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ESSENTIAL DIMENSION OF ALGEBRAIC GROUPS AND INTEGRAL REPRESENTATIONS OF WEYL GROUPS

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Abstract. Upper bounds on the essential dimension of algebraic groups can be found by examining related questions about the integral representation theory of lattices for their Weyl groups. We examine these questions in detail for all simple affine algebraic groups, expanding on work of Lorenz and Reichstein for PGL_n. This results in upper bounds on the essential dimensions of these simple affine algebraic groups which match or improve on the previously known upper bounds.

1. Introduction

The purpose of this paper is to use integral representation theory to obtain upper bounds on the essential dimension of certain linear algebraic groups. Throughout the paper, we will work over a fixed algebraically closed field k of characteristic zero. Let G be a fixed linear algebraic group over k.

A *G-variety* X is an algebraic variety with a regular G-action. X is a *generically free G-variety* if G acts freely (with trivial stabilizers) on a dense open subset of X. A dominant G-equivariant rational map of generically free G-varieties X and Y is called a *G-compression* and is denoted by $X \dashrightarrow Y$. Then the essential dimension of an algebraic group is

$$\mathrm{ed}(G) = \min \dim(Y/G),$$

where Y is taken over the set of generically free G-varieties for which there exists a G-compression from a linear generically free G-variety X to Y. Essential dimension comes up naturally in a number of interesting situations. It is a numerical invariant of the group G which is often the number of independent parameters needed to describe certain algebraic objects associated to G. In [Re], it is shown that:

- $\mathrm{ed}(\mathrm{PGL}_n)$ is the minimum positive integer d such that every division algebra of degree n over k can be defined over a field K_0 with $\mathrm{trdeg}_k K_0 \leqslant d$;
- $\mathrm{ed}(\mathrm{O}_n)$ (respectively $\mathrm{ed}(\mathrm{SO}_n)$) is the minimum positive integer d such that every quadratic form in n variables (respectively of determinant 1) is equivalent to one defined over a field of transcendence degree $\leqslant d$;

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- $\mathrm{ed}(\mathrm{G}_2)$ (respectively $\mathrm{ed}(\mathrm{F}_4)$) is the minimum positive integer d such that every octonion algebra (respectively exceptional 27 dimensional Jordan algebra) can be defined over a field of $\mathrm{trdeg}_k(K_0) \leqslant d$.

The methods used for computing, or at least estimating, the essential dimension of an algebraic group G are geometric, algebraic and cohomological. In a few cases, the value of the essential dimension is known. The following proposition, due to Reichstein, gives a summary of most of the known exact values for $\mathrm{ed}(G)$:

Proposition 1.1. ([Re])

(a) $\mathrm{ed}(G) = 0$ *if and only if* G *is a connected algebraic group with Levi subgroup isomorphic to* $L_1 \times \cdots \times L_r$ *with* $L_i \cong k^\times$, Sp_{2n} *or* SL_n *for some* n;
(b) $\mathrm{ed}(\mathrm{O}_n) = n$;
(c) $\mathrm{ed}(\mathrm{SO}_n) = n - 1$;
(d) $\mathrm{ed}(\mathrm{G}_2) = 3$.

The exact values for a few other simple groups are known. Rost [Ro1] has calculated the essential dimension of the Spin_n groups for $n \leqslant 14$. The essential dimension of PGL_n is known for $n = 2, 3, 4, 6$. The result for $n = 4$ is due to Rost [Ro2], the others are due to Reichstein [Re]. Kordonskiĭ [K] showed that the essential dimension of F_4 is 5. For the other simple algebraic groups, both upper and lower bounds are known (see [Re]), but naturally one seeks to make improvements.

The case of PGL_n has attracted some attention due to its connection with generic division algebras. Let A be a central simple algebra with centre K containing k. Define

$$\tau(A) = \min\{\mathrm{trdeg}_k(K_0) \mid A \cong K \otimes_{K_0} A_0,\ A_0 \text{ is a central simple } K_0\text{-algebra}\}.$$

Let $UD(n)$ be the universal division algebra, the k-subalgebra of $M_n(x_{ij}, y_{ij})$ generated by 2 generic n by n matrices $X = (x_{ij})$ and $Y = (y_{ij})$. Then $\mathrm{ed}(\mathrm{PGL}_n) \equiv \tau(UD(n))$, see [Re]. Procesi found the initial bound of n^2 for $\mathrm{ed}(\mathrm{PGL}_n)$, see [Pr]. The following proposition summarizes currently known bounds on the essential dimension of PGL_n:

Proposition 1.2.

- $\mathrm{ed}(\mathrm{PGL}_n) = 2,\ n = 2, 3, 6$, *see* [Re];
- $\mathrm{ed}(\mathrm{PGL}_4) = 5$, *see* [Ro2];
- $\mathrm{ed}(\mathrm{PGL}_n) \leqslant n^2 - 2n$, *see* [Re];
- $\mathrm{ed}(\mathrm{PGL}_n) \leqslant \mathrm{ed}(\mathrm{PGL}_{mn}) \leqslant \mathrm{ed}(\mathrm{PGL}_n) + \mathrm{ed}(\mathrm{PGL}_m),\ (m, n) = 1$, *see* [Re];
- $\mathrm{ed}(\mathrm{PGL}_{n^r}) \geqslant 2r$, *see* [Re];
- $\mathrm{ed}(\mathrm{PGL}_n) \leqslant (n-1)(n-2)/2$, $n \geqslant 5$ *odd, see* [LR], [LRRS].

In the last result, due to Lorenz and Reichstein, lattice-theoretic techniques were used. Note that this result was reproved in [LRRS] using different methods. In this paper, we will expand upon the techniques used in [LR] in order to improve on bounds for the essential dimension of certain adjoint algebraic groups.

Let G be a reductive algebraic k-group. Let T be a maximal torus, let N be the normalizer $N_G(T)$ of T, and let W be the Weyl group of G with respect to T. Let $X(T) = \mathrm{Hom}(T, k^*)$ be the character lattice which has a natural action of W. Reichstein showed in [Re] that

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(N).$$

Following the work of Saltman in [Sa2], we construct generically free N-varieties from information about the integral representation theory of the associated Weyl group. This analysis leads to some new bounds on the essential dimension of certain simple algebraic groups and produces an interesting class of irreducible Weyl group lattices.

The inequality $\mathrm{ed}(G) \leqslant \mathrm{ed}(N)$ is often strict. The bounds that we get from integral representations of Weyl groups seem to be most effective in the case of simple adjoint groups. In Section 2, we will define $\mathrm{ed}_W(X(T))$, which will be the best possible bound on $\mathrm{ed}(N)$ that can be obtained from our construction of generically free N-varieties from integral representations of the associated Weyl group. So we will have

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(N) \leqslant \mathrm{ed}_W(X(T)).$$

It is not known whether the last inequality is strict.

The first main result of the paper is the following theorem.

Theorem 1.3. *For a simple adjoint algebraic group G of rank n, not of type $\mathsf{A}_1, \mathsf{B}_n$,*

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(N) \leqslant \mathrm{ed}_W(\mathbb{Z}\Phi) \leqslant |\Phi_0| - n - 1,$$

where N is the normalizer of the maximal torus T and Φ (respectively Φ_0) is the (short) root system associated to G and T. If G is of type $\mathsf{A}_1, \mathsf{B}_n$, then $\mathrm{ed}(G) \leqslant \mathrm{ed}(N) \leqslant \mathrm{ed}_W(\mathbb{Z}\Phi) \leqslant 2n$.

Corollary 1.4. *We have the following bounds on the essential dimension of the simple adjoint algebraic groups:*

- *Type* A_1*:* $\mathrm{ed}(\mathrm{PGL}_2) \leqslant 2$;
- *Type* $\mathsf{A}_n, n \geqslant 2$*:* $\mathrm{ed}(\mathrm{PGL}_n) \leqslant n^2 - 2n$;
- *Type* $\mathsf{B}_n : \mathrm{ed}(\mathrm{SO}_{2n+1}) \leqslant 2n$;
- *Type* $\mathsf{C}_n, n \geqslant 3 : \mathrm{ed}(\mathrm{PSp}_{2n}) \leqslant 2n^2 - 3n - 1$;
- *Type* $\mathsf{D}_n, n \geqslant 4 : \mathrm{ed}(\mathrm{PO}_n) \leqslant 2n^2 - 3n - 1$;
- $\mathrm{ed}(\mathsf{E}_6(\mathrm{adj})) \leqslant 65$;
- $\mathrm{ed}(\mathsf{E}_7(\mathrm{adj})) \leqslant 112$;
- $\mathrm{ed}(\mathrm{E}_8) \leqslant 231$;
- $\mathrm{ed}(\mathrm{F}_4) \leqslant 19$;
- $\mathrm{ed}(\mathrm{G}_2) \leqslant 3$.

Remark 1.5. For an arbitrary simple algebraic group G, Kordonskiĭ obtained by different means the bound $\mathrm{ed}(G) \leqslant \dim(G) - 2\mathrm{rank}\,(G) - 1$ in [K2]. Since $\dim(G) - 2\mathrm{rank}\,(G) - 1 = |\Phi| - n - 1$ where $\mathrm{rank}\,(G) = n$ and Φ is the root system associated to G and a maximal torus T, we see that our bound matches this one in the simple adjoint simply laced cases and improves upon it in the simple adjoint nonsimply laced cases. For the adjoint group of type C_n, our bound of $2n^2 - 3n - 1$ improves on Kordonskiĭ 's bound of $2n^2 - n - 1$. In later work [K], Kordonskiĭ found a bound of 5 for the group of type F_4 which is considerably better than our corresponding bound of 19. The essential dimension of the remaining simple nonsimply connected adjoint groups of type G_2, A_1, and B_n were determined by Reichstein in [Re]. In these cases, our bound matches Reichstein's values.

The second main result produces a new bound for $\mathrm{ed}(\mathrm{PGL}_n), n \geqslant 4$, as follows.

Proposition 1.6. *For $n \geqslant 4$, $\mathrm{ed}(\mathrm{PGL}_n) \leqslant \mathrm{ed}(N) \leqslant \mathrm{ed}_{S_n}(\mathbb{Z}\mathsf{A}_{n-1}) \leqslant n^2 - 3n + 1$.*

Although this bound is not an improvement in the case of odd n covered by Lorenz and Reichstein, it gives the best currently known bound in the case where n is a power of 2, or at least divisible by a large power of 2.

The rest of this paper is structured as follows. In Section 2, we discuss the connection between the determination of upper bounds on the essential dimension of a simple algebraic group and a lattice-theoretic question about the associated Weyl group. We will use this discussion to define $\mathrm{ed}_W(X(T))$. In Section 3, we will prove Theorem 1.3 and Corollary 1.4 after some preliminaries on root systems and permutation resolutions. In Sections 4 and 5, we examine some of the lattice-theoretic questions posed in Section 2 which are motivated by the study of essential dimension and find upper bounds on $\mathrm{ed}_W(X(T))$ for the simple algebraic groups.

2. Essential dimension and Weyl group lattices

Let G be a reductive algebraic k-group where k is an algebraically closed field of characteristic 0. Let T be a maximal torus, $N \equiv N_G(T)$, the normalizer of T, W the Weyl group. Let $X(T) = \mathrm{Hom}(T, k^*)$ be the character lattice.

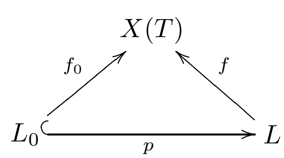
A construction due to Saltman [Sa2] produces a generically free linear N-variety from an exact sequence of W-lattices

$$0 \longrightarrow M \longrightarrow P \xrightarrow{f} X(T) \longrightarrow 0,$$

with $M = \mathrm{Ker}(f)$ a faithful W-lattice and P a permutation W-lattice. We will generalise his construction here, and then we will make connections to upper bounds on the essential dimension of N extending results in [LR] for PGL_n to an arbitrary reductive algebraic group G.

Proposition 2.1. *To any map of W-lattices $f : L \to X(T)$ which extends to a map of W-lattices $\hat{f} : P \to X(T)$ where P is a permutation W-lattice, one can associate an irreducible N-variety X_f having the following properties:*

(a) *Functoriality: A commutative diagram*



of W-lattices, where $f : L \to X(T)$ extends to a W-map $\hat{f} : P \to X(T)$ for a permutation lattice P, leads to a dominant rational map of N-varieties $X_p : X_f \dashrightarrow X_{f_0}$;

(b) $\dim(X_f/N) = \mathrm{rank}\,(\mathrm{Ker}(f))$*;*

(c) *X_f is generically free if and only if f is surjective and* $\mathrm{Ker}(f)$ *is a faithful W-lattice;*

(d) *If $L = P$ is a permutation lattice, then X_f is birationally equivalent to a linear N-variety.*

Proof. Let $f : L \to X(T)$ be a W-map of lattices. Let $k[L]$ be the group algebra of L with k-basis $\{e^l \mid l \in L\}$ written exponentially. We will show that there is a twisted N-action on $k[L]$ and hence on the irreducible variety $\mathrm{Spec}(k[L])$ which satisfies

$$n \cdot (ke^l) = ke^{(nT)\cdot l}; \qquad t \cdot e^l = f(l)(t)e^l,\ n \in N, t \in T, l \in L \tag{1}$$

where $nT \in W$ is the image of $n \in N$ in $W = N/T$. In [Sa2], Saltman does this for a permutation lattice $L = P$. We will extend his construction to a more general class of W-lattices. For example, for any permutation projective lattice L, we may extend $f : L \to X(T)$ to $\hat{f} : P \to X(T)$ where L is a direct summand of the permutation lattice P.

Let the permutation W-lattice P have permutation basis $p_i, i = 1, \dots, r$. We will define a kN space $V_{\hat{f}}$ with k-basis $\{e^{p_i} \mid i = 1, \dots, r\}$, which satisfies

$$n \cdot (ke^{p_i}) = ke^{(nT)\cdot p_i}; \qquad t \cdot e^{p_i} = \hat{f}(p_i)(t)e^{p_i}, n \in N, t \in T, i = 1, \dots, r. \tag{2}$$

The permutation lattice P can be decomposed into a direct sum of transitive components $P = \oplus_{j=1}^m P_j$. Let $\hat{f}_j : P_j \to X(T)$ be the restriction of P to P_j. First observe that if we determine for each $j = 1, \dots, m$, a kN space $V_{\hat{f}_j}$ which satisfies (2) for $\hat{f}_j : P_j \to X(T)$, then $V_{\hat{f}} = \oplus_{j=1}^m V_{\hat{f}_j}$ would satisfy (2) for $\hat{f} : P \to X(T)$. So it suffices to construct $V_{\hat{f}}$ for a transitive permutation lattice $P = \mathbb{Z}Wq = \mathbb{Z}[W/W_q]$ and a W-map $\hat{f} : P \to X(T)$ satisfying (2).

Note that, although the extension of groups

$$1 \longrightarrow T \longrightarrow N \xrightarrow{\pi} W \longrightarrow 1$$

does not in general split, Saltman shows in [Sa2, 2.5] that for each $\varphi \in X(T)$,

$$1 \longrightarrow T/\mathrm{Ker}(\varphi) \longrightarrow N_\varphi/\mathrm{Ker}(\varphi) \longrightarrow W_\varphi \longrightarrow 1 \tag{3}$$

does split as a sequence of W modules where W_φ is the stabilizer subgroup of φ in W and $N_\varphi = \pi^{-1}(W_\varphi)$.

We may apply this result to $\hat{f}(q) \in X(T)$. Let V_0 be the permutation space $k[W_{\hat{f}(q)}/W_q]$ with permutation basis e^{q_k}, $k = 1, \dots, d$ (written exponentially). (3) shows that we can define an action of $N_{\hat{f}(q)}$ on V_0 which extends the permutation action of $W_{\hat{f}(q)}$ and such that $t \in T$ acts on V_0 via

$$t \cdot e^{q_k} = \hat{f}(q_k)(t)e^{q_k}.$$

Indeed, we have

$$tw \cdot e^{q_k} = \hat{f}(w \cdot q_k)(t)e^{w\cdot q_k} = w \cdot [\hat{f}(q_k)](t)e^{w\cdot q_k} = \hat{f}(q_k)(w^{-1}tw)e^{w\cdot q_k} = w(w^{-1}tw) \cdot e^{q_k}$$

for $t \in T, w \in W_{\hat{f}(q)}, k = 1, \dots, d$, so the actions of T and $W_{\hat{f}(q)}$ combine to define a locally free action of $N_{\hat{f}(q)}$ on V_0.

Let $V_{\hat{f}} = \mathrm{Ind}_{N_{\hat{f}(q)}}^N V_0$. Note that the dimension of $V_{\hat{f}}$ is $r = \mathrm{rank}\,(P)$. Let $s = [N : N_{\hat{f}(q)}] = [W : W_{\hat{f}(q)}]$. For a transversal $n_j, j = 1, \dots, s$ of $N_{\hat{f}(q)}$ in N,

$$\{n_j \otimes e^{q_k} \mid j = 1, \dots, s;\ k = 1, \dots, d\}$$

is a k-basis of $V_{\hat{f}}$ and, correspondingly,

$$\{n_j T \cdot q_k \mid j = 1, \ldots, s;\ k = 1, \ldots, d\}$$

is a permutation basis of P. Relabelling this basis of P as $p_i, i = 1, \ldots, r$ and the corresponding basis of $V_{\hat{f}}$ as $e^{p_i}, i = 1, \ldots, r$, we may observe that $\{e^{p_i} \mid i = 1, \ldots, r\}$ satisfies (2). Indeed, $t \in T$ acts as required on this basis, since for $j = 1, \ldots, s$, $k = 1, \ldots, d$, we have

$$\begin{aligned} t \cdot [n_j \otimes e^{q_k}] &= n_j \otimes (n_j^{-1} t n_j) \cdot e^{q_k} = n_j \otimes \hat{f}(q_k)(n_j^{-1} t n_j) \cdot e^{q_k} \\ &= [n_j \cdot (\hat{f}(q_k))](t)] n_j \otimes e^{q_k} = [\hat{f}(n_j T \cdot q_k)(t)] n_j \otimes e^{q_k}. \end{aligned}$$

We also find that $n \in N$ acts as required on this basis: Since there exists $n' \in N_{\hat{f}(q)}$ which satisfies $nn_j = n_l n'$ for some $l \in \{1, \ldots, s\}$, we have

$$n \cdot [kn_j \otimes e^{q_k}] = n_l \otimes n' \cdot ke^{q_k} = kn_l \otimes e^{n'T \cdot q_k} = ke^{n_l T \cdot (n'T \cdot q_k)} = ke^{nT \cdot (n_j T \cdot q_k)}$$

using the identification $e^{n_j T \cdot q_k} \equiv n_j \otimes e^{q_k}$.

Let $S_{\hat{f}}$ be the multiplicative subgroup of $k(V_{\hat{f}})^\times$ generated by $k^\times$ and the $e^{p_i}, i = 1, \ldots, r$. Note that T acts trivially on $S_{\hat{f}}/k^\times$, which is then a $W = N/T$ lattice isomorphic to P. As Abelian groups, $S_f \cong k^\times \oplus P = k[P]^\times$, the units of the group algebra of P. So the N-action on S_f induces an extension of N-modules

$$[\alpha] = [0 \longrightarrow k^\times \longrightarrow k[P]^\times \longrightarrow P \longrightarrow 0] \in \mathrm{Ext}^1_N(P, k^\times) \cong H^1(N, \mathrm{Hom}(P, k^\times)).$$

Note that the action of N on $k[P]^\times$ can be completely defined by the 1-cocycle α as

$$n \cdot e^p = \alpha_n(n^{-1}(p)) e^{nT \cdot p}. \tag{4}$$

The N-action on the units $k[P]^\times$ extends naturally to $k[P]$ and hence to the k-variety $\mathrm{Spec}(k[P])$. We will write $X_{\hat{f}} = \mathrm{Spec}(k_\alpha[P])$ to remind us of the twisted action by α where $[\alpha] \in \mathrm{Ext}^1_N(P, k^\times)$ defines the N-action via (2). Note that the N-variety $X_{\hat{f}}$ with the N-action induced by (2) satisfies (1) for $L = P$.

Now for the W-map of lattices $f : L \to X(T)$ and its extension $\hat{f} : P \to X(T)$, the inclusion $L \hookrightarrow P$ induces an inclusion of N-varieties $\mathrm{Spec}(k[L]) \hookrightarrow \mathrm{Spec}(k[P]) = X_{\hat{f}}$. Define X_f as $\mathrm{Spec}(k[M])$ with this induced N-action. Let $[\beta]$ be the image of $[\alpha]$ under the induced map

$$\mathrm{Ext}^1_N(P, k^\times) \to \mathrm{Ext}^1_N(L, k^\times).$$

Then X_f is $\mathrm{Spec}(k_\beta[L])$ as an N-variety.

Now

$$k(X_{\hat{f}}/T) = k(X_{\hat{f}})^T = k_{\alpha'}(\mathrm{Ker}(\hat{f}))$$

as a $W = N/T$-field where

$$[\alpha'] \in \mathrm{Ext}^1_W(\mathrm{Ker}(\hat{f}), k^\times) = \mathrm{Ext}^1_N(\mathrm{Ker}(\hat{f}), k^\times)$$

is the image of $[\alpha]$ under

$$\mathrm{Ext}^1_N(L, k^\times) \to \mathrm{Ext}^1_N(\mathrm{Ker}(f), k^\times).$$

Then

$$k(X_f/T) = k(X_f)^T = k(X_f) \cap k(X_{\hat{f}})^T = k_{\beta'}(\mathrm{Ker}(f))$$

as a W-field where

$$[\beta'] \in \mathrm{Ext}^1_W(\mathrm{Ker}(f), k^\times) = \mathrm{Ext}^1_N(\mathrm{Ker}(f), k^\times)$$

is the image of $[\beta]$ under

$$\mathrm{Ext}^1_N(L, k^\times) \longrightarrow \mathrm{Ext}^1_N(\mathrm{Ker}(f), k^\times)$$

so that

$$k(X_f/N) = k(X_f/T)^W = k_{\beta'}(\mathrm{Ker}(f))^W.$$

Since the transcendence degree of $k(X_f/N) = k_{\beta'}(\mathrm{Ker}(f))^W$ over k is $\mathrm{rank}\,(\mathrm{Ker}(f))$, we see that $\dim(X_f/N) = \mathrm{rank}\,(\mathrm{Ker}(f))$.

X_f is a generically free T-variety if and only if f is surjective [OV, Theorem 3.2.5] and X_f/T is a generically free W-variety if and only if $\mathrm{Ker}(f)$ is a faithful W-lattice. So X_f is a generically free N-variety if and only if f is surjective and $\mathrm{Ker}(f)$ is a faithful W-lattice [LR, Lemma 2.1].

A commutative diagram as in Proposition 2.1(a) induces an injective W-equivariant map from $k_{\beta_0}[L_0] \to k_\beta[L]$, and hence a dominant rational N-equivariant map

$$X_p : X_f = \mathrm{Spec}(k_\beta[L]) \dashrightarrow X_{f_0} = \mathrm{Spec}(k_{\beta_0}[L_0]).$$

If $L = P$ is a permutation W-lattice, the N-variety $X_{\hat{f}} = \mathrm{Spec}(k_\alpha[P])$ is birationally equivalent to a linear N-variety. □

Corollary 2.2. *If there exists a commutative diagram of W-lattices*

$$\begin{array}{ccccc} & & X(T) & & \\ & \overset{f_0}{\nearrow} & & \overset{f}{\nwarrow} & \\ P_0 & & \underset{p}{\hookrightarrow} & & P \end{array}$$

with P permutation and $\mathrm{Ker}(f_0)$ *faithful, then*

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(N) \leqslant \mathrm{rank}\,(P_0) - \mathrm{rank}\,(G).$$

Proof. By Proposition 2.1, the diagram above induces a dominant N-equivariant map $X_p : X_f \dashrightarrow X_{f_0}$ from a generically free linear N-variety X_f to a generically free N-variety X_{f_0}. Then

$$\mathrm{ed}(N) \leqslant \dim(X_f/N) = \mathrm{rank}\,(\mathrm{Ker}(f_0)) = \mathrm{rank}\,(P_0) - \mathrm{rank}\,(G)$$

as required. □

The following general definition for an H-lattice Y will give an upper bound on $\mathrm{ed}(N)$ from representation-theoretic information about the W-lattice $X(T)$.

