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THE GERBE OF HIGGS BUNDLES

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Abstract. The purpose of this work is to describe the (category of) Higgs bundles on a scheme $X/\mathbb{C}$ having a given cameral cover $\widetilde{X}$. We show that this category is a $T_{\widetilde{X}}$-gerbe, where $T_{\widetilde{X}}$ is a certain sheaf of abelian groups on X, and we describe the class of this gerbe precisely. In particular, it follows that the set of isomorphism classes of Higgs bundles with a fixed cameral cover $\widetilde{X}$ is a torsor over the group $H^1(X, T_{\widetilde{X}})$, which itself parametrizes $T_{\widetilde{X}}$-torsors on X. This underlying group $H^1(X, T_{\widetilde{X}})$ can be described as a generalized Prym variety, whose connected component is either an abelian variety or a degenerate abelian variety. The main part of the work deals with abstract Higgs bundles; in the last two sections we derive the applications to Higgs bundles valued in a line bundle K and to bundles on elliptic fibrations.

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0. Introduction

The purpose of this work is to describe the (category of) Higgs bundles on a scheme $X/\mathbb{C}$ having a given cameral cover $\widetilde{X}$. We show that this category is a $T_{\widetilde{X}}$-gerbe, where $T_{\widetilde{X}}$ is a certain sheaf of abelian groups on X, and we describe the class of this gerbe precisely. In particular, it follows that the set of isomorphism classes of Higgs bundles with a fixed cameral cover $\widetilde{X}$ is a torsor over the group $H^1(X, T_{\widetilde{X}})$, which itself parametrizes $T_{\widetilde{X}}$-torsors on X. This underlying group $H^1(X, T_{\widetilde{X}})$ can be described as a generalized Prym variety, whose connected component is either an abelian variety or a degenerate abelian variety.

The hardest part of our work goes into identifying precisely the $H^1(X, T_{\widetilde{X}})$-torsor we get, or in other words, identifying the class of the gerbe. This class is surprisingly complicated. One piece of it can be identified as a twist along the ramification divisors of $\widetilde{X}$ over X, and is present for all groups G. A second piece is a shift which can be present even for unramified covers. While the twist along the ramification expresses properties of the cameral cover, this shift expresses the non-vanishing of a certain group cohomology element, specifically, the extension class $[N]$ of the normalizer $N = N_G(T)$, which is an element in the cohomology group $H^2(W, T)$ of the Weyl group acting on the maximal torus. It vanishes for some groups, such as $\mathrm{GL}(n), \mathrm{PGL}(n), \mathrm{SL}(2n+1), \mathrm{SO}(n)$, but not for others such as $\mathrm{SL}(2n)$. Yet a third piece is present only for the groups $\mathrm{SO}(2n+1)$ (or groups containing them as direct factors); this piece expresses the existence of non-primitive coroots, which amounts to the non-vanishing of an element in another cohomology group. We give several examples to illustrate these individual ingredients as well as their combined effect.

Throughout this work, we let G be a connected reductive group, and let X be a scheme over the complex numbers. A Higgs bundle over X is a principal G-bundle plus some additional data. We describe this additional data next: first for $G = \mathrm{GL}(n)$, and then for all G, in subsection 0.1. In the remainder of this introduction we will outline our results 0.2, discuss some examples and applications 0.3, and review some related results in the literature 0.4. The notation we employ is summarized in 0.5.

0.1. Abelianization: Higgs bundles and cameral covers. It is especially easy to spell out the definition when $G = \mathrm{GL}(n)$. In this case a G-bundle is the same as a vector bundle E over X, and a Higgs structure on it is a subbundle of commutative associative algebras $\mathbf{c}_X \subset \mathrm{End}_{\mathcal{O}_X}(E)$, which has rank n over X and such that $\mathbf{c}_X$ is locally generated by one section. In this case the relative spectrum of $\mathbf{c}_X$ over X is a flat n-sheeted cover of X, called the *spectral cover* corresponding to our Higgs bundle. We will denote it by $\overline{X}$.

How can we classify Higgs bundles with a given spectral cover $\overline{X}$? The answer is simple: these are in bijection with line bundles on $\overline{X}$. Thus, by asking not just for principal G-bundles, but rather for G-bundles endowed with a Higgs structure with a fixed spectral cover, we go from a *non-abelian* problem to an *abelian* one.

The natural question now is how to extend the above discussion to other reductive groups. It turns out that the notion of an abstract Higgs bundle is quite easy to generalize. Namely, a Higgs bundle is a pair $(E_G, \mathbf{c}_X)$, where E_G is a principal G-bundle over X and $\mathbf{c}_X$ is a subbundle of the associated bundle of Lie algebras $\mathfrak{g}_{E_G}$, whose fibers are *regular centralizers*. The precise definition is given in Section 2. Here we only recall that a regular centralizer in the Lie algebra $\mathfrak{g}$ is an abelian subalgebra $\mathbf{c} \subset \mathfrak{g}$ which is the centralizer of some regular (but not necessarily semisimple) element $g \in \mathfrak{g}$. In particular, taking g to be regular semisimple, we see that every Cartan subalgebra (i.e., the Lie algebra of a maximal torus) is a regular centralizer. In fact, we will see in Section 1 that the set of regular centralizers in $\mathfrak{g}$ is parametrized by an algebraic variety $\overline{G/N}$ which is a partial compactification of the parameter space G/N for the maximal tori. The simplest Higgs bundles are the *unramified* ones, i.e., Higgs bundles $(E_G, \mathbf{c}_X)$ for which all the fibers of $\mathbf{c}_X$ are maximal tori.

The situation is less transparent with spectral covers. In fact, we do not know a good definition of a spectral cover that would work for any G and reproduce for $\mathrm{GL}(n)$ the old object.

Instead, we use the notion of a *cameral cover* introduced in [8]. By definition, the latter is a finite flat map $p : \widetilde{X} \to X$ such that the Weyl group W of G acts on $\widetilde{X}$ and certain restrictions on the ramification behaviour are satisfied (cf. Section 2). When $G = \mathrm{GL}(n)$, we will note below that this notion is *different* from that of a spectral cover, though equivalent to it.

It turns out that every Higgs bundle determines in a canonical way a cameral cover, so one is led naturally to the problem of classification of Higgs bundles with a given cameral cover. This is the problem we solve in the present paper. Given a cameral cover $\widetilde{X}$, we will describe the corresponding Higgs bundles in terms of the "abelian" data consisting of the maximal torus $T \subset G$, the W-action on T, and the ramification pattern of $\widetilde{X}$ over X. The "non-abelian" data involving the group G itself is not needed.

0.2. Outline of the results. We formulate the above classification problem in the categorical framework, in terms of the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ of Higgs bundles together with an isomorphism between the induced cameral cover and $\widetilde{X}$. Our first result shows that this classification problem is indeed abelian.

Namely, starting from $\widetilde{X}$ we define a sheaf of abelian groups $T_{\widetilde{X}}$. We assert in Theorem 4.4 that $\mathrm{Higgs}_{\widetilde{X}}(X)$ is a *gerbe* bound by the Picard category of $T_{\widetilde{X}}$-torsors. (These notions are reviewed for the reader's convenience in Section 3.) This result has two immediate consequences.

First, the set of isomorphism classes of objects in our category $\mathrm{Higgs}_{\widetilde{X}}(X)$, i.e., the set of isomorphism classes of Higgs bundles with the given cameral cover $\widetilde{X}$, if non-empty, carries a simply transitive action of the abelian group $H^1(X, T_{\widetilde{X}})$ (Corollary 4.6), and is therefore non-canonically isomorphic to it. It is thus a generalized Prym variety; cf. [9]: depending on the circumstances, this may appear as a Jacobian of a spectral curve, or as an ordinary Prym, or as various types of Prym–Tyurin varieties [21], and so on.

The second consequence allows us to determine when Higgs bundles with the given cameral cover $\widetilde{X}$ actually exist. This happens if and only if the gerbe is trivial: the cameral cover $\widetilde{X}$ determines an obstruction class in $H^2(X, T_{\widetilde{X}})$, and Higgs bundles with the given $\widetilde{X}$ exist if and only if this class vanishes (Corollary 4.5).

In the above, the sheaf $T_{\widetilde{X}}$ is defined in terms of the slightly larger sheaf $\overline{T}_{\widetilde{X}}$ (on X) of W-equivariant maps $\widetilde{X} \to T$, i.e., $\overline{T}_{\widetilde{X}}(U) := \mathrm{Mor}_W(\widetilde{U}, T)$, where $\widetilde{U}$ is the induced cameral cover of U. For each positive root $\alpha : T \to \mathbb{G}_m$, let s_α be the corresponding reflection acting on $\widetilde{X}$, and let $D^\alpha_{\widetilde{X}} \subset \widetilde{X}$ be its fixed point scheme. Any section t of $\overline{T}_{\widetilde{X}}(U)$ determines a function $\alpha \circ t : \widetilde{U} \to \mathbb{G}_m$ which goes to its own inverse under the reflection s_α. In particular, its restriction to the ramification locus D^α_X *equals* its inverse, so it equals ± 1. The subsheaf $T_{\widetilde{X}} \subset \overline{T}_{\widetilde{X}}$ is given by the positive choice:

$$T_{\widetilde{X}}(U) := \{t \in \overline{T}_{\widetilde{X}}(U) \mid (\alpha \circ t)|_{D^\alpha_U} = +1 \text{ for each root } \alpha\}.$$

Although Theorem 4.4 is quite useful, it is not a completely satisfactory result by itself, as it does not describe which $T_{\widetilde{X}}$-gerbe we get. Our main result, Theorem 6.4, gives a complete description of the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ as the gerbe parametrizing certain "R-twisted, N-shifted W-equivariant T-bundles on $\widetilde{X}$". The "twist" here is along the ramification divisors, and the "shift" is by the extension class of the normalizer N.

Our description of this gerbe is based on an explicit description of the underlying Picard category $\mathrm{Tors}_{T_{\widetilde{X}}}$ which appears in the statement of Theorem 4.4. An object in this category, i.e., a $T_{\widetilde{X}}$-torsor, consists of:

- a (weakly W-equivariant) T-bundle $\mathcal{L}_0$ on $\widetilde{X}$,
- a group homomorphism $\gamma_0 : N_0 \to \mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$, commuting with the projections to W, and
- for every simple root α_i, a trivialization

$$\beta_{i,0} : \alpha_i(\mathcal{L}_0)|_{D^{\alpha_i}_X} \simeq \mathcal{O}_{D^{\alpha_i}_X}.$$

The data of γ_0 and β_0 must satisfy some compatibility conditions, which are described in detail in Section 16. (Roughly, these say that the collection β_0 of isomorphisms $\beta_{i,0}$ is W-equivariant, and β_0, γ_0 are related by the compatibility condition $\gamma_{0|D^\alpha_X} = \check{\alpha} \circ \beta_0$.) Morphisms in this category are T-bundle maps that are compatible with the data of γ_0 and β_0.

Our notation here is as follows. An element of the group $\mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$, for a T-bundle $\mathcal{L}_0$ on $\widetilde{X}$, consists of an element $w \in W$ together with an isomorphism $w^*(\mathcal{L}_0) \to \mathcal{L}_0$. The bundle $\mathcal{L}_0$ is weakly W-equivariant if $\mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$ surjects onto W, in which case $\mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$ is an extension of W by $\mathrm{Mor}(\widetilde{X}, T)$. Now the semidirect product N_0 of T and W induces one such extension, and γ_0 is supposed to induce an isomorphism of this extension with $\mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$. We think of the root α as a homomorphism $T \to \mathbb{G}_m$, so $\alpha(\mathcal{L}_0)$ is the line bundle associated to $\mathcal{L}_0$ via this homomorphism. Similarly, the coroot $\check{\alpha}$ is a homomorphism $\mathbb{G}_m \to T$.

In describing our gerbe, we replace each linear feature in the description of $\mathrm{Tors}_{T_{\widetilde{X}}}$ by an affine variant. We start with the equivariance: the T-bundles $\mathcal{L}_0$ were weakly W-equivariant (which means that $w^*(\mathcal{L}_0)$ was isomorphic to $\mathcal{L}_0$, for each $w \in W$), and in fact strongly W-equivariant (which simply means that W itself, and hence also the semidirect product N_0, acted on them).

Our variant of the weakly W-equivariant T-bundles $\mathcal{L}_0$ involves T-bundles $\mathcal{L}$ which are R-twisted weakly W-equivariant, meaning that now $w^*(\mathcal{L}) \otimes \mathcal{R}^w_X$ is isomorphic to $\mathcal{L}$, for each $w \in W$. Here $\mathcal{R}^w_X$ is a T-bundle on $\widetilde{X}$ which encodes the ramification pattern of $\widetilde{X}$ over X. In the simplest case, when $\widetilde{X}$ is integral and w is the reflection s_α corresponding to a simple root α, we have $\mathcal{R}^w_X = \mathcal{R}^\alpha_X = \check{\alpha}(R^\alpha_X)$, where R^α_X is the line bundle $\mathcal{O}_{\widetilde{X}}(D^\alpha_X)$. The precise definitions are given in Section 6.

Next, we need a substitute for the strong equivariance. We replace $\mathrm{Aut}(\mathcal{L}_0, \widetilde{X}/X)$ by the group $\mathrm{Aut}_R(\mathcal{L}, \widetilde{X}/X)$ of isomorphisms $w^*(\mathcal{L})\otimes\mathcal{R}^w_X \to \mathcal{L}$, and the semidirect product N_0 by the normalizer N, so we demand that γ should map N to $\mathrm{Aut}_R(\mathcal{L}, \widetilde{X}/X)$.

Finally, β_i needs to be twisted by the ramification, so it now sends $\alpha_i(\mathcal{L})|_{D_X^{\alpha_i}} \to \mathcal{R}^{\alpha_i}|_{D_X^{\alpha_i}}$. One final complication is that β_i now depends (linearly) on the choice of a lift of w_i to an element $n_i \in N$. (This choice of a lift is unnecessary in the linear version, since W is a subgroup of N_0, so the w_i's have a canonical lift.)

We can now give an almost complete statement of our main result, Theorem 6.4. It says that a Higgs bundle with given cameral cover $\widetilde{X}$ is equivalent to:

- an R-twisted, weakly W-equivariant T-bundle $\mathcal{L}$ on $\widetilde{X}$,
- a group homomorphism $\gamma : N \to \mathrm{Aut}_R(\mathcal{L}, \widetilde{X}/X)$, and
- for every simple root α_i and lift $n_i \in N$ of the reflection $s_i \in W$ into N, the data of an isomorphism

$$\beta_i(n_i) : \alpha_i(\mathcal{L})|_{D_X^{\alpha_i}} \simeq \mathcal{R}^{\alpha_i}|_{D_X^{\alpha_i}}.$$

The data of γ and β must satisfy several compatibility conditions, which are described in detail in Section 6. (Roughly, these say that the collection β of isomorphisms $\beta_i(n_i)$ is N-equivariant, and β, γ are related by the compatibility condition $\gamma_{|D_X^{\alpha}} = \check{\alpha} \circ \beta$.) In fact, the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ is equivalent to the category $\mathrm{Higgs}'_{\widetilde{X}}(X)$ whose objects are the triples $(\mathcal{L}, \gamma, \beta)$ as above. Morphisms in this category are again T-bundle maps that are compatible with the data of γ and β.

Note that the possible nontriviality of our gerbe can be attributed to three separate causes: the twist along the ramification R; the shift resulting from nontriviality of the extension class of N; or the extra complication involved in choosing the β_i. In subsection 6.5 we give a simplified version of our theorem, which avoids this last complication. It applies in all cases except when the group G has $\mathrm{SO}(2n+1)$ as a direct summand.

0.3. Some examples and applications

0.3.1. *The unramified case.* The cameral cover $\widetilde{X} \to X$ is unramified if and only if the Higgs bundle $(E_G, \mathbf{c}_X)$ is unramified, i.e., if and only if the bundle of regular centralizers $\mathbf{c}_X$ is actually a bundle of Cartan subalgebras. In this case the classification (given in [9]) is easy: specifying a Higgs bundle $(E_G, \mathbf{c}_X)$ with the unramified cameral cover $\widetilde{X}$ is equivalent to giving an N-bundle E_N over X together with an identification of the quotient E_N/T with $\widetilde{X}$. In this case, our T-bundle $\mathcal{L}$ is just E_N, considered as a T-bundle over $E_N/T = \widetilde{X}$. Since there is no ramification, there is no R-twist; similarly, there is no β; and $\mathrm{Aut}_R(\mathcal{L}, \widetilde{X}/X)$ is just $\mathrm{Aut}(E_N, \widetilde{X}/X)$, which is induced from the extension N, so γ is the tautological map.

0.3.2. $GL(n)$. Consider first the case of $G = \mathrm{GL}(n)$. The spectral cover $\overline{X}$ is then of degree n over X, while the cameral cover $\widetilde{X}$ is of degree $n!$. The n points of $\overline{X}$ above each point x of X correspond to the n simultaneous eigenvectors (in the standard representation) of the corresponding centralizer $\mathbf{c}_x$, while the $n!$ points of $\widetilde{X}$ above x correspond to the ways of ordering these eigenvectors. In a generic situation, e.g., when the Higgs bundle is unramified or only simply ramified, it is clear that $\widetilde{X}$ is precisely the Galois closure of the spectral cover $\overline{X}$. Conversely, $\overline{X}$ is recovered as the quotient of $\widetilde{X}$ by S_{n-1}, the stabilizer in the permutation group $W = S_n$ of one of the n eigenvectors. Following [8], we study the relation between the two types of covers in Section 9. In particular, we show that the above correspondence actually extends to an equivalence between cameral and spectral covers, even when we are very far from the generic situation.

0.3.3. *The universal objects.* The set of all maximal tori $T \subset G$, or equivalently, the set of Cartan subalgebras in $\mathfrak{g}$, is parametrized by the quotient G/N. Over $X = G/N$ we have the tautological, unramified Higgs bundle: the underlying G-bundle is the trivial one, $X \times G$, and the regular centralizers are the universal family of Cartan subgroups. The corresponding (unramified) cameral cover in this case is $G/T \to G/N$. Note that a point of G/T is determined by a Cartan subgroup together with a Borel subgroup containing it.

The cover $G/T \to G/N$ admits a natural partial compactification $\overline{G/T} \to \overline{G/N}$. Here $\overline{G/N}$ paramatrizes *regular centralizers* in the Lie algebra $\mathfrak{g}$, and $\overline{G/T}$ is the ramified W-cover of $\overline{G/N}$ parametrizing pairs consisting of a regular centralizer together with a Borel subgroup containing it; cf. Section 1 and Section 10. The map $\overline{G/T} \to \overline{G/N}$ is the cameral cover of the tautological Higgs bundle on $\overline{G/N}$: the underlying G-bundle is still $\overline{G/N} \times G$, and the regular centralizers form the universal group scheme $\mathcal{C}$ of centralizers over $\overline{G/N}$. We refer to these as universal objects; every Higgs bundle on X is locally the pullback of the tautological one via some map $X \to \overline{G/N}$, and every cameral cover of X is locally the pullback of $\overline{G/T} \to \overline{G/N}$ via the same map $X \to \overline{G/N}$.

Although our ultimate results are concerned with Higgs bundles on arbitrary schemes, much of our work boils down to a group-theoretic analysis of these universal objects $\overline{G/N}$ and $\overline{G/T}$. For instance, we will see that the ramification divisors are indexed by the positive roots α of G. In fact, one of the key points of this paper is that the tautological group scheme $\mathcal{C}$ can be completely recovered by looking at the ramification pattern of $\overline{G/T}$ over $\overline{G/N}$. In a strong sense, this says that a regular centralizer can be recovered from the scheme parametrizing those Borel subgroups which contain it. This is our Theorem 11.6. We emphasize that it is the phenomenon described in Theorem 11.6 which is “responsible” for the abelianization.

0.3.4. $SL(2)$, $PGL(2)$. We saw that in the general case, the final form of the answer is quite involved. A main source of technical difficulties is the possible presence in G of non-primitive coroots (cf. [25]).

From the classification of reductive groups we know that this can occur only when G has $\mathrm{SO}(2n+1)$ as a direct factor. So the simplest case where this extra complication occurs is for $G = \mathrm{SO}(3) = \mathrm{PGL}(2)$. In an attempt to illustrate the effect of these non-primitive coroots, we will, in Section 8, work out explicitly and contrast the examples of $G = \mathrm{SL}(2)$, for which no β's are necessary because all coroots are primitive, versus $G = \mathrm{PGL}(2)$, for which the roots are non-primitive. For these groups, both the spectral cover and the cameral cover are double covers of X, so the entire analysis can be made much more concrete than for a general group. In particular, there are very explicit descriptions of the universal objects $\overline{G/T}, \overline{G/N}, G/T, G/N$; cf. Subsection 1.6.

0.3.5. *K-valued Higgs bundles.* The point of our abstract notion of a Higgs bundle is that it provides a uniform approach to the analysis of various more concrete objects. In the literature, the most common notion of a Higgs bundle is that of a *K-valued Higgs bundle* on X, where K is a fixed line bundle on X. By definition, this means a pair (E_G, s), where E_G is a principal G-bundle on X and s is a section of $\mathfrak{g}_{E_G} \otimes K$. Starting with one of our "abstract" Higgs bundles $(E_G, \mathbf{c}_X)$, we get a K-valued Higgs bundle by choosing a section of $\mathbf{c}_X \otimes K$. Conversely, a K-valued Higgs bundle (E_G, s) on X determines a unique "abstract" Higgs bundle on the open subset $X_0 \subset X$ where s is regular. We say that a K-valued Higgs bundle is *regular* if $X_0 = X$.

