Transformation Groups, Vol. 9, No. 2, 2004, pp. 167–209

A UNIVERSAL CONSTRUCTION FOR MODULI SPACES OF DECORATED VECTOR BUNDLES OVER CURVES

ALEXANDER SCHMITT

Universität Duisburg-Essen
FB6 Mathematik & Informatik
D-45117 Essen, Deutschland
alexander.schmitt@uni-essen.de

Abstract. Let X be a smooth projective curve over the field of complex numbers, and fix a homogeneous representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$. Then one can associate to every vector bundle E of rank r over X a vector bundle E_ρ with fibre V. We would like to study triples (E, L, φ) where E is a vector bundle of rank r over X, L is a line bundle over X, and $\varphi\colon E_\rho \to L$ is a nontrivial homomorphism. This setup comprises well known objects such as framed vector bundles, Higgs bundles, and conic bundles. In this paper, we will formulate a general (parameter dependent) semistability concept for such triples, which generalizes the classical Hilbert–Mumford criterion, and we establish the existence of moduli spaces for the semistable objects. In the examples which have been studied so far, our semistability concept reproduces the known ones. Therefore, our results give in particular a unified construction for many moduli spaces considered in the literature.

Introduction

The present paper is devoted to the study of vector bundles with an additional structure from a unified point of view. We have picked the name “decorated vector bundles” suggested in [23].

Before we outline our paper, let us give some background. The first problem to treat is the problem of classifying vector bundles over an algebraic curve X, assumed here to be smooth, projective and defined over $\mathbb{C}$. From the point of view of projective geometry, this is important because it is closely related to classifying projective bundles over X, so-called *ruled manifolds.* The basic invariants of a vector bundle E are its rank and its degree. They determine E as a topological $\mathbb{C}$-vector bundle. The problem of classifying all vector bundles of fixed degree d and rank r is generally accessible only in a few cases:

- The case $r = 1$, i.e., the case of line bundles which is covered by the theory of Jacobian varieties.
- The case $X = \mathbb{P}_1$ where Grothendieck’s splitting theorem [18] provides the classification.
- The case $g(X) = 1$. In this case, the classification has been worked out by Atiyah [1].

As is clear from the theory of line bundles, over a curve of genus $g \geq 1$, vector bundles of degree d and rank r cannot be parameterized by discrete data. Therefore, one seeks

Received April 21, 2003. Accepted July 29, 2003.

a variety parameterizing all vector bundles of given degree d and rank r characterized by a universal property like the Jacobian. Such a universal property was formulated by Mumford in his definition of a *coarse moduli space* [29]. However, one checks that the family of all vector bundles of degree d and rank r is not bounded which implies that a coarse moduli space cannot exist. For this reason, one has to restrict one's attention to suitable bounded subfamilies of the family of all vector bundles of degree d and rank r. Motivated by his general procedure to construct moduli spaces via his Geometric Invariant Theory [29], Mumford suggested that these classes should be the classes of stable and semistable vector bundles. His definition, given in [28], is the following: A vector bundle E is called *(semi)stable* if for every nontrivial, proper subbundle $F \subset E$

$$\mu(F) := \frac{\deg F}{\operatorname{rk} F} (\leq) \mu(E).$$

Here, "$(\leq)$" means that "$\leq$" is to be used for defining "semistable" and "$<$" for stable. Seshadri then succeeded to give a construction of the coarse moduli space of stable vector bundles, making use of Geometric Invariant Theory [42]. This moduli space is only a quasi-projective manifold. To compactify it, one has also to look at semistable vector bundles. Seshadri formulated the notion of *S-equivalence* of semistable bundles which agrees with isomorphy for stable bundles but is coarser for properly semistable ones. The moduli space of S-equivalence classes exists by the same construction and is a normal projective variety compactifying the moduli space of stable bundles. Later Gieseker, Maruyama, and Simpson generalized the results to higher dimensions [14], [27], [43]. Their constructions also apply to curves and replace Seshadri's (see [24]). Narasimhan and Seshadri related stable bundles to unitary representations of fundamental groups, a framework in which vector bundles had been formerly studied [31], [32].

The next step is to consider vector bundles with extra structures. Let us mention a few sources for this kind of problems:

• **Classification of algebraic varieties**. We have already mentioned that the classification of vector bundles is related to the classification of projective bundles via the assignment $E \mapsto \mathbb{P}(E)$. Suppose, for example, that we want to study divisors in projective bundles. For this, let E be a vector bundle, $\mathbb{P}(E)$ its associated projective bundle, k a positive integer, and M a line bundle on X. To give a divisor D in the linear system $|\mathcal{O}_{\mathbb{P}(E)}(k) \otimes \pi^* M|$ we have to give a section $\sigma\colon \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_{\mathbb{P}(E)}(k) \otimes \pi^* M$ which is the same as giving a nonzero homomorphism $\mathcal{O}_X \to S^k E \otimes M$, or $S^k E^\vee \to M$. Thus, we are led to classify triples (E, M, τ) where E is a vector bundle over X, M a line bundle, and $\tau\colon S^k E \to M$ a nontrivial homomorphism. In case the rank of E is three and k is two, this is the theory of *conic bundles*, recently studied by Gómez and Sols [15].

• **Dimensional reduction**. Here one looks at vector bundles $\mathcal{G}$ on $X \times \mathbb{P}_1$ which can be written as extensions

$$0 \to \pi_X^* F \to \mathcal{G} \to \pi_X^* E \otimes \pi_{\mathbb{P}_1}^* \mathcal{O}_{\mathbb{P}_1}(2) \to 0$$

where E and F are vector bundles on X. These extensions are parameterized by $H^0(E^\vee \otimes F) = \operatorname{Hom}(E, F)$. The study of such vector bundles is thus related to the study of triples (E, F, φ) where E and F are vector bundles on X and $\varphi\colon E \to F$ is a nonzero homomorphism. These are the *holomorphic triples* of Bradlow and García–Prada [13] and [7]. They were also studied from the algebraic point of view by the author [39]. For the special case $E = \mathcal{O}_X$, we find the problem of vector bundles with a section, so-called *Bradlow pairs* [4]. An important application of Bradlow pairs was given by Thaddeus in his proof of the Verlinde formula [45].

• **Representations of fundamental groups**. *Higgs bundles* are pairs (E, φ), consisting of a vector bundle E and a twisted endomorphism $\varphi\colon E \to E \otimes \omega_X$. Simpson used in [43] the higher dimensional analogues of these objects to study representations of fundamental groups of projective manifolds. This ties up nicely with the work of Narasimhan and Seshadri.

• **Gauge theory**. Here, one starts with differentiable vector bundles together with an additional structure and considers certain differential equations associated to these data. The solutions of the equations then have — via a Kobayashi–Hitchin correspondence — interpretations as holomorphic decorated vector bundles over X, satisfying certain stability conditions. Again, the first case where this arose was the theory of Hermite–Einstein equations and stable vector bundles (see [26]) and was later studied in more complicated situations as in the above examples. Recently, Banfield [2] and Mundet i Riera [30] investigated this in a broad context. We will come back to this again.

Now, for all of these problems and many more, there exist notions of semistability, depending on a rational parameter. The task of projective geometry is then to generalize the construction of Seshadri and successors to obtain moduli spaces for the respective semistable and stable objects. These constructions, where existent, were done case by case and follow a certain pattern inspired by Gieseker's, Maruyama's, and Simpson's constructions. One is therefore led to ask for a single unifying construction incorporating the known examples. This would complete the algebraic counterpart to the work of Banfield and Mundet i Riera.

We will consider this problem in the present article. Our framework is as follows: We fix a representation $\rho\colon \operatorname{GL}(r) \to \operatorname{GL}(V)$, such that the restriction to the centre $\mathbb{C}^* \subset \operatorname{GL}(r)$ is $z \mapsto z^\alpha \cdot \mathrm{id}_V$ for some integer α. Then to any vector bundle E, we can associate a vector bundle E_ρ of rank $\dim V$. The objects we will treat are triples (E, M, τ) where E is a vector bundle of rank r, M is a line bundle, and $\tau\colon E_\rho \to M$ is a nonzero homomorphism. E.g., for $\rho\colon \operatorname{GL}(3) \to \operatorname{GL}(S^2\mathbb{C}^3)$, we recover conic bundles. The list of problems we then have to solve is the following.

- Formulate an appropriate notion of semistability for the above objects!
- Prove boundedness of the semistable triples (E, M, τ) where $\deg E$ and $\deg M$ are fixed!
- Construct a parameter space $\mathfrak{P}$ for the semistable objects together with an action of a general linear group G, such that the equivalence relation induced by this action is the natural equivalence relation on those triples!
- Show that the categorical quotient $\mathfrak{P}/\!\!/G$ exists!

The latter space will then be the moduli space. As one sees from this list, especially in view of the existing constructions, Geometric Invariant Theory will play a central rôle. Let us explain how one can find the semistability concept. First, assume that we are given a bounded family of triples (E, M, τ). Using the theory of quot-schemes it is by now not too hard a task to construct a parameter space $\mathfrak{P}$ for the members of the family in such a way that we have a group action as required together with a family of linearizations — depending on a rational parameter — in line bundles over $\mathfrak{P}$. Therefore, we have realized the input for the GIT process. The Hilbert–Mumford criterion now tells us how to find the semistable points. Thus, it is clear that our notion of semistability should mimic the Hilbert–Mumford criterion as closely as possible. Such an approach was also taken in gauge theory [2] and [30]. The structure of one-parameter subgroups of the special linear group suggests that one-parameter subgroups should be replaced by *weighted filtrations of vector bundles.* For weighted filtrations, one then defines the necessary numerical quantities resembling Mumford's "μ" and arrives at the desired semistability concept.

Our paper is organized as follows: In the first section, we collect the necessary background material from representation theory and GIT. Then we come to the definition of semistability for the triples (E, M, τ) which depends on a positive rational parameter and describe the associated moduli functors. We state the main result, namely the existence of moduli spaces, and proceed to the proofs along the lines outlined before. The paper concludes with a long discussion of examples in order to show that the known problems in that context can be recovered from our results and that, in some cases, additional light is shed on them. The reader will notice that our general semistability concept is in the known cases more complicated than the existing ones and has to be simplified to recover the known ones. This is one of the key points of the paper: The notion of semistability should be simplified after doing the GIT construction and not before. This is why a unifying construction is feasible. However, we will present a general method to simplify the semistability concept in terms of the representation ρ. This method enables us to write down in every concrete situation the semistability concept in a more classical form. Applying this procedure, e.g., to framed bundles or conic bundles immediately reproduces the known semistability concepts. This provides us with a mechanism for finding the correct notion of semistability without guessing or referring to gauge theory.

Finally, we remark that we have confined ourselves to the case of curves in order to have a nice moduli functor associated to every representation of the general linear group. However, if one restricts to direct sums of tensor powers, the construction can also be performed over higher dimensional manifolds [16]. These higher dimensional versions have, in turn, important applications in the problem of compactifying moduli spaces of principal bundles with *singular objects* [40], [17]. Finally, there is now also a version for product groups $\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_s)$ over base manifolds of arbitrary dimension [41] the construction of which is based on the results of the present paper.

Acknowledgments. In November 1999, Professor Okonek organized a workshop on the gauge-theoretic aspects of decorated vector bundles. I would like to thank him for the invitation to that workshop and financial support through his Swiss National Funds project and the other speakers, Professor Lübke and Professor Andrei Teleman, for their enlightening talks which provided crucial impulses for the present work. The author acknowledges support by the DFG through the "Schwerpunkt" program "Globale Methoden in der Komplexen Geometrie — Global Methods in Complex Geometry".

This paper is an English version of Chapters I–III of the author's work "*Universelle Konstruktionen für Modulräume dekorierter Vektorbündel über Kurven*" which was accepted as his habilitation thesis by the University of Essen.

Notation and conventions

• All schemes will be defined over the field of complex numbers, X will be a smooth projective curve of genus $g \geq 2$. We denote by $\underline{\mathrm{Sch}}_{\mathbb{C}}$ the category of separated schemes of finite type over $\mathbb{C}$. A *point* will be a closed point unless otherwise mentioned.

• For a vector bundle E over a scheme S, we denote by $\mathbb{P}(E)$ the projective bundle of hyperplanes in the fibres of E.

• Given a product $X \times Y$ of schemes, π_X and π_Y stand for the projections from $X \times Y$ onto the respective factors.

• Let V be a finite-dimensional $\mathbb{C}$-vector space and $\rho\colon G \to \mathrm{GL}(V)$ a representation of the algebraic group G. This yields an action of G on $\mathbb{P}(V)$ and a linearization $G \times \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathcal{O}_{\mathbb{P}(V)}(1)$. We will denote this linearization again by ρ.

• Let E be a vector bundle of rank r. Then the associated $\mathrm{GL}(r)$-principal bundle is given as $\mathfrak{P}(E) = \bigcup_{x\in X} \mathrm{Isom}(\mathbb{C}^r, E_x) \subset \underline{\mathrm{Hom}}(\mathcal{O}_X^{\oplus r}, E)$. If we are furthermore given an action $\Gamma\colon\ \mathrm{GL}(r)\times F \to F$ of $\mathrm{GL}(r)$ on a quasi-projective manifold F, we set $\mathfrak{P}(E)\times^{\mathrm{GL}(r)} F := (\mathfrak{P}(E)\times F)/\,\mathrm{GL}(r)$. Here, $\mathrm{GL}(r)$ acts on $\mathfrak{P}(E)\times F$ by $(x, y)\cdot g = (x\cdot g, g^{-1}\cdot y)$. If F is a vector space and the action Γ comes from a representation $\rho\colon\ \mathrm{GL}(r) \to \mathrm{GL}(F)$, we write E_ρ for the vector bundle $\mathfrak{P}(E) \times^{\mathrm{GL}(r)} F$.

• For any $x \in \mathbb{R}$, we set $[x]_+ := \max\{\, 0, x \,\}$.

1. Preliminaries

1.1. Representations of the general linear group

First, let $\rho\colon\ \mathrm{GL}(r) \to \mathrm{GL}(V)$ be an irreducible representation on the finite-dimensional $\mathbb{C}$-vector space V.

Theorem 1.1. *There are integers $a_1, \dots, a_r$ with $a_i \geq 0$ for $i = 1, \dots, r-1$, such that ρ is a direct summand of the natural representation of* $\mathrm{GL}(r)$ *on*

$$S^{a_1}\bigl(\mathbb{C}^r\bigr) \otimes \cdots \otimes S^{a_{r-1}}\bigl(\textstyle\bigwedge^{r-1} \mathbb{C}^r\bigr) \otimes \bigl(\textstyle\bigwedge^{r} \mathbb{C}^r\bigr)^{\otimes a_n}.$$

Proof. See [12], Proposition 15.47. □

For any vector space W, the representations of $\mathrm{GL}(W)$ on $S^i(W)$ and $\bigwedge^i W$ are direct summands of the representation of $\mathrm{GL}(W)$ on $W^{\otimes i}$. Setting $a := a_1 + \cdots + a_{r-1}(r-1)$ and $b := a_n$, we see that ρ is a direct summand of the representation $\rho_{a,b}$ of $\mathrm{GL}(r)$ on $(\mathbb{C}^r)^{\otimes a} \otimes (\bigwedge^r \mathbb{C}^r)^{\otimes b}$.

Corollary 1.2. *Let $\rho\colon\ \mathrm{GL}(r) \to \mathrm{GL}(V)$ be a (not necessarily irreducible) representation of $\mathrm{GL}(r)$ on the finite-dimensional $\mathbb{C}$-vector space V, such that the centre $\mathbb{C}^* \subset \mathrm{GL}(r)$ acts by $z \mapsto z^\alpha \cdot \mathrm{id}_V$ for some $\alpha \in \mathbb{Z}$. Then there exist $a, b, c \in \mathbb{Z}_{\geq 0}$, $c > 0$, such that ρ is a direct summand of the natural representation $\rho_{a,b,c}$ of* $\mathrm{GL}(r)$ *on*

$$V_{a,b,c} := \Bigl((\mathbb{C}^r)^{\otimes a} \otimes \bigl(\textstyle\bigwedge^r \mathbb{C}^r\bigr)^{\otimes -b} \Bigr)^{\oplus c}.$$

Proof. We can decompose $\rho = \rho_1 \oplus \cdots \oplus \rho_c$ where the ρ_i's are irreducible representations. By what we have said before, there are integers a_i, b_i, $i = 1, \ldots, c$, with $a_i \geq 0$, $i = 1, \ldots, c$, such that ρ is a direct summand of $\rho_{a_1,b_1} \oplus \cdots \oplus \rho_{a_c,b_c}$. Our assumption on the action of $\mathbb{C}^*$ implies that $a_1 + rb_1 = \cdots = a_c + rb_c$. Let b be a positive integer which is so large that $b_i + b > 0$ for $i = 1, \ldots, c$. Then ρ_{a_i,b_i} is the natural representation of $\mathrm{GL}(r)$ on

$$\left(\mathbb{C}^r\right)^{\otimes a_i} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes b_i + b} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes -b}, i = 1, \ldots, c.$$

Now, the $\mathrm{GL}(r)$-module $\left(\mathbb{C}^r\right)^{\otimes a_i} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes b_i + b}$ is a direct summand of $\left(\mathbb{C}^r\right)^{\otimes a}$, $a := a_1 + r(b_1 + b) = \ldots = a_c + r(b_c + b)$, and we are done. □

1.2. Basic concepts from GIT

We briefly summarize the main steps in Geometric Invariant Theory to fix the notation. References are [29] and [33].

The GIT-process. Let G be a reductive algebraic group and $G \times F \to F$ an action of G on the projective scheme F. Let L be an ample line bundle on F. A *linearization* of the given action in L is a lifting of that action to an action $\rho\colon G \times L \to L$, such that for every $g \in G$ and $x \in F$ the induced map $L_x \to L_{g \cdot x}$ is a linear isomorphism. Taking tensor powers, ρ provides us with linearizations of the action in any power $L^{\otimes k}$, $k > 0$, and actions of G on $H^0(F, L^{\otimes k})$ for any $k > 0$. A point $x_0 \in F$ is called *semistable* if there exist an integer $k > 0$ and a G-invariant section $\sigma \in H^0(F, L^{\otimes k})$ not vanishing in x_0. If, moreover, the action of G on the set $\{\, x \in F \mid \sigma(x) \neq 0 \,\}$ is closed and $\dim G \cdot x_0 = \dim G$, x_0 is called *stable*. The sets $F^{(s)s}$ of (semi)stable points are open G-invariant subsets of F. Finally, a point $x \in F$ is called *polystable* if it is semistable and its G-orbit is closed in F^{ss}. Using this definition, the stable points are precisely the polystable points with finite stabilizer. The core of Mumford's Geometric Invariant Theory is that the categorical quotients $F^{ss}/\!\!/G$ and $F^s/\!\!/G$ do exist and that $F^{ss}/\!\!/G$ is a projective scheme whose closed points are in one-to-one correspondence to the orbits of polystable points, so that $F^s/\!\!/G$ is in particular an orbit space.

A finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ provides an action of G on $\mathbb{P}(V)$ and a linearization of this action in $\mathcal{O}_{\mathbb{P}(V)}(1)$, called again ρ. A point $[v] \in \mathbb{P}(V)$ represented by $v \in V^\vee$ is then semistable if and only if the closure of the orbit of v in $V^\vee$ does not contain 0, stable if, furthermore, its orbit is closed and the dimension of this orbit equals the dimension of G, and polystable if the orbit of v in $V^\vee$ is closed.

Around the Hilbert–Mumford criterion. Let F be a projective variety on which the reductive group G acts. Suppose this action is linearized in the line bundle L. Call the linearization ρ. Then given a one-parameter subgroup λ of G and $x \in F$, we can form

$$x_\infty = \lim_{z \to \infty} \lambda(z) \cdot x.$$

The point x_∞ is clearly a fix point for the $\mathbb{C}^*$-action on F induced by λ. Thus, $\mathbb{C}^*$ acts on the fibre of L over x_∞, say, with weight γ. One defines

$$\mu_\rho(\lambda, x) := -\gamma.$$

Theorem 1.3. (Hilbert–Mumford criterion [29]) *A point $x \in F$ is (semi)stable if and only if for every nontrivial one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$*

$$\mu_\rho\big(\lambda, x\big)\ (\geq)\ 0.$$

Moreover, a point $x \in F$ is polystable if and only if it is semistable and, for every one parameter subgroup λ of G with $\mu_\rho(\lambda, x) = 0$, there is a $g \in G$ with $x_\infty = g \cdot x$.

As we have explained in the introduction, our concept of stability for decorated vector bundles is basically a Hilbert–Mumford criterion. To define the necessary numerical invariants, we need the following preparatory result.

Lemma 1.4. *Let S be a scheme and $\sigma\colon S \to F$ a morphism. Suppose the G-action on F is linearized in the ample line bundle L. Then*

$$\mu_\rho(\lambda, \sigma) := \max\big\{\, \mu_\rho(\lambda, \sigma(s)) \mid s \in S \,\big\}\, \textit{exists.}$$

Proof. We may assume that L is a very ample line bundle. Set $V := H^0(F, L)$. The linearization ρ provides us with a representation $\rho\colon G \to \mathrm{GL}(V)$ and a G-equivariant embedding $\iota\colon F \hookrightarrow \mathbb{P}(V)$. Since obviously $\mu_\rho(\lambda, x) = \mu_{\mathrm{id}_{\mathrm{GL}(V)}}(\lambda, \iota(x))$ for all points $x \in F$ and all one-parameter subgroups λ of G, we can assume $F = \mathbb{P}(V)$. Now, there are a basis $v_1, \ldots, v_n$ of V and integers $\gamma_1 \leq \cdots \leq \gamma_n$ with $\lambda(z) \cdot \sum_{i=1}^n c_i v_i = \sum_{i=1}^n z^{\gamma_i} c_i v_i$. A point $[l] \in \mathbb{P}(V)$ can be thought of as the equivalence class of a linear form $l\colon V \to \mathbb{C}$. Then $\mu_\rho\big(\lambda, [l]\big) = -\min\big\{\, \gamma_i \mid l(v_i) \neq 0 \,\big\}$. Therefore, $\mu_\rho(\lambda, \sigma(s)) \in \{\, -\gamma_1, \ldots, -\gamma_n \,\}$, and this implies the assertion. □

Remark 1.5. Let $F \subset \mathbb{P}(V)$ and λ a one-parameter subgroup of G. Choose a basis $v_1, \ldots, v_n$ of V and $\gamma_1 \leq \cdots \leq \gamma_n$ as before. Suppose $\mu_\rho(\lambda, \sigma) = -\gamma_{i_0}$ and let $V^0 \subset V$ be the eigenspace for the weight γ_{i_0}. Let $U \subset S$ be the open set where the rational map $S \xrightarrow{\sigma} F \hookrightarrow \mathbb{P}(V) \dashrightarrow \mathbb{P}(V^0)$ is defined. Then $\mu_\rho(\lambda, \sigma(s)) = -\gamma_{i_0}$ for all $s \in U$. In other words, if S is irreducible, $\mu_\rho(\lambda, \sigma)$ is just the generic weight occurring for a point $\sigma(s)$, $s \in S$.

Semistability for actions coming from direct sums of representations. Let G be a reductive algebraic group and $V_1, \ldots, V_s$ finite-dimensional vector spaces. Suppose we are given representations $\rho_i\colon G \to \mathrm{GL}(V_i)$, $i = 1, \ldots, s$. The direct sum $\rho_1 \oplus \cdots \oplus \rho_s$ provides us with a linear action of G on $\mathbb{P}(V)$, $V := V_1 \oplus \cdots \oplus V_s$. Furthermore, for any $\underline{\iota} = (\iota_1, \ldots, \iota_t)$ with $0 < t \leq s$, $\iota_1, \ldots, \iota_t \in \{\, 1, \ldots, s \,\}$, and $\iota_1 < \cdots < \iota_t$, the ρ_i's yield an action $\sigma_{\underline{\iota}}$ of G on $\mathbb{P}_{\underline{\iota}} := \mathbb{P}(V_{\iota_1}) \times \cdots \times \mathbb{P}(V_{\iota_t})$, and, for any sequence of positive integers $k_1, \ldots, k_t$, a linearization of $\sigma_{\underline{\iota}}$ in the very ample line bundle $\mathcal{O}(k_1, \ldots, k_t)$. The computation of the semistable points in $\mathbb{P}(V)$ can be reduced to the computation of the semistable points in the $\mathbb{P}_{\underline{\iota}}$'s by means of the following statement.

