + D'\omega'^{2\bar{2}}, \quad (6)$$

where $B' = (\bar{A}B - A\bar{C})/A$ and $D' = \bar{A}(AD - BC)/A$.

The case $D \neq 0$ is reduced to the previous one if we interchange ω^1 with ω^2 , and change the sign of ω^3 . Notice that in this case we get (6) with $B' = (\bar{B}D - C\bar{D})/D$, and $D' = \bar{D}(AD - BC)/D$.

Let us suppose $A = D = 0$ in equations (4). The change of basis given by $\omega^1 = \omega''^1 + \omega''^2$, $\omega^2 = \omega''^1 - \omega''^2$, $\omega^3 = -2\omega''^3$, transforms (4) into equations of the form

$$d\omega''^1 = d\omega''^2 = 0, \quad d\omega''^3 = \rho\omega''^{12} + A''\omega''^{1\bar{1}} + B''\omega''^{1\bar{2}} + C''\omega''^{2\bar{1}} + D''\omega''^{2\bar{2}},$$

where $D'' = -A'' = (B + C)/2$ and $B'' = -C'' = (B - C)/2$. Therefore, if $B + C \neq 0$, then we can again reduce these equations to the form (6) with $B' = (|B|^2 - |C|^2)/(B + C)$ and $D' = -BC(\bar{B} + \bar{C})/(B + C)$. Finally, if $A = D = B + C = 0$, then using the change of basis given by $\omega^1 = \omega''^1 + i\omega''^2$, $\omega^2 = i\omega''^1 + \omega''^2$, and $\omega^3 = 2\omega''^3$, we arrive at equations of the form

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\omega'^{12} + iB(\omega'^{1\bar{1}} - \omega'^{2\bar{2}}).$$

Now, if J is not complex parallelizable, then the coefficient $B \neq 0$ and we can apply the argument used in the case “ $A \neq 0$ ” above to get equations of the form (6), with $B' = 0$ and $D' = -|B|^2$. \square

Lemma 12. *Let J be a complex structure on an NLA \mathfrak{g} with reduced equations (5). Then, the dimension of the center of \mathfrak{g} is ≥ 3 if and only if $|B| = \rho$ and $D = 0$.*

Proof. Let $\{Z_j\}$ be the dual basis of $\{\omega^j\}$. From equations (5) it is clear that $\Re Z_3$ and $\Im Z_3$ belong to the center of \mathfrak{g} . Now, if $T = \sum_{j=1}^2 (\lambda_j Z_j + \bar{\lambda}_j \bar{Z}_j)$ is a central element in \mathfrak{g} for some $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, then the condition

$$0 = [T, Z_1] = (\rho\lambda_2 + \bar{\lambda}_1 + B\bar{\lambda}_2)Z_3 - \bar{\lambda}_1\bar{Z}_3$$

implies that λ_1 must be zero. In addition, there is a solution $\lambda_2 \neq 0$ of the equation $B\bar{\lambda}_2 + \rho\lambda_2 = 0$ if and only if $|B| = \rho$. Moreover, the condition $0 = [T, Z_2] = D\bar{\lambda}_2 Z_3 - \bar{D}\bar{\lambda}_2 \bar{Z}_3$ implies that $D = 0$ if $\lambda_2 \neq 0$. Therefore, there is an element T in the center of \mathfrak{g} such that $\{\Re Z_3, \Im Z_3, T\}$ are linearly independent if and only if $|B| = \rho$ and $D = 0$. \square

We finish this section with a general result showing which are, in the sense of Proposition 4(b), the NLAs underlying the reduced equations (5) in terms of the coefficients ρ , B , and D .

Proposition 13. *Let J be a complex structure on an NLA \mathfrak{g} given by (5), and let us denote $x = \Re D$ and $y = \Im D$. Then:*

- (i) *If $|B| = \rho$, then the Lie algebra \mathfrak{g} is isomorphic to:*
 - (i.1) \mathfrak{h}_2 for $y \neq 0$;
 - (i.2) \mathfrak{h}_3 for $\rho = y = 0$ and $x \neq 0$;
 - (i.3) \mathfrak{h}_4 for $\rho = 1$, $y = 0$ and $x \neq 0$;
 - (i.4) \mathfrak{h}_6 for $\rho = 1$ and $x = y = 0$;
 - (i.5) \mathfrak{h}_8 for $\rho = x = y = 0$.
- (ii) *If $|B| \neq \rho$, then the Lie algebra \mathfrak{g} is isomorphic to:*
 - (ii.1) \mathfrak{h}_2 for $4y^2 > (\rho - |B|^2)(4x + \rho - |B|^2)$;
 - (ii.2) \mathfrak{h}_4 for $4y^2 = (\rho - |B|^2)(4x + \rho - |B|^2)$;
 - (ii.3) \mathfrak{h}_5 for $4y^2 < (\rho - |B|^2)(4x + \rho - |B|^2)$.

Proof. From Proposition 10, a Lie algebra \mathfrak{g} underlying (5) must be isomorphic to \mathfrak{h}_2 , \mathfrak{h}_3 , \mathfrak{h}_4 , \mathfrak{h}_5 , \mathfrak{h}_6 , or \mathfrak{h}_8 . Notice that the dimension of the center is 4 for \mathfrak{h}_8 , 3 for \mathfrak{h}_6 , and 2 for the rest. The first Betti number is 5 for \mathfrak{h}_3 and \mathfrak{h}_8 , and 4 for \mathfrak{h}_2 , \mathfrak{h}_4 , \mathfrak{h}_5 , and \mathfrak{h}_6 .

From Lemma 12, \mathfrak{g} is isomorphic to \mathfrak{h}_6 or \mathfrak{h}_8 if and only if $|B| = \rho$ and $D = 0$. Moreover, under these conditions the first Betti number is 4 if $\rho = 1$, and 5 if $\rho = 0$. So, (i.4) and (i.5) are proved.

Notice that \mathfrak{g} has first Betti number equal to 5 if and only if $B = \rho = 0$ and $D \in \mathbb{R}$ in equations (5). Therefore, \mathfrak{g} is isomorphic to \mathfrak{h}_3 for $B = \rho = y = 0$ and $x \neq 0$, which proves (i.2).

For the remaining cases $|B| \neq \rho$, $|B| = \rho$, and $y \neq 0$, or $|B| = \rho = 1, y = 0$, and $x \neq 0$, the NLA \mathfrak{g} has always the two-dimensional center by Lemma 12, and its first Betti number is equal to 4. Therefore, $\mathfrak{g} \cong \mathfrak{h}_2, \mathfrak{h}_4$, or \mathfrak{h}_5 . In order to decide which one is the corresponding Lie algebra in terms of the coefficients ρ , B , and D , we observe the following fact. Let $\alpha(\mathfrak{g})$ be the number of linearly independent elements τ in $\bigwedge^2(\mathfrak{g}^*)$ such that $\tau \in d(\mathfrak{g}^*)$ and $\tau \wedge \tau = 0$. It is straightforward to check that $\alpha(\mathfrak{h}_k)$ for $k = 2, 4, 5$, equals the number of linearly independent exact 2-forms which are decomposable, that is, $\alpha(\mathfrak{h}_2) = 2$, $\alpha(\mathfrak{h}_4) = 1$, and $\alpha(\mathfrak{h}_5) = 0$.

Let $\tau = \lambda d\omega^3 + \mu d\bar{\omega}^3$, where $\lambda, \mu \in \mathbb{C}$, be any exact 2-form on \mathfrak{g} . Since τ is real, $\mu = \bar{\lambda}$ and, therefore,

$$\tau = \rho\lambda\omega^{12} + (\lambda - \bar{\lambda})\omega^{1\bar{1}} + B\lambda\omega^{12} - \bar{B}\bar{\lambda}\omega^{2\bar{1}} + (D\lambda - \bar{D}\bar{\lambda})\omega^{2\bar{2}} + \rho\bar{\lambda}\omega^{1\bar{2}}.$$

A direct calculation shows that

$$\tau \wedge \tau = 2((\rho^2 - |B|^2)|\lambda|^2 - (\lambda - \bar{\lambda})(D\lambda - \bar{D}\bar{\lambda}))\omega^{12\bar{1}\bar{2}}.$$

Thus, if we denote $p = \Re \lambda$ and $q = \Im \lambda$, then $\tau \wedge \tau = 0$ if and only if

$$(\rho - |B|^2)p^2 + 4ypq + (\rho - |B|^2 + 4x)q^2 = 0. \quad (7)$$

If $|B| = \rho$, then (7) becomes $4q(yq + xq) = 0$. Therefore, $\tau_1 = d(\Re \omega^3)$ is an exact 2-form on \mathfrak{g} which is nonzero if $\rho = 1$ or $y \neq 0$, and it satisfies $\tau_1 \wedge \tau_1 = 0$. Moreover, when $\rho = 1, y = 0$, and $x \neq 0$, it follows from (7) that $q = 0$ and any exact 2-form τ satisfying $\tau \wedge \tau = 0$ must be a multiple of τ_1 , thus $\alpha(\mathfrak{g}) = 1$ and \mathfrak{g} is isomorphic to \mathfrak{h}_4 , which proves (i.3). But when $y \neq 0$, the exact 2-form $\tau_2 = -(x/y)d(\Re \omega^3) - d(\Im \omega^3)$ satisfies $\tau_2 \wedge \tau_2 = 0$. Since τ_1, τ_2 are linearly independent, we have that $\alpha(\mathfrak{g}) = 2$ and $\mathfrak{g} \cong \mathfrak{h}_2$, so (i.1) is proved. This completes the proof of (i).

