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UNIPOTENT ELEMENTS IN SMALL CHARACTERISTIC

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Dedicated to Vladimir Drinfeld on the occasion of his 50th birthday

Abstract. Let G be a reductive connected algebraic group over an algebraically closed field of characteristic exponent $p \geqslant 1$. One of the aims of this paper is to present a picture of the unipotent elements of G which should apply for arbitrary p and is as close as possible to the picture for $p = 1$. Another aim is the study of $\mathcal{B}_u$, the variety of Borel subgroups of G containing a unipotent element u. It is known [Sp] that when p is a good prime, the l-adic cohomology spaces of $\mathcal{B}_u$ are pure. We would like to prove a similar result in the case where p is a bad prime. We present a method by which this can be achieved in a number of cases.

Introduction

0.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic exponent $p \geqslant 1$. Let G be a reductive connected algebraic group over $\mathbf{k}$. Let $\mathcal{U}$ be the variety of unipotent elements of G. The unipotent classes of G are the orbits of the conjugation action of G on $\mathcal{U}$. The theory of Dynkin and Kostant [Ko] provides a classification of unipotent classes of G assuming that $p = 1$. It is known that this classification remains valid when $p \geqslant 2$ is assumed to be a good prime for G. But the analogous classification problem in the case where p is a bad prime for G is more complicated. In every case a classification of unipotent classes is known: see [W] for classical groups and [E], [Sh], and [M] for exceptional groups, but from these works it is difficult to see the general features of the classification.

One of the aims of this paper is to present a picture of the unipotent elements which should apply for arbitrary p and is as close as possible to the picture for $p = 1$.

In 1.4 we observe that the set of unipotent classes in G can be parametrized by a set $\mathcal{S}_{\mathbf{W}}^p$ of irreducible representations of the Weyl group $\mathbf{W}$, which can be described a priori purely in terms of the root system. This explains clearly why the classification is different for small p.

In 1.1 we restate in a more precise form an observation of [L2] according to which $\mathcal{U}$ is naturally partitioned into finitely many "unipotent pieces" which are locally closed subvarieties stable under conjugation by G; the classification of unipotent pieces is in-

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dependent of p. For $p = 1$ or a good prime, each unipotent piece is a single conjugacy class. When p is a bad prime, a unipotent piece is in general a union of several conjugacy classes. Also, each unipotent piece has some topological properties which are independent of p (for example, over a finite field, the number of points of a unipotent piece is given by a formula independent of the characteristic).

Another aim of this paper is the study of $\mathcal{B}_u$, the variety of Borel subgroups of G containing a unipotent element u. It is known [Sp] that when p is a good prime, the l-adic cohomology spaces of $\mathcal{B}_u$ are pure. We would like to prove a similar result in the case where p is a bad prime. We present a method by which this can be achieved in a number of cases. Our strategy is to extend a technique from [DLP] in which (assuming that $p = 1$), $\mathcal{B}_u$ is analyzed by first partitioning it into finitely many smooth locally closed subvarieties using relative position of a point in $\mathcal{B}_u$ with a canonical parabolic attached to u. Much of our effort is concerned with trying to eliminate reference to the linearization procedure of Bass–Haboush (available only for $p = 1$), which was used in an essential way in [DLP]. Our approach is based on a list of properties $\mathfrak{P}_1$–$\mathfrak{P}_8$ of unipotent elements of which the first five (respectively last three) are expected to hold in general (respectively in many cases). All these properties are verified for general linear and symplectic groups (any p) in Sections 2 and 3. In writing Section 3 (on symplectic groups mostly with $p = 2$), I found that the treatment in [W] is not sufficient for this paper's purposes; I therefore included a treatment which does not rely on [W].

Notation. When $p > 1$, we denote by $\mathbf{k}_p$ an algebraic closure of the field with p elements. Let $\mathcal{B}$ the variety of Borel subgroups of G. If Γ' is a subgroup of a group Γ and $x \in \Gamma$, let $Z_{\Gamma'}(x) = \{z \in \Gamma' \mid zx = xz\}$. For a finite set Z, let $|Z|$ be the cardinal of Z. Let l be a prime number invertible in $\mathbf{k}$. For $a, b \in \mathbf{Z}$ let $[a, b] = \{z \in \mathbf{Z} \mid a \leqslant z \leqslant b\}$.

1. Some properties of unipotent elements

1.1. G acts by conjugation on $\mathrm{Hom}(\mathbf{k}^*, G)$ (homomorphisms of algebraic groups). The set of orbits $\mathrm{Hom}(\mathbf{k}^*, G)/G$ is naturally in bijection with the analogous set $\mathrm{Hom}(\mathbf{C}^*, G')/G'$, where G' is a connected reductive group over $\mathbf{C}$ of the same type as G. (Both sets may be identified with the set of Weyl group orbits on the group of 1-parameter subgroups of some maximal torus.) Let $\widetilde{D}_{G'}$ be the set of all $\omega \in \mathrm{Hom}(\mathbf{C}^*, G')$ such that there exists a homomorphism of algebraic groups $\widetilde{\omega}\colon \mathrm{SL}_2(\mathbf{C}) \to G'$ with $\widetilde{\omega}\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right] = \omega(t)$ for all $t \in \mathbf{C}^*$. Now $\widetilde{D}_{G'}$ is G'-stable; it has been described explicitly by Dynkin. Let $\widetilde{D}_G$ be the unique G-stable subset of $\mathrm{Hom}(\mathbf{k}^*, G)$ whose image in $\mathrm{Hom}(\mathbf{k}^*, G)/G$ corresponds under the bijection $\mathrm{Hom}(\mathbf{k}^*, G)/G \leftrightarrow \mathrm{Hom}(\mathbf{C}^*, G')/G'$ (as above) to the image of $\widetilde{D}_{G'}$ in $\mathrm{Hom}(\mathbf{C}^*, G')/G'$. Let D_G be the set of sequences $\triangle = (G_0^\triangle \supset G_1^\triangle \supset G_2^\triangle \supset \cdots)$ of closed connected subgroups of G such that for some $\omega \in \widetilde{D}_G$ we have (for $n \geqslant 0$):

$$\mathrm{Lie}\ G_n^\triangle = \{x \in \mathrm{Lie}\ G \mid \lim_{t \in \mathbf{k}^*;\ t \to 0} t^{1-n}\mathrm{Ad}\omega(t)x = 0\}.$$

Now G acts on D_G by conjugation, and the obvious map $\widetilde{D}_G \to D_G$ induces a bijection $\widetilde{D}_G/G \xrightarrow{\sim} D_G/G$ on the set of orbits. If $\triangle \in D_G$ and $g \in G$, then $G_n^{g\triangle g^{-1}} = gG_n^\triangle g^{-1}$ for $n \geqslant 0$. $G_0^\triangle$ is a parabolic subgroup of G with unipotent radical $G_1^\triangle$ and $G_n^\triangle$ is normalized by $G_0^\triangle$ for any n. Moreover,

(a) $G_2^{\triangle}/G_3^{\triangle}$ is a commutative connected unipotent group;
(b) the conjugation action of $G_0^{\triangle}$ on $G_2^{\triangle}/G_3^{\triangle}$ factors through an action of $\overline{G}_0^{\triangle} := G_0^{\triangle}/G_1^{\triangle}$ on $G_2^{\triangle}/G_3^{\triangle}$.

Note also that $G_n^{\triangle}$ for $n \neq 0, 2$ are uniquely determined by $G_0^{\triangle}, G_2^{\triangle}$.

Let $\blacktriangle$ be a G-orbit in D_G. Then $\widetilde{H}^{\blacktriangle} := \cup_{\triangle \in \blacktriangle} G_2^{\triangle}$ is a closed irreducible subset of $\mathcal{U}$ (since for $\triangle \in \blacktriangle$, $G_2^{\triangle}$ is a closed irreducible subset of $\mathcal{U}$ stable under conjugation by $G_0^{\triangle}$ and $G/G_0^{\triangle}$ is projective). Let

$$H^{\blacktriangle} = \widetilde{H}^{\blacktriangle} - \bigcup_{\blacktriangle' \in D_G/G;\ \widetilde{H}^{\blacktriangle'} \subsetneq \widetilde{H}^{\blacktriangle}} \widetilde{H}^{\blacktriangle'}.$$

For $\triangle \in D_G$, let $X^{\triangle} = G_2^{\triangle} \cap H^{\blacktriangle}$, where $\blacktriangle$ is the G-orbit of $\triangle$. Then $H^{\blacktriangle}$ is an open dense subset of $\widetilde{H}^{\blacktriangle}$ stable under conjugation by G, and $X^{\triangle}$ is an open dense subset of $G_2^{\triangle}$ stable under conjugation by $G_0^{\triangle}$. (We use the fact that D_G/G is finite.) Hence $H^{\blacktriangle}$ is locally closed in $\mathcal{U}$. The subsets $H^{\blacktriangle}(\blacktriangle \in D_G/G)$ are called the *unipotent pieces* of G.

We state the following properties $\mathfrak{P}_1$–$\mathfrak{P}_5$.

$\mathfrak{P}_1$. The sets $X^{\triangle}(\triangle \in D_G)$ form a partition of $\mathcal{U}$.
$\mathfrak{P}_2$. Let $\blacktriangle \in D_G/G$. The sets $X^{\triangle}(\triangle \in \blacktriangle)$ form a partition of $H^{\blacktriangle}$. More precisely, $H^{\blacktriangle}$ is a fibration over $\blacktriangle$ with smooth fibers isomorphic to $X^{\triangle}$ ($\triangle \in \blacktriangle$); in particular, $H^{\blacktriangle}$ is smooth.
$\mathfrak{P}_3$. The locally closed subets $H^{\blacktriangle}(\blacktriangle \in D_G/G)$ form a (finite) partition of $\mathcal{U}$.
$\mathfrak{P}_4$. Let $\triangle \in D_G$. We have $G_3^{\triangle} X^{\triangle} = X^{\triangle} G_3^{\triangle} = X^{\triangle}$.
$\mathfrak{P}_5$. Assume that $\mathbf{k} = \mathbf{k}_p$. Let $F\colon G \to G$ be the Frobenius map corresponding to a split $\mathbf{F}_q$-rational structure with $q-1$ sufficiently divisible. Let $\triangle \in D_G$ be such that $F(G_n^{\triangle}) = G_n^{\triangle}$ for all $n \geqslant 0$ and let $\blacktriangle$ be the G-orbit of $\triangle$. Then $|H^{\blacktriangle}(\mathbf{F}_q)|, |X^{\triangle}(\mathbf{F}_q)|$ are polynomials in q with integer coefficients independent of p.

Assume first that $p = 1$ or $p \gg 0$. By the theory of Dynkin–Kostant, for $\triangle \in D_G$ there is a unique open $G_0^{\triangle}$-orbit $X'^{\triangle}$ in $G_2^{\triangle}$; we then have a bijection of D_G/G with the set of unipotent classes on G, which to the G-orbit $\blacktriangle$ of $\triangle \in D_G$ associates the unique unipotent class $H'^{\blacktriangle}$ of G that contains $X'^{\triangle}$. Moreover, if $g \in X'^{\triangle}$, then $Z_{G_0^{\triangle}}(g) = Z_G(g)$. As stated by Kawanaka [Ka], the same holds when p is a good prime of G (but his argument is rather sketchy). To show that $\mathfrak{P}_1 - \mathfrak{P}_3$ holds when p is a good prime, it then suffices to show that $X^{\triangle} = X'^{\triangle}$ for any $\triangle$. It also suffices to show that $X'^{\triangle} = G_2^{\triangle} \cap H'^{\blacktriangle}$ for $\triangle \in \blacktriangle$ as above. (Assume that $g \in G_2^{\triangle} \cap H'^{\blacktriangle}, g \notin X'^{\triangle}$. Let $g' \in X'^{\triangle}$. By the definition of $X'^{\triangle}$ and the irreducibility of $G_2^{\triangle}$, the dimension of the $G_0^{\triangle}$-orbit of g is strictly smaller than the dimension of the $G_0^{\triangle}$-orbit of g'. Hence $\dim Z_{G_0^{\triangle}}(g) > \dim Z_{G_0^{\triangle}}(g')$. We have $\dim Z_G(g) \geqslant \dim Z_{G_0^{\triangle}}(g)$, $\dim Z_{G_0^{\triangle}}(g') = \dim Z_G(g')$, hence $\dim Z_G(g) > \dim Z_G(g')$. This contradicts the fact that g, g' are G-conjugate.) In this case we have $H^{\blacktriangle} = H'^{\blacktriangle}$ and $\widetilde{H}^{\blacktriangle}$ is the closure of $H'^{\blacktriangle}$.

In 2.9 (respectively 3.13, 3.14) we shall verify that $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for any p when G is a general linear (respectively symplectic) group. We will show elsewhere that $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold when G is a special orthogonal group (any p). If G is of type E_n (any p), one can deduce $\mathfrak{P}_1$–$\mathfrak{P}_5$ from the various lemmas in [M], or, rather, from the extensive computations (largely omitted) on which those lemmas are based; it would be desirable to have an independent verification of these properties.

We note the following consequence of $\mathfrak{P}_1$.

(c) *If* $\triangle \in D_G$ *and* $u \in X^\triangle$*, then* $Z_G(u) \subset G_0^\triangle$*.*

Let $g \in G$. Then $gug^{-1} \in X^{g\triangle}$. Hence if $g \in Z_G(u)$, we have $u \in X^{g\triangle}$. Thus, $X^{g\triangle} \cap X^\triangle \neq \varnothing$. From $\mathfrak{P}_1$ we see that $g\triangle = \triangle$. In particular $gG_0^\triangle g^{-1} = G_0^\triangle$ and $g \in G_0^\triangle$, as required.

1.2. Let $\triangle \in D_G$. We assume that $\mathfrak{P}_1$–$\mathfrak{P}_4$ hold for G. Let $\pi^\triangle : G_2^\triangle \to G_2^\triangle / G_3^\triangle$ be the obvious homomorphism. By $\mathfrak{P}_4$ we have $X^\triangle = (\pi^\triangle)^{-1}(\overline{X}^\triangle)$, where $\overline{X}^\triangle$ is a well defined open dense subset of $G_2^\triangle / G_3^\triangle$ stable under the action of $\overline{G}_0^\triangle$. We wish to consider some properties of the sets $\overline{X}^\triangle$, which may or may not hold for G.

- $\mathfrak{P}_6$. If $u \in X^\triangle$, then $uG_3^\triangle = G_3^\triangle u$ is contained in the $G_0^\triangle$-conjugacy class of u. Hence $\gamma \mapsto (\pi^\triangle)^{-1}(\gamma)$ is a bijection between the set of $\overline{G}_0^\triangle$-orbits in $\overline{X}^\triangle$ and the set of $G_0^\triangle$-conjugacy classes in $X^\triangle$.
- $\mathfrak{P}_7$. Let γ be a $\overline{G}_0^\triangle$-orbit in $\overline{X}^\triangle$. Let $\hat{\gamma}$ be the union of all $\overline{G}_0^\triangle$-orbits in $\overline{X}^\triangle$ whose closure contains γ. Thus, $\hat{\gamma}$ is an open subset of $\overline{X}^\triangle$ and γ is a closed subset of $\hat{\gamma}$. There exist a variety γ_1 and a morphism $\rho\colon \hat{\gamma} \to \gamma_1$ such that the restriction of ρ to γ is a finite bijective morphism $\sigma\colon \gamma \to \gamma_1$ and the map of sets $\sigma^{-1}\rho\colon \hat{\gamma} \to \gamma$ is compatible with the actions of $\overline{G}_0^\triangle$.
- $\mathfrak{P}_8$. There exist a finite set I and a bijection $J \mapsto \Phi_J$ between the set of subsets of I and the set of $G_0^\triangle$-orbits in $X^\triangle$ such that for any $J \subset I$, the closure of Φ_J in $X^\triangle$ is $\cup_{J';\, J \subset J'} \Phi_{J'}$. Moreover, if $\mathbf{k}, q$ are as in $\mathfrak{P}_5$, then there exists a function $I \to \{2, 4, 6, \ldots\}, i \mapsto c_i$ such that $|\Phi_J(\mathbf{F}_q)| = \prod_{i \in J}(q^{c_i} - 1)|\Phi_\varnothing(\mathbf{F}_q)|$ for any $J \subset I$.

When $p = 1$ or $p \gg 0$, property $\mathfrak{P}_6$ can be deduced from the theory of Dynkin–Kostant; properties $\mathfrak{P}_7, \mathfrak{P}_8$ are trivial. In the case where $G = \mathrm{GL}_n(\mathbf{k})$ (any p), the validity of $\mathfrak{P}_6$ follows from 2.9; properties $\mathfrak{P}_7, \mathfrak{P}_8$ are trivial. In the case where G is a symplectic group (any p), the validity of $\mathfrak{P}_6$–$\mathfrak{P}_8$ follows from 3.14. $\mathfrak{P}_6$ is false for G of type G_2, $p = 3$.

1.3. Let $\mathbf{V}$ be a finite dimensional $\mathbf{Q}$-vector space. Let $R \subset \mathbf{V}^* = \mathrm{Hom}(\mathbf{V}, \mathbf{Q})$ be a (reduced) root system, let $\check{R} \subset \mathbf{V}$ be the corresponding set of coroots, and let $\mathbf{W} \subset \mathrm{GL}(\mathbf{V})$ be the Weyl group of R. Let $\beta \leftrightarrow \check{\beta}$ be the canonical bijection $R \leftrightarrow \check{R}$. Let Π be a set of simple roots for R and let $\check{\Pi} = \{\check{\alpha} \mid \alpha \in \Pi\}$. Let

$$\Theta = \{\beta \in R \mid \beta - \alpha \notin R \quad \text{for all } \alpha \in \Pi\},$$
$$\widetilde{\Theta} = \{\beta \in R \mid \check{\beta} - \check{\alpha} \notin \check{R} \quad \text{for all } \alpha \in \Pi\},$$
$$\mathcal{A} = \{J \subset \Pi \cup \Theta \mid J \text{ linearly independent in } \mathbf{V}^*\},$$
$$\widetilde{\mathcal{A}} = \{J \subset \Pi \cup \widetilde{\Theta} \mid J \text{ linearly independent in } \mathbf{V}^*\}.$$

For any prime number r, let $\mathcal{A}_r$ be the set of all $J \in \mathcal{A}$ such that

$$\sum_{\alpha \in \Pi} \mathbf{Z}\alpha \Big/ \sum_{\beta \in J} \mathbf{Z}\beta$$

is finite of order r^k for some $k \in \mathbf{N}$.

For any $J \in \mathcal{A}$ or $J \in \widetilde{\mathcal{A}}$, let $\mathbf{W}_J$ be the subgroup of $\mathbf{W}$ generated by the reflections with respects to roots in J. For $W' = \mathbf{W}$ or $\mathbf{W}_J$, let $\mathrm{Irr}(W')$ be the set of (isomorphism classes) of irreducible representations of W' over $\mathbf{Q}$. For $E \in \mathrm{Irr}(W')$, let b_E be the smallest integer $\geqslant 0$ such that E appears with nonzero multiplicity in the b_E-th symmetric power of $\mathbf{V}$ regarded as a W'-module; if this multiplicity is 1, we say that E is good. If J is as above and $E \in \mathrm{Irr}(W')$ is good, then there is a unique $\widetilde{E} \in \mathrm{Irr}(\mathbf{W})$ such that $\widetilde{E}$ appears in $\mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}} E$ and $b_{\widetilde{E}} = b_E$; moreover, $\widetilde{E}$ is good. We set $\widetilde{E} = j_{\mathbf{W}_J}^{\mathbf{W}} E$.

Let $\mathcal{S}_{\mathbf{W}} \subset \mathrm{Irr}(\mathbf{W})$ be the set of special representations of $\mathbf{W}$ (see [L1]). Now any $E \in \mathcal{S}_{\mathbf{W}}$ is good. Following [L1], let $\mathcal{S}^1_{\mathbf{W}}$ be the set of all $E \in \mathrm{Irr}(\mathbf{W})$ such that $E = j_{\mathbf{W}_J}^{\mathbf{W}} E_1$ for some $J \in \widetilde{\mathcal{A}}$ and some $E_1 \in \mathcal{S}_{\mathbf{W}_J}$. (Note that $\mathbf{W}_J$ is like $\mathbf{W}$ with the same $\mathbf{V}$ and with R replaced by the root system with J as the set of simple roots; hence $\mathcal{S}_{\mathbf{W}_J}$ is defined.) Now any $E \in \mathcal{S}^1_{\mathbf{W}}$ is good.

For any prime number r, let $\mathcal{S}^r_{\mathbf{W}}$ be the set of all $E \in \mathrm{Irr}(\mathbf{W})$ such that $E = j_{\mathbf{W}_K}^{\mathbf{W}} E_1$ for some $K \in \mathcal{A}_r$ and some $E_1 \in \mathcal{S}^1_{\mathbf{W}_K}$. (Note that $\mathbf{W}_K$ is like $\mathbf{W}$ with the same $\mathbf{V}$ and with R replaced by the root system with K as set of simple roots; hence $\mathcal{S}^1_{\mathbf{W}_K}$ is defined.)

We have $\mathcal{S}^1(\mathbf{W}) \subset \mathcal{S}^r(\mathbf{W})$. We have $\mathcal{S}^1(\mathbf{W}) = \mathcal{S}^r(\mathbf{W})$, if r is a good prime for $\mathbf{W}$ and also in the following cases: $\mathbf{W}$ of type $\mathsf{G}_2, r = 2$; $\mathbf{W}$ of type F_4, $r = 3$; $\mathbf{W}$ of type E_6; $\mathbf{W}$ of type E_7, $r = 3$; $\mathbf{W}$ of type E_8, $r = 5$. If $\mathbf{W}$ is of type G_2 and $r = 3$, then $\mathcal{S}^r(\mathbf{W}) - \mathcal{S}^1(\mathbf{W})$ consists of a single representation of dimension 1 coming under $j_{\mathbf{W}_J}^{\mathbf{W}}$ from a $\mathbf{W}_J$ of type A_2. If $\mathbf{W}$ is of type F_4 and $r = 2$, then $\mathcal{S}^r(\mathbf{W}) - \mathcal{S}^1(\mathbf{W})$ consists of four representations of dimensions 9/4/4/2 coming under $j_{\mathbf{W}_J}^{\mathbf{W}}$ from a $\mathbf{W}_J$ of type $\mathsf{C}_3\mathsf{A}_1/\mathsf{C}_3\mathsf{A}_1/\mathsf{B}_4/\mathsf{B}_4$. If $\mathbf{W}$ is of type E_7 and $r = 2$, then $\mathcal{S}^r(\mathbf{W}) - \mathcal{S}^1(\mathbf{W})$ consists of a single representation of dimensions 84 coming under $j_{\mathbf{W}_J}^{\mathbf{W}}$ from a $\mathbf{W}_J$ of type $\mathsf{D}_6\mathsf{A}_1$. If $\mathbf{W}$ is of type E_8 and $r = 2$, then $\mathcal{S}^r(\mathbf{W}) - \mathcal{S}^1(\mathbf{W})$ consists of four representations of dimensions 1050/840/168/972 coming under $j_{\mathbf{W}_J}^{\mathbf{W}}$ from a $\mathbf{W}_J$ of type $\mathsf{E}_7\mathsf{A}_1/\mathsf{D}_5\mathsf{A}_3/\mathsf{D}_8/\mathsf{E}_7\mathsf{A}_1$. If $\mathbf{W}$ is of type E_8 and $r = 3$, then $\mathcal{S}^r(\mathbf{W}) - \mathcal{S}^1(\mathbf{W})$ consists of a single representation of dimensions 175 coming under $j_{\mathbf{W}_J}^{\mathbf{W}}$ from a $\mathbf{W}_J$ of type $\mathsf{E}_6\mathsf{A}_2$.

1.4. Let $\mathbf{W}$ be the Weyl group of G. Let u be a unipotent element in G. Springer's correspondence (generalized to arbitrary characteristic) associates to u and to the trivial representation of $Z_G(u)/Z_G(u)^0$ a representation $\rho_u \in \mathrm{Irr}(\mathbf{W})$. Moreover $u \mapsto \rho_u$ defines an injective map from the set of unipotent classes in G to $\mathrm{Irr}(\mathbf{W})$. Let $\mathcal{X}^p(\mathbf{W})$ be the image of this map (p as in 0.1). We state:

(a) *if* $p = 1$, *we have* $\mathcal{X}^1(\mathbf{W}) = \mathcal{S}^1(\mathbf{W})$ (*see* [L1]);
(b) *if* $p > 1$, *we have* $\mathcal{X}^p(\mathbf{W}) = \mathcal{S}^p_{\mathbf{W}}$.

The proof of (b) follows from the explicit description of the Springer correspondence for small p given in [LS], [S2].

1.5. In this subsection we assume that G is adjoint and that $p > 1$. Let $G'(p)$ be the set of all $g \in G'$ (as in 1.1) such that $Z_{G'}(g_s)^0$ is semisimple and $g_s^{p^k} = 1$ for some $k \in \mathbf{N}$; here g_s is the semisimple part of g. One can reformulate 1.4(b) as follows: there is a natural surjective map

$$\Phi\colon \{G'\text{-conjugacy classes in } G'(p)\} \to \{\text{unipotent conjugacy classes in } G\}$$

which preserves the dimension of a conjugacy class.

Indeed, via the Springer correspondence, we may identify the source (respectively target) of Φ with $\bigoplus_J \mathcal{X}^1(\mathbf{W}_J)$, where J runs over the subsets in $\mathcal{A}_p$, modulo the action of the fundamental group of G (respectively with $\mathcal{X}^p(\mathbf{W})$), and then Φ is given by the maps $j_{\mathbf{W}_J}^{\mathbf{W}}$.

Although the definition of Φ is indirect, we can think of Φ as a process of "reduction mod p".

2. General linear groups

2.1. Let $\bar{\mathcal{C}}$ be the category whose objects are $\mathbf{Z}$-graded $\mathbf{k}$-vector spaces $\overline{V} = \bigoplus_{a\in\mathbf{Z}} \overline{V}_a$ such that $\dim \overline{V} < \infty$; the morphisms are linear maps respecting the grading. Let $\overline{V} \in \bar{\mathcal{C}}$. For $j \in \mathbf{Z}$, let $\mathrm{End}_j(\overline{V}) = \{T \in \mathrm{Hom}(\overline{V}, \overline{V}) \mid T(\overline{V}_a) \subset \overline{V}_{a+j} \text{ for all } a\}$. Let $\mathrm{End}_2^0(\overline{V})$ be the set of all $\nu \in \mathrm{End}_2(\overline{V})$ that satisfy the *Lefschetz condition*: $\nu^n\colon \overline{V}_{-n} \to \overline{V}_n$ is an isomorphism for any $n \geqslant 0$. Let $\nu \in \mathrm{End}_2^0(\overline{V})$. Define a graded subspace $P^\nu = \overline{V}^{\mathrm{prim}}$ of $\overline{V}$ by $P_a^\nu = \{x \in \overline{V}_a \mid \nu^{1-a}x = 0\}$ for $a \leqslant 0$, $P_a^\nu = 0$ for $a > 0$. A standard argument shows that $N^{(a-c)/2}\colon P_c^\nu \to \overline{V}_a$ is injective if $c \in a + 2\mathbf{Z}, c \leqslant a \leqslant -c$, and we have

(a) $\bigoplus_{c\in a+2\mathbf{Z};\ c\leqslant a\leqslant -c} P_c^\nu \xrightarrow{\sim} \overline{V}_a, (z_c) \mapsto \sum_{c\in a+2\mathbf{Z};\ c\leqslant a\leqslant -c} N^{(a-c)/2} z_c.$

We show:

(b) *Let* $j \in \mathbf{N}$, $R \in \mathrm{End}_{j+2}(\overline{V})$. *Then* $R = T\nu - \nu T$ *for some* $T \in \mathrm{End}_j(\overline{V})$.

