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SYMPLECTIC IMPLOSION

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Abstract. Let K be a compact Lie group. We introduce the process of symplectic implosion, which associates to every Hamiltonian K-manifold a stratified space called the imploded cross-section. It bears a resemblance to symplectic reduction, but instead of quotienting by the entire group, it cuts the symmetries down to a maximal torus of K. We examine the nature of the singularities and describe in detail the imploded cross-section of the cotangent bundle of K, which turns out to be identical to an affine variety studied by Gelfand, Popov, Vinberg, and others. Finally we show that "quantization commutes with implosion".

1. Introduction

According to Cartan and Weyl, a finite-dimensional representation of a compact connected Lie group K is determined up to isomorphism by its highest weight vectors. In the language of the orbit method, the classical analogue of a representation is a symplectic manifold M equipped with a Hamiltonian action of the group K. The classical analogue of the collection of highest weights is then the moment, or Kirwan, polytope of M. This paper deals with the question: what is the classical analogue of the set of highest weight vectors?

In answer to this question we construct a space called the *imploded cross-section* of M. It is defined by fixing a Weyl chamber of K, taking its inverse image under the moment map, and quotienting the resulting subset of M by a certain equivalence relation. While this subquotient construction is reminiscent of symplectic reduction, the imploded cross-section is almost always singular, whereas symplectic quotients often are

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not. For example, the imploded cross-section of the cotangent bundle T^*K is singular unless the commutator subgroup of K is a product of copies of $\mathbf{SU}(2)$. It is not even an orbifold unless the universal cover of $[K, K]$ is a product of copies of $\mathbf{SU}(2)$. (See Section 6.) The singularities are however not completely arbitrary. Like singular symplectic quotients, imploded cross-sections stratify naturally into symplectic manifolds and the structure of the singularities is locally conical. The imploded cross-section is defined in Section 2 and its stratification is studied in Section 5.

Moreover, there is a residual action on M_{impl} of a maximal torus T of K, which is Hamiltonian in a suitable sense. The image of the T-moment map on M_{impl} is equal to the moment polytope of M. The classical analogue of the Cartan-Weyl theorem is then the following assertion: for each λ in the Weyl chamber the quotients $M/\!\!/_\lambda K$ and $M_{\mathrm{impl}}/\!\!/_\lambda T$ are symplectomorphic. (The notation $/\!\!/_\lambda$ stands for symplectic reduction at level λ. If λ is in the Weyl chamber, it makes sense to reduce with respect to either a T-action or a K-action. The reason is that the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ has a natural complementary subspace V, namely the sum of the root spaces. By identifying the dual $\mathfrak{t}^*$ with the annihilator of V in $\mathfrak{k}^*$ we can view λ as an element of $\mathfrak{t}^*$ or alternatively of $\mathfrak{k}^*$.) In this way the process of symplectic implosion abelianizes Hamiltonian K-manifolds at the cost of introducing singularities. This issue is dealt with in Section 3.

The imploded cross-section of the cotangent bundle T^*K enjoys two special properties. The first, which is explored in Section 4, is that $(T^*K)_{\mathrm{impl}}$ carries an additional K-action which commutes with the T-action, and that the imploded cross-section of any Hamiltonian K-manifold M can be obtained as a symplectic quotient of the product $M \times (T^*K)_{\mathrm{impl}}$ with respect to K. For this reason we call $(T^*K)_{\mathrm{impl}}$ the *universal* imploded cross-section. The second property, which is investigated in Section 6, says that $(T^*K)_{\mathrm{impl}}$ possesses the structure of an irreducible complex affine variety and that the symplectic structure is Kähler. The K-action extends to an algebraic action of the complexified group $G = K^{\mathbb{C}}$ and the G-orbits are identical to the symplectic strata. The open orbit is of type G/N, where N is a maximal unipotent subgroup of G. In fact, $(T^*K)_{\mathrm{impl}}$ can be identified with the basic affine variety introduced by Bernstein et al. [3]. Thus implosion is the symplectic counterpart of taking the quotient of a G-variety by the action of N.

The result of Section 7 reinforces further the analogy between imploded cross-sections and highest weight vectors. It asserts that "quantization commutes with implosion" in the following sense. Assuming that M is prequantizable, we define its quantization to be the K-equivariant Riemann–Roch number $\mathrm{RR}(M)$ with coefficients in the prequantum line bundle. The quantization of M_{impl} is defined as the T-equivariant Riemann–Roch number of a suitable desingularization. We then show that $\mathrm{RR}(M_{\mathrm{impl}})$ is equal as a virtual T-module to the space of highest weight vectors in $\mathrm{RR}(M)$.

For the reader's convenience we have provided an index of notation at the end of the paper.

Many of the results in Sections 2–6 are taken from an unpublished manuscript dating from 1993. They have recently found an application in the theory of vector bundles on Riemann surfaces (see Hurtubise and Jeffrey [10]), which is why we are making available this updated and expanded version. More precisely, this application involves symplectic implosion for group-valued moment maps, which we hope to take up in a sequel to this paper.

2. The construction

Let (M,ω) be a connected symplectic manifold and let K be a compact connected Lie group acting on M in a Hamiltonian fashion with equivariant moment map $\Phi\colon M\to\mathfrak{k}^*$, where $\mathfrak{k}=\operatorname{Lie}K$. Our sign convention for the moment map is $d\Phi^\xi=\iota(\xi_M)\omega$, where ξ_M denotes the vector field on M induced by $\xi\in\mathfrak{k}$, and $\Phi^\xi=\langle\Phi,\xi\rangle$, the component of the moment map along ξ.

We choose once and for all a maximal torus T of K and a closed fundamental Weyl chamber $\mathfrak{t}^*_+$ in the dual of the Cartan subalgebra $\mathfrak{t}=\operatorname{Lie}T$. The Weyl chamber is the disjoint union of 2^r open faces (sometimes called walls), where r is the rank of the commutator subgroup $[K,K]$. The *principal face* σ_{prin} for M is the minimal face σ such that $\Phi(M)\cap\mathfrak{t}^*_+\subseteq\bar\sigma$. In most cases of interest it is equal to $(\mathfrak{t}^*_+)^\circ$, the interior of the Weyl chamber. The symplectic cross-section theorem (see below) says that $\Phi^{-1}(\sigma_{\mathrm{prin}})$ is a T-stable symplectic submanifold of M. We want to "complete" this submanifold by adding lower dimensional symplectic strata. An obvious guess is to take $\Phi^{-1}(\mathfrak{t}^*_+)=\Phi^{-1}(\bar\sigma_{\mathrm{prin}})$, but to turn this into a symplectic object, we need to collapse the boundary components along certain directions in the following manner. Let K_λ denote the centralizer (i.e., the stabilizer for the coadjoint action) of an element $\lambda\in\mathfrak{k}^*$. Two points m_1 and m_2 in $\Phi^{-1}(\mathfrak{t}^*_+)$ are *equivalent*, $m_1\sim m_2$, if there exists $k\in[K_{\Phi(m_1)},K_{\Phi(m_1)}]$ such that $m_2=km_1$. By equivariance of the moment map, $m_1\sim m_2$ implies $\Phi(m_1)=\Phi(m_2)$, so $\sim$ is an equivalence relation. We shall see shortly that when $\Phi(m)$ is in the principal face σ_{prin}, then m is equivalent only to itself. (See the remark following Theorem 2.5.)

2.1. Definition. The *imploded cross-section* of M is the quotient space $M_{\mathrm{impl}}=\Phi^{-1}(\mathfrak{t}^*_+)/{\sim}$, equipped with the quotient topology. The quotient map $\Phi^{-1}(\mathfrak{t}^*_+)\to M_{\mathrm{impl}}$ is denoted by π. The *imploded moment map* Φ_{impl} is the continuous map $M_{\mathrm{impl}}\to\mathfrak{t}^*_+$ induced by Φ.

The image of Φ_{impl} is equal to $\Phi(M)\cap\mathfrak{t}^*_+$. All points in a face σ have the same centralizer, denoted K_σ, and therefore M_{impl} is set-theoretically a disjoint union of orbit spaces,

$$M_{\mathrm{impl}}=\coprod_{\sigma\in\Sigma}\Phi^{-1}(\sigma)/[K_\sigma,K_\sigma]. \tag{2.2}$$

Here Σ denotes the collection of faces of $\mathfrak{t}^*_+$. We define a partial order on Σ by putting $\sigma\le\tau$ if $\sigma\subseteq\bar\tau$.

2.3. Lemma. *The projection π is proper and M_{impl} is Hausdorff, locally compact, and second countable. If M is compact, then so is M_{impl}. Each subspace in the decomposition* (2.2) *is locally closed in M_{impl}.*

Proof. First we show that π is closed. Let $C\subseteq\Phi^{-1}(\mathfrak{t}^*_+)$ be closed. We need to show that

$$\pi^{-1}(\pi(C))=\coprod_{\sigma\in\Sigma}\Phi^{-1}(\sigma)\cap[K_\sigma,K_\sigma]\cdot C$$

is closed. Let $\{m_i\}$ be a sequence in $\pi^{-1}(\pi(C))$ converging to $m\in\Phi^{-1}(\mathfrak{t}^*_+)$. Let σ be the face containing $\Phi(m)$. After passing to a subsequence we may assume that all $\Phi(m_i)$ are in the same face, say τ. Then $\sigma\le\tau$, so $K_\tau\subseteq K_\sigma$ and $[K_\tau,K_\tau]\subseteq[K_\sigma,K_\sigma]$. Since $[K_\tau,K_\tau]$ is compact, $[K_\tau,K_\tau]\cdot C$ is closed, so $m\in[K_\tau,K_\tau]\cdot C$. Hence

$$m\in\Phi^{-1}(\sigma)\cap[K_\tau,K_\tau]\cdot C\subseteq\Phi^{-1}(\sigma)\cap[K_\sigma,K_\sigma]\cdot C\subseteq\pi^{-1}(\pi(C)),$$

i.e., $\pi^{-1}(\pi(C))$ is closed. The fact that π has compact fibres now implies that it is proper. The stated properties of M_{impl} follow easily, and the local closedness of $\Phi^{-1}(\sigma)/[K_\sigma, K_\sigma] = \pi\big(\Phi^{-1}(\sigma)\big)$ follows from the observation that $\Phi^{-1}(\sigma)$ is equal to $\Phi^{-1}(\bar{\sigma}) \setminus \bigcup_{\tau<\sigma} \Phi^{-1}(\tau)$. $\square$

The moment map is determined up to an additive constant vector in $\mathfrak{z}^*$, where $\mathfrak{z}$ is the Lie algebra of Z, the unit component of the centre of K. The choice of this constant does not affect the imploded cross-section. This is most easily seen from the direct sum decomposition $\mathfrak{k} = \mathfrak{z} \oplus [\mathfrak{k}, \mathfrak{k}]$, from which, by identifying $\mathfrak{z}^*$ with the annihilator of $[\mathfrak{k}, \mathfrak{k}]$, one gets a dual decomposition $\mathfrak{k}^* = \mathfrak{z}^* \oplus [\mathfrak{k}, \mathfrak{k}]^*$. Correspondingly, the Weyl chamber decomposes into a product of a vector space and a proper cone, $\mathfrak{t}^*_+ = \mathfrak{z}^* \times (\mathfrak{t}^*_+ \cap [\mathfrak{k}, \mathfrak{k}]^*)$. In fact, this argument proves the first assertion of the following lemma. The second assertion is clear.

2.4. Lemma. *The imploded cross-section of M with respect to the K-action is the same as the imploded cross-section with respect to the $[K, K]$-action. Likewise, replacing K by a finite cover does not alter the imploded cross-section.*

To obtain more detailed information we invoke the cross-section theorem, which is essentially due to Guillemin and Sternberg; cf. [7]. The version stated below incorporates some refinements made by Lerman et al. [16]. Consider a face σ of $\mathfrak{t}^*_+$ and the K_σ-stable subset $\mathfrak{S}_\sigma = K_\sigma \cdot \operatorname{star}\sigma$ of $\mathfrak{k}^*$, where $\operatorname{star}\sigma$ denotes the open star $\bigcup_{\tau\geq\sigma} \tau$ of σ. It is well known that $\mathfrak{S}_\sigma$ is a slice for the coadjoint action (i.e., $K\mathfrak{S}_\sigma$ is open and K-equivariantly diffeomorphic to the associated bundle $K \times^{K_\sigma} \mathfrak{S}_\sigma$), in fact the largest possible slice containing all points of orbit type (K_σ). The *symplectic cross-section* of M over σ is the subset

$$M_\sigma = \Phi^{-1}(\mathfrak{S}_\sigma).$$

Note that $\Phi(M_\sigma) \subseteq \mathfrak{S}_\sigma \subseteq \mathfrak{k}^*_\sigma$ and that the saturation KM_σ of M_σ is open in M. The cross-section $M_{\sigma_{\mathrm{prin}}}$ is called *principal*.

2.5. Theorem (symplectic cross-sections). *Let σ be an open face of $\mathfrak{t}^*_+$.*

(i) *The cross-section M_σ is a connected K_σ-stable symplectic submanifold of M. The K_σ-action on M_σ is Hamiltonian with moment map $\Phi_\sigma = \Phi|_{M_\sigma}$.*
(ii) *The multiplication map $K \times M_\sigma \to M$ induces a symplectomorphism $K \times^{K_\sigma} M_\sigma \to KM_\sigma$. If M_σ is nonempty, then KM_σ is dense in M.*
(iii) *The commutator subgroup of $K_{\sigma_{\mathrm{prin}}}$ acts trivially on $M_{\sigma_{\mathrm{prin}}}$.*

It follows from (iii) that if $\Phi(m_1) \in \sigma_{\mathrm{prin}}$, then $m_1 \sim m_2$ implies $m_1 = m_2$. From (i) it then follows that $\pi\big(\Phi^{-1}(\sigma_{\mathrm{prin}})\big)$ is connected and open and from (ii) that it is dense.

2.6. Corollary. *The restriction of π to $\Phi^{-1}(\sigma_{\mathrm{prin}})$ is a homeomorphism onto its image. The image is connected, and open and dense in M_{impl}. Hence M_{impl} is connected.*

By Theorem 2.5(i), the composition of Φ_σ with the projection $\mathfrak{k}^*_\sigma \to [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]^*$ is a moment map for the action of $[K_\sigma, K_\sigma]$ on M_σ. Its zero fibre is

$$\Phi^{-1}(\mathfrak{z}^*_\sigma) \cap M_\sigma = \Phi^{-1}(\mathfrak{z}^*_\sigma \cap \mathfrak{S}_\sigma) = \Phi^{-1}(\sigma).$$

The decomposition (2.2) can therefore be written more insightfully as follows.

2.7. Corollary. *The imploded cross-section partitions into symplectic quotients, each of which is locally closed,*

$$M_{\mathrm{impl}} = \coprod_{\sigma \in \Sigma} M_\sigma /\!\!/ [K_\sigma, K_\sigma]. \tag{2.8}$$

Here the notation $/\!\!/_\lambda$ indicates symplectic reduction at level λ, the subscript being usually omitted when it is 0. For instance, the piece corresponding to the smallest face of the Weyl chamber is $\Phi^{-1}(\mathfrak{z}^*)/[K,K] = M/\!\!/[K,K]$.

Not all the pieces of the partition (2.8) are symplectic manifolds, but a decomposition into symplectic manifolds can be obtained by subdividing each of the pieces according to orbit type. For any $\sigma \in \Sigma$ and any closed subgroup H of $K' = [K_\sigma, K_\sigma]$, let

$$M_{\sigma,(H)} = \{\, m \in M_\sigma \mid K'_m \text{ is conjugate within } K' \text{ to } H \,\}$$

be the stratum of orbit type (H) in the K'-manifold M_σ. Here K'_m is the stabilizer of m with respect to the K'-action. By [22, Theorem 2.1], the intersection $\Phi^{-1}(\sigma) \cap M_{\sigma,(H)}$ is a smooth K'-stable submanifold of M_σ and the null-foliation of the symplectic form restricted to this submanifold is exactly given by the K'-orbits. Hence the quotient

$$\bigl(\Phi^{-1}(\sigma) \cap M_{\sigma,(H)}\bigr)/K' \tag{2.9}$$

is a symplectic manifold in a natural way. It is more convenient to work with the connected components of these manifolds instead. Let $\{\, X_i \mid i \in \mathcal{I} \,\}$ be the collection of components of all manifolds of the form (2.9), where σ ranges over all faces of $\mathfrak{t}^*_+$ and (H) over all conjugacy classes of subgroups of $[K_\sigma, K_\sigma]$. We call the X_i *strata*, although it will not be proved until Section 5 that they form a stratification of M_{impl}. There is a partial order on the index set $\mathcal{I}$ defined by $i \leq j$ if $X_i \subseteq \bar{X}_j$. By Corollary 2.6, $\mathcal{I}$ has a unique maximal element, corresponding to the face $\sigma = \sigma_{\mathrm{prin}}$ and the subgroup $H = [K_{\sigma_{\mathrm{prin}}}, K_{\sigma_{\mathrm{prin}}}]$. It is well known that the orbit type decomposition of any manifold with a smooth action of a compact Lie group is locally finite. Together with the fact that the quotient map π is proper (Lemma 2.3), this implies that the collection of strata is locally finite. We have proved the following result.