To prove (ii), we consider (7) as a second-degree equation in the variable p . Notice that the discriminant is $\Delta = 4q^2(4y^2 - (\rho - |B|^2)(4x + \rho - |B|^2))$, and that $q \neq 0$ because otherwise (7) reduces to $p = 0$ and therefore λ would be zero. Therefore, if $4y^2 > (\rho - |B|^2)(4x + \rho - |B|^2)$, then $\Delta > 0$ and for each $q \neq 0$ there exist two distinct solutions p_1 and p_2 of (7). In this case we have $\alpha(\mathfrak{g}) = 2$ and therefore the underlying Lie algebra is isomorphic to \mathfrak{h}_2 , which proves (ii.1). A similar argument gives (ii.2) and (ii.3). \square

1.4. Equivalence of complex structures

Let \mathfrak{g} be a Lie algebra endowed with two complex structures J and J' . We recall that J and J' are said to be *equivalent* if there is an automorphism $F: \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra

such that $J' = F^{-1} \circ J \circ F$, that is, F is a linear automorphism such that $F^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ commutes with the Chevalley–Eilenberg differential d and F commutes with the complex structures J and J' . The latter condition is equivalent to saying that F^* , extended to the complexified exterior algebra, preserves the bigraduations induced by J and J' .

It is clear that the nilpotency condition for a complex structure is invariant under equivalence, that is, if J' is equivalent to J , then J is nilpotent if and only if J' is.

Proposition 14. *Any nilpotent (resp., nonnilpotent) complex structure on an NLA of dimension 6 is equivalent to a complex structure defined by (4) (resp., by (3)) in the sense of Proposition 4.*

Proof. Notice that if $\mathfrak{g}_J^{1,0}$ and $\mathfrak{g}_{J'}^{1,0}$ denote the $(1,0)$ -subspaces of $\mathfrak{g}_\mathbb{C}^*$ associated to two complex structures J and J' , then they are equivalent if and only if there is a \mathbb{C} -linear isomorphism $F^* : \mathfrak{g}_J^{1,0} \rightarrow \mathfrak{g}_{J'}^{1,0}$ such that $d \circ F^* = F^* \circ d$. Therefore, the result follows from Proposition 2. \square

Lemma 11 states that any complex structure (not complex-parallelizable) on a 2-step NLA with first Betti number ≥ 4 is equivalent to one defined by (5) in the sense of Proposition 4. Moreover:

Corollary 15. *On the Lie algebras \mathfrak{h}_6 and \mathfrak{h}_8 , any two complex structures are equivalent.*

Proof. From (i.5) in Proposition 13 we have that any complex structure on \mathfrak{h}_8 is equivalent to the one defined by (5) with $\rho = B = D = 0$, and (i.4) shows that any complex structure on \mathfrak{h}_6 is equivalent to one defined by (5) with $\rho = |B| = 1$ and $D = 0$. Since $|B| = 1$ there exists a nonzero λ satisfying $\bar{\lambda}B = \lambda$, and the change of basis given by $\omega'^1 = \lambda\omega^1$, $\omega'^2 = \bar{\lambda}\omega^2$, and $\omega'^3 = |\lambda|^2\omega^3$ allows us to consider the coefficient $B = 1$. \square

Let J_0^+ and J_0^- be the abelian complex structures on the Lie algebra \mathfrak{h}_3 defined by

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}} \pm \omega^{2\bar{2}}.$$

Corollary 16. *Any complex structure on \mathfrak{h}_3 is equivalent to J_0^+ or J_0^- .*

Proof. By Proposition 13(i.2) any complex structure on \mathfrak{h}_3 is equivalent to one defined by (5) with $\rho = B = 0$ and $D \in \mathbb{R} - \{0\}$, and we can normalize D to be 1 or -1 depending on the sign of D . \square

Notice that the orientation induced by J_0^+ is opposite to the one induced by the structure J_0^- .

2. Strong Kähler with torsion geometry in six dimensions

Let (J, g) be a Hermitian structure on a $2n$ -dimensional manifold M , that is, J is a complex structure on M which is orthogonal relative to the Riemannian metric g . We denote by Ω the *fundamental 2-form* of (J, g) , which is defined by $\Omega(X, Y) = g(JX, Y)$ for any differentiable vector fields X, Y on M .

It is well known that the integrability of J produces a decomposition of the exterior differential d of M as $d = \partial + \bar{\partial}$, where $\partial = \pi_{*+1,*} \circ d$ and $\bar{\partial}$ is the conjugate of ∂ . Since $d^2 = 0$, we have $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

A Hermitian structure (J, g) is called *strong Kähler with torsion* (SKT for short) if $\partial\bar{\Omega}$ is a nonzero $\bar{\partial}$ -closed form. In this case, we shall refer to g as an *SKT metric*. Notice that a Hermitian structure (J, g) is SKT if and only if $Jd\Omega$ is nonzero and closed, because $\partial\bar{\partial}$ acts as $\frac{1}{2}i dJd$ on forms of bidegree $(1, 1)$.

Let J be a complex structure on a Lie algebra \mathfrak{g} . An inner product g on \mathfrak{g} such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ will be called a *J-Hermitian metric* on \mathfrak{g} , and we shall refer to the associated Ω as the *fundamental form of the Hermitian structure* (J, g) on \mathfrak{g} .

Since J is integrable on \mathfrak{g} , the extended Chevalley–Eilenberg differential $d: \bigwedge^* \mathfrak{g}_{\mathbb{C}}^* \rightarrow \bigwedge^{*+1} \mathfrak{g}_{\mathbb{C}}^*$ also decomposes as $d = \partial + \bar{\partial}$, where $\partial = \pi_{p+1, q} \circ d: \bigwedge^{p, q}(\mathfrak{g}^*) \rightarrow \bigwedge^{p+1, q}(\mathfrak{g}^*)$ and $\bar{\partial}$ is the conjugate of ∂ . Any J -Hermitian metric g on \mathfrak{g} for which $\partial\Omega$ is a nonzero $\bar{\partial}$ -closed form will be called an *SKT metric* on \mathfrak{g} , and we shall refer to the pair (J, g) as an *SKT structure* on \mathfrak{g} .

If the simply connected nilpotent Lie group G corresponding to an NLA \mathfrak{g} has a discrete subgroup Γ such that $M = \Gamma \backslash G$ is compact, then any Hermitian (resp., SKT) structure (J, g) on \mathfrak{g} will pass to a Hermitian (resp., SKT) structure on the nilmanifold M . Such a structure on M will also be denoted by (J, g) and we shall refer to it as an *invariant Hermitian* (resp., *invariant SKT*) structure on M .

Suppose that the NLA \mathfrak{g} has dimension 6 and fix a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$. Then, in terms of this basis, any J -Hermitian metric g on \mathfrak{g} is expressed as

$$\begin{aligned} g = & r(\omega^1 \otimes \omega^{\bar{1}} + \omega^{\bar{1}} \otimes \omega^1) + s(\omega^2 \otimes \omega^{\bar{2}} + \omega^{\bar{2}} \otimes \omega^2) + t(\omega^3 \otimes \omega^{\bar{3}} + \omega^{\bar{3}} \otimes \omega^3) \\ & - iu(\omega^1 \otimes \omega^{\bar{2}} + \omega^{\bar{2}} \otimes \omega^1) + i\bar{u}(\omega^2 \otimes \omega^{\bar{1}} + \omega^{\bar{1}} \otimes \omega^2) \\ & - iv(\omega^2 \otimes \omega^{\bar{3}} + \omega^{\bar{3}} \otimes \omega^2) + i\bar{v}(\omega^3 \otimes \omega^{\bar{2}} + \omega^{\bar{2}} \otimes \omega^3) \\ & - iz(\omega^1 \otimes \omega^{\bar{3}} + \omega^{\bar{3}} \otimes \omega^1) + i\bar{z}(\omega^3 \otimes \omega^{\bar{1}} + \omega^{\bar{1}} \otimes \omega^3), \end{aligned} \quad (8)$$

where $r, s, t \in \mathbb{R}$ and $u, v, z \in \mathbb{C}$ must satisfy restrictions that guarantee that g is positive definite, i.e., $g(Z, \bar{Z}) > 0$ for any nonzero $Z \in (\mathfrak{g}^{1,0})^*$. Therefore, $r > 0$, $s > 0$, $t > 0$, $rs > |u|^2$, $st > |v|^2$, $rt > |z|^2$, and $rst + 2\Re(i\bar{u}\bar{v}z) > t|u|^2 + r|v|^2 + s|z|^2$.