Our philosophy is to think of a regular K-valued Higgs bundle as involving two separate pieces of data. The first requires specifying the basis of "eigenvectors" of the Higgs field, i.e., it amounts to specifying the underlying abstract Higgs bundle. The other piece of the data corresponds to the "eigenvalues"; in our case this amounts to specifying the section s of $\mathbf{c}_x \otimes K$. Our point is that this second part of the data is irrelevant for the abelianization process, so we focus on the "eigenvectors" encoded in the abstract Higgs bundle. One obvious advantage of this approach is that it allows the bundle K of "values" to be replaced by various other objects, as we will see below.

A little more generally, we can work with the concept of a *regularized* K-valued Higgs bundle on X, which means a triple $(E_G, \mathbf{c}_X, s)$, with $(E_G, \mathbf{c}_X)$ a Higgs bundle in our abstract sense, and s a (not necessarily regular!) section of $\mathbf{c}_X \otimes K$. The moduli space of regular K-valued Higgs bundles is open in the moduli of all K-valued Higgs bundles (for X projective), and is also open inside the moduli space of regularized K-valued Higgs bundles. For a "general" Higgs bundle, we can expect the complement of X_0 to have codimension 3, so if X is projective of dimension 1 or 2, we expect the open subset of regular Higgs bundles to be nonempty.

In Section 17, we apply our results to show that the algebraic stack $\mathbf{Higgs}(X, K)$ of regularized K-valued Higgs bundles on X fibers over the space $\mathbf{B}(X, K)$ which parametrizes K-valued cameral covers, i.e., pairs $(\widetilde{X}, v)$ where v is a W-equivariant map $v : \widetilde{X} \to \mathfrak{t} \otimes K$ (of schemes over X). The fibers can be identified with the gerbe $Higgs_{\widetilde{X}}(X)$ which we studied in the abstract case. In accordance with our general philosophy, the fiber is independent of the bundle K or the way $\widetilde{X}$ maps to K: it depends only on the abstract cameral cover $\widetilde{X}$.

In case X is a smooth, projective curve and K is its canonical bundle, we thus recover a version of Hitchin's integrable system [20]. (The main difference is that we work with regularized K-valued Higgs bundles while Hitchin uses semistable K-valued Higgs bundles.) As an application, our results can be used to establish a duality between the fibers of the Hitchin map for a group G and those corresponding to its Langlands dual group $\check{G}$.

0.3.6. *Bundles on elliptic fibrations.* Essentially no new phenomena are encountered if we allow our Higgs bundle to take its "values" in a vector bundle K. But we can go further and try to take K to be any abelian group scheme over X, such as the relative Picard scheme of some (projective, integral) family $f : Y \to X$. This leads us in Section 18 to define a *regularized G-bundle on Y* to be the data $(\widetilde{X}, E_G, \mathbf{c}_X)$, with $\widetilde{X} \to X$ a cameral cover of X, and $(E_G, \mathbf{c}_X) \in \mathrm{Higgs}_{\widetilde{Y}}(Y)$ a Higgs bundle on Y with cameral cover $\widetilde{Y} := f^*\widetilde{X}$. This notion is most natural in case f is an *elliptic* fibration since then we know what it means for a bundle (on Y) to be regular above a point (of X). As in the situation for K-valued Higgs bundles, "most" G-bundles on an elliptic curve are indeed regular, and a regular bundle has a unique regularization.

In Theorem 18.5 we apply our results for abstract Higgs bundles to obtain a complete spectral description of regularized G-bundles on Y. In the most interesting case, when f is an elliptic fibration, this is the main result of [10]. Letting $\mathbf{Reg}(X,Y)$ denote the algebraic stack of regularized G-bundles on Y, we obtain a "spectral map" $h : \mathbf{Reg}(X,Y) \to \mathbf{B}(X,Y)$, sending a regularized bundle to its $\mathrm{Pic}(Y/X)$-valued cameral cover, the fibers now being a slightly twisted version of our gerbe $\mathrm{Higgs}_{\widetilde{X}}(X)$.

0.4. Some history. The idea of abelianization has its source in quantum field theory and has been extensively exploited by both physicists and mathematicians. This idea was originally applied not to our notion of an abstract Higgs bundle, but rather to K-valued Higgs bundles. These were considered by Hitchin [20] in case X is a curve and K its canonical bundle. Other line bundles, on $X = P^1$, were considered by Adams, Harnad and Hurtubise [1] and Beauville [2]. Several aspects of spectral covers of P^1 and their Prym–Tyurin varieties were considered by Kanev in [21]. The abelianization of K-valued Higgs bundles on other curves was considered by Beilinson and Kazhdan, Bottacin, Donagi and Markman, Faltings, Markman, and Scognamillo [3, 4, 11, 14, 24, 25], among others. In the case that the base X is a curve, these Higgs bundles are related to representations of the fundamental group of a punctured Riemann surface, as well as to integrable systems arising from loop algebras. The notion of a cameral cover was introduced in [8], where its relation to the various spectral covers was analyzed.

The main point of many of the works cited above is to show, in various interesting special cases, that the fiber of the Hitchin map, i.e., the family of Higgs bundles with given spectral (or cameral) cover, "is" generically a Jacobian or a Prym variety, depending on the group. A description of this fiber in the general setting was announced in [9].

In particular, the generalized Prym was described there as a certain quotient of $H^1(\overline{T}_{\widetilde{X}})$. (This could be off by a finite isogeny: we have seen that the correct description involves $H^1(T_{\widetilde{X}})$.) It was also noted there that the fiber is canonically identified not with the generalized Prym variety itself, but with a certain torsor over it. The class of this torsor was described there in terms of the "twist" arising from the ramification divisor and the "shift" by the class of the normalizer N in $H^2(W,T)$. The additional complication which arises only for $\mathrm{SO}(2n+1)$ was first noted in [25]. This is encoded in the present work in our β's.

Higgs bundles on higher dimensional varieties X, valued in the cotangent bundle $K := T^*X$, were introduced by Simpson [26]. Through work of Corlette and Simpson, their moduli spaces are related to those of local systems on X. The version where K is replaced by an elliptic fibration was developed in [10] and [12]. These elliptically valued Higgs bundles are of interest because of their relevance to the construction and parametrization of bundles on elliptic fibrations. These have attracted attention recently because of their importance to understanding the conjectured duality between F-theory and the heterotic string; cf. [15, 16, 17, 18, 10, 5, 13].

0.5. Notation. We work throughout with a fixed connected reductive group G over $\mathbb{C}$ and we let $\mathfrak{g}$ denote its Lie algebra. We fix a Borel subgroup $B \subset G$ and denote by $\mathcal{F}l$ the flag variety G/B. By definition, $\mathcal{F}l$ classifies Borel subalgebras in $\mathfrak{g}$.

Let U be the unipotent radical of B and T the Cartan quotient B/U; we will fix a splitting $T \to B$. We will denote by $\mathfrak{b}$ and $\mathfrak{t}$ the Lie algebras of B and T, respectively. The rank r of G is by definition the dimension of T. By N we will denote the normalizer of T (not the nilpotent subgroup!), and by W the Weyl group N/T.

The set of positive roots will be denoted by Δ^+. For $\alpha \in \Delta^+$, let $\mathfrak{t}^\alpha \subset \mathfrak{t}$ denote the corresponding root hyperplane and $s_\alpha \in W$ the corresponding reflection. The set of simple roots we will denote by I. For $i \in I$, we will use the notation s_i instead of s_{α_i}.

Part I. Main Results on Higgs Bundles and Cameral Covers

1. Regular centralizers

1.1. Recall that an element $x \in \mathfrak{g}$ is called *regular* if its centralizer $Z_{\mathfrak{g}}(x)$ has the smallest possible dimension, namely r (the rank of $\mathfrak{g}$). Note that with this definition, a regular element need not be semisimple. The set of all regular elements forms an open subvariety of $\mathfrak{g}$, which we will denote by $\mathfrak{g}_{\mathrm{reg}}$.

A Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called a *regular centralizer* if $\mathfrak{a} = Z_{\mathfrak{g}}(x)$ for some $x \in \mathfrak{g}_{\mathrm{reg}}$. Note that such $\mathfrak{a}$ is automatically abelian. Our first goal is to introduce a variety which parametrizes all regular centralizers in $\mathfrak{g}$.

1.2. Let Ab^r be the closed subvariety in the Grassmannian of r-planes $\mathrm{Gr}^r_{\mathfrak{g}}$ that classifies abelian subalgebras in $\mathfrak{g}$ of dimension r. Let $\Gamma \subset \mathrm{Ab}^r \times \mathfrak{g}$ be the incidence correspondence, i.e., the closed subvariety defined by the condition:

$$(\mathfrak{a}, x) \in \Gamma \text{ if } x \in \mathfrak{a}.$$

Let Γ_{reg} be the intersection $\Gamma \cap (\mathrm{Ab}^r \times \mathfrak{g}_{\mathrm{reg}})$.

Proposition 1.3. *There is a smooth morphism $\phi : \mathfrak{g}_{\mathrm{reg}} \to \mathrm{Ab}^r$ whose graph is Γ_{reg}.*

The proof is postponed until Section 10.

Let $\overline{G/N}$ denote the image of the map ϕ. The above proposition implies that $\overline{G/N}$ is smooth and irreducible. It is clear that $\mathbb{C}$-points of $\overline{G/N}$ are exactly the regular centralizers in $\mathfrak{g}$.

By definition, the group G acts on both Ab^r and $\mathfrak{g}_{\mathrm{reg}}$. Therefore, the variety $\overline{G/N}$ acquires a natural G-action and the map ϕ is G-equivariant.

Consider the quotient G/N; it classifies Cartan subalgebras in $\mathfrak{g}$. These are the centralizers in $\mathfrak{g}$ of regular semisimple elements. Hence G/N embeds into $\overline{G/N}$ as an open subvariety. Obviously, the action of G on G/N by left multiplication is the restriction of its action on $\overline{G/N}$.

1.4. Consider the closed subvariety of $\overline{G/N} \times \mathcal{Fl}$ defined by the condition: for $\mathfrak{a} \in \overline{G/N}$ and $\mathfrak{b}' \in \mathcal{Fl}$,

$$(\mathfrak{a}, \mathfrak{b}') \in \overline{G/T} \text{ if } \mathfrak{a} \subset \mathfrak{b}'.$$

We will denote this variety by $\overline{G/T}$ and the natural projection $\overline{G/T} \to \overline{G/N}$ by π. It follows from the definitions that we have a natural G-action on $\overline{G/T}$.

The quotient G/T can clearly be identified with the open subscheme $\pi^{-1}(G/N)$ of $\overline{G/T}$. We have a natural action of the Weyl group $W = T\backslash N$ on G/T; this action is free and the quotient can be identified with G/N.

In what follows, by a W-cover of a scheme X we will mean a finite flat scheme $p : \widetilde{X} \to X$, acted on by W such that $p_* \mathcal{O}_{\widetilde{X}}$ is locally isomorphic as a coherent sheaf with a W-action to $\mathcal{O}_X \otimes \mathbb{C}[W]$. A basic example is $\mathfrak{t} \to \mathfrak{t}/W$: as is well known, it is ramified along the complexified walls of the Cartan subalgebra $\mathfrak{t}$.

The following assertion will be proved in Section 10.

Proposition 1.5. *The variety $\overline{G/T}$ is smooth and connected. The W-action on G/T extends to the whole of $\overline{G/T}$ and it makes the latter a W-cover of $\overline{G/N}$. Moreover, the two W-covers $\overline{G/T} \to \overline{G/N}$ and $\mathfrak{t} \to \mathfrak{t}/W$ are étale-locally isomorphic.*

1.6. Here is an explicit description of $\overline{G/N}$ and $\overline{G/T}$ for $G = \mathrm{SL}(2)$. In this case $\overline{G/N}$ is the space of all lines in $\mathfrak{g}$, i.e., $\overline{G/N} \simeq \mathbb{P}^2$. We have a natural map $\overline{G/T} \to \mathbb{P}^1 \times \mathbb{P}^1$, where the first projection is the natural map $\overline{G/T} \to \mathcal{Fl} \simeq \mathbb{P}^1$ and the second projection is a composition of the first one with the action of $-1 \in S_2 \simeq W$ on $\overline{G/T}$.

It is easy to see that this map is an isomorphism. Under the identification, $\pi : \overline{G/T} \to \overline{G/N}$ is the symmetrization map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$.

1.7. *G*-orbits. For each root α, let $D^\alpha \subset \overline{G/T}$ denote the fixed point set of s_α on $\overline{G/T}$. This is a smooth codimension 1 subscheme of $\overline{G/T}$. Indeed, using the étale-local isomorphism between $\overline{G/T} \to \overline{G/N}$ and $\mathfrak{t} \to \mathfrak{t}/W$ given in Proposition 1.5, it is enough to prove this statement on $\mathfrak{t}$. However, $\mathfrak{t}^{s_\alpha}$ is just the corresponding root hyperplane $\mathfrak{t}^\alpha \subset \mathfrak{t}$.

Proposition 1.8. *The G-orbits in $\overline{G/T}$ are precisely the locally closed subsets*

$$D^{\Delta'} := \bigcap_{\alpha \in \Delta'} (D^\alpha) \setminus \bigcup_{\beta \notin \Delta'} (D^\beta)$$

where $\Delta' \subset \Delta$ is a subset of the set of roots, closed under linear combinations. The G-orbits in $\overline{G/N}$ are the images of the $D^{\Delta'}$; they are indexed by the Δ' modulo the action of W.

The proof will be given in Section 10.

2. Higgs bundles and cameral covers

2.1. Higgs bundles. A family of Cartan subalgebras parametrized by a scheme X is given by a map from X to G/N. Equivalently, it is given by a G-equivariant map from the trivial G-bundle over X to G/N. An advantage of this latter description is that there is a natural way to twist it: given any principal G-bundle E_G over X, we specify a family of Cartan subalgebras *in the adjoint bundle* $\mathfrak{g}_{E_G} := E_G \underset{G}{\times} \mathfrak{g}$ by a G-equivariant map from E_G to the variety G/N. By generalizing this, we define:

Definition 2.2. A Higgs bundle over a scheme X is a pair (E_G, σ), where E_G is a principal G-bundle over X and σ is a G-equivariant map $\sigma : E_G \to \overline{G/N}$.

Therefore, according to Proposition 1.3, a Higgs structure in a given G-bundle E_G is the same as a vector subbundle $\mathbf{c}_X$ of $\mathfrak{g}_{E_G}$ of rank r such that $[\mathbf{c}_X, \mathbf{c}_X] = 0$ and such that locally in the étale topology $\mathbf{c}_X$ is the sheaf of centralizers of a section of $E_G \underset{G}{\times} \mathfrak{g}_{\mathrm{reg}}$.

The restriction of a Higgs bundle to an open subset $U \subset X$ over which E_G is trivialized can be specified more simply by a map $U \to \overline{G/N}$. In particular, the *universal Higgs bundle* over $\overline{G/N}$ corresponds to the identity map $\overline{G/N} \to \overline{G/N}$.

2.3. The Higgs category and stack. Higgs bundles over X form a category, denoted $\mathrm{Higgs}(X)$. By definition, an element of $\mathrm{Hom}((E_G^1,\sigma^1),(E_G^2,\sigma^2))$ is a G-bundle map $s: E_G^1 \to E_G^2$ such that $\sigma^2 \circ s = \sigma^1$.

One can say that $\mathrm{Higgs}(X)$ is the category of maps from X to *the stack* $G\backslash(\overline{G/N})$. Additionally, for a fixed X, we can consider the functor on the category of schemes, which attaches to a scheme S the category $\mathrm{Higgs}(S\times X)$. When X is projective, this functor is representable by an algebraic stack, which we will denote by $\mathbf{Higgs}(X)$. (The representability follows because the stack $\mathbf{Bun}_G(X)$ classifying principal G-bundles on X is an algebraic stack. We have: $G\backslash(\overline{G/N}) = \mathbf{Higgs}(\mathrm{Spec}(\mathbb{C}))$.

2.4. Cameral covers. We will now introduce our second basic object.

Definition 2.5. A W-cover of a scheme X is a scheme $\widetilde{X} \xrightarrow{\pi} X$ finite and flat over X such that as an $\mathcal{O}_X$-module with a W-action, $\pi_*(\mathcal{O}_{\widetilde{X}})$ is locally isomorphic to $\mathcal{O}_X\otimes\mathbb{C}[W]$.

Definition 2.6. A cameral cover of X is a W-cover $\widetilde{X}\to X$, such that locally with respect to the étale topology on X, $\widetilde{X}$ is a pullback of the W-cover $\mathfrak{t}\to\mathfrak{t}/W$.

As an example, we note that any W-cover is cameral when $G=\mathrm{SL}(2)$, i.e., $W=S_2$. On the other hand, not every $W=S_3$-cover is cameral: the stabilizer of each point must be a Weyl subgroup of W, so, for example, an A_3 stabilizer is not allowed.

2.7. Openness. It is easy to see that the condition for a W-cover $\widetilde{X}\to X$ to be cameral is open on X. Indeed, $\pi:\widetilde{X}\to X$ is cameral if and only if, locally on X, we can find a W-equivariant embedding $\widetilde{X}\hookrightarrow X\times\mathfrak{t}$. (Note that the space of W-equivariant maps of X-schemes $\widetilde{X}\to X\times\mathfrak{t}$ is isomorphic to the space of sections of the sheaf $\mathrm{Hom}^W_{\mathcal{O}_X}(\mathfrak{t}^*\otimes\mathcal{O}_X,\pi_*(\mathcal{O}_{\widetilde{X}}))$, and the latter sheaf is non-canonically isomorphic to $\mathfrak{t}\otimes\mathcal{O}_X$, since $\widetilde{X}\to X$ was assumed to be a W-cover.)

2.8. The cameral category and stack. Cameral covers form a category in a natural way, denoted $\mathrm{Cam}(X)$. By definition, $\mathrm{Hom}(\widetilde{X}^1,\widetilde{X}^2)$ consists of all W-equivariant isomorphisms $\widetilde{X}^1\to\widetilde{X}^2$. It is easy to see that there exists an algebraic stack $\mathbf{Cam}$, such that $\mathrm{Cam}(X)$ is the category $\mathrm{Hom}(X,\mathbf{Cam})$.

Indeed, consider the space of commutative W-equivariant ring structures on the vector space $V:=\mathbb{C}[W]$. This is clearly an affine scheme, and let us denote it by $\mathbf{Cov}$. By construction, there exists a universal W-cover $\widetilde{\mathbf{Cov}}\to\mathbf{Cov}$. Let $\mathbf{Cam}'$ be the maximal open subscheme of $\mathbf{Cov}$, over which $\widetilde{\mathbf{Cov}}$ is cameral. Let $\mathrm{Aut}^W(V)$ be the algebraic group of automorphisms of V as a W-module. Clearly, $\mathrm{Aut}^W(V)$ acts on $\mathbf{Cam}'$ and the action lifts on $\widetilde{\mathbf{Cov}}|_{\mathbf{Cam}'}$. We can now let $\mathbf{Cam}$ be the stack-theoretic quotient $\mathrm{Aut}^W(V)\backslash\mathbf{Cam}'$.

As for Higgs bundles, for a fixed X we can consider the functor $S\mapsto\mathrm{Cam}(S\times X)$. For X projective this functor is representable by an algebraic stack $\mathbf{Cam}(X)$.

Proposition 2.9. *There is a natural functor* $F:\mathrm{Higgs}(X)\to\mathrm{Cam}(X)$. *In particular, for a projective scheme* X, *we obtain a map between algebraic stacks* $\mathbf{Higgs}(X)\to\mathbf{Cam}(X)$.

Proof. Any map $\sigma: E_G\to\overline{G/N}$ determines a cameral cover $\widetilde{E}_G$ of E_G, namely $\overline{G/T}\underset{\overline{G/N}}{\times}E_G$; cf. Proposition 1.5.

For a Higgs bundle, which involves a G-equivariant map σ, the cameral cover $\widetilde{E}_G \to E_G$ is itself G-equivariant, so by descent theory, it is pulled back from a unique cameral cover $\widetilde{X} \to X$. Clearly, the assignment $(E_G, \sigma) \mapsto \widetilde{X}$ constructed above is functorial. $\square$

Over an open set $U \subset X$ where E_G is trivialized, the restriction $\widetilde{U} \to U$ of the cameral cover is given in terms of σ as $\overline{G/T} \times_{\overline{G/N}} U$. For example, applying this to the universal Higgs bundle over $\overline{G/N}$ gives the cameral cover $\overline{G/T} \to \overline{G/N}$. For this reason we refer in this paper to $\overline{G/T} \to \overline{G/N}$ (rather than $\mathfrak{t} \to \mathfrak{t}/W$) as the *universal cameral cover.*

2.10. The fiber. Let us now fix a cameral cover $\widetilde{X}$. Let $\mathrm{Higgs}_{\widetilde{X}}(X)$ denote the category-fiber of the above functor $F : \mathrm{Higgs}(X) \to \mathrm{Cam}(X)$ over $\widetilde{X}$. In other words, the objects of $\mathrm{Higgs}_{\widetilde{X}}(X)$ are pairs

$$((E_G, \sigma) \in \mathrm{Higgs}(X), t : F(E_G, \sigma) \simeq \widetilde{X})$$

and $\mathrm{Hom}((E^1_G, \sigma^1, t^1), (E^2_G, \sigma^2, t^2))$ is the set of all bundle maps $s : E^1_G \to E^2_G$ with $\sigma^2 \circ s = \sigma^1$ and such that the composition

$$\widetilde{X} \xrightarrow{(t^1)^{-1}} F(E^1_G, \sigma^1) \to F(E^2_G, \sigma^2) \xrightarrow{t^2} \widetilde{X}$$

is the identity automorphism of $\widetilde{X}$.

The goal of this paper is to describe explicitly the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ in terms of the W-action on $\widetilde{X}$.

3. Gerbes

3.1. Since the objects we study have automorphisms, it is difficult to describe them adequately without the use of some categorical language. Specifically, our description requires the notion of an $\mathcal{A}$-gerbe, where $\mathcal{A}$ is a sheaf of abelian groups on X. This is a particularly useful case of the more general notion of a gerbe over a sheaf of Picard categories. In this section we review the corresponding definitions. For more details, the reader is referred to [19] or [6].