Theorem 1.6. *Let $w' := ([w_{\iota_1}, w_{\iota_2}], [w_{\iota_3}], \ldots, [w_{\iota_t}])$ be a point in the space $\mathbb{P}(V_{\iota_1} \oplus V_{\iota_2}) \times \mathbb{P}_{(\iota_3, \ldots, \iota_t)}$. Then w' is semistable (polystable) with respect to the given linearization in the line bundle $\mathcal{O}(k, k_3, \ldots, k_t)$ if and only if either $([w_{\iota_i}], [w_{\iota_3}], \ldots, [w_{\iota_t}])$ is semistable (polystable) in $\mathbb{P}_{(\iota_i, \iota_3, \ldots, \iota_t)}$ with respect to the linearization in $\mathcal{O}(k, k_3, \ldots, k_t)$ for either $i = 1$ (and $w_{\iota_2} = 0$) or $i = 2$ (and $w_{\iota_1} = 0$), or there are positive natural numbers n, k_1, and k_2, such that $k_1 + k_2 = nk$ and the point $([w_{\iota_1}], [w_{\iota_2}], [w_{\iota_3}], \ldots, [w_{\iota_t}])$ is semistable (polystable) in $\mathbb{P}_{(\iota_1, \iota_2, \iota_3, \ldots, \iota_t)}$ with respect to the linearization in $\mathcal{O}(k_1, k_2, nk_3, \ldots, nk_t)$.*

Remark 1.7. As one easily checks, for stable points only the "if" direction remains true.

Proof. This theorem can be proved with the methods developed in [35] for $s = 2$. A more elementary approach is contained in the note [38]. □

1.3. One parameter subgroups of SL(r)

Let $\mathrm{GL}(r) \times F \to F$ be an action of the general linear group on the projective manifold F. For our definition of semistability, only the induced action of $\mathrm{SL}(r) \times F \to F$ will matter. Since the Hilbert–Mumford criterion will play a central role throughout our considerations, we will have to describe the one-parameter subgroups of $\mathrm{SL}(r)$.

Given a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{SL}(r)$, we can find a basis $\underline{w} = (w_1, \ldots, w_r)$ of $\mathbb{C}^r$ and a weight vector $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ with integral entries, such that

- $\gamma_1 \le \cdots \le \gamma_r$ and $\sum_{i=1}^r \gamma_i = 0$, and
- $\lambda(z) \cdot \sum_{i=1}^r c_i w_i = \sum_{i=1}^r z^{\gamma_i} c_i w_i$.

Conversely, a basis $\underline{w}$ of $\mathbb{C}^r$ and a weight vector $\underline{\gamma}$ with the above properties define a one-parameter subgroup $\lambda(\underline{w}, \underline{\gamma})$ of $\mathrm{SL}(r)$.

To conclude, we remark that, for any vector $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of integers with $\gamma_1 \le \cdots \le \gamma_r$ and $\sum \gamma_i = 0$, there is a decomposition

$$\underline{\gamma} = \sum_{i=1}^{r-1} \frac{\gamma_{i+1} - \gamma_i}{r} \gamma^{(i)}$$

with

$$\gamma^{(i)} := \big(\underbrace{i - r, \ldots, i - r}_{i\times}, \underbrace{i, \ldots, i}_{(r-i)\times}\big), i = 1, \ldots, r - 1.$$

1.4. Estimates for the weights of some special representations

In the following, $\rho_{a,b,c}$ will stand for the induced representation of $\mathrm{GL}(r)$ on the vector space $V_{a,b,c} := \big((\mathbb{C}^r)^{\otimes a} \otimes (\bigwedge^r \mathbb{C}^r)^{\otimes -b}\big)^{\oplus c}$ where $a, b \in \mathbb{Z}_{\ge 0}$, $c \in \mathbb{Z}_{>0}$. Then $\mathbb{P}(V_{a,b,c}) = \mathbb{P}(V_{a,0,c})$ and $V_{a,b,c} \cong V_{a,0,c}$ as $\mathrm{SL}(r)$-modules.

Let $\underline{w} = (w_1, \ldots, w_r)$ be a basis for $\mathbb{C}^r$ and $\underline{\gamma} = \sum_{i=1}^{r-1} \alpha_i \gamma^{(i)}$, $\alpha_i \in \mathbb{Q}_{\ge 0}$, an integral weight vector. Let I^a be the set of all a-tuples $\underline{\iota} = (\iota_1, \ldots, \iota_a)$ with $\iota_j \in \{1, \ldots, r\}$, $j = 1, \ldots, a$. For $\underline{\iota} \in I^a$ and $k \in \{1, \ldots, c\}$, we define $w_{\underline{\iota}} := w_{\iota_1} \otimes \cdots \otimes w_{\iota_a}$, and $w_{\underline{\iota}}^k := (0, \ldots, 0, w_{\underline{\iota}}, 0, \ldots, 0)$, $w_{\underline{\iota}}$ occupying the k-th entry. The elements $w_{\underline{\iota}}^k$ with $\underline{\iota} \in I^a$ and $k \in \{1, \ldots, c\}$ form a basis for $V_{a,0,c}$. We let $w_{\underline{\iota}}^{k\vee}$, $\underline{\iota} \in I^a$, $k \in \{1, \ldots, c\}$ be the dual basis of $V_{a,0,c}^\vee$. Now, let $[l] \in \mathbb{P}(V_{a,0,c})$ where $l = \sum a_{\underline{\iota}}^k w_{\underline{\iota}}^{k\vee}$. Then there exist k_0 and $\underline{\iota}_0$ with $a_{\underline{\iota}_0}^{k_0} \neq 0$ and

$$\mu_{\rho_{a,b,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [l]\big) = \mu_{\rho_{a,0,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [l]\big) = \mu_{\rho_{a,0,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [w_{\underline{\iota}_0}^{k_0\vee}]\big),$$

and for any other k and $\underline{\iota}$ with $a_{\underline{\iota}}^k \neq 0$

$$\mu_{\rho_{a,b,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [l]\big) \ge \mu_{\rho_{a,0,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [w_{\underline{\iota}}^{k\vee}]\big).$$

We also find that for $i \in \{1, \ldots, r-1\}$

$$\mu_{\rho_{a,0,c}}\big(\lambda(\underline{w}, \gamma^{(i)}), [w_{\underline{\iota}_0}^{k_0\vee}]\big) = \nu \cdot r - a \cdot i,\ \nu = \#\{\, \iota_j \le i \mid \underline{\iota}_0 = (\iota_1, \ldots, \iota_a),\ j = 1, \ldots, a \,\}.$$

One concludes the following.

Lemma 1.8. (i) *For every basis* $\underline{w} = (w_1, \dots, w_r)$ *of* $\mathbb{C}^r$*, every integral weight vector* $\underline{\gamma} = \sum_{i=1}^{r-1} \alpha_i \gamma^{(i)}$*,* $\alpha_i \in \mathbb{Q}_{\geq 0}$*, and every point* $[l] \in \mathbb{P}(V_{a,b,c})$

$$\left(\sum_{i=1}^{r-1} \alpha_i\right) a(r-1) \geq \mu_{\rho_{a,b,c}}\big(\lambda(\underline{w}, \underline{\gamma}), [l]\big) \geq -\left(\sum_{i=1}^{r-1} \alpha_i\right) a(r-1).$$

(ii) *For every basis* $\underline{w} = (w_1, \dots, w_r)$ *of* $\mathbb{C}^r$*, every two integral weight vectors* $\underline{\gamma}_1 = \sum_{i=1}^{r-1} \alpha_i \gamma^{(i)}$*,* $\alpha_i \in \mathbb{Q}_{\geq 0}$*,* $\underline{\gamma}_2 = \sum_{i=1}^{r-1} \beta_i \gamma^{(i)}$*,* $\beta_i \in \mathbb{Q}_{\geq 0}$*, and every point* $[l] \in \mathbb{P}(V_{a,b,c})$

$$\mu_{\rho_{a,b,c}}\big(\lambda(\underline{w}, \underline{\gamma}_1 + \underline{\gamma}_2), [l]\big) \geq \mu_{\rho_{a,b,c}}\big(\lambda(\underline{w}, \underline{\gamma}_1), [l]\big) - \left(\sum_{i=1}^{r-1} \beta_i\right) a(r-1).$$

2. Decorated vector bundles

2.1. The moduli functors

In this section, we will introduce the vector bundle problems we would like to treat. The main topic will be the definition of the semistability concept. Having done this, we describe the relevant moduli functors to be studied throughout the rest of this chapter.

Semistable objects. The input data for our construction are:

- a positive integer r,
- an action of the general linear group $\mathrm{GL}(r)$ on the projective manifold F, such that the centre $\mathbb{C}^* \subset \mathrm{GL}(r)$ acts trivially.

The objects we want to classify are pairs (E, σ) where

- E is a vector bundle of rank r, and
- $\sigma\colon X \to \mathfrak{F}(E) := \mathfrak{P}(E) \times^{\mathrm{GL}(r)} F$ is a section.

Here, $\mathfrak{P}(E)$ is the principal $\mathrm{GL}(r)$-bundle associated with E. Uninspired as we are, we call (E, σ) an *F-pair.* Two F-pairs (E^1, σ^1) and (E^2, σ^2) are called *equivalent* if there exists an isomorphism $\psi\colon E^1 \to E^2$ such that $\sigma^1 = \sigma^2 \circ \widehat{\psi}$, $\widehat{\psi}\colon \mathfrak{F}(E^1) \to \mathfrak{F}(E^2)$ being the induced isomorphism.

It will be our task to formulate a suitable semistability concept for these objects and to perform a construction of the moduli spaces. Let E be a vector bundle over X. A *weighted filtration of* E is a pair $(E^\bullet, \underline{\alpha})$ consisting of a filtration $E^\bullet : 0 \subset E_1 \subset \cdots \subset E_s \subset E$ of E by nontrivial proper subbundles and a vector $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ of positive rational numbers. Given such a weighted filtration, we set

$$M\big(E^\bullet, \underline{\alpha}\big) := \sum_{j=1}^{s} \alpha_j \big(\deg(E) \operatorname{rk} E_j - \deg(E_j) \operatorname{rk} E\big).$$

Suppose we are also given a linearization ρ of the $\mathrm{GL}(r)$-action on F in an ample line bundle L. Let (E, σ) be as above and let $(E^\bullet, \underline{\alpha})$ be a weighted filtration of E. We define $\mu_\rho(E^\bullet, \underline{\alpha}; \sigma)$ as follows: Let $\underline{w} = (w_1, \dots, w_r)$ be an arbitrary basis of $W := \mathbb{C}^r$. For every $i \in \{1, \dots, r-1\}$, we set $W^{(i)}_{\underline{w}} := \langle w_1, \dots, w_i \rangle$. Define $i_j := \operatorname{rk} E_j$, $j = 1, \dots, s$. This provides a flag

$$W^\bullet : 0 \subset W^{(i_1)}_{\underline{w}} \subset \cdots \subset W^{(i_s)}_{\underline{w}} \subset W$$

and thus a parabolic subgroup $P \subset \mathrm{SL}(r)$, namely the stabilizer of the flag $W^\bullet$. Finally, set $\underline{\gamma} = \sum_{j=1}^{s} \alpha_j \gamma^{(i_j)}$. Next, let U be an open subset of X over which there is an isomorphism $\psi\colon E|_U \to W\otimes\mathcal{O}_U$ with $\psi(E^\bullet|_U) = W^\bullet\otimes\mathcal{O}_U$. Then ψ gives us an isomorphism $\mathfrak{F}(E|_U)\to U\times F$ and σ a morphism $\widetilde{\sigma}\colon U\to U\times F\to F$. If $\underline{\gamma}$ is a vector of integers, we set $\mu_\rho(E^\bullet,\underline{\alpha};\sigma) := \mu_\rho(\lambda(\underline{w},\underline{\gamma}),\widetilde{\sigma})$, as in Lemma 1.4. Otherwise, we choose $k>0$ such that $k\cdot\underline{\gamma}$ is a vector of integers and set $\mu_\rho(E^\bullet,\underline{\alpha};\sigma) := (1/k)\mu_\rho(\lambda(\underline{w},k\underline{\gamma}),\widetilde{\sigma})$. Since for an integral weight vector $\underline{\gamma}'$ and a positive integer k' one has $\mu_\rho(\lambda(\underline{w},k'\underline{\gamma}'),\sigma) = k'\mu_\rho(\lambda(\underline{w},\underline{\gamma}'),\sigma)$, this is well defined. Note that the weight vector $\underline{\gamma}$ is canonically defined by $(E^\bullet,\underline{\alpha})$, but that we have to verify that the definition does not depend on the basis $\underline{w}$ and the trivialization ψ. First, let $\underline{w}' = (w_1',\dots,w_r')$ be a different basis. Let $g\in\mathrm{GL}(r)$ be the element which maps w_i to w_i', $i=1,\dots,r$, and set $\psi' := (g\otimes\mathrm{id}_{\mathcal{O}_U})\circ\psi$. This defines the morphism $\widetilde{\sigma}'\colon U\to F$. Then $\lambda(\underline{w}',\underline{\gamma}) = g\cdot\lambda(\underline{w},\underline{\gamma})\cdot g^{-1}$ and $\widetilde{\sigma}'(x) = g\cdot\widetilde{\sigma}(x)$ for every $x\in U$. Since $\mu_\rho(\lambda(\underline{w}',\underline{\gamma}),\widetilde{\sigma}'(x)) = \mu_\rho(g\cdot\lambda(\underline{w},\underline{\gamma})\cdot g^{-1}, g\cdot\widetilde{\sigma}(x)) = \mu_\rho(\lambda(\underline{w},\underline{\gamma}),\widetilde{\sigma}(x))$, we may fix the basis $\underline{w}$. Any other trivialization $\widetilde{\psi}$ defined with respect to $\underline{w}$ differs from ψ by a map $U\to P$. Now, for every $g\in P$ and every point $x\in U$, $\mu_\rho(\lambda,\widetilde{\sigma}(x)) = \mu_\rho(g\lambda g^{-1}, g\cdot\widetilde{\sigma}(x)) = \mu_\rho(\lambda, g\cdot\widetilde{\sigma}(x))$. The last equality results from [29], Proposition 2.7, p. 57. This shows our assertion. To conclude, Remark 1.5 shows that the definition is also independent of the choice of the open subset U.

Fix also a number $\delta\in\mathbb{Q}_{>0}$. With these conventions, we call (E,σ) *δ-ρ-(semi)stable* if for every weighted filtration $(E^\bullet,\underline{\alpha})$ of E

$$M\big(E^\bullet,\underline{\alpha}\big) + \delta\cdot\mu_\rho\big(E^\bullet,\underline{\alpha};\sigma\big)(\geq)0.$$

Next, we remark that we should naturally fix the degree of E. Then the topological fibre space $\pi\colon\mathfrak{F}^{d,r}\to X$ underlying $\mathfrak{F}(E)$ will be independent of E, so that it makes sense to fix the homology class $[\sigma(X)]\in H_2(\mathfrak{F}^{d,r},\mathbb{Z})$. Given $d\in\mathbb{Z}$, $r\in\mathbb{Z}_{>0}$, and $h\in H_2(\mathfrak{F}^{d,r},\mathbb{Z})$, we say that (E,σ) is *of type* (d,r,h) if E is a vector bundle of degree d and rank r, and $[\sigma(X)] = h$. Before we define the moduli functor, we enlarge our scope.

For a given linearization of the $\mathrm{GL}(r)$-action on F in the line bundle L, we can choose a positive integer k such that $L^{\otimes k}$ is very ample. Therefore, we obtain a $\mathrm{GL}(r)$-equivariant embedding $F\hookrightarrow\mathbb{P}(V)$, $V := H^0(F,L^{\otimes k})$. Note that $\mathbb{C}^*$ acts trivially on $\mathbb{P}(V)$. Therefore, we formulate the following classification problem: The input now consists of

- a positive integer r, a finite-dimensional vector space V, and
- a representation $\rho\colon\ \mathrm{GL}(r)\to\mathrm{GL}(V)$ whose restriction to the centre $\mathbb{C}^*$ is of the form $z\mapsto z^\alpha\cdot\mathrm{id}_V$ for some integer α,

and the objects we want to classify are pairs (E,σ) where

- E is a vector bundle of rank r, and
- $\sigma\colon X\to\mathbb{P}(E_\rho)$ is a section. Here, E_ρ is the vector bundle of rank $\dim V$ associated to E via the representation ρ.

The equivalence relation is the same as before. Now, giving a section $\sigma\colon X\to\mathbb{P}(E_\rho)$ is the same as giving a line bundle M on X and a surjection $\tau\colon E_\rho\to M$. Remember that (M,τ) and (M',τ') give the same section if and only if there exists an isomorphism $M\to M'$ which carries τ into τ'. Moreover, fixing the homology class $[\sigma(X)]$ amounts to the same as fixing the degree of M. Since the condition that τ be surjective will be an open condition in a suitable parameter space, we formulate the following classification problem: The input data are

- a tuple (d,r,m) called the *type*, where d, r, and m are integers, $r>0$,

- a representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$,

and the objects to classify are triples (E, M, τ) where

- E is a vector bundle of rank r and degree d,
- M is a line bundle of degree m, and
- $\tau\colon E_\rho \to M$ is a nonzero homomorphism.

Then (E, M, τ) is called a *ρ-pair of type* (d, r, m), and (E^1, M^1, τ^1) and (E^2, M^2, τ^2) are said to be *equivalent* if there exist isomorphisms $\psi\colon E^1 \to E^2$ and $\chi\colon M^1 \to M^2$ with $\tau^1 = \chi^{-1} \circ \tau^2 \circ \psi_\rho$, where $\psi_\rho\colon E^1_\rho \to E^2_\rho$ is the induced isomorphism. Let (E, M, τ) be a ρ-pair of type (d, r, m). A *weak automorphism* of (E, M, τ) is the class $[\psi] \in \mathbb{P}(\mathrm{End}(E))$ of an automorphism $\psi\colon E \to E$ with $\tau = \tau \circ \psi_\rho$. We call (E, M, τ) *simple* if there are only finitely many weak automorphisms.

Remark 2.1. (i) A representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$ of the general linear group with $\rho(z \cdot E_n) = z^\alpha \cdot \mathrm{id}_V$ is called *homogeneous of degree* α. Every representation of $\mathrm{GL}(r)$ obviously splits into a direct sum of homogeneous representations. Some cases of inhomogeneous representations ρ can be treated within our framework. Indeed if ρ is a representation, such that its homogeneous components $\rho_1, \ldots, \rho_n$ have *positive* degrees $\alpha_1, \ldots, \alpha_n$, let κ be a common multiple of the α_i. Then we pass to the homogeneous representation

$$\rho' := \bigoplus_{\nu_1\alpha_1 + \cdots + \nu_n\alpha_n = \kappa} S^{\nu_1}\rho_1 \otimes \cdots \otimes S^{\nu_n}\rho_n.$$

The solution of the moduli problem associated with ρ' can be used to solve the moduli problem associated with ρ. This trick was used in [35] and will be recalled in the section on examples.

(ii) The identification of τ and $\lambda \cdot \tau$, or equivalently, considering sections in $\mathbb{P}(E_\rho)$ rather than in E_ρ seems a little artificial. First, this identification is mandatory to get projective moduli spaces. Second, for homogeneous representations of degree $\alpha \neq 0$, this is naturally forced upon us. Third, if we are given a homogeneous representation ρ of degree zero and are interested in the moduli problem without the identification of τ and $\lambda\tau$, we may pass to the representation ρ', obtained from ρ by adding the trivial one-dimensional representation. Then one gets from the solution of the moduli problem associated with ρ' a compactification of the moduli problem associated with ρ. This will be explained within the context of Hitchin pairs in the examples.

In order to define a functor, we first fix a Poincaré line bundle $\mathcal{L}$ on $\mathrm{Jac}^m \times X$. For every scheme S and every morphism $\kappa\colon S \to \mathrm{Jac}^m$, we define $\mathcal{L}[\kappa] := (\kappa \times \mathrm{id}_X)^*\mathcal{L}$. Now, let S be a scheme of finite type over $\mathbb{C}$. Then a *family of ρ-pairs of type* (d, r, m) *parameterized by* S is a tuple $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ with

- E_S a vector bundle of rank r having degree d on $\{s\} \times X$ for all $s \in S$,
- $\kappa_S\colon S \to \mathrm{Jac}^m$ a morphism,
- $\mathfrak{N}_S$ a line bundle on S,
- $\tau_S\colon E_{S,\rho} \to \mathcal{L}[\kappa_S] \otimes \pi_S^*\mathfrak{N}_S$ a homomorphism whose restriction to $\{s\} \times X$ is nonzero for every closed point $s \in S$.

Two families $(E^1_S, \kappa^1_S, \mathfrak{N}^1_S, \tau^1_S)$ and $(E^2_S, \kappa^2_S, \mathfrak{N}^2_S, \tau^2_S)$ are called *equivalent* if $\kappa^1_S = \kappa^2_S =: \kappa_S$ and there exist isomorphisms $\psi_S\colon E^1_S \to E^2_S$ and $\chi_S\colon \mathfrak{N}^1_S \to \mathfrak{N}^2_S$ with

$$\tau^1_S = (\mathrm{id}_{\mathcal{L}[\kappa_S]} \otimes \pi_S^*\chi_S)^{-1} \circ \tau^2_S \circ \psi_{S,\rho}.$$

To define the semistability concept for ρ-pairs, observe that for given (E, M, τ), the homomorphism $\tau\colon E \to M$ will be generically surjective; therefore we get a rational section $\sigma'\colon X \dashrightarrow \mathbb{P}(E_\rho)$ which can, of course, be prolonged to a section $\sigma\colon X \to \mathbb{P}(E_\rho)$, so that we can define for every weighted filtration $(E^\bullet, \underline{\alpha})$ of E

$$\mu_\rho\bigl(E^\bullet, \underline{\alpha}; \tau\bigr) := \mu_\rho\bigl(E^\bullet, \underline{\alpha}; \sigma\bigr).$$

We will occasionally use the following short hand notation: If E' is a nonzero, proper subbundle of E, we set

$$\mu_\rho(E', \tau) := \mu_\rho(0 \subset E' \subset E, (1); \tau).$$

Now, for fixed $\delta \in \mathbb{Q}_{>0}$, call a ρ-pair (E, M, τ) *δ-(semi)stable*, if for every weighted filtration $(E^\bullet, \underline{\alpha})$

$$M\bigl(E^\bullet, \underline{\alpha}\bigr) + \delta \cdot \mu_\rho\bigl(E^\bullet, \underline{\alpha}; \tau\bigr)(\geq)0.$$

Remark 2.2. For the F-pairs, one can formulate the semistability concept in a more intrinsic way. For this, one just has to choose a linearization ρ of the given action in an ample Then $\mu_\rho(E^\bullet, \underline{\alpha}; \Phi)$ can still be defined, and an F-pair (E, Φ) will be called *ρ-(semi)stable* if

$$M\bigl(E^\bullet, \underline{\alpha}\bigr) + \mu_\rho\bigl(E^\bullet, \underline{\alpha}; \Phi\bigr)(\geq)0.$$

In gauge theory, one would say that the notion of semistability depends only on the metric chosen on the fibre F. If ρ is a linearization in an ample line bundle L and $\delta \in \mathbb{Q}$, we can pass to the induced linearization "$\rho^{\otimes\delta}$" in the $\mathbb{Q}$-line bundle δL to recover δ-ρ-semistability. For the moduli problems associated with a representation ρ, the formulation with the parameter δ seems more appropriate and practical and, since we treat F-pairs only as special cases of ρ-pairs, we have opted for the given definition of δ-ρ-semistability.

We define the functors

$$\begin{array}{rccl} \underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}\colon & \underline{\mathrm{Sch}}_{\mathbb{C}} & \to & \underline{\mathrm{Set}} \\ & S & \mapsto & \left\{ \begin{array}{c} \text{Equivalence classes of families of } \delta\text{-(semi)stable} \\ \rho\text{-pairs of type } (d, r, m) \text{ parameterized by } S \end{array} \right\}. \end{array}$$

Remark 2.3. The definition of the moduli functor involves the choice of the Poincaré sheaf $\mathcal{L}$. Nevertheless, the above moduli functor is independent of that choice. Indeed, choosing another Poincaré line bundle $\mathcal{L}'$ on $\mathrm{Jac}^m \times X$, there is a line bundle $\mathfrak{N}_{\mathrm{Jac}^m}$ on Jac^m with $\mathcal{L} \cong \mathcal{L}' \otimes \pi^*_{\mathrm{Jac}^m}\mathfrak{N}_{\mathrm{Jac}^m}$. Therefore, assigning to a family $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ defined via $\mathcal{L}$ the family $(E_S, \kappa_S, \mathfrak{N}_S \otimes \kappa_S^*\mathfrak{N}_{\mathrm{Jac}^m}, \tau_S)$ defined via $\mathcal{L}'$ identifies the functor which is defined with respect to $\mathcal{L}$ with the one defined with respect to $\mathcal{L}'$.