The fundamental 2-form $\Omega \in \bigwedge^2 \mathfrak{g}^*$ of the Hermitian structure (J, g) is then given by

$$\Omega = i(r\omega^{1\bar{1}} + s\omega^{2\bar{2}} + t\omega^{3\bar{3}}) + u\omega^{1\bar{2}} - \bar{u}\omega^{2\bar{1}} + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}}. \quad (9)$$

The following result is proved by a direct calculation, so we omit the proof.

Lemma 17. *Let (J, g) be a Hermitian structure on a six-dimensional NLA \mathfrak{g} , and Ω its fundamental form.*

- (i) *If J is nonnilpotent, then in terms of the basis $\{\omega^j\}_{j=1}^3$ of $\mathfrak{g}^{1,0}$ satisfying (3), the $(2, 1)$ -form $\partial\Omega$ is given by*

$$\begin{aligned} \partial\Omega = & -(\bar{A}v + ibz)\omega^{12\bar{1}} - ibEv\omega^{12\bar{2}} - (i\bar{A}t - u + E\bar{u})\omega^{13\bar{1}} \\ & + (is + bt)E\omega^{13\bar{2}} + Ev\omega^{13\bar{3}} + (is - bt)\omega^{23\bar{1}}. \end{aligned}$$

- (ii) *If J is nilpotent, then in terms of the basis $\{\omega^j\}_{j=1}^3$ of $\mathfrak{g}^{1,0}$ satisfying (4), the form $\partial\Omega$ is given by*

$$\begin{aligned} \partial\Omega = & - (i\epsilon s + \rho\bar{z} + (1 - \epsilon)\bar{A}v - \bar{B}z)\omega^{12\bar{1}} - (\rho\bar{v} + \bar{C}v - (1 - \epsilon)\bar{D}z)\omega^{12\bar{2}} \\ & + i\rho t\omega^{12\bar{3}} + (\epsilon\bar{v} - i(1 - \epsilon)\bar{A}t)\omega^{13\bar{1}} - i\bar{C}t\omega^{13\bar{2}} - i\bar{B}t\omega^{23\bar{1}} - i(1 - \epsilon)\bar{D}t\omega^{23\bar{2}}. \end{aligned}$$

The theorem below is essentially given by Fino, Parton, and Salamon in [FPS, Theorems 1.2 and 3.2]. Their proof involves a direct but rather long calculation following a decision tree to eliminate $B_{13}, B_{1\bar{3}}, B_{23}, B_{2\bar{3}}$ and the five coefficients A 's in the general equations (1) under the SKT hypothesis. We give a simple proof based on our previous study of complex geometry developed in Section 1, together with the fact that the SKT condition is satisfied up to equivalence of the complex structure. Our proof also illustrates a general procedure that is useful to investigate balanced and locally conformal Kähler geometry, as is shown in the next sections. Notice that part (a) of the following theorem is a slightly stronger version of Theorem 1.2 in [FPS].

Lemma 18. *Let \mathfrak{g} be a Lie algebra endowed with a complex structure J having compatible SKT metrics. Then, any other complex structure J' equivalent to J possesses compatible SKT metrics.*

Proof. Let (J, g) be an SKT structure with fundamental form Ω , and $F \in \text{Aut}(\mathfrak{g})$ an automorphism such that $F \circ J' = J \circ F$. Then, $g' = F^*g$ is a J' -Hermitian metric on \mathfrak{g} with fundamental form $\Omega' = F^*\Omega$. Since F^* commutes with d and preserves the bidegree, we get $\bar{\partial}'\partial'\Omega' = F^*(\bar{\partial}\partial\Omega)$, where $d = \partial' + \bar{\partial}'$ is the decomposition of d with respect to J' . Therefore, $\bar{\partial}\Omega$ is a nonzero $\bar{\partial}$ -closed form if and only if $\partial'\Omega'$ is a nonzero $\bar{\partial}'$ -closed form. \square

Theorem 19. *Let \mathfrak{g} be a six-dimensional NLA admitting complex structures.*

- (a) *A Hermitian structure (J, g) on \mathfrak{g} is SKT if and only if the complex structure J is equivalent to one defined by (5) with*

$$\rho + |B|^2 = 2\Re\epsilon(D). \tag{10}$$

In particular, if (J, g) is an SKT structure, then J is nilpotent and any other J -Hermitian metric on \mathfrak{g} is SKT.

- (b) *There exists an SKT structure on \mathfrak{g} if and only if it is isomorphic to $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5,$ or \mathfrak{h}_8 .*

Proof. To prove (a), we use Lemma 18 and Proposition 14 to focus on the two special types of complex structures defined by (3) and (4). If J is a nonnilpotent complex structure defined by (3), then it follows from Lemma 17(i) that $\bar{\partial}\partial\Omega = 2i(b^2t\omega^{12\bar{1}\bar{2}} + s\omega^{13\bar{1}\bar{3}}) \neq 0$, because g is positive definite and, in particular, $s > 0$. Thus, J must necessarily be nilpotent if it has a compatible SKT metric, so it can be expressed by equations of the form (4). Now, from Lemma 17(ii) we have

$$\bar{\partial}\partial\Omega = it(\rho^2 + |B|^2 + |C|^2 - 2(1 - \epsilon)^2\Re\epsilon(A\bar{D}))\omega^{12\bar{1}\bar{2}}.$$

If $\epsilon = 1$, then we must have $\rho = B = C = 0$ because $t > 0$, so in such a case we can suppose $\epsilon = 0$ after interchanging ω^2 with ω^3 . Also notice that a complex parallelizable structure cannot satisfy the condition $\bar{\partial}\partial\Omega = 0$, unless the NLA \mathfrak{g} be abelian, in which case Ω would be closed. Therefore, we can apply Proposition 10 and Lemma 11 to get the equivalent condition (10).

In order to prove (b), we first observe that Proposition 13 implies that the possible candidates to admit an SKT structure are $\mathfrak{h}_2, \dots, \mathfrak{h}_6$ and \mathfrak{h}_8 . But, from (i.2) and (i.4)

in Proposition 13, it follows that there is no SKT structure either on \mathfrak{h}_3 or on \mathfrak{h}_6 , because (10) is never satisfied. On the other hand, (i.5) shows that any complex structure on \mathfrak{h}_8 has compatible SKT metrics. Finally, the condition (10) for $\rho = 1$ and $B = 0$ in equations (5) reduces to $D = \frac{1}{2} + iy$, so Proposition 13(ii) implies the existence of SKT structures on \mathfrak{h}_2 for $|y| > \sqrt{3}/2$, on \mathfrak{h}_4 for $y = \pm\sqrt{3}/2$, and on \mathfrak{h}_5 for $|y| < \sqrt{3}/2$. \square

Next we describe (a parametrization of) the space of SKT structures in dimension 6 up to equivalence of the underlying complex structure. In view of (a) in the theorem above, we restrict our attention to complex structures J defined by (5) with $B = p+iq, D = x+iy \in \mathbb{C}$, and $\rho = 0, 1$, and satisfying the SKT condition $x = \frac{1}{2}(\rho+p^2+q^2)$. Let us fix $\rho = 0$ or 1 , which is equivalent to requiring that J be abelian or not. Then, the complex structures having compatible SKT metric are parametrized by points (p, q, y) in the Euclidean space \mathbb{R}^3 . Now, given an NLA \mathfrak{g} admitting SKT structures, we shall find which is the region in the Euclidean space that parametrizes the space of complex structures (up to equivalence) on \mathfrak{g} satisfying the SKT condition. For that, we make use of Proposition 13 taking into account that $(\rho - |B|^2)(4x + \rho - |B|^2) = 4\rho - (\rho + p^2 + q^2)^2$ under the SKT assumption:

- First, let us suppose that J is abelian, that is, $\rho = 0$. If $p = q = 0$, then the corresponding Lie algebra is \mathfrak{h}_8 for $y = 0$, and \mathfrak{h}_2 for $y \neq 0$. If $p^2 + q^2 \neq 0$, then $4y^2 + (p^2 + q^2)^2$ is strictly positive, so the corresponding Lie algebra is again \mathfrak{h}_2 . Therefore, the SKT structures (J, g) with abelian J are parametrized by the points in the Euclidean space \mathbb{R}^3 , where the origin corresponds to SKT structures on the Lie algebra \mathfrak{h}_8 and the points in $\mathbb{R}^3 - \{0\}$ to SKT structures on \mathfrak{h}_2 .