Let $\mathrm{Sch}_{\mathrm{et}}(X)$ denote the big étale site over X. (By definition, $\mathrm{Sch}_{\mathrm{et}}(X)$ is the category of all schemes over X and the covering maps are surjective étale morphisms.)

3.2. Recall that a presheaf $\mathcal{Q}$ of categories on $\mathrm{Sch}_{\mathrm{et}}(X)$ assigns to every object $U \to X$ in $\mathrm{Sch}_{\mathrm{et}}(X)$ a category $\mathcal{Q}(U)$ and to every morphism $f : U_1 \to U_2$ in $\mathrm{Sch}_{\mathrm{et}}(X)$ a functor $f^*_{\mathcal{Q}} : \mathcal{Q}(U_2) \to \mathcal{Q}(U_1)$. Moreover, for every composition $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$ there should be a natural transformation $f^*_{\mathcal{Q}} \circ g^*_{\mathcal{Q}} \Rightarrow (g \circ f)^*_{\mathcal{Q}}$, such that an obvious compatibility relation for three-fold compositions holds.

A presheaf $\mathcal{Q}$ of categories on $\mathrm{Sch}_{\mathrm{et}}(X)$ is said to be a sheaf of categories (or a stack) if the following two axioms hold:

Axiom SC-1. *For $U \to X$ in $\mathrm{Sch}_{\mathrm{et}}(X)$ and a pair of objects $C_1, C_2 \in \mathcal{Q}(U)$, the presheaf of sets on $\mathrm{Sch}_{\mathrm{et}}(U)$ that assigns to $f : U' \to U$ the set $\mathrm{Hom}_{\mathcal{Q}(U')}(f^*_{\mathcal{Q}}(C_1), f^*_{\mathcal{Q}}(C_2))$ is a sheaf.*

Axiom SC-2. *If $f : U' \to U$ is a covering, then the category $\mathcal{Q}(U)$ is equivalent to the category of descent data on $\mathcal{Q}(U')$ with respect to f (i.e., every descent data on $\mathcal{Q}(U')$ with respect to f is canonically effective; cf.* [6], p. 221).

3.3. Here is our main example of a sheaf of categories. Fix a cameral cover $\widetilde{X} \to X$. For every object $U \in \mathrm{Sch}_{\mathrm{et}}(X)$ write $\widetilde{U} := U \underset{X}{\times} \widetilde{X}$, which is a cameral cover of U.

We define the presheaf of categories $\mathrm{Higgs}_{\widetilde{X}}$ by $\mathrm{Higgs}_{\widetilde{X}}(U) := \mathrm{Higgs}_{\widetilde{U}}(U)$ (the functors $\mathrm{Higgs}_{\widetilde{X}}(U) \to \mathrm{Higgs}_{\widetilde{X}}(U')$ for $U' \to U$ and the corresponding natural transformations are defined in a natural way).

The following is an easy exercise in descent theory:

Lemma 3.4. $\mathrm{Higgs}_{\widetilde{X}}$ *satisfies* SC-1 *and* SC-2.

3.5. Recall that a Picard category is a groupoid endowed with a a structure of a tensor category, in which every object is invertible. A basic example (and the source of the name) is the category of line bundles over a scheme.

A sheaf of categories $\mathcal{P}$ is said to be a sheaf of Picard categories if for every $(U \to X) \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\mathcal{P}(U)$ is endowed with a structure of a Picard category such that the pullback functors $f^*_{\mathcal{P}}$ are compatible with the tensor structure in an appropriate sense. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two sheaves of Picard categories, one defines (in a straightforward fashion) a notion of a tensor functor between them.

A typical and the most important example of a sheaf of Picard categories can be constructed as follows:

Let $\mathcal{A}$ be a sheaf of abelian groups over $\mathrm{Sch}_{\mathrm{et}}(X)$. For an object $f : U \to X$ of $\mathrm{Sch}_{\mathrm{et}}(X)$ let $\mathrm{Tors}_{\mathcal{A}}(U)$ denote the category of $\mathcal{A}|_U$-torsors on U. This is a Picard category and it is easy to see that the assignment $U \to \mathrm{Tors}_{\mathcal{A}}(U)$ defines a sheaf of Picard categories on $\mathrm{Sch}_{\mathrm{et}}(X)$ which we will denote by $\mathrm{Tors}_{\mathcal{A}}$.

3.6. Just as a torsor is a space on which a group acts simply transitively, a gerbe is a category on which a Picard category acts simply transitively: A category $\mathcal{Q}$ is said to be a *gerbe bound by the Picard category* $\mathcal{P}$, if $\mathcal{P}$ acts on $\mathcal{Q}$ as a tensor category and for any object $C \in \mathcal{Q}$ the functor $\mathcal{P} \to \mathcal{Q}$ given by

$$P \in \mathcal{P} \longrightarrow \mathrm{Action}(P, C) \in \mathcal{Q}$$

is an equivalence.

Now, if $\mathcal{P}$ is a sheaf of Picard categories and $\mathcal{Q}$ is another sheaf of categories, we say that $\mathcal{Q}$ is a *gerbe bound by the sheaf of Picard categories* $\mathcal{P}$, if the following holds:

- For every $(U \to X) \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\mathcal{Q}(U)$ has a structure of a gerbe bound by $\mathcal{P}(U)$. This structure is compatible with the pullback functors $f^*_{\mathcal{P}}$ and $f^*_{\mathcal{Q}}$.
- There exists a covering $U \to X$, such that $\mathcal{Q}(U)$ is non-empty.

A basic feature of gerbes is that if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are gerbes bound by $\mathcal{P}$, one can form a new gerbe $\mathcal{Q}_1 \underset{\mathcal{P}}{\otimes} \mathcal{Q}_2$, called their tensor product; cf. [6].

3.7. The basic example of a gerbe bound by an arbitrary sheaf of Picard categories $\mathcal{P}$, is $\mathcal{P}$ itself. Here is a less trivial example:

Fix a short exact sequence $0 \to \mathcal{A} \to \mathcal{A}'' \to \mathcal{A}' \to 0$ of sheaves of abelian groups on X and let $\tau_{\mathcal{A}'}$ be an $\mathcal{A}'$-torsor over X. We introduce a sheaf of categories $\mathcal{Q} = \mathcal{Q}_{\tau_{\mathcal{A}'}}$ as follows. For $U \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\mathcal{Q}(U)$ is the category of all "liftings" of $\tau_{\mathcal{A}'}|_U$ to an $\mathcal{A}''|_U$-torsor. It is easy to check that $\mathcal{Q}$ is a gerbe bound by $\mathcal{P} = \mathrm{Tors}_{\mathcal{A}}$.

In fact, gerbes bound by $\mathrm{Tors}_{\mathcal{A}}$ can be classified cohomologically:

Lemma 3.8. *There is a bijection between the set of equivalence classes of gerbes bound by* $\mathrm{Tors}_{\mathcal{A}}$ *and* $H^2(X,\mathcal{A})$*. For a given gerbe* $\mathcal{Q}$ *the corresponding class in* $H^2(X,\mathcal{A})$ *vanishes if and only if the category* $\mathcal{Q}(X)$ *of "global sections" is non-empty.*

In the above example, the class on $H^2(X,\mathcal{A})$ corresponds to the image of the class of $\tau_{\mathcal{A}'}$ under the boundary map $H^1(X,\mathcal{A}') \to H^2(X,\mathcal{A})$.

3.9. The following will be needed in Section 18.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be sheaves of Picard categories, and $\mathbf{a} : \mathcal{P}_1 \to \mathcal{P}_2$ a functor compatible with the tensor structure. We say that $\mathbf{a}$ is a monomorphism if for every $U \in \mathrm{Sch}_{\mathrm{et}}(X)$ the functor $\mathbf{a}(U) : \mathcal{P}_1(U) \to \mathcal{P}_2(U)$ is faithful.

We say that $\mathbf{a}$ is an epimorphism if, for every $U \in \mathrm{Sch}_{\mathrm{et}}(X)$ and $P, \widetilde{P} \in \mathcal{P}_1(U)$, the map of sheaves on $\mathrm{Sch}_{\mathrm{et}}(U)$: $\mathrm{Hom}_{\mathcal{P}_1(U')}(P|_{U'}, \widetilde{P}|_{U'}) \to \mathrm{Hom}_{\mathcal{P}_2(U')}(\mathbf{a}(P|_{U'}), \mathbf{a}(\widetilde{P})|_{U'})$ is an epimorphism (in the sense of sheaves), and for every $P_2 \in \mathcal{P}_2(U)$, there exists a covering $U' \to U$, such that $P_2|_{U'}$ is isomorphic to $\mathbf{a}(P_1)$ for some $P_1 \in \mathcal{P}_1(U')$.

Similarly, if we have three sheaves of Picard categories and tensor functors $\mathbf{a} : \mathcal{P}_1 \to \mathcal{P}_2$, and $\mathbf{b} : \mathcal{P}_2 \to \mathcal{P}_3$, we say that the form a short exact sequence if $\mathbf{b}$ is an epimorphism and $\mathbf{a}$ induces an equivalence between $\mathcal{P}_1$ and the category-fiber of $\mathcal{P}_2$ over the unit object in $\mathcal{P}_3$.

In this case, for every object $P_3 \in \mathcal{P}_3$, the category fiber $\mathbf{b}^{-1}(P_3)$ of $\mathcal{P}_2$ over it is, in a natural way, gerbe bound by $\mathcal{P}_1$. This generalizes the above example of $0 \to \mathcal{A} \to \mathcal{A}'' \to \mathcal{A}' \to 0$.

Now let $\mathcal{Q}_1$ be a gerbe bound $\mathcal{P}_1$, and $\mathbf{a} : \mathcal{P}_1 \to \mathcal{P}_2$ a tensor functor. In this case one can construct a canonical induced gerbe $\mathcal{Q}_2$ bound by $\mathcal{P}_2$ with the property that there exists a functor $\mathcal{Q}_1 \to \mathcal{Q}_2$, compatible with the $\mathcal{P}_1$- and $\mathcal{P}_2$-actions via $\mathbf{a}$.

Suppose now that

$$0 \to \mathcal{P}_1 \xrightarrow{\mathbf{a}} \mathcal{P}_2 \xrightarrow{\mathbf{b}} \mathcal{P}_3 \to 0$$

is a short exact sequence of Picard categories, and $\mathcal{Q}_1$ is a gerbe bound by $\mathcal{P}_1$. Let $\mathcal{Q}_2$ be the corresponding induced $\mathcal{P}_2$-gerbe.

The next lemma follows from the definitions in a straightforward way:

Lemma 3.10. *There exists a canonical functor* $\mathcal{Q}_2 \to \mathcal{P}_3$*. The category fiber of* $\mathcal{Q}_2$ *over a given object* $P_3 \in \mathcal{P}_3$ *is naturally a* $\mathcal{P}_1$*-gerbe, canonically equivalent to the tensor product* $\mathcal{Q}_1 \underset{\mathcal{P}_1}{\otimes} \mathbf{b}^{-1}(P_3)$.

4. $\mathrm{Higgs}_{\widetilde{X}}$ is a gerbe

4.1. Given a cameral cover $\widetilde{X} \to X$, let $\overline{T}_{\widetilde{X}}$ be the sheaf "of W-equivariant maps $\widetilde{X} \to T$" on the étale site over X. More precisely, for $U \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\overline{T}_{\widetilde{X}}(U) = \mathrm{Hom}_W(\widetilde{U}, T)$, where $\widetilde{U}$ is the induced cameral cover of U and the subscript "W" means maps respecting the W-action.

However, we need a slightly smaller sheaf.

4.2. Let $D^{\alpha}_{\widetilde{X}}$ (for each positive root α) be the fixed point scheme of the reflection s_α acting on $\widetilde{X}$. Locally, this is the pullback of the *universal ramification divisor*, i.e., $D^{\alpha} \subset \overline{G/T}$.

Let α be a root of G, considered as a homomorphism $\alpha : T \to \mathbb{G}_m$. Then any section t of $\overline{T}_{\widetilde{X}}(U)$ determines a function $\alpha \circ t : U \to \mathbb{G}_m$ which goes to its own inverse under the reflection s_α. In particular, its restriction to the ramification locus $D^\alpha_{\widetilde{X}}$ *equals* its inverse, so it equals ± 1. The subsheaf $T_{\widetilde{X}} \subset \overline{T}_{\widetilde{X}}$ is defined by the following condition:

$$T_{\widetilde{X}}(U) := \{t \in \overline{T}_{\widetilde{X}}(U) \mid (\alpha \circ t)|_{D^\alpha_{\widetilde{U}}} = +1 \text{ for each root } \alpha\}. \qquad (*)$$

By construction, $\overline{T}_{\widetilde{X}}/T_{\widetilde{X}}$ is a $\mathbb{Z}_2$-torsion sheaf. Note, in addition, that it suffices to impose condition $(*)$ for one representative of each orbit of W on the set of roots.

Remark. Recall that a coroot $\check{\alpha} : \mathbb{G}_m \to T$ is called primitive if $\ker(\check{\alpha}) = 1$ (this is equivalent to saying that $\check{\alpha}$ is a primitive element of the lattice of cocharacters of T.) It is clear that condition $(*)$ holds automatically for roots whose corresponding coroots are primitive. For example, when the derived group of G is simply connected, all coroots are primitive, i.e., $(*)$ is automatic and $T_{\widetilde{X}} = \overline{T}_{\widetilde{X}}$. In fact, G has non-primitive coroots if and only if it contains $\mathrm{SO}(2n+1)$ (e.g. $\mathrm{PGL}(2) = \mathrm{SO}(3)$) as a direct factor, as is easily seen from the classification of Dynkin diagrams.

4.3. Our first result can be stated as:

Theorem 4.4. $\mathrm{Higgs}_{\widetilde{X}}$ *is a gerbe bound by* $\mathrm{Tors}_{T_{\widetilde{X}}}$.

Let us list several corollaries of this theorem:

Corollary 4.5. *To a cameral cover $\widetilde{X}$ there corresponds a class in $H^2(X, T_{\widetilde{X}})$, which vanishes if and only if $\widetilde{X}$ is the cameral cover corresponding to some Higgs bundle.*

This is immediate from Lemma 3.8.

Corollary 4.6. *Suppose* $\mathrm{Higgs}_{\widetilde{X}}(X)$ *is non-empty. The set of isomorphism classes of objects in this category carries a simply transitive action of $H^1(X, T_{\widetilde{X}})$. The group of automorphisms of every object is canonically isomorphic to $T_{\widetilde{X}}(X)$.*

5. Ramification

5.1. We now proceed to the formulation of our main result, Theorem 6.4, which describes the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ completely in terms of $\widetilde{X}$. For this purpose, we need to introduce some further notation that has to do with the ramification pattern of $\widetilde{X}$ over X.

5.2. For each root α we will define a line bundle $R^\alpha_{\widetilde{X}}$ on $\widetilde{X}$. Assume first that $\widetilde{X}$ is integral. In this case the subscheme $D^\alpha_{\widetilde{X}} \subset \widetilde{X}$ is a Cartier divisor, because locally it is the pullback of $D^\alpha \subset \overline{G/T}$. We set $R^\alpha_{\widetilde{X}} = \mathcal{O}(D^\alpha_{\widetilde{X}})$.

When $\widetilde{X}$ is arbitrary we proceed as follows. The construction is local, so we may assume that X, and hence also $\widetilde{X}$, is affine. Let $I^\alpha_{\widetilde{X}}$ be a coherent sheaf on $\widetilde{X}$ generated by symbols $\{g\}$, for $\{g \in \mathcal{O}_{\widetilde{X}} \mid s_\alpha(g) = -g\}$ that satisfy the relations

$$f \cdot \{g\} = \{f \cdot g\} \text{ for all } f \text{ such that } s_\alpha(f) = f.$$

Locally, $I^\alpha_{\widetilde{X}}$ is the pullback of the sheaf of ideals of the subscheme $D^\alpha \subset \overline{G/T}$. Hence, $I^\alpha_{\widetilde{X}}$ is a line bundle. We have a natural map $I^\alpha_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$ that sends $\{g\} \mapsto g$ and, by construction, its cokernel is $\mathcal{O}_{D^\alpha_{\widetilde{X}}}$.

We define the line bundle $R^\alpha_{\widetilde{X}}$ as the inverse of $I^\alpha_{\widetilde{X}}$. We have a canonical section $\mathcal{O}_{\widetilde{X}} \to R^\alpha_{\widetilde{X}}$ whose locus of zeroes is the subscheme $D^\alpha_{\widetilde{X}}$.

5.3. Consider the T-bundle $\mathcal{R}^\alpha_X := \check{\alpha}(R^\alpha_X)$ (i.e., $\mathcal{R}^\alpha_X$ is induced from R^α_X by means of the homomorphism $\check{\alpha} : \mathbb{G}_m \to T$).

For an element $w \in W$ we introduce the T-bundle $\mathcal{R}^w_X$ on $\widetilde{X}$ as

$$\mathcal{R}^w_X := \underset{\alpha}{\otimes} \mathcal{R}^\alpha_X,$$

where α runs over those positive roots for which $w(\alpha)$ is negative. For example, for $w = s_i$ (a simple reflection), $\mathcal{R}^{s_i}_X \simeq \mathcal{R}^{\alpha_i}_X$.

Observe that given a T-bundle $\mathcal{L}$ on $\widetilde{X}$ and an element $w \in W$, there are two ways to produce a new T-bundle: we can pullback by w acting as an automorphism of $\widetilde{X}$, or we can conjugate the T-action by w. We will *always* write $w^*(\mathcal{L})$ for the combination of *both* actions. For example, for $G = \mathrm{SL}(2)$, the T-bundle $\mathcal{L}$ is equivalent to a line bundle L. The two individual actions on $\mathcal{L}$ of the nontrivial element $-1 \in S_2 = W$ send L to $(-1)^*(L)$ and L^{-1}, respectively, while $(-1)^*(\mathcal{L})$ corresponds to the line bundle $(-1)^*(L^{-1})$. In particular, we have

$$w^*(\mathcal{R}^\alpha_X) \simeq \mathcal{R}^{w^{-1}(\alpha)}_X. \tag{1}$$

Lemma 5.4. *There is a canonical isomoprhism* $\mathcal{R}^{w_1 \cdot w_2}_X \overset{\varpi(w_1,w_2)}{\longrightarrow} w_2^*(\mathcal{R}^{w_1}_X) \otimes \mathcal{R}^{w_2}_X$.

The proof follows imediately from the definition of $\mathcal{R}^w_X$ and (1). The following proposition is necessary for the formulation of Theorem 6.4.

Proposition 5.5. *Let* α_i *be a simple root and let* $w \in W$ *be such that* $w(\alpha_i) = \alpha_j$ *(another positive simple root). Then, the line bundle* $\alpha_i(\mathcal{R}^w_X)|_{D^{\alpha_i}_X}$ *admits a canonical trivialization.*

Proof. Let us observe first that, since we are using only roots rather than arbitrary weights, it is sufficient to consider the case when $[G,G]$ is simply connected.

We have $w \cdot s_i = s_j \cdot w$, hence, by Lemma 5.4

$$s_i^*(\mathcal{R}^w_X) \otimes \mathcal{R}^{s_i}_X \simeq \mathcal{R}^{w \cdot s_i}_X \simeq \mathcal{R}^{s_j \cdot w}_X \simeq w^*(\mathcal{R}^{s_j}_X) \otimes \mathcal{R}^w_X.$$

However, by definition $w^*(\mathcal{R}^{s_j}_X) \simeq \mathcal{R}^{s_i}_X$, so we obtain that $s_i^*(\mathcal{R}^w_X) \simeq \mathcal{R}^w_X$. By restricting to $D^{\alpha_i}_X$, we obtain $\check{\alpha}_i(\alpha_i(\mathcal{R}^w_X)) \simeq \check{\alpha}_i(\mathcal{O}_{D^{\alpha_i}_X})$.

Since $[G,G]$ is simply connected, every coroot is primitive. Therefore, there exists a weight λ, such that $\lambda \circ \check{\alpha}_i = \mathrm{id} : \mathbb{G}_m \to \mathbb{G}_m$. By applying λ to the above isomorphism $\check{\alpha}_i(\alpha_i(\mathcal{R}^w_X)) \simeq \check{\alpha}_i(\mathcal{O}_{D^{\alpha_i}_X})$, we obtain an isomorphism $\alpha_i(\mathcal{R}^w_X) \overset{\mathrm{isom}^\lambda}{\simeq} \mathcal{O}_{D^{\alpha_i}_X}$.

Now it only remains to check that this isomorphism is independent of the choice of λ. However, since the $\mathcal{R}^w_X$'s are locally pullbacks of the corresponding T-bundles on $\overline{G/T}$, it suffices to consider the universal situation, namely the case $X = \overline{G/N}$.

In the latter case, the T-bundle $\mathcal{R}^w|_{D^{\alpha_i}}$ itself is trivialized over an open dense part of D^{α_i}, namely over $D^{\alpha_i} - \underset{\alpha \neq \alpha_i}{\cup} (D^\alpha \cap D^{\alpha_i})$. This is because α_i is not among the set of roots which become negative under the action of w. In particular, we obtain an isomorphism $\alpha_i(\mathcal{R}^w) \overset{\mathrm{isom}'}{\simeq} \mathcal{O}_{D^{\alpha_i}}$ over $D^{\alpha_i} - \underset{\alpha \neq \alpha_i}{\cup} (D^\alpha \cap D^{\alpha_i})$.

Moreover, it is easy to see that for any λ as above, the isomorphisms isom^λ and isom' coincide. In particular, isom^λ is independent of λ over $D^{\alpha_i} - \underset{\alpha\neq\alpha_i}{\cup}(D^\alpha \cap D^{\alpha_i})$ and hence over the whole of D^{α_i}, which is what we need. □

The following notions will be used in the formulation of Theorem 6.4.