Let $c \leqslant 0$. Since $\nu^{1-c}\colon \overline{V}_{j-c} \to \overline{V}_{j+c+2}$ is surjective, the induced map

$$\mathrm{Hom}(P_{-c}^\nu, \overline{V}_{j-c}) \longrightarrow \mathrm{Hom}(P_{-c}^\nu, \overline{V}_{j+c+2})$$

is surjective. Hence there exists $\tau_c \in \mathrm{Hom}(P_{-c}^\nu, \overline{V}_{j-c})$ such that

$$\nu^{1-c}\tau_c = -\sum_{i+i'=-c} \nu^i R \nu^{i'}.$$

For $k \in [0, -c]$, we define $\tau_{c,k} \in \mathrm{Hom}(P_c^\nu, \overline{V}_{c+2k+j})$ by $\tau_{c,0} = \tau_c$ and $\tau_{c,k} = \nu\tau_{c,k-1} + R\nu^{k-1}$ for $k \in [1, -c]$. Then $\nu\tau_{c,-c} + R\nu^{-c} = 0$. Let $T\colon \overline{V} \to \overline{V}$ be the unique linear map such that $T(\nu^k x) = \tau_{c,k}(x)$ for $x \in P_c^\nu, c \leqslant 0, k \in [0, -c]$. This T has the required property.

2.2. Let $\mathcal{C}$ be the category whose objects are $\mathbf{k}$-vector spaces of finite dimension; morphisms are linear maps. Let $V \in \mathcal{C}$. A collection of subspaces $V_* = (V_{\geqslant a})_{a\in\mathbf{Z}}$ of V is said to be a *filtration* of V if $V_{\geqslant a+1} \subset V_{\geqslant a}$ for all a, and $V_{\geqslant a} = 0$ for some a, $V_{\geqslant a} = V$ for some a. We say that V is *filtered* if a filtration V_* of V is given. Assume that this is the case. We set $\mathrm{gr}\, V_* = \bigoplus_{a\in\mathbf{Z}} \mathrm{gr}_a V_* \in \bar{\mathcal{C}}$, where $\mathrm{gr}_a V_* = V_{\geqslant a}/V_{\geqslant a+1}$. For any $j \in \mathbf{Z}$, let $E_{\geqq j} V_* = \{T \in \mathrm{End}(V) \mid T(V_{\geqslant a}) \subset V_{\geqslant a+j} \quad \text{for all } a\}$. Any such T induces a linear map $\overline{T} \in \mathrm{End}_j(\mathrm{gr}V_*)$.

2.3. Let $V \in \mathcal{C}$. Let $\mathrm{Nil}(V) = \{T \in \mathrm{End}(V) \mid T \text{ nilpotent }\}$. Let $N \in \mathrm{Nil}(V)$. When $p = 1$, the Dynkin–Kostant theory associates to $1 + N$ a canonical filtration V_*^N of V; in terms of a basis of V of the form

(a) $\{N^k v_r \mid r \in [1,t], k \in [0, e_r - 1]\}$ *with* $v_r \in V, e_r \geqslant 1, N^{e_r} v_r = 0$ *for* $r \in [1,t]$,

$V^N_{\geqslant a}$ is the subspace spanned by $\{N^k v_r \mid r \in [1,t], k \in [0, e_r-1], 2k+1 \geqslant e_r + a\}$. This subspace makes sense for any p, and we denote it in general by $V^N_{\geqslant a}$; it is independent of the choice of basis: we have

$$V^N_{\geqslant a} = \sum_{j \geqslant \max(0,a)} N^j(\ker N^{2j-a+1}).$$

The subspaces $V^N_{\geqslant a}$ form a filtration V^N_* of V; thus, V becomes a filtered vector space. From the definitions we see that

(b) $N \in E_{\geqslant 2} V^N_*$ *and* $\overline{N} \in \mathrm{End}_2(\mathrm{gr} V^N_*)$ *belongs to* $\mathrm{End}^0_2(\mathrm{gr}V^N_*)$.

Note that for any $j \geqslant 1$,

(c) $\dim P^{\overline{N}}_{1-j}$ *is the number of Jordan blocks of size* j *of* $N\colon V \to V$.

From 2.1(a) we deduce that for any $n \geqslant 0$:

(d) $\dim P^{\overline{N}}_{-n} = \dim \mathrm{gr}_{-n} V^N_* - \dim \mathrm{gr}_{-n-2} V^N_*$.

2.4. According to [D2, 1.6.1],

(a) *if* V_* *is a filtration of* V *and* $N \in E_{\geqslant 2} V_*$ *induces an element* $\nu \in \mathrm{End}^0_2(\mathrm{gr}V_*)$, *then* $V_* = V^N_*$.

We show that $V_{\geqslant a} = V^N_{\geqslant a}$ for all a. Let e be the smallest integer $\geqslant 0$ such that $N^e = 0$. We argue by induction on e. If $a \geqslant e$, then $\nu^a\colon \mathrm{gr}_{-a} V_* \to \mathrm{gr}_a V_*$ is both 0 and an isomorphism, hence $V_{\geqslant -a} = V_{\geqslant 1-a}$ and $V_{\geqslant a} = V_{\geqslant a+1}$. Thus $V_{\geqslant e} = V_{\geqslant e+1} = \ldots = 0$ and $V_{\geqslant 1-e} = V_{\geqslant -e} = \ldots = V$. Similarly, $V^N_{\geqslant e} = V^N_{\geqslant e+1} = \ldots = 0$ and $V^N_{\geqslant 1-e} = V^N_{\geqslant -e} = \ldots = V$. Hence $V_{\geqslant a} = V^N_{\geqslant a}$ if $a \geqslant e$ or if $a \leqslant 1-e$. This already suffices in the case where $e \leqslant 1$. Thus we may assume that $e \geqslant 2$. Now $\nu^{e-1}\colon \mathrm{gr}_{1-e} V_* \to \mathrm{gr}_{e-1} V_*$ is an isomorphism, that is, $N^{e-1}\colon V/V_{\geqslant 2-e} \to V_{\geqslant e-1}$ is an isomorphism. We see that $V_{\geqslant e-1} = N^{e-1}V$ and $V_{\geqslant 2-e} = \ker(N^{e-1})$. Hence if $2-e \leqslant a \leqslant e-1$, we have $N^{e-1}V \subset V_{\geqslant a} \subset \ker(N^{e-1})$; let $V'_{\geqslant a}$ be the image of $V_{\geqslant a}$ under the obvious map $\rho\colon \ker(N^{e-1}) \to V' := \ker(N^{e-1})/N^{e-1}V$. For $a \leqslant 1-e$, we set $V'_{\geqslant a} = V'$, and for $a \geqslant e$ we set $V'_{\geqslant a} = 0$. Now $(V'_{\geqslant a})_{a \in \mathbf{Z}}$ is a filtration of V', satisfying a property like (a) (with N replaced by the map $N'\colon V' \to V'$ induced by N). Since $N'^{e-1} = 0$, the induction hypothesis applies to N'; it shows that $V'_{\geqslant a} = V'^{N'}_{\geqslant a}$ for all a. Since for $2-e \leqslant a \leqslant e-1$, $V_{\geqslant a} = \rho^{-1}(V'_{\geqslant a})$, it follows that $V_{\geqslant a} = \rho^{-1}(V'^{N'}_{\geqslant a})$; similarly, $V^N_{\geqslant a} = \rho^{-1}(V'^{N'}_{\geqslant a})$, hence $V_{\geqslant a} = V^N_{\geqslant a}$. This completes the proof.

With notation in the proof above we have:

$V^N_{\geqslant a} = 0$ for $a \geqslant e$,
$V^N_{\geqslant a} = V$ for $a \leqslant 1-e$,
$V^N_{\geqslant a} = \rho^{-1}(V'^{N'}_{\geqslant a}), V'^{N'}_{\geqslant a} = \rho(V^N_{\geqslant a})$ for $e \geqslant 2$ and $2-e \leqslant a \leqslant e-1$,
$V^N_{\geqslant e-1} = N^{e-1}V$ if $e \geqslant 1$,
$V^N_{\geqslant 2-e} = \ker(N^{e-1})$ if $e \geqslant 1$.
We have $\mathrm{gr}_a V^N_* = 0$ for $a \geqslant e$ and for $a \leqslant -e$.

Note also that the proof above provides an alternative (inductive) definition of $V^N_{\geqslant a}$ that does not use a choice of basis.

2.5. Let V, N be as in 2.3. Let $V_* = V_*^N$. Let $\nu = \overline{N} \in \mathrm{End}_2(\mathrm{gr}V_*)$. We can find a grading $V = \bigoplus_{a \in \mathbf{Z}} V_a$ of V such that

(a) *$NV_a \subset V_{a+2}$ and $V_{\geqslant a} = V_a \bigoplus V_{a+1} \bigoplus \cdots$ for all a.*

For example, in terms of a basis of V as in 2.3(a), we can take V_a to be the subspace spanned by $\{N^k v_r \mid r \in [1,t], k \in [0, e_r - 1], 2k+1 = e_r + a\}$. Taking direct sum of the obvious isomorphisms $V_a \xrightarrow{\sim} \mathrm{gr}_a V_*$, we obtain an isomorphism of graded vector spaces $V \xrightarrow{\sim} \mathrm{gr}V_*$ under which N corresponds to ν. It follows that:

(b) *$N \in \mathrm{End}_2^0(V)$ (defined in terms of the grading $\bigoplus_a V_a$).*

We note the following result.

(c) *Let $n \geqslant 0$ and let $x \in P_{-n}^{\nu}$. There exists a representative $\dot{x}$ of x in $V_{\geqslant -n}$ such that $N^{n+1}\dot{x} = 0$.*

Let V_a be as above. There is a unique representative $\dot{x}$ of x in $V_{\geqslant -n}$ such that $\dot{x} \in V_{-n}$. We have $N^{n+1}\dot{x} \in V_n$, and the image of $N^{n+1}\dot{x}$ under the canonical isomorphism $V_n \xrightarrow{\sim} \mathrm{gr}_n V_*^N$ is 0; hence $N^{n+1}\dot{x} = 0$.

Let $E_{\geqslant 1}^N V_* = \{S \in E_{\geqslant 1} V_* \mid SN = NS\}$, $\mathrm{End}_1^{\nu}(\mathrm{gr}V_*) = \{\sigma \in \mathrm{End}_1(\mathrm{gr}V_*) \mid \sigma\nu = \nu\sigma\}$. We show:

(d) *The obvious map $E_{\geqslant 1}^N V_* \longrightarrow \mathrm{End}_1^{\nu}(\mathrm{gr}V_*), S \mapsto \overline{S}$ is surjective.*

Let $\sigma \in \mathrm{End}_1^{\nu}(\mathrm{gr}V_*)$. Let V_a be as above. In terms of these V_a, we define $V \xrightarrow{\sim} \mathrm{gr}V_*$ as above. Under this isomorphism, σ corresponds to a linear map $S \colon V \to V$. Clearly, $S \in E_{\geqslant 1}^N V_*$ and $\overline{S} = \sigma$.

2.6. Let V, N be as in 2.3. Let $V_* = V_*^N$. Now $1 + E_{\geqslant 1}V_*$ is a subgroup of $\mathrm{GL}(V)$ acting on $N + E_{\geqslant 3}V_*$ by conjugation. We show that

(a) *the conjugation action of $1 + E_{\geqslant 1}V_*$ on $N + E_{\geqslant 3}V_*$ is transitive.*

We must show: if $S \in E_{\geqslant 3}V_*$, then there exists $T \in E_{\geqslant 1}V_*$ such that $(1+T)N = (N+S)(1+T)$, that is, $TN - NT = S + ST$. We fix subspaces V_a as in 2.5. We have $S = \sum_{j \geqslant 3} S_j$, where $S_j \in \mathrm{End}(V)$ satisfy $S_j V_a \subset V_{a+j}$ for all a. We seek a linear map $T = \sum_{j \geqslant 1} T_j$, where $T_j \in \mathrm{End}(V)$ satisfy $T_j V_a \subset V_{a+j}$ for all a and $\sum_{j \geqslant 1}(T_j N - NT_j) = \sum_{j \geqslant 3} S_j + \sum_{j' \geqslant 3, j'' \geqslant 1} S_{j'} T_{j''}$, that is,

$$T_j N - NT_j = S_{j+2} + \sum_{j' \in [1, j-1]} S_{j+2-j'} T_{j'} \text{ for } j = 1, 2, \ldots. \qquad (*)$$

We show that this system of equations in T_j has a solution. We take $T_1 = 0$. Assume that T_j has been found for $j < j_0$ for some $j_0 \geqslant 2$ so that $(*)$ holds for $j < j_0$. We set $R = S_{j_0+2} + \sum_{j' \in [1, j_0-1]} S_{j+2-j'} T_{j'}$. Then $R(V_a) \subset V_{a+j_0+2}$ for any a. The equation $T_{j_0} N - NT_{j_0} = R$ can be solved by 2.1(b) (see 2.5(b)). This shows by induction that the system $(*)$ has a solution. Thus (a) is proved.

We now show:

(b) *if $\widetilde{N} \in N + E_{\geqslant 3}V_*$, then $V_*^{\widetilde{N}} = V_*$.*

Indeed by (a) we can find $u \in 1 + E_{\geqslant 1}V_*$ such that $\widetilde{N} = uNu^{-1}$. Since V_*^N is canonically attached to N, we have $V_{\geqslant a}^{uNu^{-1}} = u(V_{\geqslant a}^N) = V_{\geqslant a}^N$, and (b) follows. For example,

(c) *if* $\widetilde{N} = c_1N + c_2N^2 + \cdots + c_kN^k$, *where* $c_i \in \mathbf{k}, c_1 \neq 0$, *then* $V_*^{\widetilde{N}} = V_*^N$.

We may assume that $c_1 = 1$. Since $c_2N^2 + \cdots + c_kN^k \in E_{\geqslant 4}V_* \subset E_{\geqslant 3}V_*$, (b) is applicable and (c) follows.

2.7. Let V, N be as in 2.3. Let $V_* = V_*^N$. Let $\nu = \overline{N} \in \mathrm{End}_2(\mathrm{gr}V_*)$. Let $r \geqslant 2$ be such that $N^r = 0$ on V. Let W be an N-stable subspace of V such that there exists an N-stable complement of W in V, $N: W \to W$ has no Jordan block of size $\neq r, r-1$, and $N^{r-2} = 0$ on V/W. Then $W_* = W_*^N$ is defined. Define a linear map $\mu: \mathrm{gr}V_* \to \mathrm{gr}W_*$ as follows. Let $x \in \mathrm{gr}_aV_*$. We have uniquely $x = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c} \nu^{(a-c)/2}x_c$, where $x_c \in P_c^\nu$; we set

$$\mu(x) = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c, c=1-r \text{ or } 2-r} \nu^{(a-c)/2}x_c.$$

Let $\mathcal{X}$ be the set of N-stable complements of W in V. Then $\mathcal{X} \neq \varnothing$. For $Z \in \mathcal{X}$, define $\Pi_Z: V \to W$ by $\Pi_Z(w + z) = w$, where $w \in W, z \in Z$. Let $\overline{\Pi}_Z: \mathrm{gr}V_* \to \mathrm{gr}W_*$ be the map induced by Π_Z. We show that

(a) $\Pi_Z(V_{\geqslant a}) \subset W_{\geqslant a}$ *for all* a *and* $\overline{\Pi}_Z = \mu$.

We have $V_{\geqslant a} = W_{\geqslant a} \bigoplus Z_{\geqslant a}$. If $x \in V_{\geqslant a}, x = w + z, w \in W_{\geqslant a}, z \in Z_{\geqslant a}$, then $\Pi_Z(x) = w$. Thus $\Pi_Z(V_{\geqslant a}) \subset W_{\geqslant a}$. We can find direct sum decompositions $W = \bigoplus_m W_m, Z = \bigoplus_m Z_m$ such that $NW_m \subset W_{m+2}$, $NZ_m \subset Z_{m+2}$, and $N^m: W_{-m} \xrightarrow{\sim} W_m$, $N^m: Z_{-m} \xrightarrow{\sim} Z_m$ for $m \geqslant 0$ (see 2.5). Let $V_a = W_a \bigoplus Z_a$. Define V_a^{prim}, W_a^{prim}, Z_a^{prim} as in 2.1 in terms of N. We have $V_a^{\mathrm{prim}} = W_a^{\mathrm{prim}} \bigoplus Z_a^{\mathrm{prim}}$. We must show that $\overline{\Pi}_Z(x) = \mu(x)$ for $x \in \mathrm{gr}_aV_*$. It suffices to show: if $w \in W_a, z \in Z_a$, and $w + z = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c} \nu^{(a-c)/2}x_c$, where $x_c \in V_c^{\mathrm{prim}}$, then

$$w = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c, 1-r \leqslant c \leqslant 2-r} \nu^{(a-c)/2}x_c.$$

We have $x_c = w_c + z_c$, where $w_c \in W_c^{\mathrm{prim}}, z_c \in Z_c^{\mathrm{prim}}$, and

$$w = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c} \nu^{(a-c)/2}w_c.$$

Now if $W_c^{\mathrm{prim}} \neq 0$, then $1 - r \leqslant c \leqslant 2 - r$. Hence

$$w = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c, 1-r \leqslant c \leqslant 2-r} \nu^{(a-c)/2}w_c.$$

Also, $Z_{1-r}^{\mathrm{prim}} = Z_{2-r}^{\mathrm{prim}} = 0$ since $N: Z \to Z$ has no Jordan blocks of size $\geqslant r - 1$. Thus if $c \in a + 2\mathbf{Z}$, $c \leqslant a \leqslant -c, 1 - r \leqslant c \leqslant 2 - r$, then $z_c = 0$ and $x_c = w_c$. Thus $w = \sum_{c \in a+2\mathbf{Z};\ c \leqslant a \leqslant -c, 1-r \leqslant c \leqslant 2-r} \nu^{(a-c)/2}x_c$, as required.

Let $Z, Z' \in \mathcal{X}$. By the previous argument, $\Pi_Z, \Pi_{Z'}: V \to W$ both map $V_{\geqslant a}$ into $W_{\geqslant a}$ and induce the same map $\mathrm{gr}V_* \to \mathrm{gr}W_*$. It follows that $\Pi_Z - \Pi_{Z'}: V \to W$ maps $V_{\geqslant a}$ into $W_{\geqslant a+1}$. In other words,

(b) *if* $x \in V_{\geqslant a}$ *and* $x = w + z = w' + z'$*, where* $w, w' \in W, z \in Z, z' \in Z'$*, then* $w - w' \in W_{\geqslant a+1}$.

Define $\Phi \in \mathrm{GL}(V)$ by $\Phi(x) = x$ for $x \in W$, $\Phi(x) = x'$ for $x \in Z$ where $x' \in Z'$ is given by $x - x' \in W$. We show:

(c) $(1 - \Phi)V_{\geqslant a} \subset V_{\geqslant a+1}$ *for any* a.

Let $x \in V_{\geqslant a}$. We have $x = w + z = w' + z'$, where $w, w' \in W, z \in Z, z' \in Z'$. We have $\Phi(x) = w + z'$, hence $(1-\Phi)(x) = (w+z)-(w+z') = z - z' = w' - w$, and this belongs to $W_{\geqslant a+1}$ by (b).

We show:

(d) $\Phi N = N\Phi$.

Indeed, for $x = x_1 + x_2, x_1 \in W, x_2 \in Z$, we have $Nx = Nx_1 + Nx_2$ with $Nx_1 \in W$, $Nx_2 \in Z$, and $x_2 - x_2' \in W$ with $x_2' \in Z'$. We have $Nx_2 - Nx_2' \in W$ with $Nx_2 \in Z$, $Nx_2' \in Z'$. Hence $\Phi(Nx) = Nx_1 + Nx_2' = N(x_1 + x_2') = N\Phi(x)$, as required.

2.8. Let V, N be as in 2.3. Let $r \geqslant 1$ be such that $N^r = 0$. A subspace W of V is said to be *r-special* if $NW \subset W$, $N \colon W \to W$ has no Jordan blocks of size $\neq r$ and $N^{r-1} = 0$ on N/W. We show:

(a) *If* W, W'*, are* r*-special subspaces, then there exists a subspace* X *of* V *such that* $NX \subset X, W \bigoplus X = V, W' \bigoplus X = V$.

We argue by induction on r. If $r = 1$, the result is obvious; we have $W = W' = V$. Assume that $r \geqslant 2$. Let $V' = \ker N^{r-1}, V'' = \ker N^{r-2}$. Let $E \subset W, E' \subset W'$ be such that $W = E \bigoplus NE \bigoplus \cdots \bigoplus N^{r-1}E$, $W' = E' \bigoplus NE' \bigoplus \cdots \bigoplus N^{r-1}E'$. Clearly, $E \cap V' = 0$, $E' \cap V' = 0$, $NE \subset V'$, $NE \cap V'' = 0$, $NE' \subset V'$, $NE' \cap V'' = 0$. Let E'' be a subspace of V' such that E'' is a complement of $NE \bigoplus V''$ in V' and a complement of $NE' \bigoplus V''$ in V'. (Such E'' exists since $\dim(NE \bigoplus V'') = \dim(NE' \bigoplus V') = \dim E + \dim V'' = \dim E' + \dim V''$.) Then

$$W_1 = (E'' \bigoplus NE) + N(E'' \bigoplus NE) + \cdots + N^{r-2}(E'' \bigoplus NE),$$
$$W_1' = (E'' \bigoplus NE') + N(E'' \bigoplus NE') + \cdots + N^{r-2}(E'' \bigoplus NE')$$

are $(r-1)$-special subspaces of V'. By the induction hypothesis we can find an N-stable subspace X_1 of V' such that $V_1 \bigoplus X_1 = V', V_1' \bigoplus X_1 = V'$. Then $X = (E'' + N(E'') + \cdots + N^{r-2}(E'')) + X_1$ has the required properties.

(b) *If* W, W' *are* r*-special subspaces, then there exists* $g \in 1 + E_{\geqslant 1}V_*$ *such that* $g(W) = W'$*,* $gN = Ng$.

Let X be as in (a). Define $g \in \mathrm{GL}(V)$ by $g(x) = x$ for $x \in X$ and $g(w) = w'$ for $w \in W$, where $w' \in W'$ is given by $w - w' \in X$. Then $g(W) = W'$, $(g-1)X = 0$, and $(g-1)W \subset X$. Clearly, $gN = Ng$. We have $V_{\geqslant a} = W_{\geqslant a} \bigoplus X_{\geqslant a}$. It suffices to show that $(g-1)(W_{\geqslant a}) \subset X_{\geqslant a+1}$. Now $X = X_{\geqslant 2-r}$. We have $W = W_{\geqslant 1-r}, W_{\geqslant 2-r} = W_{\geqslant 3-r} = NW, W_{\geqslant 4-r} = W_{\geqslant 5-r} = N^2W, \ldots$. Now if $a \leqslant 1 - r$, then $(g-1)W_{\geqslant a} = (g-1)W \subset X = X_{\geqslant a+1}$. If $a = 2 - r$ or $a = 3 - r$, then

$$(g-1)W_{\geqslant a} = (g-1)NW = N(g-1)W \subset NX = NX_{\geqslant 2-r} \subset X_{\geqslant 4-r} \subset X_{\geqslant a+1}.$$

If $a = 4 - r$ or $a = 5 - r$, then

$$(g-1)W_{\geqslant a} = (g-1)N^2W = N^2(g-1)W \subset N^2X = N^2X_{\geqslant 2-r} \subset X_{\geqslant 6-r} \subset X_{\geqslant a+1}.$$

Continuing in this way, the result follows.

2.9. Let $V \in \bar{\mathcal{C}}$. Let $G = \mathrm{GL}(V)$. For any filtration V_* of V, let

$$\xi(V_*) = \{N \in \mathrm{Nil}(V) \mid V_*^N = V_*\} = \{N \in E_{\geqslant 2}V_* \mid \overline{N} \in \mathrm{End}_2^0(\mathrm{gr}V_*)\}$$

(see 2.3(b), 2.4(a)). The following three conditions are equivalent:

(i) $\xi(V_*) \neq \varnothing$;
(ii) $\mathrm{End}_2^0(\mathrm{gr}V_*) \neq \varnothing$;
(iii) $\dim \mathrm{gr}_n V_* = \dim \mathrm{gr}_{-n} V_* \geqslant \dim \mathrm{gr}_{-n-2} V_*$ *for all* $n \geqslant 0$.

We have (i)⇒(ii) by the definition of $\xi(V_*)$; we have (ii)⇒(iii) by 2.3(d). The fact that (iii)⇒(ii) is easily checked. If (ii) holds, we pick for any a a subspace V_a of $V_{\geqslant a}$ complementary to $V_{\geqslant a+1}$ and an element in $\mathrm{End}_2^0(V)$ (defined in terms of the grading $\bigoplus_a V_a$). This element is in $\xi(V_*)$ and (i) holds.

Let $\mathfrak{F}_V$ be the set of all filtrations V_* of V that satisfy (i)–(iii). From the definitions we have a bijection

(a) $\mathfrak{F}_V \xrightarrow{\sim} D_G, V_* \mapsto \triangle$

(D_G as in 1.1), where $\triangle = (G_0^\triangle \supset G_1^\triangle \supset G_2^\triangle \supset \dots)$ is defined in terms of V_* by $G_0^\triangle = E_{\geqslant 0}V_* \cap G$ and $G_n^\triangle = 1 + E_{\geqslant n}V_*$ for $n \geqslant 1$.

The sets $\xi(V_*)$ (with $V_* \in \mathfrak{F}_V$) form a partition of $\mathrm{Nil}(V)$. (If $N \in \mathrm{Nil}(V)$, we have $N \in \xi(V_*)$, where $V_* = V_*^N$).