- Suppose now that J is nilpotent but nonabelian, i.e., $\rho = 1$. If $p^2 + q^2 = 1$, then the corresponding Lie algebra is \mathfrak{h}_4 for $y = 0$, and \mathfrak{h}_2 for $y \neq 0$. If $p^2 + q^2 \neq 1$, then the equation $4y^2 - 4 + (1 + p^2 + q^2)^2 = 0$ represents an ovaloid of revolution generated by rotating the curve illustrated in Figure 1 about the y -axis. Therefore, the SKT structures (J, g) with nonabelian J are parametrized by the points in the Euclidean 3-space, where the region outside the ovaloid corresponds to SKT structures on the Lie algebra \mathfrak{h}_2 , the points on the ovaloid correspond to SKT structures on \mathfrak{h}_4 , and the region inside the ovaloid to SKT structures on \mathfrak{h}_5 .

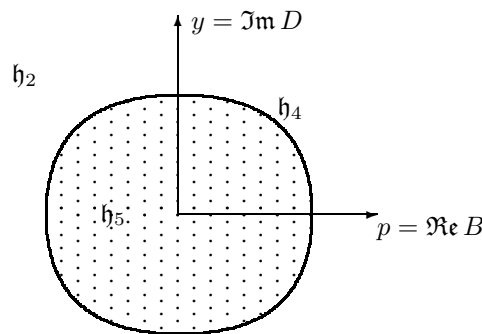


Figure 1. SKT structures in the nonabelian case.

The Lie algebras $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$, and \mathfrak{h}_8 possess abelian complex structures. The following result is a direct consequence of our study above.

Corollary 20. *Let (J, g) be an SKT structure on a six-dimensional NLA \mathfrak{g} . If J is abelian, then \mathfrak{g} is isomorphic to \mathfrak{h}_2 or \mathfrak{h}_8 . Therefore, none of the abelian complex structures on \mathfrak{h}_4 or \mathfrak{h}_5 admits a compatible SKT metric.*

Next we prove that the existence of an SKT structure on a compact nilmanifold $\Gamma \backslash G$ implies the existence of an invariant one. The proof is based on the “symmetrization” process, and we follow the ideas of [Be], [FG].

Let $M = \Gamma \backslash G$ be a compact nilmanifold and $\nu = d\tau$ a volume element on M induced by a bi-invariant one on the Lie group G [Mi]. After rescaling, we can suppose that M has volume equal to 1.

Given any covariant k -tensor field $T: \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \rightarrow \mathcal{C}^\infty(M)$ on the nilmanifold M , we define a covariant k -tensor $T_\nu: \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow \mathbb{R}$ on the NLA \mathfrak{g} of G by

$$T_\nu(X_1, \dots, X_k) = \int_{p \in M} T_p(X_1|_p, \dots, X_k|_p) \nu \quad \text{for } X_1, \dots, X_k \in \mathfrak{g},$$

where $X_j|_p$ is the value at the point $p \in M$ of the projection on M of the left-invariant vector field X_j on the Lie group G .

Obviously, $T_\nu = T$ for any tensor field T coming from a left-invariant one. In [Be] it is proved that if $T = \alpha \in \mathcal{A}^k(M)$ is a k -form on M , then $(d\alpha)_\nu = d\alpha_\nu$. A simple calculation shows that $(\alpha_\nu \wedge \beta)_\nu = \alpha_\nu \wedge \beta_\nu$ for any $\alpha \in \mathcal{A}^k(M)$ and $\beta \in \mathcal{A}^l(M)$.

Remark 5. The symmetrization process defines a linear map $\tilde{\nu}: \mathcal{A}^k(M) \rightarrow \bigwedge^k \mathfrak{g}^*$, given by $\tilde{\nu}(\alpha) = \alpha_\nu$ for any $\alpha \in \mathcal{A}^k(M)$, which commutes with the differentials. Moreover, it follows from the Nomizu theorem [N] that $\tilde{\nu}$ induces an isomorphism $H^k(\tilde{\nu}): H^k(M) \rightarrow H^k(\mathfrak{g})$ between the k th cohomology groups for each k . In particular, any closed k -form α on M is cohomologous to the invariant k -form α_ν obtained by the symmetrization process.

Let us suppose now that the nilmanifold $M = \Gamma \backslash G$ is equipped with an invariant complex structure J . If g is a J -Hermitian metric on M and Ω denotes its fundamental form, then g_ν is a J -Hermitian metric on the NLA \mathfrak{g} with fundamental form Ω_ν .

Proposition 21. *Let $(M = \Gamma \backslash G, J)$ be a compact complex nilmanifold with invariant J , and suppose that the NLA \mathfrak{g} of G is not abelian. If (J, g) is an SKT structure on M , then (J, g_ν) is an SKT structure on \mathfrak{g} .*

Proof. The fundamental form Ω of (J, g) satisfies $dJd\Omega = 0$ but $d\Omega \neq 0$. As it is observed in [FG], since J is invariant, we have that $(J\alpha)_\nu = J\alpha_\nu$ for any $\alpha \in \mathcal{A}^k(M)$. Therefore, $dJd\Omega_\nu = dJ(d\Omega)_\nu = d(Jd\Omega)_\nu = (dJd\Omega)_\nu = 0$. Moreover, since \mathfrak{g} is not abelian, it follows from [BG] that $d\Omega_\nu \neq 0$ because M has no Kähler metric. \square

Notice that the symmetrization of SKT structures on a torus gives rise to invariant Kähler metrics.

Corollary 22. *A nontoral compact nilmanifold $M = \Gamma \backslash G$ of dimension 6 admits an SKT metric compatible with an invariant complex structure if and only if the Lie algebra of G is isomorphic to $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$, or \mathfrak{h}_8 .*

The result follows directly from Theorem 19(b) and Proposition 21. In particular, the first Betti number of M must be ≥ 4 in order to admit an SKT structure (J, g) with invariant J .

Finally, Corollary 20 implies that $M = \Gamma \backslash G$ has SKT structures whose underlying complex structure is abelian if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_2 or \mathfrak{h}_8 .

3. Balanced metrics on six-dimensional nilmanifolds

Let (J, g) be a Hermitian structure on a $2n$ -dimensional manifold M with fundamental form Ω . According to [GH], the W_1 and W_2 components in the well known Gray–Hervella decomposition of $\nabla\Omega$ are identically zero, that is, $\nabla\Omega \in W_3 \oplus W_4$. In this section we are interested in Hermitian structures satisfying $\nabla\Omega \in W_3$.

Let θ be the *Lee form* associated to the Hermitian structure (J, g) , that is, $\theta = (1/(1 - n))J\delta\Omega$, where δ denotes the formal adjoint of d with respect to the metric g . If θ vanishes identically, then the Hermitian structure is called *balanced* and we shall say that g is a *balanced* metric on M . Such structures correspond to the Gray–Hervella class \mathcal{W}_3 [GH], and they are also known in the literature as *cosymplectic* or *semi-Kähler*.

A large class of balanced structures is provided by the compact *complex parallelizable* manifolds, that is, compact complex manifolds M for which the holomorphic bundle $T^{1,0}M$ is trivial. Wang [W] proved that M is a compact quotient $\Gamma \backslash G$ of a simply connected *complex* Lie group G by a discrete subgroup Γ . Therefore, G is unimodular [Mi], so any Hermitian left-invariant metric on the complex Lie group G is balanced by [AG, Theorem 2.2] and it descends to M .

Alexandrov and Ivanov prove in [AI, Remark 1] (see also [FPS, Prop. 1.4]) that the balanced condition is complementary to the SKT condition in dimension ≥ 6 . As a consequence we have:

Proposition 23. *A compact complex parallelizable manifold (not a torus) of dimension ≥ 6 has no compatible SKT metrics.*

Proof. Let $M = \Gamma \backslash G$ be a compact complex parallelizable manifold and denote by J its natural complex structure. Any Hermitian left-invariant metric on G does not satisfy the SKT condition, because it is balanced. So there are no left-invariant SKT metrics on G compatible with J .