Definition 5.6. Let $\mathcal{L}_0$ be a T-bundle on $\widetilde{X}$. We say that it is weakly W-equivariant if for every w there exists an isomorphism $w^*(\mathcal{L}_0) \to \mathcal{L}_0$.

For a weakly W-equivariant T-bundle, let $\mathrm{Aut}(\mathcal{L}_0)$ be the group whose elements are pairs: an element $w \in W$ plus an isomorphism $w^*(\mathcal{L}_0) \to \mathcal{L}_0$. By definition, $\mathrm{Aut}(\mathcal{L}_0)$ fits into a short exact sequence

$$1 \to \mathrm{Hom}(\widetilde{X}, T) \to \mathrm{Aut}(\mathcal{L}_0) \to W \to 1.$$

Definition 5.7. A strongly W-equivariant T-bundle is a weakly W-equivariant T-bundle $\mathcal{L}_0$ plus a choice of a splitting $\gamma_0 : W \to \mathrm{Aut}(\mathcal{L}_0)$.

Definition 5.8. A T-bundle on $\widetilde{X}$ is called weakly R-twisted W-equivariant if for every $w \in W$ there exists an isomorphism $w^*(\mathcal{L}) \otimes \mathcal{R}^w_X \simeq \mathcal{L}$.

For a weakly R-twisted W-equivariant T-bundle $\mathcal{L}$ we introduce the group $\mathrm{Aut}_{\mathcal{R}}(\mathcal{L})$. Its elements are pairs $w \in W$ and an ismorphism $w^*(\mathcal{L}) \otimes \mathcal{R}^w_X \simeq \mathcal{L}$. The group law is defined via the isomorphism $\varpi(w_1, w_2)$ of Lemma 5.4. By definition, $\mathrm{Aut}_{\mathcal{R}}(\mathcal{L})$ is also an extension of W by means of $\mathrm{Hom}(\widetilde{X}, T)$.

6. The main result

6.1. We need one more piece of notation. For a simple root α_i, let M_i be the corresponding minimal Levi subgroup. Under the projection $N \to W$, the intersection $N \cap [M_i, M_i]$ surjects onto $\langle s_i \rangle \simeq S_2$. Let $\mathcal{N}_i$ denote the preimage of s_i in $N \cap [M_i, M_i]$.

By definition, if n_i and n_i' are two elements in $\mathcal{N}_i$, there exists $c \in \mathbb{G}_m$ such that $n_i' = \check{\alpha}_i(c) \cdot n_i$.

6.2. Given a cameral cover $\widetilde{X} \to X$, we introduce the category $\mathrm{Higgs}'_{\widetilde{X}}(X)$ of "R-twisted, N-shifted W-equivariant T-bundles on $\widetilde{X}$". Its objects consist of:

- A weakly R-twisted W-equivariant T-bundle $\mathcal{L}$ on $\widetilde{X}$.
- A map of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{natural map}} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow \\ 1 & \longrightarrow & \mathrm{Hom}(\widetilde{X}, T) & \longrightarrow & \mathrm{Aut}_{\mathcal{R}}(\mathcal{L}) & \longrightarrow & W & \longrightarrow & 1 \end{array}.$$

- For each simple root α_i and element $n_i \in \mathcal{N}_i$, an isomorphism of line bundles on $D^{\alpha_i}_X$

$$\beta_i(n_i) : \alpha_i(\mathcal{L})|_{D^{\alpha_i}_X} \simeq R^{\alpha_i}_X|_{D^{\alpha_i}_X}.$$

These data must satisfy three compatibility conditions:
(1) If $n_i' = \check{\alpha}_i(c) \cdot n_i$ for $c \in \mathbb{G}_m$, then $\beta_i(n_i') = c \cdot \beta_i(n_i)$.
(2) Let α_i be again a simple root and $n_i \in \mathcal{N}_i$. Consider the isomorphism

$$\gamma(n_i) : s_i^*(\mathcal{L}) \otimes \mathcal{R}_{\widetilde{X}}^{s_i} \simeq \mathcal{L}.$$

When we restrict it to $D_{\widetilde{X}}^{\alpha_i}$, it induces an isomorphism

$$\check{\alpha}_i(\alpha_i(\mathcal{L})|_{D_{\widetilde{X}}^{\alpha_i}}) \simeq \check{\alpha}_i(R_{\widetilde{X}}^{\alpha_i}|_{D_{\widetilde{X}}^{\alpha_i}}),$$

by the definition of $\mathcal{R}_{\widetilde{X}}^{s_i}$. We need this isomorphism to coincide with $\check{\alpha}_i(\beta_i(n_i))$.

(3) Let α_i and α_j be two simple roots and let $w \in W$ be such that $w(\alpha_i) = \alpha_j$. Let $\tilde{w} \in N$ be an element that projects to w, and n_j an element of $\mathcal{N}_j$. By pulling back the isomorphism $\beta_j(n_j)$ with respect to w, we obtain an isomorphism $\alpha_i(w^*(\mathcal{L}))|_{D_{\widetilde{X}}^{\alpha_i}} \simeq R_{\widetilde{X}}^{\alpha_i}|_{D_{\widetilde{X}}^{\alpha_i}}$. In addition, the isomorphisms induced by $\gamma(\tilde{w})$ and Proposition 5.5 lead to a sequence of isomorphisms:

$$\alpha_i(\mathcal{L})|_{D_{\widetilde{X}}^{\alpha_i}} \xrightarrow{\gamma(\tilde{w})} \alpha_i(w^*(\mathcal{L}))|_{D_{\widetilde{X}}^{\alpha_i}} \otimes \alpha_i(\mathcal{R}_{\widetilde{X}}^{w})|_{D_{\widetilde{X}}^{\alpha_i}} \xrightarrow{\text{Proposition 5.5}} \alpha_i(w^*(\mathcal{L}))|_{D_{\widetilde{X}}^{\alpha_i}}.$$

By composing the two, we obtain an isomorphism $\alpha_i(\mathcal{L})|_{D_{\widetilde{X}}^{\alpha_i}} \simeq R_{\widetilde{X}}^{\alpha_i}|_{D_{\widetilde{X}}^{\alpha_i}}$ and our condition is that it coincides with $\beta_i(n_i)$, where $n_i = \tilde{w}^{-1} \cdot n_j \cdot \tilde{w} \in \mathcal{N}_i$.

This concludes the definition of objects of $\mathrm{Higgs}'_{\widetilde{X}}(X)$. Morphisms between $(\mathcal{L}, \gamma, \beta_i)$ and $(\mathcal{L}^1, \gamma^1, \beta_i^1)$ are T-bundle isomorphism maps $\mathcal{L}^1 \to \mathcal{L}$, which intertwine in the obvious sense γ with γ^1 and β_i with β_i^1.

6.3. It is easy to see that $\mathrm{Higgs}'_{\widetilde{X}}(X)$ can be naturally sheafified. Namely, we define the presheaf of categories $\mathrm{Higgs}'_{\widetilde{X}}$ by setting for $U \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\mathrm{Higgs}'_{\widetilde{X}}(U) := \mathrm{Higgs}'_{\widetilde{U}}(U)$. The pullback functors are defined in an evident manner and it is easy to see that $\mathrm{Higgs}'_{\widetilde{X}}$ satisfies SC-1 and SC-2.

Our main result is:

Theorem 6.4. *The sheaves of categories* $\mathrm{Higgs}'_{\widetilde{X}}$ *and* $\mathrm{Higgs}_{\widetilde{X}}$ *are naturally equivalent.*

In particular, we obtain that $\mathrm{Higgs}_{\widetilde{X}}(X)$ is equivalent to $\mathrm{Higgs}'_{\widetilde{X}}(X)$. In other words, a Higgs bundle on X with the given cameral cover $\widetilde{X}$ is equivalent to a T-bundle on $\widetilde{X}$ which is R-twisted, N-shifted W-equivariant.

6.5. Variant. Assume that all coroots in G are primitive, i.e., for every α, the corresponding 1-parameter subgroup maps injectively into T.

We claim that the definition of $\mathrm{Higgs}'_{\widetilde{X}}(X)$ is equivalent to the following (simplified) one. We introduce the category $\mathrm{Higgs}''_{\widetilde{X}}(X)$ as follows:

Objects of $\mathrm{Higgs}''_{\widetilde{X}}(X)$ are pairs:

- a weakly R-twisted W-equivariant T-bundle $\mathcal{L}$ on $\widetilde{X}$,
- a map of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 1 \\
\downarrow & & \Big\downarrow{\scriptstyle \text{natural map}} & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
1 & \longrightarrow & \mathrm{Hom}(\widetilde{X}, T) & \longrightarrow & \mathrm{Aut}_{\mathcal{R}}(\mathcal{L}) & \longrightarrow & W & \longrightarrow & 1
\end{array},$$

such that the following condition holds:

(1′) Let λ be a weight of T such that $\langle\lambda,\check{\alpha}_i\rangle = 0$, which implies that $\lambda(\mathcal{L})|_{D_X^{\alpha_i}} \simeq \lambda(s_i^*(\mathcal{L})\otimes\mathcal{R}_X^{s_i})|_{D_X^{\alpha_i}}$. Our condition is that for every $n_i \in \mathcal{N}_i$ the composition

$$\lambda(\mathcal{L})|_{D_X^{\alpha_i}} \simeq \lambda(s_i^*(\mathcal{L})\otimes\mathcal{R}_X^{s_i})|_{D_X^{\alpha_i}} \xrightarrow{\gamma(n_i)} \lambda(\mathcal{L})|_{D_X^{\alpha_i}}$$

is the identity map.

Morphisms between $(\mathcal{L},\gamma)$ and $(\mathcal{L}^1,\gamma^1)$ are T-bundle maps, which intertwine between γ and γ^1.

Let us show that $\mathrm{Higgs}'_{\widetilde{X}}(X)$ and $\mathrm{Higgs}''_{\widetilde{X}}(X)$ are naturally equivalent. Indeed, if we have an object $(\mathcal{L},\gamma,\beta_i) \in \mathrm{Higgs}'_{\widetilde{X}}(X)$, the corresponding object of $\mathrm{Higgs}''_{\widetilde{X}}(X)$ is obtained by just forgetting the β_i's.

Conversely, if $(\mathcal{L},\gamma) \in \mathrm{Higgs}''_{\widetilde{X}}(X)$, we reconstruct the β_i's as follows:

For a simple root α_i and $n_i \in \mathcal{N}_i$, consider the isomorphism $\gamma(n_i)$ restricted to $D_X^{\alpha_i}$. It yields an isomorphism

$$\check{\alpha}_i(\alpha_i(\mathcal{L}))|_{D_X^{\alpha_i}} \simeq \check{\alpha}_i(R_X^{\alpha_i})|_{D_X^{\alpha_i}}.$$

Since $\check{\alpha}_i$ is primitive, there exists a weight λ' with $\langle\lambda',\check{\alpha}_i\rangle = 1$. By evaluating λ on the above isomorphism, we obtain the required identification $\beta_i(n_i) : \alpha_i(\mathcal{L})|_{D_X^{\alpha_i}} \to R_X^{\alpha_i}|_{D_X^{\alpha_i}}$. This isomorphism does not depend on the choice of λ' because of our condition (1′) on γ.

The fact that conditions (1) and (2) hold follows from the construction. Condition (3) follows from the way in which we build the isomorphism of Proposition 5.5.

Part II. Basic Examples

7. The universal example: $\overline{G/N}$

7.1. In the category $\mathrm{Higgs}_{\overline{G/T}}(\overline{G/N})$ there is a canonical tautological object. One of the main steps in the proof of Theorem 6.4 is to exhibit the corresponding canonical object in $\mathrm{Higgs}'_{\overline{G/T}}(\overline{G/N})$. This is our goal in this section.

7.2. Consider the canonical T-bundle $\mathcal{L}_{\mathfrak{Fl}} = G/U$ over $\mathfrak{Fl} = G/B$ and let us denote by $\mathcal{L}_{\mathrm{can}}$ its pullback to $\overline{G/T}$ under the natural projection $\overline{G/T} \to \mathfrak{Fl}$. This will be the first piece in the data $(\mathcal{L}_{\mathrm{can}},\gamma_{\mathrm{can}},\beta_{i,\mathrm{can}})$.

When we restrict $\mathcal{L}_{\mathrm{can}}$ to $G/T \subset \overline{G/T}$, it becomes identified with $G \to G/T$. Hence for every element $\tilde{w} \in N$ that projects to $w \in W$, we obtain an isomorphism $\gamma_{\mathrm{can}}(\widetilde{w})$: $w^*(\mathcal{L}_{\mathrm{can}}) \simeq \mathcal{L}_{\mathrm{can}}$ over G/T, given by right multipliction by $\tilde{w}^{-1}$ on G.

However, when exended to the whole of $\overline{G/T}$, the above identification is meromorphic and the configuration of its zeroes and poles is given by a divisor on $\overline{G/T}$ with values in the cocharacter lattice of T.

Theorem 7.3. *For a simple reflection s_i, the divisor of the above meromorphic map $s_i^*(\mathcal{L}_{\mathrm{can}}) \to \mathcal{L}_{\mathrm{can}}$ is given by $-\check{\alpha}_i(D^{\alpha_i})$.*

The proof will be given in Section 15.

Since $\mathcal{R}_X^{w_1\cdot w_2} \simeq w_2^*(\mathcal{R}_X^{w_1}) \otimes \mathcal{R}_X^{w_2}$, Theorem 7.3 implies that for any element $w \in W$, the divisor of zeroes/poles of the above meromorphic map $w^*(\mathcal{L}_{\mathrm{can}}) \to \mathcal{L}_{\mathrm{can}}$ coincides with $\mathcal{R}^w_{\overline{G/T}}$. Hence, we obtain the data of $\gamma_{\mathrm{can}} : N \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{L}_{\mathrm{can}})$.

Finally, we have to specify the data of $\beta_{i,\mathrm{can}}$ and check the compatibility conditions. Let us first consider the case when $[G, G]$ is simply connected. As was explained in Section 6.5, in this case the data of $\beta_{i,\mathrm{can}}$ can be recovered from γ_{can}, once we check that condition $(1')$ holds.

Thus, let α_i be a simple root and let λ be a weight orthogonal to $\check{\alpha}_i$. It suffices to check condition $(1')$ at the generic point of D^{α_i}. Let M_i be the corresponding minimal Levi subgroup. We have a closed embedding $\overline{M_i/T} \subset \overline{G/T}$ (cf. Section 10.5) and its orbit under the G-action is the open subset of $\overline{G/T}$ equal to $G/T \cup (D^{\alpha_i} - \underset{\alpha\neq\alpha_i}{\cup} (D^\alpha \cap D^{\alpha_i}))$. In particular, it contains a dense subset of D^{α_i}.

Since all our constructions are G-equivariant, this implies that condition $(1')$ for α_i is equivalent to the corresponding statement for M_i. Moreover, we can replace M_i by an isogenous group, namely $[M_i, M_i] \times Z(M_i)$. However, in the latter case our compatibility condition becomes obvious, as λ factors through $Z(M_i)$.

Now, let G be arbitrary. Choose an isogeny $G' \to G$ such that $[G', G']$ is simply connected. The varieties $\overline{G/T}$ and $\overline{G'/T'}$ are canonically identified and the T-bundle $\mathcal{L}_{\mathrm{can}}$ is induced from the T'-bundle $\mathcal{L}'_{\mathrm{can}}$ under $T' \to T$. Therefore, once we know the data of $\beta'_{i,\mathrm{can}}$ for $\mathcal{L}'_{\mathrm{can}}$ that satisfies the compatibility conditions, it produces the corresponding data for $\mathcal{L}_{\mathrm{can}}$.

Thus, we have constructed a canonical G-equivariant object of $\mathrm{Higgs}'_{\overline{G/T}}$ over $\overline{G/N}$.

8. Some simple cases

8.1. The unramified situation. We call a Higgs bundle (E_G, σ) unramified if σ maps E_G to G/N. Such a map amounts to a reduction of the structure group from G to N. The category of unramified Higgs bundles is therefore equivalent to the category of principal N-bundles.

The functor $F : \mathrm{Higgs}(X) \to \mathrm{Cam}(X)$ sends an N-bundle E_N to $\widetilde{X} := T\backslash E_N$, which is a principal W-bundle over X (i.e., an étale W-cover).

In this case the assertion of Theorem 6.4 is quite evident.

8.2. $G = \mathrm{SL}(2)$. Fix an S_2-cover $p : \widetilde{X} \to X$ and consider the subsheaf of $p_*(\mathcal{O}_{\widetilde{X}})$ consisting of S_2-anti-invariants. We will denote it by $\mathbf{c}_X$.

It is easy to see that the category $\mathrm{Higgs}_{\widetilde{X}}(X)$ is canonically equivalent to the category of pairs (L, γ'), where L is a line bundle on $\widetilde{X}$ and γ' is an isomorphism $\det(p_*(L)) \simeq \mathcal{O}_X$.

Let $D_X \subset \widetilde{X}$ be the ramification divisor and let R_X be the corresponding line bundle (cf. Section 5). It is easy to see that the category $\mathrm{Higgs}'_{\widetilde{X}}(X)$ (which in our case is equivalent to its simplified version $\mathrm{Higgs}''_{\widetilde{X}}(X)$) consists of pairs (L, γ), where L is a line bundle on $\widetilde{X}$ and γ is an isomorphism $(-1)^*(L^{-1}) \otimes R_X \simeq L$ such that the composition

$$L \otimes (-1)^*(R_X) \simeq (-1)^*((-1)^*(L^{-1}) \otimes R_X) \xrightarrow{(-1)^*(\gamma)} (-1)^*(L) \overset{\gamma}{\simeq} L \otimes (-1)^*(R_X)$$

is minus the identity map.

Let us visualize the equivalence $\mathrm{Higgs}_{\widetilde{X}}(X) \simeq \mathrm{Higgs}'_{\widetilde{X}}(X)$ of Theorem 6.4 in this case. Indeed, for any line bundle L on $\widetilde{X}$ we have a canonical S_2-equivariant isomorphism

$$p^*(\det(p_*(L))) \otimes R_X \simeq L \otimes (-1)^*(L).$$

Therefore, a data of γ' defines the data of γ, and it is easy to see that this sets up an equivalence.

8.3. $G = \mathrm{PGL}(2)$. In this case the only coroot is non-primitive, so one has to work a little harder.

By definition, objects of $\mathrm{Higgs}'_{\widetilde{X}}(X)$ are the following data:

- a line bundle L on $\widetilde{X}$,
- an S_2-equivariant isomorphism of line bundles $\gamma : L \otimes (-1)^*(L) \simeq R_X^{\otimes 2}$,
- an identification $\beta : L|_{D_X} \to R_X|_{D_X}$, which is compatible in the obvious sense with the restriction of γ to D_X.

Let us make the statement of Theorem 6.4 explicit in this case too. Starting from an object (E_G, σ, t) in $\mathrm{Higgs}_{\widetilde{X}}(X)$ we can locally choose a principal SL(2)-bundle E^1_G, which induces E_G. Then (E^1_G, σ, t) is an SL(2)-Higgs bundle. Using the above analysis for SL(2), we can attach to it a pair (L^1, γ^1), where L^1 is a line bundle on $\widetilde{X}$ and $\gamma^1 : (-1)^*((L^1)^{-1}) \otimes R_X \simeq L^1$.

The corresponding object of $\mathrm{Higgs}'_{\widetilde{X}}(X)$ is constructed as follows. We define the line bundle L as $(L^1)^{\otimes 2}$ and $\gamma := (\gamma^1)^{\otimes 2}$. The data of β comes from the sequence of isomorphisms

$$(L^1)^{-1} \otimes R_X|_{D^X} \simeq (-1)^*((L^1)^{-1}) \otimes R_X|_{D^X} \overset{\gamma^1}{\simeq} L^1|_{D_X}.$$

If we choose a different lifting of E_G to an SL(2)-bundle, the corresponding L^1 will be modified by tensoring with $p^*(L^0)$, where L^0 is a line bundle on X with $(L^0)^{\otimes 2} \simeq \mathcal{O}$, which will not affect the resulting (L, γ, β).

It is an easy exercise to check that the above construction defines an equivalence of categories.

9. Spectral covers versus cameral covers for $G = \mathrm{GL}(n)$

9.1. Observe first that a regular centralizer in $\mathfrak{gl}(n)$ is the same as an n-dimensional associative and commutative subalgebra in $\mathrm{Mat}(n, n)$ generated by one element.

Definition 9.2. An n-sheeted spectral cover of a scheme X is a finite flat scheme $p : \overline{X} \to X$ such that $p_*(\mathcal{O}_{\overline{X}})$ has rank n and is locally uni-generated as a sheaf of algebras.

Thus, a Higgs bundle for $\mathfrak{gl}(n)$ is the same as a rank n vector bundle E and an n-sheeted spectral cover $\overline{X} \to X$ with an embedding of bundles of algebras $p_*(\mathcal{O}_{\overline{X}}) \hookrightarrow \mathrm{End}_{\mathcal{O}_X}(E)$. This is equivalent to saying that E is a line bundle over $\overline{X}$.

In this section we will analyze the connection of this description of Higgs bundles for $\mathrm{GL}(n)$ with the one given by Theorem 6.4. The starting point is the observation that the category of S_n-cameral covers of X is naturally equivalent to the category of n-sheeted spectral covers. Let us describe the functors in both directions:

Given an S_n-cameral cover $\widetilde{X} \to X$, we define the scheme $\overline{\widetilde{X}}$ as $S_{n-1}\backslash\widetilde{X}$. Conversely, given an n-sheeted spectral cover $\overline{X} \to X$, we define $\widetilde{\overline{X}}$ to be the scheme that represents the functor of orderings of the sheets of $\overline{X} \to X$. This functor attaches to a scheme S the set of data consisting of

$$(\text{A map } S \to X \text{ and } n \text{ sections } t_i : S \to \overline{S} := \overline{X} \underset{X}{\times} S),$$

such that the characteristic polynomial of the multiplication action on $p_*(\mathcal{O}_{\overline{X}})$ of any function $f \in \mathcal{O}_{\overline{S}}$ equals $\prod\limits_i (Y - f \circ t_i)$, where Y is an indeterminate.