2.10. Theorem. *The imploded cross-section is a locally finite disjoint union of locally closed connected subspaces, each of which is a symplectic manifold,*

$$M_{\mathrm{impl}} = \coprod_{i \in \mathcal{I}} X_i. \tag{2.11}$$

There is a unique open stratum, which is dense in M_{impl} and symplectomorphic to the principal cross-section of M.

3. Abelianization

The imploded cross-section of a Hamiltonian K-manifold can be viewed as its abelianization in a sense which we shall now make precise.

Let $\mathcal{X}$ be a topological space with a decomposition $\mathcal{X} = \coprod_{i \in \mathcal{I}} \mathcal{X}_i$ into connected subspaces $\mathcal{X}_i$, each of which is equipped with the structure of a smooth manifold and a symplectic form ω_i. A continuous action of a Lie group $\mathcal{G}$ on $\mathcal{X}$ is *Hamiltonian* if it preserves the decomposition and is smooth on each $\mathcal{X}_i$, and if there exists a continuous Ad^*-equivariant map $\Phi_{\mathcal{X}}\colon \mathcal{X} \to (\operatorname{Lie}\mathcal{G})^*$, the *moment map*, such that $\Phi_{\mathcal{X}}|_{\mathcal{X}_i}$ is a moment map in the usual sense for the $\mathcal{G}$-action on $\mathcal{X}_i$ for all $i \in \mathcal{I}$. The triple $\bigl(\mathcal{X}, \{\,(\mathcal{X}_i, \omega_i) \mid i \in \mathcal{I}\,\}, \Phi_{\mathcal{X}}\bigr)$ is a *Hamiltonian $\mathcal{G}$-space*. An *isomorphism* from $\mathcal{X}$ to a second Hamiltonian $\mathcal{G}$-space $\bigl(\mathcal{Y}, \{\,(\mathcal{Y}_j, \omega_j) \mid j \in \mathcal{J}\,\}, \Phi_{\mathcal{Y}}\bigr)$ is a pair (F, f), where F is a homeomorphism $\mathcal{X} \to \mathcal{Y}$ and f is a bijection $\mathcal{I} \to \mathcal{J}$ subject to the following conditions: F is equivariant, $\Phi_{\mathcal{X}} = \Phi_{\mathcal{Y}} \circ F$, F maps $\mathcal{X}_i$ symplectomorphically onto $\mathcal{Y}_{f(i)}$ for all $i \in \mathcal{I}$.

3.1. Example. Let V be a finite-dimensional unitary K-module with inner product $\langle\cdot,\cdot\rangle$. This is a Hamiltonian K-manifold with symplectic form ω_V and moment map Φ_V given by

$$\omega_V = -\operatorname{Im}\langle\cdot,\cdot\rangle \qquad \text{and} \qquad \Phi_V^\xi(v) = \frac{1}{2}\,\omega_V(\xi v, v), \tag{3.2}$$

respectively. Let $\mathcal{X}$ be a K-stable irreducible complex algebraic subvariety of V, regarded as a subspace for the classical topology on V. There is a natural minimal decomposition of $\mathcal{X}$ into nonsingular K-stable algebraic subvarieties, each of which inherits a symplectic form and a moment map from the ambient space V. Let us call the Hamiltonian K-space $\mathcal{X}$ thus obtained an *affine Hamiltonian K-space*. The topology on $\mathcal{X}$ and its decomposition into manifolds depend only on its coordinate ring $\mathfrak{A}$, and the K-action is determined by the K-module structure of $\mathfrak{A}$. The symplectic forms and the moment map, however, depend on the embedding into V and the inner product on V. When the embedding and the inner product are fixed, we will sometimes abuse notation and write $\mathcal{X} = \operatorname{Spec}\mathfrak{A}$ to indicate that $\mathcal{X}$ is the affine Hamiltonian K-space whose underlying variety is the subvariety of V with coordinate ring $\mathfrak{A}$.

For $\gamma \in (\operatorname{Lie}\mathcal{G})^*$, the *symplectic quotient* (or *reduced space*) at level γ is the topological space $\mathcal{X}/\!\!/_\gamma\mathcal{G} = \Phi_{\mathcal{X}}^{-1}(\gamma)/\mathcal{G}_\gamma$. If $\mathcal{G}$ is compact (or if $\mathcal{G}_\gamma$ acts properly on $\Phi_{\mathcal{X}}^{-1}(\gamma)$), the symplectic quotient can be decomposed into connected smooth symplectic manifolds by partitioning each of the pieces $\mathcal{X}_i$ according to $\mathcal{G}$-orbit type, $\mathcal{X}_i = \coprod_{(\mathcal{H})} \mathcal{X}_{i,(\mathcal{H})}$, forming the symplectic manifolds $\bigl(\Phi_{\mathcal{X}}^{-1}(\gamma) \cap \mathcal{X}_{i,(\mathcal{H})}\bigr)/\mathcal{G}_\gamma$ as in (2.9), and subdividing these into their connected components.

3.3. Example. Let V be as in Example 3.1 and let $\mathbb{P}(V)$ be the associated projective space. As a symplectic manifold, $\mathbb{P}(V)$ is isomorphic to the quotient $V/\!\!/_{-1}S^1$, where S^1 acts by scalar multiplication. Let $\mathcal{X}$ be a K-stable irreducible complex algebraic subvariety of $\mathbb{P}(V)$. By partitioning into a minimal set of nonsingular algebraic subvarieties as in Example 3.1, we see that $\mathcal{X}$ is a Hamiltonian K-space, called a *projective Hamiltonian K-space*. The affine cone $\tilde{\mathcal{X}} = \operatorname{Spec}\mathfrak{A} \subseteq V$ is an affine Hamiltonian K-space, and we have $\mathcal{X} \cong \tilde{\mathcal{X}}/\!\!/_{-1}S^1$ as Hamiltonian K-spaces.

Now let (M, ω, K) be an arbitrary Hamiltonian K-manifold. We claim that the imploded cross-section M_{impl}, equipped with the decomposition (2.11), is a Hamiltonian T-space. Indeed, the preimage $\Phi^{-1}(\mathfrak{t}^*_+)$ is stable under the action of the maximal torus. In addition, $m_1 \sim m_2$ implies $tm_1 \sim tm_2$ for all $t \in T$ and $m_1, m_2 \in \Phi^{-1}(\mathfrak{t}^*_+)$, because T normalizes each of the groups $[K_\sigma, K_\sigma]$. Thus the action of T descends to a continuous action on M_{impl}. This action is Hamiltonian with moment map Φ_{impl}. The easiest way to see this is to use the following alternative definition of the T-action: for each σ

the K_σ-action on M_σ descends to a $K_\sigma/[K_\sigma,K_\sigma]$-action on the quotient $M_\sigma /\!\!/ [K_\sigma,K_\sigma]$. Via the canonical surjective map $T \to K_\sigma/[K_\sigma,K_\sigma]$ this induces a (non-effective) T-action on $M_\sigma /\!\!/ [K_\sigma,K_\sigma]$. Again because T normalizes $[K_\sigma,K_\sigma]$, this action preserves the $[K_\sigma,K_\sigma]$-orbit type strata and is Hamiltonian on each such stratum. The moment maps are induced by the restrictions of Φ to the manifolds $\Phi^{-1}(\sigma) \cap M_{\sigma,(H)}$; in other words they are the restrictions of Φ_{impl} to the various strata.

Symplectic reduction of M_{impl} with respect to T turns out to be the same as symplectic reduction of M with respect to K. Namely, let $\lambda \in \mathfrak{t}^*_+$ and let σ be the face of $\mathfrak{t}^*_+$ containing λ. Then $\Phi^{-1}_{\mathrm{impl}}(\lambda) = \Phi^{-1}(\lambda)/[K_\sigma,K_\sigma]$, so there is a quotient map $\Phi^{-1}(\lambda) \to \Phi^{-1}_{\mathrm{impl}}(\lambda)$.

3.4. Theorem. *For every $\lambda \in \mathfrak{t}^*_+$, the canonical map $\Phi^{-1}(\lambda) \to \Phi^{-1}_{\mathrm{impl}}(\lambda)$ induces an isomorphism of symplectic quotients $M /\!\!/_\lambda K \cong M_{\mathrm{impl}} /\!\!/_\lambda T$.*

Proof. Let σ be the face containing λ. Assume first that all points in $\Phi^{-1}(\lambda)$ are of the same orbit type for the action of $K_\lambda = K_\sigma$, so that $M /\!\!/_\lambda K$ is a smooth symplectic manifold. By reduction in stages, it is naturally symplectomorphic to the iterated quotient

$$\big(M_\sigma /\!\!/_0 [K_\sigma,K_\sigma]\big) /\!\!/_\lambda T. \tag{3.5}$$

Since $\Phi^{-1}_{\mathrm{impl}}(\lambda) \subseteq M_\sigma /\!\!/ [K_\sigma,K_\sigma]$ and the restriction of Φ_{impl} to $M_\sigma /\!\!/ [K_\sigma,K_\sigma]$ is the moment map for the T-action, it is clear that (3.5) is naturally symplectomorphic to $M_{\mathrm{impl}} /\!\!/_\lambda T$. If $\Phi^{-1}(\lambda)$ consists of more than one stratum, the same argument, using stratified reduction in stages (see [22, §4]), shows that the quotient map $\Phi^{-1}(\lambda) \to \Phi^{-1}_{\mathrm{impl}}(\lambda)$ induces a homeomorphism $M /\!\!/_\lambda K \to M_{\mathrm{impl}} /\!\!/_\lambda T$, which maps strata symplectically onto strata. $\square$

3.6. Example. For all $\lambda \in \mathfrak{t}^*_+$, $T^*K /\!\!/_\lambda K \cong K\lambda$, the coadjoint orbit through λ, so $(T^*K)_{\mathrm{impl}} /\!\!/_\lambda T \cong K\lambda$.

4. The universal imploded cross-section

As an example we determine the cross-sections and the imploded cross-section of the cotangent bundle T^*K. It turns out that the space $(T^*K)_{\mathrm{impl}}$ has a universal property, which greatly facilitates calculations involving symplectic implosion. Another interesting feature is its homogeneous structure. We shall see that if K is semisimple, $(T^*K)_{\mathrm{impl}}$ is a cone over a compact space, which stratifies into contact manifolds.

Consider the actions of K on itself given by $\mathcal{L}_g k = gk$ and $\mathcal{R}_g k = kg^{-1}$, which both lift to a Hamiltonian actions on T^*K. Identify T^*K with $K \times \mathfrak{k}^*$ by means of left translations. Then the actions are given by $\mathcal{L}_g(k,\lambda) = (gk,\lambda)$ and $\mathcal{R}_g(k,\lambda) = (kg^{-1}, g\lambda)$, where $g\lambda$ is an abbreviation for $\mathrm{Ad}^*(g)(\lambda)$. The moment maps (with respect to the symplectic form $\omega = d\beta$, where β is the canonical one-form) are respectively

$$\Phi_{\mathcal{L}}(k,\lambda) = -k\lambda, \qquad \Phi_{\mathcal{R}}(k,\lambda) = \lambda.$$

The inversion map $k \mapsto k^{-1}$ intertwines the left and right actions. Its cotangent lift, given by $(k,\lambda) \mapsto (k^{-1}, -k\lambda)$, is a symplectic involution of T^*K, which intertwines $\Phi_{\mathcal{L}}$

and $\Phi_{\mathcal{R}}$. Therefore the cross-sections for the left action are canonically isomorphic to those for the right action. For simplicity let us use $\Phi_{\mathcal{R}}$. Clearly

$$\Phi_{\mathcal{R}}^{-1}(\mathfrak{S}_\sigma) = K \times \mathfrak{S}_\sigma, \tag{4.1}$$

$$(T^*K)_{\mathrm{impl}} = \coprod_{\sigma\in\Sigma} (K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma] = \coprod_{\sigma\in\Sigma} \frac{K}{[K_\sigma, K_\sigma]} \times \sigma, \tag{4.2}$$

so in this example the decompositions (2.8) and (2.11) are identical.

As stated in Theorem 2.5(i), the cross-section (4.1) inherits a symplectic structure from T^*K. Here is an alternative construction of the symplectic form. For each face σ the subalgebra $\mathfrak{k}_\sigma$ is equal to the subspace of Z_σ-invariants in $\mathfrak{k}$, where Z_σ is the identity component of the centre of K_σ. We therefore have a canonical K_σ-invariant projection $\mathfrak{k} \to \mathfrak{k}_\sigma$ and hence a connection θ on the principal K_σ-bundle

$$K_\sigma \to K \to K/K_\sigma. \tag{4.3}$$

A connection $\vartheta \in \Omega^1(\mathcal{P}, \operatorname{Lie}\mathcal{G})$ on a principal bundle $\mathcal{P}$ for a Lie group $\mathcal{G}$ is *fat* at $\gamma \in (\operatorname{Lie}\mathcal{G})^*$ if the two-form $\langle\gamma, d\vartheta\rangle$ is nondegenerate on the horizontal subspaces of $\mathcal{P}$. This is a necessary and sufficient condition for the closed two-form $d\langle \mathrm{pr}_2, \vartheta\rangle$ on $\mathcal{P} \times (\operatorname{Lie}\mathcal{G})^*$ to be nondegenerate at $P \times \{\gamma\}$, where $\mathrm{pr}_2 \colon \mathcal{P} \times (\operatorname{Lie}\mathcal{G})^* \to (\operatorname{Lie}\mathcal{G})^*$ is the projection.

4.4. Lemma ([7, Corollary 2.3.8]). *The canonical connection θ on the bundle* (4.3) *is fat at $\lambda \in \mathfrak{k}_\sigma^*$ if and only if $\lambda \in \mathfrak{S}_\sigma$.*

Therefore the form $d\langle \mathrm{pr}_2, \theta\rangle$ on $K \times \mathfrak{k}_\sigma^*$ is symplectic on $K \times \mathfrak{S}_\sigma$. It is straightforward to check that $\langle \mathrm{pr}_2, \theta\rangle$ is equal to the restriction to $K \times \mathfrak{S}_\sigma$ of the canonical one-form β. Hence $d\langle \mathrm{pr}_2, \theta\rangle$ is equal to the restriction of $d\beta = \omega$.

The symplectic form on the stratum $(K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma]$ of $(T^*K)_{\mathrm{impl}}$ can therefore be interpreted as the form obtained by reducing $(K \times \mathfrak{S}_\sigma, d\langle \mathrm{pr}_2, \theta\rangle)$ with respect to $[K_\sigma, K_\sigma]$. A third alternative is to note that the projection $\mathfrak{k} \to \mathfrak{k}_\sigma$ descends to a Z_σ-equivariant projection $\mathfrak{k}/[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \to \mathfrak{k}_\sigma/[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \cong \mathfrak{z}_\sigma$, whence we obtain a connection $\bar\theta \in \Omega^1\big(K/[K_\sigma, K_\sigma], \mathfrak{z}_\sigma\big)$ on the torus bundle

$$K_\sigma/[K_\sigma, K_\sigma] \to K/[K_\sigma, K_\sigma] \to K/K_\sigma. \tag{4.5}$$

Put $\beta_\sigma = \langle \mathrm{pr}_2, \bar\theta\rangle$ and $\omega_\sigma = d\langle \mathrm{pr}_2, \bar\theta\rangle$, where pr_2 now stands for the projection

$$K/[K_\sigma, K_\sigma] \times \sigma \to \sigma.$$

Lemma 4.4 implies that $\bar\theta$ is fat at all points of $\mathfrak{S}_\sigma \cap \mathfrak{z}_\sigma^* = \sigma$, and therefore ω_σ is a symplectic form on $K/[K_\sigma, K_\sigma] \times \sigma = (K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma]$. The following result asserts that this form is the same as the one defined by symplectic implosion.

4.6. Lemma. *Let p be the orbit map $K \times \sigma \to (K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma]$. Then $\beta|_{K\times\sigma} = p^*\beta_\sigma$ and $\omega|_{K\times\sigma} = p^*\omega_\sigma$.*

Proof. The second equality is immediate from the first. Because of K-invariance, the first equality need only be checked at points of the form $(1,\lambda)\in K\times\sigma$. Let $(\xi,\mu)\in T_{(1,\lambda)}(K\times\sigma)=\mathfrak{k}\times\mathfrak{z}_\sigma^*$. Then $\beta_{(1,\lambda)}(\xi,\mu)=\lambda(\xi)$ by the definition of β. (We identify $\mathfrak{z}_\sigma^*$ with the annihilator of $[\mathfrak{k}_\sigma,\mathfrak{k}_\sigma]\cap\mathfrak{t}$ in $\mathfrak{t}^*$, so that $\lambda(\xi)$ is well defined.) Moreover,

$$(p^*\beta_\sigma)_{(1,\lambda)}(\xi,\mu)=(\beta_\sigma)_{p(1,\lambda)}(\xi \bmod [\mathfrak{k}_\sigma,\mathfrak{k}_\sigma],\mu)=\lambda\big(\bar\theta(\xi \bmod [\mathfrak{k}_\sigma,\mathfrak{k}_\sigma])\big),$$

which is equal to $\lambda(\xi)$ because $\lambda\in\mathfrak{z}_\sigma^*$ and $\bar\theta(\xi \bmod [\mathfrak{k}_\sigma,\mathfrak{k}_\sigma])$ is equal to the projection of ξ onto $\mathfrak{z}_\sigma$. □

Not only is $(T^*K)_{\mathrm{impl}}$ a Hamiltonian T-space for the T-action induced by the right K-action, but the left K-action on T^*K descends to a K-action on $(T^*K)_{\mathrm{impl}}$ given by $g\pi_{\mathcal{R}}(k,\lambda)=\pi_{\mathcal{R}}(gk,\lambda)$, which is Hamiltonian as well. (Here $\pi_{\mathcal{R}}$ denotes the quotient map $\Phi_{\mathcal{R}}^{-1}(\mathfrak{t}_+^*)\to(T^*K)_{\mathrm{impl}}$.) Its moment map is induced by $\Phi_{\mathcal{L}}$ and is for simplicity also denoted by $\Phi_{\mathcal{L}}$. Clearly the two actions commute, so that $(T^*K)_{\mathrm{impl}}$ is a Hamiltonian $K\times T$-space with moment map $\Phi_{\mathcal{L}}\times(\Phi_{\mathcal{R}})_{\mathrm{impl}}$.