Definition 2.3. Let H be a finite group and Y a $\mathbb{Z}H$-lattice. Let $\mathcal{P}(Y)$ be the set of pairs (P, π) such that P is an H-permutation lattice, $\pi : P \to Y$ is an H-epimorphism with $\mathrm{Ker}(\pi)$ a faithful H-lattice, and let

$$r(Y) = \min\{\mathrm{rank}\,(P) \mid (P, \pi) \in \mathcal{P}(Y)\}.$$

Let $\mathcal{PE}(Y)$ be the set of extension classes

$$[0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow P \xrightarrow{\pi} Y \longrightarrow 0]$$

such that $(P, \pi) \in \mathcal{P}(Y)$. Let M be a faithful H-lattice such that

$$\xi = [0 \longrightarrow M \longrightarrow P \longrightarrow Y \longrightarrow 0] \in \mathcal{PE}(Y),$$

and let $\mathrm{ed}(\xi)$ be the minimum rank of a faithful H-sublattice M_0 such that

$$\xi \in \mathrm{Im}(\mathrm{Ext}^1_H(Y, M_0) \longrightarrow \mathrm{Ext}^1_H(Y, M)).$$

Then define

$$\mathrm{ed}_H(Y) = \min\{\mathrm{ed}(\xi) \mid \xi \in \mathcal{PE}(Y)\}.$$

Remark 2.4. This definition was suggested by Reichstein [Rem] and was motivated by Merkurjev's definition of the essential dimension of a functor [Me]. By Corollary 2.2, we note that

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(N) \leqslant \mathrm{ed}_W(X(T)),$$

as the existence of a diagram of the form as in Corollary 2.2 is equivalent to

$$\xi \in \mathrm{Im}(\mathrm{Ext}^1_W(X(T), M_0) \longrightarrow \mathrm{Ext}^1_W(X(T), M))$$

where $\xi \in \mathrm{Ext}^1_W(X(T), M)$ corresponds to the extension class

$$[0 \longrightarrow M \longrightarrow P \to X(T) \longrightarrow 0] \in \mathcal{PE}(X(T)).$$

Note also that $r(X(T)) - \mathrm{rank}\,(G)$ gives a first (rough) upper bound for $\mathrm{ed}_W(X(T))$.

In the case of PGL_n, Lorenz and Reichstein in [LR] showed

$$\mathrm{ed}(\mathrm{PGL}_n) \leqslant \frac{(n-1)(n-2)}{2}, \quad n \text{ odd}, n \geqslant 5,$$

by showing that there exists a commutative diagram of $\mathbb{Z}S_n$ lattices

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}\mathsf{A}_{n-1}^{\otimes 2} & \longrightarrow & \mathbb{Z}[S_n/S_{n-2}] & \longrightarrow & \mathbb{Z}\mathsf{A}_{n-1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \bigwedge^2(\mathbb{Z}\mathsf{A}_{n-1}) & \longrightarrow & X & \longrightarrow & \mathbb{Z}\mathsf{A}_{n-1} & \longrightarrow & 0 \end{array} \tag{5}$$

where $\mathbb{Z}\mathsf{A}_{n-1}$ is the root lattice and X is an S_n-lattice. They showed this by cohomological means. That is, they showed:

$$\operatorname{Ext}^1_{S_n}\big(\mathbb{Z}\mathsf{A}_{n-1}, \wedge^2(\mathbb{Z}\mathsf{A}_{n-1})\big) \twoheadrightarrow \operatorname{Ext}^1_{S_n}\big(\mathbb{Z}\mathsf{A}_{n-1}, (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2}\big)$$

for $n \geqslant 5$ odd using

$$\wedge^2(\mathbb{Z}\mathsf{A}_{n-1}) \rightarrowtail (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2} \twoheadrightarrow \operatorname{Sym}^2(\mathbb{Z}\mathsf{A}_{n-1})$$

and the fact, proved by Lemire and Lorenz in [LL], that $\operatorname{Sym}^2(\mathbb{Z}\mathsf{A}_{n-1})$ is stably permutation for n odd. So, in fact, they found a bound on $\operatorname{ed}_{S_n}(\mathbb{Z}\mathsf{A}_{n-1})$ for $n \geqslant 5$ odd. In this paper, we will show that the same reduction for the even case is impossible. However, we will now prove Proposition 1.6 to improve on the n^2-2n bound for the even case using similar techniques.

Proof of Proposition 1.6. Let U_n be the standard permutation lattice for S_n with permutation basis

$$\{e_i \mid 1 \leqslant i \leqslant n\}$$

where

$$\sigma(e_i) = e_{\sigma(i)}, \quad \sigma \in S_n, 1 \leqslant i \leqslant n.$$

Then the root lattice $\mathbb{Z}\mathsf{A}_{n-1}$ is the kernel of the augmentation map $\epsilon : U_n \to \mathbb{Z}$, $e_i \mapsto 1$. So we have the following exact sequence of S_n-lattices:

$$0 \longrightarrow \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow \mathbb{Z}[S_n/S_{n-1}] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Tensor this sequence with $\mathbb{Z}\mathsf{A}_{n-1}$ and use the fact that

$$\mathbb{Z}\mathsf{A}_{n-1} \otimes_{\mathbb{Z}} U_n \cong P := \oplus_{r \neq s}\mathbb{Z}(e_r \otimes e_s), \quad (e_s - e_r) \otimes e_s \mapsto e_r \otimes e_s$$

to obtain an exact sequence

$$0 \longrightarrow (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2} \longrightarrow P \xrightarrow{f} \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow 0$$

where $f(e_r \otimes e_s) = e_s - e_r$. Note that $P \cong \mathbb{Z}[S_n/S_{n-2}]$. Now define $g : P \to U_n$ by $g(e_r \otimes e_s) = e_s$ and put $P_0 = \operatorname{Ker}(g)$ and $M_0 = \operatorname{Ker} f \cap \operatorname{Ker} g = (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2} \cap P_0$. If $\{r, s, t\}$ are all distinct, then the element $e_r \otimes e_s - e_t \otimes e_s$ is in P_0 and maps to $e_t - e_r$ under f. Therefore, $f(P_0) = \mathbb{Z}\mathsf{A}_{n-1}$ if $n \geqslant 3$, and we obtain a commutative diagram of S_n- lattices:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2} & \longrightarrow & P & \xrightarrow{f} & \mathbb{Z}\mathsf{A}_{n-1} & \longrightarrow & 0 \\
 & & \big\uparrow & & \big\uparrow & & \big\| & & \\
0 & \longrightarrow & M_0 & \longrightarrow & P_0 & \xrightarrow{f_0} & \mathbb{Z}\mathsf{A}_{n-1} & \longrightarrow & 0
\end{array} .$$

Note that

$$\operatorname{rank} M_0 = \operatorname{rank} P - \operatorname{rank} U_n - \operatorname{rank} \mathbb{Z}\mathsf{A}_{n-1} = n^2 - 3n + 1.$$

Furthermore, if $n \geqslant 4$, then M_0 is faithful. Indeed, let $1 \neq \sigma \in S_n$, say $\sigma(i) \neq i$. Choose $j \notin \{i, \sigma(i)\}$ and choose two distinct elements $r, s \notin \{i, j\}$. Then the element

$$m = (e_s - e_r) \otimes (e_i - e_j) \in P$$

satisfies $f(m) = 0, g(m) = 0$ and $\sigma(m) \neq m$.

We can now apply Corollary 2.2 to complete the proof of the proposition. Note that in this case, we could actually apply Lemma 3.2 and 3.3 of [LR] as we are doing in the PGL_n case. □

We will show that for each irreducible root system $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$, a minimal element of $\mathcal{P}(\mathbb{Z}\Phi)$ is given by $(P(\Phi), \pi(\Phi))$, where $P(\Phi) = \oplus_{\alpha \in \Phi_0} \mathbb{Z} e_\alpha$ is the permutation lattice on the set of short roots Φ_0 and

$$\pi(\Phi) : P(\Phi) \longrightarrow \mathbb{Z}\Phi, \quad e_\alpha \mapsto \alpha.$$

Let $K(\Phi) = \mathrm{Ker}(\pi(\Phi))$. It turns out that $P(\mathsf{A}_{n-1}) \cong \mathbb{Z}[S_n/S_{n-2}]$, $K(\mathsf{A}_{n-1}) \cong \mathbb{Z}\mathsf{A}_{n-1}^{\otimes 2}$ and the Formanek–Procesi sequence above is precisely that given by $(P(\mathsf{A}_{n-1}), \pi(\mathsf{A}_{n-1}))$. There is also an analogous construction to that of (5). Let

$$P_-(\Phi) = \oplus_{\alpha \in (\Phi_0)_+} \mathbb{Z}(e_\alpha - e_{-\alpha}).$$

Then $P_-(\Phi)$ is clearly a W-sublattice of $P(\Phi)$. Let $K_-(\Phi) = P_-(\Phi) \cap K(\Phi)$. Then $K_-(\mathsf{A}_{n-1}) \cong \bigwedge^2(\mathbb{Z}\mathsf{A}_{n-1})$ and $\pi(\Phi)$ restricted to $P_-(\Phi)$ gives the exact sequence

$$0 \longrightarrow K_-(\Phi) \longrightarrow P_-(\Phi) \longrightarrow 2\mathbb{Z}\Phi \longrightarrow 0,$$

as in the case of A_{n-1}.

In Section 4, we examine the cohomological properties of the sequence

$$0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0. \tag{6}$$

We consider when it is possible to have (6) in the image of the map

$$\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K_-(\Phi)) \longrightarrow \mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$$

and find that this is possible if and only if $\Phi = \mathsf{A}_{n-1}$, n odd. More generally, we examine the question of when compressions of the above sequence are possible.

In Section 5, we determine minimal elements of $\mathcal{P}(X(T))$ and $r(X(T))$ for each simple algebraic group G with maximal torus T. This gives us rough upper bounds on $\mathrm{ed}_W(X(T))$ in each case.

3. Essential dimension of simple adjoint groups

In this section we will determine an upper bound on the essential dimension of the simple adjoint groups. We first discuss some preliminary material about permutation resolutions and root systems.

3.1. Permutation resolutions

Let H be a finite group and Y an H-lattice. We want to determine $(P,\pi) \in \mathcal{P}(Y)$. That is, we want to find an H-epimorphism $\pi : P \to Y$ from an H-permutation lattice P with $\mathrm{Ker}(\pi)$ being a faithful H-lattice.

We will begin with some general observations about homomorphisms from permutation lattices.

Remark 3.1. Let H_0 be a subgroup of H. Then $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}[H/H_0], Y)$ is in bijective correspondence with Y^{H_0}. This is just the easiest case of Shapiro's Lemma, i.e., for $n = 0$. Explicitly, any $g \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}[H/H_0], Y)$ is completely determined by its value on H_0, $g(H_0)$, which must be fixed by H_0. Conversely, for any $y \in Y^{H_0}$, the map $\mathbb{Z}[H/H_0] \to Y,\ hH_0 \mapsto hy$, is a well defined H-map.

Notation. Let $y_i \in Y$ and let H_i be a subgroup of H_{y_i} for $i = 1, \dots, k$ where H_y is the stabilizer subgroup of $y \in Y$ in H. Then $f_{y_1,\dots,y_k}$ will denote the H-homomorphism

$$f_{y_1,\dots,y_k} : \oplus_{i=1}^k \mathbb{Z}[H/H_i] \longrightarrow Y, \quad (h_1H_1, \dots, h_kH_k) \mapsto \textstyle\sum_{i=1}^k h_i y_i.$$

Remark 3.2. Let H_i, $i = 1, \dots, k$ be subgroups of the finite group H, let $y_i \in Y$ for $i = 1, \dots, k$ and let Y be an H-lattice. Then $f_{y_1,\dots,y_k}$ is an H-epimorphism from $\oplus_{i=1}^k \mathbb{Z}[H/H_i]$ onto Y if and only if

(a) $H_i \leqslant H_{y_i}$ for all $i = 1, \dots, k$;

(b) $\sum_{i=1}^k (\mathbb{Z}H) y_i = Y$.

Proposition 3.3. *Suppose H is a finite group and Y is a faithful H-lattice such that $\mathbb{Q}Y$ is an irreducible $\mathbb{Q}H$ space. Then:*

(a) *Suppose $g : P \twoheadrightarrow Y$ is an H-epimorphism. Then $M = \mathrm{Ker}(g)$ is a faithful H-lattice if and only if for every nontrivial normal subgroup N of H,* $\mathrm{rank}\,(P^N) < \mathrm{rank}\,(M)$*;*

(b) $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N) \leqslant [H : H_0]/2$ *for any subgroup H_0 of H_y, any $0 \neq y \in Y$ and any nontrivial normal subgroup N of H. In particular, $\mathbb{Z}[H/H_0]$ is a faithful H-lattice;*

(c) *If $\mathbb{Z}Hy = Y$ for some $H_0 \leqslant H_y$, then $g : \mathbb{Z}[H/H_0] \to Y, hH_0 \mapsto hy$ is an H-epimorphism such that $M = \mathrm{Ker}(g)$ is a faithful H-lattice if $[H : H_0] >$* $2\mathrm{rank}\,(Y)$*.*

Proof. (a) Let N be a normal subgroup of H. Then $0 \to M^N \to P^N \to Y^N$ is an exact sequence. Y^N is an H-sublattice of Y so that $\mathbb{Q}Y^N$ is a $\mathbb{Q}H$ submodule of $\mathbb{Q}Y$. But $\mathbb{Q}Y$ is irreducible and faithful as a $\mathbb{Q}H$ module so that $\mathbb{Q}Y^N = 0$, and hence $Y^N = 0$. So $M^N \cong P^N$, and hence $\mathrm{rank}\,(M^N) = \mathrm{rank}\,(P^N)$. Note also that $P^N \leqslant \mathrm{Ker}(\pi) = M$ as $\pi(P^N) \subset Y^N = 0$ so that $\mathrm{rank}\,(P^N) \leqslant \mathrm{rank}\,(M)$.

Let $N_0 = \mathrm{Ker}(H \to \mathrm{Aut}(M))$, then $M^{N_0} = M$ so that $\mathrm{rank}\,(M) = \mathrm{rank}\,(P^{N_0})$. Now if $\mathrm{rank}\,(P^N) < \mathrm{rank}\,(M)$ for all nontrivial normal subgroups N of H, then $N_0 = 1$ so that M is a faithful H-lattice.

Conversely, suppose M were a faithful H-lattice and N is were a nontrivial normal subgroup of H. If $\mathrm{rank}\,(P^N) = \mathrm{rank}\,(M)$, it follows that $M = P^N$ as both P/P^N and P/M are torsion free and $P^N \subset M$. This would contradict the fact that M is a faithful H-lattice. So $\mathrm{rank}\,(P^N) < \mathrm{rank}\,(M)$ for every nontrivial normal subgroup N of H.

(b) *Claim:* Given $1 \neq h_0$ in H_y, there exists an $h \in H$ such that $h^{-1}h_0h \notin H_y$.

Suppose not. Then $h_0hy = hy$ for all $h \in H$. But $\mathbb{Q}Hy = \mathbb{Q}Y$ as $\mathbb{Q}Y$ is an irreducible $\mathbb{Q}H$ module. This implies that h_0 fixes all elements of $\mathbb{Q}Y$, which contradicts the fact that $\mathbb{Q}Y$ is a faithful $\mathbb{Q}H$ module. So, by contradiction, there exists $h \in H$, $h^{-1}h_0h \notin H_y$ as claimed.

Note that it follows from the claim that H_y cannot contain a nontrivial normal subgroup N of H since otherwise for any $h_0 \in N \subset H_y$, there would exist $h \in H$ such that $h^{-1}h_0h \notin H_y$ and hence not in N, contradicting the normality of N.

Now let N be an arbitrary nontrivial normal subgroup of H and let H_0 be a subgroup of H_y. From the above remark we know that $N \cap H_y$ is a proper subgroup of N and hence $N \cap H_0$ must also be a proper subgroup of N. Using Mackey decomposition, we will determine $\mathrm{Res}^H_N(\mathbb{Z}[H/H_0])$:

$$\mathrm{Res}^H_N(\mathbb{Z}[H/H_0]) \cong \oplus_{D=NhH_0}\mathbb{Z}[N/hH_0h^{-1} \cap N]$$

where the sum is taken over the set of double cosets of N and H_0 in H. Fortunately, since N is normal, the double cosets of N and H_0 in H correspond to the left cosets of the subgroup NH_0 in H. Note also that the subgroups $hH_0h^{-1} \cap N$ are all conjugate in H since N is normal. So $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N)$ is the number of N-orbits in $\mathrm{Res}^H_N(\mathbb{Z}[H/H_0])$, or equivalently, the number of cosets of NH_0 in H. But this number is

$$\frac{|H||N \cap H_0|}{|H_0||N|} \quad .$$

Since $N \cap H_0$ is a proper subgroup of N, $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N)$ is at most $|H|/2|H_0| = [H : H_0]/2$. It follows that $\mathbb{Z}[H/H_0]$ is a faithful H-lattice.

(c) We already know that $g : \mathbb{Z}[H/H_0] \to Y, hH_0 \mapsto hy$ is an H-epimorphism. By (a), to check that $M = \mathrm{Ker}(g)$ is a faithful H-lattice, we need only show that $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N) < \mathrm{rank}\,(M)$ for all nontrivial normal subgroups N of H. But by (b), given a nontrivial normal subgroup N of H, $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N) \leqslant [H : H_0]/2$. Since $\mathrm{rank}\,(M) = [H : H_0] - \mathrm{rank}\,(Y)$, $\mathrm{rank}\,(\mathbb{Z}[H/H_0]^N) < \mathrm{rank}\,(M)$ if $[H : H_0] > 2\mathrm{rank}\,(Y)$. $\square$

3.2. Weyl groups, root lattices and weight lattices

Let Φ be a root system with Weyl group $W = W(\Phi)$. Let $(\cdot\,,\cdot)$ be a fixed W-invariant bilinear form on $\mathbb{Q}\Phi$.

If Φ is irreducible, there are at most two root lengths, and all roots of a given length are conjugate under W. If Φ has two distinct root lengths, we refer to short and long roots. The set of short roots, Φ_0, is again a root system with respect to the same root pairing $\langle\cdot\,,\cdot\rangle$. If Φ is irreducible with only one root length, then $\Phi = \Phi_0$. In the cases in which there are two root lengths, the W-orbit of a short root spans the root system over $\mathbb{Z}$.

The weight lattice of Φ is

$$\Lambda(\Phi) = \{\lambda \in \mathbb{Q}\Phi \mid \langle\lambda, \alpha\rangle \in \mathbb{Z}, \alpha \in \Phi\}$$

where $\langle v, w\rangle = 2(v, w)/(w, w)$. We will sometimes abbreviate $\Lambda(\Phi)$ by Λ where there is no confusion.

For each root system Φ, we will fix a root system base $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let s_i be the reflection in the root α_i. Let $\{\varpi_1, \dots, \varpi_n\}$ be the set of fundamental dominant weights corresponding to Δ. That is, $\langle \varpi_i, \alpha_j \rangle = \delta_{ij}$ for all i, j. Note that any weight $\lambda \in \Lambda(\Phi)$ can be written uniquely as $\lambda = \sum_{j=1}^n \langle \lambda, \alpha_j \rangle \varpi_j$ and that $\mathbb{Z}\Phi \subset \Lambda(\Phi)$. A weight is called *dominant* with respect to Δ if $\langle \lambda, \alpha_i \rangle \geqslant 0$ for all $i = 1, \dots, n$. We will refer to Humphreys' lists of bases [Hu1, pp. 64–65] and their corresponding sets of fundamental dominant weights [Hu1, p. 69] for each irreducible root system Φ.

If J is a subset of $\{1, \dots, n\}$, the parabolic subgroup

$$W_J = \langle s_j \mid j \in J \rangle$$

is the Weyl group of Φ_J, the root system with base $\Delta_J = \{\alpha_j \mid j \in J\}$.

The isotropy subgroup of $\lambda \in \Lambda$ is

$$W_\lambda = \{w \in W \mid w\lambda = \lambda\}.$$

There is a nice description of W_λ for any $\lambda \in \Lambda^+$ where $\Lambda^+ = \{\sum_{i=1}^n m_i \varpi_i \mid m_i \geqslant 0\}$ is the set of dominant weights with respect to the base Δ. That is:

$$W_\lambda = \langle s_i \mid s_i \lambda = \lambda \rangle$$

So in each case the isotropy subgroup is a proper parabolic subgroup of W. Since

$$s_i \varpi_j = \varpi_j - \delta_{ij} \alpha_i,$$

we see that $W_{\varpi_i} = \langle s_1, \dots, \hat{s}_i, \dots s_n \rangle$ corresponds to the maximal parabolic subgroups of W. Let $\lambda \in \Lambda$. Then there exists a unique $w \in W$ such that $w\lambda \in \Lambda^+$. If $w\lambda = \sum_{i=1}^n k_i \varpi_i \in \Lambda^+$,

$$wW_\lambda w^{-1} = W_{w\lambda} = \langle s_j \mid k_j = 0 \rangle = \cap_{k_j \neq 0} W_{\varpi_j}.$$

A root $\alpha \in \Phi$ can be written uniquely as $\alpha = \sum_{i=1}^n c_i \alpha_i$ where the c_i's are integers that are of the same sign. If $c_i \geqslant 0$ for all i, then α is called a positive root with respect to Δ. We denote by Φ_+ the set of all positive roots in Φ with respect to the base Δ. For a positive root $\alpha = \sum_{i=1}^n c_i \alpha_i$, the height of α is defined as $\text{height}(\alpha) = \sum_{i=1}^n c_i$. There exists a unique highest root which is always long. There also exists a unique highest short root which we will denote by $\widetilde{\alpha}$.

3.3. Proof of Theorem 1.3

Let G be a simple group of adjoint type with maximal torus T. Since G is adjoint, the character lattice $X(T)$ associated to T and G is $\mathbb{Z}\Phi$, the root lattice of the root system Φ associated to G and T. Let the Weyl group of Φ be denoted by $W = W(\Phi)$.

Set $P(\Phi)$ to be the permutation W-lattice with $\mathbb{Z}$-basis

$$\{e_\alpha \mid \alpha \in \Phi_0\}$$

in bijective correspondence with the set Φ_0 of short roots in Φ. Let W act on $P(\Phi)$ via $we_\alpha = e_{w\alpha}$, for all $w \in W, \alpha \in \Phi_0$. Then there exists a natural W-epimorphism from $P(\Phi)$ to $\mathbb{Z}\Phi$, namely

$$\pi \equiv \pi(\Phi) : P(\Phi) \longrightarrow \mathbb{Z}\Phi, \quad e_\alpha \mapsto \alpha.$$

Let $K(\Phi)$ be the kernel of this map. Note that $\eta : \mathbb{Z}[W/W_{\widetilde{\alpha}}] \to P(\Phi)$, $wW_{\widetilde{\alpha}} \mapsto e_{w\widetilde{\alpha}}$ is a W-isomorphism for which $f_{\widetilde{\alpha}} = \pi(\Phi) \circ \eta$ where $f_{\widetilde{\alpha}} : \mathbb{Z}[W/W_{\widetilde{\alpha}}] \to P(\Phi)$, $wW_{\widetilde{\alpha}} \mapsto w\widetilde{\alpha}$.

Note that $P(\mathsf{A}_n) \cong \mathbb{Z}[S_{n+1}/S_{n-1}]$ and $\pi(\mathsf{A}_n)$ corresponds to the map

$$\mathbb{Z}[S_{n+1}/S_{n-1}] \longrightarrow \mathbb{Z}\mathsf{A}_n, \quad y_{ij} \mapsto \varepsilon_i - \varepsilon_j.$$

So $K(\mathsf{A}_n) \cong (\mathbb{Z}\mathsf{A}_n)^{\otimes 2}$ and the exact sequence

$$0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0$$

is a direct analogue of the Formanek–Procesi sequence for other irreducible root systems.