Let $V_* \in \mathfrak{F}_V$. Let $\Pi = E_{\geqslant 0}V_* \cap G$. We show that $\xi(V_*)$ is a single Π-conjugacy class. Let $N, N' \in \xi(V_*)$. Since $V_*^N = V_*^{N'}$, we see from 2.3(d) that $\dim P_{1-j}^{\overline{N}} = \dim P_{1-j}^{\overline{N}'}$ for any $j \geqslant 0$. Using 2.3(c), we see that for any $j \geqslant 0$, N, N' have the same number of Jordan blocks of size j. Hence there exists $g \in G$ such that $N' = gNg^{-1}$. For any a, $gV_{\geqslant a}^N = V_{\geqslant a}^{N'} = V_{\geqslant a}^N$, hence $gV_{\geqslant a} = V_{\geqslant a}$. We see that $g \in E_{\geqslant 0}$, hence $g \in \Pi$ as required. Taking in the previous argument $N' = N$, we see that if $N \in \xi(V_*)$ and $g \in G$ satisfies $gNg^{-1} = N$, then $g \in \Pi$. Now any element in $E_{\geqslant 2}V_* - \xi(V_*)$ is in the closure of $\xi(V_*)$ (since $E_{\geqslant 2}V_*$ is irreducible and $\xi(V_*)$ is open in it (and nonempty), hence it is in the closure of the G-conjugacy class containing $\xi(V_*)$). We show that it is not contained in that G-conjugacy class. (Assume that it is. Then we can find $N \in \xi(V_*)$ and $N' \in E_{\geqslant 2}V_* - \xi(V_*)$ that are G-conjugate. Then the Π-orbit $\Pi(N)$ of N in $E_{\geqslant 2}V_*$ is $\xi(V_*)$, hence is dense in $E_{\geqslant 2}V_*$, while the Π-orbit $\Pi(N')$ of N' is contained in the proper closed subset $E_{\geqslant 2}V_* - \xi(V_*)$ of $E_{\geqslant 2}V_*$; hence $\dim \Pi(N) = \dim(E_{\geqslant 2}V_*) > \dim \Pi(N')$. It follows that $a < a'$ where a (respectively a') is the dimension of the centralizer of N (respectively N') in Π. Let $\widetilde{a}$ (respectively $\widetilde{a}'$) be the dimension of the centralizer of N (respectively N') in G. By an earlier argument we have $a = \widetilde{a}$. Obviously $a' \leqslant \widetilde{a}'$. Since N, N' are G-conjugate, we have $\widetilde{a} = \widetilde{a}'$. Thus, $\widetilde{a} = a < a' \leqslant \widetilde{a}' = \widetilde{a}$, a contradiction.) We see that $1 + \xi(V^*) = X^\triangle$, where $V_* \mapsto \triangle$ as in (a) and $X^\triangle$ is as in 1.1. Thus $\mathfrak{P}_1$ holds for G. From this, $\mathfrak{P}_2, \mathfrak{P}_3$ follow; $H^\blacktriangle$ in $\mathfrak{P}_2$ is a single conjugacy class in this case. Also, $\mathfrak{P}_8$ is trivial since $G_0^\triangle$ acts transitively on $X^\triangle$. Now $\mathfrak{P}_5$ is easily verified. $\mathfrak{P}_6$ (hence $\mathfrak{P}_4$) follows from 2.6(a); $\mathfrak{P}_7$ is trivial in this case.

3. Symplectic groups

3.1. In this section any text marked as ♠ . . . ♠ applies only in the case $p = 2$.

For $V, V' \in \mathcal{C}$, let $\mathrm{Bil}(V, V')$ be the space of all bilinear forms $V \times V' \to \mathbf{k}$. For $b \in \mathrm{Bil}(V, V')$, define $b^* \in \mathrm{Bil}(V', V)$ by $b^*(x, y) = b(y, x)$. We write $\mathrm{Bil}(V)$ instead of $\mathrm{Bil}(V, V)$. Let $\mathrm{Symp}(V)$ be the set of nondegenerate symplectic forms on V.

Let $\overline{V} \in \bar{\mathcal{C}}$. We say that $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(\overline{V})$ is *admissible* if $\langle x, y\rangle_0 = 0$ for $x \in \overline{V}_a, y \in \overline{V}_{a'}, a + a' \neq 0$. Assume that $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(\overline{V})$ is admissible and that $\nu \in \mathrm{End}_2^0(\overline{V})$ is *skew-adjoint*, that is, $\langle \nu(x), y\rangle_0 + \langle x, \nu(y)\rangle_0 = 0$ for $x, y \in \overline{V}$. For $n \geqslant 0$ we define a bilinear form $b_n \colon P^\nu_{-n} \times P^\nu_{-n} \to \mathbf{k}$ by $b_n(x, y) = \langle x, \nu^n y\rangle_0$. We show:

(a) $b_n(x, y) = (-1)^{n+1} b_n(y, x)$ *for* $x, y \in P^\nu_{-n}$.

Indeed,

$$b_n(x, y) = \langle x, \nu^n y\rangle_0 = (-1)^n \langle \nu^n x, y\rangle_0 = (-1)^{n+1}\langle y, \nu^n x\rangle_0 = (-1)^{n+1} b_n(y, x),$$

as required. We show:

(b) b_n *is nondegenerate.*

Let $y \in P^\nu_{-n}$ be such that $\langle x, \nu^n y\rangle_0 = 0$ for all $x \in P^\nu_{-n}$. If $x' \in P^\nu_{-n-2k}, k > 0$, we have $\langle \nu^k x', \nu^n y\rangle_0 = \pm\langle x, \nu^{n+k} y\rangle_0 = \pm\langle x, 0\rangle_0 = 0$. Since $\overline{V}_{-n} = \sum_{k\geqslant 0} \nu^k P^\nu_{-n-2k}$, we see that $\langle x, \nu^n y\rangle_0 = 0$ for all $x \in \overline{V}_{-n}$. Since $\langle \overline{V}_m, \nu^n y\rangle_0 = 0$ for $m \neq -n$, we see that $\langle \overline{V}, \nu^n y\rangle_0 = 0$. By the nondegeneracy of $\langle\ ,\ \rangle_0$, it follows that $\nu^n y = 0$. Since $\nu^n \colon \overline{V}_{-n} \xrightarrow{\sim} \overline{V}_n$, it follows that $y = 0$ as required. We show:

(c) *if* $n \geqslant 0$ *is even, then* b_n *is a symplectic form. Hence* $\dim P^\nu_{-n}$ *is even.*

Indeed, for $x \in P^\nu_{-n}$ we have $\langle x, \nu^n x\rangle_0 = \pm\langle \nu^{n/2} x, \nu^{n/2} x\rangle_0 = 0$.

3.2. Let $V \in \mathcal{C}$ and let $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$. Let

$$\mathrm{Sp}(\langle\ ,\ \rangle) = \{T \in GL(V) \mid T \text{ preserves } \langle\ ,\ \rangle\}.$$

For any subspace W of V, we set $W^\perp = \{x \in V \mid \langle x, W\rangle = 0\}$. A filtration V_* of V is said to be *self-dual* if $(V_{\geqslant a})^\perp = V_{\geqslant 1-a}$ for any a. It follows that

(a) $\langle V_{\geqslant a}, V_{\geqslant a'}\rangle = 0$ *if* $a + a' \geqslant 1$.

It also follows that there is a unique admissible $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(\mathrm{gr} V_*)$ such that for $x \in \mathrm{gr}_a V_*, y \in \mathrm{gr}_{-a} V_*$ we have $\langle x, y\rangle_0 = \langle \dot{x}, \dot{y}\rangle$, where $\dot{x} \in V_{\geqslant a}, \dot{y} \in V_{\geqslant -a}$ represent x, y. Moreover,

(b) *there exists a direct sum decomposition* $\bigoplus_{a\in\mathbf{Z}} V_a$ *of* V *such that* $V_{\geqslant a} = V_a \bigoplus V_{a+1} \bigoplus \cdots$ *for all* a *and* $\langle V_a, V_{a'}\rangle = 0$ *for all* a, a' *such that* $a + a' \neq 0$.

Let $\mathcal{M}_{\langle\ ,\ \rangle}$ be the set of $N \in \mathrm{Nil}(V)$ such that $\langle Nx, y\rangle + \langle x, Ny\rangle + \langle Nx, Ny\rangle = 0$ for all x, y or, equivalently, $1 + N \in \mathrm{Sp}(\langle\ ,\ \rangle)$. Define an involution $N \mapsto N^\dagger$ of $\mathcal{M}_{\langle\ ,\ \rangle}$ by $\langle x, Ny\rangle = \langle N^\dagger x, y\rangle$ for all $x, y \in V$ or, equivalently, by $N^\dagger = (1 + N)^{-1} - 1 = -N + N^2 - N^3 + \cdots$.

Let $N \in \mathcal{M}_{\langle\ ,\ \rangle}$. We set $V_* = V^N_*$. By 2.6(c) we have $V^{N^\dagger}_* = V_*$. We show:

(c) *the filtration* V_* *is self-dual.*

We argue by induction on e as in 2.4. If $a \geqslant e$, then $V_{\geqslant a} = 0, V_{\geqslant 1-a} = V$, and (c) holds. If $a \leqslant 1 - e$, then $V_{\geqslant a} = V$, $V_{\geqslant 1-a} = 0$, and (c) holds. If $e \leqslant 1$, this already suffices.

Hence we may assume that $e \geqslant 2$ and $2-e \leqslant a \leqslant e-1$, hence $2-e \leqslant 1-a \leqslant e-1$. Let $V' = \ker(N^{e-1})/\mathrm{Im}(N^{e-1})$. Let $\rho\colon \ker(N^{e-1}) \to V'$ be the canonical map. We have $N^{e-1}V = \ker((N^\dagger)^{e-1})^\perp = \ker(N^{e-1})^\perp$ since $(N^\dagger)^{e-1} = (-N)^{e-1}$. Hence $\langle\ ,\ \rangle$ induces $\langle\ ,\ \rangle' \in \mathrm{Symp}(V')$. Also N induces a linear map $N'\colon V' \to V'$ such that $N' \in \mathcal{M}_{\langle\ ,\ \rangle'}$. By the induction hypothesis, $V'^{N'}_{\geqslant 1-a}$ is the perpendicular in V' of $V'^{N'}_{\geqslant a}$. Hence $V_{\geqslant a} = \rho^{-1}(V'^{N'}_{\geqslant a})$ is the perpendicular in V of $V_{\geqslant 1-a} = \rho^{-1}(V'^{N'}_{\geqslant 1-a})$. This completes the proof.

Let $\nu \in \mathrm{End}^0_2(\mathrm{gr}V_*)$ be the endomorphism induced by N. We show that

(d) *ν is skew-adjoint (with respect to $\langle\ ,\ \rangle_0$ on $\mathrm{gr}\,V^N_*$).*

It suffices to show that, if $a+a'+2=0$ and $x \in V_{\geqslant a'}$, $y \in V_{\geqslant a}$, then $\langle Nx, y\rangle + \langle x, Ny\rangle = 0$. It suffices to show that $\langle Nx, Ny\rangle = 0$. From (a), (b) we see that $\langle V_{\geqslant -1-a}, Ny\rangle = 0$, hence it suffices to show that $Nx \in V_{\geqslant -1-a}$. We have $Nx \in V_{\geqslant a'+2} \subset V_{\geqslant -1-a}$ since $a'+2 > -1-a$. This proves (c).

3.3. ♠ In this subsection we assume that $p=2$. Let $V, \langle\ ,\ \rangle, N, \nu, \langle\ ,\ \rangle_0$ be as in 3.2. Let $V_* = V^N_*$. Then $b_n \in \mathrm{Bil}(P^\nu_{-n})$ is defined for $n \geqslant 0$, see 3.1. Let $\mathcal{L}$ be the set of all even integers $n \geqslant 2$ such that b_{n-1}, b_{n+1} are symplectic forms. Let $\mathcal{L}'$ be the set of all even integers $n \geqslant 2$ such that $b_{n-1}, b_{n+1}, b_{n+3}, \dots$ are symplectic forms or, equivalently, $\langle z, \nu^{n-1}(z)\rangle_0 = 0$ for all $z \in \mathrm{gr}_{1-n}V_*$. (Assume first that $b_{n-1}, b_{n+1}, b_{n+3}, \dots$ are symplectic forms. By 2.1(a), any $z \in \mathrm{gr}_{1-n}V_*$ is of the form $\sum_{k\geqslant 0}\nu^k z_k$, where $z_k \in P^\nu_{1-n-2k}$. For $k \geqslant 0$ we have $\langle \nu^k z_k, \nu^{n-1}(\nu^k z_k)\rangle_0 = 0$ since b_{n+2k-1} is symplectic. Since $z' \mapsto \langle z', \nu^{n-1}(z')\rangle_0$ is additive in z' it follows that $\langle z, \nu^{n-1}(z)\rangle_0 = 0$. Conversely, assume that $\langle z, \nu^{n-1}(z)\rangle_0 = 0$ for any $z \in \mathrm{gr}_{1-n}V_*$. In particular, for $k \geqslant 0$ and $z_k \in P^\nu_{1-n-2k}$, we have $\langle \nu^k z_k, \nu^{n-1}(\nu^k z_k)\rangle_0 = 0$, that is, $\langle z_k, \nu^{n-1+2k}z_k)\rangle_0 = 0$. We see that b_{n+2k-1} is symplectic.)

Clearly, $\mathcal{L}' \subset \mathcal{L}$.

For $n \in \mathcal{L}$, we define $q_n\colon P^\nu_{-n} \to \mathbf{k}$ by $q_n(x) = \langle \dot{x}, N^{n-1}\dot{x}\rangle$, where $\dot{x} \in V_{\geqslant -n}$ is a representative for $x \in P^\nu_{-n}$ such that $N^{n+1}\dot{x} = 0$ (see 2.5(c)). We show that $q_n(x)$ is well defined. It suffices to show that if $y \in V_{\geqslant 1-n}, N^{n+1}y = 0$, then $\langle \dot{x}+y, N^{n-1}(\dot{x}+y)\rangle = \langle \dot{x}, N^{n-1}\dot{x}\rangle$, that is, $\langle y, N^{n-1}(y)\rangle + \langle \dot{x}, N^{n-1}(y)\rangle + \langle y, N^{n-1}(\dot{x})\rangle = 0$. Since $N^{n+1}(\dot{x}) = 0$, we have

$$\langle \dot{x}, N^{n-1}(y)\rangle + \langle y, N^{n-1}(\dot{x})\rangle = \langle y, (N^{n-1} + (N^\dagger)^{n-1})(\dot{x})\rangle = \langle y, N^n(\dot{x})\rangle.$$

This is zero, since $y \in V_{\geqslant 1-n}, N^n(\dot{x}) \in V_{\geqslant n}$ and $1-n+n = 1$. It remains to show that $\langle y, N^{n-1}(y)\rangle = 0$. It suffices to show that $\langle z, \nu^{n-1}(z)\rangle_0 = 0$ for all $z \in \mathrm{gr}_{1-n}V_*$ such that $N^{n+1}z = 0$. By 2.1(a), any such z is of the form $z_0 + \nu z_1$, where $z_0 \in P^\nu_{1-n}, z_1 \in P^\nu_{-1-n}$. Now $z' \mapsto \langle z', \nu^{n-1}(z')\rangle_0$ is additive in z', hence it suffices to show that $\langle z_0, \nu^{n-1}(z_0)\rangle_0 = 0$ and $\langle \nu(z_1), \nu^{n-1}(\nu(z_1))\rangle_0 = 0$ for z_0, z_1 as above. This follows from our assumption that b_{n-1} and b_{n+1} are symplectic.

We show:

(a) *For $x, y \in P^\nu_{-n}$, we have $q_n(x+y) = q_n(x) + q_n(y) + b_n(x,y)$.*

Let $\dot{x}, \dot{y} \in V_{\geqslant -n}$ be representatives for x, y such that $N^{n+1}\dot{x} = 0$, $N^{n+1}\dot{y} = 0$. We must show that

$$\langle \dot{x}+\dot{y}, N^{n-1}(\dot{x}+\dot{y})\rangle = \langle \dot{x}, N^{n-1}(\dot{x})\rangle + \langle \dot{y}, N^{n-1}(\dot{y})\rangle + \langle \dot{x}, N^n(\dot{y})\rangle,$$

or that
$$\langle \dot{x}, N^{n-1}(\dot{y})\rangle + \langle \dot{y}, N^{n-1}(\dot{x})\rangle + \langle \dot{x}, N^{n}(\dot{y})\rangle = 0,$$
or that $\langle \dot{x}, ((N^{\dagger})^{n-1} + N^{n-1} + N^{n})\dot{y}\rangle = 0$. Since n is even, $(N^{\dagger})^{n-1} + N^{n-1} + N^{n}$ is a linear combination of $N^{n+1}, N^{n+2}, \dots$, and it remains to use the equality $N^{n+1}(\dot{y}) = 0$.

For $n \in \mathcal{L}'$, we define $Q_n\colon \mathrm{gr}_{-n}V_* \to \mathbf{k}$ by $Q_n(x) = \langle \dot{x}, N^{n-1}\dot{x}\rangle$, where $\dot{x} \in V_{\geqslant -n}$ is a representative for x. We show that $Q_n(x)$ is well defined. It suffices to show that if $y \in V_{\geqslant 1-n}$, then $\langle \dot{x} + y, N^{n-1}(\dot{x} + y)\rangle = \langle \dot{x}, N^{n-1}\dot{x}\rangle$, that is, $\langle y, N^{n-1}(y)\rangle + \langle \dot{x}, N^{n-1}(y)\rangle + \langle y, N^{n-1}(\dot{x})\rangle = 0$. We have
$$\langle \dot{x}, N^{n-1}(y)\rangle + \langle y, N^{n-1}(\dot{x})\rangle = \langle y, (N^{n-1} + (N^{\dagger})^{n-1})(\dot{x})\rangle,$$
and this is a linear combination of terms $\langle y, N^{n'}(\dot{x})\rangle$ with $n' \geqslant n$. Each of these terms is 0 since $y \in V_{\geqslant 1-n}, N^{n'}(\dot{x}) \in V_{\geqslant 2n'-n}$ and $1 - n + 2n' - n \geqslant 1$. It remains to show that $\langle y, N^{n-1}(y)\rangle = 0$. This follows from the fact that $\langle z, \nu^{n-1}(z)\rangle_0 = 0$ for all $z \in \mathrm{gr}_{1-n}V_*$.

For $n \in \mathcal{L}'$ we show:

(b) *if* $x, y \in \mathrm{gr}_{-n}V_*$, *then* $Q_n(x + y) = Q_n(x) + Q_n(y) + \langle x, \nu^n y\rangle$.

Let $\dot{x}, \dot{y} \in V_{\geqslant -n}$ be representatives for x, y. We must show that
$$\langle \dot{x} + \dot{y}, N^{n-1}(\dot{x} + \dot{y})\rangle = \langle \dot{x}, N^{n-1}(\dot{x})\rangle + \langle \dot{y}, N^{n-1}(\dot{y})\rangle + \langle \dot{x}, N^{n}(\dot{y})\rangle,$$
or that
$$\langle \dot{x}, N^{n-1}(\dot{y})\rangle + \langle \dot{y}, N^{n-1}(\dot{x})\rangle + \langle \dot{x}, N^{n}(\dot{y})\rangle = 0,$$
or that $\langle \dot{x}, ((N^{\dagger})^{n-1} + N^{n-1} + N^{n})\dot{y}\rangle$ is 0. Since n is even, this is a linear combination of terms $\langle \dot{x}, N^{n'}(\dot{y})\rangle$, with $n' > n$. Each of these terms is 0 since $N^{n'}(\dot{y}) \in V_{\geqslant 2n'-n}, \dot{x} \in V_{\geqslant -n}$, and $2n' - n - n \geqslant 1$.

Now let $n \in \mathcal{L}'$ and let $x \in \mathrm{gr}_{-n}V_*$. We can write $x = \sum_{k\geqslant 0} \nu^k x_k$, where $x_k \in P^{\nu}_{-n-2k}$. We show that

(c) $Q_n(x) = \sum_{k\geqslant 0} q_{n+2k}(x_k)$.

Let $\dot{x}_k$ be a representative of x_k in $V_{\geqslant -n-2k}$ such that $N^{n+2k+1}\dot{x}_k = 0$. Then $\sum_{k\geqslant 0} N^k\dot{x}_k$ is a representative of x in $V_{\geqslant -n}$ and we must show:
$$\langle \sum_{k\geqslant 0} N^k\dot{x}_k, N^{n-1}\sum_{k'\geqslant 0} N^{k'}\dot{x}_{k'}\rangle = \sum_{k\geqslant 0}\langle \dot{x}_k, N^{n+2k-1}\dot{x}_k\rangle.$$
The left-hand side is $\sum_{k,k'\geqslant 0}\langle N^k\dot{x}_k, N^{n-1+k'}\dot{x}_{k'}\rangle$. If $k \geqslant k' + 2$, we have
$$\langle N^k\dot{x}_k, N^{n-1+k'}\dot{x}_{k'}\rangle = \langle \dot{x}_k, (N^{\dagger})^k N^{n-1+k'}\dot{x}_{k'}\rangle,$$
and this is zero since $N^{n+2k'+1}\dot{x}_{k'} = 0$. If $k' \geqslant k + 2$, we have
$$\langle N^k\dot{x}_k, N^{n-1+k'}\dot{x}_{k'}\rangle = \langle (N^{\dagger})^{n-1+k'} N^k\dot{x}_k, \dot{x}_{k'}\rangle,$$
and this is zero since $N^{n+2k+1}\dot{x}_k = 0$. It suffices to show that
$$\sum_{k\geqslant 0}(\langle N^k\dot{x}_k, N^{n-1+k}\dot{x}_k\rangle + \langle N^{k+1}\dot{x}_{k+1}, N^{n-1+k}\dot{x}_k\rangle + \langle N^k\dot{x}_k, N^{n+k}\dot{x}_{k+1}\rangle)$$
$$= \sum_{k\geqslant 0}\langle \dot{x}_k, N^{n+2k-1}\dot{x}_k\rangle.$$

We have

$$\begin{aligned}\langle N^{k+1}\dot{x}_{k+1}, N^{n-1+k}\dot{x}_k\rangle + \langle N^k\dot{x}_k, N^{n+k}\dot{x}_{k+1}\rangle \\ &= \langle N^{k+1}\dot{x}_{k+1}, (N^{n-1+k} + (N^\dagger)^{n-1}N^k)\dot{x}_k\rangle \\ &= \langle N^{k+1}\dot{x}_{k+1}, (c_1N^{n+k} + c_2N^{n+k+1} + \cdots)\dot{x}_k\rangle \\ &= c_1\langle \nu^{k+1}x_{k+1}, \nu^{n+k}x_k\rangle_0 = c_1\langle x_{k+1}, \nu^{n+2k+1}x_k\rangle_0 = 0.\end{aligned}$$

(Here $c_1, c_2, \ldots \in \mathbf{k}$.) It suffices to show that $\langle N^k\dot{x}_k, N^{n-1+k}\dot{x}_k\rangle + \langle \dot{x}_k, N^{n+2k-1}\dot{x}_k\rangle$ is 0. This equals

$$\begin{aligned}\langle \dot{x}_k, (N^{n+2k-1} + (N^\dagger)^k N^{n-1+k})\dot{x}_k\rangle &= \langle \dot{x}_k, (N^{n+2k} + c'_1N^{n+2k+1} + \cdots)\dot{x}_k\rangle \\ &= \langle x_k, \nu^{n+2k}x_k\rangle_0 = \langle \nu^{k+n/2}x_k, \nu^{k+n/2}x_k\rangle_0 = 0.\end{aligned}$$

(Here $c'_1, c'_2, \ldots \in \mathbf{k}$.) This completes the proof of (c).

We say that $(q_n)_{n\in\mathcal{L}}$ are *the quadratic forms attached to* $(N, \langle\ ,\ \rangle)$. We say that $(Q_n)_{n\in\mathcal{L}'}$ are *the Quadratic forms attached to* $(N, \langle\ ,\ \rangle)$. ♠

3.4. Let $V \in \mathcal{C}$ and let V_* be a filtration of V. We fix $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(\mathrm{gr}V_*)$, which is admissible, and $\nu \in \mathrm{End}_2^0(\mathrm{gr}V_*)$, which is skew-adjoint with respect to $\langle\ ,\ \rangle_0$ (see 3.1). Then P^ν_{-n} are defined in terms of $\mathrm{gr}V_*, \nu$, and $b_n \in \mathrm{Bil}(P^\nu_{-n})$ are defined as in 3.1 for any $n \geqslant 0$. Let $\mathcal{V} = 1 + E_{\geqslant 1}V_*$, a subgroup of $\mathrm{GL}(V)$.

♠ If $p = 2$, let $\mathbf{n}$ be the smallest even integer $\geqslant 2$ such that $b_{\mathbf{n}-1}, b_{\mathbf{n}+1}, b_{\mathbf{n}+3}, \ldots$ are symplectic or, equivalently, such that $\langle z, \nu^{\mathbf{n}-1}(z)\rangle_0 = 0$ for all $z \in \mathrm{gr}_{1-\mathbf{n}}V_*$. Let $Q\colon \mathrm{gr}_{-\mathbf{n}}V_* \to \mathbf{k}$ be a quadratic form such that $Q(x+y) = Q(x) + Q(y) + \langle x, \nu^{\mathbf{n}}y\rangle$ for all $x, y \in \mathrm{gr}_{-\mathbf{n}}V_*$. ♠.

Let $\mathcal{Z}$ be the set of all pairs $(N, \langle\ ,\ \rangle)$, where $N \in \mathrm{Nil}(V)$, $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$ are such that $V^N_* = V_*$, $\langle Nx, y\rangle + \langle x, Ny\rangle + \langle Nx, Ny\rangle = 0$ for $x, y \in V$, N induces ν on $\mathrm{gr}V_*$, $\langle\ ,\ \rangle$ induces $\langle\ ,\ \rangle_0$ on $\mathrm{gr}V_*$; ♠ in the case $p = 2$, we require in addition that $Q_{\mathbf{n}}$ defined in terms of $(N, \langle\ ,\ \rangle)$, as in 3.3, is equal to Q. ♠

The proofs of Propositions 3.5, 3.6, 3.7 below are intertwined (see 3.11).

Proposition 3.5. *In the setup of* 3.4, *let* $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$ *be such that* V_* *is self-dual with respect to* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle$ *induces* $\langle\ ,\ \rangle_0$ *on* $\mathrm{gr}V_*$. *Let* $Y = Y_{\langle\ ,\ \rangle} = \{N \mid (N, \langle\ ,\ \rangle) \in \mathcal{Z}\}$. *Let* $U' = \mathcal{V} \cap \mathrm{Sp}(\langle\ ,\ \rangle)$, *a subgroup of* $\mathrm{Sp}(\langle\ ,\ \rangle)$. *Then*

(a) $Y \neq \varnothing$;
(b) *if* $N \in Y$ *and* $z \in U'$, *then* $zNz^{-1} \in Y$ *(thus* U' *acts an* Y *by conjugation);*
(c) *the action* (b) *of* U' *on* Y *is transitive.*

The proof of (a) is given in 3.8. Now (b) follows immediately from Proposition 3.7(a).

We show that (c) is a consequence of Proposition 3.7(c). Assume that Proposition 3.7(c) holds. Let $N, N' \in Y$. We have $(N, \langle\ ,\ \rangle) \in \mathcal{Z}, (N', \langle\ ,\ \rangle) \in \mathcal{Z}$, and by Proposition 3.7(c) there exists $g \in \mathcal{V}$ such that $N' = gNg^{-1}$, $\langle g^{-1}x, g^{-1}y\rangle = \langle x, y\rangle$ for $x, y \in V$. Then $g \in U'$ and (c) is proved (assuming Proposition 3.7(c)).