Moreover, since G is unimodular there exists a bi-invariant volume element, and the symmetrization of an SKT metric on M would be a left-invariant SKT metric on G . In fact, the proof of Proposition 21 is valid in this context, except that we use Theorem 2.1 in [AG], which states that if G is not abelian then there are no left-invariant Kähler metrics on G compatible with J , in order to ensure that the symmetrization of the fundamental form is not closed. \square

Let \mathfrak{g} be a Lie algebra of dimension $2n$ endowed with a Hermitian structure (J, g) , in the sense of Section 2, with Lee form $\theta \in \mathfrak{g}^*$. We say that (J, g) is a *balanced* structure, or that g is a *balanced* metric, on \mathfrak{g} if $\theta = 0$.

Fixed a complex structure J on \mathfrak{g} , we denote by $\mathcal{M}_3(\mathfrak{g}, J)$ the set of all balanced J -Hermitian metrics g on \mathfrak{g} .

Lemma 24. *If J' is a complex structure on \mathfrak{g} equivalent to J , then the metrics in $\mathcal{M}_3(\mathfrak{g}, J')$ are in one-to-one correspondence with the metrics in $\mathcal{M}_3(\mathfrak{g}, J)$.*

Proof. Let $F \in \text{Aut}(\mathfrak{g})$ be an automorphism of the Lie algebra such that $F \circ J' = J \circ F$. Given $g \in \mathcal{M}_3(\mathfrak{g}, J)$ with fundamental form Ω , let us consider the J' -Hermitian metric

$g' = F^*g$, whose fundamental form is $\Omega' = F^*\Omega$. If we denote by δ' the adjoint of d with respect to the metric g' , then $\delta'F^* = F^*\delta$, which implies that the Lee form θ' of the Hermitian structure (J', g') is given by $\theta' = F^*\theta$. Therefore, (J, g) is balanced if and only if (J', g') is. \square

When \mathfrak{g} is six-dimensional, $2*\Omega = \Omega \wedge \Omega$, where $*$ denotes the Hodge star with respect to g . So the Lee form vanishes if and only if Ω^2 is closed. But, $d\Omega^2 = 2\Omega \wedge d\Omega$ is a real 5-form which decomposes as a sum of forms of types $(3, 2)$ and $(2, 3)$, because the bidegree of Ω is equal to $(1, 1)$. Thus, Ω^2 is closed if and only if $(d\Omega^2)^{3,2} = 2\Omega \wedge (d\Omega)^{2,1} = 0$. Therefore, a Hermitian structure is balanced if and only if $\partial\Omega \wedge \Omega = 0$.

Fix a complex structure J on an NLA \mathfrak{g} of dimension 6, the set $\mathcal{M}_3(\mathfrak{g}, J)$ is then given by

$$\mathcal{M}_3(\mathfrak{g}, J) = \{J\text{-Hermitian metrics } g \mid \partial\Omega_g \wedge \Omega_g = 0\},$$

where Ω_g is the fundamental form associated to g . Our first goal is to prove that $\mathcal{M}_3(\mathfrak{g}, J) \neq \emptyset$ only for a Lie algebra \mathfrak{g} isomorphic to $\mathfrak{h}_1, \dots, \mathfrak{h}_6$ or \mathfrak{h}_{19}^- .

Proposition 25. *Let (J, g) be a Hermitian structure on a six-dimensional NLA \mathfrak{g} .*

- (a) *If J is nonnilpotent, then (J, g) is balanced if and only if the complex structure J is equivalent to one defined by (3) and the metric coefficients of g in (8) satisfy*

$$z = \frac{-iuv}{s} \quad \text{and} \quad As + b\bar{E}u + b\bar{u} = 0. \quad (11)$$

- (b) *If J is nilpotent but not complex parallelizable, then (J, g) is balanced if and only if J is equivalent to one defined by (5) and the metric coefficients of g in (8) satisfy*

$$st - |v|^2 + D(rt - |z|^2) = B(it\bar{u} - v\bar{z}). \quad (12)$$

- (c) *If J is a complex parallelizable structure, then any J -Hermitian metric is balanced.*

Proof. Suppose first that J is nonnilpotent. From Lemma 24 and Proposition 14, we can restrict our attention to fundamental 2-forms Ω given by (9) in terms of a basis $\{\omega^j\}_{j=1}^3$ satisfying (3). Using Lemma 17(i), a direct calculation shows that

$$\partial\Omega \wedge \Omega = (\bar{A}(st - |v|^2) + b(tu - i\bar{v}z) + bE(t\bar{u} + i\bar{v}\bar{z}))\omega^{123\bar{1}\bar{2}} + (uv - isz)\omega^{123\bar{1}\bar{3}}.$$

Therefore, a metric g given by (8) is balanced if and only if $z = -iuv/s$ and

$$0 = \bar{A}(st - |v|^2) + b(tu - i\bar{v}z) + bE(t\bar{u} + i\bar{v}\bar{z}) = \frac{st - |v|^2}{s}(\bar{A}s + bu + bE\bar{u}).$$

Since g is positive definite, the latter condition is equivalent to $As + b\bar{E}u + b\bar{u} = 0$ because s and b are real numbers, so part (a) of the proposition is proved.

To prove (b) we can focus, again by Lemma 24 and Proposition 14, on nilpotent complex structures J defined by equations of the form (4). For any Ω given by (9), from Lemma 17(ii) we get, by a simple calculation, that

$$\begin{aligned} \partial\Omega \wedge \Omega &= ((1-\epsilon)\bar{A}(st - |v|^2) + \bar{B}(itu + \bar{v}z) - \bar{C}(it\bar{u} - v\bar{z}) + (1-\epsilon)\bar{D}(rt - |z|^2))\omega^{123\bar{1}\bar{2}} \\ &\quad - \epsilon(st - |v|^2)\omega^{123\bar{1}\bar{3}}. \end{aligned}$$

Since g is positive definite, the coefficient of $\omega^{123\bar{1}\bar{3}}$ vanishes if and only if $\epsilon = 0$. Thus, if J is not complex parallelizable, then Proposition 10 and Lemma 11 imply that J is equivalent to one defined by (5), and so the form $\partial\Omega \wedge \Omega$ is zero if and only if (12) holds.

Finally, if $\epsilon = A = B = C = D = 0$, then $\partial\Omega \wedge \Omega$ vanishes identically, so (c) is clear. It also follows directly from [AG]. \square

Theorem 26. *Let \mathfrak{g} be an NLA of dimension 6. Then, there exists a balanced structure on \mathfrak{g} if and only if it is isomorphic to \mathfrak{h}_k for $1 \leq k \leq 6$, or \mathfrak{h}_{19}^- . Moreover:*

- (a) *Any complex structure on \mathfrak{h}_6 and \mathfrak{h}_{19}^- has compatible metrics which are balanced.*
- (b) *A complex structure on \mathfrak{h}_3 has balanced compatible metrics iff it is equivalent to J_0^- .*
- (c) *On the Lie algebras \mathfrak{h}_2 , \mathfrak{h}_4 , and \mathfrak{h}_5 there exist complex structures having balanced compatible metrics, but there also exist complex structures not admitting such metrics.*

Proof. If there exists a balanced structure (J, g) on \mathfrak{g} such that J is nonnilpotent, then it follows from (11) by complex conjugation that $\bar{A}s + bu + bE\bar{u} = 0$. On the other hand, if we multiply the second equation in (11) by E , then taking into account that $|E| = 1$ we get $AEs + bu + bE\bar{u} = 0$. Therefore, $s(\bar{A} - AE) = 0$, that is, $\bar{A} = AE$ because g is positive definite. Now Proposition 9 implies that \mathfrak{g} cannot be isomorphic to \mathfrak{h}_{26}^+ .

Now suppose that \mathfrak{g} has a balanced structure (J, g) such that J is nilpotent. Propositions 10 and 25 imply that, up to isomorphism, the possible candidates for such a Lie algebra are $\mathfrak{h}_1, \dots, \mathfrak{h}_6$ and \mathfrak{h}_8 . But the Lie algebra \mathfrak{h}_8 is excluded by Proposition 13(i.5), because (12) reduces to $st - |v|^2 = 0$ for $B = D = 0$, which contradicts that g is positive definite. Therefore, \mathfrak{g} cannot be isomorphic to \mathfrak{h}_k for $7 \leq k \leq 16$.

Notice that for the “canonical” metric g given by $r = s = t = 1$ and $u = v = z = 0$, the balanced condition (12) reduces to $D = -1$. From Proposition 13 it follows that there is a balanced structure on \mathfrak{h}_2 for $|B| < 1 = \rho$, on \mathfrak{h}_4 for $\rho = |B| = 1$, on \mathfrak{h}_5 for $|B| > 1 = \rho$, and on \mathfrak{h}_3 for the complex structure J_0^- , i.e., for $\rho = B = 0$.