Proof of Theorem 1.3. For the first statement, let $\rho : P(\Phi) \to \mathbb{Z}$, $e_\alpha \mapsto 1$ be the augmentation map. Assume $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$ and set $P_0(\Phi) = \operatorname{Ker}(\rho)$ and $K_0 = \operatorname{Ker}(\rho) \cap K(\Phi)$. Since $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$ there exist $\alpha, \beta \in \Phi_0$ with $\langle \alpha, \beta \rangle = -1$. So $\alpha + \beta = s_\beta(\alpha) \in \Phi_0$. But then $\pi(P_0(\Phi)) = \mathbb{Z}\Phi$ since $\alpha + \beta \in \rho(P_0(\Phi))$ and the W-span of a short root is $\mathbb{Z}\Phi$.

For the $\alpha, \beta \in \Phi_0$ above, $\alpha + \beta \in \Phi_0$, $x = e_\alpha + e_\beta - e_{\alpha+\beta} \in K(\Phi)$ and $\rho(x) = 1$. This shows that $\rho(K(\Phi)) = \mathbb{Z}$.

So we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
K_0(\Phi) & \hookrightarrow & P_0(\Phi) & \twoheadrightarrow & \mathbb{Z}\Phi \\
\downarrow & & \downarrow & & \| \\
K(\Phi) & \hookrightarrow & P(\Phi) & \twoheadrightarrow & \mathbb{Z}\Phi \\
\downarrow & & \downarrow & & \\
\mathbb{Z} & = & \mathbb{Z} & &
\end{array} .$$

In order to apply Corollary 2.2 to this diagram, we need to verify that $K(\Phi)$ and $K_0(\Phi)$ are faithful W-lattices.

We have that $P(\Phi) \cong \mathbb{Z}[W/W_{\widetilde{\alpha}}]$ and that, under this isomorphism, the map $\pi = \pi(\Phi)$ corresponds to

$$f_{\widetilde{\alpha}} : \mathbb{Z}[W/W_{\widetilde{\alpha}}] \longrightarrow \mathbb{Z}\Phi, \quad wW_{\widetilde{\alpha}} \mapsto w\widetilde{\alpha}.$$

Since $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$, $\operatorname{rank}(P(\Phi)) = [W : W_{\widetilde{\alpha}}] = |\Phi_0| > 2n$, and hence we may use Proposition 3.3(c) to conclude that $K(\Phi)$ is a faithful W-lattice. Now, since

$$0 \longrightarrow K_0(\Phi) \longrightarrow K(\Phi) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact, $K_0(\Phi)$ must also be a faithful W-lattice, otherwise there would exist a nontrivial normal subgroup N which fixes both $K_0(\Phi)$ and $\mathbb{Z}$ and hence also fixes $K(\Phi)$. By contradiction, $K_0(\Phi)$ is also a faithful W-lattice. Then the result follows by applying Corollary 2.2 since $\operatorname{rank}(P_0(\Phi)) = |\Phi_0| - 1$.

For the second statement, note that for $\Phi = \mathsf{A}_1, \mathsf{B}_n$ there exists a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P(\Phi) \oplus K(\Phi) & \longrightarrow & P(\Phi) \oplus P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & P_-(\Phi) \oplus K(\Phi) & \longrightarrow & P_-(\Phi) \oplus P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0
\end{array}$$

where $P_-(\Phi)$ is the W-sublattice of $P(\Phi)$ with $\mathbb{Z}$-basis $\{e_\alpha - e_{-\alpha} \mid \alpha \in (\Phi_0)_+\}$. Since $\Phi = \mathsf{A}_1, \mathsf{B}_n$, we have $P_-(\Phi) \cong \mathbb{Z}\Phi$ and hence it is W-irreducible and thus a faithful W-lattice. This shows that $P_-(\Phi) \oplus K(\Phi)$ is W-faithful. Since $K(\Phi)$ and $P_-(\Phi)$ both have rank n in this case, and $P(\Phi) \oplus K(\Phi)$ is W-faithful, the second statement follows from an application of Corollary 2.2. □

Remark 3.4. Note that Corollary 1.4 now follows directly from Proposition 3.3 and the cardinalities of the corresponding short root systems. Observe also that we will later prove in Corollary 5.9 that $(P(\Phi), \pi(\Phi))$ is a minimal element of $\mathcal{P}(\mathbb{Z}\Phi)$ if $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$ and $(P(\Phi) \oplus P(\Phi), (\pi(\Phi), 0))$ is a minimal element of $P(\mathbb{Z}\Phi)$ if $\Phi = \mathsf{A}_1, \mathsf{B}_n$. Then $r(\mathbb{Z}\Phi) = |\Phi_0|$ if $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$ and $r(\mathbb{Z}\Phi) = 2|\Phi_0|$ if $\Phi = \mathsf{A}_1, \mathsf{B}_n$.

4. Existence of N-compressions of $X_{\pi(\Phi)}$

In this section, we look for N-compressions of $X_{\pi(\Phi)}$ where

$$\pi(\Phi) : P(\Phi) \longrightarrow \mathbb{Z}\Phi, \quad e_\alpha \mapsto \alpha,$$

for an irreducible root system Φ. We first discuss some preliminaries about the cohomology and W-lattice structure of the root and weight lattice of Φ.

4.1. Cohomology and representation theoretic properties of the root and weight lattices

We will use the notation and definitions of Section 3.2. Let $W = W(\Phi)$ be the Weyl group of an irreducible root system Φ and let $\mathbb{Z}\Phi$ be its root lattice, Φ_0 its set of short roots and Λ its weight lattice. Let $\widetilde{\alpha}$ be the highest short root of Φ.

Let $\mathrm{Supp}\{\widetilde{\alpha}\} = \{j \mid \langle \widetilde{\alpha}, \alpha_j \rangle \neq 0\}$ and let $I = \{1, \dots, n\} \setminus \mathrm{Supp}\{\widetilde{\alpha}\}$. Set Φ_I to be the subroot system of Φ with base $\{\alpha_i \mid i \in I\}$ and set $W_I = W(\Phi_I)$. Then, by the above discussion, $W_{\widetilde{\alpha}} = W_I$.

The following lemma gives a list of the highest short root $\widetilde{\alpha}$, I and $W_{\widetilde{\alpha}}$ for each irreducible root system.

Lemma 4.1. *Let the notation be given as above. For any irreducible root system Φ, the highest short root $\widetilde{\alpha}$ is a dominant weight so that $W_{\widetilde{\alpha}} = W(\Phi_I)$. The following is a list of $\widetilde{\alpha}$, $W_{\widetilde{\alpha}}$ and Φ_I for each irreducible root system Φ.*

- *For $\Phi = \mathsf{A}_1$, $\widetilde{\alpha} = 2\varpi_1$, $I = \varphi$ and $W_{\widetilde{\alpha}} = 1$.*
- *For $\Phi = \mathsf{A}_n$, $n \geqslant 2$, $\widetilde{\alpha} = \varpi_1 + \varpi_n$, $\Phi_I = \mathsf{A}_{n-2}$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{A}_{n-2})$.*
- *For $\Phi = \mathsf{B}_n$, $\widetilde{\alpha} = \varpi_1$, $\Phi_I = \mathsf{B}_{n-1}$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{B}_{n-1})$.*
- *For $\Phi = \mathsf{C}_n$, $\widetilde{\alpha} = \varpi_2$, $\Phi_I = \mathsf{A}_1 \cup \mathsf{C}_{n-2}$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})$.*
- *For $\Phi = \mathsf{D}_n$, $\widetilde{\alpha} = \varpi_2$, $\Phi_I = \mathsf{A}_1 \cup \mathsf{D}_{n-2}$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})$.*
- *For $\Phi = \mathsf{E}_6$, $\widetilde{\alpha} = \varpi_2$, $\Phi_I = \mathsf{A}_5$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{A}_5)$.*
- *For $\Phi = \mathsf{E}_7$, $\widetilde{\alpha} = \varpi_1$, $\Phi_I = \mathsf{D}_6$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{D}_6)$.*
- *For $\Phi = \mathsf{E}_8$, $\widetilde{\alpha} = \varpi_8$, $\Phi_I = \mathsf{E}_7$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{E}_7)$.*
- *For $\Phi = \mathsf{F}_4$, $\widetilde{\alpha} = \varpi_4$, $\Phi_I = \mathsf{B}_3$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{B}_3)$.*
- *For $\Phi = \mathsf{G}_2$, $\widetilde{\alpha} = \varpi_1$, $\Phi_I = \mathsf{A}_1$ and $W_{\widetilde{\alpha}} \cong W(\mathsf{A}_1)$.*

Proof. Suppose $\langle\widetilde{\alpha},\alpha_j\rangle < 0$ for some j. Then $s_{\alpha_j}(\widetilde{\alpha}) = \widetilde{\alpha}+\alpha_j$ is another short root with height larger than that of $\widetilde{\alpha}$. By contradiction, $\langle\widetilde{\alpha},\alpha_i\rangle \geqslant 0$ for all $i = 1,\ldots,n$ so that $\widetilde{\alpha}$ is a dominant weight. By [Hu2, pp. 22–23], $W_{\widetilde{\alpha}} = \langle s_{\alpha_i} \mid \langle\widetilde{\alpha},\alpha_i\rangle = 0\rangle = W(\Phi_I) \equiv W_I$. The rest of the lemma is simply a list of the highest short roots [Hu1, p. 66] expressed in terms of the fundamental dominant weights for the convenience of the reader. □

The following technical lemma gives cohomological information about the W-lattice $\mathbb{Z}\Phi$. It was first proved in [Kl]. The proof is supplied for the convenience of the reader as the above reference is not easily accessible.

Lemma 4.2. *Let Φ be an irreducible root system. Then*

$$H^{-1}(W,\mathbb{Z}\Phi) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \textit{if } \Phi = \mathsf{A}_1, \mathsf{B}_n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By definition,

$$H^{-1}(W,\mathbb{Z}\Phi) = \mathrm{Ker}_{\mathbb{Z}\Phi}(N_W)/\textstyle\sum_{w\in W}\mathrm{Im}_{\mathbb{Z}\Phi}(w-1).$$

For every $\alpha \in \Phi$,

$$N_W(\alpha) = \textstyle\sum_{w\in W} w\alpha = |W_\alpha|\sum_{\beta\in W\alpha}\beta = 0$$

where the last equality follows from the fact that $W\alpha = W(-\alpha) = -W\alpha$. Since the root lattice is spanned by Φ, this shows that $\mathrm{Ker}_{\mathbb{Z}\Phi}(N_W) = \mathbb{Z}\Phi$. Since Δ is a $\mathbb{Z}$-basis for the root lattice, we have $\mathrm{Ker}_{\mathbb{Z}\Phi}(N_W) = \oplus_{i=1}^n \mathbb{Z}\alpha_i$.

Recall that if a finite group H is generated by a subset S, and X is an H module, we have $\sum_{h\in H}\mathrm{Im}_X(h-1) = \sum_{s\in S}\mathrm{Im}_X(s-1)$. Applying this fact to the generating set $\{s_i \mid i = 1,\ldots,n\}$ for W, we find that

$$\textstyle\sum_{w\in W}\mathrm{Im}_{\mathbb{Z}\Phi}(w-1) = \sum_{i=1}^n \mathrm{Im}_{\mathbb{Z}\Phi}(s_i-1) = \sum_{i=1}^n\big(\sum_{j=1}^n \mathbb{Z}\langle\alpha_j,\alpha_i\rangle\big)\alpha_i.$$

By examining the Cartan matrix for each irreducible root system, we observe that

$$\underset{j=1,\ldots,n}{\gcd}\{\langle\alpha_j,\alpha_i\rangle\} = \begin{cases} 2, & \text{if } i = n \text{ and } \Phi = \mathsf{A}_1 \text{ or } \Phi = \mathsf{B}_n; i = n, \\ 1, & \text{else.} \end{cases}$$

Together with the first paragraph this gives the result of Lemma 4.2. □

We now need some more information about the structure of $\Lambda(\Phi_0)$ as a W-lattice. Note that

$$\Lambda(\Phi_0) = \{v \in \mathbb{Q}\Phi_0 = \mathbb{Q}\Phi \mid \langle v,\beta\rangle \in \mathbb{Z} \text{ for all } \beta \in \Phi_0\}$$

where $\langle v,w\rangle = 2(v,w)/(w,w)$ is still defined using the fixed $W(\Phi)$-invariant bilinear form $(\cdot,\cdot)$. This shows that even if $\Phi_0 \neq \Phi$, the lattice $\Lambda(\Phi_0)$ is still a $W(\Phi)$-lattice. In fact, when $\Phi_0 \neq \Phi$, $W(\Phi) = \mathrm{Aut}(\Phi_0)$.

Lemma 4.3. $(\mathbb{Z}\Phi)^*$ *is isomorphic as a $W(\Phi)$ lattice to $\Lambda(\Phi_0)$.*

Proof. Let $\beta_1,\dots,\beta_n$ be a root system base for Φ_0. Let $\lambda_1,\dots,\lambda_n$ be the corresponding fundamental dominant weights. Then $\beta_1,\dots,\beta_n$ is a $\mathbb{Z}$-basis for the root lattice for Φ. Let $(\cdot\,,\cdot)$ be the $W(\Phi)$-invariant bilinear form on $\mathbb{Q}\Phi$ used to define $\langle\cdot\,,\cdot\rangle$. Let $r=(\widetilde{\alpha},\widetilde{\alpha})$. Note that $r=(\alpha,\alpha)$ for all $\alpha\in\Phi_0$. Define

$$\theta:\Lambda(\Phi_0)\to(\mathbb{Z}\Phi)^*,\quad \lambda\mapsto\langle\lambda,\cdot\rangle.$$

Since $\theta(\lambda)=\langle\lambda,\cdot\rangle=\frac{2}{r}(\lambda,\cdot)$, it is clear that $\theta(\lambda)\in(\mathbb{Z}\Phi)^*$ and that θ is a $\mathbb{Z}$-linear map from the bilinearity of $(\cdot\,,\cdot)$. Now $\theta(w\lambda)(\alpha)=\frac{2}{r}(w\lambda,\alpha)=\frac{2}{r}(\lambda,w^{-1}\alpha)=(w\theta)(\alpha)$ for all $\lambda\in\Lambda(\Phi_0)$ and $\alpha\in\Phi$. Moreover, $\frac{2}{r}(\lambda_i,\beta_j)=\langle\lambda_i,\beta_j\rangle=\delta_{ij}=\beta_i^*(\beta_j)$ for all i,j so that $\theta(\lambda_i)=\beta_i^*$ for all i. This shows that θ is a $W(\Phi)$-isomorphism as required. $\square$

The next lemma describes the structure of $\Lambda(\Phi_0)$ as a $W_{\widetilde{\alpha}}$ lattice. It requires some more notation and conventions about the structure of the base chosen for Φ_0.

Notation. We will denote the base of the short root system Φ_0 by $\Delta_0=\{\beta_1,\dots,\beta_n\}$ and its corresponding fundamental dominant weights by $\{\lambda_1,\dots,\lambda_n\}$, which will give a $\mathbb{Z}$-basis for $\Lambda(\Phi_0)$. If $\Phi=\Phi_0$, we will choose $\beta_i=\alpha_i$ for all $i=1,\dots,n$ so that $\Delta_0=\Delta$ and $\lambda_i=\varpi_i$ for all $i=1,\dots,n$. When $\Phi\neq\Phi_0$, we will choose the order of Δ_0 to match that of the base of the root system Φ_0 that we chose before (i.e., [Hu1, pp. 64–65]). The order of the fundamental dominant weights $\lambda_1,\dots,\lambda_n$ will then correspond. Note that the highest short root $\widetilde{\alpha}$ of Φ is the highest root of Φ_0 and hence is a dominant weight with respect to Δ_0. If $\Phi=\mathsf{B}_n$, then Φ_0 is of type A_1^n; if $\Phi=\mathsf{C}_n$, then Φ_0 is of type D_n; if $\Phi=\mathsf{F}_4$, then Φ_0 is of type D_4; if $\Phi=\mathsf{G}_2$, then Φ_0 is of type A_1. In each case $W(\Phi)=\mathrm{Aut}(\Phi_0)$.

Lemma 4.4. *For any subset J of $\{1,\dots,n\}$,*

$$0\longrightarrow\Lambda(\Phi)^{W(\Phi_J)}\longrightarrow\Lambda(\Phi)\longrightarrow\Lambda(\Phi_J)\longrightarrow 0$$

is an exact sequence of $W_J=W(\Phi_J)$-lattices. It is also an exact sequence of $\mathrm{Aut}(\Phi_J)$-lattices.

Proof. Note that

$$s_i\varpi_j=\varpi_j-\delta_{ij}\alpha_j=\varpi_j-\delta_{ij}\textstyle\sum_{k=1}^n\langle\alpha_j,\alpha_k\rangle\varpi_k,\quad 1\leqslant i,j\leqslant n.$$

Then since $W(\Phi_J)=\langle s_j\mid j\in J\rangle$, we have that $\Lambda(\Phi)^{W(\Phi_J)}=\oplus_{j\notin J}\mathbb{Z}\varpi_j$. Let $\varpi_j^J, j\in J$ be the set of fundamental dominant weights with respect to $\Delta_J=\{\alpha_j\mid j\in J\}$. Then

$$s_i\varpi_j^J=\varpi_j^J-\delta_{ij}\textstyle\sum_{k\in J}\langle\alpha_j,\alpha_k\rangle\varpi_k^J,\quad i,j\in J.$$

Now $\overline{\varpi}_j\equiv\varpi_j+\Lambda(\Phi)^{W(\Phi_J)}, j\in J$ is a $\mathbb{Z}$-basis for $\Lambda(\Phi)/\Lambda(\Phi)^{W(\Phi_J)}$, and from the above, it is clear that

$$\Lambda(\Phi)/\Lambda(\Phi)^{W(\Phi_J)}\longrightarrow\Lambda(\Phi_J),\quad \overline{\varpi}_j\mapsto\varpi_j^J, j\in J,$$

is an isomorphism of $W(\Phi_J)$-lattices. In fact, it is also an isomorphism of $\mathrm{Aut}(\Phi_J)=W(\Phi_J)\rtimes\mathrm{Diag}(\Delta_J)$-lattices where $\mathrm{Diag}(\Delta_J)$ is the diagram automorphism group of Δ_J, since $\mathrm{Diag}(\Delta_J)$ permutes $\varpi_j, j\in J$. $\square$

4.2. Compressions of the analogue of the Formanek–Procesi sequence

In this section, for the Weyl group $W = W(\Phi)$ of an irreducible root system $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$, we ask when there exists a commutative diagram of W-lattices

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\Phi) & \longrightarrow & P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & K_-(\Phi) & \longrightarrow & X & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \end{array} \tag{7}$$

or, equivalently, when

$$[0 \to K(\Phi) \to P(\Phi) \stackrel{\pi(\Phi)}{\to} \mathbb{Z}\Phi \to 0] \in \operatorname{Im}(\operatorname{Ext}^1_W(\mathbb{Z}\Phi, K_-(\Phi)) \to \operatorname{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))). \tag{8}$$

Recall that $P(\Phi)$ is the W-lattice with $\mathbb{Z}$-basis $\{e_\alpha \mid \alpha \in \Phi_0\}$ on which W acts by permuting the short roots and $P_-(\Phi)$ is the W-sublattice

$$P_-(\Phi) = \oplus_{\alpha\in\Phi_0}\mathbb{Z}(e_\alpha - e_{-\alpha}).$$

The W-map

$$\pi \equiv \pi(\Phi) : P(\Phi) \longrightarrow \mathbb{Z}\Phi, \quad e_\alpha \mapsto \alpha,$$

is surjective with kernel $K(\Phi)$ and $K_-(\Phi) = K(\Phi) \cap P_-(\Phi)$. As we have discussed before, in the A_{n-1}, $n \geqslant 3$ case we have

$$K(\mathsf{A}_{n-1}) = (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2}, \quad P(\mathsf{A}_{n-1}) = \mathbb{Z}[S_n/S_{n-2}], K_-(\Phi) = \wedge^2(\mathbb{Z}\mathsf{A}_{n-1}),$$

and the sequence in the top row of (7) is the Formanek–Procesi sequence. In the A_{n-1} case, Lorenz and Reichstein in [LR] found a better upper bound on $\operatorname{ed}(\mathrm{PGL}_n)$, $n \geqslant 5$ odd, using a special case of Corollary 2.2, by showing that a commutative diagram as above exists in this case. It is natural to ask whether we could find an analogous bound on the essential dimension of the other simple adjoint groups by answering the analogous question for the corresponding root systems. Unfortunately, we will find in Proposition 4.13 that there exists a diagram of the form (7) if and only if $\Phi = \mathsf{A}_{n-1}$, $n \geqslant 5$ odd. We will do this by answering the cohomological question (8). In order to do this, we need to first look at the W-lattice structure and the cohomology of the lattices in the exact sequences

$$0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0$$

and

$$0 \longrightarrow K_-(\Phi) \longrightarrow P_-(\Phi) \longrightarrow 2(\mathbb{Z}\Phi) \longrightarrow 0$$

where the second map in the second sequence is the restriction of $\pi(\Phi)$ to $P_-(\Phi)$.

Now

$$P_-(\Phi) \cong \mathbb{Z}W \otimes_{W_{\tilde{\alpha}} \times \langle s_{\tilde{\alpha}}\rangle} \mathbb{Z}_\xi$$

as a W-lattice where $\mathbb{Z}_\xi$ is the rank 1 $\mathbb{Z}W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle$ lattice on which $W_{\widetilde{\alpha}}$ acts trivially and $s_{\widetilde{\alpha}}$ acts as multiplication by -1. Let $P_+(\Phi)$ be the W-sublattice of $P(\Phi)$ with $\mathbb{Z}$-basis

$$\{e_\alpha + e_{-\alpha} \mid \alpha \in (\Phi_0)_+\}$$

so that

$$P_+(\Phi) \cong \mathbb{Z}W \otimes_{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle} \mathbb{Z}$$

as a W-lattice. There exist natural surjective W-lattice homomorphisms from

$$P(\Phi) \longrightarrow P_\pm(\Phi), \quad e_\alpha \mapsto e_\alpha \pm e_{-\alpha},$$

with kernels $P_\mp(\Phi)$. Note that $P_-(\mathsf{A}_1) \cong \mathbb{Z}\mathsf{A}_1$ and that $P_-(\mathsf{B}_n) \cong \mathbb{Z}\mathsf{B}_n$. In both cases, $e_{\beta_i} - e_{-\beta_i} \mapsto \beta_i$ defines the isomorphism. This shows that $K(\Phi) \cong P_+(\Phi)$ is a permutation lattice for $\Phi = \mathsf{A}_1, \mathsf{B}_n$. Note that in general, for all Φ, $P_+(\Phi)$ is a sublattice of $K(\Phi)$.

We need to determine $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$ and $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K_-(\Phi))$ in order to ascertain when the induced map

$$\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K_-(\Phi)) \longrightarrow \mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$$

is surjective. Applying cohomology to the analogue of the Formanek–Procesi sequence above, we obtain the exact sequence

$$\mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) \longrightarrow \mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi)) \longrightarrow \mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)).$$

We will first determine $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi))$ and find that it is usually zero.

Recall that $P(\Phi) \cong \mathbb{Z}[W/W_{\widetilde{\alpha}}]$ where $\widetilde{\alpha}$ is the highest short root of Φ. Note that $W_{\widetilde{\alpha}}$ is the parabolic subgroup W_I where $I = \{1, \ldots, n\} \setminus \mathrm{Supp}\{\widetilde{\alpha}\}$.

Proposition 4.5.

$$\mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)) \cong H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \textit{if } \Phi = \mathsf{B}_n, \\ 0 & \textit{else.} \end{cases}$$

Proof. The first isomorphism follows from the fact that $P(\Phi) \cong \mathbb{Z}[W/W_{\widetilde{\alpha}}]$ and then

$$\begin{aligned} \mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)) &\cong H^1(W, \mathrm{Hom}(\mathbb{Z}\Phi, P(\Phi))) \\ &\cong H^1(W_{\widetilde{\alpha}}, \mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{Z})) \\ &\cong H^1(W_{\widetilde{\alpha}}, (\mathbb{Z}\Phi)^*) \\ &\cong H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) \end{aligned}$$

where the first isomorphism follows from Shapiro's Lemma and the third from Lemma 4.3. So it suffices to determine $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0))$ in each case.