Proposition 3.6. *In the setup of* 3.4, *let* $N \in \mathrm{Nil}(V)$ *be such that* $V^N_* = V_*$ *and* N *induces* ν *on* $\mathrm{gr}V_*$. *Let* $X = X_N = \{\langle\ ,\ \rangle \mid (N, \langle\ ,\ \rangle) \in \mathcal{Z}\}$. *Let* $U = U_N = \{T \in \mathcal{V} \mid TN = NT\}$, *a subgroup of* $\mathrm{GL}(V)$. *Then:*

(a) *$X \neq \varnothing$;*
(b) *if $\langle\ ,\ \rangle \in X$ and $u \in U$, then the symplectic form $\langle\ ,\ \rangle'$ on V given by $\langle x,y\rangle' = \langle u^{-1}x, u^{-1}y\rangle$ belongs to X (thus U acts naturally an X);*
(c) *the action* (b) *of U on X is transitive.*

We show that (a) is a consequence of Proposition 3.7(a). By Proposition 3.7(a) there exists $(N', \langle\ ,\ \rangle') \in \mathcal{Z}$. By 2.6(a) there exists $g \in \mathcal{V}$ such that $N = gN'g^{-1}$. Define $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$ by $\langle\ ,\ \rangle = \langle g^{-1}x, g^{-1}y\rangle'$. From 3.7(a) we see that $(N, \langle\ ,\ \rangle) \in \mathcal{Z}$ hence $\langle\ ,\ \rangle \in X_N$. Thus $X_N \neq \varnothing$, as required.

Now (b) follows immediately from 3.7(b). The proof of (c) is given in 3.9, 3.10.

Proposition 3.7. *In the setup of* 3.4*,*

(a) *$\mathcal{Z} \neq \varnothing$;*
(b) *if $(N, \langle\ ,\ \rangle) \in \mathcal{Z}$, $g \in \mathcal{V}$, and $(N', \langle\ ,\ \rangle')$ is defined by $N' = gNg^{-1}$, $\langle x,y\rangle' = \langle g^{-1}x, g^{-1}y\rangle$, then $(N', \langle\ ,\ \rangle') \in \mathcal{Z}$ (thus $\mathcal{V}$ acts naturally on $\mathcal{Z}$);*
(c) *the action* (b) *of $\mathcal{V}$ on $\mathcal{Z}$ is transitive.*

Clearly (a) is a consequence of Proposition 3.5(a).

We prove (b). We have $V^{N'}_{\geqslant a} = gV^{N}_{\geqslant a} = V^{N}_{\geqslant a} = V_{\geqslant a}$. Next we must show that we have $\langle gNg^{-1}x, y\rangle' + \langle x, gNg^{-1}y\rangle' + \langle gNg^{-1}x, gNg^{-1}y\rangle' = 0$ for $x, y \in V$, that is, $\langle Ng^{-1}x, g^{-1}y\rangle + \langle g^{-1}x, Ng^{-1}y\rangle + \langle Ng^{-1}x, Ng^{-1}y\rangle = 0$ for $x, y \in V$. This follows from $\langle Nx', y'\rangle + \langle x', Ny'\rangle + \langle Nx', Ny'\rangle = 0$ for $x', y' \in V$. Next we must show that gNg^{-1}, N induce the same map $\mathrm{gr}V_* \to \mathrm{gr}V_*$. (We must show that if $x \in V_{\geqslant a}$, then $gNg^{-1}(x) - Nx \in V_{\geqslant a+3}$; this follows from $g \in \mathcal{V}$.) Next we must show that for $x \in V_{\geqslant -a}, y \in V_{\geqslant a}$ we have $\langle x,y\rangle' = \langle x,y\rangle$, that is, $\langle g^{-1}x, g^{-1}y\rangle = \langle x,y\rangle$. Set $g^{-1} = 1 + S$, where $S \in E_{\geqslant 1}V_*$. We must show that $\langle Sx, y\rangle + \langle x, Sy\rangle + \langle Sx, Sy\rangle = 0$. But $Sx \in V_{\geqslant 1-a}, y \in V_{\geqslant a}$ implies $\langle Sx, y\rangle = 0$. Similarly $\langle x, Sy\rangle = 0, \langle Sx, Sy\rangle = 0$.

♠ In the case where $p = 2$ we see that the number $\mathbf{n}$ defined in terms of $N, \langle\ ,\ \rangle$ is the same as that defined in terms of $N', \langle\ ,\ \rangle'$, and we must check that for $x \in V_{\geqslant -\mathbf{n}}$ we have $\langle x, (gNg^{-1})^{\mathbf{n}-1}x\rangle' = \langle x, N^{\mathbf{n}-1}x\rangle$, that is, $\langle g^{-1}x, N^{\mathbf{n}-1}g^{-1}x\rangle = \langle x, N^{\mathbf{n}-1}x\rangle$, that is, $\langle Sx, N^{\mathbf{n}-1}x\rangle + \langle x, N^{\mathbf{n}-1}Sx\rangle + \langle Sx, N^{\mathbf{n}-1}Sx\rangle = 0$. We have $\langle Sx, N^{\mathbf{n}-1}x\rangle + \langle x, N^{\mathbf{n}-1}Sx\rangle = \langle x, (N^{\mathbf{n}-1} + (N^{\dagger})^{\mathbf{n}-1})Sx\rangle$. This is a linear combination of terms $\langle x, N^{n'}Sx\rangle$, where $n' \geqslant \mathbf{n}$; each of these terms is zero since $x \in V_{\geqslant -\mathbf{n}}, N^{n'}Sx \in V_{\geqslant 2n'-\mathbf{n}+1}$, and $-\mathbf{n} + 2n' - \mathbf{n} + 1 \geqslant 1$. Next we have $\langle Sx, N^{\mathbf{n}-1}Sx\rangle = 0$ since $\langle y, N^{\mathbf{n}-1}y\rangle = 0$ for all $y \in V_{\geqslant 1-\mathbf{n}}$ by the definition of $\mathbf{n}$. ♠

This completes the proof of (b).

We show that (c) is a consequence of Proposition 3.6(c). Let $(N, \langle\ ,\ \rangle) \in \mathcal{Z}, (N', \langle\ ,\ \rangle') \in \mathcal{Z}$. By 2.6(a), since $V^N_* = V^{N'}_*$ and N, N' induce the same ν, we can find $S \in E_{\geqslant 1}V_*$ such that $R = 1 + S$ satisfies $N'R = RN$. Define $\langle\ ,\ \rangle'' \in \mathrm{Symp}(V)$ by $\langle x,y\rangle'' = \langle Rx, Ry\rangle'$. From (b) we see that $(R^{-1}N'R, \langle\ ,\ \rangle'') \in \mathcal{Z}$, that is, $(N, \langle\ ,\ \rangle'') \in \mathcal{Z}$. Thus $\langle\ ,\ \rangle \in X_N, \langle\ ,\ \rangle'' \in X_N$. By Proposition 3.6(c) we can find $S' \in E_{\geqslant 1}V_*$ such that $R' = 1 + S'$ satisfies $R'N = NR'$ and $\langle x,y\rangle = \langle R'x, R'y\rangle''$ for all x, y, that is, $\langle x,y\rangle = \langle RR'x, RR'y\rangle'$. Then $RR' \in U'$ and $RR'N = RNR' = N'RR'$. Thus under the action (b), RR' carries $(N, \langle\ ,\ \rangle)$ to $(N', \langle\ ,\ \rangle')$. This proves (c) (assuming Proposition 3.6(c) holds).

3.8. *Proof of* 3.5(a). We choose a direct sum decomposition $\bigoplus_{a\in\mathbf{Z}} V_a$ of V as in 3.2(b). Define $N_2 \in \mathrm{End}_2(V)$ by the requirement that $N_2\colon V_a \to V_{a+2}$ corresponds to $\nu\colon \mathrm{gr}_a V_* \to \mathrm{gr}_{a+2}V_*$ under the obvious isomorphisms $V_a \xrightarrow{\sim} \mathrm{gr}_a V_*$, $V_{a+2} \xrightarrow{\sim} \mathrm{gr}_{a+2}V_*$.

♠. If $p=2$, we regard Q as a quadratic form on $V_{-\mathbf{n}}$ via the obvious isomorphism $V_{-\mathbf{n}} \xrightarrow{\sim} \mathrm{gr}_{-\mathbf{n}} V_*$. ♠

We will construct a linear map $N = \sum_{j\geqslant 1} N_{2j}$, where N_2 is as above, and for $j \geqslant 2$, $N_{2j} \in \mathrm{End}(V)$ satisfy $N_{2j} V_a \subset V_{a+2j}$ for all a and

$$\langle \sum_{j\geqslant 1} N_{2j}x, y\rangle + \langle x, \sum_{j\geqslant 1} N_{2j} y\rangle + \langle \sum_{j'\geqslant 1} N_{2j'}x, \sum_{j''\geqslant 1} N_{2j''} y\rangle = 0$$

for any a, c and any $x \in V_a, y \in V_c$, that is,

$$\langle N_{2j}x, y\rangle + \langle x, N_{2j}y\rangle + \sum_{j',j''\geqslant 1;\ j'+j''=j} \langle N_{2j'}x, N_{2j''}y\rangle = 0 \tag{a}$$

for any $j \geqslant 1$, any a, c such that $a + c + 2j = 0$, and any $x \in V_a, y \in V_c$.

♠ If $p = 2$, we require in addition that $\langle x, N^{\mathbf{n}-1}x\rangle = Q(x)$ for all $x \in V_{-\mathbf{n}}$, that is, $\sum_{i+i'=\mathbf{n}-2} \langle x, N_2^i N_4 N_2^{i'} x\rangle = Q(x)$ for all $x \in V_{-\mathbf{n}}$. ♠

We shall determine N_j by induction on j. For $j = 1$ the equation (a) is just $\langle N_2x, y\rangle + \langle x, N_2y\rangle = 0$ for any a, c such that $a + c + 2 = 0$ and any $x \in V_a, y \in V_c$; this holds automatically by our choice of N_2. For $x \in V_a$, with $a < -2$, we set $N_4(x) = 0$. Then the equation (a) for $j = 2$ becomes:

(b) $\langle N_4x, y\rangle + \langle x, N_4y\rangle = -\langle N_2x, N_2y\rangle$ for any $x \in V_{-2}, y \in V_{-2}$, $\langle N_4x, y\rangle = -\langle N_2x, N_2y\rangle$ for any $a > -2, x \in V_a, y \in V_{-a-4}$.

The second equation in (b) determines uniquely $N_4(x)$ for $x \in V_a, a > -2$. Since $\langle N_2x, N_2y\rangle$ is a symplectic form on V_{-2}, we can find $[\ ,\ ] \in \mathrm{Bil}(V_{-2})$ such that $[x, y] - [y, x] = -\langle N_2x, N_2y\rangle$ for any $x, y \in V_{-2}$. There is a unique linear map $N_4 : V_{-2} \to V_2$ such that $\langle N_4x, y\rangle = [x, y]$ for any $x, y \in V_{-2}$. Then equation (a) for $j = 2$ is satisfied.

♠ If $p = 2$, the N_4 just determined satisfies $\sum_{i+i'=\mathbf{n}-2} \langle x, N_2^i N_4 N_2^{i'} x\rangle = Q'(x)$ for all $x \in V_{-\mathbf{n}}$, for some quadratic form $Q' : V_{-\mathbf{n}} \to \mathbf{k}$ not necessarily equal to Q. For $x, y \in V_{-\mathbf{n}}$ we have (by the choice of N_4):

$$\begin{aligned} Q'(x+y) - Q'(x) - Q'(y) &= \sum_{i+i'=\mathbf{n}-2} \langle x, N_2^i N_4 N_2^{i'} y\rangle + \sum_{i+i'=\mathbf{n}-2} \langle y, N_2^i N_4 N_2^{i'} x\rangle \\ &= \sum_{i+i'=\mathbf{n}-2} \langle N_2^i x, N_4 N_2^{i'} y\rangle + \sum_{i+i'=\mathbf{n}-2} \langle N_4 N_2^i x, N_2^{i'} y\rangle \\ &= \sum_{i+i'=\mathbf{n}-2} \langle N_2 N_2^i x, N_2 N_2^{i'} y\rangle = \sum_{i+i'=\mathbf{n}-2} \langle x, N_2^{\mathbf{n}} y\rangle = \langle x, N_2^{\mathbf{n}} y\rangle \\ &= Q(x+y) - Q(x) - Q(y). \end{aligned}$$

It follows that $Q'(x) = Q(x) + \theta(x)^2$, where $\theta \in \mathrm{Hom}(V_{-\mathbf{n}}, \mathbf{k})$. We try to find $\zeta \in \mathrm{End}(V)$ with $\zeta(V_a) \subset V_{a+4}$ for all a in such a way that (a) (for $j = 2$) remains true when N_4 is replaced by $N_4 + \zeta$ and $\sum_{i+i'=\mathbf{n}-2} \langle x, N_2^i (N_4 + \zeta) N_2^{i'} x\rangle = Q(x)$ for $x \in V_{-\mathbf{n}}$. (Then $N_4 + \zeta$ will be our new N_4.) Thus we are seeking ζ such that

$$\langle \zeta(x), y\rangle + \langle x, \zeta(y)\rangle = 0 \text{ for any } a, c \text{ with } a + c + 4 = 0 \text{ and } x \in V_a, y \in V_c,$$
$$\sum_{i+i'=\mathbf{n}-2} \langle x, N_2^i \zeta N_2^{i'} x\rangle = \theta(x)^2 \text{ for } x \in V_{-\mathbf{n}}.$$

The first of these two equations can be satisfied for $(a,c)\neq(-2,-2)$ by defining $\zeta(x)=0$ for $x\in V_a, a\neq -2$. Then in the second equation the terms corresponding to i' such that $2i'-\mathbf{n}\neq -2$ are 0. Thus it remains to find a linear map $\zeta\colon V_{-2}\to V_2$ such that

$$\langle\zeta(x),y\rangle+\langle x,\zeta(y)\rangle=0 \text{ for any } x,y\in V_{-2},$$
$$\langle N_2^t x,\zeta N_2^t x\rangle=\theta(x)^2 \text{ for } x\in V_{-\mathbf{n}}, \text{ where } t=(\mathbf{n}-2)/2.$$

Since $N_2^t\colon V_{-\mathbf{n}}\to V_{-2}$ is injective (by the Lefschetz condition), there exists $\theta_1\in\mathrm{Hom}(V_{-2},\mathbf{k})$ such that $\theta_1(N_2^t x)=\theta(x)$ for all $x\in V_{-\mathbf{n}}$. We see that it suffices to find $\zeta\in\mathrm{Hom}(V_{-2},V_2)$ such that

$$\langle\zeta(x),y\rangle+\langle x,\zeta(y)\rangle=0 \text{ for any } x,y\in V_{-2},$$
$$\langle x',\zeta x'\rangle=\theta_1(x')^2 \text{ for } x'\in V_{-2}.$$

It also suffices to find $b_0\in\mathrm{Bil}(V_{-2})$ such that $b_0=b_0^*$ and $b_0(x,x)=\theta_1(x)^2$ for $x\in V_{-2}$. Such b_0 clearly exists. ♠.

This completes the determination of N_4.

Now assume that $j\geqslant 3$ and that $N_{2j'}$ is already determined for $j'<j$. For $x\in V_a$ with $a<-j$ we set $N_{2j}(x)=0$. Then equation (a) for our j determines uniquely $N_{2j}(x)$ for $x\in V_a$ with $a>-j$. Next we can find $[\ ,\ ]\in\mathrm{Bil}(V_{-j})$ such that

$$[x,y]-[y,x]=-\sum_{j',j''\geqslant 1|j'+j''=j}\langle N_{2j'}x,N_{2j''}y\rangle.$$

To see this we observe that the right hand side is a symplectic form, that is, $\sum_{j',j''\geqslant 1;\ j'+j''=j}\langle N_{2j'}x,N_{2j''}x\rangle=0$. There is a unique $N_{2j}\in\mathrm{Hom}(V_{-j},V_j)$ such that $\langle N_{2j}x,y\rangle=[x,y]$ for any $x,y\in V_{-j}$. Then equation (a) for our j is satisfied. This completes the inductive construction of N. We have $V_*^N=V_*$ by 2.4(a). We see that $N\in Y$. This completes the proof.

3.9. In this subsection we prove Proposition 3.6(c) in a special case. Let $n\in\mathbf{Z}_{>0}$. We have $[-n,n]=I_0\sqcup I_1$, where $I_\epsilon=\{i\in[-n,n]\mid i=\epsilon \mod 2\}$ for $\epsilon\in\{0,1\}$. For $i\in[-n,n]$, define $|i|\in\{0,1\}$ by $i=|i| \mod 2$, that is, by $i\in I_{|i|}$. Let $F_0,F_1\in\mathcal{C}$. Let $V=\bigoplus_{i\in[-n,n]}F_i$, where $F_i=F_{|i|}$. A typical element of V is of the form $(x_i)_{i\in[-n,n]}$ where $x_i\in F_{|i|}$. Define $N\colon V\to V$ by $(x_i)\mapsto(x'_i)$, where $x'_i=x_{i-2}$ for $i\in[2-n,n]$, $x'_{-n}=0, x'_{1-n}=0$. We fix $\langle\ ,\ \rangle_0\in\mathrm{Symp}(V)$ such that $\langle(x_i),(y_i)\rangle_0=\sum_{i\in[-n,n]}(-1)^{(i-|i|)/2}b^{|i|}(x_i,y_{-i})$, where $b^\epsilon\in\mathrm{Bil}(F_\epsilon)$ $(\epsilon\in\{0,1\})$ satisfy $b^{\epsilon*}=(-1)^{1-\epsilon}b^\epsilon$, b^ϵ is nondegenerate, $b^0\in\mathrm{Symp}(F_0)$. Note that $\langle Nx,y\rangle_0+\langle x,Ny\rangle_0=0$ for $x,y\in V$.

We assume: if $p\neq 2$, then either $F_0=0$ or $F_1=0$; ♠ if $p=2, b^1$ is symplectic and $n\geqslant 2$, then we are given a quadratic form $Q\colon F_0\to\mathbf{k}$ such that $Q(x+y)=Q(x)+Q(y)+b^0(x,y)$ for $x,y\in F_0$. ♠

Let X be the set of all $\langle\ ,\ \rangle\in\mathrm{Symp}(V)$ such that $\langle Nx,y\rangle+\langle x,Ny\rangle+\langle Nx,Ny\rangle=0$ for $x,y\in V$ and $\langle x,y\rangle=\langle x,y\rangle_0$ if there exists i such that $x_j=0$ for $j\neq i$ and $y_j=0$ for $j\neq -i$; ♠ if $p=2, b^1$ is symplectic and $n\geqslant 2$, we require also that $\langle x,Nx\rangle=Q(x_{-2})$ if $x\in V$ is such that $x_j=0$ for $j\neq -2$. ♠

Setting $\langle(x_i),(y_i)\rangle=\sum_{i,j}b_{ij}(x_i,y_j)$ identifies X with the set of all families $(b_{ij})_{i,j\in[-n,n]}$, where $b_{ij}\in\mathrm{Bil}(F_{|i|},F_{|j|})$ are such that

$b_{i-2,j}+b_{i,j-2}+b_{ij}=0$ if $i,j\in[2-n,n]$,
$b_{i,-i}=(-1)^{(i-|i|)/2}b^{|i|}$ for all $i\in[-n,n]$,
$b_{ii}\in \mathrm{Symp}(F_{|i|})$ for all $i\in[-n,n]$,
$b^*_{ij}=-b_{ji}$ for all $i,j\in[-n,n]$,
$b_{-2,0}(x,x)=Q(x)$ for $x\in F_0$ if $p=2,F_1=0$ and n is even, $\geqslant 2$.

(We have automatically $b_{ij}=0$ if $i+j\geqslant 1$.)

Let $\Delta=\{T\in \mathrm{GL}(V)\mid TN=NT\}$, a subgroup of $\mathrm{GL}(V)$; equivalently Δ is the set of linear maps $T\colon V\to V$ of the form

(a) $T\colon (x_i)\mapsto(x'_i), x'_i=\sum_{j\in[-n,i]}T^{|i|,|j|}_{i-j}x_j$,

where $T^{\epsilon,\delta}_r\in \mathrm{Hom}(F_\delta,F_\epsilon)$ ($r\in[0,2n],\epsilon,\delta\in\{0,1\},r+\delta=\epsilon \mod 2$) are such that T^{00}_0,T^{11}_0 are invertible and $T^{1-|n|,1-|n|}_{2n}=0$. Now Δ acts on X by $T\colon\langle\ ,\ \rangle\mapsto\langle\ ,\ \rangle'$, where $\langle Tx,Ty\rangle'=\langle x,y\rangle$, or equivalently by $T\colon (b_{ij})\mapsto(b'_{ij})$, where

$$b_{ij}(x,y)=\sum_{i'\in[i,n],j'\in[j,n]}b'_{i'j'}(T^{|i'|,|i|}_{i'-i}(x),T^{|j'|,|j|}_{j'-j}(y)).$$

Let $\Delta_u=\{T\in\Delta\mid T^{00}_0=1,T^{11}_0=1\}$, a subgroup of Δ. We show:

(b) *Let $k\in[1-n,0]$ and let $(\widetilde{b}_{ij}),(b_{ij})$ be two points of X such that $b_{ij}=\widetilde{b}_{ij}$ for $i+j\geqslant 2k$. Then there exists $T\in\Delta_u$ such that $T(b_{ij})=(b'_{ij})$ and $b'_{ij}=\widetilde{b}_{ij}$ for $i+j\geqslant 2k-2$.*

For $\epsilon\in\{0,1\}$ we set $a^\epsilon=\widetilde{b}_{ij}$ for $i,j\in[-n,n],i+j=-1,i=\epsilon \mod 2$. Then a^ϵ are independent of choices; they are 0 unless $p=2$. We have $a^{\epsilon*}=a^{1-\epsilon}$. For $h\in\{2k-2,2k-1\}$ we set $c^\epsilon_h=(-1)^{(i-\epsilon)/2}(b_{ij}-\widetilde{b}_{ij})$, where $i,j\in[-n,n],i+j=h,i=\epsilon \mod 2$. Then c^ϵ_h is independent of i,j. We have $c^\epsilon_{2k-1}=0$ unless $p=2$. We have $c^{\epsilon*}_{2k-2}=(-1)^{k-\epsilon}c^\epsilon_{2k-2}$, $c^{\epsilon*}_{2k-1}=c^{1-\epsilon}_{2k-1}$. Since $b_{k-1,k-1}-\widetilde{b}_{k-1,k-1}$ is symplectic, c^ϵ_{2k-2} is symplectic, where $\epsilon=k-1 \mod 2$.

Case 1*: $p\neq 2$.* Let $\epsilon\in\{0,1\}$ be such that $F_{1-\epsilon}=0$. Since $c^{\epsilon*}_{2k-2}=(-1)^{k-\epsilon}c^\epsilon_{2k-2}$, we can find $\widetilde{c}\in \mathrm{Bil}(F_\epsilon)$ such that $c^\epsilon_{2k-2}=\widetilde{c}+(-1)^{k-\epsilon}\widetilde{c}^*$. Since b^ϵ is nondegenerate, we can find $\tau\in \mathrm{End}(F_\epsilon)$ such that $\widetilde{c}(x,y)=b^\epsilon(x,\tau(y))$ for $x,y\in F_\epsilon$. For $i,j\in[-n,n],i+j=2k-2,i=\epsilon \mod 2$ and $x,y\in F_\epsilon$ we have

$$\begin{aligned}b_{ij}(x,y)-\widetilde{b}_{ij}(x,y)&=(-1)^{(i-\epsilon)/2}(\widetilde{c}(x,y)+(-1)^{k-\epsilon}\widetilde{c}(y,x))\\&=\widetilde{b}_{i,j+2-2k}(x,\tau(y))-\widetilde{b}_{j,i+2-2k}(y,\tau(x))\\&=\widetilde{b}_{i,j+2-2k}(x,\tau(y))+\widetilde{b}_{i+2-2k,j}(\tau(x),y).\end{aligned}$$

Let T be as in (a) with $T^{00}_0=1,T^{11}_0=1$, $T^{\epsilon,\epsilon}_{2-2k}=\tau$ and the other components 0. Define (b'_{ij}) by $T(b_{ij})=(b'_{ij})$. Then (b'_{ij}) has the required properties.

♠ *Case* 2*: $p=2,k=0$.* Since b^0 is nondegenerate we can find $T^{0,1}_1\in \mathrm{Hom}(F_1,F_0)$ such that $c^0_{-1}(x,y)=\widetilde{b}^0(x,T^{0,1}_1(y))$ for all $x\in F_0,y\in F_1$. Then $c^1_{-1}(x,y)=\widetilde{b}^0(T^{0,1}_1(x),y)$ for all $x\in F_1,y\in F_0$. Thus for $i\in I_0,j\in I_1,i+j=-1$ and $x\in F_0,y\in F_1$

we have $b_{ij}(x,y)+\widetilde{b}_{ij}(x,y)=\widetilde{b}_{i,-i}(x,T_1^{0,1}(y))$; for $i\in I_1, j\in I_0, i+j=-1$ and $x\in F_1, y\in F_0$ we have

$$b_{ij}(x,y)+\widetilde{b}_{ij}(x,y)=\widetilde{b}_{-j,j}(T_1^{0,1}(x),y).$$

Since $c_{-2}^{0*}=c_{-2}^{0}, b^{1*}=b^1$, we have $c_{-2}^0(y,y)=\theta(y)^2$ for $y\in F_0$, $b^1(x,x)=\theta_1(x)^2$ for $x\in F_1$, where $\theta\in \mathrm{Hom}(F_0,\mathbf{k})$, $\theta_1\in\mathrm{Hom}(F_1,\mathbf{k})$. If b^1 is not symplectic, we have $\theta_1\neq 0$. Hence there exists $T_1^{1,0}\in\mathrm{Hom}(F_0,F_1)$ such that $\theta(y)=\theta_1(T_1^{1,0}(y))$ for all $y\in F_0$. Then $c_{-2}^0(y,y)+b^1(T_1^{1,0}(y),T_1^{1,0}(y))=0$ for all $y\in F_0$. Thus $c_{-2}^0+b^1(T_1^{1,0}\otimes T_1^{1,0})$ is symplectic. This also holds if b^1 is symplectic (in which case we have $c_{-2}^0(y,y)=b_{-2,0}(y,y)-\widetilde{b}_{-2,0}(y,y)=Q(y)-Q(y)=0$ for $y\in F_0$) and we take $T_1^{1,0}=0$. Now c_{-2}^1 is also symplectic.