4.7. Example. Let $K=\mathbf{SU}(2)$, which we shall identify with $S^3\subseteq\mathbb{H}$, the unit quaternions. The Lie algebra is then the set of imaginary quaternions, the unit circle $S^1\subseteq\mathbb{C}$ is a maximal torus, and the fibration (4.3) is none other than the Hopf fibration $S^1\to S^3\to S^2$. The symplectic form on $K\times(\mathfrak{t}_+^*)^\circ=S^3\times(0,\infty)$ is $d(t\theta)$, with t being the standard coordinate on $(0,\infty)$. The map $(z,t)\mapsto\sqrt{2t}\,z$ is a symplectomorphism from $S^3\times(0,\infty)$ onto $\mathbb{H}\setminus\{0\}$ with its standard symplectic structure. The remaining stratum, corresponding to $\sigma=\{0\}$, consists of a single point. Consequently, the continuous map $F\colon T^*K\to\mathbb{H}=\mathbb{C}^2$ defined by $F(k,\lambda)=\sqrt{2\,\|\lambda\|}\,k$ induces a continuous bijection $(T^*K)_{\mathrm{impl}}\to\mathbb{C}^2$, which is a homeomorphism because $(T^*K)_{\mathrm{impl}}$ is locally compact Hausdorff. In this sense, the isolated singularity is removable and the imploded cross-section is a symplectic $\mathbb{C}^2$. Modulo this identification, the left K-action on $(T^*K)_{\mathrm{impl}}$ is the standard representation of $\mathbf{SU}(2)$ on $\mathbb{C}^2$ and the right T-action is given by $t\cdot z=t^{-1}z$. This example will be generalized to arbitrary K in Section 6.

Now let (M,ω,Φ) be any Hamiltonian K-manifold and define $j\colon M\to M\times T^*K$ by $j(m)=(m,1,\Phi(m))$.

4.8. Lemma. (i) *The map j is a symplectic embedding and induces an isomorphism of Hamiltonian K-manifolds, $\bar\jmath\colon M\to(M\times T^*K)/\!\!/K$. Here the right-hand side is the quotient with respect to the diagonal K-action, where K acts on the left on T^*K. The K-action on $(M\times T^*K)/\!\!/K$ is the one induced by the right action on T^*K.*

(ii) *For every face σ, j maps M_σ into $M\times K\times\mathfrak{S}_\sigma$ and induces an isomorphism of Hamiltonian K_σ-manifolds, $\bar\jmath_\sigma\colon M_\sigma\to(M\times K\times\mathfrak{S}_\sigma)/\!\!/K$, where the quotient is taken as in* (i).

Proof. The map $m\mapsto(1,\Phi(m))$ sends M to the Lagrangian $\mathfrak{k}^*\subseteq T^*K$, so j is a symplectic embedding. The moment map for the diagonal K-action on $M\times T^*K$ is given by $\Psi(m,k,\lambda)=\Phi(m)-k\lambda$, so j maps M into $\Psi^{-1}(0)$. One checks readily that the induced map $\bar\jmath$ is a diffeomorphism. It is a symplectomorphism because j is a symplectic embedding. Moreover, $j(km)=(km,1,k\Phi(m))=\mathcal{L}_k\mathcal{R}_kj(m)$ and $\Phi_{\mathcal{R}}(j(m))=\Phi(m)$, so $\bar\jmath$ is K-equivariant and intertwines the K-moment maps on M and $(M\times T^*K)/\!\!/K$. This proves (i). (ii) is proved in exactly the same way. □

Observe that j maps $\Phi^{-1}(\mathfrak{t}^*_+)$ into $\coprod_\sigma M \times K \times \sigma$ and therefore induces a continuous map $\Phi^{-1}(\mathfrak{t}^*_+) \to M \times (T^*K)_{\mathrm{impl}}$. Because of the following result we call $(T^*K)_{\mathrm{impl}}$ the *universal imploded cross-section*. See Section 3 for the definition of an isomorphism of Hamiltonian T-spaces.

4.9. Theorem. *The map j induces an isomorphism of Hamiltonian T-spaces,*

$$\bar{\jmath}_{\mathrm{impl}} \colon M_{\mathrm{impl}} \to \big(M \times (T^*K)_{\mathrm{impl}}\big) /\!\!/ K,$$

where the quotient is taken with respect to the diagonal K-action.

Proof. The K-moment map on $M \times (T^*K)_{\mathrm{impl}}$ is given by $\Psi(m, \pi_{\mathcal{R}}(k,\lambda)) = \Phi(m) - k\lambda$, so j maps $\Phi^{-1}(\mathfrak{t}^*_+)$ into $\Psi^{-1}(0)$. Moreover, $m_1 \sim m_2$ implies $j(m_1) = j(m_2)$, so j induces a continuous map $M_{\mathrm{impl}} \to \Psi^{-1}(0)$. Upon quotienting by K we obtain the map $\bar{\jmath}_{\mathrm{impl}}$. As in the proof of Lemma 4.8 one checks that $\bar{\jmath}_{\mathrm{impl}}$ is a homeomorphism which is T-equivariant and intertwines the T-moment maps on M_{impl} and $\big(M \times (T^*K)_{\mathrm{impl}}\big) /\!\!/ K$. Note that $\bar{\jmath}_{\mathrm{impl}}$ restricts to a map

$$M_\sigma /\!\!/ [K_\sigma, K_\sigma] \to \big(M \times (K \times \mathfrak{S}_\sigma) /\!\!/ [K_\sigma, K_\sigma]\big) /\!\!/ K. \tag{4.10}$$

This is none other than the map induced, upon reduction with respect to $[K_\sigma, K_\sigma]$, by the map $\bar{\jmath}_\sigma$ defined in Lemma 4.8(ii). Because $\bar{\jmath}_\sigma$ is an isomorphism of Hamiltonian K_σ-manifolds, it preserves the $[K_\sigma, K_\sigma]$-orbit types and therefore (4.10) maps strata onto strata and is a symplectomorphism on each stratum. ☐

4.11. *Remark.* Let $S_\sigma = \Phi_{\mathcal{L}}^{-1}(\sigma)$ denote the stratum of $(T^*K)_{\mathrm{impl}}$ corresponding to a face $\sigma \in \Sigma$. Furthermore, let $\tau = \sigma_{\mathrm{prin}}$ be the principal face of M. Then the closure of S_τ is equal to $\coprod_{\sigma \leq \tau} S_\sigma$. Since the moment polytope of M is contained in $\bar{\tau} = \coprod_{\sigma \leq \tau} \sigma$, the proof of Theorem 4.9 shows that j induces an isomorphism $M_{\mathrm{impl}} \cong (M \times \bar{S}_\tau) /\!\!/ K$.

4.12. Example. Let M be a point. Then the theorem asserts that $(T^*K)_{\mathrm{impl}} /\!\!/_\lambda K$ consists of a single point for all $\lambda \in \mathfrak{k}^*$. In particular, $(T^*K)_{\mathrm{impl}}$ is a multiplicity-free K-space.

As an application, consider a Hamiltonian action of a second compact Lie group H on M, which commutes with K. The H-moment map on M is K-invariant and therefore induces a continuous map on M_{impl} (implosion with respect to the K-action). This map is a moment map for the H-action induced on M_{impl}. Using Theorem 4.9 and reduction in stages we conclude that reduction commutes with implosion.

4.13. Corollary. *The Hamiltonian T-spaces $(M /\!\!/_\eta H)_{\mathrm{impl}}$ and $M_{\mathrm{impl}} /\!\!/_\eta H$ are isomorphic for every $\eta \in \mathfrak{l}^*$.*

4.14. Example. Consider the $K \times K$-space $M \times T^*K$, where the first copy of K acts diagonally (and by left multiplication on T^*K), and the second copy acts by right multiplication on T^*K. Reducing with respect to the first copy, imploding with respect to the second, and applying Lemma 4.8(i) we find that

$$M_{\mathrm{impl}} \cong \big((M \times T^*K) /\!\!/ K\big)_{\mathrm{impl}} \cong \big(M \times (T^*K)_{\mathrm{impl}}\big) /\!\!/ K.$$

In other words, Corollary 4.13 is equivalent to Theorem 4.9.

4.15. Example. If H is finite, then $(M/H)_{\mathrm{impl}} \cong M_{\mathrm{impl}}/H$. In particular, let Γ be the finite central subgroup $Z \cap [K,K]$ of K. Then $K = (Z \times [K,K])/\Gamma$ and $T^*K = T^*Z \times^{\Gamma} T^*[K,K]$. Since implosion relative to K is the same as implosion relative to $[K,K]$, we see that

$$(T^*K)_{\mathrm{impl}} \cong (T^*Z \times T^*[K,K])_{\mathrm{impl}}/\Gamma \cong T^*Z \times^{\Gamma} (T^*[K,K])_{\mathrm{impl}}, \tag{4.16}$$

a bundle with fibre $(T^*[K,K])_{\mathrm{impl}}$ over the cotangent bundle of the torus Z/Γ. In turn, $(T^*[K,K])_{\mathrm{impl}}$ can be written as $(T^*[K,K]^{\sim})_{\mathrm{impl}}/\Delta$, where Δ is the fundamental group of $[K,K]$ and $[K,K]^{\sim}$ its universal cover. For instance, if $K = \mathbf{SO}(3) = \mathbf{SU}(2)/\{\pm 1\}$, then Example 4.7 shows that $(T^*K)_{\mathrm{impl}}$ is the symplectic orbifold $\mathbb{C}^2/\{\pm\,\mathrm{id}\}$, and if $K = \mathbf{U}(2)$ we find that $(T^*K)_{\mathrm{impl}} = \mathbb{C}^{\times} \times^{\{\pm\,\mathrm{id}\}} \mathbb{C}^2$.

Because of (4.16), to describe the singularities of $(T^*K)_{\mathrm{impl}}$ we can focus our attention on the fibre $(T^*[K,K])_{\mathrm{impl}}$. So let us assume for the remainder of this section that K is semisimple. For $t \geq 0$ let $\mathcal{A}_t$ be fibrewise multiplication by t in T^*K. Let

$$T^{\times}K = T^*K \setminus \{\text{zero section}\} = K \times (\mathfrak{k}^* \setminus \{0\})$$

be the punctured cotangent bundle and $\mathbb{R}^{\times} = (0,\infty)$ the multiplicative group of positive reals. Then for $t > 0$, $\mathcal{A}_t$ defines a proper free action of $\mathbb{R}^{\times}$ on $T^{\times}K$. Let ζ be its infinitesimal generator. Then ζ is a Liouville vector field, i.e., $\mathcal{L}(\zeta)\omega = \omega$, and therefore we call $\mathcal{A}$ a *Liouville action.* We can write the canonical one-form as $\beta = \iota(\zeta)\omega$. Recall that if ν is a contact form on a manifold, then the associated *Reeb vector field* is the vector field χ defined by $\iota(\chi)d\nu = 0$ and $\iota(\chi)\nu = 1$.

4.17. Lemma. *Let ζ be a global Liouville vector field on a symplectic manifold (M,ω) and let β be the potential one-form $\iota(\zeta)\omega$. Let Ξ be a symplectic vector field on M that commutes with ζ.*

(i) *The function $\varphi = -\iota(\Xi)\beta$ is a Hamiltonian for Ξ and satisfies $\mathcal{L}(\zeta)\varphi = \varphi$.*
(ii) *Any $c \neq 0$ is a regular value of φ, the restriction of β to the hypersurface $\varphi^{-1}(c)$ is a contact form, and the restriction of $-c^{-1}\Xi$ is its Reeb vector field.*

Proof. By assumption $\mathcal{L}(\Xi)\beta = \mathcal{L}(\Xi)\iota(\zeta)\omega = \big(\iota(\zeta)\mathcal{L}(\Xi) + \iota([\Xi,\zeta])\big)\omega = 0$, and hence

$$d\varphi = -d\iota(\Xi)\beta = -\mathcal{L}(\Xi)\beta + \iota(\Xi)d\beta = \iota(\Xi)d\beta = \iota(\Xi)\omega,$$
$$\mathcal{L}(\zeta)\varphi = \iota(\zeta)d\varphi = \iota(\zeta)\iota(\Xi)\omega = -\iota(\Xi)\beta = \varphi,$$

so (i) holds. It follows that $d\varphi_m(\zeta) = \mathcal{L}(\zeta)\varphi_m = \varphi(m)$, so any $c \neq 0$ is a regular value and ζ is transverse to $\varphi^{-1}(c)$. It is now easy to see that β is a contact form on $\varphi^{-1}(c)$; see, e.g., McDuff and Salamon [17, Proposition 3.57]. Furthermore, on $\varphi^{-1}(c)$ we have $\iota(\Xi)d\beta = \iota(\Xi)\omega = d\varphi = 0$ and $\iota(\Xi)\beta = -\varphi(m) = -c$, so $-c^{-1}\Xi|_{\varphi^{-1}(c)}$ is the Reeb vector field, which proves (ii). ☐

4.18. Example. Let $M = \mathbb{C}^n$ with its standard symplectic form and let

$$\zeta = \frac{1}{2}\sum_i \Big(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\Big) \qquad \text{and} \qquad \Xi = \sum_i \Big(-y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}\Big).$$

Then ζ is a Liouville vector field and generates the action $(t,z) \mapsto \sqrt{t}\,z$, and Ξ generates the standard circle action. Moreover, $\varphi(z) = -\frac{1}{2}\|z\|^2$, so for $c < 0$ the hypersurfaces $\varphi^{-1}(c)$ are spheres and the orbits of the Reeb vector field on $\varphi^{-1}(c)$ are the fibres of the Hopf fibration.

4.19. Example. Let $M = T^\times K$ and choose $\Xi \in \mathfrak{t}$. Let T act on M from the right and consider the vector field on M induced by Ξ, which for brevity we will also denote by Ξ. Then $\varphi(k,\lambda) = \lambda(\Xi)$ and Ξ commutes with $\zeta = d\mathcal{A}_t/dt|_{t=0}$. Therefore $\varphi^{-1}(-1) = K \times \{\, \lambda \mid \lambda(\Xi) = -1 \,\}$ is a hypersurface of contact type with Reeb vector field Ξ.

In Example 4.18 the hypersurface $\varphi^{-1}(-1)$ is compact and M is topologically a cone over $\varphi^{-1}(-1)$. This is obviously not the case in Example 4.19, but we shall now show that this can be remedied by imploding M. Since K is semisimple, $\Phi_{\mathcal{R}}^{-1}(\mathfrak{z}^*) = \Phi_{\mathcal{R}}^{-1}(0)$ is the zero section of T^*K, and therefore

$$(T^\times K)_{\mathrm{impl}} = (T^*K)_{\mathrm{impl}} \setminus \{*\} = \textstyle\coprod_{\sigma\in\Sigma\setminus\{0\}} F_\sigma,$$

where $F_\sigma = (K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma]$ and $\{*\} = F_{\{0\}}$ is the *vertex* of $(T^*K)_{\mathrm{impl}}$. Because $\Phi_{\mathcal{R}}$ is homogeneous and equivariant, the action $\mathcal{A}$ descends to an action on $(T^*K)_{\mathrm{impl}}$, denoted also by $\mathcal{A}$, which off the vertex is proper and free, and on each stratum F_σ is a Liouville action. For each F_σ, let ω_σ be the symplectic form, ζ_σ the Liouville vector field, β_σ the one-form $\iota(\zeta_\sigma)\omega_\sigma$, and Ξ_σ the vector field induced by $\Xi \in \mathfrak{t}$. The Hamiltonian $\varphi = -\iota(\Xi)\beta$ descends to a continuous function on $(T^*K)_{\mathrm{impl}}$, also denoted by φ. The functions $\varphi_\sigma = \varphi|_{F_\sigma}$ satisfy $\varphi_\sigma = -\iota(\Xi_\sigma)\beta_\sigma$. The subset $\varphi^{-1}(-1)$ of $(T^*K)_{\mathrm{impl}}$ is called the *link* of the vertex and denoted by $\mathrm{lk}(*)$. The infinite cone $\bigl(X \times [\,0,\infty)\bigr)/\bigl(X \times \{0\}\bigr)$ over a space X is denoted by $C^\circ(X)$.