If $\Phi = \mathsf{A}_n, n = 1, 2$, then $W_{\widetilde{\alpha}} = 1$ so that the result is trivial in this case.

If $\Phi = \mathsf{A}_n, n \geqslant 3; \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, then $\Phi_0 = \Phi$, so by Lemma 4.4,

$$0 \longrightarrow \Lambda(\Phi)^{W_{\widetilde{\alpha}}} \longrightarrow \Lambda(\Phi) \longrightarrow \Lambda(\Phi_I) \longrightarrow 0$$

is an exact sequence of $W_{\widetilde{\alpha}} = W_I$ lattices. Then

$$0 = H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)^{W_{\widetilde{\alpha}}}) \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_I))$$

is exact. Now $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_I)) = H^1(W_I, \Lambda(\Phi_I)) \cong H^{-1}(W_I, \mathbb{Z}\Phi_I)$. For A_n, $n \geqslant 4$; E_6, E_7, E_8, Φ_I is an irreducible root system not of type A_1 or B_n, so we have $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_I)) \cong H^{-1}(W_I, \mathbb{Z}\Phi_I) = 0$, and hence $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) = 0$ in these cases.

For A_3, $W_{\widetilde{\alpha}} = \langle s_2 \rangle$ and s_2 fixes ϖ_1, ϖ_3, but $s_2(\varpi_2) = \varpi_1 - \varpi_2 + \varpi_3$. Since $\{\varpi_1 - \varpi_2 + \varpi_3, \varpi_2, \varpi_3\}$ is an alternate basis for $\Lambda(\Phi)$, we see that $\Lambda(\Phi) \cong \mathbb{Z}W_{\widetilde{\alpha}} \oplus \mathbb{Z}$ so that $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) = 0$.

For D_4, $W_{\widetilde{\alpha}} = \langle s_1, s_3, s_4 \rangle \cong (\mathsf{C}_2)^3$. Now for $i = 1, 3, 4$, $s_i(\varpi_i) = -\varpi_i + \varpi_2$; $s_i(\varpi_j) = \varpi_j$, $j \neq i$, so that s_i permutes the basis $\varpi_i, -\varpi_i + \varpi_2, \varpi_j, \varpi_k$, where $\{2, i, j, k\} = \{1, 2, 3, 4\}$. Hence $H^1(\langle s_i \rangle, \Lambda(\Phi)) = 0$ for $i = 1, 3, 4$. Applying the inflation-restriction sequence to $\Lambda(\Phi)$, we note that

$$H^1(W_{\widetilde{\alpha}}/\langle s_1 \rangle, \Lambda(\Phi)^{\langle s_1 \rangle}) \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \longrightarrow H^1(\langle s_1 \rangle, \Lambda(\Phi)) = 0$$

is exact. So $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \cong H^1(\langle s_3, s_4 \rangle, \oplus_{i=2}^4 \mathbb{Z}\varpi_i)$. Applying the inflation-restriction sequence again, we see that $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \cong H^1(\langle s_3 \rangle, \oplus_{i=2}^3 \mathbb{Z}\varpi_i) = 0$, since s_3 permutes $-\varpi_3 + \varpi_2, \varpi_2$.

For $\mathsf{D}_n, n \geqslant 5$, $W_{\widetilde{\alpha}} = \langle s_1, s_3, \ldots, s_n \rangle \cong W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})$. Since $W(\mathsf{D}_{n-2}) = \langle s_3, \ldots, s_n \rangle$ is a normal subgroup of $W_{\widetilde{\alpha}}$, we may apply inflation-restriction to obtain the exact sequence

$$0 \longrightarrow H^1(W_{\widetilde{\alpha}}/W(\mathsf{D}_{n-2}), \Lambda(\Phi)^{W(\mathsf{D}_{n-2})}) \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \longrightarrow H^1(W(\mathsf{D}_{n-2}), \Lambda(\Phi)).$$

Applying Lemma 4.4 and cohomology to $J = \{3, \ldots, n\}$ and $W(\mathsf{D}_{n-2}) = W(\Phi_J)$ for $W = W(\mathsf{D}_n)$, we find that

$$0 = H^1(W(\mathsf{D}_{n-2}), \Lambda(\Phi)^{W(\mathsf{D}_{n-2})}) \longrightarrow H^1(W(\mathsf{D}_{n-2}), \Lambda(\Phi)) \longrightarrow H^1(W(\mathsf{D}_{n-2}), \Lambda(\Phi_J)).$$

But by Lemmas 4.3 and 4.2, $H^1(W(\mathsf{D}_{n-2}), \Lambda(\Phi_J)) \cong H^{-1}(W(\mathsf{D}_{n-2}), \mathbb{Z}\Phi_J) = 0$. So we may conclude that

$$H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi)) \cong H^1(\langle s_1 \rangle, \oplus_{i=1}^2 \mathbb{Z}\varpi_2) = 0,$$

since s_1 permutes the basis $-\varpi_1 + \varpi_2, \varpi_1$.

For C_n, $n \geqslant 3$, $W_{\widetilde{\alpha}} = \langle s_1, s_3, \ldots, s_n \rangle \cong W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})$. Note that $W(\Phi_0)_{\widetilde{\alpha}} \cong W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})$ is a normal subgroup of $W_{\widetilde{\alpha}}$ with $\Lambda(\Phi_0)^{W(\Phi_0)_{\widetilde{\alpha}}} = \mathbb{Z}\widetilde{\alpha}$. But since $W_{\widetilde{\alpha}}/W(\Phi_0)_{\widetilde{\alpha}}$ is generated by an element which permutes λ_{n-1} and λ_n and fixes all other $\lambda_i, i \neq n-1, n$ (in particular $\widetilde{\alpha} = \lambda_2$), we see that

$$H^1(W_{\widetilde{\alpha}}/W(\Phi_0)_{\widetilde{\alpha}}, \Lambda(\Phi_0)^{W(\Phi_0)_{\widetilde{\alpha}}}) = 0,$$

so that by inflation-restriction,

$$0 \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) \longrightarrow H^1(W(\Phi_0)_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = H^1(W(\mathsf{D}_n)_{\widetilde{\alpha}}, \Lambda(\mathsf{D}_n)) = 0$$

is exact, where the last equality follows from our calculations for $\mathsf{D}_n, n \geqslant 4$ and $\mathsf{D}_3 = \mathsf{A}_3$. So $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = 0$.

For F_4, $W_{\widetilde{\alpha}} = \langle s_1, s_2, s_3\rangle \cong W(\mathsf{B}_3)$ and $\Phi_0 = \mathsf{D}_4$. Note that $W(\Phi_0)_{\widetilde{\alpha}}$ is a normal subgroup of $W_{\widetilde{\alpha}}$ with $\Lambda(\Phi_0)^{W(\Phi_0)_{\widetilde{\alpha}}} = \mathbb{Z}\widetilde{\alpha}$. But since $W_{\widetilde{\alpha}}/W(\Phi_0)_{\widetilde{\alpha}}$ is isomorphic to a subgroup which permutes $\lambda_1, \lambda_3, \lambda_4$ and fixes $\widetilde{\alpha} = \lambda_2$, we see that

$$H^1(W_{\widetilde{\alpha}}/W(\Phi_0)_{\widetilde{\alpha}}, \Lambda(\Phi_0)^{W(\Phi_0)_{\widetilde{\alpha}}}) = 0,$$

so that by inflation-restriction,

$$0 \longrightarrow H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) \longrightarrow H^1(W(\Phi_0)_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = H^1(W(\mathsf{D}_4)_{\widetilde{\alpha}}, \Lambda(\mathsf{D}_4)) = 0$$

is exact, where the last equality follows from our calculations for D_4. So $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = 0$.

For G_2, $W_{\widetilde{\alpha}} = \langle s_{\alpha_2}\rangle \cong \mathsf{C}_2$. We can see directly that $s_{\alpha_2}(\lambda_1) = -\lambda_1$ and $s_{\alpha_2}(\lambda_2) = \lambda_2 - \lambda_1$ so that s_{α_2} permutes the basis $\lambda_2 - \lambda_1, \lambda_2$ of $\Lambda(\Phi_0)$. So $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = 0$.

Lastly, for B_n, $n \geqslant 2$, $W_{\widetilde{\alpha}} = \langle s_2, \dots, s_n\rangle \cong W(\mathsf{B}_{n-1})$, $\Phi_0 = \mathsf{A}_1^n$ and $\Lambda(\Phi_0) = \oplus_{i=1}^n \mathbb{Z}\lambda_i$ where $\lambda_i = \frac{1}{2}e_i$. So $\Lambda(\Phi_0) = \mathbb{Z}\lambda_1 \oplus \oplus_{i=1}^n \mathbb{Z}\lambda_i \cong \mathbb{Z} \oplus \mathbb{Z}\mathsf{B}_{n-1}$ as $W_{\widetilde{\alpha}} \cong W(\mathsf{B}_{n-1})$ lattices. But then $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = H^1(W(\mathsf{B}_{n-1}), \mathbb{Z}\mathsf{B}_{n-1})$. Now C_2^{n-1} is a normal subgroup of $W(\mathsf{B}_{n-1})$ with $\mathbb{Z}\mathsf{B}_{n-1}^{\mathsf{C}_2^{n-1}} = 0$ and $W(\mathsf{B}_{n-1})/\mathsf{C}_2^{n-1} \cong S_{n-1}$, so by the inflation-restriction sequence,

$$\begin{aligned} H^1(W(\mathsf{B}_{n-1}), \mathbb{Z}\mathsf{B}_{n-1}) &\cong H^1(\mathsf{C}_2^{n-1}, \mathbb{Z}\mathsf{B}_{n-1})^{S_{n-1}} \\ &\cong [H^1(\mathsf{C}_2, \mathbb{Z}_-)^{n-1}]^{S_{n-1}} \cong [(\mathbb{Z}/2\mathbb{Z})^{n-1}]^{S_{n-1}} = \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

since S_{n-1} permutes the $n-1$ copies of $H^1(\mathsf{C}_2, \mathbb{Z}_-)$. So $H^1(W_{\widetilde{\alpha}}, \Lambda(\Phi_0)) = \mathbb{Z}/2\mathbb{Z}$ in this case. □

Then, by Proposition 4.5, for all $\Phi \neq \mathsf{B}_n$, we find that

$$\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi)) \cong \mathrm{Coker}(\mathrm{Hom}_W(\mathbb{Z}\Phi), P(\Phi)) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)).$$

To determine $\mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ and $\mathrm{Hom}_W(\mathbb{Z}\Phi, P_+(\Phi))$, we need to find the $W_{\widetilde{\alpha}}$-orbits on Φ_0.

Notation. Let the irreducible components of Φ_I be denoted by $\Phi_{I_k}, k = 1, \dots, r$. We will denote $\Phi_{I_k} \cap \Phi_0$ by $\Phi^0_{I_k}$. Define

$$\mathcal{O}(\alpha)_i = \{\beta \in \Phi_0 \mid \langle \beta, \alpha\rangle = i\}$$

for $\alpha \in \Phi_0, i = 0, \pm 1, \pm 2$. Lastly, set

$$P_1(\widetilde{\alpha}) = \{j \mid \alpha_j \in \mathcal{O}(\widetilde{\alpha})_1\}.$$

Lemma 4.6. *The orbits of $W_{\widetilde{\alpha}} = W_I$ on Φ_0 are $\Phi^0_{I_k}, k = 1, \dots, r$; $\pm(W_I \cdot \alpha_j)$, $j \in P_1(\widetilde{\alpha})$; $\{\widetilde{\alpha}\}$ and $\{-\widetilde{\alpha}\}$. Moreover,*

$$\mathcal{O}(\widetilde{\alpha})_0 = \cup_{k=1}^r \Phi^0_{I_k}, \quad \mathcal{O}(\widetilde{\alpha})_{\pm 1} = \cup_{j \in P_1(\widetilde{\alpha})} W_I \cdot (\pm\alpha_j), \quad \mathcal{O}(\widetilde{\alpha})_{\pm 2} = \{\pm\widetilde{\alpha}\}.$$

Proof. $W_{\widetilde{\alpha}}$ stabilizes the sets

$$\mathcal{O}(\widetilde{\alpha})_i = \{\beta \in \Phi_0 \mid \langle \beta, \widetilde{\alpha} \rangle = i\}, \quad i = 0, \pm 1, \pm 2,$$

since for $w \in W_{\widetilde{\alpha}}$, $\langle w\beta, \widetilde{\alpha} \rangle = \langle w\beta, w\widetilde{\alpha} \rangle = \langle \beta, \widetilde{\alpha} \rangle$. Note that Φ_0 is the disjoint union of the sets $\mathcal{O}(\widetilde{\alpha})_i$, $i = 0, \pm 1, \pm 2$. We need now determine how the $\mathcal{O}(\widetilde{\alpha})_i$ split into $W_{\widetilde{\alpha}}$-orbits. It is clear that $\mathcal{O}(\widetilde{\alpha})_2 = \{\widetilde{\alpha}\}$ and $\mathcal{O}(\widetilde{\alpha})_{-2} = \{-\widetilde{\alpha}\}$. Since these are each one element sets, they are $W_{\widetilde{\alpha}}$-orbits.

Observe that for all roots α, β, we have $\langle \alpha, \beta \rangle = 0$ if and only if $\langle \beta, \alpha \rangle = 0$ and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \geqslant 0$. The first observation implies that for $\beta = \sum_{j=1}^n m_i \alpha_i \in \mathcal{O}(\widetilde{\alpha})_i$ we have

$$\langle \beta, \widetilde{\alpha} \rangle = \textstyle\sum_{j \in \mathrm{Supp}\{\widetilde{\alpha}\}} m_j \langle \alpha_j, \widetilde{\alpha} \rangle = i. \tag{9}$$

Using the second observation above and the fact that $\widetilde{\alpha}$ is dominant, we see that $\beta \in \mathcal{O}(\widetilde{\alpha})_0$ if and only if $\beta \in \Phi_I \cap \Phi_0$. So the W_I-orbits of $\mathcal{O}(\widetilde{\alpha})_0$ are in bijection with the connected components of Φ_I. That is, the W_I-orbits of $\mathcal{O}(\widetilde{\alpha})_0$ are $\Phi^0_{I_k}$, $k = 1, \ldots, r$ as required.

Observe that for A_1 and B_n we have $P_1(\widetilde{\alpha}) = \varnothing$. But by (9), we see that $\mathcal{O}(\widetilde{\alpha})_1 = \varnothing$ as well in these cases, as required. For all other Φ, $P_1(\widetilde{\alpha}) = \mathrm{Supp}(\widetilde{\alpha})$. For $\mathsf{A}_n, n \geqslant 2$, $P_1(\widetilde{\alpha})$ has cardinality 2, and for all other cases $P_1(\widetilde{\alpha})$ has cardinality 1.

Assume $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$ so that $P_1(\widetilde{\alpha}) = \mathrm{Supp}(\widetilde{\alpha})$. We claim that $\mathcal{O}(\widetilde{\alpha})_1 = \cup_{j \in P_1(\widetilde{\alpha})} W_{\widetilde{\alpha}} \alpha_j$. From (9), we see that if $\beta = \sum_{j=1}^n m_j \alpha_j \in \mathcal{O}(\widetilde{\alpha})_1$, then

$$\beta = \alpha_j + \textstyle\sum_{k \notin \mathrm{Supp}(\widetilde{\alpha})} m_k \alpha_k \tag{10}$$

where $j \in \mathrm{Supp}(\widetilde{\alpha}) = P_1(\widetilde{\alpha})$, so that $\beta \in \Phi_+ \cap \Phi_0$. We proceed by induction on the height of $\beta \in \mathcal{O}(\widetilde{\alpha})_1$ to show that $\beta \in W_{\widetilde{\alpha}} \alpha_j$. If $\mathrm{height}(\beta) = 1$, then $\beta = \alpha_j \in W_{\widetilde{\alpha}} \alpha_j$. Let $\beta \in \mathcal{O}(\widetilde{\alpha})_1$ have $\mathrm{height}(\beta) > 1$. Then $\beta \in \Phi_+ \cap \Phi_0$ is not a simple root. This means that there exists a simple root α_k such that $\langle \beta, \alpha_k \rangle > 0$ [Hu1, p. 50]. If this holds for α_j, it implies that $\langle \beta, \alpha_j \rangle = 1$ as both roots are short and not equal. Then $\gamma = s_j(\beta) = \beta - \alpha_j \in \Phi_I \cap \Phi_0$ and $\langle \beta, \gamma \rangle = 1$ so that $s_\gamma(\beta) = \alpha_j$ or $\beta = s_\gamma \alpha_j \in W_I \alpha_j$. If $\langle \beta, \alpha_j \rangle \leqslant 0$, then there exists α_k, $k \neq j$, with $s_k(\beta) = \beta - \alpha_k \in \mathcal{O}(\widetilde{\alpha})_1$ with $\mathrm{height}(s_k(\beta)) = \mathrm{height}(\beta) - 1$. Note that $s_k(\beta) \in \Phi_+$, so that by (10), we have $k \notin \mathrm{Supp}(\widetilde{\alpha})$, and hence $\alpha_k \in \Phi_I$. By the induction hypothesis, $s_k(\beta) \in W_I \alpha_j$, and so $\beta \in W_I \alpha_j$ as $s_k \in W_I$. So $\mathcal{O}(\widetilde{\alpha})_1 = \cup_{j \in P_1(\widetilde{\alpha})} W_I \alpha_j$, as required.

Since $\mathcal{O}(\widetilde{\alpha})_{-1} = -\mathcal{O}(\widetilde{\alpha})_1$, we see that this implies that $\mathcal{O}(\widetilde{\alpha})_{-1} = \cup_{j \in P_1(\widetilde{\alpha})} W_I(-\alpha_j)$. □

Corollary 4.7. *If $\Phi \neq \mathsf{A}_n$, then $P(\Phi)^{W_{\widetilde{\alpha}}}$ has $\mathbb{Z}$-basis*

$$\{\textstyle\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_1} e_\beta, \sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_{-1}} e_\beta, e_{\widetilde{\alpha}}, e_{-\widetilde{\alpha}}\} \cup \{\sum_{\beta \in \Phi^0_{I_k}} e_\beta \mid k = 1, \ldots, r\}$$

and $P_+(\Phi)^{W_{\widetilde{\alpha}}}$ has $\mathbb{Z}$-basis

$$\{\textstyle\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_1} (e_\beta + e_{-\beta}), e_{\widetilde{\alpha}} + e_{-\widetilde{\alpha}}\} \cup \{\sum_{\beta \in \Phi^0_{I_k}} e_\beta \mid k = 1, \ldots, r\}.$$

Proof. If $\Phi \neq \mathsf{A}_n$, then $\mathcal{O}(\widetilde{\alpha})_1$ is itself a $W_{\widetilde{\alpha}}$-orbit. The rest of the first statement follows from Lemma 4.6 and the fact that $P(\Phi)$ is a W-permutation lattice with basis $\{e_\alpha \mid \alpha \in \Phi_0\}$. The second statement follows from Lemma 4.6 and the fact that $P_+(\Phi)^{W_{\widetilde{\alpha}}} = P(\Phi)^{W_{\widetilde{\alpha}}} \cap P_+(\Phi)$. □

If $\Phi = \mathsf{A}_n$, $n \geqslant 2$, then $K_-(\mathbb{Z}\mathsf{A}_n) = \bigwedge^2(\mathbb{Z}\mathsf{A}_n)$ is an irreducible S_{n+1}-lattice. The next lemma proves the analogous result for the W-lattice $K_-(\mathbb{Z}\Phi)$ for $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$.

Lemma 4.8. *$K_-(\Phi)$ is an irreducible W-lattice if $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$.*

Proof. Recall that $K_-(\Phi) = 0$ for $\Phi = \mathsf{A}_1, \mathsf{B}_n$. For $\Phi = \mathsf{A}_n, n \geqslant 2$, $K_-(\Phi) \cong \bigwedge^2(\mathbb{Z}\mathsf{A}_n)$ which is an irreducible $\mathbb{Z}S_{n+1}$-lattice [FH, Ex. 4.6, p. 48]. Now assume $\Phi \neq \mathsf{A}_n, \mathsf{B}_n$. Note that

$$\mathbb{Q}P(\Phi) = \mathbb{Q}P_-(\Phi) \oplus \mathbb{Q}P_+(\Phi)$$

as $\mathbb{Q}W$-modules. Note also that from Corollary 4.7,

$$\mathbb{Q}P(\Phi)^{W_{\widetilde{\alpha}}} = \mathbb{Q}e_{\widetilde{\alpha}} \oplus \mathbb{Q}e_{-\widetilde{\alpha}} \oplus \mathbb{Q}\textstyle\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_1} e_\beta + \mathbb{Q}\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_{-1}} e_\beta \oplus \oplus_{i=1}^r \mathbb{Q}\sum_{\beta \in \Phi^0_{I_k}} e_\beta$$

so that

$$\mathbb{Q}P(\Phi)^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle} = (\mathbb{Q}P(\Phi)^{W_{\widetilde{\alpha}}})^{\langle s_{\widetilde{\alpha}} \rangle} = \mathbb{Q}(e_{\widetilde{\alpha}} + e_{-\widetilde{\alpha}}) \oplus \mathbb{Q}\textstyle\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_{\pm 1}} e_\beta \oplus \oplus_{i=1}^r \mathbb{Q}\sum_{\beta \in \Phi^0_{I_k}} e_\beta,$$

since $s_{\widetilde{\alpha}}(\widetilde{\alpha}) = -\widetilde{\alpha}$; $\beta \in \mathcal{O}(\widetilde{\alpha})_{\pm 1}$ if and only if $s_{\widetilde{\alpha}}(\beta) \in \mathcal{O}(\widetilde{\alpha})_{\mp 1}$ and $s_{\widetilde{\alpha}}(\beta) = \beta$ if $\beta \in \Phi_I$. Now

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Q}W}(\mathbb{Q}P_-(\Phi), \mathbb{Q}P_+(\Phi)) &\cong \mathrm{Hom}_{\mathbb{Q}W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle}(\mathbb{Q}P_-(\Phi), \mathbb{Q}) \\
&\cong (\mathbb{Q}P_-(\Phi))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle} \\
&= (\mathbb{Q}P(\Phi))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle} \cap \mathbb{Q}P_-(\Phi) = 0.
\end{aligned}$$

The last equality follows from the fact that if $x \in (\mathbb{Q}P(\Phi))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle} \cap P_-(\Phi)$, then

$$x = p(e_{\widetilde{\alpha}} + e_{-\widetilde{\alpha}}) + q\big(\textstyle\sum_{\beta \in \mathcal{O}(\widetilde{\alpha})_{\pm 1}} e_\beta\big) + \sum_{k=1}^r s_k\big(\sum_{\beta \in \Phi^0_{I_k}} e_\beta\big) = \sum_{\beta \in \Phi_0} m_\beta e_\beta$$

for some $p, q, s_k \in \mathbb{Q}$ and $m_{-\beta} = -m_\beta$ for all $\beta \in \Phi_0$. Now $p = m_{-\widetilde{\alpha}} = -m_{\widetilde{\alpha}} = -p$ implies that $p = 0$. Since $\beta \in \mathcal{O}(\widetilde{\alpha})_{\pm 1}$ implies $-\beta \in \mathcal{O}(\widetilde{\alpha})_{\mp 1}$, we see that $q = 0$, and since $\beta \in \Phi^0_{I_k}$ if and only if $-\beta \in \Phi^0_{I_k}$, we see that $s_k = 0, k = 1, \ldots, r$. So we see that $\mathbb{Q}P_-(\Phi)$ and $\mathbb{Q}P_+(\Phi)$ have no common irreducible $\mathbb{Q}W$-submodules. Then by Corollary 4.7, we have

$$\begin{aligned}
\dim\big(\mathrm{Hom}_{\mathbb{Q}W}(\mathbb{Q}P_-(\Phi), \mathbb{Q}P_-(\Phi))\big) &= \dim\big(\mathrm{Hom}_{\mathbb{Q}W}(\mathbb{Q}P(\Phi), P(\Phi))\big) \\
&\quad - \dim\big(\mathrm{Hom}_{\mathbb{Q}W}(\mathbb{Q}P_+(\Phi), P_+(\Phi))\big) \\
&= \dim\big((\mathbb{Q}P(\Phi))^{W_{\widetilde{\alpha}}}\big) - \dim\big((\mathbb{Q}P_+(\Phi))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}} \rangle}\big) = 2.
\end{aligned}$$

But $\dim(\mathrm{Hom}_{\mathbb{Q}W}(\mathbb{Q}P_-(\Phi), \mathbb{Q}P_-(\Phi)))$ is $\sum_{i=1}^k n_i^2$ where k is the number of irreducible components of $\mathbb{Q}P_-(\Phi)$ and n_i is the number of irreducible components of $\mathbb{Q}P_-(\Phi)$ of the ith type. Since

$$\mathbb{Q}P_-(\Phi) = \mathbb{Q}K_-(\Phi) \oplus \mathbb{Q}\Phi$$

is a decomposition of $\mathbb{Q}W$-modules with $\mathbb{Q}\Phi$ irreducible, then $\mathbb{Q}K_-(\Phi)$ must be an irreducible $\mathbb{Q}W$-module. Hence, $K_-(\Phi)$ is an irreducible W-lattice as required. □

The following lemma will be useful for computing $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$.