Since $a^{0*}=a^1$, $a^1(T_1^{1,0}\otimes 1)+a^0(1\otimes T_1^{1,0})$ is symplectic. Similarly $a^0(T_1^{0,1}\otimes 1)+a^1(1\otimes T_1^{0,1})$ is symplectic. Hence $c_{-2}^0+b^1(T_1^{1,0}\otimes T_1^{1,0})+a^1(T_1^{1,0}\otimes 1)+a^0(1\otimes T_1^{1,0})$ is symplectic and $c_{-2}^1+a^0(T_1^{0,1}\otimes 1)+a^1(1\otimes T_1^{0,1})$ is symplectic. Hence we can find $\widetilde{c}^0\in\mathrm{Bil}(F_0),\widetilde{c}^1\in\mathrm{Bil}(F_1)$ such that

$$c_{-2}^0+b^1(T_1^{1,0}\otimes T_1^{1,0})+a^1(T_1^{1,0}\otimes 1)+a^0(1\otimes T_1^{1,0})=\widetilde{c}^0+\widetilde{c}^{0*},$$
$$c_{-2}^1+a^0(T_1^{0,1}\otimes 1)+a^1(1\otimes T_1^{0,1})=\widetilde{c}^1+\widetilde{c}^{1*}.$$

Since b^0 and b^1 are nondegenerate, we can find $T_2^{0,0}\in\mathrm{End}(F_0)$, $T_2^{1,1}\in\mathrm{End}(F_1)$ such that $\widetilde{c}^0(x,y)=b^0(x,T_2^{0,0}(y))$ for $x,y\in F_0$, $\widetilde{c}^1(x,y)=b^1(x,T_2^{1,1}(y))$ for $x,y\in F_1$. For $x,y\in F_0$ we have

$$\begin{aligned}c_{-2}^0(x,y)+b^1(T_1^{1,0}(x)\otimes T_1^{1,0}(x))+a^1(T_1^{1,0}(x),y)+a^0(x,T_1^{1,0}(y))\\ =b^0(x,T_2^{0,0}(y))+b^0(T_2^{0,0}(x),y).\end{aligned}$$

For $x,y\in F_1$ we have

$$c_{-2}^1(x,y)+a^0(T_1^{0,1}(x),y)+a^1(x,T_1^{0,1}(y))=b^1(x,T_2^{1,1}(y))+b^1(T_2^{1,1}(x),y).$$

Thus for $i,j\in I_0, i+j=-2$ and $x,y\in F_0$ we have

$$\begin{aligned}b_{ij}(x,y)=\widetilde{b}_{ij}(x,y)+\widetilde{b}_{i+1,j}(T_1^{1,0}(x),y)+\widetilde{b}_{i,j+1}(x,T_1^{1,0}(y))\\ +\widetilde{b}_{i+1,j+1}(T_1^{1,0}(x),T_1^{1,0}(y))+\widetilde{b}_{i,-i}(x,T_2^{0,0}(y))+\widetilde{b}_{-j,j}(T_2^{0,0}(x),y);\end{aligned}$$

for $i,j\in I_1, i+j=-2$ and $x,y\in F_1$ we have

$$\begin{aligned}b_{ij}(x,y)=\widetilde{b}_{ij}(x,y)+\widetilde{b}_{i+1,j}(T_1^{0,1}(x),y)+\widetilde{b}_{i,j+1}(x,T_1^{0,1}(y))\\ +\widetilde{b}_{i,-i}(x,T_2^{1,1}(y))+\widetilde{b}_{-j,j}(T_2^{1,1}(x),y).\end{aligned}$$

Let T be as in (a) with $T_0^{00}=1,T_0^{11}=1,T_1^{1,0},T_1^{0,1},T_2^{1,1}$, $T_2^{0,0}$ as above and the other components 0. Define (b'_{ij}) by $T(b_{ij})=(b'_{ij})$. Then (b'_{ij}) has the required properties.

Case 3*:* $p=2,k=-1$. In this case we have $n\geqslant 2$. We first show that there exists $\sigma\in\mathrm{End}(F_1)$ such that

$$b^1(x,\sigma(y))=b^1(\sigma(x),y),\quad c_{-4}^1(x,x)=b^1(x,\sigma(x))+b^1(\sigma(x),\sigma(x))\qquad (*)$$

for $x, y \in F_1$. The functions $F_1 \to \mathbf{k}, x \mapsto b^1(x,x), x \mapsto c^1_{-4}(x,x)$ are additive and homogeneous of degree 2, hence are of the form $x \mapsto \theta(x)^2, x \mapsto \theta_1(x)^2$, where $\theta, \theta_1 \in \mathrm{Hom}(F_1, \mathbf{k})$. We can find a direct sum decomposition $F_1 = F' \bigoplus F''$, where $b^1(x', x'') = 0$ for all $x' \in F', x'' \in F''$, $\theta|_{F'} = 0$, $F' = F_1$ if $\theta = 0$, $\dim F'' \in \{1, 2\}$ if $\theta \neq 0$. Define $\sigma' \in \mathrm{End}(F')$ by $\theta_1(x)\theta_1(y) = b^1(x, \sigma'(y))$ for $x, y \in F'$. Then $b^1(x, \sigma'(y)) = b^1(\sigma'(x), y)$ for $x, y \in F'$, $\theta_1(x)^2 = b^1(x, \sigma'(x)) + \theta(\sigma'(x))^2$ for $x \in F'$.

If $\dim F'' = 1$, we have $\theta|_{F''} \neq 0$ and there is a unique $v \in F''$ such that $\theta(v) = 1$. Let $\sigma'' \colon F'' \to F''$ be multiplication by a, where $a \in \mathbf{k}$ satisfies $a^2 + a = \theta_1(v)^2$. Then $\theta_1(x)^2 = b^1(x, \sigma''(x)) + \theta(\sigma''(x))^2$ for $x \in F''$ and $b(x, \sigma''(y)) = b(\sigma''(x), y)$ for $x, y \in F''$.

If $\dim F'' = 2$, we can find a basis $\{v, v'\}$ of F'' such that $\theta(v') = 0, \theta(v'') = 1$. We set $b(v', v'') = f$. We have $f \neq 0$. Define $\sigma'' \in \mathrm{End}(F'')$ by $\sigma''(v') = \widetilde{a} f^{-1} v' + \widetilde{a} v''$, $\sigma''(v'') = \theta_1(v'')^2 f^{-1} v'$, where $\widetilde{a} \in \mathbf{k}$ satisfies $\widetilde{a}^2 + \widetilde{a} = \theta_1(v')^2$. Then $b(x, \sigma''(y)) = b(\sigma''(x), y)$ for $x, y \in F''$, $\theta_1(x)^2 = b(x, \sigma''(x)) + \theta(\sigma''(x))^2$ for $x \in F''$.

If $F'' = 0$, let $\sigma'' \colon F'' \to F''$ be the 0 map.

Define $\sigma \in \mathrm{End}(F_1)$ by $\sigma(x) = \sigma'(x)$ if $x \in F'$, $\sigma(x) = \sigma''(x)$ if $x \in F''$. Then σ satisfies $(*)$. Since b^0 is nondegenerate, we can find $T_3^{0,1} \in \mathrm{Hom}(F_1, F_0)$ such that

$$c^0_{-3}(x, y) + a^0(x, \sigma(y)) = b^0(x, T_3^{0,1}(y))$$

for $x \in F_0, y \in F_1$. For any $i \in I_0, j \in I_1, i + j = -3$, and $x \in F_{|i|}$, $y \in F_{|j|}$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + \widetilde{b}_{i,j+2}(x, \sigma(y)) + \widetilde{b}_{i,j+3}(x, T_3^{0,1}(y)).$$

It follows that for any $i \in I_1, j \in I_0, i + j = -3$, and $x \in F_{|i|}, y \in F_{|j|}$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + \widetilde{b}_{i+2,j}(\sigma(x), y) + \widetilde{b}_{i+3,j}(T_3^{0,1}(x), y).$$

Define $d_1 \in \mathrm{Bil}(F_1)$ by $d_1(x, y) = \widetilde{b}_{i,j+2}(x, \sigma(y)) + \widetilde{b}_{i+2,j}(\sigma(x), y)$, where $i, j \in I_1, i + j = -4$. Using the first equality in $(*)$ we see that d_1 is independent of the choice of i, j. Define $d \in \mathrm{Bil}(F_1)$ by

$$d(x, y) = c^1_{-4}(x, y) + d_1(x, y) + b^1(\sigma(x), \sigma(y)) + a^0(T_3^{0,1}(x), y) + a^1(x, T_3^{0,1}(y)).$$

We have $d(x, x) = 0$ for $x \in F_1$. (We use $(*)$ and the identity $\widetilde{b}_{i,j+2} + \widetilde{b}_{j,i+2} = b^1$ for $i, j \in I_1, i + j = -4$.) Thus d is symplectic, hence we can find $d' \in \mathrm{Bil}(F_1)$ such that $d = d' + d'^*$. Since b^1 is nondegenerate we can find $T_4^{1,1} \in \mathrm{End}(F_1)$ such that $d'(x, y) = b^1(x, T_4^{1,1}(y))$ for $x, y \in F_1$. We have

$$d(x, y) = b^1(x, T_4^{1,1}(y)) + b^1(y, T_4^{1,1}(x)) = b^1(x, T_4^{1,1}(y)) + b^1(T_4^{1,1}(x), y).$$

Hence for $i, j \in I_1, i + j = -4$, and $x, y \in F_1$ we have

$$\begin{aligned} b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) &+ \widetilde{b}_{i,j+2}(x, \sigma(y)) + \widetilde{b}_{i+2,j}(\sigma(x), y) \\ &+ \widetilde{b}_{i+2,j+2}(\sigma(x), \sigma(y)) + \widetilde{b}_{i+3,j}(T_3^{0,1}(x), y) + \widetilde{b}_{i,j+3}(x, T_3^{0,1}(y)) \\ &+ b_{i,j+4}(x, T_4^{1,1}(y)) + b_{i+4,j}(T_4^{1,1}(x), y). \end{aligned}$$

For $i, j \in I_1, i + j = -2$, and $x, y \in F_1$, we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + \widetilde{b}_{i,j+2}(x, \sigma(y)) + \widetilde{b}_{i+2,j}(\sigma(x), y) = \widetilde{b}_{ij}(x, y).$$

Define $f \in \mathrm{Bil}(F_0)$ by $f(x, y) = c^0_{-4}(x, y)$. Then f is symplectic. (We use the fact that c^0_{-4} is symplectic.) Hence we can find $f' \in \mathrm{Bil}(F_0)$ such that $f = f' + f'^*$. Since b^0 is nondegenerate we can find $T^{0,0}_4 \in \mathrm{End}(F_0)$ such that $f'(x, y) = b^0(x, T^{0,0}_4(y))$ for $x, y \in F_0$. We have

$$f(x, y) = b^0(x, T^{0,0}_4(y)) + b^0(y, T^{0,0}_4(x)) = b^0(x, T^{0,0}_4(y)) + b^0(T^{0,0}_4(x), y),$$

hence for $i, j \in I_0, i + j = -4$, and $x, y \in F_0$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + b_{i,j+4}(x, T^{0,0}_4(y)) + b_{i+4,j}(T^{0,0}_4(x), y).$$

For $i, j \in I_0, i + j = -2$, and $x, y \in F_0$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y).$$

Let T be as in (a) with $T^{00}_0 = 1, T^{11}_0 = 1$, $T^{0,1}_3, T^{1,1}_4, T^{0,0}_4$, $T^{1,1}_2 = \sigma$ as above and the other components 0. Define (b'_{ij}) by $T(b_{ij}) = (b'_{ij})$. Then (b'_{ij}) has the required properties.

Case 4: $p = 2, k < -1$. In this case we have $n \geqslant 3$. Define $\epsilon, \delta \in \{0, 1\}$ by $\epsilon = k - 1 \bmod 2, \delta = 1 - \epsilon$. Since b^δ is nondegenerate, we have $c^\delta_{2k-2}(x, y) = b^\delta(x, \sigma(y))$ for $x, y \in F_\delta$, where $\sigma \in \mathrm{End}(F_\delta)$ is well defined. Since $b^{\delta*} = b^\delta, c^{\delta*}_{2k-2} = c^\delta_{2k-2}$, we have $b^\delta(x, \sigma(y)) = b^\delta(\sigma(x), y)$. Since b^ϵ is nondegenerate we can find $T^{\epsilon,\delta}_{1-2k} \in \mathrm{Hom}(F_\delta, F_\epsilon)$ such that

$$c^\epsilon_{2k-1}(x, y) + a^\epsilon(x, \sigma(y)) = b^\epsilon(x, T^{\epsilon,\delta}_{1-2k}(y))$$

for $x \in F_\epsilon, y \in F_\delta$. For any $i \in I_\epsilon, j \in I_\delta, i + j = 2k - 1$, and $x \in F_\epsilon$, $y \in F_\delta$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + \widetilde{b}_{i,j-2k}(x, \sigma(y)) + \widetilde{b}_{i,j+1-2k}(x, T^{\epsilon,\delta}_{1-2k}(y)).$$

It follows that for any $i \in I_\delta, j \in I_\epsilon, i + j = 2k - 1$, and $x \in F_\delta, y \in F_\epsilon$ we have

$$b_{ij}(x, y) = \widetilde{b}_{ij}(x, y) + \widetilde{b}_{i-2k,j}(\sigma(x), y) + \widetilde{b}_{i+1-2k,j}(T^{\epsilon,\delta}_{1-2k}(x), y).$$

Define $d_1 \in \mathrm{Bil}(F_\delta)$ by $d_1(x, y) = \widetilde{b}_{i,j-2k}(x, \sigma(y)) + \widetilde{b}_{i-2k,j}(\sigma(x), y)$, where $i, j \in I_\delta, i + j = 2k - 2$. Using $b^\delta(1 \otimes \sigma) = b^\delta(\sigma \otimes 1)$ we see that d_1 is independent of the choice of i, j. Define $d \in \mathrm{Bil}(F_\delta)$ by

$$d(x, y) = c^\delta_{2k-2}(x, y) + d_1(x, y) + a^\epsilon(T^{\epsilon,\delta}_{1-2k}(x), y) + a^\delta(x, T^{\epsilon,\delta}_{1-2k}(y)).$$

We have $d(x, x) = 0$ for $x \in F_\delta$. (This follows from the choice of σ and the identity $\widetilde{b}_{i,j-2k} + \widetilde{b}_{j,i-2k} = b^\delta$ for $i, j \in I_\delta, i + j = 2k - 2$.) Thus d is symplectic, hence we can find $d' \in \mathrm{Bil}(F_\delta)$ such that $d = d' + d'^*$. Since b^δ is nondegenerate we can find $T^{\delta,\delta}_{2-2k} \in \mathrm{End}(F_\delta)$ such that $d'(x,y) = b^\delta(x, T^{\delta,\delta}_{2-2k}(y))$ for $x, y \in F_\delta$. We have

$$d(x,y) = b^\delta(x, T^{\delta,\delta}_{2-2k}(y)) + b^\delta(y, T^{\delta,\delta}_{2-2k}(x)) = b^\delta(x, T^{\delta,\delta}_{2-2k}(y)) + b^\delta(T^{\delta,\delta}_{2-2k}(x), y).$$

Hence for $i, j \in I_\delta, i + j = 2k - 2$, and $x, y \in F_\delta$ we have

$$\begin{aligned} b_{ij}(x,y) = \widetilde{b}_{ij}(x,y) &+ \widetilde{b}_{i,j-2k}(x,\sigma(y)) + \widetilde{b}_{i-2k,j}(\sigma(x),y) + \widetilde{b}_{i+1-2k,j}(T^{\epsilon,\delta}_{1-2k}(x),y) \\ &+ \widetilde{b}_{i,j+1-2k}(x, T^{\epsilon,\delta}_{1-2k}(y)) + b_{i,j+2-2k}(x, T^{\delta,\delta}_{2-2k}(y)) \\ &+ b_{i+2-2k,j}(T^{\delta,\delta}_{2-2k}(x), y). \end{aligned}$$

For $i, j \in I_\delta, i + j = 2k$, and $x, y \in F_\delta$ we have

$$b_{ij}(x,y) = \widetilde{b}_{ij}(x,y) + \widetilde{b}_{i,j-2k}(x,\sigma(y)) + \widetilde{b}_{i-2k,j}(\sigma(x),y) = \widetilde{b}_{ij}(x,y).$$

Define $f \in \mathrm{Bil}(F_\epsilon)$ by $f(x,y) = c^\epsilon_{2k-2}(x,y)$. Then f is symplectic. (We use the fact that c^ϵ_{2k-2} is symplectic.) Hence we can find $f' \in \mathrm{Bil}(F_\epsilon)$ such that $f = f' + f'^*$. Since b^ϵ is nondegenerate we can find $T^{\epsilon,\epsilon}_{2-2k} \in \mathrm{End}(F_\epsilon)$ such that $f'(x,y) = b^\epsilon(x, T^{\epsilon,\epsilon}_{2-2k}(y))$ for $x, y \in F_\epsilon$. We have

$$f(x,y) = b^\epsilon(x, T^{\epsilon,\epsilon}_{2-2k}(y)) + b^\epsilon(y, T^{\epsilon,\epsilon}_{2-2k}(x)) = b^\epsilon(x, T^{\epsilon,\epsilon}_{2-2k}(y)) + b^\epsilon(T^{\epsilon,\epsilon}_{2-2k}(x), y),$$

hence for $i, j \in I_\epsilon, i + j = 2k - 2$, and $x, y \in F_\epsilon$ we have

$$b_{ij}(x,y) = \widetilde{b}_{ij}(x,y) + b_{i,j+2-2k}(x, T^{\epsilon,\epsilon}_{2-2k}(y)) + b_{i+2-2k,j}(T^{\delta,\delta}_{2-2k}(x), y).$$

For $i, j \in I_\delta, i + j = 2k$, and $x, y \in F_\epsilon$ we have $b_{ij}(x,y) = \widetilde{b}_{ij}(x,y)$. Let T be as in (a) with $T^{00}_0 = 1, T^{11}_0 = 1$, $T^{\epsilon,\delta}_{1-2k}, T^{\epsilon,\epsilon}_{2-2k}, T^{\delta,\delta}_{2-2k}$, $T^{\delta,\delta}_{-2k} = \sigma$ as above and the other components 0. Define (b'_{ij}) by $T(b_{ij}) = (b'_{ij})$. Then (b'_{ij}) has the required properties. ♠

This completes the proof of (b).

We now verify the following special case of Proposition 3.6(c).

(c) *Let $(\widetilde{b}_{ij}), (b_{ij})$ be two points of X. Then there exists $T \in \Delta_u$ such that $T(b_{ij}) = (\widetilde{b}_{ij})$.*

We first prove the following statement by induction on $k \in [-n, 0]$.

(P_k) *Assume in addition that $b_{ij} = \widetilde{b}_{ij}$ for any i, j with $i + j \geqslant 2k$. Then there exists $T \in \Delta_u$ such that $T(b_{ij}) = (\widetilde{b}_{ij})$.*

If $k = -n$, the result is obvious. Assume now that $k \in [1 - n, 0]$. By (b) we can find $T' \in \Delta_u$ such that $T'(b_{ij}) = (b'_{ij})$ and $b'_{ij} = \widetilde{b}_{ij}$ for $i + j \geqslant 2k - 2$. By the induction hypothesis we can find $T'' \in \Delta_u$ such that $T''(b'_{ij}) = (\widetilde{b}_{ij})$. Let $T = T''T' \in \Delta_u$. Then $T(b_{ij}) = (\widetilde{b}_{ij})$. This completes the proof of (P_k) for $k \in [-n, 0]$. In particular (P_0) holds and (c) is proved.

3.10. *Proof of Proposition* 3.6(c). Let $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$ be two elements of the set X in 3.6. We must show that $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$ are in the same U-orbit. We argue by induction on e, the smallest integer $\geqslant 0$ such that $N^e = 0$. If $e = 0$, we have $V = 0$ and the result is obvious. If $e = 1$, we have $N = 0$. Then $V = \mathrm{gr}V_*$ canonically, $U = \{1\}$, and both $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$ are the same as $\langle\ ,\ \rangle_0$, hence the result is clear. We now assume that $e \geqslant 2$.

♠. Assume first that $p = 2$. For $n \in \mathcal{L}$, let $q_n : P^\nu_{-n} \to \mathbf{k}$ be the quadratic forms attached to $(N, \langle\ ,\ \rangle)$ in 3.3 and let $q'_n : P^\nu_{-n} \to \mathbf{k}$ be the analogous quadratic forms defined in terms of $(N, \langle\ ,\ \rangle')$. We show:

(a) *there exists $T \in U$ such that if $\langle\ ,\ \rangle'' \in \mathrm{Symp}(V)$ is given by $\langle x, y\rangle'' = \langle Tx, Ty\rangle$, then for $n \in \mathcal{L}$ the quadratic form q''_n defined as in* 3.3 *in terms of $(N, \langle\ ,\ \rangle'')$ satisfies $q''_n = q'_n$.*

We are seeking an $S \in E_{\geqslant 1}V_*$ such that $SN = NS$ and $\langle (1+S)\dot{x}, (1+S)N^{n-1}\dot{x}\rangle = \langle \dot{x}, N^{n-1}\dot{x}\rangle'$, that is, $\langle (1+S)\dot{x}, N^{n-1}(1+S)\dot{x}\rangle = \langle \dot{x}, N^{n-1}\dot{x}\rangle'$, that is,

$$\langle S\dot{x}, N^{n-1}\dot{x}\rangle + \langle \dot{x}, N^{n-1}S\dot{x}\rangle + \langle S\dot{x}, N^{n-1}S\dot{x}\rangle = \langle \dot{x}, N^{n-1}\dot{x}\rangle' + \langle \dot{x}, N^{n-1}\dot{x}\rangle$$

for any $n \in \mathcal{L}$ and any $\dot{x} \in V_{\geqslant -n}$ such that $N^{n+1}\dot{x} = 0$. Now

$$\langle S\dot{x}, N^{n-1}\dot{x}\rangle + \langle \dot{x}, N^{n-1}S\dot{x}\rangle = \langle S\dot{x}, (N^{n-1} + (N^\dagger)^{n-1})\dot{x}\rangle$$

is a linear combination of terms $\langle S\dot{x}, N^{n'}\dot{x}\rangle$, with $n' \geqslant n$; each of these terms is 0 since $S\dot{x} \in V_{\geqslant 1-n}, N^{n'}\dot{x} \in V_{\geqslant 2n'-n}$, and $1 - n + 2n' - n \geqslant 1$. Moreover $\langle S\dot{x}, N^{n-1}S\dot{x}\rangle = \langle \overline{S}x, \nu^{n-1}\overline{S}x\rangle_0$, where $x \in P^\nu_{-n}$ is the image of $\dot{x}$ and $\langle \dot{x}, N^{n-1}\dot{x}\rangle' + \langle \dot{x}, N^{n-1}\dot{x}\rangle = q'_n(x) + q_n(x)$. By the surjectivity of the map $S \mapsto \overline{S}$ in 2.5(d) we see that it suffices to show that there exists $\sigma \in \mathrm{End}^\nu_1(\mathrm{gr}V_*)$ (that is, $\sigma \in \mathrm{End}_1(\mathrm{gr}V_*)$ such that $\sigma\nu = \nu\sigma$), with $\langle \sigma x, \nu^{n-1}\sigma x\rangle_0 = q'_n(x) + q_n(x)$ for any $n \in \mathcal{L}$ and any $x \in P^\nu_{-n}$.

For $n \in \mathcal{L}'$ the last equation is automatically satisfied for any σ. (The left-hand side is zero by the definition of $\mathcal{L}'$. The right-hand side is equal by 3.3(c) to $Q'_{\mathbf{n}}(\nu^{(n-\mathbf{n})/2}x) + Q_{\mathbf{n}}(\nu^{(n-\mathbf{n})/2}x)$, where $Q_{\mathbf{n}}$ is the quadratic form attached as in 3.3 to $(N, \langle\ ,\ \rangle)$ and $Q'_{\mathbf{n}}$ is the analogous quadratic form defined in terms of $(N, \langle\ ,\ \rangle')$. The last sum is zero since $Q_{\mathbf{n}} = Q'_{\mathbf{n}} = Q$.)

We see that it suffices to show there exists $\sigma \in \mathrm{End}^\nu_1(\mathrm{gr}V_*)$ such that $\langle \sigma x, \nu^{n-1}\sigma x\rangle_0 = q'_n(x) + q_n(x)$ for any $n \in \mathcal{L} - \mathcal{L}'$ and any $x \in P^\nu_{-n}$.

For $n \in \mathcal{L} - \mathcal{L}'$, the quadratic forms q'_n, q_n have the same associated symplectic form (see 3.3(a)); hence there exists $\theta_n \in \mathrm{Hom}(P^\nu_{-n}, \mathbf{k})$ such that $q'_n(x) + q_n(x) = \theta_n(x)^2$ for all $x \in P^\nu_{-n}$. Hence it suffices to show that the linear map

$$\rho : \mathrm{End}^\nu_1(\mathrm{gr}V_*) \longrightarrow \textstyle\bigoplus_{n \in \mathcal{L}-\mathcal{L}'} \mathrm{Hom}(P^\nu_{-n}, \mathbf{k})$$

given by $\sigma \mapsto (\theta_n)$, where $\theta_n(x) = \sqrt{\langle \sigma x, \nu^{n-1}\sigma x\rangle_0}$ for $x \in P^\nu_{-n}$ is surjective. Let $\mathcal{E} = \bigoplus_{n \geqslant 0} \mathrm{Hom}(P^\nu_{-n}, \mathrm{gr}_{1-n}V_*)$. We have an isomorphism $\pi : \mathrm{End}^\nu_1(\mathrm{gr}V_*) \xrightarrow{si} \mathcal{E}$ given by $\sigma \mapsto (\sigma_n)$, where $\sigma_n \in \mathrm{Hom}(P^\nu_{-n}, \mathrm{gr}_{1-n}V_*)$ is the restriction of σ. Define a linear map

$$\rho' : \mathcal{E} \to \textstyle\bigoplus_{n \in \mathcal{L}-\mathcal{L}'} \mathrm{Hom}(P^\nu_{-n}, \mathbf{k})$$

by $(\sigma_n) \mapsto (\theta_n)$, where $\theta_n(x) = \sqrt{\langle \sigma_n x, \nu^{n-1}\sigma_n x\rangle_0}$ for $x \in P^\nu_{-n}$. We have $\rho'\pi = \rho$. Hence it suffices to show that ρ' is surjective. It also suffices to show that for any $n \in \mathcal{L} - \mathcal{L}'$, the linear map

$$\rho'_n \colon \mathrm{Hom}(P^\nu_{-n}, \mathrm{gr}_{1-n}V_*) \to \mathrm{Hom}(P^\nu_{-n}, \mathbf{k})$$

given by $f \mapsto \theta$, where $\theta(x) = \sqrt{\langle fx, \nu^{n-1}fx\rangle_0}$ for $x \in P^\nu_{-n}$, is surjective. Define $g \in \mathrm{Hom}(\mathrm{gr}_{1-n}V_* \to \mathbf{k})$ by $h \mapsto \sqrt{\langle h, \nu^{n-1}h\rangle_0}$. Then $\rho'_n(f) = g \circ f$ for $f \in \mathrm{Hom}(P^\nu_{-n}, \mathrm{gr}_{1-n}V_*)$. Hence to show that ρ'_n is surjective, it suffices to show that $g \neq 0$. Since $n \in \mathcal{L} - \mathcal{L}'$, there exists m odd such that $m \geqslant n+3$ and b_m is not symplectic. Hence there exists $u' \in P^\nu_{-m}$ such that $\langle u', \nu^m u'\rangle_0 \neq 0$. We have $m = (n-1) + 2k$, where k is an integer $\geqslant 2$. Let $u = N^k u' \in \mathrm{gr}_{1-n}V_*$ and

$$\langle u, \nu^{n-1}u\rangle_0 = \langle \nu^k u', \nu^{n-1+k}u'\rangle_0 = \langle u', \nu^m u'\rangle_0 \neq 0.$$

Thus $g(u) \neq 0$. We see that $g \neq 0$ as required. This proves (a).