4.20. Proposition. *Assume that K is semisimple and that Ξ is in the interior of the cone spanned by the negative coroots. Then*

(i) *φ is proper on $(T^*K)_{\mathrm{impl}}$, $\mathrm{lk}(*)$ is compact, $\varphi \le 0$, and $\varphi^{-1}(0) = \{*\}$.*
(ii) *For all $\sigma \ne 0$, the intersection $\mathrm{lk}(*)_\sigma = \mathrm{lk}(*) \cap F_\sigma$ is a smooth manifold, β_σ restricts to a contact form, and Ξ_σ to the Reeb vector field on $\mathrm{lk}(*)_\sigma$.*
(iii) *The link of $*$ is a global section of the principal $\mathbb{R}^\times$-bundle $(T^\times K)_{\mathrm{impl}}$. The map $f\colon \mathrm{lk}(*) \times \mathbb{R}^\times \to (T^\times K)_{\mathrm{impl}}$ given by $f(\pi_{\mathcal{R}}(k,\lambda),t) = \mathcal{A}_t(\pi_{\mathcal{R}}(k,\lambda)) = \pi_{\mathcal{R}}(k,t\lambda)$ is a stratum-preserving homeomorphism, which on every stratum is a diffeomorphism, and satisfies $f^*\omega_\sigma = d(t(\beta_\sigma|_{\mathrm{lk}(*)_\sigma}))$.*
(iv) *f extends uniquely to a homeomorphism $C^\circ(\mathrm{lk}(*)) \to (T^*K)_{\mathrm{impl}}$.*

Proof. The cone spanned by the *positive* coroots is the dual of the cone $\mathfrak{t}^*_+$. Since $-\Xi$ is in its interior and K is semisimple, the linear function $\lambda \mapsto \lambda(\Xi)$ is proper on $\mathfrak{t}^*_+$. It follows that $\varphi(\pi_{\mathcal{R}}(k,\lambda)) = \lambda(\Xi)$ is proper on $(T^*K)_{\mathrm{impl}}$. The other assertions in (i) are now obvious. To prove (ii), apply Lemma 4.17 to each stratum F_σ. (iii) follows from the observation that the simplex $\{\, \lambda \in \mathfrak{t}^*_+ \mid \lambda(\Xi) = -1 \,\}$ is a global section of the $\mathbb{R}^\times$-action $\lambda \mapsto t\lambda$ on the punctured Weyl chamber $\mathfrak{t}^*_+ \setminus \{0\}$. The equality $f^*\omega_\sigma = d(t(\beta_\sigma|_{\mathrm{lk}(*)_\sigma}))$ is readily checked on the tangent space T_mF_σ at any $m \in \mathrm{lk}(*)_\sigma$ and therefore holds globally by homogeneity. To prove (iv), observe that $(\pi_{\mathcal{R}}(k,\lambda),t) \mapsto \pi_{\mathcal{R}}(k,t\lambda)$ defines a map $\mathrm{lk}(*) \times [\,0,\infty) \to (T^*K)_{\mathrm{impl}}$ which sends $\mathrm{lk}(*) \times \{0\}$ to $*$ and therefore descends to a homeomorphism $C^\circ(\mathrm{lk}(*)) \to (T^*K)_{\mathrm{impl}}$. □

The analogy between $\mathbb{C}^n$ and $(T^*K)_{\mathrm{impl}}$ goes even further: if Ξ is integral, it generates a circle action, which turns out to be locally free. This will be proved in greater generality in the next section.

5. The stratification

We show now that the symplectic decomposition of the imploded cross-section of a Hamiltonian K-manifold M is a *stratification* in the sense that it is locally finite, satisfies the frontier condition (i.e., $X_i \cap \bar{X}_j \neq \varnothing$ implies $X_i \subseteq \bar{X}_j$) and a certain regularity condition, which is sometimes called local normal triviality. This means that locally at every point, in the direction transverse to the stratum, M_{impl} is a cone over a lower dimensional stratified space, called the *link* of the point. The link carries a locally free circle action such that the quotient space, the *symplectic link*, decomposes naturally into symplectic manifolds. (In fact, the symplectic link is the imploded cross-section of a singular symplectic quotient.)

This is analogous to results about singular symplectic quotients proved in [22, §5]. (There is however one aspect in which imploded cross-sections are different from symplectic quotients: they do not appear to have a naturally defined algebra of functions equipped with a Poisson bracket. Imploded cross-sections are therefore strictly speaking not "stratified symplectic spaces" in the sense of [22].) The strategy of the proof is the same: write a local normal form for an open neighbourhood of any point in $\Phi^{-1}(\mathfrak{t}_+^*)$ and carry out all computations in this model. We shall do this in three steps. The final result is summarized in Theorem 5.10 at the end of this section. Let $x \in M_{\mathrm{impl}}$ and choose $m \in \Phi^{-1}(\mathfrak{t}_+^*)$ such that $x = \pi(m)$.

Step 1. Assume that K is semisimple and $\Phi(m) = 0$. Then the orbit Km is isotropic. The *symplectic slice* at m is

$$V = T_m(Km)^\omega / T_m(Km).$$

The natural linear action of the isotropy subgroup $H = K_m$ on V is symplectic and has a moment map given by (3.2). Hence we can form the quotient

$$F(K,H,V) = (T^*K \times V) /\!\!/ H, \tag{5.1}$$

where we let H act on T^*K from the left. It is a symplectic vector bundle with fibre V over the base $T^*K/\!\!/H = T^*(K/H)$, and it carries a Hamiltonian K-action induced by the right K-action on T^*K. Let $[k,\lambda,v] \in F(K,H,V)$ denote the point corresponding to a point $(k,\lambda,v) \in T^*K \times V = K \times \mathfrak{k}^* \times V$ in the zero fibre of the H-moment map. The symplectic slice theorem of Marle and Guillemin–Sternberg says that there exists a map from a K-stable open neighbourhood of m in M to a K-stable open neighbourhood of $[1,0,0]$ in $F(K,H,V)$ which is an isomorphism of Hamiltonian K-manifolds and sends m to $[1,0,0]$. This means that for the purpose of investigating M_{impl} near $x = \pi(m)$ we can replace M by $F(K,H,V)$. Corollary 4.13 implies

$$F(K,H,V)_{\mathrm{impl}} = \big((T^*K)_{\mathrm{impl}} \times V\big) /\!\!/ H.$$

Since K is semisimple, the lowest dimensional stratum of $(T^*K)_{\mathrm{impl}}$ consists of a single point. Therefore the stratum of $\pi([1,0,0])$ in $F(K,H,V)_{\mathrm{impl}}$ is the subspace of fixed vectors V^H, which shows once again that the stratum of x in M_{impl} is a locally closed subspace and a symplectic manifold. Let $W = (V^H)^\omega$ be the skew complement of V^H. Then $W^H = \{0\}$ and $V = W \oplus V^H$ as a symplectic H-module. Thus we have a splitting

$$\begin{aligned} F(K,H,V)_{\mathrm{impl}} &= \big((T^*K)_{\mathrm{impl}} \times W \times V^H\big) /\!\!/ H \\ &= \big(((T^*K)_{\mathrm{impl}} \times W) /\!\!/ H\big) \times V^H = F(K,H,W)_{\mathrm{impl}} \times V^H, \end{aligned}$$

which shows that, near x, M_{impl} is symplectically the product of a neighbourhood of x inside its stratum and a neighbourhood of $\pi([1,0,0])$ inside $F(K,H,W)_{\mathrm{impl}}$.

The space $F(K,H,W)_{\mathrm{impl}} = \big((T^*K)_{\mathrm{impl}} \times W\big) /\!\!/ H$ carries information on the nature of the singularity at x. Let

$$F(K,H,W)_{\mathrm{impl}} = \coprod_{j\in\mathcal{J}} F_j \tag{5.2}$$

be its symplectic decomposition, let $* = F_{j_0} = \pi([1,0,0])$ be the lowest stratum, called the *vertex*, and put

$$F^\times(K,H,W)_{\mathrm{impl}} = \coprod_{j\in\mathcal{J}\setminus\{j_0\}} F_j.$$

Define a continuous $\mathbb{R}^\times$-action $\hat{\mathcal{A}}$ on $(T^*K)_{\mathrm{impl}} \times W$ by

$$\hat{\mathcal{A}}_t(\pi_{\mathcal{R}}(k,\lambda), w) = \big(\pi_{\mathcal{R}}(k, t\lambda), \sqrt{t}\, w\big).$$

This action commutes with the H-action, preserves the strata of $(T^*K)_{\mathrm{impl}} \times W$, and on each stratum is a Liouville action. Moreover, the H-moment map is homogeneous of degree 1 with respect to $\hat{\mathcal{A}}$, so $\hat{\mathcal{A}}$ descends to an action $\mathcal{A}$ on $F(K,H,W)_{\mathrm{impl}}$ which preserves the stratification and on each stratum is a Liouville action. Let ω_j be the symplectic form, ζ_j the Liouville vector field, and β_j the one-form $\iota(\zeta_j)\omega_j$ on F_j.

Now choose an H-invariant ω-compatible complex structure on the H-module W and a circle subgroup of T with infinitesimal generator $\Xi \in \mathfrak{t}$. Consider the S^1-action on $(T^*K)_{\mathrm{impl}} \times W$ given by Ξ acting from the right on $(T^*K)_{\mathrm{impl}}$ and complex scalar multiplication on W. It is generated by the Hamiltonian $\hat{\varphi}(\pi_{\mathcal{R}}(k,\lambda), w) = \lambda(\Xi) - \frac{1}{2}\|w\|^2$, it commutes with the H-action, and therefore descends to a Hamiltonian S^1-action on $F(K,H,W)_{\mathrm{impl}}$. The reduced Hamiltonian is $\varphi(\pi([k,\lambda,w])) = \lambda(\Xi) - \frac{1}{2}\|w\|^2$. For each $j \in \mathcal{J}$, let Ξ_j be the vector field on F_j induced by $\Xi \in \mathfrak{t}$. The functions $\varphi_j = \varphi|_{F_j}$ satisfy $\varphi_j = -\iota(\Xi_j)\beta_j$. The subspace $\mathrm{lk}(x) = \varphi^{-1}(-1)$ of $F(K,H,W)_{\mathrm{impl}}$ is called the *link* of x. The following result says that, stratum by stratum, $F(K,H,W)_{\mathrm{impl}}$ is the symplectification of $\mathrm{lk}(x)$ in the sense of Arnold [1, Appendix 4]. It is proved in the same way as Proposition 4.20.

5.3. Proposition. *Assume that K is semisimple and that Ξ is in the interior of the cone spanned by the negative coroots. Then*

(i) *φ is proper on $F(K,H,W)_{\mathrm{impl}}$, $\mathrm{lk}(x)$ is compact, $\varphi \le 0$, and $\varphi^{-1}(0) = *$.*

(ii) *For all $j \ne j_0$, $\mathrm{lk}(x)_j = \mathrm{lk}(x) \cap F_j$ is a smooth manifold, β_j restricts to a contact form, and Ξ_j to the Reeb vector field on $\mathrm{lk}(x)_j$.*

(iii) *The map $f\colon \mathrm{lk}(x) \times \mathbb{R}^\times \to F^\times(K,H,W)_{\mathrm{impl}}$ defined by*

$$f\big(\pi([k,\lambda,w]), t\big) = \mathcal{A}_t\big(\pi([k,\lambda,w])\big)$$

is a stratum-preserving homeomorphism, which on every stratum is a diffeomorphism, and satisfies $f^*\omega_j = d(t(\beta_j|_{\mathrm{lk}(x)_j}))$.

(iv) *f extends uniquely to a homeomorphism $C^\circ(\mathrm{lk}(x)) \to F(K,H,W)_{\mathrm{impl}}$.*

Consequently, if the link is a (homology) sphere, then M_{impl} is a topological (homology) manifold at x. The quotient

$$\mathrm{slk}(x) = \mathrm{lk}(x)/S^1 = F(K,H,W)_{\mathrm{impl}} /\!\!/_{-1} S^1 = \big((T^*K)_{\mathrm{impl}} \times W\big) /\!\!/_{(0,-1)} H \times S^1$$

is the *symplectic link* of x. For instance, the symplectic link of the vertex in $(T^*K)_{\mathrm{impl}}$ is $(T^*K)_{\mathrm{impl}}/\!\!/_{-1}S^1$. To be definite, let us henceforth take Ξ to be the sum of the simple coroots. (This choice is motivated by Proposition 6.10 in the next section.) The following result says that, stratum by stratum, the link is the contactification of the symplectic link.

5.4. Proposition. *Assume that K is semisimple. Then*

(i) *The S^1-action on $F^\times(K,H,W)_{\mathrm{impl}}$ is locally free.*

(ii) *For each $j \in \mathcal{J} \setminus \{j_0\}$ the space $\mathrm{slk}(x)_j = \mathrm{lk}(x)_j/S^1$ is a symplectic orbifold, and $\beta_j|_{\mathrm{lk}(x)_j}$ is a connection on the orbibundle $\mathrm{lk}(x)_j \to \mathrm{slk}(x)_j$, whose curvature is the reduced symplectic form on $\mathrm{slk}(x)_j$.*

Proof. Let us denote the element $(\pi_{\mathcal{R}}(k,\lambda), w)$ of $(T^*K)_{\mathrm{impl}} \times W$ by $(\!(k,\lambda,w)\!)$ and its image in the orbit space $\big((T^*K)_{\mathrm{impl}} \times W\big)/S^1$ by $[\![k,\lambda,w]\!]$. To prove (i) we need to show that for every $(\!(k,\lambda,w)\!) \neq (\!(1,0,0)\!)$ in the zero level set of the H-moment map, the infinitesimal stabilizers $\mathfrak{l}_{(\!(k,\lambda,w)\!)}$ and $\mathfrak{l}_{[\![k,\lambda,w]\!]}$ are equal. There exists an infinitesimal character $\chi \in \mathfrak{l}^*_{[\![k,\lambda,w]\!]}$ such that for all $\eta \in \mathfrak{l}_{[\![k,\lambda,w]\!]}$

$$\exp\eta \cdot (\!(k,\lambda,w)\!) = \big(\!\big(k\exp(-2\pi i\chi(\eta)\Xi), \lambda, e^{2\pi i\chi(\eta)}w\big)\!\big). \tag{5.5}$$

Let σ be the face containing λ. Then (5.5) boils down to

$$\exp(k^{-1}\eta) = \exp(-2\pi i\chi(\eta)\Xi) \bmod [K_\sigma, K_\sigma] \quad \text{and} \quad (\exp\eta)w = e^{2\pi i\chi(\eta)}w.$$

Differentiating at $\eta = 0$ yields

$$k^{-1}\eta + \chi(\eta)\Xi = [\xi_1, \xi_2] \quad \text{for certain } \xi_1,\, \xi_2 \in \mathfrak{k}_\sigma, \tag{5.6}$$

$$\eta w = \chi(\eta) w. \tag{5.7}$$

Since $(\!(k,\lambda,w)\!)$ is in the zero fibre of the H-moment map, $0 = \Phi^\eta_{\mathcal{L}}(k,\lambda) + \Phi^\eta_W(w) = -\lambda(k^{-1}\eta) + \frac{1}{2}\omega_W(\eta w, w)$. Combined with (5.7) this gives $\lambda(k^{-1}\eta) = \frac{1}{2}\omega_W(\eta w, w) = \frac{\chi(\eta)}{2}\omega_W(w,w) = 0$. Applying λ to both sides of (5.6) we then obtain

$$\chi(\eta)\lambda(\Xi) = \lambda([\xi_1,\xi_2]) = -(\mathrm{ad}^*(\xi_1)\lambda)(\xi_2) = 0,$$

because $\xi_1 \in \mathfrak{k}_\sigma$ and $\lambda \in \sigma$. Since $\Xi < 0$ on $\mathfrak{t}^*_+$, $\lambda(\Xi) < 0$ and hence $\chi(\eta) = 0$. From (5.5) we conclude that $\eta \in \mathfrak{l}_{(\!(k,\lambda,w)\!)}$, and therefore $\mathfrak{l}_{[\![k,\lambda,w]\!]}$ is contained in $\mathfrak{l}_{(\!(k,\lambda,w)\!)}$. The reverse inclusion is obvious. This proves (i). (ii) follows immediately from (i). $\square$

Theorem 2.10 implies that the symplectic link is a union of symplectic manifolds. In fact, the manifold decomposition of $\mathrm{slk}(x)$ is a refinement of the orbifold decomposition $\mathrm{slk}(x) = \coprod \mathrm{slk}(x)_j$.

Step 2. Next consider the case where K may have a positive dimensional centre and $\Phi(m)$ is contained in $\mathfrak{z}^*$, the fixed point set of the coadjoint action. This case reduces immediately to the previous one by replacing K with its semisimple part $[K,K]$. This works because the $[K,K]$-moment map sends m to $0 \in [\mathfrak{k},\mathfrak{k}]^*$ and if U is a $[K,K]$-stable open neighbourhood of m, then $U \cap \Phi^{-1}(\mathfrak{t}^*_+)$ is saturated under the equivalence relation $\sim$, so that $\pi(U \cap \Phi^{-1}(\mathfrak{t}^*_+))$ is an open neighbourhood of $\pi(m)$ in M_{impl}.