To complete the proof it remains to show that any complex structure on \mathfrak{h}_6 and \mathfrak{h}_{19}^- has a compatible balanced metric, and that there exists a complex structure on each one of the Lie algebras $\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4$, and \mathfrak{h}_5 admitting no compatible balanced metric.

Let g_u be the metric defined by $r = 1 + |u|^2, s = t = 1$, and $v = z = 0$. If $u = -\bar{A}/(2b)$, then we have a metric g_u on \mathfrak{h}_{19}^- compatible with the complex structure J defined by (3) with $E = \bar{A}/A$, according to Proposition 9. Since g_u satisfies (11), from Lemma 24 we conclude that any other complex structure on \mathfrak{h}_{19}^- has a balanced compatible metric. Moreover, if $u = i$, then the metric g_u on \mathfrak{h}_6 is J -Hermitian for the complex structure J defined by (5) with $\rho = B = 1$ and $D = 0$, according to Proposition 13(i.4), and it is clear that (12) holds. From Corollary 15 and Lemma 24 it follows that any other complex structure on \mathfrak{h}_6 possesses a balanced compatible metric.

On the other hand, for the complex structure J_0^+ on \mathfrak{h}_3 given in Corollary 16 the balanced condition (12) reduces to $st - |v|^2 + rt - |z|^2 = 0$, so g cannot be positive definite. Therefore, there is no balanced compatible metric.

Finally, if $\rho = 1$ and $B = x = 0$ in Proposition 13, then (12) reduces to $st - |v|^2 + iy(rt - |z|^2) = 0$, so the metric cannot be positive definite, and depending on the value of y we get complex structures on the Lie algebras $\mathfrak{h}_2, \mathfrak{h}_4$, and \mathfrak{h}_5 having no balanced compatible metric. \square

Remark 6. In [FPS] it is shown that the metric given by $r = s = t = 1/2$, $u = v = z = 0$ is balanced with respect to one particular complex structure on \mathfrak{h}_k for $1 \leq k \leq 6$.

By the symmetrization process, Fino and Grantcharov prove in [FG] that if J is an invariant complex structure on a compact nilmanifold $M = \Gamma \backslash G$ admitting a balanced metric g , then there is a balanced structure (J, \hat{g}) on the Lie algebra \mathfrak{g} of G . Therefore:

Corollary 27. *A compact nilmanifold $M = \Gamma \backslash G$ of dimension 6 admits a balanced metric compatible with an invariant complex structure if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_{19}^- or \mathfrak{h}_k for some $1 \leq k \leq 6$.*

Since \mathfrak{h}_3 is the Lie algebra underlying the compact nilmanifold $N(2, 1) \times S^1$, where $N(2, 1)$ is a quotient of the five-dimensional generalized Heisenberg group $H(2, 1)$, we have that an invariant complex structure J on $N(2, 1) \times S^1$ has compatible balanced metrics if and only if J is equivalent to J_0^- .

Let (J, g) be a Hermitian structure on a manifold M and denote by ∇^B its *Bismut connection*, i.e., the unique connection for which g and J are parallel and the torsion T^B is given by $g(X, T^B(Y, Z)) = Jd\Omega(X, Y, Z)$. Gutowski, Ivanov, and Papadopoulos pose in [GIP] the following version of the Calabi conjecture in the non-Kähler setting: on *any* $2n$ -dimensional compact complex manifold with vanishing first Chern class there exists a Hermitian structure with restricted holonomy of the Bismut connection contained in $SU(n)$. They prove that this property holds for Moishezon manifolds, for compact complex manifolds with vanishing first Chern class which are cohomologically Kähler, and for connected sums of $k \geq 2$ copies of $S^3 \times S^3$.

Now, let $M = \Gamma \backslash G$ be a compact nilmanifold of dimension 6 equipped with an invariant complex structure J . It follows from (1) that ω^{123} is a holomorphic non-vanishing $(3, 0)$ -form. Therefore, Theorem 4.1 in [FG] asserts that if the Ricci tensor of the Bismut connection of some J -Hermitian metric g on M vanishes, then there is a metric \tilde{g} conformal to g such that (J, \tilde{g}) is a balanced structure on M , so there is an invariant balanced structure (J, \hat{g}) on M by the symmetrization process. Conversely, given an invariant balanced Hermitian structure on M there is a conformal change of metric such that the Ricci tensor of the Bismut connection of the resulting metric vanishes (see [FPS, Prop. 6.1]).

Using this result, Fino and Grantcharov provide counterexamples to the above conjecture by showing that there exist invariant complex structures on the Iwasawa manifold which do not admit compatible invariant balanced metrics. In the following result we describe the general situation for six-dimensional nilmanifolds.

Corollary 28. *Let $M = \Gamma \backslash G$ be a six-dimensional compact nilmanifold, and \mathfrak{g} the Lie algebra of G . Then:*

- (a) *If \mathfrak{g} is isomorphic to \mathfrak{h}_6 or \mathfrak{h}_{19}^- , then any invariant complex structure on M has a Hermitian structure with restricted holonomy of the Bismut connection contained in $SU(3)$.*
- (b) *If \mathfrak{g} is isomorphic to \mathfrak{h}_2 , \mathfrak{h}_3 , \mathfrak{h}_4 , or \mathfrak{h}_5 , then there are invariant complex structures on M having a Hermitian structure with restricted holonomy of the Bismut connection contained in $SU(3)$, but there also exist invariant complex structures on M for which the restricted holonomy of the Bismut connection of any Hermitian metric is not contained in $SU(3)$.*

- (c) If \mathfrak{g} is isomorphic to \mathfrak{h}_{26}^+ or \mathfrak{h}_k for some $7 \leq k \leq 16$, then any invariant complex structure on M does not possess Hermitian structures with restricted holonomy of the Bismut connection contained in $SU(3)$.

Observe that on the compact nilmanifold $N(2, 1) \times S^1$, an invariant complex structure J has a Hermitian structure for which the holonomy of its Bismut connection reduces to $SU(3)$ if and only if J is equivalent to J_0^- .

We finish this section with some remarks about the stability of the balanced condition under small deformations of the complex structure. As a consequence of Theorem 26, the nilmanifolds corresponding to \mathfrak{h}_6 and \mathfrak{h}_{19}^- are stable in the sense that given a balanced structure (J_0, g_0) , with J_0 invariant, then along any deformation J_α of J_0 consisting of invariant complex structures, there always exists a balanced J_α -Hermitian metric g_α for each value of α .

However, it is shown in [FG, Theorem 4.2] that such stability does not hold for the Iwasawa manifold. Next we show that for the compact nilmanifold $\Gamma \backslash (H^3 \times H^3)$, where H^3 is the Heisenberg group, the balanced condition is not stable under small deformations.

Since the Lie algebra of $H^3 \times H^3$ is \mathfrak{h}_2 , we consider the equations (5) with $\rho = 1$ and $B = 0$, and denote by $J_{x,y}$ the complex structure on \mathfrak{h}_2 given by

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + (x + iy)\omega^{2\bar{2}},$$

where $x, y \in \mathbb{R}$ satisfy $4y^2 - 4x - 1 > 0$, according to Proposition 13. Observe that the balanced condition (12) is satisfied for $J_{x,y}$ if and only if $x < -\frac{1}{4}$ and $y = 0$. In particular, for $x = -1$ the family of complex structures $J_{-1,y}$ with $y \in \mathbb{R}$ only admits balanced metrics for $y = 0$.

Corollary 29. *The property “vanishing Ricci tensor for the Bismut connection” is not stable under small deformations on the nilmanifold $\Gamma \backslash (H^3 \times H^3)$.*

4. Locally conformal Kähler geometry

A Hermitian structure (J, g) on a $2n$ -dimensional manifold M is called *locally conformal Kähler* (LCK for short) if M has an open cover $\{U_i\}$ and a family $\{f_i\}$ of differentiable functions $f_i: U_i \rightarrow \mathbb{R}$ such that each local metric $g_i = \exp f_i g|_{U_i}$ is Kähler. If Ω denotes the fundamental form, then the Hermitian structure is LCK if and only if $d\Omega = \theta \wedge \Omega$ with closed Lee form θ . Notice that the class of LCK structures corresponds to the Gray–Hervella class \mathcal{W}_4 [GH].

An interesting special class of locally conformal Kähler metrics is the one consisting of those for which θ is a nowhere vanishing parallel form. A manifold endowed with such a metric is called a *Vaisman manifold* [DO], [V1], [V2].