Note that $\langle\ ,\ \rangle''$ in (a) is in X (in fact in the U-orbit of $\langle\ ,\ \rangle$). Replacing if necessary $\langle\ ,\ \rangle$ by $\langle\ ,\ \rangle''$, we see that

(b) *we may assume that* $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$ *are such that* $q_n = q'_n$ *for all* $n \in \mathcal{L}$. ♠

We now return to a general p. Let $r \geqslant e$. Let F be a complement of $V_{\geqslant 2-r} = \ker N^{r-1}$ in $V_{\geqslant 1-r} = V$, and let F' be a complement of $V_{\geqslant 3-r} = \ker N^{r-2} + NV$ in $V_{\geqslant 2-r} = \ker N^{r-1}$. Consider the linear map α of $F \bigoplus F' \bigoplus F \bigoplus \cdots \bigoplus F' \bigoplus F$ ($2r-1$ summands) into V given by

$$(x_{1-r}, x_{2-r}, \dots, x_{r-2}, x_{r-1}) \mapsto \\ x_{1-r} + Nx_{3-r} + \cdots + N^{r-1}x_{r-1} + x_{2-r} + Nx_{4-r} + \cdots + N^{r-2}x_{r-2}.$$

(Here $x_i \in F$, if $i = r+1 \bmod 2$, and $x_i \in F'$, if $i = r \mod 2$.) Let W be the image of α. We show that

(c) $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *are nondegenerate on* W.

We prove this only for $\langle\ ,\ \rangle$; the proof for $\langle\ ,\ \rangle'$ is the same. Assume that $w = x_{1-r} + Nx_{3-r} + \cdots + N^{r-1}x_{r-1} + x_{2-r} + Nx_{4-r} + \cdots + N^{r-2}x_{r-2}$ with x_i as above satisfies $\langle w, W\rangle = 0$. We show that each x_i is 0. We have $0 = \langle w, N^{r-1}F\rangle = \langle x_{1-r}, N^{r-1}F\rangle = 0$. Using the nondegeneracy of b_{r-1}, we see that $x_{1-r} = 0$ and $w = Nx_{3-r} + \cdots + N^{r-1}x_{r-1} + x_{2-r} + Nx_{4-r} + \cdots + N^{r-2}x_{r-2}$. We have $0 = \langle w, N^{r-2}F'\rangle = \langle x_{2-r}, N^{r-2}F'\rangle$. Using the nondegeneracy of b_{r-2}, we see that $x_{2-r} = 0$ and $w = Nx_{3-r} + \cdots + N^{r-1}x_{r-1} + Nx_{4-r} + \cdots + N^{r-2}x_{r-2}$. We have $0 = \langle w, N^{r-2}F\rangle = \langle Nx_{3-r}, N^{r-2}F\rangle = -\langle x_{3-r}, N^{r-1}F\rangle$. Using the nondegeneracy of b_{r-1}, we see that $x_{3-r} = 0$. Continuing in this way we see that each x_i is 0. This proves (c).

The proof also shows that α is injective.

Let $Z = \{x \in V \mid \langle x, W\rangle = 0\}$, $Z' = \{x \in V \mid \langle x, W\rangle' = 0\}$. From (c) we see that $V = W \bigoplus Z = W \bigoplus Z'$.

Clearly W is N-stable, hence $(1+N)$-stable. Since $1+N$ is an isometry of $\langle\ ,\ \rangle$ it follows that Z is $(1+N)$-stable hence N-stable. Similarly Z' is N-stable. Define $\Phi \in \mathrm{GL}(V)$ by $\Phi(x) = x$ for $x \in W$, $\Phi(x) = x'$ for $x \in Z$, where $x' \in Z'$ is given by

$x-x'\in W$. We have $\Phi\in U$ (see 2.7(c),(d)). Define $'\langle\ ,\ \rangle\in \mathrm{Symp}(V)$ by $'\langle x,y\rangle=\langle\Phi(x),\Phi(y)\rangle'$. By Proposition 3.6(a) we have $'\langle\ ,\ \rangle\in X$.

Let $'Z=\{x\in V\mid {}'\langle x,W\rangle=0\}$. We show that $'Z=Z$. Let $x=x_1+x_2$, where $x_1\in W, x_2\in Z$. We have $x_2=w+x_2'$, $w\in W, x_2'\in Z'$. For $w'\in W$ we have $\langle\Phi(x),w'\rangle'=\langle x_1+x_2',w'\rangle'=\langle x_1,w'\rangle'$. The condition $\langle\Phi(x),W\rangle'=0$ is that $\langle x_1,W\rangle'=0$ or that $x_1=0$ (using (c)) or that $x\in Z$. Thus $'Z=\{x\in V\mid \langle\Phi(x),\Phi(W)\rangle'=0\}=\{x\in V\mid\langle\Phi(x),W\rangle'=0\}=Z$ as required.

♠ In the case where $p=2$ we show that, for any $n\in\mathcal{L}$, the quadratic form q_n' attached to $(N,\langle\ ,\ \rangle')$ as in 3.3 is equal to the analogous quadratic form attached to $(N,{}'\langle\ ,\ \rangle)$. We must show that if $x\in V_{\geqslant -n}$, $N^{n+1}x=0$, then $\langle\Phi x,\Phi N^{n-1}x\rangle'=\langle x,N^{n-1}x\rangle'$ that is, $\langle\Phi x,N^{n-1}\Phi x\rangle'=\langle x,N^{n-1}x\rangle'$. Both sides are additive in x. We can write $x=x_1+x_2$, where $x_1\in W, x_2\in Z$ satisfy $x_1,x_2\in V_{\geqslant -n}$, $N^{n+1}x_1=0, N^{n+1}x_2=0$. We may assume that $x=x_1$ or $x=x_2$. When $x=x_1$, the desired equality is obvious. Hence we may assume that $x\in Z$. Write $x=x'+w$, $x'\in Z', w\in W$. We must show that $\langle x+w,N^{n-1}x+N^{n-1}w\rangle'=\langle x,N^{n-1}x\rangle'$, that is, $\langle x,N^{n-1}w\rangle'+\langle w,N^{n-1}x\rangle'+\langle w,N^{n-1}w\rangle'=0$, that is, $\langle x,(N^{n-1}+(N^\dagger)^{n-1})w\rangle'+\langle w,N^{n-1}w\rangle'=0$, that is, $\langle x,N^nw\rangle'+\langle w,N^{n-1}w\rangle'=0$ (we use $N^{n+1}w=0$), that is, $\langle x'+w,N^nw\rangle'+\langle w,N^{n-1}w\rangle'=0$, that is, $\langle w,N^nw\rangle'+\langle w,N^{n-1}w\rangle'=0$. Now $w\in W_{\geqslant 1-n}$ (see Proposition 2.7(b)), $N^nw\in W_{\geqslant n+1}$, hence $\langle w,N^nw\rangle'=0$. It remains to show $\langle w,N^{n-1}w\rangle'=0$. Since $w\in W_{\geqslant 1-n}$ and $N^{n+1}w=0$, it suffices to show $\langle y,\nu^{n-1}y\rangle_0=0$ for any $y\in \mathrm{gr}_{1-n}V_*$ such that $\nu^{n+1}y=0$. This has already been seen in the proof in 3.3 that q_n is well defined. ♠

Replacing $\langle\ ,\ \rangle'$ by $'\langle\ ,\ \rangle$ (which is in the same U-orbit) we see that condition (b) is preserved (for $p=2$).

Thus we may assume that $\langle\ ,\ \rangle,\langle\ ,\ \rangle'$ satisfy $Z=Z'$ and that for $p=2$ condition (b) holds. Thus $V=W\bigoplus Z$ is an othogonal decomposition with respect to either $\langle\ ,\ \rangle$ or $\langle\ ,\ \rangle'$. Let $\langle\ ,\ \rangle_W,\langle\ ,\ \rangle_Z$ be the restrictions of $\langle\ ,\ \rangle$ to W,Z. Let $\langle\ ,\ \rangle_W',\langle\ ,\ \rangle_Z'$ be the restrictions of $\langle\ ,\ \rangle'$ to W,Z. Let U_1 (respectively U_2) be the analogue of U for W (respectively Z) defined in terms of N and W_*^N (respectively Z_*^N). We have naturally $U_1\times U_2\subset U$.

We consider 5 cases.

Case 1*:* $p\neq 2$. Take $r=e+1$ (thus $F=0$). By the induction hypothesis $\langle\ ,\ \rangle_Z$ is carried to $\langle\ ,\ \rangle_Z'$ by some $u_2\in U_2$. By Proposition 3.9 $\langle\ ,\ \rangle_W$ is carried to $\langle\ ,\ \rangle_W'$ by some $u_1\in U_1$. Then $\langle\ ,\ \rangle$ is carried to $\langle\ ,\ \rangle'$ by $(u_1,u_2)\in U$.

♠ *Case* 2*:* $p=2$*, e is odd, and b_{e-2} is symplectic.* Take $r=e+1$ (thus, $F=0$). We have $e-1\in\mathcal{L}$. The sets $\mathcal{L}$ attached to $\langle\ ,\ \rangle_Z,\langle\ ,\ \rangle_Z'$ are the same as $\mathcal{L}$ for $\langle\ ,\ \rangle,\langle\ ,\ \rangle'$. The quadratic forms attached to $\langle\ ,\ \rangle_Z$, $\langle\ ,\ \rangle_Z'$, and $n\in\mathcal{L}-\{e-1\}$ are the same as those attached to $\langle\ ,\ \rangle,\langle\ ,\ \rangle'$, and n, hence they coincide. The quadratic forms attached to $\langle\ ,\ \rangle_Z,\langle\ ,\ \rangle_Z'$, and $n=e-1$ also coincide: they are both 0. Hence the Quadratic forms attached to $\langle\ ,\ \rangle_Z,\langle\ ,\ \rangle_Z'$ coincide (see 3.3(c)). The quadratic forms attached to $\langle\ ,\ \rangle_W,\langle\ ,\ \rangle_W'$ coincide: for $e-1$ they are the same as those attached to $\langle\ ,\ \rangle,\langle\ ,\ \rangle'$, and $e-1$. For other n they are zero. Hence the Quadratic forms attached to $\langle\ ,\ \rangle_W,\langle\ ,\ \rangle_W'$ coincide. By the induction hypothesis $\langle\ ,\ \rangle_Z$ is carried to $\langle\ ,\ \rangle_Z'$ by some $u_2\in U_2$. By Proposition 3.9, $\langle\ ,\ \rangle_W$ is carried to $\langle\ ,\ \rangle_W'$ by some $u_1\in U_1$. Then $\langle\ ,\ \rangle$ is carried to $\langle\ ,\ \rangle'$ by $(u_1,u_2)\in U$.

Case 3*:* $p=2$*, e is even, and b_{e-1} is symplectic.* Take $r=e+1$ (thus, $F=0$). The sets $\mathcal{L}$ attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$ are the same as $\mathcal{L}$ for $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. The quadratic forms attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$ and $n \in \mathcal{L}$ are the same as those attached to $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$, and n, hence they coincide. Thus the Quadratic forms attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$ coincide. By the induction hypothesis $\langle\ ,\ \rangle_Z$ is carried to $\langle\ ,\ \rangle'_Z$ by some $u_2 \in U_2$. By Proposition 3.9 $\langle\ ,\ \rangle_W$ is carried to $\langle\ ,\ \rangle'_W$ by some $u_1 \in U_1$. Then $\langle\ ,\ \rangle$ is carried to $\langle\ ,\ \rangle'$ by $(u_1, u_2) \in U$.

Case 4: $p = 2$, e is even, and b_{e-1} is not symplectic. Take $r = e$. The sets $\mathcal{L}$ attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$ are the same as $\mathcal{L}$ for $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. The quadratic forms attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$, and $n \in \mathcal{L}$ are the same as those attached to $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$, and n, hence they coincide. Thus the Quadratic forms attached to $\langle\ ,\ \rangle_Z, \langle\ ,\ \rangle'_Z$ coincide. By the induction hypothesis $\langle\ ,\ \rangle_Z$ is carried to $\langle\ ,\ \rangle'_Z$ by some $u_2 \in U_2$. By Proposition 3.9 $\langle\ ,\ \rangle_W$ is carried to $\langle\ ,\ \rangle'_W$ by some $u_1 \in U_1$. Then $\langle\ ,\ \rangle$ is carried to $\langle\ ,\ \rangle'$ by $(u_1, u_2) \in U$.

Case 5: $p = 2$, e is odd, $\geqslant 3$, and b_{e-2} is not symplectic. Take $r = e$. By Proposition 3.9 $\langle\ ,\ \rangle_W$ is carried to $\langle\ ,\ \rangle'_W$ by some $u_1 \in U_1$. Replacing $\langle\ ,\ \rangle'$ by a translate under $(u_1, 1) \in U$ we see that we may assume in addition that $\langle\ ,\ \rangle_W = \langle\ ,\ \rangle'_W$. Let $\widetilde{W} = F + NF + \cdots + N^{r-1}F$. Let $W' = \{w \in W \mid \langle w, \widetilde{W}\rangle = 0\} = \{w \in W \mid \langle w, \widetilde{W}\rangle' = 0\}$. Then $W = \widetilde{W} \bigoplus W'$, the orthogonal direct sum for both $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. Let $\widetilde{Z} = W' \bigoplus Z$, the orthogonal direct sum for both $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. Then $V = \widetilde{W} \bigoplus \widetilde{Z}$, the orthogonal direct sum for both $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. Let $\langle\ ,\ \rangle_{\widetilde{Z}}, \langle\ ,\ \rangle'_{\widetilde{Z}}$ be the restrictions of $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$ to $\widetilde{Z}$. Let $\widetilde{U}_1$ (respectively $\widetilde{U}_2$) be the analogue of U for $\widetilde{W}$ (respectively $\widetilde{Z}$) defined in terms of N and $\widetilde{W}^N_*$ (respectively $\widetilde{Z}^N_*$). We have naturally $\widetilde{U}_1 \times \widetilde{U}_2 \subset U$. The sets $\mathcal{L}$ attached to $\langle\ ,\ \rangle_{\widetilde{Z}}$, $\langle\ ,\ \rangle'_{\widetilde{Z}}$ are the same as $\mathcal{L}$ for $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$. The quadratic forms attached to $\langle\ ,\ \rangle_{\widetilde{Z}}, \langle\ ,\ \rangle'_{\widetilde{Z}}$, and $n \in \mathcal{L}$ are the same as those attached to $\langle\ ,\ \rangle, \langle\ ,\ \rangle'$, and n, hence they coincide. Thus the Quadratic forms attached to $\langle\ ,\ \rangle_{\widetilde{Z}}, \langle\ ,\ \rangle'_{\widetilde{Z}}$ coincide. By the induction hypothesis, $\langle\ ,\ \rangle_{\widetilde{Z}}$ is carried to $\langle\ ,\ \rangle'_{\widetilde{Z}}$ by some $\widetilde{u}_2 \in \widetilde{U}_2$. Then $\langle\ ,\ \rangle$ is carried to $\langle\ ,\ \rangle'$ by $(1, \widetilde{u}_2) \in U$. ♠

This completes the proof of Proposition 3.6(c), hence also those of Propositions 3.5, 3.6, and 3.7.

3.11. Here is the order of the proof of the various assertions in Propositions 3.5–3.7: 3.5(a), see 3.8; 3.7(a), see 3.7; 3.6(a), see 3.6; 3.7(b), see 3.7; 3.6(b), see 3.6; 3.5(b), see 3.5; 3.6(c), see 3.9, 3.10; 3.7(c), see 3.7; 3.5(c), see 3.5.

3.12. Let $V \in \mathcal{C}$ and let $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$. The following result can be deduced from [S1, I, 2.10].

Let C, C_0 be two $\mathrm{GL}(V)$-conjugacy classes in $\mathrm{Nil}(V)$ such that $C \cap \mathcal{M}_{\langle\ ,\ \rangle} \neq \varnothing$, $C_0 \cap \mathcal{M}_{\langle\ ,\ \rangle} \neq \varnothing$, and C is contained in the closure of C_0 in $\mathrm{GL}(V)$. Then $C \cap \mathcal{M}_{\langle\ ,\ \rangle}$ is contained in te closure of $C_0 \cap \mathcal{M}_{\langle\ ,\ \rangle}$ in $\mathcal{M}_{\langle\ ,\ \rangle}$.

3.13. Let $V \in \mathcal{C}$ and let $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$. Let $G = \mathrm{Sp}(\langle\ ,\ \rangle)$. For any self-dual filtration V_* of V and for $n \geqslant 1$, let $E^{\langle\ ,\ \rangle}_{\geqslant n} V_* = E_{\geqslant n} V_* \cap \mathcal{M}_{\langle\ ,\ \rangle}$, a unipotent algebraic group with multiplication $T * T' = T + T' + TT'$. Let

$$\widetilde{\xi}(V_*) = \xi(V_*) \cap \mathcal{M}_{\langle\ ,\ \rangle} = \{N \in \mathcal{M}_{\langle\ ,\ \rangle} \mid V^N_* = V_*\} = \{N \in E^{\langle\ ,\ \rangle}_{\geqslant 2} V_* \mid \overline{N} \in \mathrm{End}^0_2(\mathrm{gr} V_*)\}$$

(see 2.9). The following three conditions are equivalent:

(i) $\widetilde{\xi}(V_*) \neq \varnothing$;
(ii) there exists $\nu \in \mathrm{End}_2^0(\mathrm{gr}V_*)$ which is skew-adjoint with respect to the symplectic form on $\mathrm{gr}V_*$ induced by $\langle\ ,\ \rangle$;
(iii) $\dim \mathrm{gr}_n V_* = \dim \mathrm{gr}_{-n} V_* \geqslant \dim \mathrm{gr}_{-n-2} V_*$ for all $n \geqslant 0$, and $\dim \mathrm{gr}_{-n} V_* = \dim \mathrm{gr}_{-n-2} V_* \mod 2$ for all $n \geqslant 0$ even.

We have (i)⇒(ii) by the definition of $\widetilde{\xi}(V_*)$; we have (ii)⇒(iii) by 2.3(d) and 3.1(c). Now (iii)⇒(ii) is easily checked. We have (ii)⇒(iii) by Proposition 3.5(a).

Let $\mathfrak{F}_{\langle\ ,\ \rangle}$ be the set of all self-dual filtrations V_* of V that satisfy (i)–(iii). From the definitions we have a bijection

(a) $\mathfrak{F}_{\langle\ ,\ \rangle} \xrightarrow{\sim} D_G, V_* \mapsto \triangle$

(D_G as in 1.1), where $\triangle = (G_0^\triangle \supset G_1^\triangle \supset G_2^\triangle \supset \dots)$ is defined in terms of V_* by $G_0^\triangle = E_{\geqslant 0} V_* \cap G$ and $G_n^\triangle = 1 + E_{\geqslant n}^{\langle\ ,\ \rangle} V_*$ for $n \geqslant 1$. The sets $\widetilde{\xi}(V_*)$ (with $V_* \in \mathfrak{F}_{\langle\ ,\ \rangle}$) form a partition of $\mathcal{M}_{\langle\ ,\ \rangle}$. (If $N \in \mathcal{M}_{\langle\ ,\ \rangle}$, we have $N \in \widetilde{\xi}(V_*)$, where $V_* = V_*^N$.)

Let $V_* \in \mathfrak{F}_{\langle\ ,\ \rangle}$. Let C_0 be the unique $\mathrm{GL}(V)$-conjugacy class in $\mathrm{Nil}(V)$ that contains $\xi(V_*)$. We have

$$E_{\geqslant 2}^{\langle\ ,\ \rangle} V_* - \widetilde{\xi}(V_*) = (E_{\geqslant 2} V_* - \xi(V_*)) \cap \mathcal{M}_{\langle\ ,\ \rangle} = E_{\geqslant 2} V_* \cap (\cup_C C) \cap \mathcal{M}_{\langle\ ,\ \rangle}$$

(the last equality follows from 2.9; C runs over all $\mathrm{GL}(V)$-orbits in $\mathrm{Nil}(V)$ such that $C \subset \overline{C}_0 - C_0$). Using 3.12 we see that

$$E_{\geqslant 2}^{\langle\ ,\ \rangle} V_* - \widetilde{\xi}(V_*) = E_{\geqslant 2}^{\langle\ ,\ \rangle} V_* \cap (\cup_C (C \cap \mathcal{M}_{\langle\ ,\ \rangle})),$$

where C runs over all $\mathrm{GL}(V)$-orbits in $\mathrm{Nil}(V)$ such that $C \cap \mathcal{M}_{\langle\ ,\ \rangle} \neq \varnothing$ and $C \subset \overline{C_0 \cap \mathcal{M}_{\langle\ ,\ \rangle}} - (C_0 \cap \mathcal{M}_{\langle\ ,\ \rangle})$. We see that if $V_* \mapsto \triangle$ (as in (a)) and $\blacktriangle$ is the G-orbit of $\triangle$ in D_G, then (with the notation of 1.1) $\widetilde{H}^\blacktriangle$ is the union of G-conjugacy classes in $1 + \mathcal{M}_{\langle\ ,\ \rangle}$ contained in $1 + \overline{C}_0$, $H^\blacktriangle$ is the union of G-conjugacy classes in $1 + \mathcal{M}_{\langle\ ,\ \rangle}$ contained in $1 + C_0$, and $X^\triangle = 1 + \widetilde{\xi}(V_*) = 1 + (E_{\geqslant 2}^{\langle\ ,\ \rangle} V_* \cap C_0)$. We see that $\mathfrak{P}_1$–$\mathfrak{P}_3$ hold.

3.14. We preserve the setup of 3.13. Let $V_* \in \mathfrak{F}_{\langle\ ,\ \rangle}$ and let $\triangle \in D_G$ be the corresponding element. Define $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(\mathrm{gr}V_*)$ as in 3.2. The map $E_{\geqslant 2}^{\langle\ ,\ \rangle} V_* \to \mathrm{End}_2^0(\mathrm{gr}V_*), N \mapsto \overline{N}$ restricts to a map

$$\pi \colon \widetilde{\xi}(V_*) \to E := \{\nu \in \mathrm{End}_2^0(\mathrm{gr}V_*) \mid \nu \text{ skew-adjoint with respect to } \langle\ ,\ \rangle_0\}.$$

We show:

(a) *The group* $E_{\geqslant 3}^{\langle\ ,\ \rangle} V_*$ (*see* 3.13) *acts freely on* $\widetilde{\xi}(V^*)$ *by* $T, N \mapsto T * N$ (*see* 3.13) *and the orbit space of this action may be identified with* E *via* π.

We show this only at the level of sets. If $T \in E_{\geqslant 3}^{\langle\ ,\ \rangle} V_*, N \in \widetilde{\xi}(V_*)$, then $T * N \in E_{\geqslant 2}^{\langle\ ,\ \rangle} V_*$ and $T * N, N$ induce the same map in $\mathrm{End}_2(\mathrm{gr}V_*)$; hence $T * N \in \widetilde{\xi}(V_*)$. Thus $T, N \mapsto T * N$ is an action of $E_{\geqslant 3}^{\langle\ ,\ \rangle} V_*$ on $\widetilde{\xi}(V_*)$. This action is free: it is the restriction of the action of $E_{\geqslant 3}^{\langle\ ,\ \rangle} V_*$ on $E_{\geqslant 2}^{\langle\ ,\ \rangle} V_*$ by left multiplication for the group structure in

3.13. If $N, N' \in \tilde{\xi}(V_*)$ induce the same map in $\mathrm{End}_2^0(\mathrm{gr}V_*)$, then $N' - N \in E_{\geqslant 3}V_*$. Set $T = (N' - N)(1+N)^{-1} \in E_{\geqslant 3}V_*$. Then $(1+T)(1+N) = 1 + N'$ and we have automatically $T \in E_{\geqslant 3}^{\langle\,,\,\rangle}V_*$ and $T + N = N'$. Thus the orbits of the $E_{\geqslant 3}^{\langle\,,\,\rangle}V_*$-action on $\tilde{\xi}(V^*)$ are exactly the nonempty fibers of π. It remains to show that π is surjective. This follows from Proposition 3.5(a).

Now let $N, N' \in \tilde{\xi}(V_*)$ be such that $\overline{N} = \overline{N}' = \nu \in \mathrm{End}_2^0(\mathrm{gr}V_*)$. We show:

(b) *there exists $g \in E_{\geqslant 0}V_* \cap G$ such that $N' = gNg^{-1}$.*

First assume that $p = 2$. The set $\mathcal{L} \subset 2\mathbf{N}$ defined in 3.3 in terms of N is the same as that defined in terms of N'. Let $q_n \colon P^{\nu}_{-n} \to \mathbf{k}$ be the quadratic form defined in terms of N (for $n \in \mathcal{L}$) as in 3.3 and let $q'_n \colon P^{\nu}_{-n} \to \mathbf{k}$ be the analogous quadratic form defined in terms of N'. From 3.3(a) we see that for any $n \in \mathcal{L}$ there exists an automorphism $h_n \colon P^{\nu}_{-n} \to P^{\nu}_{-n}$ which preserves the symplectic form $x, y \mapsto b_n(x,y)$ (see 3.1) and satisfies $q'_n(x) = q_n(h_n x)$ for any $x \in P^{\nu}_{-n}$. There is a unique $h \in \mathrm{Sp}(\langle\,,\,\rangle_0)$ such that $h(x) = h_n(x)$ for $x \in P^{\nu}_{-n}, n \in \mathcal{L}$, $h(x) = x$ for $x \in P^{\nu}_{-n}$, $n \in \mathbf{Z} - \mathcal{L}$, and $h\nu = \nu h$. Let $V = \bigoplus_a V_a$ be a direct sum decomposition as in 3.2(b). Then $\mathrm{End}_0(V)$ is defined and we define $\tilde{h} \in \mathrm{End}_0(V)$ by the requirement that for any a, $\tilde{h} \colon V_a \to V_a$ corresponds to $h \colon \mathrm{gr}_a V_* \to \mathrm{gr}_a V_*$ under the obvious isomorphism $V_a \xrightarrow{\sim} \mathrm{gr}_a V_*$. Then $\tilde{h} \in E_{\geqslant 0}V_* \cap G$ and $\tilde{h}N\tilde{h}^{-1} = N''$, where $N'' \in E_{\geqslant 2}^{\langle\,,\,\rangle}V_*$ satisfies $\overline{N}'' = \nu$. Moreover, the quadratic form $P^{\nu}_{-n} \to \mathbf{k}$ defined as in 3.3 in terms of N'' (instead of N) for $n \in \mathcal{L}$ is $x \mapsto h_n(x)$, that is, q'_n. From 3.3(c) we see that the Quadratic form Q_n defined for $n \in \mathcal{L}'$ in terms of N'' is the same as that defined in terms of N'. From Proposition 3.5(c) we see that there exists $h' \in 1 + E_{\geqslant 1}^{\langle\,,\,\rangle}V_*$ such that $h'N''h'^{-1} = N'$. Setting $g = h'\tilde{h} \in E_{\geqslant 0}V_* \cap G$, we have $gNg^{-1} = N'$.

Next assume that $p \neq 2$. From 3.5(c) we see that there exists $g \in 1 + E_{\geqslant 1}^{\langle\,,\,\rangle}V_*$ such that $gNg^{-1} = N'$. This proves (b).

We see that $\mathfrak{P}_6$ (hence $\mathfrak{P}_4$) holds.

From (a) we see that the $G_0^{\triangle}$-action on $\tilde{\xi}(V^*)$ (conjugation) induces an action of $\overline{G}_0^{\triangle} = G_0^{\triangle}/G_1^{\triangle}$ on E and from (b) we see that this gives rise to a bijection between the set of $G_0^{\triangle}$-orbits on $\tilde{\xi}(V^*)$ and the set of $\overline{G}_0^{\triangle}$-orbits on E. We describe this last set of orbits. We identify $\overline{G}_0^{\triangle}$ with $\mathrm{End}_0(\mathrm{gr}V_*) \cap \mathrm{Sp}(\langle\,,\,\rangle_0)$ with the action on E given by $g \colon \nu \mapsto \nu'$, where $\nu'(x) = g\nu(g^{-1}x)$ for $x \in \mathrm{gr}V_*$.