Step 3. Finally we reduce the general case to the case $\Phi(m) \in \mathfrak{z}^*$. Let $\sigma \in \Sigma$ be the face containing $\Phi(m)$. Then $m \in M_\sigma$ and $\pi(m) \in M_\sigma /\!\!/ [K_\sigma, K_\sigma]$. The *standard open neighbourhood* of $M_\sigma /\!\!/ [K_\sigma, K_\sigma]$ in M_{impl} is the T-stable open set

$$O_\sigma = \Phi^{-1}(\operatorname{star}\sigma)/{\sim} = \Phi_{\mathrm{impl}}^{-1}(\operatorname{star}\sigma) = \textstyle\coprod_{\tau \geq \sigma} M_\tau /\!\!/ [K_\tau, K_\tau].$$

Let $R \subseteq \mathfrak{t}^*$ be the root system of (K, T) and S the set of roots which are simple relative to the chamber $\mathfrak{t}^*_+$. The root system of (K_σ, T) is then

$$R_\sigma = \{\, \alpha \in R \mid \lambda(\alpha^\vee) = 0 \text{ for all } \lambda \in \sigma \,\}, \tag{5.8}$$

and its set of simple roots is $S_\sigma = R_\sigma \cap S$. The corresponding positive Weyl chamber is denoted by $\mathfrak{t}^*_{+,\sigma}$. Both $\mathfrak{z}^*_\sigma$ and $\operatorname{star}\sigma$ are contained in $\mathfrak{t}^*_{+,\sigma}$. Let $(M_\sigma)_{\mathrm{impl}} = \Phi_\sigma^{-1}(\mathfrak{t}^*_{+,\sigma})/{\sim_\sigma}$ be the imploded cross-section of M_σ with respect to the K_σ-action, $\pi_\sigma \colon \Phi_\sigma^{-1}(\mathfrak{t}^*_{+,\sigma}) \to (M_\sigma)_{\mathrm{impl}}$ the quotient map, and $(\Phi_\sigma)_{\mathrm{impl}}$ the associated imploded moment map.

5.9. Lemma. (i) $\operatorname{star}\sigma$ *is open in* $\mathfrak{t}^*_{+,\sigma}$.

(ii) O_σ *is isomorphic to the open subset* $(\Phi_\sigma)_{\mathrm{impl}}^{-1}(\operatorname{star}\sigma)$ *of* $(M_\sigma)_{\mathrm{impl}}$.

Proof. (i) follows immediately from $\mathfrak{t}^*_{+,\sigma} = \{\, \lambda \in \mathfrak{t}^* : \lambda(\alpha^\vee) \geq 0 \text{ for all } \alpha \in S_\sigma \,\}$, $\operatorname{star}\sigma = \{\, \lambda \in \mathfrak{t}^* : \lambda(\alpha^\vee) \geq 0 \text{ for all } \alpha \in S_\sigma,\ \lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in S \setminus S_\sigma \,\}$.

From (i) it follows that $(\Phi_\sigma)_{\mathrm{impl}}^{-1}(\operatorname{star}\sigma)$ is open in $(M_\sigma)_{\mathrm{impl}}$. For $\lambda \in \operatorname{star}\sigma$, $K_\lambda \subseteq K_\sigma$, so for m_1, $m_2 \in \Phi^{-1}(\operatorname{star}\sigma)$, $m_1 \sim m_2$ is equivalent to $m_1 \sim_\sigma m_2$. Hence $(\Phi_\sigma)_{\mathrm{impl}}^{-1}(\operatorname{star}\sigma) = \Phi_\sigma^{-1}(\operatorname{star}\sigma)/{\sim_\sigma} = \Phi^{-1}(\operatorname{star}\sigma)/{\sim} = O_\sigma$. □

To examine the structure of M_{impl} near $\pi(m)$ we can therefore resort to the space $(M_\sigma)_{\mathrm{impl}}$ and the point $\pi_\sigma(m)$. But here the argument of steps 1 and 2 applies, because $\Phi(m) \in \mathfrak{z}^*_\sigma$ is fixed under the coadjoint action of K_σ. We can summarize this discussion as follows.

5.10. Theorem. *Let (M, ω, Φ) be a Hamiltonian K-manifold and let $x \in M_{\mathrm{impl}}$. Select $m \in \Phi^{-1}(\mathfrak{t}^*_+)$ such that $x = \pi(m)$. Let σ be the face of $\mathfrak{t}^*_+$ containing $\Phi(m)$. Let $H = [K_\sigma, K_\sigma]_m$ be the stabilizer of m and V the symplectic slice at m for the $[K_\sigma, K_\sigma]$-action on M_σ. Put $W = (V^H)^\omega$. Then x has an open neighbourhood which is isomorphic to a product $U_1 \times U_2$, where U_1 is a neighbourhood of x in its stratum and U_2 is a neighbourhood of the vertex in $F([K_\sigma, K_\sigma], H, W)_{\mathrm{impl}}$. The space $F([K_\sigma, K_\sigma], H, W)_{\mathrm{impl}}$ is the stratified symplectification of the link* $\operatorname{lk}(x)$ *in the sense of Proposition* 5.3, *and* $\operatorname{lk}(x)$ *is the stratified contactification of the symplectic link* $\operatorname{slk}(x)$ *in the sense of Proposition* 5.4. *The symplectic decomposition of M_{impl} is locally finite and satisfies the frontier condition.*

Proof. Everything has been proved except for the frontier condition. Let X_{i_0} be the stratum of x and suppose that x is in the closure of a stratum X_i. Put $Y = \{\, y \in X_{i_0} \mid y \in \bar{X}_i \,\}$. Then $x \in Y$ and Y is closed in X_{i_0}. In the local model $F([K_\sigma, K_\sigma], H, V)_{\mathrm{impl}}$ around x, the stratum X_{i_0} is of the form $F_{j_0} \times V^H = \{*\} \times V^H$, whereas X_i is of the form $F_j \times V^H$ for some $j \in \mathcal{J}$. Here the notation is as in (5.2). Every stratum F_j in $F^\times(K, H, W)_{\mathrm{impl}}$ is stable under the $\mathbb{R}^\times$-action and therefore has the vertex $*$ as a limit point. It follows that $X_{i_0} \cap U \subseteq \bar{X}_i \cap U$, where U is an appropriate open neighbourhood of x in M_{impl}. Hence Y is open, and therefore $Y = X_{i_0}$. We have shown that $X_{i_0} \subseteq \bar{X}_i$. □

In the next section we shall prove that the link and the symplectic link of every point in M_{impl} are connected.

6. Kähler structures

In this section we show that the strata of the universal imploded cross-section fit together in an unexpectedly nice way. It turns out that if K is semisimple and simply connected, $(T^*K)_{\mathrm{impl}}$ can be embedded into a unitary K-module E in such a manner that the symplectic form on each stratum is the pullback of the flat Kähler form on E. The image of the embedding is a closed K-stable affine subvariety with coordinate ring $\mathbb{C}[G]^N$, where $G = K^{\mathbb{C}}$ is the complexification of K and N is a maximal unipotent subgroup of G. This happy state of affairs permits us to calculate $(T^*K)_{\mathrm{impl}}$ for groups other than $\mathbf{SU}(2)$ and to prove that the imploded cross-section of *every* Kähler Hamiltonian K-manifold is a Kähler space.

Assume first that K is an arbitrary compact connected Lie group. Let $\Lambda = \ker(\exp|_{\mathfrak{t}})$ be the exponential lattice in $\mathfrak{t}$ and $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ the weight lattice in $\mathfrak{t}^*$. Then $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}^*_+$ is the monoid of dominant weights. For any dominant weight λ let V_λ be the K-module with highest weight λ. Select a minimal set of generators Π of Λ^*_+, put

$$E = \bigoplus\nolimits_{\varpi \in \Pi} V_\varpi, \tag{6.1}$$

and fix a highest weight vector v_ϖ in each V_ϖ. To each face $\sigma \in \Sigma$ is associated a parabolic subgroup P_σ of G with Lie algebra

$$\mathfrak{p}_\sigma = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus\nolimits_{\alpha \in R_\sigma} \mathfrak{g}_\alpha \oplus \bigoplus\nolimits_{R_+ \setminus R_{+,\sigma}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space for α and R_σ is the root system of K_σ defined in (5.8).

6.2. Lemma. *For each face σ let $v_\sigma = \sum_{\varpi \in \bar\sigma} v_\varpi$. Then the stabilizer of v_σ for the G-action is equal to $[P_\sigma, P_\sigma]$.*

Proof. Let $[v_\sigma]$ denote the equivalence class of v_σ in the product of projective spaces $\prod_{\varpi \in \bar\sigma} \mathbb{P}(V_\varpi)$. Then $G_{[v_\sigma]} = P_\sigma$ and $G_{v_\sigma} = \{\, g \in P_\sigma \mid \lambda(g) = 1 \text{ for all } \lambda \in \Lambda^* \cap \bar\sigma \,\}$. (Here we identify λ with the character of P_σ to which it exponentiates.) Let $G_\sigma = (K_\sigma)^{\mathbb{C}}$ and let U_σ be the unipotent radical of P_σ. The corresponding Lie algebras are

$$\mathfrak{g}_\sigma = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus\nolimits_{\alpha \in R_\sigma} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{u}_\sigma = \bigoplus\nolimits_{\alpha \in R_+ \setminus R_{+,\sigma}} \mathfrak{g}_\alpha, \tag{6.3}$$

and we have Levi decompositions

$$P_\sigma = G_\sigma U_\sigma \qquad \text{and} \qquad [P_\sigma, P_\sigma] = [G_\sigma, G_\sigma]\, U_\sigma. \tag{6.4}$$

Since characters of P_σ vanish on U_σ, we have $G_{v_\sigma} = Q_\sigma U_\sigma$ with $Q_\sigma = G_{v_\sigma} \cap G_\sigma$. Thus both $[G_\sigma, G_\sigma]$ and Q_σ are semisimple groups with root system R_σ. To finish the proof it suffices to show that Q_σ is connected. The elements of $\Lambda^* \cap \bar\sigma$ being G_σ-invariant, Q_σ is stable under conjugation by the connected group G_σ. Therefore we need only show that the intersection of Q_σ with the maximal torus T is connected. Observe that

$$Q_\sigma \cap T = \{\, t \in T \mid \lambda(t) = 1 \text{ for all } \lambda \in \Lambda^* \cap \bar\sigma \,\}.$$

If C is any top-dimensional polyhedral cone in a vector space V and Γ is a lattice in V, then the set $\Gamma \cap C$ contains a $\mathbb{Z}$-basis for Γ. Therefore $\Lambda^* \cap \bar\sigma$ contains a $\mathbb{Z}$-basis for the lattice $\Lambda^* \cap \mathfrak{z}^*_\sigma$. This basis can be extended to a basis of Λ^*, and this implies that $Q_\sigma \cap T$ is connected. $\square$

Let N be the maximal unipotent subgroup of G with Lie algebra $\mathfrak{n} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ and let A be the (real) subgroup with Lie algebra $\mathfrak{a} = i\mathfrak{t}$. Then $T^{\mathbb{C}} = TA$ and $B = TAN$ is a Borel subgroup of G. The Borel–Weil Theorem implies that the ring of N-invariants is a multigraded direct sum,

$$\mathbb{C}[G]^N = \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda.$$

In particular it is generated by the finite-dimensional subspace E^*. It follows that there exists a closed G-equivariant algebraic embedding $G_N \hookrightarrow E$. (Following Kraft [14, Kapitel III], for any affine G-variety X, we denote by X_N the affine variety with coordinate ring $\mathbb{C}[X]^N$.) By results of Popov and Vinberg, cf. [23, §3], G_N consists of finitely many G-orbits, which are labelled by the faces of the cone $\mathfrak{t}^*_+$. In addition, G_N contains the subspace E^N and $G_N = GE^N$. The stabilizer of the orbit corresponding to the face σ is the group of all $g \in P_\sigma$ such that $\lambda(g) = 1$ for all $\lambda \in \Lambda^* \cap \bar{\sigma}$, which is equal to $[P_\sigma, P_\sigma]$ by Lemma 6.2. In particular, the open orbit is of type G/N. Moreover, the embedding $G_N \hookrightarrow E$ is uniquely determined by sending 1 mod N to the sum of the highest weight vectors $\sum_{\varpi \in \Pi} v_\varpi$. We shall identify G_N with its image in E. We turn E into a $K \times T$-module by letting T act on V_ϖ with weight $-\varpi$. Observe that a different choice of highest weight vectors leads to a new embedding $G_N \hookrightarrow E$ which differs from the old by multiplication by an element of the complex torus TA.

Let $\langle \cdot, \cdot \rangle$ be the unique K-invariant Hermitian inner product on E satisfying $\|v_p\| = 1$ for all p. We regard E as a flat Kähler manifold with the Kähler form $\omega_E = -\operatorname{Im}\langle \cdot, \cdot \rangle$. It is convenient to write $\omega_E = d\beta_E$ with

$$(\beta_E)_v(w) = -\frac{1}{2}\operatorname{Im}\langle v, w\rangle \tag{6.5}$$

for v, $w \in E$.

Now assume that K is semisimple and simply connected. The set Π is then uniquely determined: it is the set of fundamental weights $\{\varpi_1, \varpi_2, \dots, \varpi_r\}$, which form a $\mathbb{Z}$-basis of the weight lattice. Let α_1, $\alpha_2, \dots, \alpha_r$ be the corresponding simple roots. Put $V_p = V_{\varpi_p}$ and $v_p = v_{\varpi_p}$. Since $\lambda(\alpha^\vee) \geq 0$ if $\lambda \in \mathfrak{t}^*_+$,

$$s(\lambda) = \frac{1}{\sqrt{\pi}} \sum_{p=1}^{r} \sqrt{\lambda(\alpha_p^\vee)}\, v_p \tag{6.6}$$

defines a continuous map from $\mathfrak{t}^*_+$ into E^N.

6.7. *Remark.* The subspace E^N inherits a symplectic form from E and according to (3.2) the moment map for the T-action on E^N is given by

$$\Phi_{E^N}\left(\textstyle\sum_p c_p v_p\right) = -\pi \textstyle\sum_p |c_p|^2 \varpi_p.$$

Observe that, K being semisimple and simply connected, the T-action on E^N is equivalent to the $(\mathbb{C}^\times)^r$-action on $\mathbb{C}^r$ and so is effective and multiplicity-free. The map Φ_{E^N} separates the T-orbits and its image is the opposite chamber $-\mathfrak{t}^*_+$. Therefore E^N is nothing but the symplectic toric manifold (multiplicity-free T-manifold or Delzant space) associated with the polyhedron $-\mathfrak{t}^*_+$. Finally note that $\Phi_{E^N}(s(\lambda)) = -\lambda$, that is, s is a section of $-\Phi_{E^N}$.

The map s extends uniquely to a $K \times T$-equivariant map $\mathcal{F}\colon K \times \mathfrak{t}^*_+ \to E$.

6.8. Proposition. *Assume that K is semisimple and simply connected.*

(i) *$\mathcal{F}$ induces a map $f\colon (T^*K)_{\mathrm{impl}} \to E$ which is continuous and closed (for the classical topology on E), and injective.*

(ii) *The restriction of f to each stratum is a smooth symplectomorphism.*

(iii) *The image of $(T^*K)_{\mathrm{impl}}$ under f is identical to G_N. Thus $f\colon (T^*K)_{\mathrm{impl}} \to G_N$ is an isomorphism of Hamiltonian K-spaces in the sense of Section 3.*

Proof. It is plain from (6.6) that $\mathcal{F}$ is continuous and closed. Furthermore, by Lemma 6.2 the stabilizer of $\mathcal{F}(1,\lambda)=s(\lambda)$ for the K-action is equal to $K\cap[P_\sigma,P_\sigma]=[K_\sigma,K_\sigma]$, where σ is the face containing λ. This implies (i).

It is clear from (6.6) that $\mathcal{F}$ is smooth on $K\times\sigma$ for every σ, and therefore f restricted to $K/[K_\sigma,K_\sigma]\times\sigma$ is a smooth embedding. We check that it preserves the symplectic form by showing that $f^*\beta_E=\beta_\sigma$, where β_σ is the one-form on $K/[K_\sigma,K_\sigma]\times\sigma$ considered in Lemma 4.6. Because $\mathcal{F}$ is K-equivariant we need only show this at the points $(\bar{1},\lambda)$, where $\bar{k}$ denotes the coset $k\,[K_\sigma,K_\sigma]$. By Lemma 4.6,

$$(\beta_\sigma)_{(\bar{1},\lambda)}(\bar{\xi},\mu)=\lambda(\xi) \tag{6.9}$$

for all $\xi\in\mathfrak{k}$ and $\mu\in\mathfrak{z}_\sigma$. On the other hand,

$$(f^*\beta_E)_{(\bar{1},\lambda)}(\bar{\xi},\mu)=(\beta_E)_{f(\bar{1},\lambda)}(f_*(\bar{\xi},\mu))=(\beta_E)_{s(\lambda)}(\mathcal{F}_*(\xi,\mu)).$$

Here

$$\mathcal{F}_*(\xi,\mu)=\frac{d}{dt}\exp(t\xi)s(\lambda+t\mu)\bigg|_{t=0}=\xi_E(s(\lambda))+\sum_{p=1}^{r}\frac{\mu(\alpha_p^\vee)}{2\sqrt{\pi\,\lambda(\alpha_p^\vee)}}\,v_p.$$

Together with (6.5) this yields

$$\begin{aligned}(f^*\beta_E)_{(\bar{1},\lambda)}(\bar{\xi},\mu)&=\frac{1}{2}\operatorname{Im}\Big\langle \xi_E(s(\lambda))+\sum_p\frac{\mu(\alpha_p^\vee)}{2\sqrt{\pi\,\lambda(\alpha_p^\vee)}}\,v_p,s(\lambda)\Big\rangle\\&=\frac{1}{2\pi}\operatorname{Im}\Big\langle\sum_p\sqrt{\lambda(\alpha_p^\vee)}\,\xi_E(v_p),s(\lambda)\Big\rangle+\frac{1}{2\pi}\operatorname{Im}\sum_p\frac{\mu(\alpha_p^\vee)}{2}\\&=\sum_p\lambda(\alpha_p^\vee)\varpi_p(\xi)=\lambda(\xi),\end{aligned}$$

where we have used $\langle\xi_E(v_p),v_q\rangle=2\pi i\varpi_p(\xi)\delta_{pq}$. Comparing with (6.9) we conclude that $f^*\beta_E=\beta_\sigma$. This proves (ii).