We also define the open subfunctors $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m/\,\mathrm{surj}}$ of equivalence classes of families $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ where $\tau_{S|\{s\}\times X}$ is surjective for all $s \in S$.

Next, let (E, M, τ) be a ρ-pair where τ is surjective, and let $\mathbb{P}^{d,r}$ be the oriented topological projective bundle underlying $\mathbb{P}(E_\rho)$. This is independent of E, and as explained before, the degree of M determines the cohomology class $h_m := [\sigma(X)] \in H_2(\mathbb{P}^{d,r}, \mathbb{Z})$ where σ is the section associated with τ. Set $h := h_m \cap [\mathfrak{F}^{d,r}] \in H_2(\mathfrak{F}^{d,r}, \mathbb{Z})$. We can now define $\underline{\mathrm{M}}(F, \rho)^{\delta-(s)s}_{d/r/h}$ as the closed subfunctor of $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m/\,\mathrm{surj}}$ of equivalence classes of families $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ for which the section $S \times X \to \mathbb{P}(E_{S,\rho})$ factorizes over $\mathfrak{P}(E_S) \times^{\mathrm{GL}(r)} F$.

Polystable pairs. Fix a basis $\underline{w} = (w_1, \dots, w_r)$ of $\mathbb{C}^r$. Let (E, M, τ) be a δ-semistable ρ-pair of type (d, r, m). We call (E, M, τ) *δ-polystable* if for every weighted filtration $(E^\bullet, \underline{\alpha})$, $E^\bullet : 0 =: E_0 \subset E_1 \subset \dots \subset E_s \subset E_{s+1} := E$, with

$$M\big(E^\bullet, \underline{\alpha}\big) + \delta \cdot \mu_\rho\big(E^\bullet, \underline{\alpha}; \tau\big) = 0$$

the following holds true:

- $E \cong \bigoplus_{j=1}^{s+1} E_j/E_{j-1}$,
- the ρ-pair (E, M, τ) is equivalent to the ρ-pair $(E, M, \tau|_{E^\gamma})$, $\gamma := -\mu_\rho(E^\bullet, \underline{\alpha}; \tau)$. Here, one uses the fact that giving an isomorphism $E \to \bigoplus_{j=1}^{s+1} E_j/E_{j-1}$ is the same as giving a cocycle for E in the group

$$Z\big(\lambda(\underline{w}, \underline{\gamma})\big) := \big\{\, g \in \mathrm{GL}(r) \mid g \cdot \lambda(\underline{w}, \underline{\gamma})(z) = \lambda(\underline{w}, \underline{\gamma})(z) \cdot g \ \forall z \in \mathbb{C}^* \,\big\}.$$

  It follows that $E_\rho \cong E^{\gamma_1} \oplus \dots \oplus E^{\gamma_t}$ where $\gamma_1, \dots, \gamma_t$ are the weights of $\lambda(\underline{w}, \underline{\gamma})$ on V and E^{γ_i} is the "eigenbundle" for the weight γ_i, $i = 1, \dots, t$.

As before, $W^\bullet : 0 \subset W_{\underline{w}}^{(\mathrm{rk}\, E_1)} \subset \dots \subset W_{\underline{w}}^{(\mathrm{rk}\, E_s)} \subset W$ and $\underline{\gamma} := \sum_{j=1}^{s} \alpha_j \gamma^{(\mathrm{rk}\, E_j)}$. The stated condition is again independent of the involved choices.

Remark 2.4. (i) If (E, M, τ) is δ-stable, the stated condition is void, so that (E, M, τ) is also δ-polystable.

(ii) It will follow from our GIT construction that (E, M, τ) is δ-stable if and only if it is δ-polystable and has only finitely many weak automorphisms.

(iii) For the description of S-equivalence in the case of $\rho = \rho_{a,b,c}$ for some $a, b, c \in \mathbb{Z}_{\geq 0}$, the reader may consult [16].

2.2. The main result

Theorem 2.5. (i) *There exist a projective scheme $\mathcal{M}(\rho)^{\delta-ss}_{d/r/m}$ and an open subscheme $\mathcal{M}(\rho)^{\delta-s}_{d/r/m} \subset \mathcal{M}(\rho)^{\delta-ss}_{d/r/m}$ together with natural transformations*

$$\vartheta^{(s)s} : \underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m} \longrightarrow h_{\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/m}}$$

with the following properties:

1. *For every scheme $\mathcal{N}$ and every natural transformation $\vartheta' : \underline{\mathrm{M}}(\rho)^{\delta-ss}_{d/r/m} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi : \mathcal{M}(\rho)^{\delta-ss}_{d/r/m} \to \mathcal{N}$ with $\vartheta' = h(\varphi) \circ \vartheta^{ss}$.*
2. *$\mathcal{M}(\rho)^{\delta-s}_{d/r/m}$ is a coarse moduli space for the functor $\underline{\mathrm{M}}(\rho)^{\delta-s}_{d/r/m}$.*
3. *$\vartheta^{ss}(\mathrm{Spec}\,\mathbb{C})$ induces a bijection between the set of equivalence classes of δ-polystable ρ-pairs of type (d, r, m) and the set of closed points of $\mathcal{M}(\rho)^{\delta-ss}_{d/r/m}$.*

(ii) *There exist a locally closed subscheme $\mathcal{M}(F, \rho)^{\delta-s}_{d/r/h}$ of $\mathcal{M}(\rho)^{\delta-s}_{d/r/m}$ and a natural transformation*

$$\vartheta_F : \underline{\mathrm{M}}(F, \rho)^{\delta-s}_{d/r/h} \longrightarrow h_{\mathcal{M}(F,\rho)^{\delta-s}_{d/r/h}}$$

which turns $\mathcal{M}(F, \rho)^{\delta-s}_{d/r/h}$ into the coarse moduli space for $\underline{\mathrm{M}}(F, \rho)^{\delta-s}_{d/r/h}$.

2.3. The proof of the main result

Given any homogeneous representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$, we have seen in Section 1.1 that we can find integers $a, b \geq 0$ and $c > 0$, such that ρ is a direct summand of the representation $\rho_{a,b,c}$. Write $\rho_{a,b,c} = \rho \oplus \overline{\rho}$. For every vector bundle E of rank r, we find $E_{\rho_{a,b,c}} \cong E_\rho \oplus E_{\overline{\rho}}$. Every ρ-pair (E, M, τ) can therefore also be viewed as a $\rho_{a,b,c}$-pair. Since $\mu_\rho(E^\bullet, \underline{\alpha}; \tau) = \mu_{\rho_{a,b,c}}(E^\bullet, \underline{\alpha}; \tau)$ for every weighted filtration $(E^\bullet, \underline{\alpha})$, the triple (E, M, τ) is δ-(semi)stable as ρ-pair if and only if it is δ-(semi)stable as $\rho_{a,b,c}$-pair. More precisely, we can recover $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}$ as closed subfunctor of $\underline{\mathrm{M}}(\rho_{a,b,c})^{\delta-(s)s}_{d/r/m}$. Indeed, for every scheme of finite type over $\mathbb{C}$,

$$\begin{aligned}\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}(S) \quad = \quad & \big\{ [E_S, \kappa_S, \mathfrak{N}_S, \tau_S] \in \underline{\mathrm{M}}(\rho_{a,b,c})^{\delta-(s)s}_{d/r/m}(S) \mid \\ & \tau_S\colon E_{S,\rho_{a,b,c}} \to \mathcal{L}[\kappa_S] \otimes \pi_S^* \mathfrak{N}_S \text{ vanishes on } E_{S,\overline{\rho}} \big\}.\end{aligned}$$

Therefore, we will assume from now on that $\rho = \rho_{a,b,c}$ for some a, b, c.

Boundedness.

Theorem 2.6. *There is a nonnegative constant C_1, depending only on r, a, and δ, such that for every δ-semistable $\rho_{a,b,c}$-pair (E, M, τ) of type (d, r, m) and every nontrivial proper subbundle E' of E*

$$\mu(E') \leq \frac{d}{r} + C_1.$$

Proof. Let $0 \subsetneq E' \subsetneq E$ be any subbundle. By Lemma 1.8 i), $\mu_{\rho_{a,b,c}}(E', \tau) \leq a(r-1)$, so that δ-semistability gives $d \operatorname{rk} E' - \deg(E')r + \delta \cdot a \cdot (r-1) \geq d \operatorname{rk} E' - \deg(E')r + \delta \cdot \mu_{\rho_{a,b,c}}\big(E', \tau\big) \geq 0$, i.e.,

$$\mu(E') \leq \frac{d}{r} + \frac{\delta \cdot a \cdot (r-1)}{r \cdot \operatorname{rk} E'} \leq \frac{d}{r} + \frac{\delta \cdot a \cdot (r-1)}{r},$$

so that the theorem holds for $C_1 := \delta \cdot a \cdot (r-1)/r$. □

Construction of the parameter space. Recall that for a scheme S of finite type over $\mathbb{C}$, a family of $\rho_{a,b,c}$-pairs parameterized by S is a quadruple $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ where E_S is a vector bundle of rank r on $S \times X$ with $\deg(E_{S|\{s\}\times X}) = d$ for all $s \in S$, $\kappa_S\colon S \to \mathrm{Jac}^m$ is a morphism, $\mathfrak{N}_S$ is a line bundle on S, and $\tau_S\colon E_S^{\otimes a \oplus c} \to \det(E_S)^{\otimes b} \otimes \mathcal{L}[\kappa_S] \otimes \pi_S^* \mathfrak{N}_S$ is a homomorphism which is nonzero on every fibre $\{s\} \times X$.

Pick a point $x_0 \in X$, and write $\mathcal{O}_X(1)$ for $\mathcal{O}_X(x_0)$. According to 2.6, we can choose an integer n_0, such that for every $n \geq n_0$ and every δ-semistable $\rho_{a,b,c}$-pair (E, M, τ) of type (d, r, m),

- $H^1(E(n)) = 0$ and $E(n)$ is globally generated,
- $H^1(\det(E)(rn)) = 0$ and $\det(E)(rn)$ is globally generated,
- $H^1(\det(E)^{\otimes b} \otimes M \otimes \mathcal{O}_X(na)) = 0$ and $\det(E)^{\otimes b} \otimes M \otimes \mathcal{O}_X(na)$ is globally generated.

Choose some $n \geq n_0$ and set $p := d + rn + r(1-g)$. Let U be a complex vector space of dimension p. We define $\mathfrak{Q}^0$ as the quasi-projective scheme parameterizing equivalence classes of quotients $q\colon U\otimes\mathcal{O}_X(-n)\to E$ where E is a vector bundle of rank r and degree d on X and $H^0(q(n))$ is an isomorphism. Then there exists a universal quotient

$$q_{\mathfrak{Q}^0}\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_{\mathfrak{Q}^0}$$

on $\mathfrak{Q}^0\times X$. Let

$$q_{\mathfrak{Q}^0\times \mathrm{Jac}^m}\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_{\mathfrak{Q}^0\times \mathrm{Jac}^m}$$

be the pullback of $q_{\mathfrak{Q}^0}$ to $\mathfrak{Q}^0\times \mathrm{Jac}^m\times X$. Set $U_{a,c}:=U^{\otimes a^{\oplus c}}$. By our assumption, the sheaf

$$\underline{\mathrm{Hom}}\Big(U_{a,c}\otimes\mathcal{O}_{\mathfrak{Q}^0\times\mathrm{Jac}^m},\pi_{\mathfrak{Q}^0\times\mathrm{Jac}^m\,*}\big(\det(E_{\mathfrak{Q}^0\times\mathrm{Jac}^m})^{\otimes b}\otimes\mathcal{L}[\pi_{\mathrm{Jac}^m}]\otimes\pi_X^*\mathcal{O}_X(na)\big)\Big)$$

is locally free, call it $\mathcal{H}$, and set $\mathfrak{H}:=\mathbb{P}(\mathcal{H}^\vee)$. We let

$$q_{\mathfrak{H}}\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_{\mathfrak{H}}$$

be the pullback of $q_{\mathfrak{Q}^0\times\mathrm{Jac}^m}$ to $\mathfrak{H}\times X$. Now, on $\mathfrak{H}\times X$, there is the tautological homomorphism

$$s_{\mathfrak{H}}\colon U_{a,c}\otimes\mathcal{O}_{\mathfrak{H}}\longrightarrow\det(E_{\mathfrak{H}})^{\otimes b}\otimes\mathcal{L}[\kappa_{\mathfrak{H}}]\otimes\pi_X^*\mathcal{O}_X(na)\otimes\pi_{\mathfrak{H}}^*\mathcal{O}_{\mathfrak{H}}(1).$$

Here, $\kappa_{\mathfrak{H}}\colon\mathfrak{H}\to\mathfrak{Q}^0\times\mathrm{Jac}^m\to\mathrm{Jac}^m$ is the natural morphism. Let $\mathfrak{T}$ be the closed subscheme defined by the condition that $s_{\mathfrak{H}}\otimes\pi_X^*\,\mathrm{id}_{\mathcal{O}_X(-na)}$ vanish on

$$\ker\big(U_{a,c}\otimes\pi_X^*\mathcal{O}_X(-na)\longrightarrow E_{\mathfrak{H}}^{\otimes a^{\oplus c}}\big).$$

Let

$$q_{\mathfrak{T}}\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_{\mathfrak{T}}$$

be the restriction of $q_{\mathfrak{H}}$ to $\mathfrak{T}\times X$. By definition, there is a universal homomorphism

$$\tau_{\mathfrak{T}}\colon E_{\mathfrak{T}}^{\otimes a^{\oplus c}}\longrightarrow\det(E_{\mathfrak{T}})^{\otimes b}\otimes\mathcal{L}[\kappa_{\mathfrak{T}}]\otimes\pi_{\mathfrak{T}}^*\mathfrak{N}_{\mathfrak{T}}.$$

Here, $\mathfrak{N}_{\mathfrak{T}}$ and $\kappa_{\mathfrak{T}}$ are the restrictions of $\mathcal{O}_{\mathfrak{H}}(1)$ and $\kappa_{\mathfrak{H}}$ to $\mathfrak{T}$. Note that the parameter space $\mathfrak{T}$ is equipped with a universal family $(E_{\mathfrak{T}},\kappa_{\mathfrak{T}},\mathfrak{N}_{\mathfrak{T}},\tau_{\mathfrak{T}})$.

Remark 2.7. Let S be a scheme of finite type over $\mathbb{C}$. Call a tuple $(q_S,\kappa_S,\mathfrak{N}_S,\tau_S)$ where

- $q_S\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\to E_S$ is a family of quotients, such that its restriction to $\{s\}\times X$ lies in $\mathfrak{Q}^0$ for every $s\in S$,
- $\kappa_S\colon S\to\mathrm{Jac}^m$ is a morphism,
- $\mathfrak{N}_S$ is a line bundle on S, and
- $\tau_S\colon E_S^{\otimes a^{\oplus c}}\to\det(E_S)^{\otimes b}\otimes\mathcal{L}[\kappa_S]\otimes\pi_S^*\mathfrak{N}_S$ is a homomorphism which is nontrivial on all fibres $\{s\}\times X$, $s\in S$,

a *quotient family of $\rho_{a,b,c}$-pairs of type (d,r,m) parameterized by S*. We say that the families $(q_S^1,\kappa_S^1,\mathfrak{N}_S^1,\tau_S^1)$ and $(q_S^2,\kappa_S^2,\mathfrak{N}_S^2,\tau_S^2)$ are *equivalent* if $\kappa_S^1=\kappa_S^2=:\kappa_S$ and there are isomorphisms $\psi_S\colon E_S^1\to E_S^2$ and $\chi_S\colon\mathfrak{N}_S^1\to\mathfrak{N}_S^2$ with $q_S^2=\psi_S\circ q_S^1$ and

$$\tau_S^1=\big(\mathrm{id}_{\mathcal{L}[\kappa_S]}\otimes\pi_S^*(\chi_S)\big)^{-1}\circ\tau_S^2\circ\Big(\psi_S^{\otimes a^{\oplus c}}\Big).$$

It can be easily inferred from the construction of $\mathfrak{T}$ and the base change theorem that $\mathfrak{T}$ represents the functor which assigns to a scheme S of finite type over $\mathbb{C}$ the set of equivalence classes of quotient families of $\rho_{a,b,c}$-pairs of type (d,r,m) parameterized by S.

Proposition 2.8. (Local universal property) *Let S be a scheme of finite type over $\mathbb{C}$, and $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ a family of δ-semistable $\rho_{a,b,c}$-pairs parameterized by S. Then there exist an open covering S_i, $i \in I$, of S, and morphisms $\beta_i \colon S_i \to \mathfrak{T}$, $i \in I$, such that the restriction of the family $(E_S, \kappa_S, \mathfrak{N}_S, \tau_S)$ to $S_i \times X$ is equivalent to the pullback of $(E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}})$ via $\beta_i \times \mathrm{id}_X$, for all $i \in I$.*

Proof. By our assumptions, the sheaf $\pi_{S*}(E_S \otimes \pi_X^* \mathcal{O}_X(n))$ is locally free of rank p. Therefore, we can choose a covering S_i, $i \in I$, of S, such that it is free over S_i for all $i \in I$. For each i, we can choose a trivialization $U \otimes \mathcal{O}_{S_i} \cong \pi_{S*}\big(E_S \otimes \pi_X^*\mathcal{O}_X(n)|_{S_i}\big)$, so that we obtain a surjection $q_{S_i} \colon U \otimes \pi_X^* \mathcal{O}_X(-n) \longrightarrow E_{S|S_i \times X}$ on $S_i \times X$. Therefore, $(q_{S_i}, \kappa_{S|S_i}, \mathfrak{N}_{S|S_i}, \tau_{S|S_i \times X})$ is a quotient family of $\rho_{a,b,c}$-pairs of type (d, r, m) parameterized by S_i, and we can conclude by Remark 2.7. □

The group action. Let $m \colon U \otimes \mathcal{O}_{\mathrm{SL}(U)} \to U \otimes \mathcal{O}_{\mathrm{SL}(U)}$ be the universal automorphism over $\mathrm{SL}(U)$. Let $(E_{\mathrm{SL}(U)\times\mathfrak{T}}, \kappa_{\mathrm{SL}(U)\times\mathfrak{T}}, \mathfrak{N}_{\mathrm{SL}(U)\times\mathfrak{T}}, \tau_{\mathrm{SL}(U)\times\mathfrak{T}})$ be the pullback of the universal family on $\mathfrak{T} \times X$ to $\mathrm{SL}(U) \times \mathfrak{T} \times X$. Define

$$q_{\mathrm{SL}(U)\times\mathfrak{T}} \colon U \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\pi_{\mathrm{SL}(U)}^*(m^{-1}) \otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(-n)}} U \otimes \pi_X^*\mathcal{O}_X(-n) \longrightarrow E_{\mathrm{SL}(U)\times\mathfrak{T}}.$$

Thus, $(q_{\mathrm{SL}(U)\times\mathfrak{T}}, \kappa_{\mathrm{SL}(U)\times\mathfrak{T}}, \mathfrak{N}_{\mathrm{SL}(U)\times\mathfrak{T}}, \tau_{\mathrm{SL}(U)\times\mathfrak{T}})$ is a quotient family of $\rho_{a,b,c}$-pairs parameterized by $\mathrm{SL}(U) \times \mathfrak{T}$, and hence, by 2.7, defines a morphism

$$\Gamma \colon \mathrm{SL}(U) \times \mathfrak{T} \longrightarrow \mathfrak{T}.$$

It is not hard to see that Γ is indeed a group action. Note that this action descends to a $\mathrm{PGL}(U)$-action!

Remark 2.9. By construction, the universal family $(E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}})$ comes with a linearization, i.e., with an isomorphism

$$\big(\Gamma \times \mathrm{id}_X\big)^*\big(E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}}\big) \longrightarrow \big(\pi_{\mathfrak{T}} \times \mathrm{id}_X\big)^*\big(E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}}\big).$$

Therefore, elements of the $\mathrm{PGL}(U)$-stabilizer of a point $t \in \mathfrak{T}$ correspond to weak automorphisms of the $\rho_{a,b,c}$-pair $(E_t, M_t, \tau_t) := (E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}})|_{\{t\}\times X}$. In particular, the $\mathrm{SL}(U)$-stabilizer of t is finite if and only if (E_t, M_t, τ_t) has only finitely many weak automorphisms.

Proposition 2.10. *Let S be a scheme of finite type over $\mathbb{C}$ and $\beta_{1,2} \colon S \to \mathfrak{T}$ two morphisms, such that the pullbacks of $(E_{\mathfrak{T}}, \kappa_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}}, \tau_{\mathfrak{T}})$ via $\beta_1 \times \mathrm{id}_X$ and $\beta_2 \times \mathrm{id}_X$ are equivalent. Then there exist an étale covering $\eta \colon T \to S$ and a morphism $\Xi \colon T \to \mathrm{SL}(U)$, such that the morphism $\beta_2 \circ \eta \colon T \to \mathfrak{T}$ equals the morphism*

$$T \xrightarrow{\Xi \times (\beta_1 \circ \eta)} \mathrm{SL}(U) \times \mathfrak{T} \xrightarrow{\Gamma} \mathfrak{T}.$$

Proof. The two morphisms β_1 and β_2 provide us with quotient families $(q_S^1, \kappa_S^1, \mathfrak{N}_S^1, \tau_S^1)$ and $(q_S^2, \kappa_S^2, \mathfrak{N}_S^2, \tau_S^2)$ of $\rho_{a,b,c}$-pairs parameterized by S. By hypothesis, $\kappa_S^1 = \kappa_S^2 =: \kappa_S$, and we have isomorphisms $\psi_S \colon E_S^1 \to E_S^2$ and $\chi_S \colon \mathfrak{N}_S^1 \to \mathfrak{N}_S^2$ with

$$\tau_S^1 = (\mathrm{id}_{\mathcal{L}[\kappa_S]} \otimes \pi_S^*\chi_S)^{-1} \circ \tau_S^2 \circ (\psi_S^{\otimes a^{\oplus c}}).$$

In particular, there is an isomorphism

$$\begin{aligned} U\otimes\mathcal{O}_S &\xrightarrow{\pi_{S*}(q_S^1\otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(n)})} \pi_{S_*}\big(E_S^1\otimes\pi_X^*\mathcal{O}_X(n)\big)\longrightarrow \\ &\xrightarrow{\pi_{S*}(\psi_S\otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(n)})} \pi_{S_*}\big(E_S^2\otimes\pi_X^*\mathcal{O}_X(n)\big)\xrightarrow{\pi_{S*}(q_S^2\otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(n)})^{-1}} U\otimes\mathcal{O}_S.\end{aligned}$$

This yields a morphism $\Xi_S\colon S\to \mathrm{GL}(U)$ and $\Delta_S:=(\det)\circ\Xi_S\colon S\to\mathbb{C}^*$. Let $T:=S\times_{\mathbb{C}^*}\mathbb{C}^*$ be the fibre product taken with respect to Δ_S and $\mathbb{C}^*\to\mathbb{C}^*$, $z\mapsto z^p$. The morphism $\eta\colon T\to S$ is then a p-sheeted étale covering coming with the projection map $\widetilde{\Delta}\colon T\to\mathbb{C}^*$. In the following, we set $\widetilde{\Delta}^e:=(z\mapsto z^e)\circ\widetilde{\Delta}$, $e\in\mathbb{Z}$. One has $\widetilde{\Delta}^p=\Delta_S\circ\eta$. By construction, the morphism

$$T\xrightarrow{\widetilde{\Delta}^{-1}\times(\Xi_S\circ\eta)}\mathbb{C}^*\times\mathrm{GL}(U)\xrightarrow{\mathrm{mult}}\mathrm{GL}(U)$$

factorizes over a morphism $\Xi\colon T\to\mathrm{SL}(U)$. The quotient family defined by the morphism

$$T\xrightarrow{\Xi\times(\beta_1\circ\eta)}\mathrm{SL}(U)\times\mathfrak{T}\xrightarrow{\Gamma}\mathfrak{T}$$

is just $(\widetilde{q}_S^1,\kappa_S\circ\eta,\eta^*\mathfrak{N}_S^1,(\eta\times\mathrm{id}_X)^*\tau_S^1)$ with

$$\widetilde{q}_S^1\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\xrightarrow{\Xi^*(m^{-1})\otimes\mathrm{id}_{\pi_X^*\mathcal{O}_X(-n)}}U\otimes\pi_X^*\mathcal{O}_X(-n)\xrightarrow{(\eta\times\mathrm{id}_X)^*q_S^1}(\eta\times\mathrm{id}_X)^*E_S^1.$$

The assertion of the proposition is that this family is equivalent to the quotient family $((\eta\times\mathrm{id}_X)^*q_S^2,\kappa_S\circ\eta,\eta^*\mathfrak{N}_S^2,(\eta\times\mathrm{id}_X)^*\tau_S^2)$. But this is easily seen, using

$$\widetilde{\psi}_T:=\widetilde{\Delta}\cdot((\eta\times\mathrm{id}_X)^*\psi_S^{-1})\colon(\eta\times\mathrm{id}_X)^*E_S^2\to(\eta\times\mathrm{id}_X)^*E_S^1$$

and $\widetilde{\chi}_T:=\widetilde{\Delta}^{a-rb}\cdot(\eta^*\chi_S^{-1})\colon\eta^*\mathfrak{N}_S^2\to\eta^*\mathfrak{N}_S^1$. □

The Gieseker space and map. Choose a Poincaré sheaf $\mathcal{P}$ on $\mathrm{Jac}^d\times X$. By our assumptions on n, the sheaf

$$\mathcal{G}_1:=\underline{\mathrm{Hom}}\big(\textstyle\bigwedge^r U\otimes\mathcal{O}_{\mathrm{Jac}^d},\pi_{\mathrm{Jac}^d\,*}\big(\mathcal{P}\otimes\pi_X^*\mathcal{O}_X(rn)\big)\big)$$

is locally free. We set $\mathbb{G}_1:=\mathbb{P}(\mathcal{G}_1^\vee)$. By replacing $\mathcal{P}$ with $\mathcal{P}\otimes\pi_{\mathrm{Jac}^d}^*$ (sufficiently ample), we may assume that $\mathcal{O}_{\mathbb{G}_1}(1)$ is very ample. Let $\mathfrak{d}\colon\mathfrak{T}\to\mathrm{Jac}^d$ be the morphism associated with $\bigwedge^r E_{\mathfrak{T}}$, and let $\mathfrak{A}_{\mathfrak{T}}$ be a line bundle on $\mathfrak{T}$ with $\bigwedge^r E_{\mathfrak{T}}\cong(\mathfrak{d}\times\mathrm{id}_X)^*\mathcal{P}\otimes\pi_{\mathfrak{T}}^*\mathfrak{A}_{\mathfrak{T}}$. Then

$$\textstyle\bigwedge^r\big(q_{\mathfrak{T}}\otimes\mathrm{id}_{\pi_X^*\mathcal{O}_X(n)}\big)\colon\bigwedge^r U\otimes\mathcal{O}_{\mathfrak{T}}\longrightarrow(\mathfrak{d}\times\mathrm{id}_X)^*\mathcal{P}\otimes\pi_X^*\mathcal{O}_X(rn)\otimes\pi_{\mathfrak{T}}^*\mathfrak{A}_{\mathfrak{T}}$$

defines a morphism $\iota_1\colon\mathfrak{T}\to\mathbb{G}_1$ with $\iota_1^*\mathcal{O}_{\mathbb{G}_1}(1)=\mathfrak{A}_{\mathfrak{T}}$.