Lemma 4.9. *For any $\alpha \in \Phi_0$,*

$$\sum_{\beta\in\Phi_0}\langle\alpha,\beta\rangle\beta = 2h\alpha$$

where h is the Coxeter number of the irreducible component Φ_0' of Φ_0 to which α belongs. In particular,

$$\sum_{\beta\in\mathcal{O}(\alpha)_1}\beta = (h-2)\alpha.$$

Proof. Observe that

$$\sum_{\beta\in\Phi_0}\langle\alpha,\beta\rangle\beta = \sum_{i=-2}^{2}\sum_{\beta\in\mathcal{O}(\alpha)_i}\langle\alpha,\beta\rangle\beta = \sum_{\beta\in\mathcal{O}(\alpha)_1}\beta + \sum_{\gamma\in\mathcal{O}(\alpha)_{-1}}(-\gamma) + 4\alpha.$$

Since $\mathcal{O}(\alpha)_{-1} = -\mathcal{O}(\alpha)_1$, we see that

$$\sum_{\beta\in\Phi_0}\langle\alpha,\beta\rangle\beta = 2\big(\sum_{\beta\in\mathcal{O}(\alpha)_1}\beta + 2\alpha\big).$$

So the two statements are equivalent.

Note that

$$\sum_{\beta\in\Phi_0}\langle\alpha,\beta\rangle\beta = \sum_{\beta\in\Phi_0'}\langle\alpha,\beta\rangle\beta,$$

since $\langle\alpha,\beta\rangle = 0$ for all $\beta \in \Phi_0 \setminus \Phi_0'$.

Now

$$\mathbb{Z}\Phi_0' \to \mathbb{Z}\Phi_0', \quad \alpha \mapsto \sum_{\beta\in\Phi_0'}\langle\alpha,\beta\rangle\beta$$

is an element of $\mathrm{Hom}_W(\mathbb{Z}\Phi_0', \mathbb{Z}\Phi_0') = \mathbb{Z}\mathrm{id}$. So $\sum_{\beta\in\Phi_0'}\langle\alpha,\beta\rangle\beta = N\alpha$ for some $N \in \mathbb{Z}$. Applying $\langle\cdot,\alpha\rangle$ to both sides of this equation, we see that

$$N = \tfrac{1}{2}\sum_{\beta\in\Phi_0}\langle\alpha,\beta\rangle^2.$$

By symmetry,

$$N = |\mathcal{O}(\alpha)_1| + 4.$$

Moreover, since every $\alpha \in \Phi_0$ can be expressed as $\alpha = w\widetilde{\alpha}$ for some $w \in W$ and $\mathcal{O}(w\widetilde{\alpha})_1 = w\mathcal{O}(\widetilde{\alpha})_1$, we have

$$N = |\mathcal{O}(\widetilde{\alpha})_1| + 4.$$

Since $\widetilde{\alpha}$ is the highest root in the irreducible component of Φ_0 to which it belongs, we have by [Hu2, p. 84] that $\mathrm{height}(\widetilde{\alpha}) = h - 1$.

We will compute length of an element of $W(\Phi_0)$ with respect to the base $\Delta_0 \subset \Phi^+$ of Φ_0. Note that by [Hu2, p. 14],

$$\mathrm{length}(s_{\widetilde{\alpha}}) = |\{\beta \in (\Phi_0)^+ \mid s_{\widetilde{\alpha}}(\beta) \in (\Phi_0)^-\}| = |\{\beta \in \Phi_0 \mid \langle\beta,\widetilde{\alpha}\rangle > 0\}| = |\mathcal{O}(\widetilde{\alpha})_1| + 1.$$

The following claim will complete the proof:

Claim: $\mathrm{length}(s_{\widetilde{\alpha}}) = 2\mathrm{height}(\widetilde{\alpha}) - 1$.

Assuming this claim, we see that $|\mathcal{O}(\widetilde{\alpha})_1| + 1 = 2h - 3$ so that $N = 2h$, as required.

To complete the proof we prove a slightly stronger statement than the claim. Namely, we show that

$$\mathrm{length}(s_\alpha) = 2\mathrm{height}(\alpha) - 1 \text{ for all } \alpha \in (\Phi_0)^+$$

by induction on the height of α. For a simple root α, it is clear that $\mathrm{length}(s_\alpha) = 1 = 2\cdot 1 - 1 = 2\mathrm{height}(\alpha) - 1$. Assume $\mathrm{height}(\alpha) > 1$. Then α is a positive nonsimple root. So there exists a simple root β with $\langle\alpha,\beta\rangle > 0$, [Hu1, p. 50]. Then $s_\beta(\alpha) = \alpha - \beta \in (\Phi_0)^+$ has $\mathrm{height}(s_\beta(\alpha)) = \mathrm{height}(\alpha) - 1$, whereas by [Hu1, p. 43] and [Hu2, p. 12], we have $\mathrm{length}(s_{s_\beta\alpha}) = \mathrm{length}(s_\beta s_\alpha s_\beta) = \mathrm{length}(s_\beta s_\alpha) - 1 = \mathrm{length}(s_\alpha) - 2$. Applying the induction hypothesis to $s_\beta(\alpha)$, we get $\mathrm{length}(s_\alpha) - 2 = 2(\mathrm{height}(\alpha) - 1) - 1$ so that $\mathrm{length}(s_\alpha) = 2\mathrm{height}(\alpha) - 1$ as required. □

Lemma 4.10.

$$\operatorname{Coker}(\operatorname{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)) \cong \begin{cases} \mathbb{Z}/h\mathbb{Z}, & \Phi = \mathsf{A}_n, \mathsf{B}_n, \\ \mathbb{Z}/2h\mathbb{Z}, & \textit{otherwise.} \end{cases}$$

Under the map induced by the commutative square

$$\begin{array}{ccccc} P_-(\Phi) & \longrightarrow & 2\mathbb{Z}\Phi & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \end{array},$$

the image of

$$\operatorname{Coker}\big(\operatorname{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi)) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi)\big)$$

in

$$\operatorname{Coker}\big(\operatorname{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$$

is a subgroup of index 2 *for all* $\Phi \neq \mathsf{A}_n$*,* n *even. For* $\Phi = \mathsf{A}_n$*,* n *even, the map between these cokernels is an isomorphism.*

Proof. First note that since $\mathbb{Q}$ is the splitting field for the irreducible module $\mathbb{Q}\Phi$, we have $\operatorname{Hom}_{\mathbb{Q}W}(\mathbb{Q}\Phi, \mathbb{Q}\Phi) = \mathbb{Q}\,\mathrm{id}$, and hence $\operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) = \mathbb{Z}\,\mathrm{id}$.

Now we have

$$\begin{aligned} \operatorname{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) &\overset{(\eta_*)^{-1}}{\cong} \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}W \otimes_{\mathbb{Z}W_I} \mathbb{Z}) \\ &\overset{(\rho_*)^{-1}}{\cong} \operatorname{Hom}_W\big(\mathbb{Z}\Phi, \operatorname{Hom}_{\mathbb{Z}W_I}(\mathbb{Z}W, \mathbb{Z})\big) \\ &\overset{(\psi)^{-1}}{\cong} \operatorname{Hom}_{\mathbb{Z}W_I}(\mathbb{Z}\Phi, \mathbb{Z}) \\ &= \big(\operatorname{Hom}(\mathbb{Z}\Phi, \mathbb{Z})\big)^{W_I} \\ &\overset{(\theta)^{-1}}{\cong} \big(\Lambda(\Phi_0)\big)^{W_I} \\ &\cong \begin{cases} \mathbb{Z}(\frac{1}{2}\widetilde{\alpha}), & \Phi = \mathsf{A}_1, \mathsf{B}_n, \\ \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_n, & \Phi = \mathsf{A}_n, n \geqslant 2, \\ \mathbb{Z}\widetilde{\alpha}, & \text{otherwise.} \end{cases} \end{aligned}$$

We need to unravel these isomorphisms in reverse in order to determine the map $\operatorname{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \to \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)$. First of all, the isomorphism $\eta : \mathbb{Z}W \otimes_{\mathbb{Z}W_I} \mathbb{Z} \to P(\Phi)$ is defined by $w \otimes 1 \mapsto e_{w\widetilde{\alpha}}$. Next, the isomorphism ρ is defined as

$$\rho : \operatorname{Hom}_{\mathbb{Z}W_I}(\mathbb{Z}W, \mathbb{Z}) \longrightarrow \mathbb{Z}W \otimes_{\mathbb{Z}W_I} \mathbb{Z}, \quad f \mapsto \textstyle\sum_{w \in W/W_I} w \otimes f(w^{-1}).$$

It is the natural isomorphism between coinduced and induced modules [Br, p. 70]. Thirdly, the isomorphism [Br, p. 64]

$$\psi : \operatorname{Hom}_{\mathbb{Z}W_I}(\mathbb{Z}\Phi, \mathbb{Z}) \longrightarrow \operatorname{Hom}_W\big(\mathbb{Z}\Phi, \operatorname{Hom}_{\mathbb{Z}W_I}(\mathbb{Z}W, \mathbb{Z})\big)$$

is given by

$$g \mapsto [\alpha \to [w \mapsto g(w\alpha)]].$$

Lastly, the W-isomorphism $\theta : \Lambda(\Phi_0) \to \mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{Z})$ is given by

$$\theta(\lambda)(\alpha) = \langle \lambda, \alpha \rangle$$

where $\lambda \in \Lambda(\Phi_0)$ and $\alpha \in \Phi_0$. Note that, since $\mathbb{Z}\Phi_0 = \mathbb{Z}\Phi$, a $\mathbb{Z}$-linear homomorphism from $\mathbb{Z}\Phi$ to any other $\mathbb{Z}$-module can be determined from its restriction to Φ_0.

Now we have to work backwards. Let $\mu : (\Lambda(\Phi_0))^{W_I} \to \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ be the composite map $\eta_* \circ \rho_* \circ \varphi \circ \theta$. Let $\lambda \in \Lambda(\Phi_0)^{W_I}$ and $\alpha \in \Phi_0$. Then

$$\begin{aligned} \theta(\lambda)(\alpha) &= \langle \lambda, \alpha \rangle, \\ \psi(\theta(\lambda))(\alpha) &= [w \mapsto \langle \lambda, w\alpha \rangle], \\ \rho_*(\psi(\theta(\lambda)))(\alpha) &= \textstyle\sum_{w \in W/W_I} w \otimes \langle \lambda, w^{-1}\alpha \rangle. \end{aligned}$$

Lastly, $\mu(\lambda)(\alpha) = \eta_*(\rho_*(\psi(\theta(\lambda))))(\alpha) = \sum_{w \in W/W_I} \langle \lambda, w^{-1}\alpha \rangle e_{w\widetilde{\alpha}}$. Evaluating this at $\lambda = \widetilde{\alpha}$, we get $\sum_{w \in W/W_I} \langle w\widetilde{\alpha}, \alpha \rangle e_{w\widetilde{\alpha}} = \sum_{\beta \in \Phi_0} \langle \beta, \alpha \rangle e_\beta$. For $\Phi \neq \mathsf{A}_n$, B_n, the element $\widetilde{\alpha}$ generates $\Lambda(\Phi_0)^{W_I}$ and, if $\Phi = \mathsf{A}_1, \mathsf{B}_n$, then $\frac{1}{2}\widetilde{\alpha}$ generates $\Lambda(\Phi_0)^{W_I}$. Now $\mu(\widetilde{\alpha})$ is the homomorphism $[\alpha \mapsto \sum_{\beta \in \Phi_0} \langle \beta, \alpha \rangle e_\beta] \in \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ which, according to Lemma 4.9, maps to $2h\,\mathrm{id} \in \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)$ under the map

$$\mathrm{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi).$$

This proves the first result for $\Phi \neq \mathsf{A}_n, n \geqslant 2$. Note that $\mu(\widetilde{\alpha})(\mathbb{Z}\Phi) \subset P_-(\Phi)$ so that, in fact, $\mathrm{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi)) = \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ in these cases. This shows that

$$\mathrm{Coker}\big(\mathrm{Hom}_W\big(\mathbb{Z}\Phi, P_-(\Phi) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi)\big)\big) \cong \mathbb{Z}/h\mathbb{Z}$$

as required.

For $\Phi = \mathsf{A}_n$, $n \geqslant 2$, recall that $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ and $\Delta = \Delta_0 = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n\}$. We have $\widetilde{\alpha} = \varpi_1 + \varpi_n = \varepsilon_1 - \varepsilon_{n+1}$. We need to determine the image of ϖ_1 and ϖ_n under the isomorphism from $\Lambda(\Phi)^{W_I} \to \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$. Evaluating μ at $\lambda = \varpi_k$, $k = 1, n$, we get $\mu(\varpi_k) = [\alpha \mapsto \sum_{w \in W/W_I} \langle \varpi_k, w^{-1}\alpha \rangle e_{w\widetilde{\alpha}}]$. It is sufficient to determine the value of this map at $\alpha = \widetilde{\alpha}$ to determine its image in $\mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)$. So for each root $\alpha = \varepsilon_i - \varepsilon_j$, we need an element $w \in S_{n+1}$ such that $\alpha = w\widetilde{\alpha}$. We have:

$$\begin{aligned} (1,i)(n+1,j)\widetilde{\alpha} &= \varepsilon_i - \varepsilon_j,\ \{i,j\} \cap \{1, n+1\} = \varphi, \\ (j, n+1)\widetilde{\alpha} &= \varepsilon_1 - \varepsilon_j,\ i = 1;\ j \neq n+1, \\ (i,1)\widetilde{\alpha} &= \varepsilon_i - \varepsilon_{n+1},\ i \neq 1,\ j = n+1, \\ (1, n+1, j)\widetilde{\alpha} &= \varepsilon_{n+1} - \varepsilon_j, \\ (1, n+1, j)^{-1}\widetilde{\alpha} &= (1, j, n+1)\widetilde{\alpha} = \varepsilon_j - \varepsilon_1,\ i = n+1,\ j \neq 1, \\ (1, i, n+1)\widetilde{\alpha} &= \varepsilon_i - \varepsilon_1, \\ (1, i, n+1)^{-1}\widetilde{\alpha} &= (1, n+1, i)\widetilde{\alpha} = \varepsilon_{n+1} - \varepsilon_i,\ i \neq n+1,\ j = 1, \\ \mathrm{id}(\widetilde{\alpha}) &= \widetilde{\alpha}, \\ (1, n+1)\widetilde{\alpha} &= -\widetilde{\alpha}. \end{aligned}$$

This implies that

$$\mu(\varpi_1)(\widetilde{\alpha}) = \sum_{j=2}^{n}(e_{\varepsilon_1-\varepsilon_j} - e_{\varepsilon_{n+1}-\varepsilon_j}) + e_{\varepsilon_1-\varepsilon_{n+1}} - e_{\varepsilon_{n+1}-\varepsilon_1}$$

and

$$\mu(\varpi_n)(\widetilde{\alpha}) = \sum_{j=2}^{n}(e_{\varepsilon_j-\varepsilon_{n+1}} - e_{\varepsilon_j-\varepsilon_1}) + e_{\varepsilon_1-\varepsilon_{n+1}} - e_{\varepsilon_{n+1}-\varepsilon_1}.$$

In either case, we see that for $k = 1, n$, $\mu(\varpi_k) \in \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ is mapped to $(n+1)\mathrm{id} \in \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)$ as required for the first result. Let $a\mu(\varpi_1) + b\mu(\varpi_n) \in \mathrm{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi))$. Then $(a\mu(\varpi_1) + b\mu(\varpi_n))(\widetilde{\alpha}) \in P_-(\Phi)$ implies that $a = b$. So $\mathrm{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi))$ is generated by $\mu(\varpi_1 + \varpi_n) = \mu(\widetilde{\alpha})$, which has image $h([\alpha \mapsto 2\alpha]) \in \mathrm{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi)$. So

$$\mathrm{Coker}\big(\mathrm{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi)) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi)\big) \cong \mathbb{Z}/h\mathbb{Z}$$

as required. However, the induced map between the cokernels is equivalent to the map between $2\mathbb{Z}/2h\mathbb{Z}$ and $\mathbb{Z}/h\mathbb{Z}$ induced by the inclusion $2\mathbb{Z} \rightarrowtail \mathbb{Z}$. This is a surjection (and hence an isomorphism) if and only if $h = n+1$ is odd. □

Lemma 4.11. $\mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, P_-(\Phi)\big) = 0$.

Proof. For $\mathsf{A}_1, \mathsf{B}_n$, we have that $P_-(\Phi) \cong \mathbb{Z}\Phi$ so that

$$\begin{aligned}
\mathrm{Ext}^1_W(P_-(\Phi), P_-(\Phi)) &\cong \mathrm{Ext}^1_{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle}(P_-(\Phi)\,|_{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle}, \mathbb{Z}_\xi) \\
&\cong \mathrm{Ext}^1_{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle}((\mathbb{Z}_\xi)^n, \mathbb{Z}_\xi) \\
&\cong H^1(W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle, ((\mathbb{Z}_\xi)^n)^* \otimes \mathbb{Z}_\xi) \\
&\cong H^1(W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle, \mathbb{Z}^n) = 0.
\end{aligned}$$

Now assume that $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$. We have the exact sequence of W-lattices

$$0 \longrightarrow P_-(\Phi) \longrightarrow P(\Phi) \longrightarrow P_+(\Phi) \longrightarrow 0.$$

Now

$$\begin{aligned}
\mathrm{Hom}_W(\mathbb{Z}\Phi, P_+(\Phi)) &\cong \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}W \otimes_{\mathbb{Z}W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle} \mathbb{Z}) \\
&\cong \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathrm{Hom}_{\mathbb{Z}W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle}(\mathbb{Z}W, \mathbb{Z})) \\
&\cong \mathrm{Hom}_{\mathbb{Z}W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle}(\mathbb{Z}\Phi, \mathbb{Z}) \\
&= (\mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{Z}))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle} \\
&\cong ((\mathbb{Z}\Phi)^*)^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle} \\
&\cong (\Lambda(\Phi_0))^{W_{\widetilde{\alpha}} \times \langle s_{\widetilde{\alpha}}\rangle} \\
&\cong ((\Lambda(\Phi_0)^{W_{\widetilde{\alpha}}})^{s_{\widetilde{\alpha}}} \\
&= \begin{cases} \mathbb{Z}(\varpi_1 - \varpi_n), & \Phi = \mathsf{A}_n, n \geqslant 2, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

If $\Phi = \mathsf{A}_n$, $n \geqslant 2$, the generators $\mu(\varpi_1)$ and $\mu(\varpi_n)$ of $\mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi))$ map to a generator of $\mathrm{Hom}_W(\mathbb{Z}\Phi, P_+(\Phi))$.

So for all Φ, $\mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \to \mathrm{Hom}_W(\mathbb{Z}\Phi, P(\Phi)/P_-(\Phi))$ is surjective so that the map $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, P_-(\Phi)) \to \mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi))$ is injective. But for $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$, we have by Proposition 4.5 that $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)) = 0$ so that $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, P_-(\Phi)) = 0$ as well. □

Proposition 4.12. *Let h be the Coxeter number of the connected component of Φ_I to which $\widetilde{\alpha}$ belongs.*

$$\xi = [0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0] \in \mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$$

is an element of $\mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$ of order m where $m = h$ if $\Phi = \mathsf{A}_n, \mathsf{B}_n$, and $m = 2h$ if $\Phi \neq \mathsf{A}_n, \mathsf{B}_n$. In each case, ξ generates the image of $\mathrm{Coker}\big(\mathrm{Hom}(\mathbb{Z}\Phi, P(\Phi)\big) \to \mathrm{Hom}\big(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$ in $\mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$. If $\Phi \neq \mathsf{B}_n$, ξ generates $\mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$. In fact,

$$\mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big) \cong \begin{cases} \mathbb{Z}/h\mathbb{Z}, & \text{if } \Phi = \mathsf{A}_n, \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } \Phi = \mathsf{B}_n, n \geqslant 2, \\ \mathbb{Z}/2h\mathbb{Z}, & \text{if } \Phi \neq \mathsf{A}_n, \mathsf{B}_n. \end{cases}$$

Proof. We will show that

$$\mathrm{Coker}\big(\mathrm{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$$

maps isomorphically onto the cyclic group generated by the extension class of

$$[0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0]$$

in $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$.

Since the map $\overline{\partial}$ from

$$\mathrm{Coker}\big(\mathrm{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$$

to $\mathrm{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$ is that induced by the connecting homomorphism

$$\partial : \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) \longrightarrow \mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big),$$

it suffices to show that ∂ is surjective as $\overline{\partial}$ is, by construction, injective.

Note that

$$\begin{array}{ccc} \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) & \xrightarrow{\ \partial\ } & \mathrm{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big) \\ & \searrow^{\delta} & \downarrow \cong \\ & & H^1\big(W, \mathrm{Hom}(\mathbb{Z}\Phi, K(\Phi))\big) \end{array}$$

is a commutative diagram.

Now $\mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) = \mathbb{Z}\,\mathrm{id}$. Under the map

$$\delta : \mathrm{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi) \ni \mathrm{id} \mapsto [w \to (w\sigma) - \sigma] \in H^1\big(W, \mathrm{Hom}(\mathbb{Z}\Phi, K(\Phi))\big)$$

where $\sigma : \mathbb{Z}\Phi \to P(\Phi)$ is any $\mathbb{Z}$-splitting of the sequence

$$0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0.$$

But under the isomorphism

$$H^1\big(W, \operatorname{Hom}(\mathbb{Z}\Phi, K(\Phi))\big) \cong \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big),$$

$[w \to (w\sigma) - \sigma]$ is mapped to the extension class

$$[0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0] \in \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big).$$

This implies that $\overline{\partial}$ is surjective and completes the proof of the second statement. For the first statement, we need only apply Lemma 4.10.

Since $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)) = 0$ for all $\Phi \neq \mathsf{B}_n$, we have that

$$\operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \cong \operatorname{Coker}\big(\operatorname{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$$

in this case.