(iii) is a consequence of the Iwasawa decomposition. Put $\mathfrak{a}=i\mathfrak{t}$, $A=\exp\mathfrak{a}$ and, for each face σ, $\mathfrak{n}_\sigma=\bigoplus_{\alpha\in R_{+,\sigma}}\mathfrak{g}_\alpha$ and $N_\sigma=\exp\mathfrak{n}_\sigma$. Then $G=KAN$ and $G_\sigma=K_\sigma AN_\sigma$. Recall also that $P_\sigma=G_\sigma U_\sigma$. Here G_σ and U_σ are as in (6.3). Let

$$\mathfrak{a}_\sigma=\mathfrak{a}\cap[\mathfrak{g}_\sigma,\mathfrak{g}_\sigma]=\bigoplus\nolimits_{\alpha\in S_\sigma}i\mathbb{R}\alpha^\vee\quad\text{and}\quad\mathfrak{a}_\sigma^\perp=i\mathfrak{z}\oplus\bigoplus\nolimits_{\alpha\in S\setminus S_\sigma}i\mathbb{R}\alpha^\vee,$$

so that $\mathfrak{a}=\mathfrak{a}_\sigma\oplus\mathfrak{a}_\sigma^\perp$. Writing $A_\sigma=\exp\mathfrak{a}_\sigma$ and $A_\sigma^\perp=\exp\mathfrak{a}_\sigma^\perp$ we find that $A=A_\sigma\times A_\sigma^\perp$ and, using (6.4),

$$[P_\sigma,P_\sigma]=[G_\sigma,G_\sigma]U_\sigma=[K_\sigma,K_\sigma]A_\sigma N_\sigma U_\sigma=[K_\sigma,K_\sigma]A_\sigma N.$$

Hence

$$G/[P_\sigma, P_\sigma] = KAN/\big([K_\sigma, K_\sigma]A_\sigma^\perp N\big) \cong K/[K_\sigma, K_\sigma] \times A_\sigma^\perp$$

as smooth K-manifolds. To finish the proof it suffices to show that $s(\sigma)$ is equal to the $A_\sigma^\perp$-orbit through v_σ for all σ. For $\lambda \in \mathfrak{t}_+^*$ put

$$\psi(\lambda) = \frac{1}{4\pi i} \sum_p \big(\log \lambda(\alpha_p^\vee) - \log \pi\big)\alpha_p^\vee \in \mathfrak{a},$$

where the sum is over all p such that $\lambda(\alpha_p^\vee) \neq 0$. For each face σ, ψ defines a diffeomorphism from σ to $\mathfrak{a}_\sigma^\perp$, and therefore $\exp \circ \psi \colon \sigma \to A_\sigma^\perp$ is also a diffeomorphism. Moreover, for $\lambda \in \sigma$

$$\begin{aligned}
\exp \psi(\lambda) \cdot v_\sigma &= \sum_{\substack{p \\ \alpha_p \in S \setminus S_\sigma}} \exp\left(\frac{1}{4\pi i} \sum_{\substack{q \\ \alpha_q \in S \setminus S_\sigma}} \big(\log \lambda(\alpha_q^\vee) - \log \pi\big)\alpha_q^\vee\right) \cdot v_p \\
&= \sum_p \exp\left(\frac{1}{2} \sum_q \big(\log \lambda(\alpha_q^\vee) - \log \pi\big)\varpi_p(\alpha_q^\vee)\right) \cdot v_p \\
&= \sum_p \exp \frac{1}{2}\big(\log \lambda(\alpha_p^\vee) - \log \pi\big) \cdot v_p = \mathcal{F}(\lambda).
\end{aligned}$$

Hence $s(\sigma) = A_\sigma^\perp v_\sigma$. $\square$

In a similar fashion the symplectic link of the vertex in $(T^*K)_{\mathrm{impl}}$ can be identified with a *projective* variety. Observe that the subvariety G_N of E is conical, i.e., preserved by the standard $\mathbb{C}^\times$-action on E. The easiest way to see this is to consider the one-parameter subgroup of T generated by $\Xi = -\sum_{p=1}^r \alpha_p^\vee \in \mathfrak{t}$. As $\varpi_p(\Xi) = -1$ for all p and T acts with weight $-\varpi_p$ on V_p, Ξ generates the standard S^1-action on E. Since G_N is affine and is preserved under the action of T, it is a cone. Let us denote the subvariety $(G_N \setminus \{0\})/\mathbb{C}^\times$ of $\mathbb{P}(V)$ by $\mathbb{P}(G_N)$. As before, $*$ denotes the vertex in $(T^*K)_{\mathrm{impl}}$, and $\mathrm{slk}(*) = (T^*K)_{\mathrm{impl}}/\!\!/_{-1} S^1$ its symplectic link. The following result is now immediate from Proposition 6.8 and Example 3.3.

6.10. Proposition. *Assume that K is semisimple and simply connected. The isomorphism $f\colon (T^*K)_{\mathrm{impl}} \to G_N$ induces an isomorphism of Hamiltonian K-spaces* $\mathrm{slk}(*) \cong \mathbb{P}(G_N)$.

Now let K be a torus. Then $\Lambda_+^* = \Lambda^*$ and $(T^*K)_{\mathrm{impl}} = T^*K$. Let us take $\Pi = \{\pm\kappa_1, \pm\kappa_2, \dots, \pm\kappa_s\}$, where $\{\kappa_1, \kappa_2, \dots, \kappa_s\}$ is a $\mathbb{Z}$-basis of the lattice Λ^*. Let $\eta_1, \eta_2, \dots, \eta_s$ be the dual basis of Λ, $V_p = V_{\kappa_p}$, and $V_{-p} = V_{-\kappa_p}$. For $\lambda \in \mathfrak{k}^*$ set

$$s(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{p=1}^s \left(\chi(\lambda(\eta_p))\, v_p + \frac{1}{\chi(\lambda(\eta_p))}\, v_{-p}\right)$$

with $\chi(t) = \sqrt{t + \sqrt{t^2+1}}$. Then s extends uniquely to an equivariant map $\mathcal{F}$ from T^*K into E and it is straightforward to check that this is a symplectic embedding.

The main points of this discussion can be restated as follows.

6.11. Theorem. *Assume that K is the product of a torus and a semisimple simply connected group. There exists a $K \times T$-equivariant embedding f of $(T^*K)_{\mathrm{impl}}$ into the unitary $K \times T$-module E whose image is the Zariski-closed affine subvariety G_N. Hence the action of $K \times T$ on $(T^*K)_{\mathrm{impl}}$ extends to an action of the complexified group $K^{\mathbb{C}} \times T^{\mathbb{C}} = G \times TA$. The strata of $(T^*K)_{\mathrm{impl}}$ coincide with the orbits of G:*

$$f\big((K \times \mathfrak{S}_\sigma)/\!\!/[K_\sigma, K_\sigma]\big) = G/[P_\sigma, P_\sigma]$$

for all faces σ. The symplectic form on each stratum is the restriction of the flat Kähler form on E.

6.12. Example. Let $K = \mathbf{SU}(2)$. Write an arbitrary element of $G = \mathbf{SL}(2, \mathbb{C})$ as $g = (x_{ij})$. Let N be the subgroup consisting of upper triangular unipotent matrices. The N-invariants of degree 1 are the entries in the first column, x_{11} and x_{21}. These two elements freely generate $\mathbb{C}[G]^N$. Therefore G_N is the affine plane $\mathbb{C}^2$, which confirms the computation in Example 4.7.

For general compact connected K there is a similar embedding of $(T^*K)_{\mathrm{impl}}$ into E, but we have not been able to find one that is symplectic with respect to a natural Kähler structure on E. Instead we proceed as follows. Consider the universal cover $[K, K]^\sim$ of $[K, K]$ and the group $\tilde{K} = Z \times [K, K]^\sim$. Then $K = \tilde{K}/\Upsilon$, where Υ is a finite central subgroup of $\tilde{K}$. Let $\tilde{G}$ be the complexification of $\tilde{K}$ and $\tilde{N}$ the preimage of N in $\tilde{G}$. Then $(T^*K)_{\mathrm{impl}} \cong (T^*\tilde{K})_{\mathrm{impl}}/\Upsilon$ (see Example 4.15) and likewise $G_N \cong \tilde{G}_{\tilde{N}}/\Upsilon$. It follows that f descends to a homeomorphism $(T^*K)_{\mathrm{impl}} \to G_N$. We use this map to identify $(T^*K)_{\mathrm{impl}}$ with G_N, thus defining a structure of an affine variety on $(T^*K)_{\mathrm{impl}}$ and Kähler structures on the orbits of G_N.

By virtue of this result we can bring the machinery of algebraic geometry to bear on the universal imploded cross-section. For instance, it now makes sense to talk about algebraic subvarieties of $(T^*K)_{\mathrm{impl}}$. Each stratum, being an orbit of G, is a quasi-affine subvariety and its closure in the classical topology is the same as its Zariski closure. The following is an algebraic slice theorem for G_N, valid for arbitrary reductive G.

6.13. Lemma. *For every face σ the point v_σ has a G-stable Zariski-open neighbourhood in G_N which is equivariantly isomorphic to $G \times^{[P_\sigma, P_\sigma]} [P_\sigma, P_\sigma]_N$.*

Proof. Let $E_\sigma = \bigoplus_{\varpi \in \bar{\sigma}} V_\varpi$ and let $\mathrm{pr}\colon E \to E_\sigma$ be the orthogonal projection. Then pr is G-equivariant and $\mathrm{pr}(G_N)$ is the closure of Gv_σ. For any face τ, $\mathrm{pr}(v_\tau) = \sum_{\varpi \in \bar{\sigma} \cap \bar{\tau}} v_\varpi = v_{\sigma \wedge \tau}$, where $\sigma \wedge \tau \in \Sigma$ is the interior of $\bar{\sigma} \cap \bar{\tau}$. Hence

$$\mathrm{pr}(v_\tau) = v_\sigma \iff \tau \geq \sigma. \tag{6.14}$$

Consider the subsets of E given by

$$X_\sigma = \textstyle\coprod_{\tau \geq \sigma} [P_\sigma, P_\sigma]\, v_\tau \qquad \text{and} \qquad O_\sigma = GX_\sigma = \coprod_{\tau \geq \sigma} Gv_\tau.$$

Then X_σ is $[P_\sigma, P_\sigma]$-stable and (6.14) implies that $X_\sigma = \mathrm{pr}^{-1}(v_\sigma) \cap G_N$. Hence X_σ is Zariski-closed. Similarly, O_σ is equal to $\mathrm{pr}^{-1}(Gv_\sigma) \cap G_N$, and it is a G-stable Zariski-open neighbourhood of v_σ. If x and gx are in X_σ, then $gv_\sigma = g\,\mathrm{pr}(x) = \mathrm{pr}(gx) = v_\sigma$, so that $g \in [P_\sigma, P_\sigma]$ by Lemma 6.2. It follows that the multiplication map $G \times X_\sigma \to O_\sigma$

induces a G-equivariant isomorphism $G \times^{[P_\sigma, P_\sigma]} X_\sigma \to O_\sigma$. The affine $[P_\sigma, P_\sigma]$-variety X_σ is the union of all orbits $[P_\sigma, P_\sigma]/[P_\tau, P_\tau]$ with $\tau \geq \sigma$. The groups P_τ are exactly the parabolic subgroups of P_σ that contain B, and therefore it follows from the corollary to [23, Theorem 6] that $O_\sigma \cong [P_\sigma, P_\sigma]_N$. □

As an application we determine the smooth (nonsingular) locus of G_N. Since G_N is smooth at x if and only if it is smooth at all points in Gx, it suffices to consider $x = v_\sigma$.

6.15. Proposition. *Let σ be any face of $\mathfrak{t}^*_+$.*

(i) *G_N is smooth at v_σ if and only if $[G_\sigma, G_\sigma] \cong \mathbf{SL}(2,\mathbb{C})^k$ for some k. The slice $[P_\sigma, P_\sigma]_N$ is then $\mathbf{SL}(2,\mathbb{C})^k$-equivariantly isomorphic to the standard $\mathbf{SL}(2,\mathbb{C})^k$-representation on $(\mathbb{C}^2)^k$.*

(ii) *G_N has an orbifold singularity at v_σ if and only if $[G_\sigma, G_\sigma] \cong \mathbf{SL}(2,\mathbb{C})^k/\Upsilon$ for some k and some central subgroup Υ of $\mathbf{SL}(2,\mathbb{C})^k$. Then $[P_\sigma, P_\sigma]_N \cong (\mathbb{C}^2)^k/\Upsilon$ as an $\mathbf{SL}(2,\mathbb{C})^k/\Upsilon$-variety.*

Proof. It follows from Lemma 6.13 that G_N is smooth at v_σ if and only if $[P_\sigma, P_\sigma]_N \cong [G_\sigma, G_\sigma]_{N_\sigma}$ is smooth at the vertex. Pauer [19, Hilfssatz 2.4] proved that this is the case if and only if $[G_\sigma, G_\sigma] \cong \mathbf{SL}(2,\mathbb{C})^k$ and $[G_\sigma, G_\sigma]_{N_\sigma} \cong (\mathbb{C}^2)^k$. This proves (i).

Likewise, G_N has an orbifold singularity at v_σ if and only if $[G_\sigma, G_\sigma]_{N_\sigma}$ has an orbifold singularity at the vertex. This is the case precisely when $\mathcal{G}_\mathcal{N}$ has an orbifold singularity at the vertex, where $\mathcal{G}$ is the universal cover of $[G_\sigma, G_\sigma]$ and $\mathcal{N}$ is the preimage of N_σ in $\mathcal{G}$. Since $\mathcal{G}$ is simply connected and $\mathcal{N}$ is connected, $\mathcal{G}/\mathcal{N}$ is simply connected. The complement of $\mathcal{G}/\mathcal{N}$ inside $(\mathcal{G}_\mathcal{N})_{\mathrm{reg}}$ has complex codimension at least 2, and therefore $(\mathcal{G}_\mathcal{N})_{\mathrm{reg}}$ is also simply connected. Moreover, $(\mathcal{G}_\mathcal{N})_{\mathrm{reg}}$ is a union of $\mathcal{G}$-orbits and therefore homogeneous, i.e., stable under dilations on $\mathcal{G}_\mathcal{N}$. We conclude that the vertex has a basis of open neighbourhoods O such that the nonsingular part O_{reg} is simply connected. A result of Prill [20, Theorem 1] implies that if a complex analytic space X has a finite-quotient singularity at x and if x has a basis of neighbourhoods O such that O_{reg} is simply connected, then X is smooth at x. We conclude that $\mathcal{G}_\mathcal{N}$ has an orbifold singularity at the vertex if and only if it is smooth. Therefore (ii) follows from (i). □

In particular G_N is smooth if and only if $[G,G]$ is a product of copies of $\mathbf{SL}(2,\mathbb{C})$; it is an orbifold if and only if the universal cover of $[G,G]$ is a product of copies of $\mathbf{SL}(2,\mathbb{C})$.

6.16. Example. Let $K = \mathbf{SU}(3)$. Let $(x_{ij}) \in G = \mathbf{SL}(3,\mathbb{C})$ and let N be the subgroup consisting of upper triangular unipotent matrices. The N-invariants of degree 1 are the entries in the first column, $w_1 = x_{11}$, $w_2 = x_{21}$, and $w_3 = x_{31}$. The invariants of degree 2 are the minors extracted from the first two columns,

$$z_1 = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \quad z_2 = \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}, \quad z_3 = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

These six elements generate $\mathbb{C}[G]^N$ and the only relation is $\sum_k w_k z_k = 0$. Thus $(T^*K)_{\mathrm{impl}} = G_N$ is the quadric in $\mathbb{C}^6 = \mathbb{C}^3 \times \mathbb{C}^3$ given by this equation. The four strata corresponding to the faces of $\mathfrak{t}^*_+$ are $\{0\}$, $(\mathbb{C}^3 \setminus \{0\}) \times \{0\}$, $\{0\} \times (\mathbb{C}^3 \setminus \{0\})$ and the open stratum. Only the vertex $\{0\}$ is singular.

As an application of the foregoing results let us show that the imploded cross-section of an affine Hamiltonian K-space X (as defined in Example 3.1) is an affine variety. The following result says that implosion is the symplectic analogue of taking the quotient of a variety by the maximal unipotent subgroup of a reductive group. Recall that X_N denotes the affine variety with coordinate ring $\mathbb{C}[X]^N$.