Set $J^{d,m}:=\mathrm{Jac}^d\times\mathrm{Jac}^m$. The sheaf

$$\mathcal{G}_2:=\underline{\mathrm{Hom}}\big(U_{a,c}\otimes\mathcal{O}_{J^{d,m}},\pi_{J^{d,m}*}\big(\pi_{\mathrm{Jac}^d\times X}^*(\mathcal{P})^{\otimes b}\otimes\pi_{\mathrm{Jac}^m\times X}^*(\mathcal{L})\otimes\pi_X^*\mathcal{O}_X(na)\big)\big)$$

on $J^{d,m}$ is also locally free. Set $\mathbb{G}_2 := \mathbb{P}(\mathcal{G}_2^\vee)$. Making use of Remark 2.3, it is clear that we can assume $\mathcal{O}_{\mathbb{G}_2}(1)$ to be very ample. The homomorphism

$$\begin{aligned} U_{a,c} \otimes \mathcal{O}_{\mathfrak{T}} &\longrightarrow E_{\mathfrak{T}}^{\otimes a \oplus c} \otimes \pi_X^* \mathcal{O}_X(na) \longrightarrow \\ &\longrightarrow (\mathfrak{d} \times \mathrm{id}_X)^* \mathcal{P}^{\otimes b} \otimes \mathcal{L}[\kappa_{\mathfrak{T}}] \otimes \pi_X^* \mathcal{O}_X(na) \otimes \pi_{\mathfrak{T}}^* \big(\mathfrak{A}_{\mathfrak{T}}^{\otimes b} \otimes \mathfrak{N}_{\mathfrak{T}} \big) \end{aligned}$$

provides a morphism $\iota_2 \colon \mathfrak{T} \to \mathbb{G}_2$ with $\iota_2^* \mathcal{O}_{\mathbb{G}_2}(1) = \mathfrak{A}_{\mathfrak{T}}^{\otimes b} \otimes \mathfrak{N}_{\mathfrak{T}}$. Altogether, setting $\mathbb{G} := \mathbb{G}_1 \times \mathbb{G}_2$ and $\iota := \iota_1 \times \iota_2$, we have an injective and $\mathrm{SL}(U)$-equivariant morphism

$$\iota \colon \mathfrak{T} \longrightarrow \mathbb{G}.$$

Linearize the $\mathrm{SL}(U)$-action on $\mathbb{G}$ in $\mathcal{O}_{\mathbb{G}}(\varepsilon, 1)$ with

$$\varepsilon := \frac{p - a \cdot \delta}{r\delta},$$

and denote by $\mathbb{G}^{\varepsilon-(s/p)s}$ the sets of points in $\mathbb{G}$ which are $\mathrm{SL}(U)$-(semi/poly)stable with respect to the given linearization.

Theorem 2.11. *For n large enough, the following two properties hold true:*

(i) *The preimages $\iota^{-1}(\mathbb{G}^{\varepsilon-(s/p)s})$ consist exactly of those points $t \in \mathfrak{T}$ for which (E_t, M_t, τ_t) (notation as in Rem. 2.9) is a δ-(semi/poly)stable $\rho_{a,b,c}$-pair of type (d, r, m).*

(ii) *The restricted morphism $\iota|_{\iota^{-1}(\mathbb{G}^{\varepsilon-ss})} \colon \iota^{-1}\big(\mathbb{G}^{\varepsilon-ss}\big) \to \mathbb{G}^{\varepsilon-ss}$ is proper.*

The proof of this theorem will be given in a later section.

Proof of Theorem 2.5. Set $\mathfrak{T}^{\delta-(s)s} := \iota^{-1}\big(\mathbb{G}^{\varepsilon-(s)s}\big)$. Theorem 2.11 now shows that the categorical quotients

$$\mathcal{M}(\rho_{a,b,c})_{d/r/m}^{\delta-(s)s} := \mathfrak{T}^{\delta-(s)s} /\!\!/ \mathrm{SL}(U)$$

exist and that $\mathcal{M}(\rho_{a,b,c})_{d/r/m}^{\delta-s}$ is an orbit space. Proposition 2.8 and Proposition 2.10 tell us that we have a natural transformation of the functor $\underline{\mathrm{M}}(\rho_{a,b,c})_{d/r/m}^{\delta-(s)s}$ into the functor of points of $\mathcal{M}(\rho_{a,b,c})_{d/r/m}^{\delta-(s)s}$. The asserted minimality property of $\mathcal{M}(\rho_{a,b,c})_{d/r/m}^{\delta-ss}$ and $\mathcal{M}(\rho_{a,b,c})_{d/r/m}^{\delta-s}$'s being a coarse moduli space follow immediately from the universal property of the categorical quotient. Finally, the assertion about the closed points is a consequence of the "polystable" part of 2.11. Therefore, Theorem 2.5 is settled for representations of the form $\rho_{a,b,c}$.

For an arbitrary representation ρ, we may find a, b, c and a decomposition $\rho_{a,b,c} = \rho \oplus \overline{\rho}$. Define $\mathfrak{T}(\rho)$ as the closed subscheme of $\mathfrak{T}$ where the homomorphism

$$\widetilde{\tau}_{\mathfrak{T}} \colon E_{S,\rho_{a,b,c}} = E_S^{\otimes a \oplus c} \otimes \big(\textstyle\bigwedge^r E_S \big)^{\otimes -b} \longrightarrow \mathcal{L}[\kappa_{\mathfrak{T}}] \otimes \pi_{\mathfrak{T}}^* \mathfrak{N}_{\mathfrak{T}}$$

vanishes on $E_{S,\overline{\rho}}$. Set $\mathfrak{T}(\rho)^{\delta-(s)s} := \mathfrak{T}(\rho) \cap \mathfrak{T}^{\delta-(s)s}$. It follows that the categorical quotients

$$\mathcal{M}(\rho)_{d/r/m}^{\delta-(s)s} := \mathfrak{T}(\rho)^{\delta-(s)s} /\!\!/ \mathrm{SL}(U)$$

also exist. By our characterization of $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}$ as a closed subfunctor of the functor $\underline{\mathrm{M}}(\rho_{a,b,c})^{\delta-(s)s}_{d/r/m}$, the theorem follows likewise for ρ.

Next, we let $\mathfrak{T}_{\mathrm{surj}}$ be the open subscheme of $\mathfrak{T}$ consisting of those points t for which $\widetilde{\tau}_{\mathfrak{T}|\{t\}\times X}$ is surjective and set $\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}} := \mathfrak{T}(\rho)^{\delta-s}\cap\mathfrak{T}_{\mathrm{surj}}$. Thus, there is a section

$$\sigma_{\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}}\colon\ \mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}\times X\longrightarrow\mathbb{P}(E_{S,\rho}).$$

Moreover, $\mathfrak{P}(E_S)\times^{\mathrm{GL}(r)}F$ is a closed subscheme of $\mathbb{P}(E_{S,\rho})$. Now, we define $\mathfrak{T}(F,\rho)^{\delta-\rho-s}$ as the closed subscheme of those points $t\in\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}$ for which the restricted morphism $\sigma_{\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}|\{t\}\times X}$ factorizes over $\mathfrak{P}(E_S)\times^{\mathrm{GL}(r)}F$. Since the action of $\mathrm{SL}(U)$ on $\mathfrak{T}(\rho)^{\delta-s}$ is closed, the categorical quotient $\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}/\!\!/\,\mathrm{SL}(U)$ exists as an open subscheme of the moduli space $\mathcal{M}(\rho)^{\delta-s}_{d/r/m}$, whence

$$\mathcal{M}(F,\rho)^{\delta-\rho-s}_{d/r/h} := \mathfrak{T}(F,\rho)^{\delta-\rho-s}/\!\!/\,\mathrm{SL}(U)$$

exists as a closed subscheme of $\mathfrak{T}(\rho)^{\delta-s}_{\mathrm{surj}}/\!\!/\,\mathrm{SL}(U)$ and hence as a locally closed subscheme of $\mathcal{M}(\rho)^{\delta-s}_{d/r/m}$ as asserted. □

Proof of Theorem 2.11.

Notation and preliminaries. The remarks about one-parameter subgroups of $\mathrm{SL}(r)$ in Section 1.3 naturally apply to one-parameter subgroups of $\mathrm{SL}(U)$. We set

$$\gamma_p^{(i)} := \big(\underbrace{i-p,\dots,i-p}_{i\times},\underbrace{i,\dots,i}_{(p-i)\times}\big),\quad i=1,\dots,p-1.$$

Given a basis $\underline{u}=(u_1,\dots,u_p)$ of U and a weight vector $\underline{\widetilde{\gamma}}=\sum_{i=1}^{p-1}\beta_i\gamma_p^{(i)}$, we denote the corresponding one-parameter subgroup of $\mathrm{SL}(U)$ by $\lambda(\underline{u},\underline{\widetilde{\gamma}})$. We hope that these conventions will not give rise to too much confusion. Having fixed a basis $\underline{u}=(u_1,\dots,u_p)$ of U and an index $l\in\{1,\dots,p\}$, we set $U^{(l)}_{\underline{u}} := \langle u_1,\dots,u_l\rangle$.

Let $\rho_{\mathbb{G}_1}$ be the natural linearization of the $\mathrm{SL}(U)$-action on $\mathbb{G}_1$ in $\mathcal{O}_{\mathbb{G}_1}(1)$. Then we write $\mu_{\mathbb{G}_1}(.\,,.)$ instead of $\mu_{\rho_{\mathbb{G}_1}}(.\,,.)$. In the same way, $\mu_{\mathbb{G}_2}(.\,,.)$ is to be read. Finally, $\mu^{\varepsilon}_{\mathbb{G}}(.\,,.) := \varepsilon\mu_{\mathbb{G}_1}(.\,,.)+\mu_{\mathbb{G}_2}(.\,,.)$, i.e., $\mu^{\varepsilon}_{\mathbb{G}}(.\,,.)=\mu_{\rho^{\varepsilon}_{\mathbb{G}}}(.\,,.)$, where $\rho^{\varepsilon}_{\mathbb{G}}$ stands for the linearization of the $\mathrm{SL}(U)$-action on $\mathbb{G}$ in $\mathcal{O}(\varepsilon,1)$, $\varepsilon\in\mathbb{Q}_{>0}$.

Let $q\colon U\otimes\mathcal{O}_X(-n)\to E$ be a generically surjective homomorphism and E a vector bundle of degree d and rank r. Set $Z := H^0(\det(E)(rn))$. Then $h := \bigwedge^r(q\otimes\mathrm{id}_{\mathcal{O}_X(n)})\in\mathrm{Hom}(\bigwedge^r U,Z)$ is nontrivial, and we can look at $[h]\in\mathbb{P}(\mathrm{Hom}(\bigwedge^r U,Z)^\vee)$. On this space, there is a natural $\mathrm{SL}(U)$-action. Then it is well known (e.g., [21]) that for any basis $\underline{u}=(u_1,\dots,u_p)$ and any two weight vectors $\underline{\gamma}^i=(\gamma^i_1,\dots,\gamma^i_p)$ with $\gamma^i_1\le\dots\le\gamma^i_p$ and $\sum\gamma^i_j=0$, $i=1,2$,

$$\mu\big(\lambda(\underline{u},\underline{\gamma}^1),[h]\big)+\mu\big(\lambda(\underline{u},\underline{\gamma}^2),[h]\big)=\mu\big(\lambda(\underline{u},\underline{\gamma}^1+\underline{\gamma}^2),[h]\big)$$

and for every $l\in\{1,\dots,p-1\}$

$$\mu\big(\lambda(\underline{u},\underline{\gamma}^{(l)}),[h]\big)=p\,\mathrm{rk}\,E_l-lr. \qquad (1)$$

Here, $E_l\subset E$ stands for the subbundle generated by $q\big(U^{(l)}_{\underline{u}}\otimes\mathcal{O}_X(-n)\big)$.

Sectional semistability.

Theorem 2.12. *Fix the tuple (d, r, m) and a, b, c as before. Then there exists an n_1, such that for every $n \geq n_1$ and every δ-(semi)stable $\rho_{a,b,c}$-pair (E, M, τ), the following holds true: For every weighted filtration $(E^\bullet, \underline{\alpha})$, $E^\bullet : 0 \subset E_1 \subset \cdots \subset E_s \subset E$, of E*

$$\sum_{j=1}^{s} \alpha_{i_j} \big(\chi(E(n)) \operatorname{rk} E_i - h^0(E_i(n)) \operatorname{rk} E\big) + \delta \cdot \mu_\rho\big(E^\bullet, \underline{\alpha}; \tau\big)(\geq)0.$$

Proof. First, suppose we are given a weighted filtration $(E^\bullet, \underline{\alpha})$, $E^\bullet : 0 \subset E_1 \subset \cdots \subset E_s \subset E$, such that $E_i(n)$ is globally generated and $H^1(E_i(n)) = 0$ for $i = 1, \ldots, s$. Then for $i = 1, \ldots, s$,

$$\begin{aligned} \chi(E(n)) \operatorname{rk} E_i - & h^0(E_i(n)) \operatorname{rk} E \\ &= \big(d + r(n+1-g)\big) \operatorname{rk} E_i - \big(\deg(E_i) + \operatorname{rk} E_i (n+1-g)\big) r \\ &= d \operatorname{rk} E_i - \deg(E_i) r, \end{aligned}$$

so that the claimed condition follows from (E, M, τ) being δ-(semi)stable.

Next, recall that we have found a universal positive constant C_1 depending only on r, a, and δ, such that for every d, every semistable $\rho_{a,b,c}$-pair (E, M, τ), and every nontrivial subbundle E' of E,

$$\mu(E') \leq \frac{d}{r} + C_1.$$

If we fix another positive constant C_2, then the set of isomorphy classes of vector bundles E' such that $\mu(E') \geq (d/r) - C_2$, $\mu_{\max}(E') \leq (d/r) + C_1$, and $1 \leq \operatorname{rk} E' \leq r - 1$ is bounded. From this, we infer that there is a natural number $n(C_2)$, such that for every $n \geq n(C_2)$, every semistable $\rho_{a,b,c}$-pair (E, M, τ) of type (d, r, m), and every proper subbundle E' of E

- either $\mu(E') < (d/r) - C_2$
- or $E'(n)$ is globally generated and $H^1(E'(n)) = 0$.

Moreover, the Le Potier–Simpson estimate (see [24], Lemma 7.1.2 and proof of 7.1.1, p. 106) gives in the first case

$$h^0\big(E'(n)\big) \;\leq\; \operatorname{rk} E' \cdot \left(\frac{\operatorname{rk} E' - 1}{\operatorname{rk} E'} \Big[\frac{d}{r} + C_1 + n + 1\Big]_+ + \frac{1}{\operatorname{rk} E'} \Big[\frac{d}{r} - C_2 + n + 1\Big]_+ \right),$$

i.e., for large n,

$$h^0\big(E'(n)\big) \leq \operatorname{rk} E' \left(\frac{d}{r} + n + 1 + (r-2) C_1 - \frac{C_2}{r} \right),$$

and thus

$$\begin{aligned} \chi\big(E(n)\big) \operatorname{rk} E' - h^0\big(E'(n)\big) r \;&\geq\; K(g, r, C_1, C_2) \\ &:= \; -r(r-1)g - r(r-1)(r-2)C_1 + C_2. \end{aligned}$$

Our contention is now that for C_2 with $K(g,r,C_1,C_2) > \delta\cdot a\cdot(r-1)$ and $n_1 := n(C_2)$, the theorem holds true.

So, assume that we are given a weighted filtration $(E^\bullet,\underline{\alpha})$ with $E^\bullet: 0\subset E_1\subset\cdots\subset E_s\subset E$ and $\underline{\alpha}=(\alpha_1,\dots,\alpha_s)$. Let $j_1,\dots,j_t$ be the indices such that $\mu(E_{j_i})\ge d/r-C_2$, for $i=1,\dots,t$, so that $E_{j_i}(n)$ is globally generated and $H^1(E_{j_i}(n))=0$, $i=1,\dots,t$. We let $\widetilde{\jmath}_1,\dots,\widetilde{\jmath}_{s-t}$ be the indices in $\{1,\dots,s\}\setminus\{j_1,\dots,j_t\}$ in increasing order. We introduce the weighted filtrations $(E_1^\bullet,\underline{\alpha}_1)$ and $(E_2^\bullet,\underline{\alpha}_2)$ with $E_1^\bullet: 0\subset E_{j_1}\subset\cdots\subset E_{j_t}\subset E$, $\underline{\alpha}_1 := (\alpha_{j_1},\dots,\alpha_{j_t})$ and $E_2^\bullet: 0\subset E_{\widetilde{\jmath}_1}\subset\cdots\subset E_{\widetilde{\jmath}_{s-t}}\subset E$, $\underline{\alpha}_2=(\alpha_{\widetilde{\jmath}_1},\dots,\alpha_{\widetilde{\jmath}_{s-t}})$. Lemma 1.8 ii) yields $\mu_\rho\big(E^\bullet,\underline{\alpha};\tau\big)\ge\mu_\rho\big(E_1^\bullet,\underline{\alpha}_1;\tau\big)-\Big(\sum_{i=1}^{s-t}\alpha_{\widetilde{\jmath}_i}\Big)\cdot\delta\cdot a\cdot(r-1)$, whence

$$\begin{aligned}
&\textstyle\sum_{j=1}^s\alpha_j\big(\chi(E(n))\operatorname{rk}E_j-h^0(E_j(n))\operatorname{rk}E\big)+\delta\cdot\mu_\rho\big(E^\bullet,\underline{\alpha};\tau\big)\\
\ge\ &\textstyle\sum_{i=1}^t\alpha_{j_i}\big(\chi(E(n))\operatorname{rk}E_{j_i}-h^0(E_{j_i}(n))\operatorname{rk}E\big)+\delta\cdot\mu_\rho\big(E_1^\bullet,\underline{\alpha}_1;\tau\big)\\
&\textstyle+\sum_{i=1}^{s-t}\alpha_{\widetilde{\jmath}_i}\big(\chi(E(n))\operatorname{rk}E_{\widetilde{\jmath}_i}-h^0(E_{\widetilde{\jmath}_i}(n))\operatorname{rk}E\big)-\Big(\sum_{i=1}^{s-t}\alpha_{\widetilde{\jmath}_i}\Big)\cdot\delta\cdot a\cdot(r-1)\\
\ge\ &\textstyle\sum_{i=1}^t\alpha_{j_i}\big(\chi(E(n))\operatorname{rk}E_{j_i}-h^0(E_{j_i}(n))\operatorname{rk}E\big)+\delta\cdot\mu_\rho\big(E_1^\bullet,\underline{\alpha}_1;\tau\big)\\
&\textstyle+\Big(\sum_{i=1}^{s-t}\alpha_{\widetilde{\jmath}_i}\Big)K(g,r,C_1,C_2)-\Big(\sum_{i=1}^{s-t}\alpha_{\widetilde{\jmath}_i}\Big)\cdot\delta\cdot a\cdot(r-1).
\end{aligned}$$

Since this last expression is positive by assumption, we are done. □

The implication $t\in\iota^{-1}(\mathbb{G}^{\varepsilon-(s)s})\Rightarrow(E_t,M_t,\tau_t)$ is δ-(semi)stable. To begin with, we fix a constant K with the property that

$$rK>\max\big\{\,d(s-r)+\delta\cdot a\cdot(r-1)\mid s=1,\dots,r-1\,\big\}.$$

Now, let $t=[q\colon U\otimes\mathcal{O}_X(-n)\to E_t,M_t,\tau_t]$ be a point with $\iota(t)\in\mathbb{G}^{\varepsilon-(s)s}$.

We first claim that there can be no subbundle $E'\subset E_t$ with $\deg(E')\ge d+K$. Let E' be such a subbundle. Then for every natural number n,

$$h^0\big(E'(n)\big)\ge d+K+\operatorname{rk}E'\big(n+1-g\big).$$

Let $\widetilde{E}$ be the subbundle of E_t which is generated by $\operatorname{Im}(\mathrm{ev}\colon H^0(E'(n))\otimes\mathcal{O}_X(-n)\to E_t)$. Thus, $H^0(\widetilde{E}(n))=H^0(E'(n))$ and $\widetilde{E}$ is generically generated by its global sections. Now, choose a basis $u_1,\dots,u_i$ for $H^0(E'(n))$, complete it to a basis $\underline{u}:=(u_1,\dots,u_p)$ of U, and set $\lambda:=\lambda(\underline{u},\gamma_p^{(i)})$. Then we have seen that

$$\mu_{\mathbb{G}_1}\big(\lambda,\iota_1(t)\big)=p\cdot\operatorname{rk}\widetilde{E}-h^0(\widetilde{E}(n))\cdot r\le p\cdot\operatorname{rk}E'-h^0(E'(n))\cdot r.$$

Our discussion preceding Lemma 1.8 applies to $\mathrm{SL}(U)$ as well, whence $\mu_{\mathbb{G}_2}\big(\lambda,\iota_2(t)\big)\le a\cdot(p-i)$. Therefore,

$$\begin{aligned}
\mu_{\mathbb{G}}^{\varepsilon}\big(\lambda,\iota(t)\big)&=\varepsilon\cdot\mu_{\mathbb{G}_1}\big(\lambda,\iota_1(t)\big)+\mu_{\mathbb{G}_2}\big(\lambda,\iota_2(t)\big)\\
&\le\frac{p-a\cdot\delta}{r\cdot\delta}\big(p\cdot\operatorname{rk}E'-h^0(E'(n))\cdot r\big)+a\cdot(p-i)\\
&=\frac{p^2\operatorname{rk}E'}{r\delta}-\frac{pa\operatorname{rk}E'}{r}-\frac{ph^0(E'(n))}{\delta}+pa.
\end{aligned}$$

Next, we multiply the last expression by the positive number $r\delta/p$ in order to obtain

$$\begin{aligned} p\,\mathrm{rk}\,E' - rh^0(E'(n)) + \delta a\bigl(r - \mathrm{rk}\,E'\bigr) \\ &\le \bigl(d + r(n+1-g)\bigr)\,\mathrm{rk}\,E' - r\bigl(d + K + \mathrm{rk}\,E'(n+1-g)\bigr) + \delta a\bigl(r-1\bigr) \\ &= d\bigl(\mathrm{rk}\,E' - r\bigr) + \delta a\bigl(r-1\bigr) - rK < 0, \end{aligned}$$

by our choice of K. This obviously contradicts the assumption $\iota(t) \in \mathbb{G}^{\varepsilon-ss}$. We can also assume that $d + K > 0$. Set $C_3 := (r-1)d/r + K$. Then our arguments show that $\iota(t) \in \mathbb{G}^{\varepsilon-ss}$ implies

$$\mu_{\max}(E_t) \le \frac{d}{r} + C_3,$$

independently of the number n with which we performed the construction of $\mathbb{G}$. An argument similar to the one used in the proof of Theorem 2.12 shows that a $\rho_{a,b,c}$-pair (E, M, τ) is δ-(semi)stable if and only if for every weighted filtration $(E^\bullet, \underline{\alpha})$, such that

$$\mu(E_j) \ge \frac{d}{r} - \frac{\delta \cdot a \cdot (r-1)}{r} = \frac{d}{r} - C_1, \qquad j = 1, \ldots, s,$$

one has

$$M\bigl(E^\bullet, \underline{\alpha}\bigr) + \delta \cdot \mu_{\rho_{a,b,c}}\bigl(E^\bullet, \underline{\alpha}; \tau\bigr)(\ge)0.$$

Therefore, we choose n so large that for every vector bundle E' with $d/r + C_3 \ge \mu_{\max}(E')$, $\mu(E') \ge d/r - C_1$, and $1 \le \mathrm{rk}\,E' \le r-1$, one has that $E'(n)$ is globally generated and $H^1(E'(n))$ vanishes.