Let $I = \{n \in 2\mathbf{N}+1 \mid \dim \mathrm{gr}_{-n}V_* - \dim \mathrm{gr}_{-n-2}V_* \in \{2,4,6,\dots\}\}$. For any subset $J \subset I$ let E_J be the set of all $\nu \in E$ such that for any $n \in I$ we have

$$\{x \in \mathrm{gr}_{-n}V_* \mid \nu^{n+1}x = 0, \langle x, \nu^n x\rangle_0 \neq 0\} \neq \varnothing \leftrightarrow n \in J.$$

Let $\overline{E}$ be the set of all direct sum decompositions $\mathrm{gr}V_* = \bigoplus_{n \geqslant 0} W^n$, where $W^n \in \bar{\mathcal{C}}$ (see 2.1) are such that $\langle W^n, W^{n'}\rangle_0 = 0$ for $n \neq n'$, and for $n \geqslant 0$, $\dim W^n_a$ is $\dim \mathrm{gr}_{-n}V_* - \dim \mathrm{gr}_{-n-2}V_*$ if $a \in [-n, n], a = n \mod 2$ and is 0 for other a. Define $\phi \colon E \to \overline{E}$ by $\nu \mapsto (W^n)$, where $W^n = \sum_{k \geqslant 0} \nu^k P^{\nu}_{-n}$. Then ϕ is $\overline{G}_0^{\triangle}$-equivariant, where $\overline{G}_0^{\triangle}$ acts on $\overline{E}$ in an obvious way (transitively).

Let $w = (W^n) \in \overline{E}$. Let G^w be the stabilizer of w in $\overline{G}_0^{\triangle}$. Let $E^w = \phi^{-1}(w)$. Now E^w may be identified with $\prod_{n \geqslant 0} E^w_n$, where E^w_n is the set of all skew-adjoint elements in $\mathrm{End}_2^0(W^n)$ with respect to $\langle\ ,\ \rangle_0|_{W^n}$. Moreover G^w may be identified with $\prod_{n\geqslant 0} G_n^w$, where $G_n^w = \mathrm{End}_0(W^n)\cap \mathrm{Sp}(\langle\ ,\ \rangle_0|_{W^n})$. Furthermore, we may identify $E_n^w = E_n^{w1}\times E_n^{w2}$ where E_n^{w1} consists of all sequences of isomorphisms

(c) $W_{-n}^n \xrightarrow{\sim} W_{-n+2}^n \xrightarrow{\sim} W_{-n+4}^n \xrightarrow{\sim} \cdots \xrightarrow{\sim} W_{-\delta}^n$

($\delta = 0$ if n is even and $\delta = 1$ if n is odd) and E_n^{w2} is the set of nondegenerate symmetric bilinear forms $W_{-1}^n\times W_{-1}^n \to \mathbf{k}$, if n is odd, and is a point if n is even. (This identification is obtained by attaching to $\nu \in E_n^w$ the isomorphisms (c) induced by ν and, if n is odd, the bilinear form $x, x' \mapsto \langle x, \nu x'\rangle_0$ on W_{-1}^n.)

We claim that if $p = 2$, the subsets E_J are precisely the orbits of $\overline{G}_0^\triangle$ on E whereas if $p \neq 2$, E is a single orbit of $\overline{G}_0^\triangle$. Using the transitivity of the $\overline{G}_0^\triangle$ action on $\overline{E}$, we see that it suffices to prove the following: if $p = 2$, the subsets $E_J^w = E_J \cap E^w$ are precisely the G^w-orbits on E, whereas if $p \neq 2$, E^w is a single G^w-orbit. If $n \notin I$, G_n' acts transitively on E_n'. If $n \in I$, $\mathrm{pr}_2\colon E_n^w \to E_n^{w2}$ induces a bijection between the set of G_n^w-orbits on E_n^w and the set of $\mathrm{GL}(W_{-1}^n)$-orbits on the set of nondegenerate symmetric bilinear forms on W_{-1}^n. The last set of orbits consists of one element if $p \neq 2$ and of two elements (the symplectic forms and the nonsymplectic forms) if $p = 2$. This verifies our claim.

We see that the first assertion of $\mathfrak{P}_8$ holds.

As above, we identify E with the set of triples (w, α, j), where $w \in \overline{E}$, α is a collection of isomorphisms as in (c) (for each $n \geqslant 0$) and j is a sequence $(j_n)_{n\in I}$, where $j_n \in \mathrm{Bil}(W_{-1}^n)$ is symmetric and nondegenerate.

Assume that $p = 2$. Let $J \subset J' \subset I$. From the previous discussion we see that the $\overline{G}_0^\triangle$-orbits on E that contain E_J in their closure and are contained in the closure of $E_{J'}$ are those of the form E_K, where $J \subset K \subset J'$. Let $E_{J,J'} = \cup_{K;\ J\subset K\subset J'} E_K$. We identify E_J with the set of $(w, \alpha, j) \in E$ such that j_n is not symplectic for $n \in J$ and symplectic for $n \in I - J$. We identify $E_{J,J'}$ with the set of $(w, \alpha, j) \in E$ such that j_n is not symplectic for $n \in J$ and symplectic for $n \in I - J'$. Let $\widetilde{E}_J$ be the set of all triples $(w, \alpha, \widetilde{j})$, where w, α are as above, and $\widetilde{j} = (\widetilde{j}_n)_{n\in I}$, where for $n \in J$, $\widetilde{j}_n \in \mathrm{Bil}(W_{-1}^n)$ is a symmetric nonsymplectic nondegenerate form and, for $n \in I - J$, $\widetilde{j}_n\colon W_{-1}^n \times W_{-1}^n \to \mathbf{k}$ is the square of a symplectic nondegenerate form.

Now E_J, $E_{J,J'}$, and $\widetilde{E}_J$ are naturally algebraic varieties. Define a finite bijective morphism $\sigma\colon E_J \to \widetilde{E}_J$ by $(w, \alpha, j) \mapsto (w, \alpha, \widetilde{j})$, where $\widetilde{j}_n = j_n$ for $n \in J$, $\widetilde{j}_n = j_n^2$ for $n \in I - J$. Define $\rho\colon E_{J,J'} \to \widetilde{E}_J$ by $(w, \alpha, j) \mapsto (w, \alpha, \widetilde{j})$, where $\widetilde{j}_n = j_n$ for $n \in J$ and $\widetilde{j}_n(x, x') = j_n(x, x')^2 + j_n(x, x)j_n(x', x')$ for $n \in I - J$, $x, x' \in W_{-1}^n$. (To see that this is well defined, we must check that for $n \in I - J$, the symplectic form $x, x' \mapsto j_n(x, x') + \sqrt{j_n(x, x)j_n(x', x')}$ on W_{-1}^n is nondegenerate. Let R be the radical of this symplectic form. Let $H = \{x \in W_{-1}^n \mid j_n(x, x) = 0\}$. If $x \in R\cap H$, then $j_n(x, x')$ for all x', hence $x = 0$. Thus $R \cap H = 0$. Since H is either W_{-1}^n or a hyperplane in W_{-1}^n, we see that $R\cap H$ is either R or a hyperplane in R. It follows that $\dim R$ is 0 or 1. Since $R = \dim W_{-1}^n \bmod 2$, we see that $\dim R$ is even. Hence $R = 0$, as required.)

Taking here $J' = I$, we see that $\mathfrak{P}_7$ holds. We now return to a general J'. We consider the fiber $\mathcal{F}$ of ρ at $(w, \alpha, \widetilde{j}) \in \widetilde{E}_J$. We may identify $\mathcal{F}$ with the set of all collections $(j_n)_{n\in I-J}$, where $j_n \in \mathrm{Bil}(W_{-1}^n)$ is symmetric nondegenerate for all n, j_n is symplectic for $n \in I - J'$, and $\widetilde{j}_n(x, x') = j_n(x, x')^2 + j_n(x, x)j_n(x', x')$ for $n \in I - J$,

$x, x' \in W^n_{-1}$. Let $\mathcal{F}'$ be the set of all collections $(h_n)_{n\in I-J}$, where h_n is a linear form $W^n_{-1} \to \mathbf{k}$, zero for $n \in I - J'$. We define a map $\mathcal{F} \to \mathcal{F}'$ by $(j_n)_{n\in I-J} \mapsto (h_n)_{n\in I-J}$, where $h_n(x) = \sqrt{j_n(x,x)}$ for $x \in W^n_{-1}$. We define a map $\mathcal{F}' \to \mathcal{F}$ by $(h_n)_{n\in I-J} \mapsto (j_n)_{n\in I-J}$, where $j_n(x,x') = \sqrt{\tilde{j}_n(x,x') + h_n(x)h_n(x')}$ for $x, x' \in W^n_{-1}$. (We show that this is well defined. We must show that j_n given by the last equality is nondegenerate. Let R' be the radical of j_n. Define $v \in W^n_{-1}$ by $h_n(y) = \sqrt{\tilde{j}_n(v,y)}$ for all $y \in W^n_{-1}$. If $x \in R', y \in W^n_{-1}$, we have $\sqrt{\tilde{j}_n(x,y)} = h_n(x)h_n(y) = h_n(x)\sqrt{\tilde{j}_n(v,y)}$, hence $\sqrt{\tilde{j}_n(x - h_n(x)v, y)} = 0$. Since $\sqrt{\tilde{j}_n}$ is nondegenerate, we have $x - h_n(x)v = 0$. Hence $x = h_n(x)v = h_n(h_n(x)v)v = h_n(x)h_n(v)v$. This is 0, since $h_n(v) = \sqrt{\tilde{j}_n(v,v)} = 0$. Thus $R' = 0$.) Clearly, $\mathcal{F} \to \mathcal{F}', \mathcal{F}' \to \mathcal{F}$ are inverse to each other. We see that $\mathcal{F}$ is in natural bijection with a vector space of dimension $\sum_{n\in J'-J} c_n$, where $c_n = \dim W^n_{-1}$. Hence if $\mathbf{k}, q$ are as in $\mathfrak{P}_5$, we have

$$\sum_{K;\ J\subset K\subset J'} |E_K(\mathbf{F}_q)| = |E_{J,J'}(\mathbf{F}_q)| = \prod_{n\in J'-J} q^{c_n}|E_J(\mathbf{F}_q)|.$$

From this we see that $|E_K(\mathbf{F}_q)| = \prod_{n\in K}(q^{c_n} - 1)|E_\varnothing(\mathbf{F}_q)|$ for any $K \subset I$. Using this and $\mathfrak{P}_6$ we see that the second assertion of $\mathfrak{P}_8$ holds.

For $\mathbf{k}, q$ as in $\mathfrak{P}_5$ we have

$$|H^\blacktriangle(\mathbf{F}_q)| = |X^\vartriangle(\mathbf{F}_q)||G(\mathbf{F}_q)/G^\vartriangle_0(\mathbf{F}_q)|, \quad |X^\vartriangle(\mathbf{F}_q)| = q^{\dim G^\vartriangle_3}|E(\mathbf{F}_q)|.$$

Hence to verify $\mathfrak{P}_5$, it suffices to show that $|E(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p. Using the $\overline{G}^\vartriangle_0$-equivariant fibration $\phi\colon E \to \overline{E}$, we see that $|E(\mathbf{F}_q)| = |\overline{E}(\mathbf{F}_q)||E^w(\mathbf{F}_q)|$ for any $w \in \overline{E}$. Since $|\overline{E}(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p, it suffices to show that for any $w \in \overline{E}(\mathbf{F}_q)$, $|E^w(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p, or that $|E^w_n(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p for any $w \in \overline{E}(\mathbf{F}_q)$ and any $n \geqslant 0$. Using the identification $E^w_n = E^{w1}_n \times E^{w2}_n$ and the fact that $|E^{w1}_n(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p, we see that it suffices to show that $|E^{w2}_n(\mathbf{F}_q)|$ is a polynomial in q with integer coefficients independent of p. Thus it suffices to check the following statement.

Let W be an $\mathbf{F}_q$-vector space of dimension d. Let $b(W)$ be the set of nondegenerate symmetric bilinear forms $W \times W \to \mathbf{F}_q$. Then $|b(W)|$ is a polynomial in q with integer coefficients independent of p.

We argue by induction on d. For $d = 0$ the result is obvious. Assume that $d \geqslant 1$. We write $|b(W)| = f(d,q)$. The set of all symmetric bilinear forms $W \times W \to \mathbf{F}_q$ has cardinal $q^{d(d+1)/2}$; it is a disjoint union $\sqcup_X b_X(W)$, where X runs over the linear subspaces of W and $b_X(W)$ is the set of symmetric bilinear forms $W \times W \to \mathbf{F}_q$, with radical equal to X. Thus

$$q^{d(d+1)/2} = \sum_X |b_X(W)| = \sum_X |b(W/X)| = \sum_{d'\in[0,d]} g(d,d',q)f(d-d',q),$$

where $g(d,d',q) = |\{X \subset W \mid \dim X = d'\}|$. We see that

$$f(d,q) = q^{d(d+1)/2} - \sum_{d' \in [1,d]} g(d,d',q) f(d-d',q).$$

Since $g(d,d',q)$ is a polynomial in q with integer coefficients independent of p and the same holds for $f(d-d',q)$, with $d' \in [1,d]$ (by the induction hypothesis), it follows that $f(d,q)$ is as required.

We see that $\mathfrak{P}_5$ holds.

4. The group $A^1(u)$

4.1. In this section we assume that $p \geqslant 2$ and that $\mathfrak{P}_1$ holds. Let $u \in \mathcal{U}$. According to $\mathfrak{P}_1$ there is a unique $\triangle \in D_G$ such that $u \in X^\triangle$. Let $A^1(u) = Z_{G_1^\triangle}(u)/Z_{G_1^\triangle}(u)^0$, a finite p-group.

The image of $A^1(u)$ in $Z_G(u)/Z_G(u)^0$ is a normal subgroup (since $Z_G(u) = Z_{G_0^\triangle}(u)$ — see 1.1(c) — and $Z_{G_1^\triangle}(u)$ is normal in $Z_{G_0^\triangle}(u)$).

In this section we describe the finite group $A^1(u)$ in some examples assuming that $p = 2$ and G is a symplectic group.

Let $n \geqslant 1$. Let $I = \{i \in [-n,n] \mid i = n \mod 2\}$. Let $F \in \mathcal{C}, F \neq 0$. Let $V = \bigoplus_{i \in I} F_i$, where $F_i = F$. Define $N \colon V \to V$ by $(x_i) \mapsto (x'_i)$, where $x'_i = x_{i-2}$ for $i \in I - \{-n\}, x'_{-n} = 0$. We fix $\langle\ ,\ \rangle_0 \in \mathrm{Symp}(V)$ such that $\langle (x_i),(y_i)\rangle_0 = \sum_{i\in I} b(x_i, y_{-i})$, where $b \in \mathrm{Bil}(F)$ satisfies $b^* = b$, b is nondegenerate, and $b \in \mathrm{Symp}(F)$ if n is even.

Let $\langle\ ,\ \rangle \in \mathrm{Symp}(V)$ be such that $\langle Nx, y\rangle + \langle x, Ny\rangle + \langle Nx, Ny\rangle = 0$ for $x, y \in V$ and $\langle x, y\rangle = \langle x, y\rangle_0$ if there exists i such that $x_j = 0$ for $j \neq i$ and $y_j = 0$ for $j \neq -i$. We have $\langle (x_i),(y_i)\rangle = \sum_{i,j\in I} b_{ij}(x_i, y_j)$, where $b_{ij} \in \mathrm{Bil}(F)$ are such that

$b_{i-2,j} + b_{i,j-2} + b_{ij} = 0$ if $i, j \in I - \{-n\}$,
$b_{i,-i} = b$ for all $i \in I$,
$b_{ii} \in \mathrm{Symp}(F)$ for all $i \in I$,
$b^*_{ij} = -b_{ji}$ for all $i, j \in I$.

(We automatically have $b_{ij} = 0$ if $i + j \geqslant 1$.)

Let $\Delta' = \{T \in \mathrm{GL}(V) \mid TN = NT, \langle x, y\rangle = \langle Tx, Ty\rangle$ for all $x, y \in V\}$, a subgroup of $\mathrm{Sp}(\langle\ ,\ \rangle)$; equivalently, Δ' is the set of linear maps $T \colon V \to V$ of the form

$$T \colon (x_i) \mapsto (x'_i), x'_i = \sum_{j \in I;\ j \leqslant i} T_{i-j} x_j,$$

where $T_r \in \mathrm{End}(F)$ ($r \in \{0, 2, 4, \dots, 2n\}$) are such that

$$b_{ij}(x,y) = \sum_{i',j' \in I;\ i' \geqslant i, j' \geqslant j} b_{i'j'}(T_{i'-i}(x), T_{j'-j} y) \qquad (E_{ij})$$

for $i, j \in I, i+j \leqslant 0$, and $x, y \in F$. Now $(E_{ij}), (E_{i+2,j-2})$, with $i+j = 2k$, are equivalent if (E_{ab}) is assumed for $a + b = 2k+2$ (the sum of those two equations is $E_{i+2,j}$). Thus the conditions that T must satisfy are E_{ii} and $E_{i-2,i}$. Setting $x = y$ in these equations, we obtain equations (E^0_{ii}), $(E^0_{i-2,i})$. Note that the equation (E^0_{ii}) is $0 = 0$, hence can be omitted; the equation (E_{ii}) is a consequence of $(E^0_{i-2,i})$ (if it is defined). Hence the equations satisfied by the components of T are as follows:

$$(E^0_{-2,0}),(E_{-2,0}),(E^0_{-4,-2}),(E_{-4,-2}),\dots,(E^0_{-n,-n+2}),(E_{-n,-n+2}),(E_{-n,-n}) \tag{a}$$

(for n even),

$$\begin{aligned}(E_{-1,1}),(E^0_{-3,-1}),(E_{-3,-1}),(E^0_{-5,-3}),(E_{-5,-3}),\\ \dots,(E^0_{-n,-n+2}),(E_{-n,-n+2}),(E_{-n,-n})\end{aligned} \tag{b}$$

(for n odd). Assume first that n is even. The solutions T_0 of the first equation in (a) form an even orthogonal group, a variety with two connected components. For any such T_0, the solutions T_2 of the second equation in (a) form an affine space of dimension independent of T_0. For any T_0, T_2 already determined, the solutions T_4 of the third equation in (a) form an affine space of dimension independent of T_0, T_2. Continuing in this way, we see that the solutions of the equations (a) form a variety with two connected components. Moreover, the solutions in which T_0 is specified to be 1 form a connected variety.

Next assume that n is odd and b is symplectic. The solutions T_0 of the first equation in (b) form a symplectic group (a connected variety). For any such T_0, the solutions T_2 of the second equation in (b) form an affine space of dimension independent of T_0. For any T_0, T_2 already determined, the solutions T_4 of the third equation in (b) form an affine space of dimension independent of T_0, T_2. Continuing in this way, we see that the solutions of equations (b) form a connected variety. Moreover, the solutions in which T_0 is specified to be 1 form a connected variety.

One can show that if n is odd, $n \geqslant 3$, and b is not symplectic, then the solutions of equations (b) form a variety with two connected components. Moreover, the solutions in which T_0 is specified to be 1 form a disconnected variety.

In solving the equations above, we repeatedly use the statement (c) below. Let $\mathfrak{Q}$ be the vector space of quadratic forms $F \to \mathbf{k}$. Define linear maps a_1, a_2, and a_3 as follows:

$$\begin{aligned} a_1\colon \operatorname{End}(F) &\longrightarrow \mathfrak{Q}(F) \text{ is } \tau \mapsto q, q(x) = b(x,\tau(x));\\ a_2\colon \{\tau \in \operatorname{End}(F) \mid b(\tau(x),y) &= b(x,\tau(y) \text{ for all } x,y \in F\}\\ &\longrightarrow \operatorname{Hom}(F,\mathbf{k}) \text{ is } \tau \mapsto \theta, \theta(x) = \sqrt{b(x,\tau(x))};\\ a_3\colon \{b' \in \operatorname{Bil}(F) \mid b'^* = b'\} &\longrightarrow \operatorname{Hom}(F,\mathbf{k}) \text{ is } b' \mapsto \theta, \theta(x) = \sqrt{b'(x,x)}.\end{aligned}$$

Then

(c) *a_1, a_2, and a_3 are surjective.*

For a_3 this is clear. Consider now a_2. Let $\theta \in \operatorname{Hom}(F,\mathbf{k})$. By (c), for a_3 we can find $b' \in \operatorname{Bil}(F), b'^* = b'$ such that $\theta(x) = \sqrt{b'(x,x)}$. We can find a unique $\tau \in \operatorname{End}(F)$ such that $b(x,\tau(y)) = b'(x,y)$. Then $a_2(t) = \theta$. Consider now a_1. Let $q \in \mathfrak{Q}$. Let b^0 be a symplectic form on F. We can write $b^0 = d + d^*$, where $d \in \operatorname{Bil}(F)$. We can write $d(x,y) = b(x,\sigma(y))$ for some $\sigma \in \operatorname{End}(F)$. Then $b(x,\sigma(y)) + b(y,\sigma(x)) = b^0(x,y)$. Apply this to the symplectic form $b^0(x,y) = q(x+y) + q(x) + q(y)$. Then

$$b(x+y,\sigma(x+y))+b(x,\sigma(x))+b(y,\sigma(y)) = b(x,\sigma(y))+b(y,\sigma(x)) = q(x+y)+q(x)+q(y).$$

Hence $b(x,\sigma(x)) + q(x) = \theta(x)^2$ for some $\theta \in \mathrm{Hom}(F,\mathbf{k})$. By (c), for a_2 we can find $\tau \in \mathrm{End}(F)$ such that $b(x,\tau(x)) = \theta(x)^2$. Then $b(x,\sigma(x)) + b(x,\tau(x)) = q(x)$, that is, $b(x,(\sigma+\tau)(x)) = q(x)$. Thus a_1 is surjective. This proves (c).

4.2. Let V, $\langle\ ,\ \rangle$ be as in 3.2. Let $N \in \mathcal{M}_{\langle\ ,\ \rangle}, V_* = V_*^N$. Let e be as in 2.4. We show:

(a) *If W, W' are e-special subspaces of V (see 2.8), then there exists $g \in 1 + E_{\geqslant 1}^{\langle\ ,\ \rangle}V_*$ such that $g(W) = W'$, $gN = Ng$.*

By 2.8(b) we can find $g_1 \in 1 + E_{\geqslant 1}V_*$ such that $g_1(W) = W'$, $g_1N = Ng_1$. Then g_1 carries $\langle\ ,\ \rangle$ to a symplectic form $\langle\ ,\ \rangle'$, which induces the same symplectic form as $\langle\ ,\ \rangle$ on $\mathrm{gr}V_*$ and has the same associated quadratic forms as $\langle\ ,\ \rangle$ (see Proposition 3.6(b)). By the proof in 3.10 (cases 2 and 3), we see that there exists $g_2 \in 1 + E_{\geqslant 1}V_*$ such that $g_2(W') = W'$, $g_2N = Ng_2$, and g_2 carries $\langle\ ,\ \rangle'$ to $\langle\ ,\ \rangle$. Then $g = g_2g_1$ has the required properties.

4.3. Let V, $\langle\ ,\ \rangle$, N, V_*, and e be as in 4.2.

(a) *If $\langle x, Nx\rangle = 0$ for any $x \in V_{\geqslant -1}$, then $\mathcal{V} := \{g \in E_{\geqslant 1}^{\langle\ ,\ \rangle}V_* \mid gN = Ng\}$ is connected. Hence $A^1(1+N) = \{1\}$.*

We argue by induction on e. Let $\mathcal{X}$ be the set of all e-special subspaces (see 2.8) of V. By 2.8(b) the group $\{g \in 1 + E_{\geqslant 1}V_* \mid gN = Ng\}$ acts transitively on $\mathcal{X}$. This group is connected (it may be identified as a variety with the vector space $\{\xi \in E_{\geqslant 1}V_* \mid \xi N = N\xi\}$); hence $\mathcal{X}$ is connected. By 4.2(a), $\mathcal{V}$ acts transitively on $\mathcal{X}$. Since $\mathcal{X}$ is connected, it suffices to show that the stabilizer $\mathcal{V}_W$ of some $W \in \mathcal{X}$ in $\mathcal{V}$ is connected. This stabilizer is $\mathcal{V}' \times \mathcal{V}''$, where $\mathcal{V}', \mathcal{V}''$ are defined like $\mathcal{V}$ in terms of $W, W^\perp$ instead of V. By the results of 4.2, $\mathcal{V}'$ is connected. By the induction hypothesis applied to $W^\perp$, $\mathcal{V}''$ is connected. Hence $\mathcal{V}' \times \mathcal{V}''$ is connected. Hence $\mathcal{V}$ is connected.

5. Study of the varieties $\mathcal{B}_u$

5.1. We assume that $\mathbf{k} = \mathbf{k}_p$.

We say that an algebraic variety V over $\mathbf{k}$ has the *purity property* if for some/any $\mathbf{F}_q$-rational structure on V (where $\mathbf{F}_q$ is a finite subfield of $\mathbf{k}$) with Frobenius map $F\colon V \to V$ and any $n \in \mathbf{Z}$, any complex absolute value of any eigenvalue of $F^*\colon H_c^n(V, \overline{\mathbf{Q}}_l) \to H_c^n(V, \overline{\mathbf{Q}}_l)$ is $q^{n/2}$.

In this section we show that for certain $u \in \mathcal{U}$, the varieties $\mathcal{B}_u$ (see 0.1) have the purity property. We assume that properties $\mathfrak{P}_1$–$\mathfrak{P}_4$, $\mathfrak{P}_6$, $\mathfrak{P}_7$ hold for G.

Let $\triangle \in D_G$. Let $\Pi^\triangle$ be the (finite) set of orbits for the conjugation action of $G_0^\triangle$ on $\mathcal{B}$. Let $\bar{\mathcal{B}} = \{B \in \mathcal{B} \mid B \subset G_0^\triangle\}$. For any $\mathcal{O} \in \Pi^\triangle$, define a morphism $\xi^{\mathcal{O}}\colon \mathcal{O} \to \bar{\mathcal{B}}$ by $B \mapsto (B \cap G_0^\triangle)G_1^\triangle$. We show:

(a) *The fibers of $\xi^{\mathcal{O}}\colon \mathcal{O} \to \bar{\mathcal{B}}$ are exactly the orbits of $G_1^\triangle$ acting on $\mathcal{O}$ by conjugation.*

If $B, B' \in \mathcal{O}$, $\xi^{\mathcal{O}}(B) = \xi^{\mathcal{O}}(B')$, then $B' = g^{-1}Bg$, with $g \in G_0^\triangle$, $(B' \cap G_0^\triangle)G_1^\triangle = (B \cap G_0^\triangle)G_1^\triangle = g^{-1}(B \cap G_0^\triangle)G_1^\triangle g$. Hence $g \in (B \cap G_0^\triangle)G_1^\triangle$. Writing $g = g'g''$, $g' \in B \cap G_0^\triangle$, $g'' \in G_1^\triangle$, we have $B' = g^{-1}Bg = g''^{-1}Bg''$. This proves (a).