So we need only consider $\Phi = \mathsf{B}_n$. Now $K(\Phi) \cong P_+(\Phi)$ in this case. So

$$\begin{aligned} \operatorname{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi)) &\cong \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, P_+(\Phi)\big) \\ &\cong \operatorname{Ext}^1_{\mathbb{Z}W_{\tilde{\alpha}} \times \langle s_{\tilde{\alpha}} \rangle}(\mathbb{Z}\Phi, \mathbb{Z}) \\ &\cong H^1\big(W_{\tilde{\alpha}} \times \langle s_{\tilde{\alpha}} \rangle, \Lambda(\Phi_0)\big). \end{aligned}$$

Now $W_{\tilde{\alpha}} \times \langle s_{\tilde{\alpha}} \rangle \cong (C_2)^n \rtimes S_{n-1}$ and $\Lambda(\Phi_0) \cong \mathbb{Z}\mathsf{B}_n$ as a $W(\mathsf{B}_n)$ lattice. Since $\mathbb{Z}\mathsf{B}_n^{(C_2)^n} = 0$ and $H^1((C_2)^n, \mathbb{Z}\mathsf{B}_n) \cong (\mathbb{Z}/2\mathbb{Z})^n$, we see by inflation-restriction that $H^1(W_{\tilde{\alpha}} \times \langle s_{\tilde{\alpha}} \rangle, \Lambda(\Phi_0)) \cong H^1((C_2)^n, \mathbb{Z}\mathsf{B}_n)^{S_{n-1}} \cong (\mathbb{Z}/2\mathbb{Z})^2$ as required. □

Corollary 4.13. *There exists a commutative diagram of the form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\Phi) & \longrightarrow & P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & K_-(\Phi) & \longrightarrow & X & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \end{array}$$

if and only if $\Phi = \mathsf{A}_n$, *n even.*

Proof. Since $\xi = [0 \to K(\Phi) \to P(\Phi) \to \mathbb{Z}\Phi \to 0]$ generates the subgroup

$$\operatorname{Coker}\big(\operatorname{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big)$$

of $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$, then ξ has a preimage in

$$\operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K_-(\Phi)\big) \longrightarrow \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$$

if and only if

$$\begin{aligned} &\operatorname{Coker}\big(\operatorname{Hom}_W\big(\mathbb{Z}\Phi, P_-(\Phi)\big) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi)\big) \\ &\qquad \longrightarrow \operatorname{Coker}\big(\operatorname{Hom}_W\big(\mathbb{Z}\Phi, P(\Phi)\big) \longrightarrow \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi)\big) \end{aligned}$$

is surjective. By Lemma 4.10, $\operatorname{Coker}(\operatorname{Hom}_W(\mathbb{Z}\Phi, P_-(\Phi)) \to \operatorname{Hom}_W(\mathbb{Z}\Phi, 2\mathbb{Z}\Phi))$ maps into $\operatorname{Coker}(\operatorname{Hom}_W(\mathbb{Z}\Phi, P(\Phi)) \to \operatorname{Hom}_W(\mathbb{Z}\Phi, \mathbb{Z}\Phi))$ as a subgroup of index 2 unless $\Phi = \mathsf{A}_n$, n even, in which case this map is an isomorphism. □

Corollary 4.14. *Let K_0 be a W-sublattice of $K(\Phi)$.*

If Φ is not of type B_n, there exists a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\Phi) & \longrightarrow & P(\Phi) & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & K_0 & \longrightarrow & X & \longrightarrow & \mathbb{Z}\Phi & \longrightarrow & 0 \end{array}$$

if and only if the map $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, K_0) \longrightarrow \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$ is surjective.

Proof. Such a diagram exists if and only if

$$[0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0] \in \operatorname{Ext}^1_W\big(\mathbb{Z}\Phi, K(\Phi)\big)$$

has a preimage under the map $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, K_0) \to \operatorname{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$. Note that for Φ not of type B_n, we have $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, P(\Phi)) = 0$ so that by Proposition 4.12,

$$[0 \to K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0]$$

generates the group $\operatorname{Ext}^1_W(\mathbb{Z}\Phi, K(\Phi))$. □

5. Minimal permutation resolutions of the character lattice

Let G be a simple algebraic group with maximal torus T, corresponding Weyl group W, and character lattice $X(T)$. Let $\Phi \equiv \Phi(G,T)$ be the root system attached to G and the maximal torus T. We will use the notation and definitions introduced in Section 3.2 for the set of short roots Φ_0, the root lattice $\mathbb{Z}\Phi$, its weight lattice $\Lambda(\Phi)$, its Weyl group $W = W(\Phi)$ and W_λ, the isotropy subgroup of W fixing $\lambda \in \Lambda(\Phi)$. In this section, we will determine a minimal permutation W-lattice P of minimal rank such that there exists a W-epimorphism $\pi : P \twoheadrightarrow X(T)$ with $\operatorname{Ker}(\pi)$ being a faithful W-lattice. That is, with the notation of Section 3.1, we will find a minimal element (P, π) of $\mathcal{P}(X(T))$ together with $r(X(T)) = \operatorname{rank}(P)$. This discussion is motivated by Corollary 2.2.

We first discuss some preliminaries on W-sublattices of character lattices and minimal permutation resolutions.

5.1. *W*-sublattices of character lattices and minimal permutation resolutions

Recall that if Φ is the root system of W and $\Lambda(\Phi)$ is its weight lattice, we have $\mathbb{Z}\Phi \subset X(T) \subset \Lambda(\Phi)$, and all are W-lattices of the same rank, say n.

The following two technical lemmas are useful for determining the W-span of an element $\chi \in X(T)$ and for determining conditions on $\chi \in X(T)$ such that $\mathbb{Z}W\chi = X(T)$.

Lemma 5.1.

(a) $\Lambda(\Phi)/\mathbb{Z}\Phi$ *is a (finite) trivial W-module.*
(b) *For any $\chi \in X(T)$, $\mathbb{Z}W\chi \subset \mathbb{Z}\chi + \mathbb{Z}\Phi$.*
(c) $\mathbb{Z}W\varpi_i = \mathbb{Z}\varpi_i + \mathbb{Z}W\alpha_i$ *for all $i = 1, \ldots, n$. In particular, if α_i is a short root, then $\mathbb{Z}W\varpi_i = \mathbb{Z}\varpi_i + \mathbb{Z}\Phi$.*

Proof. (a) Note that $\mathbb{Z}\Phi$ is a W-sublattice of Λ. Let $\lambda \in \Lambda(\Phi)$. For $i = 1, \dots, n$,

$$s_i\lambda - \lambda = \langle \lambda, \alpha_i \rangle \alpha_i \in \mathbb{Z}\Phi$$

so that $s_i(\lambda + \mathbb{Z}\Phi) = \lambda + \mathbb{Z}\Phi$. But then, since the simple reflections generate W, $w(\lambda + \mathbb{Z}\Phi) = \lambda + \mathbb{Z}\Phi$ for all $w \in W$.

(b) Since $\mathbb{Z}\Phi \subset X(T) \subset \Lambda(\Phi)$, it follows from (a) that $X(T)/\mathbb{Z}\Phi$ is a trivial W-module. Now (b) follows immediately.

(c) $\mathbb{Z}W\alpha_i$ is a W-sublattice of Λ. Since the simple reflections generate W, and $s_j\varpi_i = \varpi_i - \delta_{ij}\alpha_j$ for all $j = 1, \dots, n$, we see that $w \in W$ fixes $\varpi_i + \mathbb{Z}W\alpha_i$. Hence, $\mathbb{Z}W\varpi_i \subset \mathbb{Z}\varpi_i + \mathbb{Z}W\alpha_i$. But $\varpi_i \in \mathbb{Z}W\varpi_i$, and $\varpi_i - \alpha_i = s_i\varpi_i \in \mathbb{Z}W\varpi_i$, so that $\mathbb{Z}\varpi_i + \mathbb{Z}W\alpha_i \subset \mathbb{Z}W\varpi_i$ as required. If α_i is a short root, then $\mathbb{Z}W\alpha_i = \mathbb{Z}\Phi$. □

Let $\lambda \in \Lambda(\Phi)$. Denote by λ^+ the unique element of Λ^+ in the W-orbit of λ. Note that $\mathbb{Z}W\lambda = \mathbb{Z}W\lambda^+$ and that if $\lambda^+ = w\lambda$, then $wW_\lambda w^{-1} = W_{\lambda^+}$ so that $|W_\lambda| = |W_{\lambda^+}|$.

Lemma 5.2. *Assume that the rank of G is $n > 1$.*

(a) *If $\lambda_i = m_i\varpi_{j_i} \in X(T), i = 1, \dots, r$ and d_j is the order of $\varpi_j + X(T)$ in $\Lambda/X(T)$, then if $\sum_{i=1}^r \mathbb{Z}W\lambda_i = X(T)$, we have*

$$\gcd\{d_{j_i} \mid i = 1, \dots, r\} = \gcd\{m_i \mid i = 1, \dots, r\} = 1.$$

(b) *If $\sum_{i=1}^r \mathbb{Z}W\lambda_i = X(T)$, then $\langle \lambda_i + \mathbb{Z}\Phi \mid i = 1, \dots, r\rangle$ must generate $X(T)/\mathbb{Z}\Phi$. In particular, if $X(T)/\mathbb{Z}\Phi$ is cyclic of prime power order, then there exists i such that $\lambda_i + \mathbb{Z}\Phi$ generates $X(T)/\mathbb{Z}\Phi$.*

Proof. (a) Note that since $m_i\varpi_{j_i} \in X(T)$, then d_{j_i} must divide m_i for all $i = 1, \dots, r$. So $\gcd\{d_{j_i} \mid i = 1, \dots, r\}$ divides $m = \gcd\{m_i \mid i = 1, \dots, r\}$. Now $X(T) = \sum_{i=1}^r \mathbb{Z}Wm_i\varpi_{j_i} \in m\Lambda \cap X(T)$, so that $X(T) \subset m\Lambda$. But if $m > 1$, then $m^n = [\Lambda : m\Lambda] > n + 1 \geqslant [\Lambda : X(T)]$. By contradiction, $m = 1$.

(b) By Lemma 5.1(b), $X(T) = \sum_{i=1}^r \mathbb{Z}W\lambda_i \subset \sum_{i=1}^r \mathbb{Z}\lambda_i + \mathbb{Z}\Phi$ implies that $\lambda_i + \mathbb{Z}\Phi$, $i = 1, \dots, r$, generate $X(T)$. □

Recall the following notation from Section 3.1 for an H-lattice Y: For a finite group H with subgroups $H_1, \dots, H_k$ and an H-lattice Y with $y_1, \dots, y_k$, $f_{y_1,\dots,y_k}$ is the H-map

$$f_{y_1,\dots,y_k} : \oplus_{i=1}^k \mathbb{Z}[H/H_i] \longrightarrow Y$$

which maps H_i to y_i, $i = 1, \dots, k$. The following technical proposition will help us to determine minimal permutation resolutions under certain conditions. That is, it will help to find $(P, \pi) \in \mathcal{P}(Y)$ with $\operatorname{rank}(P) = r(Y)$ where

$$r(Y) = \min\{\operatorname{rank}(P) \mid (P, \pi) \in \mathcal{P}(Y)\}.$$

Proposition 5.3. *Suppose the following conditions hold:*

(i) *The intersection of nontrivial normal subgroups is nontrivial .*

(ii) *$\mathbb{Z}[H/H_y]$ is a minimal faithful transitive permutation lattice for H, but $\operatorname{Ker}(f_y)$ is not a faithful H-lattice.*

(iii) *If $\mathbb{Z}Hz = Y$ and $\operatorname{Ker}(f_z)$ is a faithful H-lattice, then $[H : H_z] \geqslant 2[H : H_y]$.*

Then $(\mathbb{Z}[H/H_y] \oplus \mathbb{Z}[H/H_y], f_{y,0})$ is a minimal element of $\mathcal{P}(Y)$.

Proof. Note that $(\mathbb{Z}[H/H_y]\oplus\mathbb{Z}[H/H_y], f_{y,0})\in\mathcal{P}(Y)$ since $f_{y,0}$ is surjective, and $\mathrm{Ker}(f_{y,0}) = \mathrm{Ker}(f_y)\oplus\mathbb{Z}[H/H_y]$ is faithful since $\mathbb{Z}[H/H_y]$ is.

Note that the kernel of the action of H on a direct sum of lattices is the intersection of the kernels of the actions of H on each lattice. So if we assume that the intersection of nontrivial normal subgroups is nontrivial , a direct sum of nonfaithful H-lattices is also nonfaithful. This shows that if $\mathbb{Z}[H/H_y]$ is a minimal faithful transitive permutation lattice, then it is a minimal faithful permutation lattice. Now suppose $(Q,p)\in\mathcal{P}(Y)$ with $Q=\oplus_{i=1}^r\mathbb{Z}[H/H_i]$ and $p=f_{y_1,\ldots,y_r}$. If s is the number of faithful components, then $\mathrm{rank}\,(Q)\geqslant s[H:H_y]$, so we may assume that $\mathbb{Z}[H/H_1]$ is the only faithful component and $p=f_{y_1,0\ldots,0}$. So $\mathbb{Z}Hy_1=Y$, $\mathrm{Ker}(p)=\mathrm{Ker}(f_{y_1})\oplus_{i=2}^r\mathbb{Z}[H/H_i]$ is faithful, and by (iii), $\mathrm{rank}\,(Q)\geqslant[H:H_{y_1}]\geqslant 2[H:H_y]$ as required. □

Lemma 5.4. *If H_0 is a subgroup of H, then $r(Y|_{H_0})\leqslant r(Y)$.*

Proof. Let $0\to K\to P\xrightarrow{\pi} Y\to 0$ be an exact sequence of H-lattices with P being permutation and K faithful. Restricting this sequence to H_0 proves the statement. □

The following proposition can be used to determine a minimal element of $\mathcal{P}(X(T))$ in most cases.

Proposition 5.5. *Suppose G is not of type A_n and suppose that there exists a permutation lattice $P=\mathbb{Z}[W/W_\chi]$ with $\mathbb{Z}W\chi=X(T)$ which satisfies the following conditions:*

(i) $\mathrm{rank}\,(P)\leqslant[W:W_{\varpi_i}]$ *for all* $\varpi_i\in X(T)$*;*
(ii) *Let* $I(X(T))=\{i\mid[W:W_{\varpi_i}]<\mathrm{rank}\,(P)\}$*, then* $\mathrm{rank}\,(P)\leqslant[W:W_{\varpi_i+\varpi_j}]$ *for all* $i,j\in I(X(T)), i\neq j$*;*
(iii) $\mathrm{rank}\,(P)>2n$.

Then (P,f_χ) *is a minimal element of* $\mathcal{P}(X(T))$ *and* $r(X(T))=\mathrm{rank}\,(P)$.

Proof. Suppose that $P=\mathbb{Z}[W/W_\chi]$ satisfies the conditions (i),(ii) and (iii). Then by Proposition 3.3(c), $(P,f_\chi)\in\mathcal{P}(X(T))$.

Let (Q,p) be an arbitrary element of $\mathcal{P}(X(T))$ where $Q=\oplus_{i=1}^r\mathbb{Z}[W/W_i]$ and $p=f_{\lambda_1,\ldots,\lambda_r}$. Note that $\mathrm{rank}\,(Q)\geq\sum_{i=1}^k[W:W_{\lambda_i^+}]$ and $\sum_{i=1}^r\mathbb{Z}W\lambda_i^+=X(T)$. We need to show that $\mathrm{rank}\,(Q)\geqslant\mathrm{rank}\,(P)$.

Case I*:* Assume that all λ_i^+ are nonnegative multiples of fundamental dominant weights with at least one nonzero. Set $\lambda_i^+=m_i\varpi_{j_i}$, $i=1,\ldots,k$. We wish to show that there then must exist a λ_i^+ which is a positive multiple of a fundamental dominant weight contained in $X(T)$. If G is simply connected so that $X(T)=\Lambda$, this is clear. Assume then that G is not simply connected so that $X(T)$ is a proper sublattice of Λ.

Now if λ_i^+ are all positive multiples of fundamental dominant weights not in $X(T)$, then $m=\gcd\{m_{ij_i}\mid i=1,\ldots,k\}$ is divisible by $\gcd\{d_j\mid\varpi_j\notin X(T)\}$. The latter is greater than 1, since each $1\neq d_j$ must divide $[\Lambda:X(T)]$ which is a prime or at least a prime power. Then, by contradiction with Lemma 5.2(a), there must exist at least one λ_i^+ which is a positive multiple of a fundamental dominant weight in $X(T)$, say ϖ_j. Then, in this case, $\mathrm{rank}\,(Q)\geqslant[W:W_{\lambda_i}]=[W:W_{\varpi_j}]\geqslant\mathrm{rank}\,(P)$ by (i).

Case II*:* Suppose there exists at least one λ_i^+ which is a nontrivial linear combination of 2 or more fundamental dominant weights. Let $\lambda_i^+=\sum_{j=1}^n r_{ij}\varpi_j$. If there exists j such that $r_{ij}\neq 0$ and $j\notin I(X(T))$, then $\mathrm{rank}\,(Q)\geqslant[W:W_{\varpi_j}]\geqslant\mathrm{rank}\,(P)$. Otherwise, there exists $s, t \in I(X(T))$ with $r_{is} \neq 0$, $r_{it} \neq 0$. Then $\operatorname{rank}(Q) \geqslant [W : W_{\varpi_s+\varpi_t}] \geqslant \operatorname{rank}(P)$ by (ii).

Since an arbitrary element (Q, p) of $\mathcal{P}(X(T))$ must fall into one of these two cases, the rank of P is indeed minimal among those in $\mathcal{P}(X(T))$ so that $r(X(T)) = \operatorname{rank}(P)$. □

5.2. Minimal elements of $\mathcal{P}(X(T))$

For the remainder of this section we will determine minimal elements of $\mathcal{P}(X(T))$ and $r(X(T))$ for all simple groups G with maximal torus T. This will give us the rough upper bound of $r(X(T)) - \operatorname{rank}(X(T))$ for $\operatorname{ed}_W(X(T))$ in each case.

Proposition 5.6. *Suppose $G = \mathrm{SL}_{n+1}/C_d$ where d divides $n+1$.*

Let $T_{n+1,d}$ be the maximal torus of SL_{n+1}/C_d where d is a divisor of $n+1$. Let $W = W(\mathsf{A}_n)$ be the Weyl group. Note that $\mathrm{SL}_{n+1}/C_{n+1} = \mathrm{PGL}_{n+1}$ is the adjoint group of type A_n with $X(T_{n+1,n+1}) = \mathbb{Z}\mathsf{A}_n$, and $\mathrm{SL}_{n+1}/C_1 = \mathrm{SL}_{n+1}$ is the simply connected group of type A_n with $X(T_{n+1,1}) = \Lambda(\mathsf{A}_n)$. In each case, the element of $\mathcal{P}(X(T_{n+1,d}))$ listed below is minimal:

- *$r(\Lambda(\mathsf{A}_n)) = 2(n+1)$, $\operatorname{ed}_W(\Lambda(\mathsf{A}_n)) \leqslant n+2$, and*

$$\big(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}\big(\Lambda(\mathsf{A}_n)\big);$$

- *$r(\mathbb{Z}\mathsf{A}_1) = 4$, $\operatorname{ed}_W(\mathbb{Z}\mathsf{A}_1) \leqslant 3$, and $(\mathbb{Z}[W(\mathsf{A}_1)]^2, f_{\alpha_1,0}) \in \mathcal{P}(\mathbb{Z}\mathsf{A}_1)$;*
- *if $n \geqslant 2$, then $r(\mathbb{Z}\mathsf{A}_n) = n(n+1)$, $\operatorname{ed}_W(\mathbb{Z}\mathsf{A}_n) \leqslant n^2$ and*

$$\big(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-2})], f_{\varpi_1+\varpi_n}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$

- *if $(n, d) = (3, 2)$, then $r(X(T_{4,2})) = 10$, $\operatorname{ed}_W(X(T_{4,2})) \leqslant 7$ and*

$$\big(\mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_1) \times W(\mathsf{A}_1)] \oplus \mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_2)], f_{\varpi_2,0}\big) \in \mathcal{P}\big(X(T_{4,2})\big);$$

- *if $(n, d) = (2k+1, 2), k \geqslant 2$ or $(n, d) = (5, 3)$, then $r(X(T_{n+1,d})) = \binom{n+1}{d}$, $\operatorname{ed}_W(X(T_{n+1,d})) \leqslant \binom{n+1}{d} - n$ and*

$$\big(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{d-1}) \times W(\mathsf{A}_{n-d})], f_{\varpi_k}\big) \in \mathcal{P}\big(X(T_{n+1,d})\big);$$

- *if $n \geqslant 6$ and d is a proper divisor of $n+1$, then $r(X(T_{n+1,d})) = n(n+1)$, $\operatorname{ed}_W(X(T_{n+1,d})) \leqslant n^2$ and*

$$\big(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-2})], \pi\big) \in \mathcal{P}\big(X(T_{n+1,d})\big)$$

  where $\pi = f_{\varpi_1+k\varpi_2}$ if $d = 2k+1 \geqslant 3$ and $\pi = f_{(k-1)\varpi_1+\varpi_2}$ if $d = 2k > 2$.

Proof. The maximal parabolic subgroups of $W(\mathsf{A}_n) = S_{n+1}$ are $W_{\varpi_i} = W(\mathsf{A}_{n-i}) \times W(\mathsf{A}_{i-1})$ where $|W_{\varpi_i}| = (n+1-i)!i!$. So for all $i = 1, \dots, n$, $[W : W_{\varpi_i}] \geqslant n+1$. The minimum value $n+1$ is attained by ϖ_1 and ϖ_n and $W_{\varpi_1} = W_{\varpi_n} = W(\mathsf{A}_{n-1})$.

We first consider $\Lambda = \Lambda(\mathsf{A}_n)$. Note that $\Lambda(\mathsf{A}_n)/\mathbb{Z}\mathsf{A}_n \cong \mathbb{Z}/(n+1)\mathbb{Z}$ and $\varpi_k + \mathbb{Z}\mathsf{A}_n = k\varpi_1 + \mathbb{Z}\mathsf{A}_n$ for all $k = 1, \dots, n$. This shows that $\mathbb{Z}W\varpi_1 = \mathbb{Z}W\varpi_n = \Lambda$. Since $\operatorname{rank}(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^{W(\mathsf{A}_n)}) = 1 = \operatorname{rank}(\operatorname{Ker}(f_{\varpi_1})) = \operatorname{rank}(\operatorname{Ker}(f_{\varpi_n}))$, we see that $\mathrm{Ker}(f_{\varpi_1})$ (respectively $\mathrm{Ker}(f_{\varpi_n})$) is a trivial W-lattice and hence is not faithful. We wish to apply Proposition 5.3 to demonstate that $(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^2, f_{\varpi_1,0})$ (respectively $(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^2, f_{\varpi_n,0})$ is a minimal element of $\mathcal{P}(\Lambda)$.

Now, the intersection of nontrivial normal subgroups of S_n is S_2 if $n = 2$, A_n if $n \neq 2, 4$, and V_4 if $n = 4$. In all cases, it is nontrivial, verifying (i). Suppose $\mathbb{Z}[W/W_0]$ is a transitive permutation lattice of rank $r < n$. The action of W on $\mathbb{Z}[W/W_0]$ of rank $r < n$ induces a homomorphism $\varphi : S_n \to S_r$. Since φ cannot be injective, $\mathbb{Z}[W/W_0]$ cannot be faithful. So $\mathbb{Z}[W/W_{\varpi_1}]$ is a minimal faithful transitive permutation lattice verifying (ii).

Now suppose $\lambda \in \Lambda$ satisfies $\mathbb{Z}W\lambda = \Lambda$ and $\mathrm{Ker}(f_\lambda)$ is faithful. Then W_λ is a proper parabolic subgroup of S_{n+1}. If $W_\lambda = S_n$, then $\mathrm{Ker}(f_\lambda) \cong \mathbb{Z}$ is not faithful. So W_λ is a proper parabolic subgroup of S_{n+1} which is not S_n. So $n \geqslant 3$. For $n > 3$, any proper parabolic subgroup $W_\lambda \neq S_n$ would satisfy $[W : W_\lambda] \geqslant \frac{n(n+1)}{2} \geqslant 2(n+1) = 2[W : W_{\varpi_1}]$, verifying (iii). For $n = 2$, there is no $\lambda \in \Lambda$ such that $\mathbb{Z}W\lambda = \Lambda$ and $\mathrm{Ker}(f_\lambda)$ is faithful since any $\lambda \in \Lambda$ must be a multiple of ϖ_1. So (iii) is verified trivially in this case. Lastly, for $n = 3$, λ satisfying $\mathbb{Z}W\lambda = \Lambda$ and $\mathrm{Ker}(f_\lambda)$ faithful cannot have λ^+ be a multiple of a fundamental dominant weight since $\mathrm{Ker}(f_{\varpi_1})$ and $\mathrm{Ker}(f_{\varpi_3})$ are not faithful and $\mathbb{Z}W\varpi_2 \neq \Lambda$. So W_λ has rank at most 1, and so $[W : W_\lambda] \geqslant 12 \geqslant 2[W : W_{\varpi_1}]$, verifying (iii) for $n = 3$. Hence, by Proposition 5.3, $(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^2, f_{\varpi_1,0})$ is a minimal element of $\mathcal{P}(\Lambda)$ and $r(\Lambda) = 2(n + 1)$.