Now, let $(E^\bullet, \underline{\alpha})$ be a weighted filtration with

$$\mu(E_j) \ge \frac{d}{r} - C_1, \qquad j = 1, \ldots, s.$$

Fix a basis $\underline{w} = (w_1, \ldots, w_r)$ of $W := \mathbb{C}^r$, and let $W^\bullet : 0 \subset W_{\underline{w}}^{(i_1)} \subset \cdots \subset W_{\underline{w}}^{(i_s)} \subset W$ be the associated flag, $i_j := \mathrm{rk}\,E_j$, $j = 1, \ldots, s$. Let $\underline{u} = (u_1, \ldots, u_p)$ be a basis of U such that there are indices $l_1, \ldots, l_s$ with $U_{\underline{u}}^{(l_j)} = H^0(E_j(n))$, $j = 1, \ldots, s$. Define

$$\underline{\widetilde{\gamma}} := \textstyle\sum_{j=1}^{s} \alpha_j \gamma_p^{(l_j)}.$$

We also set, for $j = 1, \ldots, s+1$, $l_{s+1} := p$, $l_0 := 0$, $i_{s+1} := r$, $i_0 := 0$,

$$\mathrm{gr}_j(U, \underline{u}) := U_{\underline{u}}^{(l_j)}/U_{\underline{u}}^{(l_{j-1})} = H^0(E_j/E_{j-1}(n)), \quad \text{and} \quad \mathrm{gr}_j(W, \underline{w}) := W_{\underline{w}}^{(i_j)}/W_{\underline{w}}^{(i_{j-1})}.$$

The fixed bases $\underline{w}$ for W and $\underline{u}$ for U provide us with isomorphisms

$$U \cong \textstyle\bigoplus_{j=1}^{s+1} \mathrm{gr}_j(U, \underline{u}) \quad \text{and} \quad W \cong \textstyle\bigoplus_{j=1}^{s+1} \mathrm{gr}_j(W, \underline{w}).$$

Let $J^a := \{\, 1, \ldots, s \,\}^{\times a}$. For every index $\underline{\iota} \in J^a$, we set

$$W_{\underline{\iota}, \underline{w}} := \mathrm{gr}_{\iota_1}(W, \underline{w}) \otimes \cdots \otimes \mathrm{gr}_{\iota_a}(W, \underline{w}).$$

Analogously, we define $U_{\underline{\iota},\underline{w}}$. Moreover, for $k \in \{1,\dots,c\}$ and $\underline{\iota} \in J^a$, we let $W^k_{\underline{\iota},\underline{w}}$ be the subspace of $W_{a,c} := W^{\otimes a^{\oplus c}}$ which is $W_{\underline{\iota},\underline{w}}$ living in the k-th copy of $W^{\otimes a}$ in $W_{a,c}$, and similarly we define $U^k_{\underline{\iota},\underline{u}}$. The spaces $W^k_{\underline{\iota},\underline{w}}$ and $U^k_{\underline{\iota},\underline{u}}$, $k \in \{1,\dots,c\}$ and $\underline{\iota} \in J^a$, are eigenspaces for the actions of the one-parameter subgroups $\lambda(\underline{w},\gamma^{(i_j)})$ and $\lambda(\underline{u},\gamma_p^{(l_j)})$, respectively, $j = 1,\dots,s$. Define

$$\nu_j(\underline{\iota}) := \#\{\, \iota_i \le j \mid \underline{\iota} = (\iota_1,\dots,\iota_a),\ i = 1,\dots,a \,\}. \tag{2}$$

Then $\lambda(\underline{w},\gamma^{(i_j)})$ acts on $W^k_{\underline{\iota},\underline{w}}$ with weight $\nu_j(\underline{\iota})\cdot r - a\cdot i_j$, and $\lambda(\underline{u},\gamma_p^{(l_j)})$ acts on $U^k_{\underline{\iota},\underline{u}}$ with weight $\nu_j(\underline{\iota})\cdot p - a\cdot l_j$.

Let $Z_t := H^0(\det(E_t)^{\otimes b}\otimes M_t\otimes \mathcal{O}_X(na))$. Then $\iota_2(t) \in \mathbb{P}(\mathrm{Hom}(U_{a,c},Z_t)^\vee)$ can be represented by a homomorphism

$$L_t\colon U_{a,c} \to Z_t.$$

One readily verifies

$$\begin{aligned}&\mu_{\mathbb{G}_1}\big(\lambda(\underline{u},\widetilde{\underline{\gamma}}),[L_t]\big)\\ &= -\min\big\{ \textstyle\sum_{j=1}^s \alpha_j\big(\nu_j(\underline{\iota})\cdot p - a\cdot l_j\big) \mid k\in\{1,..,c\}, \underline{\iota}\in J^a : U^k_{\underline{\iota},\underline{u}} \not\subset \ker L_t \big\}.\end{aligned} \tag{3}$$

Next, we observe that we can choose a small open subset $X_0 \subset X$ over which E_t and M_t are trivial and there is an isomorphism $\psi\colon E_{t|X_0} \cong W\otimes\mathcal{O}_{X_0}$ with $\psi(E^\bullet|_{X_0}) = W^\bullet\otimes\mathcal{O}_{X_0}$. This trivialization and the $\rho_{a,b,c}$-pair (E_t, M_t, τ_t) provide us with

$$l_t\colon W_{a,c}\otimes\mathcal{O}_{X_0} \to \big(\textstyle\bigwedge^r W\big)^{\otimes b}\otimes\mathcal{O}_{X_0}.$$

We observe that for every $k \in \{1,\dots,c\}$ and every $\underline{\iota}\in J^a$,

$$W^k_{\underline{\iota},\underline{w}}\otimes\mathcal{O}_{X_0}\not\subset \ker l_t \Leftrightarrow U^k_{\underline{\iota},\underline{u}}\not\subset\ker L_t, \tag{4}$$

and that

$$\begin{aligned}&\mu_{\rho_{a,b,c}}\big(E^\bullet,\underline{\alpha};\tau_t\big)\\ &= -\min\big\{ \textstyle\sum_{j=1}^s \alpha_j\big(\nu_j(\underline{\iota})\cdot r - a\cdot i_j\big) \mid k\in\{1,..,c\}, \underline{\iota}\in J^a : W^k_{\underline{\iota},\underline{w}}\otimes\mathcal{O}_{X_0} \not\subset \ker l_t \big\}.\end{aligned} \tag{5}$$

Now, let $k_0 \in \{1,\dots,c\}$ and $\underline{\iota_0}\in J^a$ be such that the minimum in (3) is achieved by $\sum_{j=1}^s \alpha_j(\nu_j(\underline{\iota_0})\cdot p - a\cdot l_j)$ and $U^{k_0}_{\underline{\iota_0},\underline{u}}\not\subset\ker L_t$. We obtain

$$\begin{aligned} 0 \ (\le)\ & \mu^\varepsilon_{\mathbb{G}}\big(\lambda(\underline{u},\widetilde{\underline{\gamma}}),\iota(t)\big)\\ =\ & \varepsilon\cdot\mu_{\mathbb{G}_1}\big(\lambda(\underline{u},\widetilde{\underline{\gamma}}),\iota_1(t)\big) + \mu_{\mathbb{G}_2}\big(\lambda(\underline{u},\widetilde{\underline{\gamma}}),\iota_2(t)\big)\\ =\ & \varepsilon\cdot\textstyle\sum_{j=1}^s\alpha_j\big(p\,\mathrm{rk}\,E_j - h^0(E_j(n))r\big) + \sum_{j=1}^s\alpha_j\big(\nu_j(\underline{\iota_0})\cdot p - a\cdot l_j\big)\\ =\ & \tfrac{p-a\delta}{r\delta}\textstyle\sum_{j=1}^s\alpha_j\big(p\,\mathrm{rk}\,E_j - h^0(E_j(n))r\big) + \sum_{j=1}^s\alpha_j\big(\nu_j(\underline{\iota_0})\cdot p - a\cdot h^0(E_j(n))\big)\\ =\ & \textstyle\sum_{j=1}^s\alpha_j\Big(\frac{p^2\,\mathrm{rk}\,E_j}{r\delta} - \frac{pa\,\mathrm{rk}\,E_j}{r} - \frac{ph^0(E_j(n))}{\delta}\Big) + \sum_{j=1}^s\alpha_j\nu_j(\underline{\iota_0})\cdot p.\end{aligned}$$

We multiply this inequality by $r\delta/p$ and find

$$0(\le)\textstyle\sum_{j=1}^s\alpha_j\big(p\,\mathrm{rk}\,E_j - rh^0(E_j(n))\big) + \delta\sum_{j=1}^s\alpha_j\big(\nu_j(\underline{\iota_0})r - a\,\mathrm{rk}\,E_j\big).$$

Since $h^1(E_j(n)) = 0$, $j = 1,\dots,s$, we have $p\,\mathrm{rk}\,E_j - rh^0(E_j(n)) = d\,\mathrm{rk}\,E_j - r\deg(E_j)$, $j = 1,\dots,s$. Moreover, we have $\mathrm{rk}\,E_j = i_j$, by definition, and $\mu_{\rho_{a,b,c}}(E^\bullet,\underline{\alpha};\tau_t) \ge \sum_{j=1}^s\alpha_j(\nu_j(\underline{\iota_0})r - ai_j)$, by (4) and (5), whence we finally see

$$M\big(E^\bullet,\underline{\alpha}\big) + \delta\cdot\mu_{\rho_{a,b,c}}\big(E^\bullet,\underline{\alpha};\tau_t\big)(\ge)0,$$

as required. □

The implication (E_t, M_t, τ_t) *is* δ-*(semi)stable* $\Rightarrow$ $t \in \iota^{-1}(\mathbb{G}^{\varepsilon-(s)s})$. By the Hilbert–Mumford criterion, we have to show that for every basis $\underline{u} = (u_1, \dots, u_p)$ of U and every weight vector $\underline{\widetilde{\gamma}} = (\gamma_1, \dots, \gamma_p)$ with $\gamma_1 \le \dots \le \gamma_p$ and $\sum_{i=1}^p \gamma_i = 0$

$$\mu_{\mathbb{G}}^{\varepsilon}\big(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota(t)\big) = \varepsilon\mu_{\mathbb{G}_1}\big(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota_1(t)\big) + \mu_{\mathbb{G}_2}\big(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota_2(t)\big)(\ge)0.$$

So, let $\underline{u} = (u_1, \dots, u_p)$ be an arbitrary basis for U and $\underline{\widetilde{\gamma}} = \sum_{i=1}^{p-1} \beta_i \gamma_p^{(i)}$ a weight vector. Let $l_1, \dots, l_v$ be the indices with $\beta_{l_h} \ne 0$, $h = 1, \dots, v$. For each $h \in \{1, \dots, v\}$, let E_{l_h} be the subbundle of E_t generated by $\mathrm{Im}(U_{\underline{u}}^{(l_h)} \otimes \mathcal{O}_X(-n) \to E_t)$. Note that for $h' \ge h$ we will have $E_{l_{h'}} = E_{l_h}$ if and only if $U_{\underline{u}}^{(l_{h'})} \subset H^0(E_{l_h}(n))$. We let $E^\bullet : 0 =: E_0 \subset E_1 \subset \dots \subset E_s \subset E_{s+1} := E$ be the filtration by the distinct vector bundles occurring among the E_{l_h}'s.

Recall that we know (1),

$$\mu_{\mathbb{G}_1}\big(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota_1(t)\big) = \textstyle\sum_{h=1}^v \beta_{l_h}\big(p \operatorname{rk} E_{l_h} - l_h r\big) \ge \sum_{h=1}^v \beta_{l_h}\big(p \operatorname{rk} E_{l_h} - h^0(E_{l_h}(n)) r\big).$$

Set, for $j = 1, \dots, s$,

$$\alpha_j := \textstyle\sum_{h: E_{l_h} = E_j} \beta_{l_h},$$

so that we see

$$\mu_{\mathbb{G}_1}\big(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota_1(t)\big) \ge \textstyle\sum_{j=1}^s \alpha_j\big(p \operatorname{rk} E_j - h^0(E_j(n)) r\big). \qquad (6)$$

Next, we define for $j = 0, \dots, s$

$$h(j) := \max\big\{\, h = 1, \dots, v \mid U_{\underline{u}}^{(l_h)} \subset h^0(E_j(n)) \big\}.$$

With these conventions, $h = h(j) + 1$ is the minimal index, such that $U_{\underline{u}}^{(l_h)} \otimes \mathcal{O}_X(-n)$ generically generates E_{j+1}, $j = 0, \dots, s$. We now set

$$\widetilde{\mathrm{gr}}_j\big(U, \underline{u}\big) := U_{\underline{u}}^{(l_{h(j-1)+1})} / U_{\underline{u}}^{(l_{h(j-1)})}, \qquad j = 1, \dots, s+1.$$

The space $\bigoplus_{j=1}^{s+1} \widetilde{\mathrm{gr}}_j(U, \underline{u})$ can be identified with a subspace of U, via $\widetilde{\mathrm{gr}}_j(U, \underline{u}) \cong \langle\, l_{h(j-1)} + 1, \dots, l_{h(j-1)+1} \,\rangle$, $j = 1, \dots, s$.

For any index tuple $\underline{\iota} = (\iota_1, \dots, \iota_a) \in J^a := \{1, \dots, s\}^{\times a}$, we define

$$\widetilde{U}_{\underline{\iota}, \underline{u}} := \widetilde{\mathrm{gr}}_{\iota_1}(U, \underline{u}) \otimes \dots \otimes \widetilde{\mathrm{gr}}_{\iota_a}(U, \underline{u}).$$

Again, for $\underline{\iota} \in J^a$ and $k \in \{1, \dots, c\}$, $\widetilde{U}_{\underline{\iota}, \underline{u}}^k$ will be $\widetilde{U}_{\underline{\iota}, \underline{u}}$ viewed as a subspace of the k-th summand of $U_{a,c}$.

The effect of our definition of the $h(j)$'s is that the spaces $\widetilde{U}_{\underline{\iota}, \underline{u}}^k$, $\underline{\iota} \in J^a$ and $k \in \{1, \dots, c\}$, are eigenspaces *for all* the one-parameter subgroups $\lambda(\underline{u}, \gamma_p^{(l_h)})$, $h = 1, \dots, v$, with respect to the weight $\nu_j(\underline{\iota})p - a l_h$, $\nu_j(\underline{\iota})$ as in (2).

Now, let $\underline{w} = (w_1, \dots, w_r)$ be a basis for W and $W^\bullet : 0 \subset W_{\underline{w}}^{(i_1)} \subset \dots \subset W_{\underline{w}}^{(i_s)} \subset W$, $i_j := \operatorname{rk} E_j$, $j = 1, \dots, s$, the corresponding flag. Then the spaces $W_{\underline{\iota}, \underline{w}}^k$, $\underline{\iota} \in J^a$ and $k \in \{1, \dots, c\}$, are defined as before. We can find a small open set $X_0 \subset X$, such that

- M_t and E_t are trivial over X_0,
- there is an isomorphism $\psi\colon E_{t|X_0} \to W \otimes \mathcal{O}_{X_0}$ with $\psi(E^\bullet|_{X_0}) = W^\bullet \otimes \mathcal{O}_{X_0}$,
- $E_{t|X_0} \cong \bigoplus_{j=1}^{s+1} (E_j/E_{j-1})|_{X_0}$,
- the homomorphism $\big(\bigoplus_{j=1}^{s+1} \widetilde{\mathrm{gr}}_j(U, \underline{u})\big) \otimes \mathcal{O}_{X_0}(-n) \to \bigoplus_{j=1}^{s+1} (E_j/E_{j-1})|_{X_0}$ is surjective.

As before, let $Z_t := H^0(\det(E_t)^{\otimes b} \otimes M_t \otimes \mathcal{O}_X(na))$, so that $\iota_2(t) \in \mathbb{P}(\mathrm{Hom}(U_{a,c}, Z_t)^\vee)$ induces a homomorphism

$$\widetilde{L}_t \colon \bigoplus \widetilde{U}^k_{\underline{\iota},\underline{u}} \to Z_t.$$

Letting

$$l_t \colon W_{a,c} \otimes \mathcal{O}_{X_0} \to \left(\bigwedge^r W\right)^{\otimes b} \otimes \mathcal{O}_{X_0}$$

be the resulting homomorphism, we find that for every $k \in \{1, \dots, c\}$ and every $\underline{\iota} \in J^a$

$$W^k_{\underline{\iota},\underline{w}} \otimes \mathcal{O}_{X_0} \not\subset \ker l_t \Leftrightarrow \widetilde{U}^k_{\underline{\iota},\underline{u}} \not\subset \ker \widetilde{L}_t. \tag{7}$$

By Theorem 2.12, we have

$$\textstyle\sum_{j=1}^s \alpha_j \bigl(p \operatorname{rk} E_j - h^0(E_j(n))r\bigr) + \delta \cdot \mu_{\rho_{a,b,c}}\bigl(E^\bullet, (\alpha_1, \dots, \alpha_s); \tau_t\bigr)(\geq) 0. \tag{8}$$

Now, we choose $k_0 \in \{1, \dots, c\}$ and $\underline{\iota}_0 \in J^a$ with $W^{k_0}_{\underline{\iota}_0,\underline{w}} \otimes \mathcal{O}_{X_0} \not\subset \ker l_t$ and $\mu_{\rho_{a,b,c}}(E^\bullet, (\alpha_1, \dots, \alpha_s); \tau_t) = \sum_{j=1}^s \alpha_j(\nu_j(\underline{\iota}_0)r - a \operatorname{rk} E_j)$. Plugging this into (8) and multiplying by $p/(r\delta)$ yields

$$\begin{aligned} 0 \ (\leq) \ & \textstyle\sum_{j=1}^s \alpha_j \Bigl(\frac{p^2 \operatorname{rk} E_j}{r\delta} - \frac{pa \operatorname{rk} E_j}{r} - \frac{p h^0(E_j(n))}{\delta}\Bigr) + \sum_{j=1}^s \alpha_j \nu_j(\underline{\iota}) \cdot p \\ = \ & \varepsilon \textstyle\sum_{j=1}^s \alpha_j \bigl(p \operatorname{rk} E_j - h^0(E_j(n))r\bigr) + \sum_{j=1}^s \alpha_j \bigl(\nu_j(\underline{\iota}_0) \cdot p - a \cdot h^0(E_j(n))\bigr). \end{aligned}$$

By our definition of the α_j, and (7), we know

$$\mu_{\mathbb{G}_2}\bigl(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota_2(t)\bigr) \geq \textstyle\sum_{h=1}^v \beta_{l_h}\bigl(\nu_{j(h)}(\underline{\iota}_0)p - a l_h\bigr) \geq \sum_{j=1}^s \alpha_j \bigl(\nu_j(\underline{\iota}_0)p - a h^0(E_j(n))\bigr).$$

Here, we have set $j(h)$ to be the element $j \in \{1, \dots, s\}$ with $E_{l_h} = E_j$. This together with (6) finally shows $\mu^\varepsilon_{\mathbb{G}}(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota(t))(\geq) 0$. □

The identification of the polystable points. By the Hilbert–Mumford criterion, a point $\iota(t)$ is polystable if and only if it is semistable and, for every one parameter subgroup λ of $\mathrm{SL}(U)$ with $\mu^\epsilon_{\mathbb{G}}(\lambda, \iota(t)) = 0$, $\lim_{z \to \infty} \lambda(z) \cdot \iota(t)$ lies in the orbit of $\iota(t)$.

Now, let $\underline{u} = (u_1, \dots, u_p)$ be a basis for U and $\underline{\widetilde{\gamma}} = \sum_{j=1}^s \beta_{l_j} \gamma_p^{(l_j)}$ be a weight vector with $\beta_{l_j} \neq 0$ and $l_j \in \{1, \dots, p-1\}$ such that $\mu^\varepsilon_{\mathbb{G}}(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota(t)) = 0$. Then our previous considerations show that the following must be satisfied:

- $U^{(l_j)}_{\underline{u}} = H^0(E_{l_j}(n))$, $j = 1, \dots, s$,
- $E_{l_j}(n)$ is generated by global sections and $H^1(E_{l_j}(n)) = 0$.

Set $E_j := E_{l_j}$, $i_j := \operatorname{rk} E_j$, $\alpha_j := \beta_{l_j}$, $j = 1, \dots, s$, and choose a basis $w_1, \dots, w_r$ for W. As before, we associate to these data a flag $W^\bullet$. Consider the weighted filtration $(E^\bullet, \underline{\alpha})$ with $E^\bullet : 0 \subset E_1 \subset \cdots \subset E_s \subset E_t$ and $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$, so that the condition $\mu^\varepsilon_{\mathbb{G}}(\lambda(\underline{u}, \underline{\widetilde{\gamma}}), \iota(t)) = 0$ becomes equivalent to

$$M\bigl(E^\bullet, \underline{\alpha}\bigr) + \delta \cdot \mu_{\rho_{a,b,c}}\bigl(E^\bullet, \underline{\alpha}; \tau_t\bigr) = 0.$$

Let $t_\infty := \lim_{z \to \infty} \lambda(z) \cdot t$ and $(E_{t_\infty}, M_{t_\infty}, \tau_{t_\infty})$ be the corresponding $\rho_{a,b,c}$-pair. Then clearly $M_{t_\infty} \cong M_t$, and it is well known that $E_{t_\infty} \cong \bigoplus_{j=1}^{s+1} E_j/E_{j-1}$. Let $U_{a,c} := \bigoplus U^{\widetilde{g}_i}$

be the decomposition of $U_{a,c}$ into eigenspaces with respect to the $\mathbb{C}^*$-action coming from $\lambda(\underline{u},\widetilde{\gamma})$, and $\widetilde{g}_{i_0} = -\mu_{\mathbb{G}_2}(\lambda(\underline{u},\widetilde{\gamma}),\iota_2(t))$. If $L_t\colon U_{a,c}\to Z_t$ and $L_{t_\infty}\colon U_{a,c}\to Z_{t_\infty}=Z_t$ are the homomorphisms representing t and t_∞, respectively, then L_{t_∞} is just the restriction of L_t to $U^{\widetilde{g}_{i_0}}$ extended by zero to the other weight spaces. As we have seen before, the condition that L_{t_∞} be supported only on $U^{\widetilde{g}_{i_0}}$ is equivalent to the fact that over each open subset X_0 over which τ_{t_∞} is surjective and we have a trivialization $\psi\colon E_{t_\infty|X_0}\to W\otimes\mathcal{O}_{X_0}$ with $\psi(E^\bullet|_{X_0}) = W^\bullet\otimes\mathcal{O}_{X_0}$, the induced morphism $X_0\to\mathbb{P}(W_{a,c})$ lands in $\mathbb{P}(W^{g_0})$, where W^{g_0} is the eigenspace for the weight $g_0 := -\mu_{\rho_{a,b,c}}(E^\bullet,\underline{\alpha};\tau_{t_\infty})$. Thus, we have shown that (E_t,M_t,τ_t) being δ-polystable implies that $\iota(t)$ is a polystable point. The converse is similar. □

The properness of the Gieseker map. In this section, we will prove that the Gieseker morphism ι is proper, using the (discrete) valuative criterion.