Let $Y^\triangle = \{(u, B) \in X^\triangle \times \mathcal{B} \mid u \in B\}$. We have a partition $Y^\triangle = \cup_{\mathcal{O} \in \Pi^\triangle} Y_{\mathcal{O}}^\triangle$, where $Y_{\mathcal{O}}^\triangle = \{(u, B) \in X^\triangle \times \mathcal{O} \mid u \in B\}$. Let $\mathcal{O} \in \Pi^\triangle$. We show:

(b) $Y^{\triangle}_{\mathcal{O}}$ *is smooth.*

Let $\widetilde{B} \in \mathcal{O}$. Let $Y' = \{(u,g) \in X^{\triangle} \times G^{\triangle}_0 \mid g^{-1}ug \in \widetilde{B} \cap X^{\triangle}\}$. We have a fibration $Y' \to Y^{\triangle}_{\mathcal{O}}$ with smooth fibers isomorphic to $G^{\triangle}_0 \cap \widetilde{B}$. Hence it suffices to show that Y' is smooth. Let $Y'' = (\widetilde{B} \cap X^{\triangle}) \times G^{\triangle}_0$. Define $Y' \xrightarrow{\sim} Y''$ by $(u,g) \mapsto (g^{-1}ug, g)$. It suffices to show that Y'' is smooth, or that $\widetilde{B} \cap X^{\triangle}$ is smooth. But $\widetilde{B} \cap X^{\triangle}$ is open in $\widetilde{B} \cap G^{\triangle}_2$ which is smooth, being an algebraic group. This proves (b).

For any $\beta \in \bar{\mathcal{B}}$, let $\mathcal{G}^{\mathcal{O}}_{\beta} = ((B \cap G^{\triangle}_2)G^{\triangle}_3)/G^{\triangle}_3$, where $B \in \mathcal{O}$ is such that $\xi^{\mathcal{O}}(B) = \beta$. Note that $\mathcal{G}^{\mathcal{O}}_{\beta}$ is a closed connected subgroup of $G^{\triangle}_2/G^{\triangle}_3$ independent of the choice of B. (To verify the last statement it suffices, by (a), to show that, for B as above and $v \in G^{\triangle}_1$, we have $(vBv^{-1} \cap G^{\triangle}_2)G^{\triangle}_3 = (B \cap G^{\triangle}_2)G^{\triangle}_3$. This follows from 1.1(b).) Now $G^{\triangle}_0$ acts on $\bar{\mathcal{B}}$ and on $G^{\triangle}_2/G^{\triangle}_3$ by conjugation. From the definitions we see that for $g \in G^{\triangle}_0$ and $\beta \in \bar{\mathcal{B}}$, we have $\mathcal{G}^{\mathcal{O}}_{g\beta g^{-1}} = g\mathcal{G}^{\mathcal{O}}_{\beta}g^{-1}$.

Let $\overline{Y}^{\triangle}_{\mathcal{O}} = \{(x,\beta) \in \overline{X}^{\triangle} \times \bar{\mathcal{B}} \mid x \in \mathcal{G}^{\mathcal{O}}_{\beta}\}$. We show:

(c) $\overline{Y}^{\triangle}_{\mathcal{O}}$ *is a closed smooth subvariety of* $\overline{X}^{\triangle} \times \bar{\mathcal{B}}$.

Let $\widetilde{B} \in \mathcal{O}$. We have a fibration $X^{\triangle} \times G^{\triangle}_0 \to \overline{X}^{\triangle} \times \bar{\mathcal{B}}$, $(u,g) \mapsto (\pi^{\triangle}(u), \xi^{\mathcal{O}}(g\widetilde{B}g^{-1}))$ with smooth fibers. It suffices to show that the inverse image of $\overline{Y}^{\triangle}_{\mathcal{O}}$ under this fibration is a closed smooth subvariety of $X^{\triangle} \times G^{\triangle}_0$, or that

$$\{(u,g) \in X^{\triangle} \times G^{\triangle}_0 \mid g^{-1}ug \in X^{\triangle} \cap ((\widetilde{B} \cap G^{\triangle}_2)G^{\triangle}_3)\}$$

is a closed smooth subvariety of $X^{\triangle} \times G^{\triangle}_0$, or that $(X^{\triangle} \cap ((\widetilde{B} \cap G^{\triangle}_2)G^{\triangle}_3)) \times G^{\triangle}_0$ is a closed smooth subvariety of $X^{\triangle} \times G^{\triangle}_0$, or that $X^{\triangle} \cap ((\widetilde{B} \cap G^{\triangle}_2)G^{\triangle}_3)$ is a closed smooth subvariety of $X^{\triangle}$. It is closed since $(\widetilde{B} \cap G^{\triangle}_2)G^{\triangle}_3$ is closed in $G^{\triangle}_2$. It is smooth since it is an open subset of $(\widetilde{B} \cap G^{\triangle}_2)G^{\triangle}_3$ which is smooth, being an algebraic group.

We show:

(d) *The morphism* $a\colon Y^{\triangle}_{\mathcal{O}} \to \overline{Y}^{\triangle}_{\mathcal{O}}, (u,B) \mapsto (\pi^{\triangle}(u), \xi^{\mathcal{O}}(B))$ *is a fibration with fibers isomorphic to an affine space of a fixed dimension.*

Clearly a is surjective. Let $(u,B) \in Y^{\triangle}_{\mathcal{O}}$. Let

$$Z := a^{-1}(a(u,B)) = \{(u',B') \mid u = u'f, B' = vBv^{-1} \text{ for some } v \in G^{\triangle}_1, f \in G^{\triangle}_3; u' \in B'\}.$$

We show only that Z is isomorphic to an affine space of fixed dimension. Let $\widetilde{Z} = \{(f,v) \in G^{\triangle}_3 \times G^{\triangle}_1 \mid v^{-1}uf^{-1}v \in B\}$. Then $Z = \widetilde{Z}/(B \cap G^{\triangle}_1)$, where $B \cap G^{\triangle}_1$ acts freely on $\widetilde{Z}$ by $b\colon (f,v) \mapsto (f, vb^{-1})$. Since conjugation by $G^{\triangle}_1$ acts trivially on $G^{\triangle}_2/G^{\triangle}_3$, the map $(f,v) \mapsto (f',v), f' = u^{-1}v^{-1}uf^{-1}v$ is an isomorphism

$$\begin{aligned}\widetilde{Z} \longrightarrow \widetilde{Z}' &= \{(f',v) \in G^{\triangle}_3 \times G^{\triangle}_1 \mid uf' \in B\} \\ &= \{(f',v) \in G^{\triangle}_3 \times G^{\triangle}_1 \mid f' \in B\} = (G^{\triangle}_3 \cap B) \times G^{\triangle}_1\end{aligned}$$

(we use $u \in B$) and we have $Z = (G^{\triangle}_3 \cap B) \times G^{\triangle}_1/(B \cap G^{\triangle}_1)$. Now $G^{\triangle}_3 \cap B, G^{\triangle}_1$, and $B \cap G^{\triangle}_1$ are connected unipotent groups of dimension independent of B, for $B \in \mathcal{O}$. (The connectedness follows from the fact that these unipotent groups are normalized by a maximal torus of G contained in $G_0^\vartriangle \cap B$. The fact that the dimension does not depend on B follows from the fact that $G_1^\vartriangle, G_3^\vartriangle$ are normalized by $G_0^\vartriangle$.) We see that Z is an affine space of constant dimension.

We now fix $x \in \overline{X}^\Delta$. Let $\Sigma = (\pi^\Delta)^{-1}(x) \subset X^\Delta$. Let $\mathcal{B}_\Sigma = \{(u, B) \in \Sigma \times \mathcal{B} \mid u \in B\}$. We have $\mathcal{B}_\Sigma = \sqcup_{\mathcal{O} \in \Pi^\Delta} \mathcal{O}_\Sigma$, where $\mathcal{O}_\Sigma = \{(u, B) \in \Sigma \times \mathcal{O} \mid u \in B\}$. Let $\mathcal{O} \in \Pi^\Delta$. Let $\bar{\mathcal{B}}_x^{\mathcal{O}} = \{\beta \in \bar{\mathcal{B}} \mid x \in \mathcal{G}_\beta^{\mathcal{O}}\}$. We show:

(e) *$\bar{\mathcal{B}}_x^{\mathcal{O}}$ is a closed subvariety of $\bar{\mathcal{B}}$ and $a' \colon \mathcal{O}_\Sigma \to \bar{\mathcal{B}}_x^{\mathcal{O}}$, $(u, B) \mapsto \xi^{\mathcal{O}}(B)$ is a fibration with fibers isomorphic to an affine space of a fixed dimension.*

Let $\widetilde{B} \in \mathcal{O}, u_0 \in \Sigma$. We have a locally trivial fibration $G_0^\vartriangle \to \bar{\mathcal{B}}$, $g \mapsto \xi^{\mathcal{O}}(g\widetilde{B}g^{-1})$. To show that $\bar{\mathcal{B}}_x^{\mathcal{O}}$ is closed, it suffices to show that its inverse image under this fibration is closed in $G_0^\vartriangle$, or that $\{g \in G_0^\vartriangle \mid g^{-1}u_0 g \in (\widetilde{B} \cap G_2^\vartriangle)G_3^\vartriangle\}$ is closed in $G_0^\vartriangle$. This is clear since $(\widetilde{B} \cap G_2^\vartriangle)G_3^\vartriangle$ is closed in $G_2^\vartriangle$. The second assertion of (e) follows from (d) using the cartesian diagram

$$\begin{array}{ccc} \mathcal{O}_\Sigma & \xrightarrow{a'} & \bar{\mathcal{B}}_x^{\mathcal{O}} \\ \downarrow & & \downarrow \\ Y_{\mathcal{O}}^\vartriangle & \xrightarrow{a} & \overline{Y}_{\mathcal{O}}^\Delta \end{array},$$

where the left vertical map is the obvious inclusion and the right vertical map is $\beta \mapsto (x, \beta)$.

(f) *If the closure of the $G_0^\vartriangle$-orbit in $G_2^\vartriangle$ of some/any $u \in \Sigma$ is a subgroup Γ of $G_2^\vartriangle$, then $\bar{\mathcal{B}}_x^{\mathcal{O}}$ is smooth.*

Let $\widetilde{B} \in \mathcal{O}, u_0 \in \Sigma$. As in the proof of (e), it suffices to show that $\{g \in G_0^\vartriangle \mid g^{-1}u_0 g \in (\widetilde{B} \cap G_2^\vartriangle)G_3^\vartriangle\}$ is smooth. This variety is a fibration over $R = (G_0^\vartriangle -$ conjugacy class of $u_0) \cap ((\widetilde{B} \cap G_2^\vartriangle)G_3^\vartriangle)$ with smooth fibers isomorphic to $Z_{G_0^\vartriangle}(u_0)$ (via $g \mapsto g^{-1}u_0 g$). Hence it suffices to show that R is smooth. From our assumption we see that R is open in $\Gamma \cap ((\widetilde{B} \cap G_2^\vartriangle)G_3^\vartriangle)$ which is smooth, being an algebraic group. This proves (f).

Note that the hypothesis of (f) holds at least in the case where the $G_0^\vartriangle$-conjugacy class of some/any $u \in \Sigma$ is open dense in $G_2^\vartriangle$. We show:

(g) *If the hypothesis of (f) holds, then $\mathcal{B}_\Sigma$ has the purity property.*

From (e), (f) we see that $\bar{\mathcal{B}}_x^{\mathcal{O}}$ is a smooth projective variety of pure dimension. From [D1] it then follows that $\bar{\mathcal{B}}_x^{\mathcal{O}}$ has the purity property. From this and (e), we see that, for $\mathcal{O} \in \Pi^\Delta$, $\mathcal{O}_\Sigma$ has the purity property. Using this and the partition $\mathcal{B}_\Sigma = \sqcup_{\mathcal{O} \in \Pi^\Delta} \mathcal{O}_\Sigma$, we see that (g) holds.

5.2. Let $\overline{Z}(x) = \{\overline{g} \in \overline{G}_0^\Delta \mid \overline{g}x = x\overline{g}\}$. Let $\widetilde{Z}(x)$ be the inverse image of $\overline{Z}(x)$ under $G_0^\vartriangle \to \overline{G}_0^\Delta$. Thus we have $G_1^\vartriangle \subset \widetilde{Z}(x)$ and $\widetilde{Z}(x)/G_1^\vartriangle \xrightarrow{\sim} \overline{Z}(x)$. Note that the inverse image of $\overline{Z}(x)^0$ is $\widetilde{Z}(x)^0$ and that we have $\widetilde{Z}(x)^0/G_1^\vartriangle \xrightarrow{\sim} \overline{Z}(x)^0$. Now $\widetilde{Z}(x)$ acts transitively (by conjugation) on Σ. (Indeed, let $u, u' \in \Sigma$. By $\mathfrak{P}_6$ we can find $g \in G_0^\vartriangle$ such that $u' = gug^{-1}$. Automatically we have $g \in \widetilde{Z}(x)$.) Since Σ is irreducible, it follows that $\widetilde{Z}(x)^0$ acts transitively (by conjugation) on Σ.

Let $u \in \Sigma$. Recall that $\mathcal{B}_u = \{B \in \mathcal{B} \mid u \in B\}$. Let $Z'_G(u) = Z_G(u) \cap \widetilde{Z}(x)^0$. Since $Z_G(u) \subset \widetilde{Z}(x)$ — see 1.1(c) — we see that $Z'_G(u)$ is a normal subgroup of $Z_G(u)$ containing $Z_G(u)^0$. Let $A'(u)$ be the image of $Z'_G(u)$ in $A(u) := Z_G(u)/Z_G(u)^0$. This is a normal subgroup of $A(u)$. We have $Z'_G(u)/Z_{G_1^\triangle}(u) \xrightarrow{\sim} \overline{Z}(x)^0$. Hence $Z'_G(u) = Z_{G_1^\triangle}(u)Z'_G(u)^0 = Z_{G_1^\triangle}(u)Z_G(u)^0$. It follows that

$A'(u)$ is the image of the obvious homomorphism $A^1(u) \longrightarrow A(u)$.

Now $Z_G(u)$ acts by conjugation on $\mathcal{B}_u$; this induces an action of $A(u)$ on $H^n(\mathcal{B}_u, \overline{\mathbf{Q}}_l)$ which restricts to an $A'(u)$-action on $H^n(\mathcal{B}_u, \overline{\mathbf{Q}}_l)$.

Assume that the hypothesis of 5.1(f) holds and that $A'(u)$ acts trivially on $H^n_c(\mathcal{B}_u, \overline{\mathbf{Q}}_l)$ for any n. We show:

(a) *$\mathcal{B}_u$ has the purity property.*

Define $f\colon \mathcal{B}_\Sigma \to \Sigma$ by $(g, B) \mapsto g$. For any n, $R^n f_!(\overline{\mathbf{Q}}_l)$ is an equivariant constructible sheaf for the transitive $\widetilde{Z}(x)^0$ action on Σ; hence it is a local system on Σ corresponding to a representation of $A'(u)$ (the group of components of the isotropy group of u in $\widetilde{Z}(x)^0$) on $H^n_c(\mathcal{B}_u, \overline{\mathbf{Q}}_l)$. This representation is trivial, hence $R^n f_!(\overline{\mathbf{Q}}_l)$ is a constant local system. Since Σ is an affine space of dimension, say d, we see that $H^a_c(\Sigma, R^n f_!(\overline{\mathbf{Q}}_l))$ is $H^n_c(\mathcal{B}_u, \overline{\mathbf{Q}}_l)(-d)$, if $a = 2d$, and is zero if $a \neq 2d$. It follows that the standard spectral sequence

$$E_2^{a,n} = H^a_c(\Sigma, R^n f_!(\overline{\mathbf{Q}}_l)) \Longrightarrow H^{a+n}_c(\mathcal{B}_\Sigma, \overline{\mathbf{Q}}_l)$$

is degenerate. Hence the purity property of $\mathcal{B}_\Sigma$ (see 5.1(g)) implies that any complex absolute value of any eigenvalue of the Frobenius map on

$$E_2^{2d,n} = H^n_c(\mathcal{B}_u, \overline{\mathbf{Q}}_l)(-d)$$

is $q^{d+n/2}$. Hence any complex absolute value of any eigenvalue of the Frobenius map on $H^n_c(\mathcal{B}_u, \overline{\mathbf{Q}}_l)$ is $q^{n/2}$. This proves (a).

5.3. Since the hypothesis of 5.1(f) is not satisfied in general, we seek an alternative way to prove purity.

Let γ be the $\overline{G}_0^\triangle$-orbit of x in $\overline{X}^\triangle$. Let $\hat{\gamma} \xrightarrow{\rho} \gamma_1 \xleftarrow{\sigma} \gamma$ be as in $\mathfrak{P}_7$. Let $\Xi = \rho^{-1}(\sigma(x))$. Let $\bar{\mathcal{B}}^{\mathcal{O}}_\Xi = \{(x', \beta) \in \overline{Y}^\triangle_{\mathcal{O}} \mid x' \in \Xi\}$, a closed subvariety of $\overline{Y}^\triangle_{\mathcal{O}}$. We show:

(a) *$\bar{\mathcal{B}}^{\mathcal{O}}_\Xi$ is smooth of pure dimension.*

Let $\beta_0 \in \bar{\mathcal{B}}$. Let $\mathcal{G}_0 = \mathcal{G}^{\mathcal{O}}_{\beta_0}$. It suffices to show that the inverse image of $\bar{\mathcal{B}}^{\mathcal{O}}_\Xi$ under the fibration $\Xi \times \overline{G}_0^\triangle \to \Xi \times \bar{\mathcal{B}}$, $(x', \overline{g}) \mapsto (x', \overline{g}\beta_0\overline{g}^{-1})$ (with smooth connected fibers) is smooth of pure dimension, or that $\mathfrak{S} := \{(x', \overline{g}) \in \Xi \times \overline{G}_0^\triangle \mid \overline{g}^{-1}x'\overline{g} \in \mathcal{G}_0\}$ is smooth of pure dimension. The morphism $f\colon \mathfrak{S} \to \hat{\gamma} \cap \mathcal{G}_0$, $(x', \overline{g}) \mapsto \overline{g}^{-1}x'\overline{g}$ is smooth with fibers of pure dimension. (We show only that, for any $y \in \hat{\gamma} \cap \mathcal{G}_0$, the fiber $f^{-1}(y)$ is isomorphic to $\{\overline{g} \in \overline{G}_0^\triangle \mid \overline{g}x\overline{g}^{-1} = x\}$ which is smooth of pure dimension. We have

$$\begin{aligned} f^{-1}(y) &= \{(x', \overline{g}) \in \Xi \times \overline{G}_0^\triangle \mid \overline{g}^{-1}x'\overline{g} = y\} \cong \{\overline{g} \in \overline{G}_0^\triangle \mid \overline{g}y\overline{g}^{-1} \in \Xi\} \\ &= \{\overline{g} \in \overline{G}_0^\triangle \mid \rho(\overline{g}y\overline{g}^{-1}) = \sigma(x)\} = \{\overline{g} \in \overline{G}_0^\triangle \mid \overline{g}\sigma^{-1}(\rho(y))\overline{g}^{-1} = x\} \end{aligned}$$

and it remains to use the transitivity of the $\overline{G}_0^\Delta$-action on γ.) It suffices to show that $\hat\gamma\cap\mathcal{G}_0$ is empty or smooth, connected. Now $\hat\gamma$ is open in G_2^Δ/G_3^Δ, hence $\hat\gamma\cap\mathcal{G}_0$ is open in $\mathcal{G}_0$ which is connected and smooth (being an algebraic group).

We show:

(b) *Assume that for any $\mathcal{O}\in\Pi^\Delta$, there is a $\mathbf{k}^*$-action on $\bar{\mathcal{B}}_\Xi^{\mathcal{O}}$ which is a contraction to the projective subvariety $\bar{\mathcal{B}}_x^{\mathcal{O}}$. Then $\mathcal{B}_\Sigma$ has the purity property.*

Consider an $\mathbf{F}_q$-rational structure on G such that G_a^Δ is defined over $\mathbf{F}_q$ for any a, and that $\mathcal{O}$, x, Ξ are defined over $\mathbf{F}_q$. Let ζ be an eigenvalue of Frobenius on $H^n(\bar{\mathcal{B}}_x^{\mathcal{O}},\overline{\mathbf{Q}}_l)$. By [D2, 3.3.1], any complex absolute value of ζ is $\leqslant q^{n/2}$ (since $\bar{\mathcal{B}}_x^{\mathcal{O}}$ is projective). Our assumption implies that the inclusion $\bar{\mathcal{B}}_x^{\mathcal{O}}\subset\bar{\mathcal{B}}_\Xi^{\mathcal{O}}$ induces for any n an isomorphism $H^n(\bar{\mathcal{B}}_\Xi^{\mathcal{O}},\overline{\mathbf{Q}}_l)\xrightarrow{\sim}H^n(\bar{\mathcal{B}}_x^{\mathcal{O}},\overline{\mathbf{Q}}_l)$. Hence ζ is also an eigenvalue of Frobenius on $H^n(\bar{\mathcal{B}}_\Xi^{\mathcal{O}},\overline{\mathbf{Q}}_l)$. Since $\bar{\mathcal{B}}_\Xi^{\mathcal{O}}$ is smooth of pure dimension, say d, it satisfies Poincaré duality; hence $q^d\zeta^{-1}$ is an eigenvalue of Frobenius on $H_c^{2d-n}(\bar{\mathcal{B}}_\Xi^{\mathcal{O}},\overline{\mathbf{Q}}_l)$. By [D2, 3.3.1] applied to $\bar{\mathcal{B}}_\Xi^{\mathcal{O}}$, we see that any complex absolute value of $q^d\zeta^{-1}$ is $\leqslant q^{(2d-n)/2}$, hence any complex absolute value of ζ is $\geqslant q^{n/2}$. It follows that any complex absolute value of ζ is $q^{n/2}$. We see that $\bar{\mathcal{B}}_x^{\mathcal{O}}$ has the purity property. (This argument is similar to one of Springer in [Sp].) From this and 5.1(e), we see that for $\mathcal{O}\in\Pi^\Delta$, $\mathcal{O}_\Sigma$ has the purity property. Using this and the partition $\mathcal{B}_\Sigma=\sqcup_{\mathcal{O}\in\Pi^\Delta}\mathcal{O}_\Sigma$, we see that $\mathcal{B}_\Sigma$ has the purity property.

If we assume in addition that $A'(u)$ acts trivially on $H_c^n(\mathcal{B}_u,\overline{\mathbf{Q}}_l)$ for any n, we see, as in 5.2, that $\mathcal{B}_u$ has the purity property.

5.4. Let $V,\langle\ ,\ \rangle$ be as in 3.2. Assume that $p=2$ and that $G=\mathrm{Sp}(\langle\ ,\ \rangle)$. Let $u\in\mathcal{U}$. We set $u=1+N, V_*=V_*^N$. Assume that

(a) $\langle x,Nx\rangle=0$ *for any* $x\in V_{\geqslant-1}$.

We set

$$\Gamma=1+\{N'\in E_{\geqslant2}^{\langle\ ,\ \rangle}V_*\mid\langle x,N'x\rangle=0\text{ for all }x\in V_{\geqslant-1}\}.$$

Now Γ is a subgroup of $1+E_{\geqslant2}^{\langle\ ,\ \rangle}V_*$. (Assume that $1+N',1+N''\in\widetilde{G}_2$. Let $x\in V_{\geqslant-1}$. We have $\langle x,N'x\rangle=0,\langle x,N''x\rangle=0$. We must show that $\langle x,(N'+N''+N'N'')x\rangle=0$ or that $\langle x,N'N''x\rangle=0$. This follows from $N'N''x\in V_{\geqslant3}$ and $3-1\geqslant1$.) Clearly, Γ is normal in G_0^Δ. Since Γ is a closed unipotent subgroup normalized by G_0^Δ, it must be connected. Now

$$\mathcal{J}:=1+\{N'\in\widetilde{G}_2\mid\overline{N}'\in\mathrm{End}_2^0(\mathrm{gr}V_*)\}$$

is open in Γ since it is the inverse image under $\Gamma\to\mathrm{End}_2(\mathrm{gr}V_*),1+N'\mapsto\overline{N}'$ of the open subset $\mathrm{End}_2^0(\mathrm{gr}V_*)$ of $\mathrm{End}_2(\mathrm{gr}V_*)$. Also, $\mathcal{J}\neq\varnothing$ since $1+N\in\mathcal{J}$. Hence $\mathcal{J}$ is an open dense subset of Γ. By the results in 3.14, $\mathcal{J}$ is the G_0^Δ-conjugacy class of $1+N$.

We see that the hypothesis of 5.1(f) holds. Using 5.2(a) we see that:

(b) *$\mathcal{B}_u$ has the purity property for any $u\in G$ whose conjugacy class is minimal in the unipotent piece containing it, see* 1.1*, and such that any Jordan block of even size appears an even number times.*

(For such u, $A'(u)$ is trivial by 4.3(a).)

Alternatively, one can show that, for u as in (b), the method of 5.3 is applicable (the hypothesis of 5.3(b) holds) and one obtains another proof of (b).

References

[DLP] C. De Concini, G. Lusztig, C. Procesi, *Homology of the zero set of a nilpotent vector field on the flag manifold*, J. Amer. Math. Soc. **1** (1988), 15–34.

[D1] P. Deligne, *La conjecture de Weil,* I, Publ. Math. IHES **43** (1974), 273–308.

[D2] P. Deligne, *La conjecture de Weil,* II, Publ. Math. IHES **52** (1980), 137–252.

[E] H. Enomoto, *The conjugacy classes of Chevalley groups of type* G_2 *over finite fields of characteristic* 2 *or* 3, J. Fac. Sci. Univ. Tokyo, I **16** (1970), 497–512.

[Ka] N. Kawanaka, *Generalized Gelfand–Graev representations of exceptional simple algebraic groups over a finite field*, Invent. Math. **84** (1986), 575–616.

[Ko] B. Kostant, *The principal three dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[L1] G. Lusztig, *A class of irreducible representations of a Weyl group*, Proc. Kon. Nederl. Akad. (A) **82** (1979), 323–335.

[L2] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. **1** (1997), 194–207.

[LS] G. Lusztig, N. Spaltenstein, *On the generalized Springer correspondence for classical groups*, in: *Algebraic Groups and Related Topics*, Adv. Stud. Pure Math., Vol. 6, North Holland and Kinokuniya, Amsterdam, 1985, 289–316.

[M] K. Mizuno, *The conjugate classes of unipotent elements of the Chevalley groups* E_7 *and* E_8, Tokyo J. Math. **3** (1980), 391–459.

[Sh] K. Shinoda, *The conjugacy classes of Chevalley groups of type* F_4 *over finite fields of characteristic* 2, J. Fac. Sci. Univ. Tokyo, I **21** (1974), 133–159.

[S1] N. Spaltenstein, *Classes Unipotentes et Sous-Groupes de Borel*, LNM, Vol. 946, Springer Verlag, Berlin, New York, 1980.

[S2] N. Spaltenstein, *On the generalized Springer correspondence for exceptional groups*, in: *Algebraic Groups and Related Topics*, Adv. Stud. Pure Math., Vol. 6, North Holland and Kinokuniya, Amsterdam, 1985, 317–338.

[Sp] T. A. Springer, *A purity result for fixed point sets varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo, IA **31** (1984), 271–282.

[W] G. E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc. **3** (1963), 1–62.