Let (J, g) be a Hermitian structure on a Lie algebra \mathfrak{g} , with fundamental form $\Omega \in \wedge^2 \mathfrak{g}^*$ and Lee form $\theta \in \mathfrak{g}^*$. We say that (J, g) is an *LCK structure*, or that g is an *LCK metric*, if $d\Omega = \theta \wedge \Omega$ with closed Lee form θ .

Fixed a complex structure J on \mathfrak{g} , we denote by $\mathcal{M}_4(\mathfrak{g}, J)$ the set of LCK J -Hermitian metrics on \mathfrak{g} .

Lemma 30. *If J' is a complex structure on \mathfrak{g} equivalent to J , then the metrics in $\mathcal{M}_4(\mathfrak{g}, J')$ are in one-to-one correspondence with the metrics in $\mathcal{M}_4(\mathfrak{g}, J)$.*

Proof. Following the proof of Lemma 24, we have $F^*(d\Omega - \theta \wedge \Omega) = dF^*\Omega - F^*\theta \wedge F^*\Omega = d\Omega' - \theta' \wedge \Omega'$. \square

Since θ and Ω are real forms, taking into account their bidegrees we have that in dimension ≥ 6 a Hermitian structure is LCK if and only if $\partial\Omega = \theta^{1,0} \wedge \Omega$. Therefore, if $\dim \mathfrak{g} \geq 6$, then

$$\mathcal{M}_4(\mathfrak{g}, J) = \{J\text{-Hermitian metrics } g \mid \partial\Omega_g - (\theta_g)^{1,0} \wedge \Omega_g = 0\}.$$

Theorem 31. *A six-dimensional NLA \mathfrak{g} admits an LCK structure if and only if it is isomorphic to \mathfrak{h}_1 or \mathfrak{h}_3 . Moreover, a complex structure on \mathfrak{h}_3 has a compatible LCK metric if and only if it is equivalent to J_0^+ .*

Proof. Since the Lee form θ is a real 1-form, there exist $\lambda_j \in \mathbb{C}$, $j = 1, 2, 3$, such that

$$\theta = \lambda_1 \omega^1 + \lambda_2 \omega^2 + \lambda_3 \omega^3 + \bar{\lambda}_1 \omega^{\bar{1}} + \bar{\lambda}_2 \omega^{\bar{2}} + \bar{\lambda}_3 \omega^{\bar{3}}, \quad (13)$$

with respect to any basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$. We must find the possible values of λ_j in (13) satisfying the equation $\partial\Omega = \theta^{1,0} \wedge \Omega$. From (9) it follows that

$$\begin{aligned} \theta^{1,0} \wedge \Omega &= (\lambda_1 \omega^1 + \lambda_2 \omega^2 + \lambda_3 \omega^3) \wedge \Omega \\ &= -(\bar{u}\lambda_1 + ir\lambda_2)\omega^{12\bar{1}} + (is\lambda_1 - u\lambda_2)\omega^{12\bar{2}} + (v\lambda_1 - z\lambda_2)\omega^{12\bar{3}} \\ &\quad - (\bar{z}\lambda_1 + ir\lambda_3)\omega^{13\bar{1}} - (\bar{v}\lambda_1 + u\lambda_3)\omega^{13\bar{2}} + (it\lambda_1 - z\lambda_3)\omega^{13\bar{3}} \\ &\quad - (\bar{z}\lambda_2 - \bar{u}\lambda_3)\omega^{23\bar{1}} - (\bar{v}\lambda_2 + is\lambda_3)\omega^{23\bar{2}} + (it\lambda_2 - v\lambda_3)\omega^{23\bar{3}}. \end{aligned} \quad (14)$$

We shall also use the fact that the closedness of θ is equivalent to $\partial\theta^{1,0} = 0$ and $\bar{\partial}\theta^{1,0} + \partial\theta^{0,1} = 0$.

By Lemma 30 and Proposition 14 we can restrict our attention to the two special types of complex structures defined by (3) and (4). If J is a nonnilpotent complex structure defined by (3), then $0 = \partial\theta^{1,0} = \lambda_2 E \omega^{13}$, which implies $\lambda_2 = 0$. Moreover, comparing the coefficients of $\omega^{23\bar{2}}$ in Lemma 17(i) and (14) we get that $is\lambda_3 = 0$, so $\lambda_3 = 0$ because g is positive definite. Now, if we compare the coefficients of $\omega^{23\bar{1}}$, then $is - bt = 0$, which is a contradiction to the fact that s, b, t are nonzero real numbers. Therefore, a nonnilpotent complex structure cannot have compatible LCK metrics.

Let us suppose next that J is a nilpotent complex structure defined by (4).

Notice that if the coefficient λ_3 in (13) vanishes, then comparing the coefficients of $\omega^{13\bar{3}}$ and $\omega^{23\bar{3}}$ in Lemma 17(ii) and (14) we get that $\lambda_1 = \lambda_2 = 0$, so $d\Omega = 0$ and \mathfrak{g} must be the abelian Lie algebra \mathfrak{h}_1 [BG].

On the other hand, if $\epsilon = 1$ in equations (4) then the coefficients of $\omega^{23\bar{2}}$ and $\omega^{23\bar{3}}$ in Lemma 17(ii) and (14) imply that λ_2 and λ_3 satisfy $\bar{v}\lambda_2 + is\lambda_3 = it\lambda_2 - v\lambda_3 = 0$. Since g is positive definite, $\det \begin{pmatrix} \bar{v} & is \\ it & -v \end{pmatrix} > 0$ and the unique solution is the trivial one, in particular, $\lambda_3 = 0$ and so $\mathfrak{g} \cong \mathfrak{h}_1$ again.

Suppose next that the NLA \mathfrak{g} is not abelian and that it is endowed with a nilpotent complex structure J given by (4) admitting an LCK metric. From the previous paragraphs, $\epsilon = 0$ in (4) and $\lambda_3 \neq 0$ in (13). From (4) we have $0 = \partial\theta^{1,0} = \lambda_3 \rho \omega^{12}$, therefore

$\rho = 0$ and the complex structure J must be abelian. Since $\epsilon = \rho = 0$, Proposition 10 and Lemma 11 imply that we can suppose J given by equations (5) with $\rho = 0$. But in this case one has

$$0 = \bar{\partial}\theta^{1,0} + \partial\theta^{0,1} = (\lambda_3 - \bar{\lambda}_3)\omega^{1\bar{1}} + B\lambda_3\omega^{1\bar{2}} - \bar{B}\bar{\lambda}_3\omega^{2\bar{1}} + (D\lambda_3 - \bar{D}\bar{\lambda}_3)\omega^{2\bar{2}}.$$

In order to have a solution with $\lambda_3 \neq 0$, the coefficient B must be zero and the coefficients λ_3 and D must be real. In this case, we get

$$\partial\Omega = v\omega^{12\bar{1}} + Dz\omega^{12\bar{2}} - it\omega^{13\bar{1}} - iDt\omega^{23\bar{2}}.$$

Now taking into account the coefficients of $\omega^{13\bar{1}}$ and $\omega^{13\bar{3}}$ in (14), the condition $\partial\Omega = \theta^{1,0} \wedge \Omega$ implies that $\bar{z}\lambda_1 + ir\lambda_3 = it$ and $it\lambda_1 - z\lambda_3 = 0$, so $\lambda_3 = t^2/(rt - |z|^2)$. Moreover, from the coefficients of $\omega^{23\bar{2}}$ and $\omega^{23\bar{3}}$ in (14) we get that $\lambda_3 = Dt^2/(st - |v|^2)$. Since g is positive definite, necessarily $D > 0$. Now, Corollary 16 implies that $\mathfrak{g} \cong \mathfrak{h}_3$ and the complex structure J must be equivalent to J_0^+ .

Finally, the existence of a particular LCK structure on \mathfrak{h}_3 follows from [CFL]. In fact, one solution is obtained for $D = 1$ and $r = s = t = 1$, $u = v = z = 0$, with Lee form $\theta = 2\Re\epsilon\omega^3$. \square

Remark 7. According to [BG], $\mathcal{M}_3(\mathfrak{g}, J) \cap \mathcal{M}_4(\mathfrak{g}, J) = \emptyset$ for any complex structure J on a nonabelian NLA \mathfrak{g} . From Theorems 26 and 31 we have that for any J on the Lie algebra \mathfrak{h}_3 , either $\mathcal{M}_3(\mathfrak{h}_3, J) = \emptyset$ or $\mathcal{M}_4(\mathfrak{h}_3, J) = \emptyset$, depending on the fact that J be equivalent to J_0^+ or not. Moreover, for the remaining (nonabelian) Lie algebras \mathfrak{g} of Theorem 8, one has that $\mathcal{M}_4(\mathfrak{g}, J) = \emptyset$ for any complex structure J .