6.17. Theorem. *The imploded cross-section of an affine Hamiltonian K-space X is T-equivariantly homeomorphic to the affine variety X_N. Under the homeomorphism the strata of X_{impl} correspond to algebraic subvarieties of X_N.*

Proof. Embed X into a finite-dimensional unitary K-module V as in Example 3.1. The K-action on X extends uniquely to an algebraic G-action, which preserves the stratification of X. To examine X_{impl} let us assume that K is semisimple and simply connected. (This is justified by Lemma 2.4.) Then $X \times G_N$ is a closed K-stable affine subvariety of $V \times E$, and Theorems 4.9 and 6.11 give us isomorphisms

$$X_{\mathrm{impl}} \cong (X \times (T^*K)_{\mathrm{impl}})/\!\!/K \cong (X \times G_N)/\!\!/K. \tag{6.18}$$

A well known result of Kempf and Ness [12] (see also Schwarz [21]) says that the symplectic quotient on the right is homeomorphic to the invariant-theoretic quotient of $X \times G_N$ by G, i.e., the affine variety (associated to the scheme) $\operatorname{Spec}\mathbb{C}[X \times G_N]^G$. The homeomorphism is induced by the inclusion of the zero fibre of the moment map for the K-action on $X \times G_N$ and is therefore T-equivariant. By a result of Grosshans [6] (see also [14, III.3.2]), the ring $\mathbb{C}[X \times G_N]^G$ is isomorphic to $\mathbb{C}[X]^N$. $\square$

An example of an affine Hamiltonian K-space is the local model space defined in (5.1). Thus we see that, for every Hamiltonian K-manifold M, every point x in M_{impl} has an open neighbourhood which is homeomorphic to an open subset (in the classical topology) of an affine variety of the form X_N, where X is the local model at x. Since X is smooth, the quotient X_N is normal (see, e.g., Satz 2 in Kraft [14, III.3.4]), from which it follows that the link of x is connected.

6.19. Corollary. *Let M be an arbitrary Hamiltonian K-manifold. Then the link and the symplectic link of every point in M_{impl} are connected.*

An assertion comparable to Theorem 6.17 can be made in the analytic category. Let (M, ω, Φ) be a Hamiltonian K-manifold equipped with a K-invariant complex structure J. We assume that J is compatible with ω, so that M is a Kähler manifold, and that the K-action extends to a holomorphic G-action. We wish to show that M_{impl} is a Kähler space and in particular that its strata are Kähler manifolds. However, the complex structure is not induced "directly" from M.

6.20. Example. Let $\mathbf{U}(n)$ act diagonally on p copies of $\mathbb{C}^n$. Viewing an element of $\mathbb{C}^{n\times p}$ as an $n \times p$-matrix, and identifying $\mathfrak{u}(n)^*$ with $\mathfrak{u}(n)$ by means of the trace form, we can write the moment map as $\Phi(x) = -\frac{i}{2}xx^*$. Let $x_1, x_2, \ldots, x_n$ denote the row vectors of x. The open stratum of the imploded cross-section is the set of all $x \in \mathbb{C}^{n\times p}$ such that

$$\langle x_j, x_k\rangle = 0 \quad \text{for } j \neq k, \text{ and } \quad \|x_1\| > \|x_2\| > \cdots > \|x_n\|.$$

For $p = 1$ this happens to be a complex submanifold of $\mathbb{C}^{n\times p}$ (namely the set of vectors $(z, 0, 0, \ldots, 0)$ in $\mathbb{C}^n$ with $z \neq 0$), but for $p > 1$ it is not.

Instead, the complex structure is defined indirectly, by using the isomorphism (6.18). Let Ψ be the moment map for the diagonal K-action on $M \times G_N$, so that $M_{\mathrm{impl}} \cong$

$\Psi^{-1}(0)/K$, and let $\mathcal{S}$ be the semistable set

$$\mathcal{S} = \{\, x \in M \times G_N \mid \overline{Gx} \text{ intersects } \Psi^{-1}(0)\,\}.$$

Results of Heinzner and Loose [9] show that $\mathcal{S}$ is open in M and that for every $x \in \mathcal{S}$ the intersection $\overline{Gx} \cap \Psi^{-1}(0)$ consists of a single K-orbit, so that there is a natural surjection $\mathcal{S} \to \Psi^{-1}(0)/K \cong M_{\mathrm{impl}}$. Let $\mathcal{O}$ be the pushforward to M_{impl} of the sheaf of G-invariant holomorphic functions on $\mathcal{S}$. See, e.g., [9] for the definition of a Kähler metric on an analytic space.

6.21. Theorem. *Let M be a Kähler Hamiltonian K-manifold. Then $(M_{\mathrm{impl}}, \mathcal{O})$ is an analytic space. The strata of M_{impl} are Kähler manifolds and M_{impl} possesses a unique Kähler metric which restricts to the given Kähler metrics on the strata.*

Proof. That the strata are Kähler follows from (6.18), which shows that every stratum is a symplectic quotient of a Kähler manifold. The other assertions follow from Theorem (3.3) and Remark (3.4) in [9]. □

7. Quantization and implosion

Throughout this section let (M, ω, Φ) be a compact Hamiltonian K-manifold. Suppose that M is equivariantly prequantizable and let L be an equivariant prequantum line bundle. By the *quantization* of M we mean the equivariant index of the Dolbeault–Dirac operator on M with coefficients in L. This is an element of the representation ring of K (see, e.g., [11] or [18] for the definition), and is denoted by $\mathrm{RR}(M, L)$. In this section we compare the quantization of M with that of its imploded cross-section. A priori this does not make sense because M_{impl} is not a symplectic manifold, but, following the strategy of [18], we shall define the quantization of M_{impl} to be that of a certain partial desingularization $\tilde{M}_{\mathrm{impl}}$.

Let l_1 and l_2 be points in L with basepoints m_1 and m_2, respectively, and define $l_1 \sim l_2$ if there exists $k \in [K_{\Phi(m_1)}, K_{\Phi(m_1)}]$ such that $l_2 = kl_1$. The *imploded prequantum bundle* is the quotient $L_{\mathrm{impl}} = L/{\sim}$. There is a natural projection $L_{\mathrm{impl}} \to M_{\mathrm{impl}}$ and it follows from [18, Lemma 3.11(3)] that the fibres are of the form $\mathbb{C}/\Gamma$, where Γ is finite cyclic. (In fact, it is not hard to show that the restriction of L_{impl} to each stratum in M_{impl} is a prequantum orbibundle.)

Let $\tau = \sigma_{\mathrm{prin}}$ be the principal face of M. Choose $\lambda_0 \in \tau$ and define

$$\tilde{M}_{\mathrm{impl}} = (M_{\mathrm{impl}} \times X[\tau]) /\!\!/_{\lambda_0} T \quad \text{and} \quad \tilde{L}_{\mathrm{impl}} = (L_{\mathrm{impl}} \boxtimes \mathbb{C}) /\!\!/_{\lambda_0} T, \tag{7.1}$$

where $X[\tau]$ is the symplectic toric manifold associated with the polyhedron $-\bar{\tau}$ and $\mathbb{C}$ is the trivial line bundle on $X[\tau]$. In other words, $\tilde{M}_{\mathrm{impl}}$ is the symplectic cut of M_{impl} with respect to the polyhedral cone $\lambda_0 + \bar{\tau}$, and $\tilde{L}_{\mathrm{impl}}$ the cut bundle induced by L_{impl}. (See [15] for symplectic cutting and [18, Section 5.2.1] for symplectic cutting with respect to a polytope.)

Although $\tilde{M}_{\mathrm{impl}}$ is defined as a quotient of a singular object, observe that the fibre over λ_0 of the T-moment map on $M_{\mathrm{impl}} \times X[\tau]$ misses the singularities of M_{impl}. Recall from Theorem 2.10 that the top stratum of M_{impl} is isomorphic as a Hamiltonian T-manifold to the principal cross-section M_τ. Thus we have in fact

$$\tilde{M}_{\mathrm{impl}} = (M_\tau \times X[\tau]) /\!\!/_{\lambda_0} T \quad \text{and} \quad \tilde{L}_{\mathrm{impl}} = (L|_{M_\tau} \boxtimes \mathbb{C}) /\!\!/_{\lambda_0} T.$$

This shows that for generic values of λ_0, $\tilde{M}_{\mathrm{impl}}$ is a Hamiltonian T-orbifold with moment map $\tilde{\Phi}_{\mathrm{impl}}$, whose image is equal to $\Phi(M) \cap (\lambda_0 + \bar{\tau})$. The subset $\tilde{\Phi}^{-1}_{\mathrm{impl}}(\lambda_0 + \tau)$ is a dense open submanifold, which is isomorphic as a Hamiltonian T-manifold to the open submanifold $\Phi^{-1}_{\mathrm{impl}}(\lambda_0 + \tau)$ of the top stratum $\Phi^{-1}_{\mathrm{impl}}(\tau)$ of M_{impl}. Thus, as λ_0 tends to 0, this open set approaches the top stratum of M_{impl}. It is in this sense that $\tilde{M}_{\mathrm{impl}}$ is a partial desingularization of M_{impl}, similar to Kirwan's partial desingularization [13] of a symplectic quotient. Although there is no canonical "blow-down" map $\tilde{M}_{\mathrm{impl}} \to M_{\mathrm{impl}}$, we shall see below in what way $\tilde{M}_{\mathrm{impl}}$ is a conventional desingularization of M_{impl}.

We *define* the quantization of M_{impl} to be the T-equivariant index of $\tilde{M}_{\mathrm{impl}}$ with coefficients in $\tilde{L}_{\mathrm{impl}}$. In other words,

$$\mathrm{RR}(M_{\mathrm{impl}}, L_{\mathrm{impl}}) = \mathrm{RR}(\tilde{M}_{\mathrm{impl}}, \tilde{L}_{\mathrm{impl}}). \tag{7.2}$$

Now let Ind denote the holomorphic induction functor.

7.3. Theorem. *Let τ be the principal face of M and let $\lambda_0 \in \tau$ be a sufficiently small generic element. Then* $\mathrm{RR}(M, L) = \mathrm{Ind}_T^K \mathrm{RR}(M_{\mathrm{impl}}, L_{\mathrm{impl}})$. *Hence quantization commutes with implosion in the sense that* $\mathrm{RR}(M_{\mathrm{impl}}, L_{\mathrm{impl}}) = \mathrm{RR}(M, L)^N$.

Proof. The first assertion follows immediately from [18, Theorem 6.8] and the definition (7.2). Taking the N-invariant parts of both sides we get

$$\mathrm{RR}(M, L)^N = (\mathrm{Ind}_T^K V)^N \tag{7.4}$$

as virtual characters of T, where $V = \mathrm{RR}(\tilde{M}_{\mathrm{impl}}, \tilde{L}_{\mathrm{impl}})$. By construction the moment polytope of $\tilde{M}_{\mathrm{impl}}$ lies within the fundamental chamber $\mathfrak{t}^*_+$, so, by the quantization commutes with reduction theorem [18, Theorem 2.9], the weights occurring in V are all dominant. Let λ be such a weight and $\mathbb{C}_\lambda$ the representation of T with weight λ. By the Borel–Weil–Bott Theorem $\mathrm{Ind}_T^K \mathbb{C}_\lambda \cong V_\lambda$, the irreducible representation with highest weight λ. Hence $(\mathrm{Ind}_T^K \mathbb{C}_\lambda)^N \cong \mathbb{C}_\lambda$, since only the highest weight vector v_λ is invariant under N. We conclude that $(\mathrm{Ind}_T^K V)^N \cong V$. Together with (7.4) this proves the second assertion. ☐

7.5. Example. Taking $M = T^*K$ we find that $\mathrm{RR}((T^*K)_{\mathrm{impl}}, L_{\mathrm{impl}}) = \mathrm{RR}(T^*K, L)^N$, which by the Peter–Weyl Theorem is equal to the sum of the V_λ over all dominant weights λ. Thus $(T^*K)_{\mathrm{impl}}$ is a *model* for K in the sense that every irreducible module occurs in its quantization exactly once. This application of our theorem is of course illegal, because T^*K is not compact, but the conclusion appears correct and it would be of some interest to justify it directly. Cf. also [4, 5], where it is proved that the Kähler quantization of the stratum of $(T^*K)_{\mathrm{impl}}$ corresponding to a face σ is the Hilbert direct sum of the V_λ over all λ in σ.

Now assume that M carries a K-invariant compatible complex structure. Then M is a Kähler manifold and Theorem 6.21 says that its imploded cross-section is a Kähler space. It follows from (7.1) that the orbifold $\tilde{M}_{\mathrm{impl}}$ is Kähler as well. Following Kirwan [13] we call a *partial desingularization* of an analytic space X any analytic orbifold $\tilde{X}$ such that there exists a proper surjective bimeromorphic map $\tilde{X} \to X$.

7.6. Theorem. *Let M be a compact Kähler Hamiltonian K-manifold. For sufficiently small generic values of λ_0, $\tilde{M}_{\mathrm{impl}}$ is a partial desingularization of M_{impl}.*

Proof. Let X be the Kähler space $M_{\mathrm{impl}} \times X[\tau]$, which is equipped with a Hamiltonian T-action. Denote the T-moment map on X by Ψ. We will show that for all sufficiently small $\mu \in \Psi(X)$ there exists a bimeromorphic map $X/\!\!/_\mu T \to X/\!\!/_0 T$. (Properness and surjectivity are then immediate from the compactness of $X/\!\!/_\mu T$ and the irreducibility of $X/\!\!/_0 T$.) This is well known in the algebraic category. Let us briefly indicate how the argument carries over to the analytic category thanks to results of Heinzner and Huckleberry [8].

Let $H = T^{\mathbb{C}}$. The set of *μ-semistable points* is

$$X_\mu^{\mathrm{ss}} = \{\, x \in X \mid \overline{Hx} \text{ intersects } \Psi^{-1}(\mu) \,\}.$$

It is open and dense if nonempty ([8, §9]). Two points in X_μ^{ss} are *equivalent* under the H-action if their orbit closures intersect in X_μ^{ss}. For every $x \in X_\mu^{\mathrm{ss}}$ there is a unique $y \in X_\mu^{\mathrm{ss}}$ such that Hy is closed in X_μ^{ss} and y is in the closure of Hx. This implies that the inclusion $\Psi^{-1}(\mu) \to X_\mu^{\mathrm{ss}}$ induces a homeomorphism $X/\!\!/_\mu T \to X_\mu^{\mathrm{ss}}/\sim$. (These assertions follow from the holomorphic slice theorem, [9, §2.7] or [8, §0].) A μ-semistable point is *μ-stable* if Hx is closed in X_μ^{ss} and H_x is finite. The set of stable points is denoted X_μ^{s}. It too is open and dense if nonempty. A point $x \in \Psi^{-1}(\mu)$ is stable if and only if T_x is finite. (These facts follow also from the holomorphic slice theorem.) The last fact we need is a generalization of Atiyah's result [2] that for every $x \in X$ the image $\Psi(\overline{Hx})$ is the convex hull in $\mathfrak{t}^*$ of the H-fixed points contained in $\overline{Hx}$. Furthermore, $\Psi(\overline{Hx})$ is equal to the full image $\Psi(X)$ for all x in an open dense subset X°. The convexity is proved in [8]. For the set X° we can take $X_{\mu_1}^{\mathrm{ss}} \cap X_{\mu_2}^{\mathrm{ss}} \cap \cdots \cap X_{\mu_s}^{\mathrm{ss}}$, where $\mu_1, \mu_2, \dots, \mu_s$ are the vertices of $\Psi(X)$.

Take $\mu \in \Psi(X)$ so small that $\Psi^{-1}(\mu)$ is contained in X_0^{ss}. Then $X_\mu^{\mathrm{ss}} \subseteq X_0^{\mathrm{ss}}$, and this inclusion induces an analytic map $X/\!\!/_\mu T \cong X_\mu^{\mathrm{ss}}/\!\sim\, \to X/\!\!/_0 T \cong X_0^{\mathrm{ss}}/\!\sim$. To see that this map is bimeromorphic, observe that the stable set X_0^{s} is nonempty since 0 is a regular value of the T-moment map on the manifold $M_\tau \times X[\tau]$. Let $Y = X_0^{\mathrm{s}} \cap X^\circ$. Then the image of Y in $X/\!\!/_0 T$ is open and dense and for $x \in Y$ we have $\mu \in \Psi(\overline{Gx})$, i.e., $Y \subseteq X_\mu^{\mathrm{ss}}$. Thus we obtain an analytic map $Y/\!\sim\, \to X/\!\!/_\mu T$ which inverts the previously defined map $X/\!\!/_\mu T \to X_0^{\mathrm{ss}}/\!\sim$ over an open dense set. □

There is a more illuminating construction of this partial desingularization for the universal imploded cross-section $(T^*K)_{\mathrm{impl}}$. Assume that K is semisimple and simply connected. In Proposition 6.8 we identified $(T^*K)_{\mathrm{impl}}$ with the algebraic variety G_N. We can characterize its desingularization $(T^*K)_{\mathrm{impl}}^{\sim}$ in a similar manner. Let $\tilde{G}_N$ be the homogeneous vector bundle $G \times^B E^N$ over the flag variety G/B with fibre E^N. Here $G = K^{\mathbb{C}}$ and E is as in (6.1). The multiplication map $G \times E^N \to E$ induces a proper morphism $p \colon \tilde{G}_N \to E$.