Note that $\mathbb{Z}\mathsf{A}_1 = \mathbb{Z}\alpha_1 = \mathbb{Z}2\varpi_1 = 2\Lambda(\mathsf{A}_1)$. Then the argument for $\Lambda(\mathsf{A}_1)$ shows also that $r(\mathbb{Z}\mathsf{A}_1) = 4$ and $(\mathbb{Z}[W(\mathsf{A}_1)]^2, f_{\alpha_1,0}) \in \mathcal{P}(\mathbb{Z}\mathsf{A}_1)$ is a minimal element.

We next consider the nonsimply connected cases for $n \geqslant 2$. So, $d > 1$. Note that $\Lambda/X(T_{n+1,d}) \cong \mathbb{Z}/d\mathbb{Z}$ and $\varpi_k + X(T_{n+1,d}) = k\varpi_1 + X(T_{n+1,d})$ for all $k = 1, \ldots, n$. Note that $\varpi_k \in X(T_{n+1,d})$ if and only if d divides k. This shows that no fundamental dominant weights lie in $\mathbb{Z}\mathsf{A}_n = X(T_{n+1,n+1})$.

Let d_k be the order of $\varpi_k + X(T_{n+1,d})$ in $\Lambda/X(T_{n+1,d})$. Note that $d_1 = d_n = [\Lambda : X(T_{n+1,d})] = d > 1$. Suppose that $(Q, p) \in \mathcal{P}(X(T_{n+1,d}))$ with $Q = \oplus_{i=1}^k \mathbb{Z}[W/W_i]$ and $p = f_{\lambda_1,\ldots,\lambda_k}$. Suppose each λ_i^+ is a multiple of some fundamental dominant weight not in $\mathbb{Z}\mathsf{A}_n$, i.e., $\lambda_i^+ = m_i\varpi_{j_i}$. Then by Lemma 5.2(a), we know that $\gcd\{d_{j_i} \mid i = 1, \ldots, k\} = 1$. Since all the $d_{j_i} > 1$, we see that Q cannot be transitive. Also since all d_{j_i} divide $n + 1$ and $d_1 = d_n = n + 1$, we see that there must be at least two λ_i^+ with $j_i \in \{2, \ldots, n - 1\}$. But then $\mathrm{rank}\,(Q) \geqslant 2\min\{[W : W_{\varpi_j}] \mid j = 2, \ldots, n - 1\}$. Since $|W_{\varpi_i}| = i!(n + 1 - i)!$, this implies that $\mathrm{rank}\,(Q) \geqslant n(n + 1)$. Now suppose that there exists λ_i such that $\lambda_i^+ = \sum_{j=1}^k c_{ij}\varpi_j$ has at least two nonzero coefficients, say c_{ip} and c_{iq}. Since $W_{c_{ip}\varpi_p + c_{iq}\varpi_q} = W(\mathsf{A}_{p-1}) \times W(\mathsf{A}_{q-p-1}) \times W(\mathsf{A}_{n-q})$, we see that $\mathrm{rank}\,(Q) \geqslant n(n + 1)$ also in this case.

For the case of $\mathbb{Z}\mathsf{A}_n$, the above discussion is sufficient to conclude that $r(\mathbb{Z}\mathsf{A}_n) \geqslant n(n+1)$. Note that $\mathbb{Z}W(\varpi_1 + \varpi_n) = \mathbb{Z}\mathsf{A}_n$ and $[W : W_{\varpi_1+\varpi_n}] = n(n+1) > 2n$. Hence, $(\mathbb{Z}[W/W_{\varpi_1+\varpi_n}], f_{\varpi_1+\varpi_n}) \in \mathcal{P}(\mathbb{Z}\mathsf{A}_n)$ and $r(\mathbb{Z}\mathsf{A}_n) = n(n + 1)$.

If d is a proper nontrivial divisor of $n + 1$, we must also consider the case of $(Q, p) \in \mathcal{P}(X(T_{n+1,d}))$ as above with at least one $\lambda_i^+ = m\varpi_j$ where $\varpi_j \in X(T_{n+1,d})$. (Note that this cannot occur for $\mathbb{Z}\mathsf{A}_n$). Then $\mathrm{rank}\,(Q) \geqslant \min\{[W : W_{\varpi_k}] \mid \varpi_k \in X(T_{n+1,d})\} = [W : W_{\varpi_d}] = \binom{n+1}{d}$. Note that $\mathbb{Z}W\varpi_d = X(T_{n+1,d})$.

Now if $d = 2$ and $n = 2k + 1 > 3$, or if $(n, d) = (5, 3)$, then $2n < \binom{n+1}{d} < n(n + 1)$ so that $r(X(T_{n+1,d})) = \binom{n+1}{d}$ and $(\mathbb{Z}[W/W_{\varpi_d}], f_{\varpi_d}) \in \mathcal{P}(X(T_{n+1,d}))$ is minimal.

If $(n,d) = (3,2)$, then $V = \langle (12)(34), (13)(24) \rangle$ is a normal subgroup of $W(\mathsf{A}_3) = S_4$. By the proof of Proposition 3.3 (c),

$$\begin{aligned}\operatorname{rank}\big(\mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_1)\times W(\mathsf{A}_1)]^V\big) &= \frac{|W(\mathsf{A}_3)||V\cap W(\mathsf{A}_1)\times W(\mathsf{A}_1)|}{|W(\mathsf{A}_1)\times W(\mathsf{A}_1)||V|} = 3\\ &= \operatorname{rank}\big(\operatorname{Ker}(f_{\varpi_2})\big),\end{aligned}$$

and hence $\operatorname{Ker}(f_{\varpi_2})$ is not a faithful W-lattice by Proposition 3.3 (a). But

$$(\mathbb{Z}[W/W_{\varpi_2}] \oplus \mathbb{Z}[W/W_{\varpi_1}], f_{\varpi_2,0})$$

gives an element of $\mathcal{P}(X(T_2))$ of rank 10 since $\mathbb{Z}[W/W_{\varpi_1}] = \mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_2)]$ is faithful. By the above discussion, if $Q = \oplus_{i=1}^k \mathbb{Z}[W/W_i]$ and $(Q, f_{\lambda_1,\dots,\lambda_k}) \in \mathcal{P}(X(T_{4,2}))$ had rank smaller than 10, then Q cannot be transitive, and at least one λ_i^+ must be a positive multiple of a fundamental dominant weight contained in $X(T_{4,2})$ (i.e., ϖ_2). As all nontrivial normal subgroups intersect in V, we cannot add on a nonfaithful permutation lattice onto $\mathbb{Z}[W/W_{\varpi_2}]$. So since $\mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_2)]$ is the smallest faithful permutation lattice for $W(\mathsf{A}_3)$, we must have that $(\mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_1)\times W(\mathsf{A}_1)] \oplus \mathbb{Z}[W(\mathsf{A}_3)/W(\mathsf{A}_2)], f_{\varpi_2,0})$ is a minimal element of $\mathcal{P}(X(T_{4,2}))$ and $r(X(T_{4,2})) = 10$.

If $n \geqslant 6$ and $d > 2$ is a proper divisor of $n+1$, then $2n < n(n+1) < \binom{n+1}{d}$. Since $\mathbb{Z}W(\varpi_1 + k\varpi_2) = X(T_{n+1,2k+1})$ and $\mathbb{Z}W(2(k-1)\varpi_1 + \varpi_2) = X(T_{n+1,2k})$, we find that $r(X(T_{n+1,d})) = n(n+1)$ in this case and that $(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-2})], f_{\varpi_1+k\varpi_2}) \in \mathcal{P}(X(T_{n+1,2k+1}))$ and $(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-2})], f_{(2k-1)\varpi_1+\varpi_2}) \in \mathcal{P}(X(T_{n+1,2k}))$ are minimal elements. □

Proposition 5.7. *Let $G = \mathrm{SO}_k$, $k \geqslant 5$. Let T_k be its maximal torus. If $k = 2n+1$, $n \geqslant 2$, then $W = W(\mathsf{B}_n) = \mathsf{C}_2^n \rtimes S_n$ and $X(T_{2n+1}) = \mathbb{Z}\mathsf{B}_n = \oplus_{i=1}^n \mathbb{Z}e_i$ on which $W(\mathsf{B}_n)$ acts by signed permutations. If $k = 2n$, $n \geqslant 4$, then $W = W(\mathsf{D}_n) = C_2^{n-1} \rtimes S_n$ and $X(T_{2n}) = \mathbb{Z}\mathsf{B}_n|_{W(\mathsf{D}_n)}$ is a lattice properly between $\mathbb{Z}\mathsf{D}_n$ and $\Lambda(\mathsf{D}_n)$ on which $W(\mathsf{D}_n)$ acts by even signed permutations. Then:*

- *$r(\mathbb{Z}\mathsf{B}_n) = r(X(T_{2n+1})) = 4n$, $\mathrm{ed}_W(\mathbb{Z}\mathsf{B}_n) \leqslant 3n$, and*

$$\big(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}(\mathbb{Z}\mathsf{B}_n)$$

is minimal;

- *$r(X(T_{2n})) = 4n$, $\mathrm{ed}_W(X(T_{2n})) \leqslant 3n$ and*

$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}\big(X(T_{2n})\big)$$

is minimal.

Proof. We will use Proposition 5.3 to prove the statement for $X(T_{2n}), n \geqslant 4$. Note that the intersection of nontrivial normal subgroups of $W(\mathsf{D}_n)$ is $(C_2)^{n-1}$ if n odd and the diagonal subgroup of C_2^n if n even so that in either case it is nontrivial . We claim that $\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]$ is a minimal faithful transitive permutation lattice for $W(\mathsf{D}_n)$. Suppose $\mathbb{Z}[W/W_0]$ were a permutation lattice of rank r smaller than $2n$. Then the action of W on $\mathbb{Z}[W/W_0]$ induces a homomorphism $\varphi : W(\mathsf{D}_n) \to S_r$. Then the kernel of the action of W on $\mathbb{Z}[W/W_0]$ is nontrivial. So $\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]$ is a minimal faithful transitive permutation lattice.

Note that $\varpi_1 = e_1$, $\mathbb{Z}W\varpi_1 = X(T_{2n})$ and $W_{\varpi_1} = W(\mathsf{D}_{n-1})$. Since

$$\operatorname{rank}\left(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]^{C_2^{n-1}}\right) = n = \operatorname{rank}\left(\operatorname{Ker}(f_{\varpi_1})\right),$$

we see that by the proof of Proposition 3.3(a), the normal subgroup C_2^{n-1} fixes $\operatorname{Ker}(f_{\varpi_1})$, so $\operatorname{Ker}(f_{\varpi_1})$ is not faithful.

Suppose $\lambda \in X(T_{2n})$ is such that $\mathbb{Z}W\lambda = X(T_{2n})$ and $\operatorname{Ker}(f_\lambda)$ is faithful. Then $\lambda^+ = \sum_{i=1}^n m_i\varpi_i$ cannot be a multiple of ϖ_1 since otherwise $W_\lambda = W(\mathsf{D}_{n-1})$ and $\operatorname{Ker}(f_\lambda)$ is fixed by C_2^{n-1} by the argument above. So W_λ must be a proper parabolic subgroup not equal to $W(\mathsf{D}_{n-1})$. This means that $[W : W_\lambda] \geqslant 2n(n-1) > 4n$.

So by Proposition 5.3, $r(X(T_{2n}))=4n$ and $(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]^2, f_{e_1,0}) \in \mathcal{P}(X(T_{2n}))$ is a minimal element.

Since $\mathbb{Z}W(\mathsf{B}_n)e_1 = \mathbb{Z}\mathsf{B}_n$ and $\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]$ is a faithful permutation lattice, $(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]^2, f_{e_1,0}) \in \mathcal{P}(\mathbb{Z}\mathsf{B}_n)$ with rank $4n$.

Now for $n \geqslant 4$, since $W(\mathsf{D}_n)$ is a normal subgroup of $W(\mathsf{B}_n)$ and $\mathbb{Z}\mathsf{B}_n|_{W(\mathsf{D}_n)} = X(T_{2n})$, we have $r(\mathbb{Z}\mathsf{B}_n) \geqslant r(X(T_{2n})) = 4n$ by Lemma 5.4. This shows that the element $(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]^2, f_{e_1,0})$ is minimal and that $r(\mathbb{Z}\mathsf{B}_n) = 4n$ for $n \geqslant 4$. For $n = 2$, $\mathbb{Z}\mathsf{B}_2|_{W(\mathsf{A}_1)^2} = (\mathbb{Z}\mathsf{A}_1)^2$. Since

$$\left(\mathbb{Z}[W(\mathsf{A}_1)^2/W_{e_1}] \oplus \mathbb{Z}[W(\mathsf{A}_1)^2/W_{e_2}] \oplus \mathbb{Z}[W(\mathsf{A}_1)^2]\right) \in \mathcal{P}\left((\mathbb{Z}\mathsf{A}_1)^2\right)$$

is minimal of rank 8, then $r(\mathbb{Z}\mathsf{B}_2) \geqslant 8$ by Lemma 5.4. For $n = 3$, $X(T_6) = \mathbb{Z}\mathsf{B}_3|_{W(\mathsf{A}_3)}$ is a lattice between the root and weight lattice of A_3 where $W(\mathsf{A}_3) = W(\mathsf{D}_3) = S_4$. Note that S_4 acts on $X(T_6) = \oplus_{i=1}^3 \mathbb{Z}e_i$ by permutations via $S_4/V_4 \cong S_3$. Although $\mathbb{Z}[S_4/V_4]$ is not faithful since it is fixed by V_4,

$$\left(\mathbb{Z}[S_4/\langle(12)(34)\rangle], f_{e_1}\right) \in \mathcal{P}\left(X(T_6)\right)$$

is minimal of rank 12 so that $r(\mathbb{Z}\mathsf{B}_3) \geqslant 12$ by Lemma 5.4. So, for $n \geqslant 2$, we see that $(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]^2, f_{e_1,0}) \in \mathcal{P}(\mathbb{Z}\mathsf{B}_n)$ is a minimal element and $r(\mathbb{Z}\mathsf{B}_n) = 4n$. □

The following proposition covers the adjoint case. Note that we obtained better bounds on $\operatorname{ed}_W(\mathbb{Z}\Phi)$ than $r(\mathbb{Z}\Phi) - \operatorname{rank}(\mathbb{Z}\Phi)$ in Theorem 1.3 using compressions.

Proposition 5.8. *Suppose G is adjoint. In each case, the element of $\mathcal{P}(\mathbb{Z}\Phi)$ listed is minimal:*

- *if $\Phi = \mathsf{A}_1$, then $r(\mathbb{Z}\Phi) = 4$ and*

$$\left(\mathbb{Z}[W(\mathsf{A}_1)]^2, f_{\alpha_1,0}\right) \in \mathcal{P}(\mathbb{Z}\Phi);$$

- *if $\Phi = \mathsf{A}_n$, $n \geqslant 2$, then $r(\mathbb{Z}\Phi) = n(n+1)$ and*

$$\left(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-2})], f_{\varpi_1+\varpi_n}\right) \in \mathcal{P}(\mathbb{Z}\Phi);$$

- *if $\Phi = \mathsf{B}_n$, $n \geqslant 2$, then $r(\mathbb{Z}\Phi) = 4n$ and*

$$\left(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{B}_{n-1})]^2, f_{\varpi_1,0}\right) \in \mathcal{P}(\mathbb{Z}\Phi);$$

- *if* $\Phi = \mathsf{C}_n$, $n \geqslant 3$, *then* $r(\mathbb{Z}\Phi) = 2n(n-1)$ *and*
$$\big(\mathbb{Z}[W(\mathsf{C}_n)/W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})], f_{\varpi_2}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *If* $\Phi = \mathsf{D}_n$, $n \geqslant 4$, *then* $r(\mathbb{Z}\Phi) = 2n(n-1)$ *and*
$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})], f_{\varpi_2}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *if* $\Phi = \mathsf{E}_6$, *then* $r(\mathbb{Z}\Phi) = 72$ *and*
$$\big(\mathbb{Z}[W(\mathsf{E}_6)/W(\mathsf{A}_5)], f_{\varpi_2}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *if* $\Phi = \mathsf{E}_7$, *then* $r(\mathbb{Z}\Phi) = 126$ *and*
$$\big(\mathbb{Z}[W(\mathsf{E}_7)/W(\mathsf{D}_6)], f_{\varpi_1}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *if* $\Phi = \mathsf{E}_8$, *then* $r(\mathbb{Z}\Phi) = 240$ *and*
$$\big(\mathbb{Z}[W(\mathsf{E}_8)/W(\mathsf{E}_7)], f_{\varpi_8}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *if* $\Phi = \mathsf{F}_4$, *then* $r(\mathbb{Z}\Phi) = 24$ *and*
$$\big(\mathbb{Z}[W(\mathsf{F}_4)/W(\mathsf{B}_3)], f_{\varpi_4}\big) \in \mathcal{P}(\mathbb{Z}\Phi);$$
- *if* $\Phi = \mathsf{G}_2$, *then* $r(\mathbb{Z}\Phi) = 6$ *and*
$$\big(\mathbb{Z}[W(\mathsf{G}_2)/W(\mathsf{A}_1)], f_{\varpi_1}\big) \in \mathcal{P}(\mathbb{Z}\Phi).$$

Proof. Note that $\Phi = \mathsf{A}_n$ was treated in Proposition 5.6 and $\Phi = \mathsf{B}_n$ was covered in Proposition 5.7.

For the remaining cases, we may apply Proposition 5.5. To do so, we find a fundamental dominant weight $\varpi_i \in \mathbb{Z}\Phi$ for which $[W : W_{\varpi_i}]$ is minimal and for which $\mathbb{Z}W\varpi_i = \mathbb{Z}\Phi$. Then we verify the three hypotheses of Proposition 5.5.

Let $\Phi = \mathsf{D}_n$, $n \geqslant 4$. The fundamental dominant weights contained in $\mathbb{Z}\mathsf{D}_n$ are ϖ_{2k} for $1 \leqslant k < (n-1)/2$ and $W_{\varpi_i} = W(\mathsf{A}_{i-1}) \times W(\mathsf{D}_{n-i})$. So the minimal value of $[W : W_{\varpi_i}]$ for $\varpi_i \in \mathbb{Z}\mathsf{D}_n$ is $2n(n-1)$ and is achieved by ϖ_2. Only ϖ_1 has $[W : W_{\varpi_1}] < 2n(n-1)$, so we need not check condition (ii). Now $W_{\varpi_2} = W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})$ and $[W(\mathsf{D}_n) : W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})] = 2n(n-1) > 2n$, and we see that $(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_1) \times W(\mathsf{D}_{n-2})], f_{\varpi_2}) \in \mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\mathsf{D}_n) = 2n(n-1)$.

Let $\Phi = \mathsf{C}_n$. For $n \geqslant 4$, $\mathbb{Z}\mathsf{C}_n|_{W(\mathsf{D}_n)} = \mathbb{Z}\mathsf{D}_n$, and for $n = 3$, $\mathbb{Z}\mathsf{C}_n|_{W(\mathsf{A}_3)} \cong \mathbb{Z}\mathsf{A}_3$ where $\mathsf{A}_3 = \mathsf{D}_3$ has base
$$\varepsilon_2 + \varepsilon_3,\ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3.$$
In either case, Lemma 5.4 shows that $r(\mathbb{Z}\mathsf{C}_n) \geqslant 2n(n-1)$. Since $\mathbb{Z}W\varpi_2 = \mathbb{Z}\mathsf{C}_n$, $W_{\varpi_2} = W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})$ and $[W(\mathsf{C}_n) : W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})] = 2n(n-1) > 2n$ if $n \geqslant 3$, we see that $(\mathbb{Z}[W(\mathsf{C}_n)/W(\mathsf{A}_1) \times W(\mathsf{C}_{n-2})], f_{\varpi_2}) \in \mathcal{P}(\mathbb{Z}\Phi)$ is minimal and $r(\mathbb{Z}\mathsf{C}_n) = 2n(n-1)$.

Let $\Phi = \mathsf{E}_6$. Note that $\Lambda/\mathbb{Z}\Phi = \mathbb{Z}/3\mathbb{Z}$ and ϖ_2, ϖ_4 are the only fundamental dominant weights in $\mathbb{Z}\Phi$. Since $W_{\varpi_2} = W(\mathsf{A}_5)$ and $W_{\varpi_4} = W(\mathsf{A}_2) \times W(\mathsf{A}_2) \times W(\mathsf{A}_1)$, the minimal value of $[W : W_{\varpi_i}]$ for $\varpi_i \in \mathbb{Z}\mathsf{E}_6$ is 72 and is achieved by ϖ_2. The only ϖ_i with $[W : W_{\varpi_i}] < 72$ are ϖ_1 and ϖ_6. But $W(\mathsf{D}_4) = W_{\varpi_1+\varpi_6}$ has rank $270 > 72$. Since $\mathbb{Z}W\varpi_2 = \mathbb{Z}\Phi$ and $[W : W_{\varpi_2}] = 72 > 2(n) = 12$, we see that $r(\mathbb{Z}\Phi) = 72$ and $(\mathbb{Z}[W(\mathsf{E}_6)/W(\mathsf{A}_5)], f_{\varpi_2}) \in \mathcal{P}(\mathbb{Z}\Phi)$.

Let $\Phi = \mathsf{E}_7$. Note that $\Lambda/\mathbb{Z}\Phi = \mathbb{Z}/2\mathbb{Z}$ and $\varpi_1, \varpi_3, \varpi_4, \varpi_6$ are the only fundamental dominant weights contained in $\mathbb{Z}\Phi$. Since $W_{\varpi_1} = W(\mathsf{D}_6)$, $W_{\varpi_3} = W(\mathsf{A}_1) \times W(\mathsf{A}_5)$, $W_{\varpi_4} = W(\mathsf{A}_2) \times W(\mathsf{A}_1) \times W(\mathsf{A}_3)$ and $W_{\varpi_6} = W(\mathsf{A}_1) \times W(\mathsf{D}_5)$, we see that the minimal value of $[W : W_{\varpi_i}]$ for $\varpi_i \in \mathbb{Z}\mathsf{E}_6$ is 126 and is achieved by ϖ_1. Since ϖ_7 is the only ϖ_i with $[W : W_{\varpi_i}] < 126$, $[W(\mathsf{E}_7) : W(\mathsf{D}_6)] = 126 > 2n = 14$, and $\mathbb{Z}W\varpi_1 = \mathbb{Z}\Phi$ and $W_{\varpi_1} = W(\mathsf{D}_6)$, we see by Proposition 5.5 that $r(\mathbb{Z}\Phi) = 126$ and $(\mathbb{Z}[W(\mathsf{E}_7)/W(\mathsf{D}_6)], f_{\varpi_1}) \in \mathcal{P}(\mathbb{Z}\Phi)$.

For the remaining cases, G is also simply connected so that $\Lambda = \mathbb{Z}\Phi$. Note that hypothesis (ii) of Proposition 5.5 is automatically satisfied in these cases.