Thus, let $(C,0)$ be the spectrum of a DVR R with quotient field K. Suppose we are given a morphism $h\colon C\to\mathbb{G}^{\varepsilon-ss}$ which lifts over $\operatorname{Spec}K$ to $\mathfrak{T}$. This means that we are given a quotient family $(q_K\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_K,\kappa_K,\tau_K)$ of $\rho_{a,b,c}$-pairs parameterized by $\operatorname{Spec}K$ (we left out $\mathfrak{N}_K$, because it is trivial). This can be extended to a certain family $(\widetilde{q}_C\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\to\widetilde{E}_C,\kappa_C,\tau_C)$, consisting of

- a surjection $\widetilde{q}_C$ onto the flat family $\widetilde{E}_C$, where $\widetilde{E}_{C|\{0\}\times X}$ may have torsion
- the continuation κ_C of κ_K into 0
- a homomorphism $\tau_C\colon \widetilde{E}_C^{\otimes a^{\oplus c}}\to\det(\widetilde{E})^{\otimes b}\otimes\mathcal{L}[\kappa_C]$ whose restriction to $\{0\}\times X$ is nontrivial and whose restriction to $\operatorname{Spec}K\times X$ differs from τ_K by an element in K^*.

The resulting datum $L\colon U_{a,c}\to\pi_{C*}(\det(\widetilde{E}_C)^{\otimes b}\otimes\mathcal{L}[\kappa_C]\otimes\pi_X^*\mathcal{O}_X(na))$ defines a morphism $C\to\mathbb{G}_2$ which coincides with the second component h_2 of h.

Set $E_C := \widetilde{E}_C^{\vee\vee}$. This is a reflexive sheaf on the smooth surface $C\times X$, whence it is locally free and thus flat over C. Therefore, we have a family

$$q_C\colon U\otimes\pi_X^*\mathcal{O}_X(-n)\longrightarrow E_C$$

where the kernel of the homomorphism $U\otimes\mathcal{O}_X(-n)\to E_{C|\{0\}\times X}$ is isomorphic to the torsion $\mathcal{T}$ of $\widetilde{E}_{C|\{0\}\times X}$. One gets a homomorphism $\bigwedge^r U\otimes\mathcal{O}_C\to\pi_{C*}(\det(\widetilde{E}_C)\otimes\pi_X^*\mathcal{O}_X(rn))$ which defines a morphism $C\to\mathbb{G}_1$ which coincides with the first component h_1 of h.

Set $E_0 := E_{C|\{0\}\times X}$. Our claim is that $H^0(q_{C|\{0\}\times X}\otimes\mathrm{id}_{\pi_X^*\mathcal{O}_X(n)})\colon U\to H^0(E_0(n))$ must be injective. This implies, in particular, that $\widetilde{E}_{C|\{0\}\times X}$ is torsion free and, hence, $E_C=\widetilde{E}_C$ and $q_C=\widetilde{q}_C$. If $H := \ker(H^0(q_{C|\{0\}\times X}\otimes\mathrm{id}_{\pi_X^*\mathcal{O}_X(n)}))$ is nontrivial, we choose a basis $u_1,\dots,u_j$ for H and complete it to a basis $\underline{u}=(u_1,\dots,u_p)$ of U. Set $\overline{H}=\langle\, u_{j+1},\dots,u_p\,\rangle$. We first note (1)

$$\mu_{\mathbb{G}_1}\big(\lambda(\underline{u},\gamma_p^{(j)}),h_1(0)\big) = -jr.$$

The spaces $H_l := H^{\otimes l}\otimes\overline{H}^{\otimes(a-l)}$, $l=1,\dots,a$, are the eigenspaces of $U^{\otimes a}$ for the $\mathbb{C}^*$-action coming from $\lambda(\underline{u},\gamma_p^{(j)})$. Let H_l^k be H_l embedded into the k-th component of $U_{a,c}$, $k=1,\dots,c$, $l=1,\dots,a$. For every $k\in\{\,1,\dots,c\,\}$ and every $l\in\{\,1,\dots,a\,\}$,

$H_l^k \otimes \mathcal{O}_X(-n)$ generates a torsion subsheaf of $\widetilde{E}_{C|\{0\}\times X}^{\otimes a^{\oplus c}}$, so that $H_l^k \subset \ker L$. This implies

$$\mu_{\mathbb{G}_2}\big(\lambda(\underline{u}, \gamma_p^{(j)}), h_2(0)\big) = \mu_{\mathbb{G}_2}\big(\lambda(\underline{u}, \gamma_p^{(j)}), [L]\big) = -aj,$$

and thus

$$\mu_{\mathbb{G}}^{\varepsilon}\big(\lambda(\underline{u}, \gamma_p^{(j)}), h(0)\big) = -\varepsilon jr - aj < 0,$$

in contradiction to the assumption $h(0) \in \mathbb{G}^{\varepsilon-ss}$.

We identify U with its image in $H^0(E_0(n))$. Let K be a positive constant such that $rK > \max\{\, d(s-r) + \delta a(r-1) \mid s = 1, \dots, r-1 \,\}$. We assert that for every nontrivial and proper quotient bundle Q of E_0 we must have $\deg Q \geq -K - (r-1)g$. For this, let Q be the minimal destabilizing quotient bundle. Set $E' := \ker(E \to Q)$. It suffices to show that $\deg Q < -K-(r-1)g$ implies $\dim(H^0(E'(n)) \cap U) \geq d + K + \operatorname{rk} E'(n+1-g)$, because then a previously given argument applies. Note that we have an exact sequence

$$0 \longrightarrow H^0(E'(n)) \cap U \longrightarrow U \longrightarrow H^0(Q(n)).$$

Assume first that $h^0(Q(n)) = 0$. Thus, $\dim(H^0(E'(n)) \cap U) = p = d + \operatorname{rk} E'(n+1-g) + (r - \operatorname{rk} E')(n+1-g) \geq d + \operatorname{rk} E'(n+1-g) + n + 1 - g$. Since we can assume $n + 1 - g > K$, this is impossible.

Therefore, the Le Potier–Simpson estimate gives $h^0(Q(n)) \leq \deg Q + \operatorname{rk} Q(n+1)$ and thus

$$\begin{aligned}
\dim\big(H^0(E'(n)) \cap U\big) &\geq p - h^0(Q(n)) \\
&\geq d + r(n+1-g) - \deg Q - \operatorname{rk} Q(n+1) \\
&= d - \deg Q - g \operatorname{rk} Q + (r - \operatorname{rk} Q)(n+1-g) \\
&\geq d - \deg Q - g(r-1) + \operatorname{rk} E'(n+1-g).
\end{aligned}$$

This gives the claim. We see

$$\mu_{\min}(E_0) \geq \frac{-K-(r-1)g}{\operatorname{rk} Q} \geq -K - (r-1)g.$$

This bound does not depend on n. Since the family of isomorphy classes of vector bundles G of degree d and rank r with $\mu_{\min}(G) \geq -K - (r-1)g$ is bounded, we can choose n so large that $H^1(G(n)) = 0$ for every such vector bundle. In particular, $H^1(E_0(n)) = 0$, i.e., $U = H^0(E_0(n))$. This means that the family $(\widetilde{q}_C, \kappa_C, \tau_C)$ we started with is a quotient family of $\rho_{a,b,c}$-pairs parameterized by C and thus defines a morphism from C to $\mathfrak{T}$ which lifts h. By Theorem 2.11 (i), this morphism factorizes through $\mathfrak{T}^{\delta-ss}$, and we are done. □

3. Examples

This section is devoted to the study of the known examples within our general context. First, we discuss two important methods of simplifying the stability concept. Second, we will consider some easy specializations of the moduli functors. Then we briefly discuss the variation of the stability parameter and prove an "asymptotic irreducibility" result.

Afterwards, we turn to the examples. In the examples, we will show how many of the known stability concepts and constructions of the moduli spaces over curves can be obtained via our construction. In two cases we will see that our results give a little more than previous constructions. We have also added the stability concept for conic bundles of rank 4. The main aim of the examples is to illustrate that the complexity of the stability concept only results from the complexity of the input representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$ and to illustrate how the understanding of ρ can be used to simplify the stability concept.

3.1. Simplifications of the stability concept

In this part, we will formulate several ways of restating the concept of δ-semistability in different, easier ways which will be used in the study of examples to recover the known notions of semistability. The first uses a well known additivity property to reduce the stability conditions to conditions on subbundles. The second generalizes this to a method working for all representations. This provides the mechanism alluded to in the introduction. The third one is a method to express the concept of δ-semistability for ρ-pairs associated with a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ of representations in a certain sense in terms of the semistability concepts corresponding to the summands ρ_i. Further methods of simplifying the semistability concept will be discussed in the examples.

A certain additivity property. Let $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$ be a representation such that the following property holds true: For any basis $\underline{w} = (w_1, \ldots, w_r)$ of $\mathbb{C}^r$, any two weight vectors $\underline{\gamma}_1$ and $\underline{\gamma}_2$, and any point $[l] \in \mathbb{P}(V)$

$$\mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}_1 + \underline{\gamma}_2), [l]\big) = \mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}_1), [l]\big) + \mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}_2), [l]\big). \tag{9}$$

Now, let (E, M, τ) be a ρ-pair and δ a positive rational number. For every weighted filtration $(E^\bullet, \underline{\alpha})$, $E^\bullet : 0 \subset E_1 \subset \cdots \subset E_s \subset E$, the definition of $\mu_\rho(E^\bullet, \underline{\alpha}; \tau)$ and (9) imply

$$M(E^\bullet, \underline{\alpha}) + \delta\mu_\rho(E^\bullet, \underline{\alpha}; \tau) = \textstyle\sum_{j=1}^s \alpha_j\big((d \operatorname{rk} E_j - r \deg E_j) + \delta\mu_\rho(E_j, \tau)\big).$$

We see that the semistability condition becomes a condition on subbundles of E: The ρ-pair (E, M, τ) is δ-(semi)stable if and only if for every nontrivial proper subbundle E' of E one has

$$\mu(E') \ (\leq) \ \mu(E) + \frac{\mu_\rho\big(E', \tau\big)}{\operatorname{rk} E' \operatorname{rk} E}. \tag{10}$$

The general procedure. Let $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(V)$ be a representation on V and $\rho'\colon \mathrm{SL}(r) \to \mathrm{GL}(V)$ its restriction to $\mathrm{SL}(r)$. We fix a basis $\underline{w} = (w_1, \ldots, w_r)$ of $\mathbb{C}^r$. This basis determines a maximal torus $T \subset \mathrm{SL}(r)$. First, we observe that the Hilbert–Mumford criterion can be restated in the following form: A point $[l] \in \mathbb{P}(V)$ is ρ'-(semi)stable if and only if for every element $g \in \mathrm{SL}(r)$ and every weight vector $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ with $\gamma_1 \leq \cdots \leq \gamma_r$ and $\sum \gamma_i = 0$

$$\mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}), g \cdot [l]\big) \ (\geq) \ 0. \tag{11}$$

The representation $\rho|_T\colon T \to \mathrm{GL}(V)$ yields a decomposition

$$V = \textstyle\bigoplus_{\chi \in X(T)} V_\chi$$

with
$$V_\chi := \{ v \in V \mid \rho(t)(v) = \chi(t) \cdot v \forall t \in T \}.$$

The set $\mathrm{ST}(\rho) := \{ \chi \in X(T) \mid V_\chi \neq \langle 0 \rangle \}$ is the *set of states of* ρ. We look at the rational polyhedral cone

$$C := \{ (\gamma_1, \dots, \gamma_r) \mid \gamma_1 \leq \dots \leq \gamma_r, \sum \gamma_i = 0 \} = \mathbb{R}_{\geq 0} \cdot \gamma^{(1)} + \dots + \mathbb{R}_{\geq 0} \cdot \gamma^{(r-1)}.$$

For every subset $A \subset \mathrm{ST}(\rho)$, we obtain a decomposition

$$C = \bigcup_{\chi \in A} C_A^\chi \quad \text{with} \quad C_A^\chi := \{ \underline{\gamma} \in C \mid \langle \lambda(\underline{w}, \underline{\gamma}), \chi \rangle \leq \langle \lambda(\underline{w}, \underline{\gamma}), \chi' \rangle \ \forall \chi' \in A \}.$$

Here, $\langle ., . \rangle$ is the natural pairing between one-parameter subgroups and characters. The cones C_A^χ are also rational polyhedral cones and one has

$$C_A^\chi \cap C_A^{\chi'} = C_A^\chi \ \cap \ \{ \underline{\gamma} \mid \langle \lambda(\underline{w}, \underline{\gamma}), \chi - \chi' \rangle = 0 \},$$

so that two cones intersect in a common face. Therefore, for each A, we get a fan decomposition of C. For each edge of a cone C_A^χ, there is a minimal integral generator. For $A \subset \mathrm{ST}(\rho)$ and $\chi \in A$, we let K_A^χ be the set of those generators and $K_A = \bigcup_{\chi \in A} K_A^\chi$. The set K_A obviously contains $\{ \gamma^{(1)}, \dots, \gamma^{(r-1)} \}$, and we call A *critical* if K_A is strictly bigger than $\{ \gamma^{(1)}, \dots, \gamma^{(r-1)} \}$. Now, for each point $[l] \in \mathbb{P}(V)$, we set $\mathrm{ST}(l) := \{ \chi \mid l|_{V_\chi} \not\equiv 0 \}$. Moreover, an element $g \in \mathrm{SL}(r)$ is called *critical for* $[l]$ if the set $\mathrm{ST}(g \cdot l)$ is critical.

We observe that for a point $[l] \in \mathbb{P}(V)$ and a weight vector $\underline{\gamma} \in C$ one has

$$\mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}), [l]\big) = -\min\{ \langle \lambda(\underline{w}, \underline{\gamma}), \chi \rangle \mid \chi \in \mathrm{ST}(l) \}.$$

This means that Equation (9) remains valid if there exists a character $\chi \in \mathrm{ST}(l)$, such that $C_{\mathrm{ST}(l)}^\chi$ contains both $\underline{\gamma}_1$ and $\underline{\gamma}_2$. We infer

Corollary 3.1. *A point* $[l] \in \mathbb{P}(V)$ *is* ρ'*-(semi)stable if and only if it satisfies the following two conditions:*

1. *For every element* $g \in \mathrm{SL}(r)$ *and every* $i \in \{ 1, \dots, r-1 \}$,
$$\mu_\rho\big(\lambda(\underline{w}, \gamma^{(i)}), g \cdot [l]\big)(\geq)0.$$
2. *For every* $g \in \mathrm{SL}(r)$ *which is critical for* $[l]$ *and every weight vector* $\underline{\gamma} \in K_{\mathrm{ST}(g \cdot l)} \setminus \{ \gamma^{(1)}, \dots, \gamma^{(r-1)} \}$,
$$\mu_\rho\big(\lambda(\underline{w}, \underline{\gamma}), g \cdot [l]\big) \ (\geq) \ 0.$$

In particular, it suffices to test (11) *for the weight vectors belonging to the finite set*

$$K_\rho := \bigcup_{A \subset \mathrm{ST}(\rho)} K_A.$$

Remark 3.2. A similar procedure works for all semisimple groups G. Indeed, one fixes a pair (B,T) consisting of a Borel subgroup of G and a maximal torus $T\subset B$. With analogous arguments, one obtains decompositions of the Weyl chamber $W(B,T)$. See [8] for a precise discussion.

Let's now turn to the ρ-pairs. Let $W^\bullet$ be the complete flag $0\subset\langle w_1\rangle\subset\cdots\subset\langle w_1,\dots,w_{r-1}\rangle\subset\mathbb{C}^r$. For a ρ-pair (E,M,φ) and a filtration $0\subset E_1\subset\cdots\subset E_{r-1}\subset E$ with $\operatorname{rk}E_i=i$, $i=1,\dots,r-1$, we define $\mathrm{ST}(E^\bullet)$ as follows: Choose an open subset U and a trivialization $\psi\colon E|_U\to\mathcal{O}_U^{\oplus r}$ with $\psi(E^\bullet|_U)=W^\bullet\otimes\mathcal{O}_U$. Then for each $\chi\in\mathrm{ST}(\rho)$, there is a rational map

$$U\to\mathbb{P}(E_{\rho|U})\cong\mathbb{P}(V)\times U\to\mathbb{P}(V)\dashrightarrow\mathbb{P}(V_\chi).$$

An element $\chi\in\mathrm{ST}(\rho)$ now belongs to $\mathrm{ST}(E^\bullet)$ if and only if this rational map is defined on a non-empty subset of U. As before, one verifies that $\mathrm{ST}(E^\bullet)$ is well defined. The filtration $E^\bullet$ is called *critical for* φ if $\mathrm{ST}(E^\bullet)$ is critical. Corollary 3.1 now shows

Theorem 3.3. (i) *The ρ-pair (E,M,φ) is δ-(semi)stable if and only if it meets the following two requirements:*

1. *For every proper nontrivial subbundle E' of E*

$$\mu(E')(\le)\mu(E)+\frac{\mu_\rho(E',\varphi)}{\operatorname{rk}E'\operatorname{rk}E}.$$

2. *For every filtration $E^\bullet$ which is critical for φ and every element $\sum_{j=1}^s\alpha_j\gamma^{(i_j)}\in K_{\mathrm{ST}(E^\bullet)}\setminus\{\gamma^{(1)},\dots,\gamma^{(r-1)}\}$, $\alpha_j>0$, $i_j:=\operatorname{rk}E_j$, $j=1,\dots,s$,*

$$M\big(0\subset E_1\subset\cdots\subset E_s\subset E,(\alpha_1,\dots,\alpha_s)\big)\\ +\delta\cdot\mu_\rho\big(0\subset E_1\subset\cdots\subset E_s\subset E,(\alpha_1,\dots,\alpha_s);\varphi\big)\ (\ge)\ 0.$$

Direct sums of representations. Let $\rho_i\colon\mathrm{GL}(r)\to\mathrm{GL}(V_i)$ be representations of the general linear group and assume there is an integer α with $\rho_i(z\cdot\mathrm{id}_{\mathbb{C}^r})=z^\alpha\cdot\mathrm{id}_V$ for all $z\in\mathbb{C}^*$, $i=1,\dots,t$. Define $\rho:=\rho_1\oplus\cdots\oplus\rho_t$. Note that for every rank r vector bundle E one has $E_\rho=E_{\rho_1}\oplus\cdots\oplus E_{\rho_t}$. The following result is a counterpart to Theorem 1.6 in the first part.

Proposition 3.4. *Let (E,M,τ) be a ρ-pair of type (d,r,m) and $\delta\in\mathbb{Q}_{>0}$. Then the following conditions are equivalent:*

1. *(E,M,τ) is δ-semistable (δ-polystable).*
2. *There exist pairwise distinct indices $\iota_1,\dots,\iota_s\in\{1,\dots,t\}$, $s\le t$, such that*

$$j\in\{\iota_1,\dots,\iota_s\}\ \Rightarrow\ (\Leftrightarrow)\ \tau|_{E_{\rho_j}}:E_{\rho_j}\to M\ \textit{is nonzero},$$

and positive rational numbers $\sigma_1,\dots,\sigma_s$ with $\sum_{j=1}^s\sigma_j=1$ such that for every weighted filtration $(E^\bullet,\underline{\alpha})$

$$M(E^\bullet,\underline{\alpha})+\delta\Big(\textstyle\sum_{j=1}^s\sigma_j\mu_{\rho_{\iota_j}}(E^\bullet,\underline{\alpha};\tau|_{E_{\rho_{\iota_j}}})\Big)\ \ge\ 0.$$

(*And if equality holds*

- $E \cong \bigoplus_{j=1}^{s+1} E_j/E_{j-1}$
- *the* ρ_{ι_j}*-pair* $(E, M, \tau|_{E_{\rho_{\iota_j}}})$ *is equivalent to the* ρ_{ι_j}*-pair* $(E, M, \tau|_{E^{\gamma_j}_{\rho_{\iota_j}}})$, $\gamma_j := -\mu_{\rho_{\iota_j}}(E^\bullet, \underline{\alpha}; \tau|_{E_{\rho_{\iota_j}}})$, $j = 1, \ldots, s$ *(compare with the definition of polystability on page* 179*).*

Here, $\underline{w}$ *is a basis for* $\mathbb{C}^r$, $W^\bullet : 0 \subset W^{(\mathrm{rk}\, E_1)}_{\underline{w}} \subset \cdots \subset W^{(\mathrm{rk}\, E_s)}_{\underline{w}} \subset W$, *and* $\underline{\gamma} := \sum_{j=1}^s \alpha_j \gamma^{(\mathrm{rk}\, E_j)}$*.)*

3. *There exist pairwise distinct indices* $\iota_1, \ldots, \iota_s \in \{1, \ldots, t\}$, $s \le t$, *such that*

$$j \in \{\iota_1, \ldots, \iota_s\} \;\Rightarrow\; (\Leftrightarrow)\; \tau|_{E_{\rho_j}} : E_{\rho_j} \to M \text{ is nonzero,}$$

and positive rational numbers $\sigma_1, \ldots, \sigma_s$ *with* $\sum_{j=1}^s \sigma_j = 1$ *such that for every positive integer* ν *with* $\nu\sigma_j \in \mathbb{Z}_{>0}$, $j = 1, \ldots, s$, *the associated* $(\rho_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \cdots \otimes \rho_{\iota_s}^{\otimes \nu\sigma_s})$*-pair*

$$\big(E, M^{\otimes(\nu\sigma_1 + \cdots + \nu\sigma_s)}, (\tau|_{E_{\rho_{\iota_1}}})^{\otimes \nu\sigma_1} \otimes \cdots \otimes (\tau|_{E_{\rho_{\iota_s}}})^{\otimes \nu\sigma_s}\big)$$

of type $(d, r, \nu m)$ *is* (δ/ν)*-semistable* $((\delta/\nu)$*-polystable).*

Proof. To see the equivalence between 2. and 3., observe that $\mathcal{O}(\nu\sigma_1, \ldots, \nu\sigma_s)$ provides an equivariant embedding of $\mathbb{P}(V_{\iota_1}) \times \cdots \times \mathbb{P}(V_{\iota_s})$ into $\mathbb{P}(S^{\nu\sigma_1} V_{\iota_1} \otimes \cdots \otimes S^{\nu\sigma_s} V_{\iota_s})$. Via the canonical surjection $V_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \otimes V_{\iota_s}^{\otimes \nu\sigma_s} \to S^{\nu\sigma_1} V_{\iota_1} \otimes \cdots \otimes S^{\nu\sigma_s} V_{\iota_s}$, the latter space becomes embedded into $\mathbb{P}(V_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \otimes V_{\iota_s}^{\otimes \nu\sigma_s})$, so that we have an equivariant embedding $\iota\colon \mathbb{P}(V_{\iota_1}) \times \cdots \times \mathbb{P}(V_{\iota_s}) \hookrightarrow \mathbb{P}(V_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \otimes V_{\iota_s}^{\otimes \nu\sigma_s})$. Since for every point $x = (x_1, \ldots, x_s) \in \mathbb{P}(V_{\iota_1}) \times \cdots \times \mathbb{P}(V_{\iota_s})$ and every one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(r)$

$$\sum_{j=1}^s \sigma_j \mu_{\rho_{\iota_j}}\big(\lambda, x_j\big) = \frac{1}{\nu} \cdot \mu_{\rho_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \otimes \rho_{\iota_s}^{\otimes \nu\sigma_s}}\big(\lambda, \iota(x)\big),$$

the claimed equivalence is easily seen.