It is easy to see that this functor is indeed representable by a scheme that is finite over X. The group S_n acts on $\widetilde{\overline{X}}$ by permuting the t_i's.

Proposition 9.3. *The functors $\widetilde{X} \to \overline{\widetilde{X}}$ and $\overline{X} \to \widetilde{\overline{X}}$ send cameral covers to spectral covers and spectral covers to cameral covers, respectively. Moreover, they are inverses of one another.*

Proof. Let us consider first the universal situation: $X_0 = \mathrm{Spec}(\mathbb{C}[a_0, ..., a_{n-1}])$, $\widetilde{X}_0 = \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n])$, where the x_i's satisfy

$$\prod_i (Y - x_i) = Y^n + a_{n-1} \cdot Y^{n-1} + ... + a_1 \cdot Y + a_0,$$

and $\overline{X}_0 = \mathrm{Spec}(\mathbb{C}[x_1, a_0, ..., a_{n-1}])$, where x_1 satisfies

$$x_1^n + a_{n-1} \cdot x_1^{n-1} + ... + a_1 \cdot x_1 + a_0 = 0.$$

The natural maps $\widetilde{X}_0 \to X_0$ and $\overline{X}_0 \to X_0$ are a cameral and a spectral cover, respectively, and it is easy to see that in this case $\overline{\widetilde{X}}_0 \simeq \overline{X}_0$ and $\widetilde{\overline{X}}_0 \simeq \widetilde{X}_0$.

This proves the first assertion of the proposition. Indeed, any cameral (resp., spectral) cover is locally induced from $\widetilde{X}_0$ (resp., $\overline{X}_0$).

For a spectral cover $\overline{X}$ there is a natural map $\overline{\widetilde{\overline{X}}} \to \overline{X}$ that attaches to a map $S \to \widetilde{\overline{X}}$ given by an n-tuple $\{t_1, ..., t_n\}$ of maps $t_i : S \to \overline{S}$ the composition $S \overset{t_n}{\to} \overline{S} \to \overline{X}$. The resulting map $\overline{\widetilde{\overline{X}}} \to \overline{X}$ is an isomorphism, because this is so in the universal situation, i.e., for $\overline{\widetilde{X}}_0 \to X_0$.

Similarly, we have n maps $\widetilde{X} \to \overline{\widetilde{X}}$ which correspond to the natural map $S_n/S_{n-1} \times \widetilde{X} \to \overline{X}$. We claim that they define an isomorphism $\widetilde{X} \to \widetilde{\overline{\widetilde{X}}}$.

Indeed, both the fact that these maps satisfy the condition on the characteristic polynomial and that the resulting map is an isomorphism follow from the corresponding facts for $\widetilde{X}_0$. $\square$

9.4. Thus, fixing a spectral cover and fixing an S_n-cameral cover amounts to the same thing. Now, Theorem 6.4 implies that the category $\mathrm{Higgs}'_{\widetilde{X}}(X)$ is equivalent to the category of line bundles on the corresponding spectral cover $\overline{X}$. We would like to explain how to see this equivalence explicitly.

We start with the following observation:

Let $\widetilde{X} \to X$ be an S_n-cameral cover and let $\mathrm{Pic}_{\widetilde{X},n}(X)$ be the groupoid of S_n-equivariant line bundles L on $\widetilde{X}$ for which the following condition holds: For every reflection $s_{i,j} \in S_n$ the isomorphism

$$s^*_{i,j}(L) \to L$$

is the identity map on the fixed point set of $s_{i,j}$ in $\widetilde{X}$.

Proposition 9.5. *The pullback functor establishes an equivalence between the category of line bundles on X and $\mathrm{Pic}_{\widetilde{X},n}(X)$.*

Let us see first how this proposition implies what we need.

The natural map $\widetilde{X} \to \overline{X}$ is itself an S_{n-1}-cameral cover. On the one hand, by applying the above proposition to this map we obtain that the category of line bundles on $\overline{X}$ is equivalent to $\mathrm{Pic}_{\widetilde{X},n-1}(\overline{X})$. On the other hand, we claim that $\mathrm{Pic}_{\widetilde{X},n-1}(\overline{X})$ is equivalent to $\mathrm{Higgs}''_{\widetilde{X}}(X)$.

Indeed, let us identify the Cartan group of $\mathrm{GL}(n)$ with the product of n copies of $\mathbb{G}_m$ and let $\lambda_n : T \to \mathbb{G}_m$ be the weight corresponding to the last coordinate. Then a functor $\mathrm{Higgs}''_{\widetilde{X}}(X) \to \mathrm{Pic}_{\widetilde{X},n-1}(\overline{X})$ is given by $(\mathcal{L}, \gamma) \to L := \lambda_n(\mathcal{L})$. It is easy to see that this is indeed an equivalence.

9.6. Now let us prove Proposition 9.5. The argument will be a prototype of the one we are going to use to prove Theorem 6.4.

Given an object $L \in \mathrm{Pic}_{\widetilde{X},n}(X)$ and a point $x \in X$, we must find an étale neighbourhood of x such that, when restricted to the preimage of this neighbourhood, L becomes isomorphic to the unit object in $\mathrm{Pic}_{\widetilde{X},n}(X)$ (i.e., the one for which $L = \mathcal{O}_{\widetilde{X}}$ with the tautological S_n-structure).

First, it is easy to reduce the statement to the case when the ramification over x is the maximal possible, i.e., when x has only one geometric preimage $\tilde{x}$ in $\widetilde{X}$. Further, we can assume that X (and therefore also $\widetilde{X}$) is a spectrum of a local ring.

Choose some trivialization of L. Its discrepancy with the S_n-equivariant structure is a 1-cocycle $S_n \to \mathrm{Hom}(\widetilde{X}, \mathbb{G}_m)$. We must show that this cocycle is homologous to 0.

Let K denote the kernel of the map $\mathrm{Hom}(\widetilde{X}, \mathbb{G}_m) \to \mathbb{G}_m$ given by the evaluation at $\tilde{x}$. Our condition on L implies that the above cocycle $S_n \to \mathrm{Hom}(\widetilde{X}, \mathbb{G}_m)$ takes values in K. However, since $\widetilde{X}$ is local, K is divisible and torsion free. Hence $H^1(S_n, K) = 0$, so our cocycle is cohomologically trivial.

Part III. Basic Structure Results over $\overline{G/N}$

10. The structure of $\overline{G/N}$

10.1. The next two parts of this paper are devoted to the proofs of various results announced in the previous sections. We start with the proof of Proposition 1.3.

Proof. First we need to show that the map $\phi : \mathfrak{g}_{\text{reg}} \to \text{Ab}^r$ is well defined, which is equivalent to saying that the projection $\Gamma_{\text{reg}} \to \mathfrak{g}_{\text{reg}}$ is an isomorphism. Since the latter projection is proper and $\mathfrak{g}_{\text{reg}}$ is reduced, it is enough to show that the scheme-theoretic preimage in Γ_{reg} of every $x \in \mathfrak{g}_{\text{reg}}$ is isomorphic to $\text{Spec}(\mathbb{C})$.

This is clear on the level of $\mathbb{C}$ points, since by definition of regular elements, the only abelian r-dimensional subalgebra in $\mathfrak{g}$ that contains x is its own centralizer.

For $\mathfrak{a} \in \text{Ab}^r$, the tangent space $T_{\mathfrak{a}}(\text{Ab}^r)$ can be identified with the space of maps $T : \mathfrak{a} \to \mathfrak{g}/\mathfrak{a}$ that satisfy

$$\forall y_1, y_2 \in \mathfrak{a},\ [T(y_1), y_2] + [y_1, T(y_2)] = 0 \in \mathfrak{g}.$$

We claim that the tangent space to $\Gamma_{\text{reg}} \cap (\text{Ab}^r \times x)$ at $\mathfrak{a} \times x$ is zero. Indeed, this is the space of maps $T : \mathfrak{a} \to \mathfrak{g}/\mathfrak{a}$ as above, for which, moreover $[T(y), x] = 0,\ \forall y \in \mathfrak{a}$. However, since $\mathfrak{a} = Z_{\mathfrak{g}}(x)$, any such T is identically zero.

This implies that $T_{\mathfrak{a}\times x}(\Gamma_{\text{reg}} \cap \text{Ab}^r \times x) = 0$, which means that $\Gamma_{\text{reg}} \cap (\text{Ab}^r \times x)$ is reduced, i.e., $\simeq \text{Spec}(\mathbb{C})$.

Now let us show that ϕ is smooth. Let $\mathfrak{a} \in \text{Ab}^r$ be equal to $\phi(x)$. Using the above description of the tangent space to Ab^r, it is easy to see that $d\phi$ sends an element $u \in \mathfrak{g} \simeq T_x(\mathfrak{g}_{\text{reg}})$ to the unique map $T : \mathfrak{a} \to \mathfrak{g}/\mathfrak{a}$ that satisfies

$$[T(y), x] + [y, u] = 0,\ \forall y \in \mathfrak{a}.$$

Consider now the map $\mathbf{ev} : T_{\mathfrak{a}}(\text{Ab}^r) \to \mathfrak{g}/\mathfrak{a}$ given by $T \to T(x)$. The above description of $d\phi$ implies that the composition

$$\mathfrak{g} \simeq T_x(\mathfrak{g}_{\text{reg}}) \xrightarrow{d\phi} T_{\mathfrak{a}}(\text{Ab}^r) \xrightarrow{\mathbf{ev}} \mathfrak{g}/\mathfrak{a}$$

coincides with the tautological projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$. However, since x is regular, the fact that $[T(x), y] = -[x, T(y)],\ \forall y \in \mathfrak{a}$ implies that $\mathbf{ev}$ is an injection. We conclude that $\mathbf{ev}$ is an isomorphism, hence $\text{Im}(\phi)$ is contained in the smooth locus of Ab^r. Furthermore, $d\phi$ is surjective, so ϕ is smooth as claimed. $\square$

10.2. Let $\widetilde{\mathfrak{g}}$ be the closed subvariety in $\mathfrak{g} \times \mathcal{F}l$ defined by the condition: $(x, \mathfrak{b}') \in \widetilde{\mathfrak{g}}$ if $x \in \mathfrak{b}'$. Let $\widetilde{\mathfrak{g}}_{\text{reg}}$ denote the intersection $\widetilde{\mathfrak{g}} \cap (\mathfrak{g}_{\text{reg}} \times \mathcal{F}l)$ and let $\widetilde{\pi}$ denote the projection $\widetilde{\mathfrak{g}}_{\text{reg}} \to \mathfrak{g}_{\text{reg}}$. It is clear that as a variety, $\widetilde{\mathfrak{g}}_{\text{reg}}$ is smooth and connected, since it is an open subset in a vector bundle over $\mathcal{F}l$.

Proposition 10.3. *There exists a natural G-invariant map $\widetilde{\phi} : \widetilde{\mathfrak{g}}_{\text{reg}} \to \overline{G/T}$, such that the following square is Cartesian:*

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\text{reg}} & \xrightarrow{\widetilde{\phi}} & \overline{G/T} \\ {\scriptstyle \widetilde{\pi}}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathfrak{g}_{\text{reg}} & \xrightarrow{\phi} & \overline{G/N} \end{array}.$$

Proof. Consider the fibered product $\overline{G/T} \underset{\overline{G/N}}{\times} \mathfrak{g}_{\mathrm{reg}}$. By definition of $\overline{G/T}$, there is a closed embedding $\overline{G/T} \underset{\overline{G/N}}{\times} \mathfrak{g}_{\mathrm{reg}} \to \widetilde{\mathfrak{g}}_{\mathrm{reg}}$ that sends a triple $(\mathfrak{a} \in \overline{G/N}, \mathfrak{b}' \in \mathcal{Fl}, x \in \mathfrak{g}_{\mathrm{reg}}) \in \overline{G/T} \underset{\overline{G/N}}{\times} \mathfrak{g}_{\mathrm{reg}}$ to $(x, \mathfrak{b}') \in \widetilde{\mathfrak{g}}_{\mathrm{reg}}$.

We claim that this embedding is in fact an isomorphism. Indeed, the statement is obvious over the preimage in $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ of the regular semisimple locus of $\mathfrak{g}$. Therefore, the two schemes coincide at the generic point of $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$. This implies what we need since $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ is reduced. □

Now we are ready to prove Proposition 1.5.

Proof. The map $\widetilde{\phi} : \widetilde{\mathfrak{g}}_{\mathrm{reg}} \to \overline{G/T}$ is smooth since it is a base change of a smooth map. Hence, the fact that $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ is smooth and connected implies that $\overline{G/T}$ has the same properties.

A well known theorem of Kostant (cf. [22] or [7], p. 277) says that the restriction of the Chevalley map $\mathfrak{g} \to \mathfrak{t}/W$ to $\mathfrak{g}_{\mathrm{reg}}$ is smooth and that it gives rise to a Cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\mathrm{reg}} & \longrightarrow & \mathfrak{t} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{g}_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}/W \end{array}.$$

Therefore, the natural action of W on the preimage in $\widetilde{\mathfrak{g}}$ of the regular semisimple locus in $\mathfrak{g}$ extends to the whole of $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$. The same is true for $\overline{G/T}$ because the map $\widetilde{\phi}$ is flat and surjective. The étale local isomorphism follows from comparison of our Cartesian square with that of Proposition 10.3. □

10.4. Now let us prove Proposition 1.8.

Proof. Let Δ' be as in the formulation of the proposition. Consider an element $t \in \mathfrak{t}$ such that $\alpha(t) = 0$ for $\alpha \in \Delta'$ and $\beta(t) \neq 0$ for $\beta \notin \Delta'$.

In this case $\mathfrak{m} := Z_{\mathfrak{g}}(t)$ is a Levi subalgebra of $\mathfrak{g}$. Let M be the corresponding Levi subgroup. It is well known that $\mathfrak{m} \cap \mathfrak{b}$ is a Borel subalgebra in $\mathfrak{m}$. Let u be an element in the unipotent radical of $\mathfrak{m} \cap \mathfrak{b}$, which is regular with respect to M.

We then see that $x = t + u$ is a regular element in $\mathfrak{g}$ since $Z_{\mathfrak{g}}(x) = Z_{\mathfrak{m}}(u)$. It is known that if a Borel subalgebra contains a regular element, then it also contains its centralizer (cf. Lemma 11.5). Therefore, $(Z_{\mathfrak{m}}(u), \mathfrak{b}) \in \overline{G/T}$. Moreover, it is easy to see that every pair $(\mathfrak{a}, \mathfrak{b}') \in \overline{G/T}$ is G-conjugate to one of the above form.

To conclude the proof, it remains to show that $(Z_{\mathfrak{m}}(u), \mathfrak{b}) \in \underset{\alpha \in \Delta'}{\cap} (D^{\alpha}) \setminus \underset{\beta \notin \Delta'}{\cup} (D^{\beta})$. For that, it suffices to show that the image of $(t + u, \mathfrak{b})$ as above under $\widetilde{\mathfrak{g}}_{\mathrm{reg}} \to \mathfrak{t}$ belongs to the corresponding locus of $\mathfrak{t}$. However, the above image is just t, which makes the assertion obvious. □

10.5. Levi subgroups. Let $J \subset I$ be a subset. It defines a root subsystem Δ_J and let M_J (resp., $P_J \subset G$, $W_J \subset W$) denote the corresponding standard Levi subgroup (resp., standard parabolic, Weyl subgroup). Let N_{M_J} be the intersection $M_J \cap N$, which is the normalizer of T in M_J.

It is easy to see that the natural map $M_J/N_{M_J} \to G/N$ extends to a map $i_J : \overline{M_J/N_{M_J}} \to \overline{G/N}$. In fact, $\overline{M_J/N_{M_J}}$ is a closed subvariety of $\overline{G/N}$ which corresponds to $\{\mathfrak{a} \in \overline{G/N} \mid \mathfrak{a} \subset \mathfrak{m}_J\}$.

Proposition 10.6. *There is a canonical W-equivariant isomorphism*

$$\widetilde{i}_J : W \overset{W_J}{\times} \overline{M_J/T} \simeq \overline{M_J/N_{M_J}} \underset{\overline{G/N}}{\times} \overline{G/T}.$$

Proof. First, we have a natural closed embedding

$$\overline{M_J/T} \to \overline{M_J/N_{M_J}} \underset{\overline{G/N}}{\times} \overline{G/T} \subset \overline{G/T}.$$

Its image consists of pairs $(\mathfrak{a}, \mathfrak{b}') \in \overline{G/T}$ such that $\mathfrak{a} \subset \mathfrak{m}_J$ and $\mathfrak{b}' \subset \mathfrak{p}_J := \mathrm{Lie}(P_J)$.

This map is compatible with the W_J-action. Hence, it extends to a finite map

$$\widetilde{i}_J : W \overset{W_J}{\times} \overline{M_J/T} \to \overline{M_J/N_{M_J}} \underset{\overline{G/N}}{\times} \overline{G/T}.$$

Since both varieties are smooth, in order to prove that $\widetilde{i}_J$ is an isomorphism, it suffices to do so over the open part, i.e., over M_J/N_{M_J}. However, in the latter case, the assertion becomes obvious. $\square$

It is easy to see that the G-orbit of $\overline{M_J/N_{M_J}} \underset{\overline{G/N}}{\times} \overline{G/T} \subset \overline{G/T}$ (resp., $\overline{M_J/T} \subset \overline{G/T}$) is the union of those $D^{\Delta'}$ for which Δ' is W-conjugate to a subset of Δ_J (resp., $\Delta' \subset \Delta_J$.)

11. The group scheme of centralizers

In this section we will formulate two basic theorems, Theorem 11.6 and Theorem 11.8, which will be used for the proof of our first main result, Theorem 4.4.

11.1. The universal centralizers $\mathcal{C}$ and c. Consider the constant group scheme $G \times \overline{G/N}$ over $\overline{G/N}$, and let $\mathcal{C} \subset G \times \overline{G/N}$ be its closed group subscheme of "centralizers". In other words, $\mathcal{C}$ is defined by the condition that $(g \in G, \mathfrak{a} \in \overline{G/N}) \in \mathcal{C}$ if g commutes with $\mathfrak{a}$. Clearly, $\mathcal{C}$ is equivariant with respect to the G-action on $\overline{G/N}$.

Note that the corresponding bundle $\mathbf{c}$ of Lie algebras can be identified with the tautological rank r vector bundle over $\overline{G/N}$ which comes from the embedding $\overline{G/N} \subset \mathrm{Gr}^r_{\mathfrak{g}}$. Another interpretation of this $\mathbf{c}$, considered as a subbundle of the trivial bundle $\mathfrak{g} \times \overline{G/N}$, is that it is the family $\mathbf{c}_{\overline{G/N}}$ of centralizers of the universal Higgs bundle on $\overline{G/N}$, which was studied in detail in Section 7. (Recall from Section 2 that a Higgs bundle (E_G, σ) on any X determines, and is determined by, a subbundle $\mathbf{c}_X$ consisting of regular centralizer subalgebras of the adjoint bundle $\mathfrak{g}_{E_G}$.)

Proposition 11.2. *The group scheme $\mathcal{C}$ is commutative and smooth over $\overline{G/N}$ and is irreducible as a variety.*

Proof. Let $\mathcal{C}'$ be the group subscheme of $G \times \mathfrak{g}_{\mathrm{reg}}$ over $\mathfrak{g}_{\mathrm{reg}}$ defined by the condition

$$\mathcal{C}' := \{(g, x) \in G \times \mathfrak{g}_{\mathrm{reg}} \mid \mathrm{Ad}_g(x) = x\}.$$

First, let us show that $\mathcal{C}'$ is commutative and smooth over $\mathfrak{g}_{\mathrm{reg}}$.

Let (g, x) be a $\mathbb{C}$-point of $\mathcal{C}'$. The tangent space to $\mathcal{C}'$ at (g, x) consists of pairs $(\xi, y) \in \mathfrak{g} \times \mathfrak{g}$ such that $\mathrm{Ad}_g([x,\xi]) = \mathrm{Ad}_g(y) - y$. The differential of the map $\mathcal{C}' \to \mathfrak{g}_{\mathrm{reg}}$ sends (ξ, y) to y. We claim that it is surjective.

It is known that if G is of adjoint type, then the centralizer of every regular element is connected. (In particular, each $Z_G(x)$ is commutative; this holds even if G is not of adjoint type.). Therefore, $\underset{g\in Z_G(x)}{\mathrm{Span}}\,(\mathrm{Ad}_g(y) - y) = \mathrm{Im}(\mathrm{ad}_{Z_{\mathfrak{g}}(x)})$. However, the latter, as we saw in the proof of Proposition 1.3, coincides with $\mathrm{Im}(\mathrm{ad}_x)$ since x is regular.

To prove that $\mathcal{C}'$ is smooth over $\mathfrak{g}_{\mathrm{reg}}$, it remains to observe that the fibers of $\mathcal{C}'$ are smooth (since they are algebraic groups in char.0) and all have dimension r, by the definition of $\mathfrak{g}_{\mathrm{reg}}$. The fact that $\mathcal{C}'$ is commutative was established in the course of the above argument.

Now let us prove the assertion for $\mathcal{C}$. We have a natural closed embedding $\mathcal{C} \underset{\overline{G/N}}{\times} \mathfrak{g}_{\mathrm{reg}} \to \mathcal{C}'$, which is an isomorphism over the regular semisimple locus of $\mathfrak{g}_{\mathrm{reg}}$. Hence, it is an isomorphism because $\mathcal{C}'$ is reduced. Therefore, since the map $\phi : \mathfrak{g}_{\mathrm{reg}} \to \overline{G/N}$ is flat and surjective, this shows that $\mathcal{C}$ is commutative and smooth over $\overline{G/N}$. It is irreducible because this is obviously true over G/N. $\square$

11.3. The group scheme $\mathcal{T}$. Now we will introduce another group scheme over $\overline{G/N}$, seemingly of a different nature. Recall the sheaves $\overline{T}_{\widetilde{X}}, T_{\widetilde{X}}$ introduced in section 4.