For $\Phi = \mathsf{E}_8$, the maximal parabolic subgroups are $W_{\varpi_8} = W(\mathsf{E}_7)$, $W_{\varpi_1} = W(\mathsf{D}_7)$, $W_{\varpi_2} = W(\mathsf{A}_7)$, $W_{\varpi_7} = W(\mathsf{E}_6) \times W(\mathsf{A}_1)$, $W_{\varpi_3} = W(\mathsf{A}_6) \times W(\mathsf{A}_1)$, $W_{\varpi_6} = W(\mathsf{D}_5) \times W(\mathsf{A}_2)$, $W_{\varpi_5} = W(\mathsf{A}_4) \times W(\mathsf{A}_3)$ and $W_{\varpi_4} = W(\mathsf{A}_4) \times W(\mathsf{A}_2) \times W(\mathsf{A}_1)$. In this case, $\mathbb{Z}W\varpi_i = \mathbb{Z}\Phi = \Lambda$ for all $i = 1, \ldots, 8$. By Proposition 5.5, $(\mathbb{Z}[W(\mathsf{E}_8)/W(\mathsf{E}_7)], f_{\varpi_8})$ is a minimal element of $\mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\Phi) = 240$.

For $\Phi = \mathsf{F}_4$, the maximal parabolic subgroups are $W_{\varpi_4} = W(\mathsf{B}_3)$, $W_{\varpi_1} = W(\mathsf{C}_3)$, $W_{\varpi_2} = W_{\varpi_3} = W(\mathsf{A}_2) \times W(\mathsf{A}_1)$. ϖ_4 is the only fundamental dominant weight satisfying $\mathbb{Z}W\varpi_4 = \mathbb{Z}\Phi = \Lambda$. By Proposition 5.5, $(\mathbb{Z}[W(\mathsf{F}_4)/W(\mathsf{B}_3)], f_{\varpi_4})$ is a minimal element of $\mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\Phi) = 24$.

For $\Phi = \mathsf{G}_2$, the maximal parabolic subgroups are $W_{\varpi_1} = W_{\varpi_2} = W(\mathsf{A}_1)$. ϖ_1 is the only fundamental dominant weight satisfying $\mathbb{Z}W\varpi_1 = \mathbb{Z}\Phi = \Lambda$. By Proposition 5.5, $(\mathbb{Z}[W(\mathsf{G}_2)/W(\mathsf{A}_1)], f_{\varpi_1})$ is a minimal element of $\mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\Phi) = 6$. □

The following corollary summarizes the results of Proposition 5.8 where

$$P(\Phi) = \oplus_{\alpha \in \Phi_0} \mathbb{Z}e_\alpha \cong \mathbb{Z}[W/W_{\tilde{\alpha}}].$$

Corollary 5.9. *Let $\pi(\Phi) : P(\Phi) \to \mathbb{Z}\Phi, e_\alpha \mapsto \alpha$. If $\Phi \neq \mathsf{A}_1, \mathsf{B}_n$, then $\big(P(\Phi), \pi(\Phi)\big) \in \mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\Phi) = |\Phi_0|$. If $\Phi = \mathsf{A}_1, \mathsf{B}_n$, then $\big(P(\Phi)^2, (\pi(\Phi), 0)\big) \in \mathcal{P}(\mathbb{Z}\Phi)$ and $r(\mathbb{Z}\Phi) = 2|\Phi_0|$.*

Proposition 5.10. *Let G be simply connected. In each case below, the element of $\mathcal{P}(\Lambda)$ listed is minimal:*

- *if $\Phi = \mathsf{A}_n$, then $r(\Lambda) = 2(n+1)$, $\mathrm{ed}_W(\Lambda) \leqslant n+2$ and*

$$\big(\mathbb{Z}[W(\mathsf{A}_n)/W(\mathsf{A}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}(\Lambda);$$

- *if $\Phi = \mathsf{B}_2$, then $r(\Lambda) = 8$, $\mathrm{ed}_W(\Lambda) \leqslant 6$ and*

$$\big(\mathbb{Z}[W(\mathsf{B}_2)/W(\mathsf{A}_1)]^2, f_{\varpi_2,0}\big) \in \mathcal{P}(\Lambda);$$

- *if $\Phi = \mathsf{B}_n$, where $n \geqslant 3$, then $r(\Lambda) = 2^n$, $\mathrm{ed}_W(\Lambda) \leqslant 2^n - n$ and*

$$\big(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_n}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{C}_n$, $n \geqslant 3$, *then* $r(\Lambda) = 4n$, $\mathrm{ed}_W(\Lambda) \leqslant 3n$ *and*

$$\big(\mathbb{Z}[W(\mathsf{C}_n)/W(\mathsf{C}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{D}_n$, *then* $r(\Lambda) = 2^{n-1}$, $\mathrm{ed}_W(\Lambda) \leqslant 2^{n-1} - n$ *and*

$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_n}\big) \in \mathcal{P}(\Lambda)$$

for odd n, *and* $r(\Lambda) = 2^{n-1} + 2n$, $\mathrm{ed}_W(\Lambda) \leqslant 2^{n-1} + n$ *and*

$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1}] \oplus \mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_1,\varpi_n}\big) \in \mathcal{P}(\Lambda)$$

for even n*;*
- *if* $\Phi = \mathsf{E}_6$, *then* $r(\Lambda) = 27$, $\mathrm{ed}_W(\Lambda) \leqslant 21$ *and*

$$\big(\mathbb{Z}[W(\mathsf{E}_6)/W(\mathsf{D}_5)], f_{\varpi_6}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{E}_7$, *then* $r(\Lambda) = 56$, $\mathrm{ed}_W(\Lambda) \leqslant 49$ *and*

$$\big(\mathbb{Z}[W(\mathsf{E}_7)/W(\mathsf{E}_6)], f_{\varpi_7}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{E}_8$, *then* $r(\Lambda) = 240$, $\mathrm{ed}_W(\Lambda) \leqslant 232$ *and*

$$\big(\mathbb{Z}[W(\mathsf{E}_8)/W(\mathsf{E}_7)], f_{\varpi_8}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{F}_4$, *then* $r(\Lambda) = 24$, $\mathrm{ed}_W(\Lambda) \leqslant 20$ *and*

$$\big(\mathbb{Z}[W(\mathsf{F}_4)/W(\mathsf{B}_3)], f_{\varpi_4}\big) \in \mathcal{P}(\Lambda);$$

- *if* $\Phi = \mathsf{G}_2$, *then* $r(\Lambda) = 6$, $\mathrm{ed}_W(\Lambda) \leqslant 4$ *and*

$$\big(\mathbb{Z}[W(\mathsf{G}_2)/W(\mathsf{A}_1)], f_{\varpi_1}\big) \in \mathcal{P}(\Lambda).$$

Proof. In most cases, Proposition 5.5 applies. We note that if there exists ϖ_i with $[W : W_{\varpi_i}]$ minimal but larger than $2n$ and such that $\mathbb{Z}W\varpi_i = \Lambda$, then $(\mathbb{Z}[W/W_{\varpi_i}], f_{\varpi_i})$ is a minimal element of $\mathcal{P}(X(T))$ and $r(\Lambda) = [W : W_{\varpi_i}]$. Note that condition (ii) of Proposition 5.5 is automatically satisfied in this case.

We have already dealt with $\Phi = \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$ since the adjoint groups are also simply connected in these cases so that $\Lambda = \mathbb{Z}\Phi$. We have also already covered $\Phi = \mathsf{A}_n$ in Proposition 5.6.

For $\Phi = \mathsf{E}_6$, the maximal parabolic subgroups are $W_{\varpi_1} = W_{\varpi_6} = W(\mathsf{D}_5)$, $W_{\varpi_2} = W(\mathsf{A}_5)$, $W_{\varpi_3} = W_{\varpi_5} = W(\mathsf{A}_4) \times W(\mathsf{A}_1)$ and $W_{\varpi_4} = W(\mathsf{A}_2) \times W(\mathsf{A}_2) \times W(\mathsf{A}_1)$. $\varpi_1, \varpi_3, \varpi_5, \varpi_6$ all satisfy $\mathbb{Z}W\varpi_i = \Lambda$. By Proposition 5.5, $(\mathbb{Z}[W(\mathsf{E}_6)/W(\mathsf{D}_5)], f_{\varpi_1})$ and $(\mathbb{Z}[W(\mathsf{E}_6)/W(\mathsf{D}_5)], f_{\varpi_6})$ are both minimal elements of $\mathcal{P}(\Lambda)$ so that $r(\Lambda) = 27$.

For $\Phi = \mathsf{E}_7$, the maximal parabolic subgroups are $W_{\varpi_7} = W(\mathsf{E}_6)$, $W_{\varpi_1} = W(\mathsf{D}_6)$, $W_{\varpi_2} = W(\mathsf{A}_6)$, $W_{\varpi_6} = W(\mathsf{D}_5) \times W(\mathsf{A}_1)$, $W_{\varpi_3} = W(\mathsf{A}_5) \times W(\mathsf{A}_1)$, $W_{\varpi_5} = W(\mathsf{A}_4) \times W(\mathsf{A}_2)$ and $W_{\varpi_4} = W(\mathsf{A}_3) \times W(\mathsf{A}_2) \times W(\mathsf{A}_1)$. $\varpi_2, \varpi_5, \varpi_7$ all satisfy $\mathbb{Z}W\varpi_i = \Lambda$. By Proposition 5.5, $(\mathbb{Z}[W(\mathsf{E}_7)/W(\mathsf{E}_6)], f_{\varpi_7})$ is a minimal element of $\mathcal{P}(\Lambda)$ and $r(\Lambda) = 56$.

Let $\Phi = \mathsf{B}_2$. In this case, $W_{\varpi_i} = W(\mathsf{A}_1)$ and $[W : W_{\varpi_i}] = 4 = 2n$ for $i = 1, 2$. However, only ϖ_2 satisfies $\mathbb{Z}W\varpi_2 = \Lambda$. Since for the normal subgroup C_2^2 of $W(\mathsf{B}_2)$ we have $\operatorname{rank}(\mathbb{Z}[W(\mathsf{B}_2)/W(\mathsf{A}_1)]^{C_2^2}) = 2 = \operatorname{rank}(\operatorname{Ker}(f_{\varpi_2})$, we see that $\operatorname{Ker}(f_{\varpi_2})$ is not a faithful W-lattice. Since $\mathbb{Z}[W(\mathsf{B}_2)/W(\mathsf{A}_1)]$ is faithful, we see that $(\mathbb{Z}[W(\mathsf{B}_2)/W(\mathsf{A}_1)]^2, f_{\varpi_2,0}) \in \mathcal{P}(\Lambda)$ and has rank 8. We can apply Proposition 5.3 to show that this element is minimal. Indeed, since a subgroup of $W(\mathsf{B}_2)$ of index $r < 2n$ would induce a homomorphism $W(\mathsf{B}_2) \to S_r$, we see that $\mathbb{Z}[W/W_{\varpi_2}]$ is a minimal transitive permutation lattice so that condition (ii) of Proposition 5.3 is verified. Condition (i) follows as the intersection of nontrivial normal subgroups in $W(\mathsf{B}_2)$ is nontrivial. Now let $(\mathbb{Z}[W/W_\lambda], f_\lambda) \in \mathcal{P}(\Lambda)$. By the discussion above, λ^+ cannot be a multiple of a fundamental dominant weight, so $W_\lambda = 1$ and $[W : W_\lambda] = 8$, verifying condition (iii). So Proposition 5.3 shows that the element above is minimal and $r(\Lambda) = 8$.

Let $\Phi = \mathsf{B}_n$ with $n > 2$. For this case, Proposition 5.5 does not apply. This is because the fundamental dominant weight ϖ_1, which attains the minimum value of $[W : W_{\varpi_i}]$, does not satisfy $\mathbb{Z}W\varpi_i = \Lambda$. Suppose $(Q, p) \in \mathcal{P}(\Lambda)$ where $Q = \oplus_{i=1}^k \mathbb{Z}[W/W_i]$ and $p = f_{\lambda_1,\dots,\lambda_k}$. Assume that $\lambda_i^+ \in \operatorname{Span}_\mathbb{Z}\{\varpi_j \mid j = 1, \dots, n-1\}$ for all $i = 1, \dots, k$. But then $\sum_{i=1}^k \mathbb{Z}W\lambda_i \subset \sum_{i=1}^k \mathbb{Z}\lambda_i^+ + \mathbb{Z}\Phi \subset \sum_{i=1}^{n-1} \mathbb{Z}\varpi_i + \mathbb{Z}\Phi = \mathbb{Z}\Phi$. Since $\Lambda/\mathbb{Z}\Phi \cong \mathbb{Z}/2\mathbb{Z}$, this contradicts the surjectivity of the map p. So if $\lambda_i^+ = \sum_{j=1}^n m_{ij}\varpi_j$, there must exist i such that $m_{in} \neq 0$ so that $W_{\lambda_i^+} \leqslant W_{\varpi_n}$. Then $\operatorname{rank}(Q) \geqslant \sum_{i=1}^k [W : W_{\lambda_i}] \geqslant [W : W_{\varpi_n}]$. Since $\mathbb{Z}W\varpi_n = \Lambda$, $W_{\varpi_n} = W(\mathsf{A}_{n-1})$ and $[W : W_{\varpi_n}] = 2^n > 2n$, $(\mathbb{Z}[W(\mathsf{B}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_n}) \in \mathcal{P}(\Lambda)$ and $r(\Lambda) = 2^n$.

Let $\Phi = \mathsf{C}_n$. Note that $W(\mathsf{C}_n) = W(\mathsf{B}_n)$ and $\Lambda(\mathsf{C}_n) \cong \mathbb{Z}\mathsf{B}_n$. In both cases, $\varpi_1 = e_1$. So $(\mathbb{Z}[W(\mathsf{C}_n)/W(\mathsf{C}_{n-1})]^2, f_{\varpi_1,0})$ is a minimal element of $\mathcal{P}(\Lambda)$ and $r(\Lambda) = 4n$.

Let $\Phi = \mathsf{D}_n$. Once again, Proposition 5.5 does not apply, as ϖ_1 is the fundamental dominant weight which attains the minimum value of $[W : W_{\varpi_i}]$ but $\mathbb{Z}W\varpi_1 \neq \Lambda$. Suppose $(Q, p) \in \mathcal{P}(\Lambda)$ where $Q = \oplus_{i=1}^k \mathbb{Z}[W/W_i]$, $p = f_{\lambda_1,\dots,\lambda_k}$ and $\lambda_i^+ = \sum_{j=1}^k m_{ij}\varpi_j$. Assume that $m_{ij} = 0$ for all $i = 1, \dots, k; j = n-1, n$. But then $\sum_{i=1}^k \mathbb{Z}W\lambda_i \subset \sum_{i=1}^k \mathbb{Z}\lambda_i + \mathbb{Z}\Phi \subset \sum_{i=1}^{n-2} \mathbb{Z}\varpi_i + \mathbb{Z}\Phi \subset \mathbb{Z}(\alpha_{n-1} + \alpha_n)/2 + \mathbb{Z}\Phi$. This contradicts the surjectivity of the map p since $\Lambda/\mathbb{Z}\Phi \cong \mathbb{Z}/4\mathbb{Z}$ if n is odd, and $\Lambda/\mathbb{Z}\Phi \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if n is even. So there must exist i such that $m_{i,n-1} \neq 0$ or $m_{i,n} \neq 0$, and hence $W_{\lambda_i^+} \leqslant W_{\varpi_{n-1}}$ (respectively W_{ϖ_n}). Since $W_{\varpi_{n-1}} \cong W_{\varpi_n} \cong W(\mathsf{A}_{n-1})$ of order $n!$, we have $\operatorname{rank}(Q) \geqslant [W(\mathsf{D}_n) : W(\mathsf{A}_{n-1})] = 2^{n-1}$.

Suppose n is odd. $\mathbb{Z}W\varpi_{n-1} = \mathbb{Z}W\varpi_n = \Lambda$, and for $P = \mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})]$ with $\pi = f_{\varpi_{n-1}}$, or $\pi = f_{\varpi_n}$, we have $\operatorname{rank}(P) = 2^{n-1} > 2n$. So $(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_{n-1}}) \in \mathcal{P}(\Lambda)$ and $(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_n}) \in \mathcal{P}(\Lambda)$ and $r(\Lambda) = 2^{n-1}$. Now suppose n is even. It is not possible for Q to be a transitive permutation lattice, as otherwise $\Lambda = \mathbb{Z}W\lambda_i \subset \mathbb{Z}\lambda_i + \mathbb{Z}\Phi$, which contradicts the fact that $\Lambda/\mathbb{Z}\Phi$ is not cyclic. So there must be another nonzero λ_j with, say, $W_{\lambda_j} \leqslant W_{\varpi_k}$. Since $|W_{\varpi_1}| = \max\{|W_{\varpi_j}| \mid j = 1, \dots, k\}$ and $W_{\varpi_1} = W(\mathsf{D}_{n-1})$, we see that in fact $\operatorname{rank}(Q) \geqslant [W(\mathsf{D}_n) : W(\mathsf{A}_{n-1})] + [W(\mathsf{D}_n) : W(\mathsf{D}_{n-1})]$ for n even. Let $P = \mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})] \oplus \mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})]$ and $\pi = f_{\varpi_1,\varpi_n}$. Since $\mathbb{Z}W_{\varpi_1} + \mathbb{Z}W_{\varpi_n} = \Lambda$, and for any nontrivial normal subgroup N of W, $\operatorname{rank}(P^N) \leqslant (2^{n-1} + 2n)/2 < 2^{n-1} + n = \operatorname{rank}(\operatorname{Ker}(\pi))$, then $(P, f_{\omega_1,\omega_n}) \in \mathcal{P}(\Lambda)$ and $r(\Lambda) = 2^{n-1} + 2n$. $\square$

Proposition 5.11. *Suppose G is a nonadjoint, nonsimply connected group of type*

$\mathsf{D}_n, n \geqslant 4$. *Then if n is odd, $G = \mathrm{SO}_{2n}$, and if n is even, $G = \mathrm{SO}_{2n}$, $\mathrm{Spin}^{\pm}_{2n}$. Let T_1 be a maximal torus for SO_{2n} and T_i be maximal tori for $\mathrm{Spin}^{\pm}_{4k}$ where $i = n-1$ for Spin^{-}_{4k} and $i = n$ for Spin^{+}_{4k}. Then:*

- $r(X(T_1)) = 4n$, $\mathrm{ed}_W(X(T_1)) \leqslant 3n$, *and*

$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{D}_{n-1})]^2, f_{\varpi_1,0}\big) \in \mathcal{P}\big(X(T_{2n})\big);$$

- *if $n = 4$, then $r(X(T_i)) = 16$, $\mathrm{ed}_W(X(T_i)) \leqslant 12$ for $i = n-1, n$ and*

$$\big(\mathbb{Z}[W(\mathsf{D}_4)/W(\mathsf{A}_3)]^2, f_{\varpi_i,0}\big) \in \mathcal{P}\big(X(T_i)\big);$$

- *if $n > 4$ is even, then $r(X(T_i)) = 2^{n-1}$, $\mathrm{ed}_W(X(T_i) \leqslant 2^{n-1} - n$, and*

$$\big(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_i}\big) \in \mathcal{P}\big(X(T_i)\big)$$

  for $i = n-1, n$.

Proof. For $\Phi = \mathsf{D}_n$, n odd, $\Lambda/\mathbb{Z}\Phi \cong \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}\varpi_1 + \mathbb{Z}\Phi$. So SO_{2n} is the unique intermediate group, in this case, with $X(T_1) = \mathbb{Z}W\varpi_1$.

For $\Phi = \mathsf{D}_n$, with n even, we have $\Lambda/\mathbb{Z}\Phi \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ where $\Lambda/\mathbb{Z}\Phi = \mathbb{Z}\varpi_{n-1} + \mathbb{Z}\varpi_n + \mathbb{Z}\Phi$. Now if $k \leqslant (n-2)/2$, then $\varpi_{2k} \in \mathbb{Z}\Phi$ and $\varpi_{2k+1} + \mathbb{Z}\Phi = \mathbb{Z}(\varpi_{n-1} + \varpi_n) + \mathbb{Z}\Phi$. There are three intermediate groups in this case: SO_{2n} with $X(T_1)/\mathbb{Z}\Phi = \mathbb{Z}\varpi_1 + \mathbb{Z}\Phi$; Spin^{-}_{2n} with $X(T_{n-1}) = \mathbb{Z}W\varpi_{n-1}$ and Spin^{+}_{2n} with $X(T_n) = \mathbb{Z}W\varpi_n$.

We have already covered the case of SO_{2n} in Proposition 5.7.

Suppose $n = 4$. Then the minimal value of $[W : W_{\varpi_i}]$, $2n = 8$, is attained by $\varpi_1, \varpi_3, \varpi_4$ and $W_{\varpi_i} = W(\mathsf{A}_3) = W(\mathsf{D}_3)$ in all these cases. We have already determined a minimal element $(\mathbb{Z}[W(\mathsf{D}_4)/W(\mathsf{A}_3)]^2, f_{\varpi_1,0})$ for $\mathcal{P}(X(T_{2n}))$. But $\varpi_1, \varpi_3, \varpi_4$ are permuted by the action of the automorphism group of D_4. So suppose

$$0 \longrightarrow K \longrightarrow P \longrightarrow X(T_1) = \mathbb{Z}W\varpi_1 \longrightarrow 0$$

were an exact sequence of W-lattices with K faithful and P permutation, and $\sigma_i \in \mathrm{Aut}(\Phi)$ with $\sigma_i(\varpi_1) = \varpi_i, i = n-1, n$. Then

$$0 \longrightarrow \sigma_i(K) \longrightarrow \sigma_i(P) \longrightarrow X(T_i) = \mathbb{Z}W\varpi_i \longrightarrow 0$$

would be an exact sequence of W-lattices with $\sigma(P)$ permutation and $\sigma(K)$ faithful. This shows that $r(X(T_i)) = r(X(T_1))$ for $i = n-1, n$ and a minimal element of $\mathcal{P}(X(T^{\pm}_{2n})), i = n-1, n$ can be found by applying σ_i. That is $(\mathbb{Z}[W(\mathsf{D}_4)/W(\mathsf{A}_3)]^2, f_{\varpi_i,0})$ is a minimal element of $\mathcal{P}(X(T_i)$ for $i = 1, n-1, n$ and $r(X(T_i)) = 16$.

Now let $n > 4$ be even. Let $(Q, p) \in \mathcal{P}(X(T_k)), k = n-1, n$ with $Q = \oplus_{i=1}^{r}\mathbb{Z}[W/W_i]$ and $p = f_{\lambda_1,\ldots,\lambda_r}$. Suppose that all $\lambda_i^{+} \in \mathrm{Span}_{\mathbb{Z}}\{\varpi_i \mid i \neq k\}$. Then $\varpi_k \notin \sum_{i=1}^{r}\mathbb{Z}W\lambda_i \subset \sum_{i\neq k}\mathbb{Z}\varpi_i + \mathbb{Z}\Phi$. So, by contradiction, there must be at least one $\lambda_i^{+} = \sum_{j=1}^{n}\mathbb{Z}m_{ij}\varpi_j$ with $m_{ik} \neq 0$. Hence $\mathrm{rank}\,(Q) \geqslant [W : W_{\varpi_k}] = 2^{n-1}$. Since indeed $\mathbb{Z}W\varpi_k = X(T_k)$, $W_{\varpi_k} = W(\mathsf{A}_{n-1})$ and $[W : W_{\varpi_k}] > 2n$, we see that $(\mathbb{Z}[W(\mathsf{D}_n)/W(\mathsf{A}_{n-1})], f_{\varpi_k})$ is a minimal element of $\mathcal{P}(X(T_k))$ and $r(X(T_k)) = 2^{n-1}$ for $k = n-1, n$. $\square$

Remark 5.12. There are many directions for future research here. I would like to further examine possible compressions of

$$0 \longrightarrow K(\Phi) \longrightarrow P(\Phi) \longrightarrow \mathbb{Z}\Phi \longrightarrow 0$$

in the adjoint case or more generally a minimal element of $\mathcal{P}(X(T))$. It is also possible that a non-minimal element of $\mathcal{P}(X(T))$ could be compressed further than a minimal element. That is, that non-minimal elements of $\mathcal{P}(X(T))$ could produce smaller bounds on $\mathrm{ed}_W(X(T))$. It would also be interesting to determine whether or not the inequality $\mathrm{ed}(N) \leqslant \mathrm{ed}_W(X(T))$ is strict.

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