For the equivalence between 1. and 3., we have to go into the GIT construction of the moduli space of δ-semistable ρ-pairs. We choose a, b, c, such that ρ is a direct summand of $\rho_{a,b,c}$. Therefore, ρ_i is also a direct summand of $\rho_{a,b,c}$, $i = 1, \ldots, t$, so that we can assume $\rho_i = \rho_{a,b,c}$ for $i = 1, \ldots, t$. For a tuple $(\iota_1, \ldots, \iota_s)$, positive rational numbers $\sigma_1, \ldots, \sigma_s$, and $\nu \in \mathbb{N}$ as in the statement, we thus find

$$\rho_{\iota_1}^{\otimes \nu\sigma_1} \otimes \cdots \otimes \rho_{\iota_s}^{\otimes \nu\sigma_s} = \rho_{\nu a, \nu b, c'}$$

for some $c' > 0$. Recall that in our GIT construction of the moduli space of δ-semistable $\rho_{a,b,c}$ pairs of type (d, r, m), we had to fix some natural number n which was large enough. Being large enough depended on constants C_1, C_2, C_3, and K which in turn depended only on d, r, a, and δ. One now checks that d, r, νa, and δ/ν yield exactly the same constants, so that the construction will work also — for all ν and all c' — for (δ/ν)-semistable $\rho_{\nu a, \nu b, c'}$-pairs of type $(d, r, \nu m)$. Fix such an n. We can now argue as follows. Set $p := d + r(n + 1 - g)$, and let U be a complex vector space of dimension p. Given a δ-semistable $\rho_{a,b,c}$-pair (E, M, τ) of type (d, r, m), we can write E as a quotient $q\colon U \otimes \mathcal{O}_X(-n) \to E$ where $H^0(q(n))$ is an isomorphism. Set $Z := \mathrm{Hom}(\bigwedge^r U, H^0(\det E(rn)))$ and $W := \mathrm{Hom}(U_{a,c}, H^0(\det E^{\otimes b} \otimes M \otimes \mathcal{O}_X(na)))$.

Then $(q\colon U\otimes\mathcal{O}_X(-n)\to E, M,\tau)$ defines a Gieseker point $([z],[w_1,\dots,w_t])\in\mathbb{P}(Z^\vee)\times\mathbb{P}(W^{\vee\oplus t})$ which is semistable for the linearization of the $\mathrm{SL}(U)$-action in $\mathcal{O}(\varepsilon,1)$ with $\varepsilon=(p-a\delta)/(r\delta)$. By Theorem 1.6, we find indices $\iota_1,\dots,\iota_s$ and positive rational numbers $\sigma_1,\dots,\sigma_s$ with $\sum_{j=1}^s\sigma_j=1$, such that $w_{\iota_j}\neq 0$, $j=1,\dots,s$, and the point $([z],[w_{\iota_1}],\dots,[w_{\iota_s}])\in\mathbb{P}(Z^\vee)\times\mathbb{P}(W^\vee)\times\cdots\times\mathbb{P}(W^\vee)$ is semistable with respect to the linearization of the $\mathrm{SL}(U)$-action in $\mathcal{O}(\varepsilon,\sigma_1,\dots,\sigma_s)$. As before, there is an embedding $\iota:\mathbb{P}(Z^\vee)\times\mathbb{P}(W^\vee)\times\cdots\times\mathbb{P}(W^\vee)\hookrightarrow\mathbb{P}(Z^\vee)\times\mathbb{P}(W^{\vee\otimes\nu})$ such that the pullback of $\mathcal{O}(\nu\varepsilon,1)$ is $\mathcal{O}(\nu\varepsilon,\nu\sigma_1,\dots,\nu\sigma_s)$. The point $y:=\iota([z],[w_{\iota_1}],\dots,[w_{\iota_s}])$ is thus semistable with respect to the linearization in $\mathcal{O}(\nu\varepsilon,\nu\sigma_1,\dots,\nu\sigma_s)$. Now, the second component of y is defined by the homomorphism $U_{\nu a,c'}=U_{a,c}^{\otimes\nu}\to H^0(\det E^{\otimes b}\otimes M\otimes\mathcal{O}_X(na))^{\otimes\nu}$ obtained from q and the components $\tau|_{E_{\rho_{\iota_j}}}$, $j=1,\dots,s$. Composing this homomorphism with the natural map $H^0(\det E^{\otimes b}\otimes M\otimes\mathcal{O}_X(na))^{\otimes\nu}\to H^0(\det E^{\otimes\nu b}\otimes M^{\otimes\nu}\otimes\mathcal{O}_X(n\nu a))$, we find a point $y'\in\mathbb{P}(Z^\vee)\times\mathbb{P}(W'^\vee)$, $W':=\mathrm{Hom}(U_{\nu a,c'},H^0(\det E^{\otimes\nu b}\otimes M^{\otimes\nu}\otimes\mathcal{O}_X(n\nu a)))$. The point y' is semistable with respect to the linearization of the $\mathrm{SL}(U)$-action in $\mathcal{O}(\nu\varepsilon,1)$. By construction, y' is the Gieseker point of the quotient $\rho_{\nu a,\nu b,c'}$-pair $(q\colon U\otimes\mathcal{O}_X(-n)\to E, M^{\otimes\nu},(\tau|_{E_{\rho_{\iota_1}}})^{\otimes\nu\sigma_1}\otimes\cdots\otimes(\tau|_{E_{\rho_{\iota_s}}})^{\otimes\nu\sigma_s})$. Since $\nu\varepsilon=(p-(\nu a)(\delta/\nu))/(r\delta/\nu)$, we infer that $(E,M^{\otimes\nu},(\tau|_{E_{\rho_{\iota_1}}})^{\otimes\nu\sigma_1}\otimes\cdots\otimes(\tau|_{E_{\rho_{\iota_s}}})^{\otimes\nu\sigma_s})$ is (δ/ν)-semistable. The converse and the polystable part are similar. □

3.2. Some features of the moduli spaces

Here, we will discuss several properties of the moduli spaces which we have constructed.

Trivial specializations. Let $\rho\colon\mathrm{GL}(r)\to\mathrm{GL}(V)$ be a representation. Very often, one fixes the determinant of the vector bundles under consideration. So, let L_0 be a line bundle of degree d. If we want to consider only ρ-pairs (E,M,τ) of type (d,r,m) with $\det E\cong L_0$, we say that the *type of* (E,M,τ) is (L_0,r,m). We then obtain a closed subfunctor $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{L_0/r/m}$ of $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}$. Note that our construction shows that we have a morphism $\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/m}\to\mathrm{Jac}^d$, $[E,M,\tau]\mapsto[\det E]$. Let $\mathcal{M}(\rho)^{\delta-(s)s}_{L_0/r/m}$ be the fibre over $[L_0]$. This is then the moduli space for $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{L_0/r/m}$.

In the applications, the line bundle M is traditionally fixed. Having fixed a line bundle M_0 of degree m, we will speak of *ρ-pairs* (E,τ) *of type* (d,r,M_0). This yields a moduli functor $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/M_0}$ which is also a closed subfunctor of $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/m}$. Its moduli space, denoted by $\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/M_0}$, is the fibre over $[M_0]$ of the morphism $\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/m}\to\mathrm{Jac}^m$, $[E,M,\tau]\mapsto[M]$.

If we want to fix both L_0 and M_0, we speak of *ρ-pairs* (E,τ) *of type* (L_0,r,M_0). The corresponding moduli spaces are denoted by $\mathcal{M}(\rho)^{\delta-(s)s}_{L_0/r/M_0}$.

Variation of δ. Given $\rho\colon\mathrm{GL}(r)\to\mathrm{GL}(V)$, $d,r,m\in\mathbb{Z}$, $r>0$, we get a whole family of moduli spaces $\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/m}$ parameterized by $\delta\in\mathbb{Q}_{>0}$. This phenomenon was first studied by Thaddeus in the proof of the Verlinde formula [45]. The papers [10] and [46] study the corresponding abstract GIT version. Using these, one makes the following observations.

(1) There is an increasing sequence $(\delta_\nu)_{\nu\geq 0}$, $\delta_\nu\in\mathbb{Q}_{>0}$, $\nu=0,1,2,\dots$, which is discrete in $\mathbb{R}$, such that the concept of δ-(semi)stability is constant within each interval $(\delta_\nu,\delta_{\nu+1})$, $\nu=0,1,2,\dots$, and, for given ν, δ-semistability for $\delta\in(\delta_\nu,\delta_{\nu+1})$ implies δ_ν-

and $\delta_{\nu+1}$-semistability and both δ_ν- and $\delta_{\nu+1}$-stability imply δ-stability. In particular, there are maps $\mathcal{M}(\rho)^{\delta-ss}_{d/r/m} \to \mathcal{M}(\rho)^{\delta_{\nu(+1)}-ss}_{d/r/m}$ ("chain of flips", [45]).

(2) For $\delta \in (0, \delta_0)$ and (E, M, τ) a δ-semistable ρ-pair, the vector bundle E must be semistable, and there is a morphism $\mathcal{M}(\rho)^{\delta-ss}_{d/r/m} \to \mathcal{M}^{ss}_{d/r}$ to the moduli space of semistable bundles of degree d and rank r. Conversely, if E is a stable bundle, then (E, M, τ) will be δ-stable.

(3) In the studied examples, there are only finitely many critical values, i.e., there is a δ_∞, such that the concept of δ-semistability is constant in (δ_∞, ∞). We refer to [5], [45], [35], [37] and the examples for explicit discussions of this phenomenon. It would be interesting to know whether this is true in general or not, i.e., to check it for $\rho_{a,b,c}$.

We note that in two of the examples, namely the example of oriented framed modules and the example of Hitchin pairs, only a parameter independent stability concept has been treated so far. Our discussions will therefore complete the picture in view of the above observations.

Asymptotic irreducibility. Fix the representation ρ, the integers d and r as well as the stability parameter $\delta \in \mathbb{Q}_{>0}$. Suppose that ρ is a direct summand of the representation $\rho_{a,b,c}$. Since the estimate in Theorem 2.6 does not depend on the integer m, we conclude that the set $\mathcal{S}$ of isomorphy classes of vector bundles E, such that there exist an $m \in \mathbb{Z}$ and a δ-semistable ρ-pair (E, M, τ) of type (d, r, m) is still bounded. The same goes for the set $\mathcal{S}_\rho$ of vector bundles of the form E_ρ with $[E] \in \mathcal{S}$. Thus, there is a constant m_0, such that for every $m \geq m_0$ and every δ-semistable ρ-pair (E, M, τ) of type (d, r, m), one has

$$\mathrm{Ext}^1(E_\rho, M) = H^1(E_\rho^\vee \otimes M) = 0.$$

Our construction and standard arguments [24], §8.5, now show that the natural parameter space for δ-semistable ρ-pairs of type (d, r, m) is a projective bundle over the product of a smooth, irreducible, and quasi-projective quot-scheme and the Jacobian of degree m line bundles. In particular, it is smooth and irreducible. We infer

Theorem 3.5. *Given the data ρ, d, r, and δ as above, there exists a constant m_0, such that the moduli space $\mathcal{M}(\rho)^{\delta-ss}_{d/r/m}$ is a normal and irreducible quasi-projective variety for every $m \geq m_0$.*

Remark 3.6. Given m', m with $m - m' = l > 0$ and a point $p_0 \in X$, the assignment $(E, M, \tau) \mapsto (E, M(lp_0), \tau')$ with $\tau' \colon E_\rho \to M \subset M(lp_0)$ induces a closed embedding

$$\mathcal{M}(\rho)^{\delta-ss}_{d/r/m'} \hookrightarrow \mathcal{M}(\rho)^{\delta-ss}_{d/r/m}.$$

3.3. Extension pairs

Fix positive integers $0 < s < r$, and let F be the Grassmannian of s-dimensional quotients of $\mathbb{C}^r$. An F-pair is thus a pair $(E, q \colon E \to Q)$ where E is a vector bundle of rank r and q is a homomorphism onto a vector bundle Q of rank s. Setting $K := \ker q$, we obtain a pair (E, K) with E as before and $K \subset E$ a subbundle of rank $r - s$. These objects were introduced by Bradlow and García–Prada [6] as holomorphic extensions and called (smooth) extension pairs in [9]. In that work, q is not required to be surjective.

We embed F via the Plücker embedding into $\mathbb{P}(\bigwedge^s \mathbb{C}^r)$, i.e., we consider the representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(\bigwedge^s \mathbb{C}^r)$. To describe the notion of (δ,ρ)-semistability, we observe that for points $[v] \in F \subset \mathbb{P}(\bigwedge^s \mathbb{C}^r)$, bases $\underline{w}$ of $\mathbb{C}^r$, and weight vectors $\underline{\gamma}_1$ and $\underline{\gamma}_2$, Equation (9) holds true. Furthermore, for a point $[v\colon \mathbb{C}^r \to \mathbb{C}^s] \in F$, a basis $\underline{w} = (w_1, \ldots, w_r)$ of $\mathbb{C}^r$, and $i \in \{1, \ldots, r-1\}$

$$\mu_\rho\big(\lambda(\underline{w}, \gamma^{(i)}), [v]\big) = i \dim \ker v - r \dim\big(\langle w_1, \ldots, w_i \rangle \cap \ker v\big).$$

Therefore, according to (10), an F-pair $(E, q\colon E \to Q)$ is (δ,ρ)-(semi)stable if and only if for every nontrivial proper subbundle E' of E one has

$$\mu(E') + \delta \frac{\mathrm{rk}(E' \cap \ker q)}{\mathrm{rk}\, E'} \ (\leq)\ \mu(E) + \delta \frac{\mathrm{rk} \ker q}{\mathrm{rk}\, E}.$$

This is the same notion [9] provides for the extension pair $(E, \ker q)$.

3.4. Framed modules

The case of framed modules is one of the most thoroughly studied examples of a decorated vector bundle problem (see, e.g., [4], [13], [45], [25], [21], [22]).

First, we fix a positive integer r, an integer d, and a line bundle M_0 on X and look at the ρ-pairs of type (d, r, M_0) associated with the representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(\mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r))$, i.e., at pairs (E, φ) consisting of a vector bundle E of degree d and rank r and a homomorphism $\varphi\colon E \to M_0^{\oplus s}$. For the representation ρ, the Additivity Property (9) is clearly satisfied, and given a nontrivial proper subbundle E' of E one has $\mu_\rho(E', \varphi) = -\mathrm{rk}\, E'$ or $r - \mathrm{rk}\, E'$ if $E' \subset \ker \varphi$ or $\not\subset \ker \varphi$, respectively.

Given $\delta \in \mathbb{Q}_{>0}$, Equation (10) thus shows that (E, φ) is δ-(semi)stable if for every nontrivial proper subbundle E' of E

$$\begin{aligned} \mu(E') \ (\leq)\ \mu(E) - \frac{\delta}{\mathrm{rk}\, E} \quad &\text{if} \quad E' \subset \ker \varphi, \\ \mu(E') - \frac{\delta}{\mathrm{rk}\, E'} \ (\leq)\ \mu(E) - \frac{\delta}{\mathrm{rk}\, E} \quad &\text{if} \quad E' \not\subset \ker \varphi. \end{aligned}$$

Finally, one has the following result on the stability parameter δ:

Lemma 3.7. *Fix integers d, r, $r > 0$, and a line bundle M_0. The set of isomorphy classes of vector bundles E for which there exist a parameter $\delta \in \mathbb{Q}_{>0}$ and a δ-semistable ρ-pair of type (d, r, M_0) of the form (E, φ) is bounded.*

This is proved as Proposition 2.2.2. in [35]. From this boundedness result, it follows easily that the set of isomorphy classes of vector bundles of the form $\ker \varphi$, (E, φ) a ρ-pair of type (d, r, M_0) for which there exists a $\delta \in \mathbb{Q}_{>0}$ with respect to which it becomes semistable is bounded as well. We infer

Corollary 3.8. *There exists a positive rational number δ_∞ such that for every $\delta \geq \delta_\infty$ and every ρ-pair (E, φ) of type (d, r, M_0), the following conditions are equivalent:*

1. *(E, φ) is δ-(semi)stable.*
2. *φ is injective.*

Now, fix a vector bundle E_0 on X. Recall that a *framed module of type (d, r, E_0)* is a pair (E, ψ) consisting of a vector bundle E of degree d and rank r and a nonzero homomorphism $\psi\colon E \to E_0$. Fix a sufficiently ample line bundle M_0 on X and an embedding $\iota\colon E_0 \subset M_0^{\oplus s}$ for some s. Therefore, any framed module (E,ψ) of type (d,r,E_0) gives rise to the ρ-pair $(E,\varphi := \iota\circ\psi)$ of type (d,r,M_0), and the ρ-pair (E,φ) is δ-(semi)stable if and only if (E,ψ) is a δ-(semi)stable framed module in the sense of [21]. Finally, a family of framed modules of type (d,r,E_0) parameterized by S is a triple $(E_S,\psi_S,\mathfrak{N}_S)$ consisting of a rank r vector bundle E_S on $S\times X$, a line bundle $\mathfrak{N}_S$ on S, and a homomorphism $\psi_S\colon E_S \to \pi_X^* E_0\otimes\mathfrak{N}_S$ which is nontrivial on every fibre $\{s\}\times X$, $s\in S$. Associate to such a family $(E_S,\psi_S,\mathfrak{N}_S)$ the family $(E_S,\kappa_S,\mathfrak{N}_S,\varphi_S)$ of ρ-pairs of type (d,r,M_0) where $\kappa_S(s)=[M_0]$ for all $s\in S$ and $\varphi_S=(\pi_X^*(\iota)\otimes \mathrm{id}_{\pi_S^*\mathfrak{N}_S})\circ\psi_S$. This exhibits the functor associating to a scheme S the set of equivalence classes of families of δ-(semi)stable framed modules of type (d,r,M_0) as the subfunctor of $\underline{\mathrm{M}}(\rho)^{\delta-(s)s}_{d/r/M_0}$ of those families $(E_S,\kappa_S,\mathfrak{N}_S,\varphi_S)$ where κ_S is the constant morphism $s\mapsto[M_0]$, and the composite $E_S\to\pi_X^*(M_0^{\oplus s})\otimes\pi_S^*\mathfrak{N}_S\to\pi_X^*(M_0^{\oplus s}/\iota(E_0))\otimes\pi_S^*\mathfrak{N}_S$ vanishes. Since all these conditions are closed conditions, the moduli spaces of δ-(semi)stable framed modules on curves ([45], [21]) become closed subschemes of our moduli spaces $\mathcal{M}(\rho)^{\delta-(s)s}_{d/r/M_0}$.

Remark 3.9. We have used a slightly different, more general notion of family than [21]. This choice only destroys the property of being a fine moduli space and does not affect the construction of the moduli space of framed modules.

3.5. Oriented framed modules

We begin with the representations $\rho_1\colon \mathrm{GL}(r)\to\mathrm{GL}(\mathrm{Hom}(\mathbb{C}^s,S^r\mathbb{C}^r))$ and $\rho_2\colon \mathrm{GL}(r)\to\mathrm{GL}(\bigwedge^r\mathbb{C}^r)$, and set $\rho:=\rho_1\oplus\rho_2$. Fix line bundles L_0 and M_0. Then a ρ-pair of type (L_0,r,M_0) is a triple (E,φ,σ), consisting of a vector bundle E of rank r with $\det E\cong L_0$, a homomorphism $\varphi\colon S^rE\to M_0^{\oplus s}$, and a homomorphism $\sigma\colon \det E\to M_0$. Next, assume we are given a line bundle N_0 with $N_0^{\otimes r}=M_0$ and t such that $s=\#\{\,(i_1,\dots,i_t)\mid i_j\in\{\,0,\dots,r\,\},\ j=1,\dots,t,\text{ and }\sum_{j=1}^t i_j=r\,\}$, i.e., $S^rN_0^{\oplus t}\cong M_0^{\oplus s}$. Then to any triple (E,ψ,σ) where E is a vector bundle of rank r with $\det E\cong L_0$ and $\psi\colon E\to N_0^{\oplus t}$ and $\sigma\colon\det E\to N_0^{\otimes r}$ are homomorphisms, we can associate the ρ-pair $(E,S^r\psi,\sigma)$ of type (L_0,r,M_0). Observe that for any weighted filtration $(E^\bullet,\underline{\alpha})$ one has

$$\mu_{\rho_2}\big(E^\bullet,\underline{\alpha};\sigma\big)=0 \qquad\text{and}\qquad \mu_{\rho_1}\big(E^\bullet,\underline{\alpha};S^r\psi\big)=r\cdot\mu_{\rho_1'}\big(E^\bullet,\underline{\alpha};\psi\big)$$

where $\rho_1'\colon \mathrm{GL}(r)\to\mathrm{GL}(\mathrm{Hom}(\mathbb{C}^t,\mathbb{C}^r))$. Therefore, Proposition 3.4 and the discussion of framed modules show the following.

Lemma 3.10. *Let (E,ψ,σ) be a triple where E is a vector bundle of rank r with* $\det E\cong L_0$ *and $\psi\colon E\to N_0^{\oplus t}$ and $\sigma\colon\det E\to N_0^{\otimes r}$ are homomorphisms, and $\delta\in\mathbb{Q}_{>0}$. Then the following conditions are equivalent:*

1. *The associated ρ-pair $(E,S^r\psi,\sigma)$ of type (L_0,r,M_0) is δ-semistable.*
2. *One of the following three conditions is verified:*
    i. *E is a semistable vector bundle.*
    ii. *The homomorphisms ψ and σ are nonzero and there exists a positive rational number $\delta'\le r\delta$, such that (E,ψ) is a δ'-semistable ρ_1'-pair of type (L_0,r,N_0).*
    iii. *The homomorphism σ vanishes and (E,ψ) is an $(r\cdot\delta)$-semistable ρ_1'-pair of type (L_0,r,N_0).*

We omit the "polystable version" of this lemma. In particular, for $r\delta > \delta_\infty$ (see Corollary 3.8), one finds

Corollary 3.11. *Let (E, ψ, σ) be a triple where E is a vector bundle of rank r with $\det E \cong L_0$ and $\psi\colon E \to N_0^{\oplus t}$ and $\sigma\colon \det E \to N_0^{\otimes r}$ are homomorphisms, and $\delta > \delta_\infty/r$. Then the following conditions are equivalent:*

1. *The associated ρ-pair $(E, S^r\psi, \sigma)$ of type (L_0, r, M_0) is δ-semistable.*
2. *One of the following three conditions is verified:*
   i. *E is a semistable vector bundle.*
   ii. *The homomorphisms ψ and σ are nonzero and there exists a positive rational number δ', such that (E, ψ) is a δ'-semistable ρ_1'-pair of type (L_0, r, N_0).*
   iii. *The homomorphism σ vanishes and ψ is injective.*

Now, we turn to the moduli problem we would like to treat. For this, we fix a line bundle L_0 and a vector bundle E_0. Then an *oriented framed module of type* (L_0, r, E_0) is a triple (E, ε, ψ) where E is a vector bundle of rank r with $\det E \cong L_0$ and $\varepsilon\colon \det E \to L_0$ and $\psi\colon E \to E_0$ are homomorphisms, not both zero. The corresponding moduli problem was treated in [35]. Over curves, we can recover it from our theory in the following way: If N_0 is sufficiently ample, there are embeddings $\iota_1\colon L_0 \subset N_0^{\otimes r}$ and $\iota_2\colon E_0 \subset N_0^{\oplus t}$. Thus, setting $M_0 := N_0^{\otimes r}$, we can define $\sigma := \iota_1 \circ \varepsilon\colon \det E \to M_0$ and $\varphi := S^r(\iota_2 \circ \psi)\colon S^r E \to M_0^{\oplus s} = S^r N_0^{\oplus t}$ in order to get the ρ-pair (E, φ, σ) of type (L_0, r, M_0). By Corollary 3.11, for $\delta \geq \delta_\infty/r$, the ρ-pair (E, φ, σ) is δ-semistable if and only if (E, ε, ψ) is a semistable oriented framed module in the sense of [35].

Remark 3.12. The corresponding stability concept can be recovered via Proposition 3.4 and the characterisation "stable=polystable+simple" (Remark 2.4 (ii)).

We conclude by observing that applying Lemma 3.10 yields new semistability concepts for oriented framed modules.

3.6. Hitchin pairs

The theory of Hitchin pairs or Higgs bundles is also a famous example of a decorated vector bundle problem ([20], [43], [11], [34], [47], [19], [36]).

To begin with, we fix integers d and $r > 0$, a line bundle M_0, and the representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(\mathrm{End}(\mathbb{C}^r) \oplus \mathbb{C})$. In this case, a ρ-pair of type (d, r, M_0) is a triple (E, φ, σ) consisting of a vector bundle E of degree d and rank r, a twisted endomorphism $\varphi\colon E \to E \otimes M_0$, and a section $\sigma\colon \mathcal{O}_X \to M_0$.

Lemma 3.13. *There is a positive rational number δ_∞, such that for all $\delta \geq \delta_\infty$ and all ρ-pairs (E, φ, σ) of type (d, r, M_0) the following conditions are equivalent:*

1. *(E, φ, σ) is a δ-(semi)stable ρ-pair*
2. *for every nontrivial subbundle E' of E with $\varphi(E') \subset E' \otimes M_0$*

$$\mu(E') \ (\leq)\ \mu(E),$$

and either $\sigma \neq 0$ or φ is not nilpotent, i.e., $(\varphi \otimes \mathrm{id}_{M_0^{\otimes r-1}}) \circ \cdots \circ \varphi \neq 0$.