Consider the contravariant functor Schemes $\to$ Groups which assigns to a scheme S the set of pairs

$$(\text{A map } S \to \overline{G/N}, \text{ a } W\text{-equivariant map } \widetilde{S} := S \underset{\overline{G/N}}{\times} \overline{G/T} \to T).$$

It is easy to see that this functor is representable by an abelian group scheme over $\overline{G/N}$, which we will denote by $\overline{\mathcal{T}}$. Therefore, once $S \to \overline{G/N}$ is fixed, $\mathrm{Hom}_{\overline{G/N}}(S, \overline{\mathcal{T}}) \simeq \Gamma(S, \overline{T}_{\widetilde{S}})$. In other words, $\overline{\mathcal{T}}$ represents the sheaf $\overline{T}_{\overline{G/T}}$ on $\mathrm{Sch}_{\mathrm{et}}(\overline{G/N})$. Clearly, the G-action on $\overline{G/T}$ gives rise to a G-action on $\overline{\mathcal{T}}$.

We define the open group subscheme $\mathcal{T}$ of $\overline{\mathcal{T}}$ by the following condition $(**)$:

$\mathrm{Hom}(S, \mathcal{T})$ consists of those pairs $(S \to \overline{G/N}, \widetilde{S} \to T)$ as above, for which for every root α the composition

$$S \underset{\overline{G/N}}{\times} D^\alpha \hookrightarrow \widetilde{S} \to T \overset{\alpha}{\to} \mathbb{G}_m$$

avoids $-1 \in \mathbb{G}_m$.

Since for any map $S \to \overline{\mathcal{T}}$, the above composition takes values in $\pm 1 \subset \mathbb{G}_m$, condition $(**)$ is equivalent to condition $(*)$ in the definition of the sheaf $T_{\widetilde{S}}$ (cf. Section 4.2): for a fixed map $S \to \overline{G/N}$, $\mathrm{Hom}_{\overline{G/N}}(S, \mathcal{T}) \simeq \Gamma(S, T_{\widetilde{S}})$, i.e., the group scheme $\mathcal{T}$ represents the sheaf $T_{\overline{G/T}}$ on $\mathrm{Sch}_{\mathrm{et}}(\overline{G/N})$.

11.4. A remarkable fact is that the group schemes $\mathcal{C}$ and $\mathcal{T}$ are canonically isomorphic. Here we will construct a map between them in one direction.

Let $\mathcal{B}$ denote the universal group scheme of Borel subgroups over $\mathcal{Fl}$. Let us denote by $\widetilde{\mathcal{B}}$ its pullback to $\overline{G/T}$. In addition, let us denote by $\widetilde{\mathcal{C}}$ the pullback of $\mathcal{C}$ to $\overline{G/T}$.

Both $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$ are group subschemes of the constant group scheme $G \times \overline{G/T}$.

Lemma 11.5. *$\widetilde{\mathcal{C}}$ is a closed group subscheme of $\widetilde{\mathcal{B}}$.*

Indeed, since $\widetilde{\mathcal{C}}$ is reduced and irreducible, it suffices to check that over G/N, $\widetilde{\mathcal{C}}$ is contained in $\widetilde{\mathcal{B}}$. However, this is obvious.

We have a natural projection $\mathcal{B} \to T \times \mathcal{F}l$. By composing it with the inclusion of Lemma 11.5, we obtain a map

$$\mathcal{C} \underset{G/N}{\times} \overline{G/T} \to T.$$

This map respects the group law on $\mathcal{C}$ and T and commutes with the W-action. (This is because it suffices to check both facts after the restriction to G/N, where they become obvious.)

Hence, we obtain a homomorphism of group schemes $\overline{\chi} : \mathcal{C} \to \overline{\mathcal{T}}$.

Theorem 11.6. *The above map $\overline{\chi} : \mathcal{C} \to \overline{\mathcal{T}}$ defines an isomorphism $\chi : \mathcal{C} \to \mathcal{T}$.*

The proof will be given in the next section.

11.7. Now we will formulate the second key result which will be used in the proof of Theorem 4.4.

Consider the functor that assigns to a scheme S the set of triples $(\overline{G/N}^1_S, \overline{G/N}^2_S, \nu)$, where $\overline{G/N}^1_S$ and $\overline{G/N}^2_S$ are two S-points of $\overline{G/N}$ and ν is a W-equivariant isomorphism

$$\nu : \widetilde{S}^1 \to \widetilde{S}^2,$$

where $\widetilde{S}^i$ is the W-cover of S induced by $\overline{G/N}^i_S$ from $\pi : \overline{G/T} \to \overline{G/N}$.

It is easy to see that this functor is representable. Let $\mathcal{H}$ denote the representing scheme. Since the W-cover $\overline{G/T} \to \overline{G/N}$ is G-equivariant, we obtain a natural map $\xi : G \times \overline{G/N} \to \mathcal{H}$ which covers the map $G \times \overline{G/N} \xrightarrow{\text{Action} \times \text{id}} \overline{G/N} \times \overline{G/N}$.

Theorem 11.8. *The above map $\xi : G \times \overline{G/N} \to \mathcal{H}$ is smooth and surjective.*

This theorem will be proven in Section 13.

11.9. The scheme $\mathcal{H}$ lives over $\overline{G/N} \times \overline{G/N}$. Let $\mathcal{H}_\Delta$ denote its restriction to the diagonal. By definition, $\mathcal{H}_\Delta$ is a group scheme over $\overline{G/N}$ which represents the functor of W-equivariant automorphisms of $\overline{G/T}$ over $\overline{G/N}$.

Let $\mathrm{St} \subset G \times \overline{G/N}$ be the closed group subscheme of stabilizers, i.e.,

$$(g, \mathfrak{a}) \in \mathrm{St} \text{ if } \mathrm{Ad}_g(\mathfrak{a}) = \mathfrak{a}.$$

Obviously, $\mathcal{C}$ is a closed normal group subscheme of St.

The map $\xi : G \times \overline{G/N} \to \mathcal{H}$ gives rise to a map $\xi_\Delta : \mathrm{St} \to \mathcal{H}_\Delta$.

Proposition 11.10. *$\mathcal{H}_\Delta$ represents the quotient group scheme $\mathrm{St}/\mathcal{C}$.*

Proof. Theorem 11.8 implies that the map $\xi_\Delta : \mathrm{St} \to \mathcal{H}_\Delta$ is smooth and surjective. Therefore, all we have to show is that if $S \to \mathrm{St}$ is a map such that the induced automorphism of $\widetilde{S}$ is trivial, then S maps to $\mathcal{C}$.

Observe that $\mathcal{H}_\Delta$ acts on $\mathcal{T}$ via its action on $\overline{G/T}$. Since the isomorphism $\chi : \mathcal{C} \to \mathcal{T}$ is G-equivariant, we obtain a commutative diagram of actions:

$$\begin{array}{ccc} \mathrm{St} \underset{\overline{G/N}}{\times} \mathcal{C} & \longrightarrow & \mathcal{C} \\ {\scriptstyle \xi_\Delta \times \chi}\big\downarrow & & {\scriptstyle \chi}\big\downarrow \\ \mathcal{H}_\Delta \underset{\overline{G/N}}{\times} \mathcal{T} & \longrightarrow & \mathcal{T} \end{array},$$

where the top horizontal arrow is the adjoint action. Therefore, if a map $S \to \mathrm{St}$ induces the trivial automorphism of $\widetilde{S}$, its adjoint action on $\mathcal{C}$ is trivial too. But this means that it factors through $\mathcal{C}$. $\square$

Similarly, one shows:

Corollary 11.11. *The scheme $\mathcal{H}$ represents the quotient group scheme $(G \times \overline{G/N})/\mathcal{C}$.*

11.12. Here is one more interpretation of Theorem 11.8. Clearly, the scheme $\mathcal{H}$ with its two projections to $\overline{G/N}$ is a groupoid over that latter scheme. According to Theorem 11.8, the above projections are smooth and, therefore, we can consider the algebraic stack $\mathcal{H}\backslash(\overline{G/N})$.

Corollary 11.13. *The stack $\mathcal{H}\backslash(\overline{G/N})$ is canonically isomorphic to the stack* **Cam** *of Section* 2.4.

12. Proof of Theorem 11.6

12.1. We start by establishing a result on compatibility of our objects with restrictions to Levi subgroups. We then verify the Lie-algebraic version of the theorem by restricting to an $\mathfrak{sl}(2)$ subalgebra, and finally we refine this to prove the desired group-theoretic version.

12.2. Let $M = M_J$ be a standard Levi subgroup of G (cf. Section 10.5) and let $\mathcal{C}_M$ be the corresponding sheaf of centralizers over $\overline{M/N_M}$.

On the one hand, there is a natural closed embedding

$$\mathcal{C}_M \hookrightarrow i_J^*(\mathcal{C}) := \overline{M/N_M} \underset{\overline{G/N}}{\times} \mathcal{C}.$$

On the other hand, we have the group scheme $\overline{\mathcal{T}}$ over $\overline{G/N}$, as well as the group scheme $\overline{\mathcal{T}}_M$ over $\overline{M/N_M}$. This time, by Proposition 10.6, we have a canonical isomorphism

$$\overline{\mathcal{T}}_M \simeq i_J^*(\overline{\mathcal{T}}) := \overline{M/N_M} \underset{\overline{G/N}}{\times} \overline{\mathcal{T}}.$$

Moreover, it induces an isomorphism $\mathcal{T}_M \simeq i_J^*(\mathcal{T})$, since if a root α is not W-conjugate to a root in M, then s_α has no fixed points on $W \overset{W_J}{\times} \overline{M/T}$.

Proposition 12.3. *The map $\mathcal{C}_M \to i_J^*(\mathcal{C})$ is an isomorphism. Moreover, the diagram*

$$\begin{array}{ccc} \mathcal{C}_M & \xrightarrow{\chi_M} & \overline{\mathcal{T}}_M \\ \downarrow & & \downarrow \\ i_J^*(\mathcal{C}) & \xrightarrow{i_J^*(\chi)} & i_J^*(\overline{\mathcal{T}}) \end{array}$$

is commutative.

Proof. The map $\mathcal{C}_M \hookrightarrow i_J^*(\mathcal{C})$ is an isomorphism because it is a closed embedding and at the same time an isomorphism over the generic point of $\overline{M/N_M}$. Commutativity of the diagram can be checked over the preimage of M/N_M, in which case it becomes obvious. $\square$

12.4. We will now prove the assertion of Theorem 11.6 on the Lie-algebra level.

Let $\mathbf{t}$ denote the sheaf of Lie algebras corresponding to $\overline{\mathcal{T}}$. Obviously, it is isomorphic to $\mathrm{Lie}(\mathcal{T})$ as well. By definition, we have $\mathbf{t} \simeq (\mathfrak{t} \otimes \pi_*(\mathcal{O}_{\overline{G/T}}))^W$. Since $\pi_*(\mathcal{O}(\overline{G/T}))$ is locally isomorphic to $\mathcal{O}_{\overline{G/N}} \otimes \mathbb{C}[W]$, $\mathbf{t}$ is a vector bundle of rank r over $\overline{G/N}$.

On the other hand, recall that in subsection 11.1 we defined the sheaf $\mathbf{c}$ of Lie algebras corresponding to $\mathcal{C}$. Our map $\overline{\chi} : \mathcal{C} \to \overline{\mathcal{T}}$ induces a map $d\overline{\chi} : \mathbf{c} \to \mathbf{t}$ which, for simplicity, we abbreviate as $d\chi : \mathbf{c} \to \mathbf{t}$.

Proposition 12.5. *The map $d\chi : \mathbf{c} \to \mathbf{t}$ is an isomorphism.*

Proof. The proof will consist of two steps. The first step will be a reduction to the case of $\mathrm{SL}(2)$ and the second one will be a proof of the assertion for $\mathrm{SL}(2)$.

Step 1. Both $\mathbf{c}$ and $\mathbf{t}$ are vector bundles of rank r over $\overline{G/N}$ and the map $d\chi$ is clearly an isomorphism over G/N. Since the variety $\overline{G/N}$ is smooth, it remains to show that $d\chi$ is an isomorphism on an open subset of $\overline{G/N}$ whose complement has codimension at least 2.

It follows from Section 10.5 that such an open subset is formed by the union of the G-orbits of the images of $i_J(\overline{M_J/N_{M_J}})$, where $J = \{\alpha_j\}$ for all simple roots α_j. Therefore, by G-equivariance and by Proposition 12.3, it suffices to show that the map

$$d\chi_{M_J} : \mathbf{c}_{M_J} \to \mathbf{t}_{M_J}$$

is an isomorphism. This reduces us to the case when G is a reductive group of semi-simple rank 1. Moreover, the statement is clearly invariant under isogenies, so we may replace G by $Z^0(G) \times [G, G]$. Clearly, the assertion in such a case is equivalent to the one for $[G, G]$, which in turn can be replaced by $\mathrm{SL}(2)$.

Step 2. For $G = \mathrm{SL}(2)$, the variety $\overline{G/N}$ can be identified with $\mathbb{P}^2$ in such a way that the sheaf $\mathbf{c}$ goes over to $\mathcal{O}(-1)$. Moreover, $\overline{G/T} \to \overline{G/N}$ can be identified with the S_2-cover $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$.

To prove the assertion, it is enough to show that $\mathbf{t}$ has degree -1, since any non-zero map between two line bundles of the same degree is automatically an isomorphism.

By definition, $\mathbf{t}$ is the $\mathcal{O}(\mathbb{P}^2)$-module of anti-invariants of S_2 in $\pi_*(\mathcal{O}(\mathbb{P}^1 \times \mathbb{P}^1))$. Therefore,

$$\mathbf{t} \simeq \det(\pi_*(\mathcal{O}(\mathbb{P}^1 \times \mathbb{P}^1))) \simeq \mathcal{O}(-1). \quad \square$$

12.6. Now we will check that the map χ induces an isomorphism between $\mathbb{C}$-points of $\mathcal{C}$ and $\mathcal{T}$. Evidently, this assertion, combined with Proposition 12.5 and Proposition 11.2, implies Theorem 11.6.

Let $\mathfrak{a} \in \overline{G/N}$ be the centralizer of a regular element $x \in \mathfrak{g}$. As we saw in the proof of Proposition 11.2, on the one hand, the fiber of $\mathcal{C}$ at $\mathfrak{a} = Z_{\mathfrak{g}}(x)$ can be identified with $Z_G(x)$. On the other hand, the fiber of $\mathcal{T}$ at $\mathfrak{a}$ can be identified with $\mathrm{Hom}_W(\mathcal{Fl}^x, T)^{**}$, where $\mathcal{Fl}^x$ is the fixed point scheme of the vector field induced by x on $\mathcal{Fl}$ and the superscript $**$ corresponds to the $(**)$ condition in the definition of $\mathcal{T}$.

Let $x = x^{\mathrm{ss}} + x^{\mathrm{nil}}$ be the Jordan decomposition of x. We can assume that $Z_{\mathfrak{g}}(x^{\mathrm{ss}})$ is a standard Levi subalgebra $\mathfrak{m}$ and x^{nil} is a regular nilpotent element in $\mathfrak{m}$. Using Proposition 12.3, we can replace G by M and hence we can assume that x^{ss} is a central element in $\mathfrak{g}$.

There are natural embeddings $Z(G) \times \overline{G/N} \to \mathcal{C}$ and $Z(G) \times \overline{G/N} \to \mathcal{T}$, which make the diagram

$$\begin{array}{ccc} Z(G) \times \overline{G/N} & \longrightarrow & \mathcal{C} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \chi} \\ Z(G) \times \overline{G/N} & \longrightarrow & \overline{\mathcal{T}} \end{array}$$

commute.

Proposition 12.7. *Let x be a regular nilpotent element and let*

$$Z_G(x) = Z_G(x)^{\mathrm{ss}} \times Z_G(x)^{\mathrm{nil}},$$
$$\mathrm{Hom}_W(\mathcal{Fl}^x, T)^{**} = \mathrm{Hom}_W(\mathcal{Fl}^x, T)^{\mathrm{ss},**} \times \mathrm{Hom}_W(\mathcal{Fl}^x, T)^{\mathrm{nil},**}$$

be the Jordan decompositions of the fibers of $\mathcal{C}$ and $\mathcal{T}$ at $Z_{\mathfrak{g}}(x)$. Then the embedding of $Z(G)$ induces isomorphisms

$$Z(G) \simeq Z_G(x)^{\mathrm{ss}} \text{ and } Z(G) \simeq \mathrm{Hom}_W(\mathcal{Fl}^x, T)^{\mathrm{ss},**}.$$

It is clear, first of all, that this proposition implies the theorem. Indeed, it is enough to show that χ induces an isomorphism $Z_G(x)^{\mathrm{nil}} \to \mathrm{Hom}_W(\mathcal{Fl}^x, T)^{\mathrm{nil},**}$. But since these groups are unipotent, our assertion follows from the corresponding assertion on the Lie-algebra level, which has been proved before.

Proof. The fact that $Z(G) \simeq Z_G(x)^{\mathrm{ss}}$ is an immediate consequence of the fact that in a group of adjoint type centralizers of regular elements are connected.

To prove that $Z(G) \simeq \mathrm{Hom}_W(\mathcal{Fl}^x, T)^{\mathrm{ss},**}$, let us observe that if x is a regular nilpotent element, $\mathcal{Fl}^x$ is a local non-reduced scheme. Its closed point, viewed as a point of $\overline{G/T}$, belongs to the intersection of all the D^α's.

Let $\mathrm{Hom}_W(\mathcal{Fl}^x, T)_1$ be the subgroup of $\mathrm{Hom}_W(\mathcal{Fl}^x, T)$ which corresponds to maps $\mathcal{Fl}^x \to T$ that send the closed point of $\mathcal{Fl}^x$ to the identity in T. Clearly, $\mathrm{Hom}_W(\mathcal{Fl}^x, T)_1$ is unipotent and $\mathrm{Hom}_W(\mathcal{Fl}^x, T) \simeq \mathrm{Hom}_W(\mathcal{Fl}^x, T)_1 \times T^W$ is the Jordan decomposition of $\mathrm{Hom}_W(\mathcal{Fl}^x, T)$.

The proof is concluded by the observation that $Z(G) = \{t \in T^W \mid \alpha(t) = 1, \forall\, \alpha \in \Delta\}$, which is exactly the $(**)$ condition. $\square$

13. Proof of Theorem 11.8

13.1. We will need an additional property of the isomorphism χ.

By definition, we have a canonical W-equivariant map

$$\mathbf{t} \underset{G/N}{\times} \overline{G/T} \to \mathfrak{t},$$

hence we obtain a map $\mathbf{t} \to \mathfrak{t}/W$.

Lemma 13.2. *The above map coincides with the composition*

$$\mathbf{t} \xrightarrow{\chi^{-1}} \mathbf{c} \hookrightarrow \mathfrak{g} \times \overline{G/N} \to \mathfrak{g} \to \mathfrak{t}/W,$$

where the last arrow is the Chevalley map.

The proof follows from the fact that the two maps coincide over G/N.

13.3. Since $G \times \overline{G/N}$ is smooth, to prove the theorem, we need to show that any map $S \to \mathcal{H}$ can be lifted, locally in the étale topology, to a map $S \to G \times \overline{G/N}$.

Thus, let $\mathfrak{a}^1$ and $\mathfrak{a}^2$ be two S-points of $\overline{G/N}$ and let $\nu : \widetilde{S}^1 \to \widetilde{S}^2$ be an isomorphism between the corresponding cameral covers. The maps $\mathfrak{a}^i$ give rise to vector subbundles $\mathbf{c}^i_S \subset \mathfrak{g} \otimes \mathcal{O}_S$, and Theorem 11.6 implies that

$$\mathbf{c}^i_S \simeq \mathrm{Hom}_{W,\mathcal{O}_S}(\mathfrak{t}^*, \mathcal{O}_{\widetilde{S}^i}),\ i = 1, 2.$$

Therefore, the data of ν defines an isomorphism of vector bundles $\nu' : \mathbf{c}^1_S \to \mathbf{c}^2_S$.

By Proposition 1.3 we can find a section $x^1_S \in \mathbf{c}^1_S$, such that $\mathbf{c}^1_S = Z_{\mathfrak{g}}(x^1_S)$. Let $x^2_S \in \mathbf{c}^2_S$ be the image of x^1_S under ν'. By making the choice of x^1_S sufficiently generic, we can assume that x^2_S is regular, i.e., that $\mathbf{c}^2_S = Z_{\mathfrak{g}}(x^2_S)$.

Consider x^i_S, $i = 1, 2$ as maps $S \to \mathfrak{g}_{\mathrm{reg}}$. Lemma 13.2 implies that their compositions with the Chevalley map

$$S \xrightarrow{x^i_S} \mathfrak{g}_{\mathrm{reg}} \to \mathfrak{t}/W$$

coincide. Now, we have the following general assertion that follows from smoothness of the Chevalley map restricted to $\mathfrak{g}_{\mathrm{reg}}$:

Lemma 13.4. *The adjoint action map* $G \times \mathfrak{g}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}} \underset{\mathfrak{t}/W}{\times} \mathfrak{g}_{\mathrm{reg}}$ *is smooth and surjective.*

Therefore, locally there exists a map $g_S : S \to G$ that conjugates x^1_S to x^2_S. Then this map conjugates $\mathbf{c}^1_S$ to $\mathbf{c}^2_S$, which is what we had to prove.