Next we prove that the Lee form of any invariant LCK structure is parallel with respect to the Levi-Civita connection.

Proposition 32. *For any invariant LCK metric, the nilmanifold $N(2, 1) \times S^1$ is a Vaisman manifold.*

Proof. Since the complex structure must be equivalent to J_0^+ , we consider a basis $\{\omega^j\}_{j=1}^3$ for $(\mathfrak{h}_3)^{1,0}$ satisfying $d\omega^1 = d\omega^2 = 0$ and $d\omega^3 = \omega^{1\bar{1}} + \omega^{2\bar{2}}$. It is easy to see that a J_0^+ -Hermitian metric g given by (8) is LCK if and only if $u = (i\bar{v}z)/t$ and $|v|^2 - st = |z|^2 - rt$. In this case, the associated Lee form is

$$\theta = \frac{1}{|z|^2 - rt}(itz\omega^1 + itv\omega^2 - t^2\omega^3 - it\bar{z}\omega^{\bar{1}} - it\bar{v}\omega^{\bar{2}} - t^2\omega^{\bar{3}}).$$

Let $\{Z_j\}_{j=1}^3$ be the dual basis of $\{\omega^j\}_{j=1}^3$. For any $U, V \in (\mathfrak{h}_3)_{\mathbb{C}}$, it is easy to check that

$$\theta(\nabla_V U) = \frac{it}{|z|^2 - rt}g(\nabla_V U, Z_3 + \bar{Z}_3).$$

But, $g(\nabla_{Z_k} Z_j, Z_3 + \bar{Z}_3) = g(\nabla_{Z_k} \bar{Z}_j, Z_3 + \bar{Z}_3) = 0$ for $1 \leq j \leq k \leq 3$, because $[Z_1, \bar{Z}_1] = [Z_2, \bar{Z}_2] = \bar{Z}_3 - Z_3$ are the only brackets which do not vanish. Therefore, $g(\nabla_V U, Z_3 + \bar{Z}_3)$ vanishes identically, so the Lee form θ is parallel. \square

It is well known that the orthogonal leaves to the Lee vector field of a Vaisman manifold bear a Sasakian structure, and that the product by \mathbb{R} or S^1 of a Sasakian manifold is an LCK manifold with parallel Lee form [V1]. Thus, as an immediate consequence of Proposition 32 we have that $N(2, 1)$ is essentially the only five-dimensional nilmanifold admitting invariant Sasakian structures.

Corollary 33. *Let $M = \Gamma \backslash G$ be a nontoral compact nilmanifold of dimension 5 endowed with an invariant Sasakian structure. Then, the Lie algebra of G is isomorphic to $(0, 0, 0, 0, 12 + 34)$.*

Following an idea of [Be], next we study the symmetrization of LCK structures on nilmanifolds.

Proposition 34. *Let $(M = \Gamma \backslash G, J)$ be a compact complex nilmanifold with J invariant. If (J, g) is an LCK structure on M , then there is a metric \tilde{g} globally conformal to g such that (J, \tilde{g}_ν) is an LCK structure on the Lie algebra \mathfrak{g} of G .*

Proof. The fundamental form Ω of (J, g) satisfies $d\Omega = \theta \wedge \Omega$ with closed Lee form θ . From Remark 5 we have that θ is cohomologous to the invariant 1-form θ_ν obtained by the symmetrization process. Thus, there exists a function f on M such that $\theta_\nu - \theta = df$. Since

$$d(\exp f \Omega) = \exp f df \wedge \Omega + \exp f (\theta_\nu - df) \Omega = \theta_\nu \wedge (\exp f \Omega),$$

there is an LCK structure $(J, \tilde{g} = \exp f g)$ on M with fundamental form $\tilde{\Omega} = \exp f \Omega$ and Lee form equal to θ_ν . Now, $d\tilde{\Omega}_\nu = (d\tilde{\Omega})_\nu = (\theta_\nu \wedge \tilde{\Omega})_\nu = \theta_\nu \wedge \tilde{\Omega}_\nu$, that is, (J, \tilde{g}_ν) is an LCK structure on the Lie algebra \mathfrak{g} . \square

Corollary 35. *Let $(M = \Gamma \backslash G, J)$ be a nontoral six-dimensional compact nilmanifold endowed with an invariant complex structure J . Then, M has an LCK metric if and only if the Lie algebra of G is isomorphic to \mathfrak{h}_3 and J is equivalent to J_0^+ .*

Recently, Oeljeklaus and Toma [OT] have disproved the well known Vaisman conjecture stating that any compact LCK, but not globally conformal Kähler manifold has an odd Betti number. However, in the context of nilmanifolds, Corollary 35 leads us to believe that a $2n$ -dimensional compact nilmanifold M admitting an LCK structure is the product of $N(n-1, 1)$ by S^1 , where $N(n-1, 1)$ is the quotient of the generalized Heisenberg group $H(n-1, 1)$ by a discrete subgroup, in particular, the first Betti number of M equals $2n-1$; that is to say, the only LCK nilmanifolds would be essentially those constructed in [CFL].

The following result shows a large class of complex nilmanifolds not admitting LCK structures.

Corollary 36. *A compact complex parallelizable nilmanifold (not a torus) has no LCK metrics.*

Proof. Let M be a compact complex parallelizable nilmanifold and denote by J its complex structure. Since M is not a torus and any invariant J -Hermitian metric is balanced [AG], there do not exist invariant LCK metrics on M . By Proposition 34 there are no LCK metrics on M compatible with J . \square

In [AI, Remark 1] it is proved that the SKT condition is complementary to the LCK condition. Next we give another proof of this fact for nilmanifolds, based on the nilpotency of the underlying Lie algebra.

Proposition 37. *Let $(M = \Gamma \backslash G, J)$ be a nontoral compact complex nilmanifold of dimension $2n \geq 6$, where J is invariant. A J -Hermitian metric g on M cannot be SKT and LCK at the same time.*

Proof. Let (J, g) be a Hermitian structure on M that is both SKT and LCK. From Propositions 21 and 34, there is a Hermitian structure on the Lie algebra \mathfrak{g} of G that is SKT and LCK at the same time, i.e., its fundamental form Ω satisfies $d\Omega = \theta \wedge \Omega$ and $\partial\bar{\partial}\Omega = 0$. Let us write the Lee form as $\theta = \theta^{1,0} + \theta^{0,1}$ where $\theta^{0,1} = \overline{\theta^{1,0}}$. Since $\theta^{1,0} \wedge \Omega = \partial\Omega$ and $\bar{\partial}(\theta^{1,0} \wedge \Omega) = -\partial\bar{\partial}\Omega = 0$, we have that $\theta^{1,0} \wedge \Omega$ is a closed form. Therefore, $0 = d(\theta^{1,0} \wedge \Omega) = (d\theta^{1,0} - \theta^{1,0} \wedge \theta^{0,1}) \wedge \Omega$, which implies $d\theta^{1,0} = \theta^{1,0} \wedge \theta^{0,1}$, because the dimension of \mathfrak{g} is ≥ 6 . Notice that $\theta^{1,0} \neq 0$ because \mathfrak{g} is not abelian. Now, the real 1-form $\eta = i(\theta^{1,0} - \theta^{0,1})$ satisfies $d\eta = \eta \wedge \theta$, and a standard argument shows that this cannot happen because \mathfrak{g} is nilpotent. \square

Remark 8. The proposition above does not hold for nilmanifolds of dimension 4. In fact, for any complex structure on the Lie algebra $\mathfrak{kt} = (0, 0, 0, 12)$ underlying the well known Kodaira–Thurston manifold [T], there is a basis $\{\omega^1, \omega^2\}$ of $\mathfrak{kt}^{1,0}$ such that $d\omega^1 = 0$ and $d\omega^2 = \omega^{1\bar{1}}$. For any compatible metric

$$g = r(\omega^1 \otimes \omega^{\bar{1}} + \omega^{\bar{1}} \otimes \omega^1) + s(\omega^2 \otimes \omega^{\bar{2}} + \omega^{\bar{2}} \otimes \omega^2) - iu(\omega^1 \otimes \omega^{\bar{2}} + \omega^{\bar{2}} \otimes \omega^1) + i\bar{u}(\omega^2 \otimes \omega^{\bar{1}} + \omega^{\bar{1}} \otimes \omega^2),$$

its fundamental form Ω satisfies $\partial\bar{\partial}\Omega = 0$, so g is SKT. Moreover, g is also LCK, because $d\Omega = \theta \wedge \Omega$ with closed $\theta = (2s/(|u|^2 - rs))(\Re(iu\omega^1) - s\Re\omega^2)$.

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