Proof. First, assume 1. Let $f\colon \mathbb{C}^r \to \mathbb{C}^r$ be a homomorphism. Call a vector subspace $V \subset \mathbb{C}^r$ *f-superinvariant* if $V \subset \ker f$ and $f(\mathbb{C}^r) \subset V$.

Lemma 3.14. *Let $[f,\varepsilon]\in\mathbb{P}(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)\oplus\mathbb{C})$. Given a basis $\underline{w}=(w_1,\dots,w_r)$ of W and $i\in\{1,\dots,r-1\}$, set $W^{(i)}_{\underline{w}}:=\langle w_1,\dots,w_i\rangle$. Then*

(i) $\mu_\rho\big(\lambda(\underline{w},\gamma^{(i)}),[f,\varepsilon]\big)=r$ *if $W^{(i)}_{\underline{w}}$ is not f-invariant.*
(ii) $\mu_\rho\big(\lambda(\underline{w},\gamma^{(i)}),[f,\varepsilon]\big)=-r$ *if $W^{(i)}_{\underline{w}}$ is f-superinvariant and $\varepsilon=0$.*
(iii) $\mu_\rho\big(\lambda(\underline{w},\gamma^{(i)}),[f,\varepsilon]\big)=0$ *in all the other cases.*

Now, let (E,φ,σ) be a ρ-pair of type (d,r,M_0). For any subbundle E' of E with $\varphi(E')\subset E'\otimes M_0$, we find $\mu_\rho(E',(\varphi,\sigma))\le 0$.

Corollary 3.15. *Let $\delta\in\mathbb{Q}_{>0}$ and (E,φ,σ) a δ-(semi)stable ρ-pair of type (d,r,M_0). Then $\mu(E')(\le)\mu(E)$ for every nontrivial proper subbundle E' of E with $\varphi(E')\subset E'\otimes M_0$.*

This condition implies that for every $\delta>0$, every δ-semistable ρ-pair (E,φ,σ) of type (d,r,M_0), and every subbundle E' of E

$$\mu(E')\le\max\Big\{\,\mu(E),\mu(E)+\frac{(r-1)^2}{r}\deg M_0\,\Big\}. \tag{12}$$

See, e.g., [34]. Therefore, the set of isomorphy classes of bundles E, such that there exist a positive rational number δ and a δ-semistable ρ-pair of type (d,r,M_0) of the form (E,φ,σ), is bounded.

Now, the only thing we still have to show is that for every sufficiently large positive rational number δ and every δ-semistable ρ-pair (E,φ,σ) of type (d,r,M_0), such that $\sigma=0$, the homomorphism φ can't be nilpotent. First, let (E,φ,σ) be a ρ-pair of type (d,r,M_0), such that there exists a positive rational number δ with respect to which (E,φ,σ) is semistable and such that φ is nilpotent. Then there is a filtration

$$0=:E_0\subset E_1\subset\cdots\subset E_{s-1}\subset E_s:=E$$

with $E_j\otimes M_0=\varphi(E_{j+1})$, $j=0,\dots,s-1$. It is clear by the boundedness result that the E_j's occurring in this way live in bounded families, so that we can find a positive constant C with

$$d\,\mathrm{rk}\,E_j-\deg E_j r<C,\quad j=1,\dots,s-1$$

for all such filtrations. One checks $\mu_\rho(E^\bullet,(1,\dots,1);(\varphi,\sigma))=-r$, so that the semistability assumption yields

$$0\le M\big(E^\bullet,(1,\dots,1)\big)+\delta\mu_\rho\big(E^\bullet,(1,\dots,1);(\varphi,\sigma)\big)\le(r-1)C-\delta r.$$

This is impossible if $\delta\ge C$.

To see the converse, let (E,φ,σ) be a ρ-pair satisfying 2. Let $m_0:=\max\{0,\deg M_0(r-1)^2/r\}$. Then as before, $\mu(E')\le\mu(E)+m_0$ for every nontrivial proper subbundle E' of E, i.e., $d\,\mathrm{rk}\,E'-r\deg E'\ge -m_0r\,\mathrm{rk}\,E'\ge -m_0(r-1)r$. First, consider a weighted filtration $(E^\bullet,\underline{\alpha})$ such that $\varphi(E_j)\subset E_j\otimes M_0$, $j=1,\dots,s$. Then the condition that φ be not nilpotent if $\sigma=0$ implies $\mu_\rho(E^\bullet,\underline{\alpha};(\varphi,\sigma))=0$, so that $M(E^\bullet,\underline{\alpha})(\ge)0$ follows from 2. Second, suppose that we are given a weighted filtration $(E^\bullet,\underline{\alpha})$ such that $E_{j_1},\dots,E_{j_t}$ are not invariant under φ, i.e., $\varphi(E_{j_i})\not\subset E_{j_i}\otimes M_0$, $i=1,\dots,t$, and

$t > 0$. Let $\alpha := \max\{ \alpha_{j_1}, \dots, \alpha_{j_t} \}$. One readily verifies $\mu_\rho(E^\bullet, \underline{\alpha}; (\varphi, \sigma)) \geq \alpha \cdot r$. We thus find

$$\begin{aligned} M\big(E^\bullet, \underline{\alpha}\big) + \delta\mu_\rho\big(E^\bullet, \underline{\alpha}\big) &\geq \textstyle\sum_{i=1}^t \alpha_{j_i}\big(d \operatorname{rk} E_{j_i} - r \deg E_{j_i}\big) + r\alpha\delta \\ &\geq -(r-1)rm_0 \textstyle\sum_{i=1}^t \alpha_{j_i} + r\alpha\delta \\ &\geq \big(-(r-1)^2 rm_0 + r\delta\big)\alpha, \end{aligned}$$

so that $M\big(E^\bullet, \underline{\alpha}\big) + \delta\mu_\rho\big(E^\bullet, \underline{\alpha}\big)$ will be positive if we choose $\delta > (r-1)^2 m_0$. □

Example 3.16. For small values of δ, the concept of δ-(semi)stability seems to become rather difficult. However, in the rank two case we have: A ρ-pair (E, φ, σ) of type $(d, 2, M_0)$ is δ-(semi)stable if for every line subbundle E' of E one has

1. $\deg E'(\leq) d/2 + \delta$,
2. $\deg E'(\leq) d/2$ if E' is invariant under φ,
3. $\deg E'(\leq) d/2 - \delta$ if $E' = \ker\varphi$, $\varphi(E) \subset E' \otimes M_0$, and $\sigma = 0$.

Fix a line bundle L on X. We remind the reader [36] that a *Hitchin pair of type* (d, r, L) is a triple (E, ψ, ε) where E is a vector bundle of degree d and rank r, $\psi\colon E \to E \otimes L$ is a twisted endomorphism, and ε is a complex number. Two Hitchin pairs $(E_1, \psi_1, \varepsilon_1)$ and $(E_2, \psi_2, \varepsilon_2)$ are called *equivalent* if there exist an isomorphism $h\colon E_1 \to E_2$ and a nonzero complex number λ with $\lambda\psi_1 = (h \otimes \mathrm{id}_L)^{-1} \circ \psi_2 \circ h$ and $\lambda\varepsilon_1 = \varepsilon_2$. We fix a point x_0 and choose n large enough, so that $M_0 := L(nx_0)$ has a nontrivial global section. Fix such a global section $\sigma_0\colon \mathcal{O}_X \to M_0$ and an embedding $\iota\colon L \subset M_0$. To every Hitchin pair (E, ψ, ε) of type (d, r, L), we can assign the ρ-pair (E, φ, σ) with $\varphi := (\mathrm{id}_E \otimes \iota) \circ \psi$ and $\sigma := \varepsilon \cdot \sigma_0$. Note that this assignment is compatible with the equivalence relations. By Lemma 3.13, for $\delta \geq \delta_\infty$, the ρ-pair (E, φ, σ) is δ-(semi)stable if and only if (E, ψ, ε) is a (semi)stable Hitchin pair in the sense of [36]. Again, the above assignment carries over to families, so that the general construction also yields a construction of the moduli space of semistable Hitchin pairs on curves, constructed in [36] and [19]. This space is a compactification of the "classical" Hitchin space [20], [11], [34].

As we have seen, the semistability concept for Hitchin pairs is parameter dependent in nature, though it might be difficult to describe for low values of δ. To illustrate that we get new semistable objects for small values of δ, let us look at an example.

Example 3.17. (i) Let $x_0 \in X$ be a point, and set $\mathcal{O}(1) := \mathcal{O}_X(x_0)$. Define $E := \mathcal{O} \oplus \mathcal{O}(1)$, and $\psi\colon E \to E \otimes \mathcal{O}(1) = \mathcal{O}(1) \oplus \mathcal{O}(2)$ as the homomorphism whose restriction to $\mathcal{O}$ is zero and, moreover, the induced homomorphisms $\mathcal{O}(1) \to \mathcal{O}(1)$ and $\mathcal{O}(1) \to \mathcal{O}(2)$ are the identity and zero, respectively. First, consider the Hitchin pair $(E, \psi, 1)$. Then the third condition in 3.16 is void and the second condition is satisfied. Indeed, a ψ-invariant subbundle E' of E of rank one cannot be contained in $\mathcal{O}(1)$ whence $\deg E' \leq 0 < 1/2$. Any other line subbundle E' has degree at most one, and $E' := \mathcal{O}(1)$ is a subbundle of degree exactly one. The first condition then reads $1(\leq)1/2 + \delta$. In other words, $(E, \psi, 1)$ is δ-stable for $\delta > 1/2$, properly $(1/2)$-semistable, and not semistable for $\delta < 1/2$. Finally, we claim that $(E, \psi, 0)$ is properly $(1/2)$-semistable (although ψ is nilpotent). For this, we only have to check the condition for $E' = \mathcal{O}$, i.e., $0 \leq 1/2 - 1/2$, and this is clearly satisfied.

(ii) To see the role of δ in the whole theory, let us look at Hitchin pairs of type $(1, 2, \omega_X)$. Let δ_∞ be as in Lemma 3.13. For $\delta \geq \delta_\infty$, denote by $\mathcal{H}it_{\omega_X}$ the moduli space of stable (in the usual sense) Hitchin pairs of type $(1, 2, \omega_X)$. Let $\delta_0, \ldots, \delta_m \in (0, \delta_\infty)$ be the critical values. For $0 < \delta < \delta_0$, the moduli space of δ-stable Hitchin pairs of type $(1, 2, \omega_X)$ equals $\mathbb{P}(\mathcal{O}_\mathcal{N} \oplus T_\mathcal{N})$, the compactified cotangent bundle of $\mathcal{N}$, the moduli space of stable rank two bundles of degree one. Furthermore, let $\mathcal{M}^i_{\omega_X}$ be the moduli space of δ-stable Hitchin pairs of type $(1, 2, \omega_X)$ where $\delta \in (\delta_i, \delta_{i+1})$, $i = 0, \ldots, m-1$, and $\widetilde{\mathcal{M}}^i_{\omega_X}$ the moduli space of δ_i-semistable Hitchin pairs of type $(1, 2, \omega_X)$, $i = 0, \ldots, m$. Between those spaces, we have morphisms

$$\begin{array}{ccccccccc} & \mathbb{P}(\mathcal{O}_\mathcal{N} \oplus T_\mathcal{N}) & & \cdots & & \mathcal{M}^{m-1}_{\omega_X} & & \mathcal{H}it_{\omega_X} & \\ \swarrow & & \searrow & & \swarrow & & \searrow & \swarrow & \\ \mathcal{N} & & \widetilde{\mathcal{M}}^0_{\omega_X} & \cdots & \widetilde{\mathcal{M}}^{m-1}_{\omega_X} & & \widetilde{\mathcal{M}}^m_{\omega_X} & & \end{array}$$

As in [45], this is the factorization of the birational correspondence $\mathbb{P}(\mathcal{O}_\mathcal{N} \oplus T_\mathcal{N}) \dashrightarrow \mathcal{H}it_{\omega_X}$ into flips and is thus related to the factorization into blow ups and downs (cf. [19]).

Remark 3.18 (A. Teleman). It might seem odd that we also obtain new semistability concepts for the classical Higgs bundles (E, φ) where the semistability concept is known to be parameter independent. In gauge theory, the reason is that for studying Higgs bundles, one fixes a flat metric of infinite volume on the fibre $F = \mathrm{End}(\mathbb{C}^r)$, whereas we use a metric of bounded volume induced by the embedding $\mathrm{End}(\mathbb{C}^r) \subset \mathbb{P}(\mathrm{End}(\mathbb{C}^r) \oplus \mathbb{C})$ which yields a different moment map. If we let the parameter δ tend to infinity, we approximate the flat metric and therefore recover the parameter independent semistability concept.

The related moduli problems of framed and oriented framed Hitchin pairs discussed in [44] and [37] can also be dealt with in our context. We leave this to the interested reader.

3.7. Conic bundles

Consider the representation $\rho\colon \mathrm{GL}(r) \to \mathrm{GL}(S^2\mathbb{C}^r)$ and fix a line bundle M_0 on X. A ρ-pair of type (d, r, M_0) is thus a pair (E, φ) consisting of a vector bundle E of rank r and degree d and a nonzero homomorphism $\rho\colon S^2E \to M_0$. For $r \le 3$, these objects have been studied in [15]. We apply Theorem 3.3 to analyze the notion of semistability, using slightly different notation.

To simplify the stability concept, we have to understand the weights occurring for the action of $\mathrm{SL}(r)$ on $\mathbb{P}(S^2\mathbb{C}^r)$. For this, let $[l] \in \mathbb{P}(S^2\mathbb{C}^r)$ be a point represented by the linear form $l\colon S^2\mathbb{C}^r \to \mathbb{C}$. Set $I := \{\, (i_1, i_2) \mid i_1, i_2 \in \{\, 1, \ldots, r \,\}, i_1 \le i_2 \,\}$. For a basis $\underline{w} = (w_1, \ldots, w_r)$ and $(i_1, i_2) \in I$, we set $l(\underline{w})_{i_1 i_2} := l(w_{i_1} \otimes w_{i_2})$, so that the elements $l(\underline{w})_{i_1 i_2}$, $(i_1, i_2) \in I$, form a basis for $S^2\mathbb{C}^r$. We define a partial ordering on I, by defining $(i_1, i_2) \preceq (j_1, j_2)$ if $i_1 \le j_1$ and $i_2 \le j_2$. Furthermore, we define

$$I(\underline{w}, l) := \big\{\, (i_1, i_2) \in I \mid l(\underline{w})_{i_1 i_2} \neq 0, \text{ and } (i_1, i_2) \text{ is minimal with respect to “}\preceq\text{”} \,\big\}.$$

If $\# I(\underline{w}, l) = 1$, then one has the additivity property (9) for all weight vectors $\underline{\gamma}_1$ and $\underline{\gamma}_2$. In the other case, the cone of all weight vectors $(\gamma_1, \ldots, \gamma_r)$ with $\gamma_1 \le \cdots \le \gamma_r$ and $\sum \gamma_i = 0$ becomes decomposed into subcones $C_{i_1 i_2}(\underline{w}, l)$, $(i_1, i_2) \in I(\underline{w}, l)$, where

$$C_{i_1 i_2}(\underline{w}, l) := \big\{\, (\gamma_1, \ldots, \gamma_r) \mid \gamma_{i_1} + \gamma_{i_2} \le \gamma_{i'_1} + \gamma_{i'_2} \text{ for all } (i'_1, i'_2) \in I(\underline{w}, l) \,\big\}.$$

Then (9) is still satisfied if there is such a subcone containing both $\underline{\gamma}_1$ and $\underline{\gamma}_2$. If one chooses generators for these subcones, it therefore becomes sufficient to compute the number $\mu_\rho(\lambda(\underline{w},\underline{\gamma}),[l])$ for weight vectors $\underline{\gamma}$ which are either of the form $\gamma^{(i)}$ or belong to a set of generators for a cone $C_{i_1i_2}(\underline{w},l)$. To see how this simplifies the concept of δ-(semi)stability, let us look at the cases $r=3$ and $r=4$.

In the case $r=3$, one has $\#I(\underline{w},l)=1$ unless $l(\underline{w})_{11}=0=l(\underline{w})_{12}$ and both $l(\underline{w})_{22}$ and $l(\underline{w})_{13}$ are nonzero. One checks that $C_{13}(\underline{w},l)$ is generated by $\gamma^{(1)}$ and $\gamma^{(1)}+\gamma^{(2)}$ and that $C_{22}(\underline{w},l)$ is generated by $\gamma^{(2)}$ and $\gamma^{(1)}+\gamma^{(2)}$. To transfer this to our moduli problem, let E be a vector bundle of rank 3 and $\tau\colon S^2E\to M_0$ a nonzero homomorphism. Following [15], given subbundles F_1 and F_2, we write $F_1\cdot F_2$ for the subbundle of S^2E generated by local sections of the form $f_1\otimes f_2$ where f_i is a local section of F_i, $i=1,2$. For any nontrivial proper subbundle E' of E, one sets

- $c_\tau(E'):=2$ if $\tau|_{E'\cdot E'}\neq 0$,
- $c_\tau(E'):=1$ if $\tau|_{E'\cdot E'}=0$ and $\tau|_{E'\cdot E}\neq 0$, and
- $c_\tau(E'):=0$ if $\tau|_{E'\cdot E}=0$.

One checks

$$\mu_\rho\big(E',\tau\big) \;=\; c_\tau(E')\,\mathrm{rk}\,E-2\,\mathrm{rk}\,E'. \tag{13}$$

Finally, call a filtration $E^\bullet: 0\subset E_1\subset E_2\subset E$ with $\mathrm{rk}\,E_i=i$, $i=1,2$, *critical* if $\tau|_{E_1\cdot E_2}=0$, and $\tau|_{E_1\cdot E}$ and $\tau|_{E_2\cdot E_2}$ are both nonzero. Then

$$\mu_\rho\big(E^\bullet,(1,1);\tau\big)=0.$$

Putting everything together, we find the following.

Lemma 3.19. *A ρ-pair (E,τ) of type $(d,3,M_0)$ is δ-(semi)stable if and only if it satisfies the following two conditions:*

1. *For every nonzero proper subbundle E' one has*

$$\mu(E')-\delta\frac{c_\tau(E')}{\mathrm{rk}\,E'}\ (\leq)\ \mu(E)-\delta\frac{2}{3}.$$

2. *For every critical filtration* $0\subset E_1\subset E_2\subset E$

$$\deg E_1+\deg E_2\ (\leq)\ \deg E.$$

This is the stability condition formulated by Gómez and Sols [15]. Next, we look at the case $r=4$. Set

$$\nu(\underline{w},l):=\min\big\{\,i_1+i_2\mid l(\underline{w})_{i_1i_2}\neq 0,(i_1,i_2)\in I\big\}.$$

Suppose we are given a linear form $l\colon S^2\mathbb{C}^4\to\mathbb{C}$. Then for a basis $\underline{w}=(w_1,\dots,w_4)$, we have $\#I(\underline{w},l)=1$ except for the following cases

(1) $\nu(\underline{w},l)=4$, $l(\underline{w})_{22}\neq 0$ and $l(\underline{w})_{13}\neq 0$,
(2) $\nu(\underline{w},l)=4$, $l(\underline{w})_{22}\neq 0$, $l(\underline{w})_{13}=0$, and $l(\underline{w})_{14}\neq 0$,
(3) $\nu(\underline{w},l)=5$, $l(\underline{w})_{14}\neq 0$ and $l(\underline{w})_{23}\neq 0$,
(4) $\nu(\underline{w},l)=5$, $l(\underline{w})_{14}\neq 0$, $l(\underline{w})_{23}=0$, and $l(\underline{w})_{33}\neq 0$,
(5) $\nu(\underline{w},l)=6$, $l(\underline{w})_{24}\neq 0$ and $l(\underline{w})_{33}\neq 0$.

Straightforward computations show the following.

Lemma 3.20. (i) *In case* 1, $C_{13}(\underline{w}, l)$ *is generated by* $\gamma^{(1)}$, $\gamma^{(3)}$, *and* $\gamma^{(1)} + \gamma^{(2)}$, *and* $C_{22}(\underline{w}, l)$ *by* $\gamma^{(2)}$, $\gamma^{(3)}$, *and* $\gamma^{(1)} + \gamma^{(2)}$.

(ii) *In case* 2, $C_{14}(\underline{w}, l)$ *is generated by* $\gamma^{(3)}$, $\gamma^{(1)} + \gamma^{(3)}$, *and* $\gamma^{(2)} + \gamma^{(3)}$, *and* $C_{22}(\underline{w}, l)$ *by* $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(1)} + \gamma^{(3)}$, *and* $\gamma^{(2)} + \gamma^{(3)}$.

(iii) *In case* 3, $C_{14}(\underline{w}, l)$ *is generated by* $\gamma^{(1)}$, $\gamma^{(2)}$, *and* $\gamma^{(1)} + \gamma^{(3)}$, *and* $C_{23}(\underline{w}, l)$ *by* $\gamma^{(2)}$, $\gamma^{(3)}$, *and* $\gamma^{(1)} + \gamma^{(3)}$.

(iv) *In case* 4, $C_{14}(\underline{w}, l)$ *is generated by* $\gamma^{(2)}$, $\gamma^{(3)}$, $\gamma^{(1)} + \gamma^{(2)}$, *and* $\gamma^{(1)} + \gamma^{(3)}$, *and* $C_{33}(\underline{w}, l)$ *by* $\gamma^{(1)}$, $\gamma^{(1)} + \gamma^{(2)}$, *and* $\gamma^{(1)} + \gamma^{(3)}$.

(v) *In case* 5, $C_{24}(\underline{w}, l)$ *is generated by* $\gamma^{(1)}$, $\gamma^{(2)}$, *and* $\gamma^{(2)} + \gamma^{(3)}$, *and* $C_{33}(\underline{w}, l)$ *by* $\gamma^{(1)}$, $\gamma^{(3)}$, *and* $\gamma^{(2)} + \gamma^{(3)}$.

Now, let (E, τ) be a ρ-pair of type $(d, 4, M_0)$. For any nonzero, proper subbundle E' of E, we define $c_\tau(E')$ as before. One checks that (13) remains valid. Call a filtration $0 \subset E_1 \subset E_2 \subset E_3 \subset E$ with $\operatorname{rk} E_i = i$ *critical of type* (I), (II), (III), (IV), (V) if

(I) $\tau|_{E_1 \cdot E_2} = 0$, and $\tau|_{E_1 \cdot E_3}$ and $\tau|_{E_2 \cdot E_2}$ are both nonzero;
(II) $\tau|_{E_1 \cdot E_3} = 0$, and $\tau|_{E_1 \cdot E}$ and $\tau|_{E_2 \cdot E_2}$ are both nonzero;
(III) $\tau|_{E_1 \cdot E_3} = 0$, $\tau|_{E_2 \cdot E_2} = 0$, and both $\tau|_{E_1 \cdot E}$ and $\tau|_{E_2 \cdot E_3}$ are nonzero;
(IV) $\tau|_{E_2 \cdot E_3} = 0$, and both $\tau|_{E_1 \cdot E}$ and $\tau|_{E_3 \cdot E_3}$ are nonzero;
(V) $\tau|_{E_1 \cdot E} = 0$, $\tau|_{E_2 \cdot E_3} = 0$, and both $\tau|_{E_2 \cdot E}$ and $\tau|_{E_3 \cdot E_3}$ are nonzero.

respectively. In these cases, one has

- $\mu_\rho\big(0 \subset E_1 \subset E_2 \subset E, (1,1); \tau\big) = -2$ for type (I), (IV),
- $\mu_\rho\big(0 \subset E_1 \subset E_3 \subset E, (1,1); \tau\big) = 0$ for type (II), (III), (IV),
- $\mu_\rho\big(0 \subset E_2 \subset E_3 \subset E, (1,1); \tau\big) = 2$ for type (II), (V).

Gathering all information, we find the following.

Lemma 3.21. *The* ρ*-pair* (E, τ) *of type* $(d, 4, M_0)$ *is* δ*-(semi)stable if and only if it satisfies the following two conditions:*

1. *For every nonzero proper subbundle* E' *one has*

$$\mu(E') - \delta \frac{c_\tau(E')}{\operatorname{rk} E'} (\leq) \mu(E) - \frac{\delta}{2}.$$

2. *For every critical filtration* $0 \subset E_1 \subset E_2 \subset E_3 \subset E$

- $4 \deg E_1 + 4 \deg E_2 \;(\leq)\; 3 \deg E - 2$ if it is of type (I), (IV),
- $\deg E_1 + \deg E_3 \;(\leq)\; \deg E$ if it is of type (II), (III), (IV),
- $4 \deg E_2 + 4 \deg E_3 \;(\leq)\; 5 \deg E + 2$ if it is of type (II), (V).

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