13.5. Complements. We conclude this section by two remarks regarding the assertions of Theorem 11.6 and Theorem 11.8.

First, let us fix a $\mathbb{C}$-point $\mathfrak{a} \in \overline{G/N}$ and let $\varphi : \mathcal{F}l^{\mathfrak{a}} \to \mathfrak{t}$ be a W-equivariant map, which according to Theorem 11.6, is the same as an element $x_\varphi \in \mathfrak{a} = \mathbf{c}_{\mathfrak{a}}$. One may wonder: how can one express the condition that x_φ is a regular element of $\mathfrak{a}$ in terms of φ?

Lemma 13.6. *The necessary and sufficient condition for x_φ to be a regular element of $\mathfrak{a}$ is that $\varphi : \mathcal{F}l^{\mathfrak{a}} \to \mathfrak{t}$ is a scheme-theoretic embedding.*

Proof. First, one easily reduces the assertion to the case when $\mathfrak{a}$ is the centralizer of a regular nilpotent element, which we will assume. In this case, $\mathfrak{a}$ entirely consists of nilpotents elements. Let $\mathrm{St}_G(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$. Since the nilpotent locus in $\mathfrak{g}_{\mathrm{reg}}$ is a single G-orbit, we obtain that $\mathfrak{a} \cap \mathfrak{g}_{\mathrm{reg}}$ is a single $\mathrm{St}_G(\mathfrak{a})$-orbit.

Thus, let ϕ be an embedding. To show that x_φ is regular, it is enough to show that its centralizer in $\mathrm{St}_G(\mathfrak{a})$ coincides with $\mathcal{C}_\mathfrak{a}$.

By Proposition 11.10, the quotient $\mathrm{St}_G(\mathfrak{a})/\mathcal{C}_\mathfrak{a}$ maps isomorphically to the group of W-equivariant automorphisms of $\mathcal{F}l^\mathfrak{a}$. If for some $n \in \mathrm{St}_G(\mathfrak{a})$ we have $\mathrm{Ad}_n(x_\varphi) = x_\varphi$, then n acts trivially on $\mathcal{F}l^\mathfrak{a}$ since ϕ is an embedding. Hence $n \in \mathcal{C}_\mathfrak{a}$.

To prove the implication in the other direction, let us observe that $\mathfrak{a} \cap \mathfrak{g}_{\mathrm{reg}}$ is the only $\mathrm{St}_G(\mathfrak{a})$-invariant open subset of $\mathfrak{a}$ consisting of regular elements only. However, the locus of φ that are embeddings is clearly such a subset. $\square$

Secondly, let us see how Corollary 11.11 is related to Proposition 1.8:

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two $\mathbb{C}$-points of $\overline{G/N}$. Corollary 11.11 says that they are G-conjugate if and only if $\pi^{-1}(\mathfrak{a}_1) \simeq \pi^{-1}(\mathfrak{a}_2)$ as W-schemes. The condition of Proposition 1.8 is seemingly weaker (but in fact, equivalent): it implies that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are G-conjugate if and only if $(\pi^{-1}(\mathfrak{a}_1))_{\mathrm{red}} \simeq (\pi^{-1}(\mathfrak{a}_2))_{\mathrm{red}}$ as W-schemes.

Part IV. Proofs of the Main Results

14. Proof of Theorem 4.4

14.1. We now deduce our theorem from Theorem 11.6 and Theorem 11.8 combined with the following "abstract nonsense" observation:

Lemma 14.2. *Let $\mathcal{Q}$ be a sheaf of categories on $\mathrm{Sch}_{\mathrm{et}}(X)$, and $\mathcal{A}$ be a sheaf of abelian groups on $\mathrm{Sch}_{\mathrm{et}}(X)$. Suppose that for every $(U \to X) \in \mathrm{Sch}_{\mathrm{et}}(X)$ and every $C \in \mathcal{Q}(U)$, we are given an isomorphism $\mathrm{Aut}_{\mathcal{Q}(U)}(C) \simeq \mathcal{A}(U)$ such that the following conditions hold:*

(0) *There exists a covering $U \to X$ such that $\mathcal{Q}(U)$ is non-empty.*

(1) *If $C_1 \to C_2$ is an isomorphism between two objects in $\mathcal{Q}(U)$, then the induced isomorphism $\mathrm{Aut}_{\mathcal{Q}(U)}(C_1) \simeq \mathrm{Aut}_{\mathcal{Q}(U)}(C_2)$ is compatible with the identification of both sides with $\mathcal{A}(U)$.*

(2) *If $f : U' \to U$ is a morphism in $\mathrm{Sch}_{\mathrm{et}}(X)$ and $C \in \mathcal{Q}(U)$, then the map*

$$f^*_\mathcal{Q} : \mathrm{Aut}_{\mathcal{Q}(U)}(C) \to \mathrm{Aut}_{\mathcal{Q}(U')}(f^*_\mathcal{Q}(C))$$

is compatible with the restriction map $\mathcal{A}(U) \to \mathcal{A}(U')$.

(3) *For any $U \in \mathrm{Sch}_{\mathrm{et}}(X)$ and any two $C_1, C_2 \in \mathcal{Q}(U)$, there exist a covering $f : U' \to U$ such that the objects $f^*_\mathcal{Q}(C_1)$ and $f^*_\mathcal{Q}(C_2)$ of $\mathcal{Q}(U')$ are isomorphic.*

Then $\mathcal{Q}$ has a canonical structure of a gerbe bound by $\mathrm{Tors}_\mathcal{A}$.

14.3. We claim that $\mathrm{Higgs}_{\widetilde{X}}$ satisfies the conditions of this lemma. Condition (0) is a tautology: locally the cameral cover $\widetilde{X} \to X$ is induced from the universal one by means of a map $X \to \overline{G/N}$.

Let (E_G, σ, t) be an object of $\mathrm{Higgs}_{\widetilde{X}}(U)$. We must construct an isomorphism

$$\mathrm{Aut}_{\mathrm{Higgs}_{\widetilde{X}}(U)}(E_G, \sigma, t) \simeq T_{\widetilde{X}}(U).$$

Let us first assume that E_G is trivialized and our Higgs bundle corresponds to a map $U \to \overline{G/N}$ such that $\widetilde{U} \overset{t}{\simeq} \overline{G/T} \underset{\overline{G/N}}{\times} U$. In this case, an automorphism of (E_G, σ) as an object of $\mathrm{Higgs}(U)$ is the same as a map $U \to \mathrm{St}$ (cf. Section 11.9) that covers the given map $X \to \overline{G/N}$. Now, Proposition 11.10 implies that this automorphism belongs to $\mathrm{Aut}_{\mathrm{Higgs}_{\widetilde{X}}(U)}(E_G, \sigma, t)$ if and only if the above map factors as $U \to \mathcal{C} \to \mathrm{St}$.

Now we apply Theorem 11.6 which says that $\mathrm{Hom}_{\overline{G/N}}(U, \mathcal{C}) = \mathrm{Hom}_{\overline{G/N}}(U, \mathcal{T}) = T_{\widetilde{X}}(U)$.

The fact that the map $\chi : \mathcal{C} \to \mathcal{T}$ is G-equivariant implies that our isomorphism between $\mathrm{Aut}_{\mathrm{Higgs}_{\widetilde{X}}(U)}(E_G, \sigma, t)$ and $T_{\widetilde{X}}(U)$ is independent of the choice of a trivialization of E_G. In particular, by SC-1, it defines the required isomorphism for all E_G. The fact that conditions (1) and (2) are satisfied is automatic from the construction.

Finally, let us check condition (3). Let (E^1_G, σ^1, t^1) and (E^2_G, σ^2, t^2) be two objects of $\mathrm{Higgs}_{\widetilde{U}}(U)$. Without restricting the generality we can assume that both E^1_G and E^2_G are trivialized.

In this case, the data of $(\sigma^1, \sigma^2, t^1 \circ (t^2)^{-1})$ defines a U-point of the scheme $\mathcal{H}$. By Theorem 11.8 we can locally find a map $g_U : U \to G$ which conjugates (σ^1, t^1) to (σ^2, t^2). We can regard g_U as a gauge transformation, i.e., a map $E^1_G \to E^2_G$, which defines an isomorphism between (E^1_G, σ^1, t^1) and (E^2_G, σ^2, t^2). Thus, Theorem 4.4 is proved.

15. Proof of Theorem 7.3

15.1. It remains to prove Theorem 6.4. In this section we will prove Theorem 7.3 which takes care of the universal situation.

15.2. Step 1. First we show that our map $s_i^*(\mathcal{L}_{\mathrm{can}}) \to \mathcal{L}_{\mathrm{can}}$ is an isomorphism off D^{α_i}. To do that let us analyze more closely the situation described in Section 10.5.

Let $\Delta_J \subset \Delta$ be a root subsystem and let $M = M_J$ be the corresponding standard Levi subgroup. Let $\mathcal{F}l_M$ denote the flag variety of M and $B_M = B \cap M$, $U_M = U \cap M$.

It is well known that there exists a canonical closed embedding $W_M \backslash W \times \mathcal{F}l_M \to \mathcal{F}l$:

A point $\mathfrak{b}' \in \mathcal{F}l$ belongs to $w \times \mathcal{F}l_M$, if and only if $\mathfrak{b}'$ is in relative position w with respect to $P = P_J$ (this makes sense, as P-orbits in $\mathcal{F}l$ are parametrized exactly by $W_M \backslash W$) and $\mathfrak{b}' \cap \mathfrak{m}$ is a Borel subalgebra in M.

Consider the restriction of the canonical T-bundle $\mathcal{L}_{\mathcal{F}l}$ to $W_M \backslash W \times \mathcal{F}l_M$. It is easy to see that its further restriction to the connected component $1 \times \mathcal{F}l_M$ identifies with $\mathcal{L}_{\mathcal{F}l_M}$.

Let $w \in W$ be a *minimal* representative of its coset in $W_M \backslash W$. The action of w defines a map $1 \times \mathcal{F}l_M \to w^{-1} \times \mathcal{F}l_M$. Let us consider the pullback $w^*(\mathcal{L}_{\mathcal{F}l}|_{w \times \mathcal{F}l_M})$ as a T-bundle on $1 \times \mathcal{F}l_M = \mathcal{F}l_M$. Let $\tilde{w} \in N$ be an element that projects to $w \in W$.

Lemma 15.3. *We have a canonical M-equivariant isomorphism*

$$w^*(\mathcal{L}_{\mathcal{F}l}|_{w \times \mathcal{F}l_M}) \simeq \mathcal{L}_{\mathcal{F}l_M}.$$

Proof. Both $w^*(\mathcal{L}_{\mathcal{Fl}}|_{w\times \mathcal{Fl}_M})$ and $\mathcal{L}_{\mathcal{Fl}_M}$ are M-equivariant T-bundles on $\mathcal{Fl}_M$. To prove that they are isomorphic, we must show that the two homomorphisms $B\cap M\to T$ corresponding to the base point $\mathfrak{b}\in \mathcal{Fl}_M$ coincide. However, this follows from the fact that $w^{-1}\times \mathfrak{b}=\mathrm{Ad}_{\tilde{w}^{-1}}(\mathfrak{b})$, which is true since w is minimal. $\square$

15.4. Let w be as above. Consider the map

$$\overline{M/T}\overset{\text{Proposition 10.6}}{\longrightarrow}\overline{G/T}\overset{w}{\to}\overline{G/T}\to \mathcal{Fl}.$$

The fact that $\mathrm{Ad}_{\tilde{w}^{-1}}(B)\cap M=B_M$ implies that the above map coincides with

$$\overline{M/T}\to 1\times \mathcal{Fl}_M\overset{w}{\to} w\times \mathcal{Fl}_M\to \mathcal{Fl}.$$

Therefore, from Lemma 15.3 we obtain an isomorphism

$$\gamma'_{\mathrm{can}}(\tilde{w}): w^*(\mathcal{L}_{\mathrm{can}})|_{\overline{M/T}}\simeq \mathcal{L}_{\mathrm{can}}|_{\overline{M/T}}.$$

Moreover, it is easy to see that the above isomorphism is induced by the restriction to $\overline{M/T}$ of the (meromorphic) isomorphism $\gamma_{\mathrm{can}}(\tilde{w})$. In particular, the *a priori* meromorphic isomorphism $\gamma_{\mathrm{can}}(\tilde{w})$ is regular on $\overline{M/T}$.

Let us now go back to the situation of the theorem. We must check that the meromorphic map $s_i^*(\mathcal{L}_{\mathrm{can}})\to\mathcal{L}_{\mathrm{can}}$ has no poles along D^α if $\alpha\neq\alpha_i$. Choose a minimal Levi subgroup M_j such that $w(\alpha_j)=\alpha$ for some $w\in W$. Then the fact that $\alpha\neq\alpha_i$ implies that both w and $s_i\cdot w$ are minimal representatives of the corresponding cosets in $W/\langle s_j\rangle$. Then the above discussion shows that $s_i^*(\mathcal{L}_{\mathrm{can}})\to\mathcal{L}_{\mathrm{can}}$ has no poles on $w\times\overline{M_j/T}$.

This proves what we need since the G-orbit of $w\times\overline{M_j/T}$ contains an open part of D^α (cf. Proposition 1.8).

15.5. Step 2. Thus, we have shown that the poles of the map $s_i^*(\mathcal{L}_{\mathrm{can}})\to\mathcal{L}_{\mathrm{can}}$ can occur only on D^{α_i}. Let M_i be the corresponding minimal Levi subgroup. As we have seen before, there is a natural embedding $\overline{M_i/T}\to\overline{G/T}$, and $\mathcal{L}_{\mathrm{can}}$ restricts to the corresponding T-bundle on $\overline{M_i/T}$.

Since $s_i^*(\mathcal{L}_{\mathrm{can}})\to\mathcal{L}_{\mathrm{can}}$ is G-equivariant, to determine the contribution of the divisor D^{α_i}, it is enough to perform the corresponding calculation for M_i. The latter case easily reduces to SL(2).

For SL(2), $\overline{G/T}\simeq\mathbb{P}^1\times\mathbb{P}^1$ and $\mathcal{L}_{\mathrm{can}}\simeq\mathcal{O}(1)\boxtimes\mathcal{O}$. Moreover, $-1\in S_2=W$ acts on $\overline{G/T}$ by swapping the two $\mathbb{P}^1$ factors, with the fixed point locus $\overline{G/T}^{-1}$ being the diagonal $\mathbb{P}^1$. Hence, $(-1)^*(\mathcal{L}_{\mathrm{can}})\simeq\mathcal{O}\boxtimes\mathcal{O}(-1)$.

Therefore, we have a meromorphic map between $\mathcal{O}\boxtimes\mathcal{O}(-1)$ and $\mathcal{O}(1)\boxtimes\mathcal{O}$, which is allowed to have zeroes and poles only on the diagonal. Then it must have a zero of order 1, by degree considerations.

16. Proof of Theorem 6.4

16.1. The functor. Finally, we are ready to complete the proof of the main result. First, we claim that there is a natural functor $\Upsilon:\mathrm{Higgs}_{\widetilde{X}}\to\mathrm{Higgs}'_{\widetilde{X}}$:

Let (E_G,σ,t) be an object of $\mathrm{Higgs}_{\widetilde{X}}(U)$, where $\sigma: E_G\to\overline{G/N}$ is a G-equivariant map. We can pullback the universal object of $\mathrm{Higgs}'_{\overline{G/T}}(\overline{G/N})$ (cf. Section 7) and obtain a G-equivariant object of $\mathrm{Higgs}'_{\widetilde{E}_G}(E_G)$, where $\widetilde{E}_G$ is the induced cameral cover of E_G.

By descent, it gives rise to an object of $\mathrm{Higgs}'_{\widetilde{X}}(U)$ and this assignment is clearly a functor between sheaves of categories.

The key fact now is that $\mathrm{Higgs}'_{\widetilde{X}}$ is also a gerbe bound by $\mathrm{Tors}_{T_{\widetilde{X}}}$. Condition (0) of Lemma 14.2 follows from the existence of the functor Υ and the fact that $\mathrm{Higgs}_{\widetilde{X}}$ satisfies condition (0).

Let $(\mathcal{L},\gamma,\beta_i)$ be an object of $\mathrm{Higgs}'_{\widetilde{X}}(U)$. We must identify the group of its automorphisms with $T_{\widetilde{X}}(U)$. By definition, this group consists of T-bundle automorphisms, which respect the data of γ and β_i. However, a T-bundle map $\mathcal{L}\to\mathcal{L}$ is the same as a map $\widetilde{U}\to T$ and compatibility with γ implies that this map is W-equivariant. Therefore, we obtain a section of $\overline{T}_{\widetilde{X}}(U)$. Now, compatibility with β_i is exactly condition $(*)$. (Recall that it suffices to impose condition $(*)$ for one representative in every W-orbit on the set of roots. In particular, it is sufficient to impose it for simple roots only.)

It is easy to see that conditions (1) and (2) hold for the above identification of $\mathrm{Aut}_{\mathrm{Higgs}'_{\widetilde{X}}(U)}(\mathcal{L},\gamma,\beta_i)\simeq T_{\widetilde{X}}(U)$. In addition, it follows from the construction of χ, that $\Upsilon:\mathrm{Higgs}_{\widetilde{X}}\to\mathrm{Higgs}'_{\widetilde{X}}$ respects the identifications of groups of automorphisms of objects with $T_{\widetilde{X}}(U)$.

Assume for a moment that condition (3) of Lemma 14.2 has been checked. We claim that this already implies Theorem 6.4 because of the following general fact:

Lemma 16.2. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two gerbes bound by $\mathrm{Tors}_{\mathcal{A}}$ and let $\Upsilon:\mathcal{Q}_1\to\mathcal{Q}_2$ be a functor between the corresponding sheaves of categories. Assume that for every $U\in\mathrm{Sch}_{\mathrm{et}}(X)$ and $C\in\mathcal{Q}_1(U)$ we have a commutative square:*

$$\begin{array}{ccc}\mathcal{A}(U) & \longrightarrow & \mathrm{Aut}_{\mathcal{Q}_1(U)}(C)\\ {\scriptstyle\mathrm{id}}\big\downarrow & & \Upsilon\big\downarrow\\ \mathcal{A}(U) & \longrightarrow & \mathrm{Aut}_{\mathcal{Q}_2(U)}(\Upsilon(C))\end{array}.$$

Then Υ is an equivalence of $\mathrm{Tors}_{\mathcal{A}}$-gerbes.

16.3. The homogeneous version: $\mathrm{Tors}'_{T_{\widetilde{X}}}$. It remains to prove that every two objects of $\mathrm{Higgs}'_{\widetilde{X}}(U)$ are locally isomorphic. For this purpose we will introduce a sheaf of Picard categories $\mathrm{Tors}'_{T_{\widetilde{X}}}$, which will be the "homogeneous" version of $\mathrm{Higgs}'_{\widetilde{X}}(X)$.

Objects of $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$ are triples

$$(\mathcal{L}_0,\gamma_0,\beta_{i,0}),$$

where $(\mathcal{L}_0,\gamma_0)$ is a strongly W-equivariant T-bundle on $\widetilde{U}$ and each $\beta_{i,0}$ is a trivialization of $\alpha_i(\mathcal{L}_0)|_{D_U^{\alpha_i}}$.

The following compatibility conditions must hold:

(1) For a simple root α_i, the data of $\gamma_0(s_i): s_i^*(\mathcal{L}_0)\simeq\mathcal{L}_0$ defines, after restriction to $D_U^{\alpha_i}$, a trivialization

$$\check{\alpha}_i(\alpha_i(\mathcal{L}_0)|_{D_U^{\alpha_i}})\simeq\check{\alpha}_i(\mathcal{O}_{D_U^{\alpha_i}}).$$

We need this trivialization to coincide with $\check{\alpha}_i(\beta_{i,0})$.

(2) Assume that $w \in W$ conjugates a simple root α_i to another simple root α_j. The pullback of $\beta_{j,0}$ under w is a trivialization of $\alpha_i(w^*(\mathcal{L}_0))|_{D_U^{\alpha_i}}$, which via $\gamma_0(w)$ defines a trivialization of $\alpha_i(\mathcal{L}_0)$. Our condition is that this trivialization coincides with $\beta_{i,0}$.

Morphisms in $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$ are by definition maps between strongly W-equivariant T-bundles, compatible with the data of $\beta_{i,0}$.

If $(\mathcal{L}_0^1, \gamma_0^1, \beta_{i,0}^1)$ and $(\mathcal{L}_0^2, \gamma_0^2, \beta_{i,0}^2)$ are two objects of $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$, we can form their tensor product $(\mathcal{L}_0^1 \otimes \mathcal{L}_0^2, \gamma_0^1 \otimes \gamma_0^2, \beta_{i,0}^1 \otimes \beta_{i,0}^2)$ which will be a new object of $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$. Moreover, if $(\mathcal{L}_0, \gamma_0, \beta_{i,0})$ is an object of $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$ and $(\mathcal{L}, \gamma, \beta_i)$ is an object of $\mathrm{Higgs}'_{\widetilde{X}}(U)$, we can take their tensor product and obtain another object of $\mathrm{Higgs}'_{\widetilde{X}}(U)$.

It is easy to see that the above constructions define on $\mathrm{Tors}'_{T_{\widetilde{X}}}$ a structure of a sheaf of Picard categories and on $\mathrm{Higgs}'_{\widetilde{X}}$ a structure of a gerbe bound by it. Therefore, to prove that every two objects of $\mathrm{Higgs}'_{\widetilde{X}}(U)$ are locally isomorphic, it is enough to show that any object of $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$ is locally isomorphic to the unit object, i.e., to the one with $\mathcal{L}_0$ being *the trivial* T-bundle and $(\gamma_0, \beta_{i,0})$ being the tautological maps. The last assertion is equivalent to:

Proposition 16.4. $\mathrm{Tors}'_{T_{\widetilde{X}}}$ *is equivalent as a sheaf of Picard categories to* $\mathrm{Tors}_{T_{\widetilde{X}}}$.