7.7. Proposition. *Suppose that K is semisimple and simply connected. Let $\lambda_0 \in \mathfrak{t}^*$ be regular dominant and let ω_0 be the Kähler form on G/B obtained by identifying G/B with the coadjoint K-orbit through λ_0. Let $q \colon \tilde{G}_N \to G/B$ be the bundle projection and put $\tilde{\omega}_0 = p^*\omega_E + q^*\omega_0$.*

(i) $\tilde{G}_N$ *is an equivariant desingularization of* G_N.
(ii) $\tilde{\omega}_0$ *is a Kähler form on* $\tilde{G}_N$. *It is integral if* λ_0 *is.*
(iii) $(T^*K)^{\sim}_{\mathrm{impl}}$ *is a smooth manifold and is isomorphic as a Hamiltonian* K*-manifold to* $\tilde{G}_N$.

Proof. The image of the map p is the subvariety G_N of E and we can therefore regard it as a proper morphism $\tilde{G}_N \to G_N$. The G-orbits in $\tilde{G}_N$ are in natural one-to-one correspondence with the B-orbits in E^N, which are identical to the $T^{\mathbb{C}}$-orbits in E^N. Each B-orbit in E^N passes through a unique point of the form v_σ, so each G-orbit in $\tilde{G}_N$ passes through a unique point of the form $[1, v_\sigma]$. (Here points in $\tilde{G}_N$ are written as $[g, v]$ with $g \in G$ and $v \in E^N$.) The stabilizer of $[1, v_\sigma]$ for the G-action is $G_{[1,v_\sigma]} = B_{v_\sigma} = B \cap [P_\sigma, P_\sigma]$, where the second equality follows from Lemma 6.2. Thus the fibre $p^{-1}(v_\sigma)$ is the flag variety $[P_\sigma, P_\sigma]/(B \cap [P_\sigma, P_\sigma])$. In particular, $\tilde{G}_N$ contains a Zariski-open orbit of type G/N, namely the orbit through $[1, v_\tau]$, where τ is the top face of $\mathfrak{t}^*_+$. Hence p is birational, which proves (i).

If λ_0 is integral, then ω_0 is integral on G/B. Since ω_E is exact, this implies that $\tilde{\omega}_0$ is integral. Furthermore, being a sum of pullbacks of two Kähler forms, $\tilde{\omega}_0$ is positive semidefinite. To prove that it is Kähler, it is therefore enough to show that it is non-degenerate. We shall do this by showing that $\tilde{\omega}_0$ pulls back to the symplectic form on $(T^*K)^{\sim}_{\mathrm{impl}}$ under a suitable diffeomorphism.

The principal face τ of T^*K is the top face of $\mathfrak{t}^*_+$ and its principal cut (for the right K-action) is $K \times \tau$. We noted in Remark 6.7 that the toric manifold associated to the polyhedral cone $\mathfrak{t}^*_+$ is the symplectic vector space E^N, so by (7.1) the partial desingularization of $(T^*K)_{\mathrm{impl}}$ is $(K \times \tau \times E^N)/\!\!/_{\lambda_0} T$. To see that this space is actually a manifold rather than an orbifold, observe that the moment map for the T-action on the product $K \times \tau \times E^N$ is given by $\Psi(k, \lambda, v) = \lambda + \Phi_{E^N}(v)$, where Φ_{E^N} is the T-moment map on E^N. The map $K \times E^N \to K \times \tau \times E^N$ which sends (k, v) to $(k, \lambda_0 - \Phi_{E^N}(v), v)$ is a $K \times T$-equivariant diffeomorphism onto $\Psi^{-1}(\lambda_0)$. It therefore descends to a K-equivariant diffeomorphism

$$K \times^T E^N \to (K \times \tau \times E^N)/\!\!/_{\lambda_0} T = (T^*K)^{\sim}_{\mathrm{impl}}, \tag{7.8}$$

which shows that $(T^*K)^{\sim}_{\mathrm{impl}}$ is smooth. Moreover, the inclusion map $K \to G$ induces a diffeomorphism $K \times^T E^N \to G \times^B E^N = \tilde{G}_N$. Composing with the inverse of the map (7.8) we obtain a diffeomorphism $\tilde{\mathcal{F}} \colon (T^*K)^{\sim}_{\mathrm{impl}} \to \tilde{G}_N$. To finish the proof of (ii) and (iii) we must show that $\tilde{\mathcal{F}}^*\tilde{\omega}_0$ is the symplectic form on $(T^*K)^{\sim}_{\mathrm{impl}}$. Recall that the symplectic cut $(T^*K)^{\sim}_{\mathrm{impl}}$ contains a copy of $K \times (\lambda_0 + \tau)$ as an open dense submanifold. The symplectic form on this subset is the form ω_τ of Lemma 4.6 and the embedding $\mathcal{I}_0 \colon (K \times (\lambda_0 + \tau)) \to (T^*K)^{\sim}_{\mathrm{impl}}$ is given by $\mathcal{I}_0(k, \lambda) = [k, \lambda, s_0(\lambda)] \in \Psi^{-1}(\lambda_0)/T \subseteq (K \times \tau \times E^N)/T$. Here $s_0 \colon \lambda_0 + \mathfrak{t}^*_+ \to E^N$ is any section of the map $-\Phi_{E^N} + \lambda_0$, such as for example $s_0(\lambda) = \frac{1}{\sqrt{\pi}} \sum_{p=1}^r \sqrt{(\lambda - \lambda_0)(\alpha_p^\vee)}\, v_p$. Let us denote the open embedding $\tilde{\mathcal{F}} \circ \mathcal{I}_0 \colon K \times (\lambda_0 + \tau) \to \tilde{G}_N$ by $\tilde{\mathcal{F}}_0$. We need to show that

$$\tilde{\mathcal{F}}_0^* \tilde{\omega}_0 = \omega_\tau. \tag{7.9}$$

It suffices to check this identity at points of the form $(1, \lambda)$ with $\lambda \in \lambda_0 + \tau$. For (ξ_1, μ_1) and (ξ_2, μ_2) in $T_{(1,\lambda)}(K \times (\lambda_0 + \tau)) \cong \mathfrak{k} \times \mathfrak{t}^*$ one readily checks that

$$(\omega_\tau)_{(1,\lambda)}\big((\xi_1, \mu_1) \wedge (\xi_2, \mu_2)\big) = \mu_1(\xi_2) - \mu_2(\xi_1) - \lambda([\xi_1, \xi_2]). \tag{7.10}$$

On the other hand, $\tilde{\mathcal{F}}_0^*\tilde{\omega}_0 = (p\circ\tilde{\mathcal{F}}_0)^*\omega_E + (q\circ\tilde{\mathcal{F}}_0)^*\omega_{\lambda_0}$. Now $q\circ\tilde{\mathcal{F}}_0(k,\lambda) = \bar{k}$, where $\bar{k}\in K/T = G/B$ denotes the coset of $k\in K$, so

$$((q\circ\tilde{\mathcal{F}}_0)^*\omega_{\lambda_0})_{(1,\lambda)}((\xi_1,\mu_1)\wedge(\xi_2,\mu_2)) = (\omega_{\lambda_0})_{\bar{1}}(\bar{\xi}_1\wedge\bar{\xi}_2) = -\lambda_0([\xi_1,\xi_2]). \tag{7.11}$$

A computation as in the proof of Proposition 6.8 yields

$$((p\circ\tilde{\mathcal{F}}_0)^*\omega_{\lambda_0})_{(1,\lambda)}((\xi_1,\mu_1)\wedge(\xi_2,\mu_2)) = \mu_1(\xi_2)-\mu_2(\xi_1)+\omega_E\big(\xi_{1,E}(s_0(\lambda)),\xi_{2,E}(s_0(\lambda))\big),$$

where

$$\begin{aligned}\omega_E\big(\xi_{1,E}(s_0(\lambda)),\xi_{2,E}(s_0(\lambda))\big) &= \big\{\Phi_E^{\xi_1},\Phi_E^{\xi_2}\big\}(s_0(\lambda)) = \Phi_E^{[\xi_1,\xi_2]}(s_0(\lambda))\\ &= -\frac{1}{2}\operatorname{Im}\big\langle[\xi_1,\xi_2](s_0(\lambda)),s_0(\lambda)\big\rangle\\ &= -\frac{1}{2\pi}\operatorname{Im}2\pi i\sum_{p=1}^{r}(\lambda-\lambda_0)(\alpha_p^\vee)\,\varpi([\xi_1,\xi_2])\\ &= (\lambda_0-\lambda)([\xi_1,\xi_2]).\end{aligned}$$

Combining this with (7.11) gives $(\tilde{\mathcal{F}}_0^*\tilde{\omega}_0)_{(1,\lambda)}((\xi_1,\mu_1)\wedge(\xi_2,\mu_2)) = \mu_1(\xi_2)-\mu_2(\xi_1)-\lambda([\xi_1,\xi_2])$, which together with (7.10) proves (7.9). □

Returning to the case of a Kähler Hamiltonian K-manifold M, let us denote its principal face by τ and let us assume for simplicity that τ is the top face of $\mathfrak{t}_+^*$. By Lemma 2.4 we may assume that K is semisimple and simply connected. Using Theorem 4.9, Proposition 6.8 and reduction in stages we see that

$$\begin{aligned}\tilde{M}_{\mathrm{impl}} = (M_{\mathrm{impl}}\times X[\tau])/\!\!/_{\lambda_0}T &\cong \big((M\times G_N)/\!\!/_0K\times X[\tau]\big)/\!\!/_{\lambda_0}T\\ &\cong \big(M\times(G_N\times X[\tau])/\!\!/_{\lambda_0}T\big)/\!\!/_0K\\ &\cong \big(M\times\tilde{G}_N\big)/\!\!/_0K,\end{aligned}\tag{7.12}$$

because τ is also the principal face of G_N. Thus the desingularization of G_N plays a universal role analogous to that of G_N itself. By Proposition 7.7(i) the analytic map

$$p_M = \mathrm{id}_M\times p\colon M\times\tilde{G}_N\to M\times G_N,$$

is a G-equivariant desingularization. The following is now clear from Theorem 7.6.

7.13. Corollary. *Let M be a compact Kähler Hamiltonian K-manifold. Assume that the principal face of M is the top face of $\mathfrak{t}_+^*$. Then for small generic λ_0 the map p_M induces a map $\tilde{M}_{\mathrm{impl}}\to M_{\mathrm{impl}}$ which is identical to the partial desingularization of Theorem* 7.6.

We mention without proof that a similar result holds if the principal face τ is not the top face. In this case the toric manifold $X[\tau]$ is not E^N, but the smaller symplectic vector space $E^{[P_\tau,P_\tau]}$, where P_τ is the parabolic subgroup of G associated with τ. In Proposition 7.7 and in (7.12) one needs to replace G_N by the closure of the stratum $G/[P_\tau,P_\tau]$ and $\tilde{G}_N$ by the homogeneous bundle $G\times^{P_\tau}E^{[P_\tau,P_\tau]}$ over the partial flag variety G/P_τ.

8. Notation index

K; T	compact connected Lie group; maximal torus, § 2,
Z	identity component of centre, § 2,
R; R_+; S	root system; positive roots; simple roots, § 5,
G	complexification of K, § 6,
N; B	maximal unipotent subgroup; Borel subgroup, § 6,
$\mathfrak{t}^*_+$; Σ	closed Weyl chamber in $\mathfrak{t}^*$; set of open faces of $\mathfrak{t}^*_+$, § 2,
Λ^*; Λ^*_+	weight lattice in $\mathfrak{t}^*$; set of dominant weights, § 6,
K_σ; R_σ	centralizer of face $\sigma \in \Sigma$; root system of K_σ, § 2,
P_σ; U_σ	parabolic associated to σ; unipotent radical of P_σ, § 6,
G_σ; N_σ	Levi factor of P_σ; maximal unipotent subgroup of G_σ, § 6,
$\mathfrak{S}_\sigma$	standard slice at σ for coadjoint action, § 2,
X_N	variety with coordinate ring $\mathbb{C}[X]^N$, § 6,
(M, ω); Φ	Hamiltonian K-manifold; moment map, § 2,
$M/\!\!/_\lambda K$; $M/\!\!/K$	symplectic quotient $\Phi^{-1}(\lambda)/K_\lambda$; same with $\lambda = 0$, § 2,
M_σ; σ_{prin}	cross-section $\Phi^{-1}(\mathfrak{S}_\sigma)$; principal face of M, § 2,
$\sim$; M_{impl}	equivalence relation on $\Phi^{-1}(\mathfrak{t}^*_+)$; imploded cross-section, § 2,
π; Φ_{impl}	quotient map $\Phi^{-1}(\mathfrak{t}^*_+) \to M_{\mathrm{impl}}$; imploded moment map, § 2,
$\mathcal{L}$; $\mathcal{R}$	left; resp. right action of K on itself or on T^*K, § 4,
$\pi_{\mathcal{R}}$	quotient map $\Phi_{\mathcal{R}}^{-1}(\mathfrak{t}^*_+) \to (T^*K)_{\mathrm{impl}}$, § 4,
$\mathrm{RR}(M, L)$	equivariant index of M with coefficients in L, § 7,
$C^\circ(X)$	infinite cone $\big(X \times [0, \infty)\big)/\big(X \times \{0\}\big)$, § 4,
W^ω	skew complement in symplectic vector space, § 5,
V^H	fixed-point set in H-module, § 5.

References

[1] В. И. Арнольд, *Математические методы классической механики*, Наука, М., 1989. Engl. transl.: V. Arnold, *Mathematical Methods of Classical Mechanics*, corrected reprint of second Engl. (1989) ed., Graduate Texts in Math., vol. 60, Springer-Verlag, New York, 1990, transl. from the Russian by K. Vogtmann and A. Weinstein.

[2] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.

[3] I. Bernstein, I. Gelfand, S. Gelfand, *Differential operators on the base affine space and a study of $\mathfrak{g}$-modules*, in: *Lie Groups and their Representations* (Budapest, 1971) (I. Gelfand, ed.), Bolyai János Math. Soc., Halsted, New York, 1975, pp. 21–64.

[4] M.–K. Chuah, *Kaehler structures on $K_{\mathbf{C}}/(P, P)$*, Trans. Amer. Math. Soc. **349** (1997), no. 8, 3373–3390.

[5] M.–K. Chuah, V. Guillemin, *Kaehler structures on $K_{\mathbf{C}}/N$*, in: *The Penrose Transform and Analytic Cohomology in Representation Theory* (South Hadley, MA, 1992) (M. Eastwood et al., eds.), Contemp. Math., vol. 154, Amer. Math. Soc., Prov., RI, 1993, pp. 181–195.

[6] F. Grosshans, *Observable groups and Hilbert's fourteenth problem*, Am. J. Math. **95** (1973), no. 1, 229–253.

[7] V. Guillemin, E. Lerman, S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge Univ. Press, Cambridge, 1996.

[8] P. Heinzner, A. Huckleberry, *Kählerian potentials and convexity properties of the moment map*, Invent. Math. **126** (1996), no. 1, 65–84.

[9] P. Heinzner, F. Loose, *Reduction of complex Hamiltonian G-spaces*, Geom. Funct. Anal. **4** (1994), no. 3, 288–297.

[10] J. Hurtubise, L. Jeffrey, *Representations with weighted frames and framed parabolic bundles*, Canad. J. Math. **52** (2000), 1235–1268.

[11] L. Jeffrey, F. Kirwan, *Localization and the quantization conjecture*, Topology **36** (1997), no. 3, 647–693.

[12] G. Kempf, L. Ness, *The length of vectors in representation spaces*, in: *Algebraic Geometry* (Copenhagen, 1978) (K. Lønsted, ed.), Lect. Notes Math., vol. 732, Springer-Verlag, Berlin, 1979, pp. 233–243.

[13] F. Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. (2) **122** (1985), no. 1, 41–85.

[14] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, second revised ed., Aspekte der Mathematik, vol. D1, Vieweg, Braunschweig, 1985. Russ. transl.: Х. Крафт, *Геометрические методы в теории инвариантов*, Мир, М., 1987.

[15] E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), no. 3, 247–258.

[16] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Nonabelian convexity by symplectic cuts*, Topology **37** (1998), no. 2, 245–259.

[17] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Math. Monographs, Oxford Univ. Press, New York, 1995.

[18] E. Meinrenken, R. Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762.

[19] F. Pauer, *Glatte Einbettungen von G/U*, Math. Ann. **262** (1983), no. 3, 421–429.

[20] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), 375–386.

[21] G. Schwarz, *The topology of algebraic quotients*, in: *Topological Methods in Algebraic Transformation Groups* (Rutgers Univ., 1988) (H. Kraft et al., eds.), Progr. in Math., vol. 80, Birkhäuser, Boston, 1989, pp. 135–151.

[22] R. Sjamaar, E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. (2) **134** (1991), no. 2, 375–422.

[23] Э. Б. Винберг, В. Л. Попов, *Об одном классе квазиоднородных аффинных многообразий*, Изв. АН СССР, сер. мат. **36** (1972), 749–764. Engl. transl.: È. B. Vinberg, V. L. Popov, *On a class of quasihomogeneous affine varieties*, Math. USSR Izv. **6** (1972), 743–758.