We proceed to prove this proposition by showing that any object in $\mathrm{Tors}'_{T_{\widetilde{X}}}(U)$ is locally isomorphic to the unit object.

16.5. Step 1. Without restricting the generality, we can assume that $U = X$ and we must find an étale covering $X' \to X$, over which a given object $(\mathcal{L}_0, \gamma_0, \beta_{i,0})$ becomes isomorphic to the trivial one.

Fix a $\mathbb{C}$-point $x \in X$. First, we will reduce our situation to the case when the ramification over x is maximal possible, i.e., when x belongs to the image of $\bigcap\limits_{\alpha} D_X^\alpha$, where the intersection is taken over all roots of G.

After an étale localization we can assume that we have a map $X \to \mathfrak{t}/W$ so that $\widetilde{X} = X \underset{\mathfrak{t}/W}{\times} \mathfrak{t}$. Let t be a point in $\mathfrak{t}$ which has the same image in $\mathfrak{t}/W$ as x. By conjugating t, we can assume that there exists $J \subset I$ such that $\alpha_j(t) = 0$ for $j \in \Delta_J$ and $\beta(t) \neq 0$ for $\beta \notin \Delta_J$.

We have a Cartesian square

$$\begin{array}{ccc} W \overset{W_J}{\times} (\mathfrak{t} \setminus \bigcup\limits_{\beta \notin \Delta_J} \mathfrak{t}^\beta) & \longrightarrow & \mathfrak{t} \\ \big\downarrow & & \big\downarrow \\ (\mathfrak{t} \setminus \bigcup\limits_{\beta \notin \Delta_J} \mathfrak{t}^\beta)/W_J & \longrightarrow & \mathfrak{t}/W \end{array} ;$$

in particular, the map $\mathfrak{t}/W_J \to \mathfrak{t}/W$ is étale in a neighbourhood of the image of t in $\mathfrak{t}/W$.

Therefore, the base change $X \Rightarrow X' := X \underset{\mathfrak{t}/W}{\times} \mathfrak{t}/W_J$ is étale in a neighbourhood of x.

This reduces us to the situation when the W-cover $\widetilde{X}$ is induced from a W_J-cover $\widetilde{X}_J$, i.e., $\widetilde{X} \simeq W \overset{W_J}{\times} \widetilde{X}_J$.

By restricting $(\mathcal{L}_0, \gamma_0, \beta_{i,0})$ to $\widetilde{X}_J$ we obtain an object of $\mathrm{Tors}'_{T_{\widetilde{X}_J}}(X)$, equivariant with respect to W_J. Moreover, it is easy to see that this establishes an equivalence between $\mathrm{Tors}'_{T_{\widetilde{X}}}$ and $\mathrm{Tors}'_{T_{\widetilde{X}_J}}$, thereby reducing us to the situation when $\Delta_J = \Delta$.

16.6. Step 2. According to Step 1, we may assume that there exists a unique geometric point $\widetilde{x} \in \widetilde{X}$ over x. To prove the assertion of the proposition, we can replace X by the spectrum of the local ring of X at x. In this case all the D^{α}_X's and $\widetilde{X}$ are local too.

Let us choose a trivialization of our line bundle $\mathcal{L}_0$, subject only to the condition that it is compatible with the data of $\beta_{i,0}$ at $\widetilde{x}$ for every simple root α_i. We must show that this trivialization can be modified so that it will be compatible with the structure on $\mathcal{L}_0$ of a W-equivariant T-bundle, i.e., with the data of γ_0. (The argument given below mimics the proof of Proposition 9.5).

The discrepancy between our initial trivialization and γ_0 is given by a 1-cocycle $\mu : W \to \mathrm{Hom}(\widetilde{X}, T)$.

The evaluation at $\widetilde{x}$ gives rise to a surjection of W-modules: $\mathrm{Hom}(\widetilde{X}, T) \to T$. Thus we obtain a short exact sequence

$$0 \to K \to \mathrm{Hom}(\widetilde{X}, T) \to T \to 0,$$

where K consists of maps $\widetilde{X} \to T$ that have value 1 at $\widetilde{x}$.

Now, our condition on the trivialization (i.e., its compatibility with $\beta_{i,0}$) and condition (1) in the definition of $\mathrm{Tors}'_{T_{\widetilde{X}}}$ imply that $\mu(s_i) \in K$ for every simple reflection s_i. Hence, μ takes values in K. However, since $\widetilde{X}$ is local, K is torsion-free and divisible! Hence $H^1(W, K) = 0$.

Therefore, we can choose a trivialization of $\mathcal{L}_0$ which respects the W-equivariant structure and the data of $\beta_{i,0}$ at $\widetilde{x}$. But this implies that it is compatible with the data of $\beta_{i,0}$ on the entire $D^{\alpha_i}_X$, $\forall\, i \in I$.

Indeed, a possible discrepancy takes values in ± 1, and its value is constant along every connected component of $D^{\alpha_i}_X$. However, by construction, each $D^{\alpha_i}_X$ is local with $\widetilde{x}$ being its unique closed point.

The proof of Proposition 16.4, and hence of Theorem 6.4, is now complete.

16.7. Variant. As was the case for Higgs$'$, we can give a much simplified description of $\mathrm{Tors}'_{T_{\widetilde{X}}}$ in case our group G does not have an $\mathrm{SO}(2n+1)$ direct factor. In this case the data consists of a strongly equivariant T-bundle (L_0, γ_0), such that for a simple root α_i and a weight λ orthogonal to the corresponding coroot, the isomorphism $\lambda(s_i^*(L))|_{D^{\alpha_i}} \to \lambda(L)|_{D^{\alpha_i}}$ induced by $\gamma(s_i)$, coincides with the tautological one.

Part V. Some Applications

The point of our abstract notion of a Higgs bundle, as defined in Section 2, is that it provides a uniform approach to the analysis of various more concrete objects. In the final sections we illustrate the applications to Higgs bundles with values in a line bundle or in an elliptic fibration.

17. Higgs bundles with values

17.1. In Section 2 we defined a Higgs bundle over a scheme X to be a pair (E_G, σ), where $p : E_G \to X$ is a principal G-bundle over X, and σ is a G-equivariant map $\sigma : E_G \to \overline{G/N}$. We noted there that on a given G-bundle E_G, a Higgs bundle is specified by a vector subbundle $\mathbf{c}_X$ of $\mathfrak{g}_{E_G}$ whose fibers are regular centralizers. (Recall that $\mathfrak{g}_{E_G} := E_G \underset{G}{\times} \mathfrak{g}$ is the adjoint bundle of E_G.) In subsection 11.1 we defined the universal centralizer $\mathbf{c} \subset \mathfrak{g} \times \overline{G/N}$, corresponding to the universal Higgs bundle over $\overline{G/N}$. The family of centralizers $\mathbf{c}_X$ of a general Higgs bundle (E_G, σ) over X is related to the universal $\mathbf{c}$ by $p^*\mathbf{c}_X = \sigma^*\mathbf{c}$, an equality of vector subbundles of the trivial bundle $\mathfrak{g} \times E_G$ on the total space of E_G. We recall also that by Theorem 12.5, $\mathbf{c}$ is isomorphic to $\mathbf{t} = \mathrm{Lie}(\mathcal{T})$.

Let K be a line bundle on our base X. In the literature, the most common notion of a Higgs bundle is the following:

Definition 17.2. A K-valued Higgs bundle on X is a pair (E_G, s), where E_G is a principal G-bundle on X and s is a section of $\mathfrak{g}_{E_G} \otimes K$.

The section s of $\mathfrak{g}_{E_G} \otimes K$ is called *regular* at a point $x \in X$ if the corresponding local section of $\mathfrak{g}_{E_G}$ determined by some (hence, any) trivialization of K at x is regular. We work instead with the following more general notion, which is also better adapted to our setup.

Definition 17.3. A regularized K-valued Higgs bundle on X is a triple (E_G, σ, s), with (E_G, σ) a Higgs bundle on X and s a section of $\mathbf{c}_X \otimes K$, where $\mathbf{c}_X$ is the regular centralizer subbundle of the adjoint bundle $\mathfrak{g}_{E_G}$ determined by σ.

17.4. Regular vs. regularized. A regularized K-valued Higgs bundle (E_G, σ, s) on X clearly determines the unique K-valued Higgs bundle (E_G, s) on X. Conversely, if the section s of $\mathfrak{g}_{E_G} \otimes K$ is everywhere regular, then we can recover $\mathbf{c}_X \subset \mathfrak{g}_{E_G}$ as the centralizer of s, which defines a regularized K-valued Higgs bundle. When s is generically regular, the family $\mathbf{c}_X$ of centralizers is still unique, if it exists. In general, when s is not necessarily regular, our definition adds a choice of a regular centralizer containing s to the pair (E_G, s).

We want to establish the following result:

Theorem 17.5. *A regularized K-valued Higgs bundle on X is the same as a triple:*
(a) *A cameral cover $\widetilde{X} \to X$.*
(b) *A W-equivariant map $v : \widetilde{X} \to \mathfrak{t} \otimes K$ (of schemes over X).*
(c) *An object of* $\mathrm{Higgs}'_{\widetilde{X}}(X)$.

Proof. Given Theorem 6.4, it remains to show that the data (b) of a W-equivariant "value" map $\widetilde{X} \to \mathfrak{t} \otimes K$ is the same as the data of a section s of $\mathbf{c}_X \otimes K$. And indeed, giving such a section $s : X \to \mathbf{c}_X \otimes K$ is equivalent to giving a G-equivariant section $\tilde{s} : E_G \to \sigma^*\mathbf{c} \otimes K$ of the pullback $p^*\mathbf{c}_X \otimes K = \sigma^*\mathbf{c} \otimes K$ over E_G; cf. 17.1 above. By Theorem 12.5, this is the same as a G-equivariant section $\tilde{s}' : E_G \to \sigma^*\mathbf{t} \otimes K$. Now by the definition of $\mathcal{T}$ (cf. Subsection 11.3), $\mathrm{Hom}_{\overline{G/N}}(E_G, \mathbf{t}) = \mathrm{Hom}_W(\widetilde{E}_G, \mathfrak{t})$. Here $\widetilde{E}_G := E_G \times_X \widetilde{X}$ is the G-equivariant cameral cover of E_G associated to the Higgs bundle on E_G which is p^* of our given Higgs bundle (E_G, σ) on X. The section $\tilde{s}'$, and hence also our original section s, are therefore equivalent to a W-equivariant map of X-schemes $\overline{s} : \widetilde{E}_G \to \mathfrak{t} \otimes K$ which is also G-invariant. But this is the same as a W-equivariant map of X-schemes $v : \widetilde{X} \to \mathfrak{t} \otimes K$, as claimed. $\square$

Note that in the data (E_G, σ, s), the section $s : X \to \mathbf{c}_X \otimes K$ is regular if and only if the corresponding map v is an embedding. This follows from Lemma 13.6. So we have:

Corollary 17.6. *A regular K-valued Higgs bundle on X is the same as a triple:*
(a) *A cameral cover $\widetilde{X} \to X$.*
(b) *A W-equivariant embedding $v : \widetilde{X} \to \mathfrak{t} \otimes K$ (of schemes over X).*
(c) *An object of* $\mathrm{Higgs}'_{\widetilde{X}}(X)$.

17.7. The Hitchin map. To conclude our discussion of K-valued Higgs bundles, let us note that the data (a) and (b) in the above theorem can be assembled into what can be called "a point of the Hitchin base".

Assume that X is proper, and let $\mathbf{B}(X, K)$ denote the algebraic stack which classifies the data (a) and (b) of Theorem 17.5. That is for a scheme S, $\mathrm{Hom}(S, \mathbf{B}(X, K))$ is the category of pairs $(\widetilde{X}_S, v : \widetilde{X}_S \to \mathfrak{t} \otimes K)$, where $\widetilde{X}_S$ is a cameral cover of $S \times X$, and v is a W-equivariant morphism of X-schemes.

On the other hand, let $\mathbf{Higgs}(X, K)$ denote the algebraic stack of all regularized K-valued Higgs bundles on X. The *Hitchin map* $h : \mathbf{Higgs}(X, K) \to \mathbf{B}(X, K)$ sends a regularized K-valued Higgs bundle (E_G, σ, s) given by data (a),(b) and (c) to the point of the Hitchin base given by data (a) and (b).

Corollary 17.8. *The fibers of the Hitchin map $h : \mathbf{Higgs}(X, K) \to \mathbf{B}(X, K)$ can be identified (as categories) with* $\mathrm{Higgs}'_{\widetilde{X}}(X)$*. By Corollary* 4.6*, the set of isomorphism classes of objects of this fiber is a torsor over the abelian group $H^1(X, T_{\widetilde{X}})$, and the torsor class is given in Theorem* 6.4*.*

Note that our description of the fiber of the Hitchin map is independent of the line bundle K.

17.9. Let now $\mathbf{Hitch}(X, K)$ denote the scheme of sections of the fibration $(\mathfrak{t} \otimes K)/W \to X$. In fact, $\mathbf{Hitch}(X, K)$ is non-canonically isomorphic to an affine space.

The relation between $\mathbf{Hitch}(X, K)$ and $\mathbf{B}(X, K)$ is similar in some respects to the relation between the vector space $\mathfrak{t}/W$ parametrizing semisimple adjoint orbits in the Lie algebra $\mathfrak{g}$ and the stack $\mathfrak{g}/G$ of all G-orbits in $\mathfrak{g}$. In both cases, there is an open embedding of the variety into the stack, and there is a retraction of the stack onto the variety which is the identity on the variety.

In our case, the retraction $r : \mathbf{B}(X, K) \to \mathbf{Hitch}(X, K)$ associates to $v : \widetilde{X} \to \mathfrak{t} \otimes K$ the corresponding map $X \to (\mathfrak{t} \otimes K)/W$. As for the open embedding $i : \mathbf{Hitch}(X, K) \to \mathbf{B}(X, K)$: starting with $X \to (\mathfrak{t} \otimes K)/W$, we recover $\widetilde{X}$ as

$$\widetilde{X} := X \underset{(\mathfrak{t} \otimes K)/W}{\times} (\mathfrak{t} \otimes K),$$

and $v : \widetilde{X} \to \mathfrak{t} \otimes K$ is the second projection.

Obviously, the image $i(\mathbf{Hitch}(X, K)) \subset \mathbf{B}(X, K)$ is the open substack corresponding to the condition that the map $\widetilde{X} \to \mathfrak{t} \otimes K$ is an embedding. By Corollary 17.8, the preimage of $\mathbf{Hitch}(X, K) \subset \mathbf{B}(X, K)$ under the Hitchin map is exactly the open substack of regular K-valued Higgs bundles. Let (E_G, σ, s) denote some regularized

K-valued Higgs bundle on X. Note that the image $h((E_G, \sigma, s)) \in \mathbf{B}(X, K)$ determines whether (E_G, σ, s) is regular. A point in $\mathbf{Hitch}(X, K)$, on the other hand, can be the image of both regular and irregular (E_G, σ, s)'s.

17.10. Variant. Definition 17.3, Theorem 17.5, and Corollary 17.8 remain unchanged if we allow K to be a vector bundle, as in [26] where $K = \Omega^1_X$ is the cotangent bundle. In Definition 17.2, on the other hand, commutativity is not built in, so we must impose it by hand: the components of the section s, with respect to any local decomposition of K as a sum of line bundles, must commute with each other. Equivalently, the bracket of s with itself, interpreted as a section of $\mathfrak{g}_{E_G} \otimes \wedge^2 K$, must vanish.

18. Elliptic fibrations

Let $f : Y \to X$ be a projective, flat, dominant morphism with integral (that is, reduced and irreducible) fibers. Eventually we will specialize this to the case of an elliptic fibration, but for now we will work with the general situation. We want to describe an application of our results to the study of regularized G-bundles on Y in terms of data on the base X and along the (eventually, elliptic) fibers.

By a regularization of a G-bundle E_G on Y we mean a reduction of its structure group *along each fiber* to some regular centralizer. In other words, we want a Higgs bundle (E_G, σ) on Y whose group scheme of centralizers $\mathcal{C}_Y$ (equivalently, its cameral cover $\widetilde{Y} \to Y$) is the pullback of some group scheme of centralizers $\mathcal{C}_X$ on X (respectively, of a cameral cover $\widetilde{X} \to X$). More precisely:

Definition 18.1. A regularized G-bundle on Y consists of the data $(\widetilde{X}, E_G, \sigma)$, with $\widetilde{X} \to X$ a cameral cover of X, and $(E_G, \sigma) \in \mathrm{Higgs}_{\widetilde{Y}}(Y)$ a Higgs bundle on Y with cameral cover $\widetilde{Y} := \widetilde{X} \underset{X}{\times} Y$.

In the case of an elliptic fibration there is a natural notion of what it means for a bundle (on Y) to be regular above a point (of X). In analogy with the situation for K-valued Higgs bundles considered in Subsection 17.4, "most" G-bundles on an elliptic curve are indeed regular, and a regular bundle has a unique regularization. We review these well known facts below.

18.2. In general, our current situation is the analogue of Higgs bundles with values, in which we replace the bundle K of values from Section 17 by the relative Picard scheme $\mathrm{Pic}(Y/X)$. The tensor product $\mathfrak{t} \otimes_{\mathbb{C}} K$ can be identified with $\Lambda \otimes_{\mathbb{Z}} K$, so we take its analogue to be $\Lambda \otimes_{\mathbb{Z}} \mathrm{Pic}(Y/X) =: \mathrm{Bun}_T(Y/X)$. (Here Λ is the lattice of coweights.) Similarly, we will need the analogue of $\mathbf{c}_X \otimes K$. This is the sheaf of groups $\mathrm{Tors}_{Y/X} := \mathrm{Tors}_{\mathcal{C}_Y, Y/X}$, the sheafification of the presheaf on Y given by

$$U \mapsto \{\mathcal{C}_Y - \text{torsors on } U \text{ modulo pullbacks of } \mathcal{C}_X\text{-torsors}\}.$$

(As above, $\mathcal{C}_X$ is the group scheme of regular centralizer subgroups with Lie algebra $\mathbf{c}_X$, and $\mathcal{C}_Y := f^*(\mathcal{C}_X)$.) In fancier language, we could think of $\mathrm{Tors}_{Y/X}$ as a sheaf of Picard groupoids. But its objects have no automorphisms, so we are dealing in fact with a sheaf of abelian groups. In more detail:

We introduce the sheaf of Picard categories $\mathrm{Tors}_{\mathcal{C}_Y, Y/X}$ on $\mathrm{Sch}_{\mathrm{et}}(X)$ as "$\mathcal{C}_Y$-torsors on Y modulo pullbacks of $\mathcal{C}_X$-torsors". The definition of $\mathrm{Tors}_{\mathcal{C}_Y, Y/X}$ is as follows:

First, consider the presheaf of categories $\mathrm{Tors}^{\mathrm{pre}}_{\mathcal{C}_Y,Y/X}$, whose objects over $U \to X$ are torsors over $U \underset{X}{\times} Y$ with respect to the sheaf $T_{\widetilde{Y}}$. Morphisms between two such torsors τ' and τ'' are pairs (τ_X, σ), where τ_X is a $T_{\widetilde{X}}$-torsor on U and σ is an isomorphism $\tau'' \to \tau' \otimes f^*(\tau_X)$. (Since $f : Y \to X$ is dominant, and thus $\Gamma(U, T_{\widetilde{X}}) \to \Gamma(U \underset{X}{\times} Y, T_{\widetilde{Y}})$ is an injection, it is easy to see that the morphisms defined this way form a set and not just a category.)

The presheaf $\mathrm{Tors}^{\mathrm{pre}}_{\mathcal{C}_Y,Y/X}$ satisfies the first sheaf axiom, but not the second one, i.e., not every descent data is automatically effective. By applying the standard sheafification procedure, we obtain from $\mathrm{Tors}^{\mathrm{pre}}_{\mathcal{C}_Y,Y/X}$ a sheaf of Picard categories, which we denote by $\mathrm{Tors}_{\mathcal{C}_Y,Y/X}$.

Note, however, that since the morphism $f : Y \to X$ is projective, objects of $\mathrm{Tors}_{\mathcal{C}_Y,Y/X}$ have no nontrivial automorphisms, because for every U as above, the map $\Gamma(U, T_{\widetilde{X}}) \to \Gamma(U \underset{X}{\times} Y, T_{\widetilde{Y}})$ is in fact an isomorphism. Hence, $\mathrm{Tors}_{Y/X} := \mathrm{Tors}_{\mathcal{C}_Y,Y/X}$ is in fact a sheaf of groups.

We need an explicit description of this sheaf:

Lemma 18.3. *There is a canonical identification:*

$$\mathrm{Tors}_{Y/X}(X) = \{v \in \mathrm{Mor}_W(\widetilde{X}, \mathrm{Bun}_T(Y/X)) |\ \alpha_i \circ v_{|D_X^{\alpha_i}} = 1 \in \mathrm{Pic}(Y/X), \forall \alpha_i \in I\}.$$

(As always, I denotes the set of simple roots α_i.)

Proof. We identify $\mathrm{Tors}_{Y/X}$ and $\mathrm{Tors}'_{Y/X}$ using Proposition 16.4. There is a natural map $\iota : \mathrm{Tors}'_{Y/X} \to \mathrm{Mor}_W(\widetilde{X}, \mathrm{Bun}_T(Y/X))$, sending a T-bundle on $\widetilde{Y} = \widetilde{X} \underset{X}{\times} Y$ to its classifying morphism v. This map ι is clearly injective, and its image is contained in the RHS.

We still have to prove the surjectivity of ι, i.e., to show that a morphism v in the RHS satisfies the two compatibility conditions between β's and γ's stated in 16.3. It suffices to do so locally, and then we may assume that $f : Y \to X$ has a section. In this case, we can identify $\mathrm{Tors}'_{Y/X}$ with the sheaf of T-bundles on $\widetilde{Y}$ satisfying the two compatibility conditions between β's and γ's, which additionally are trivialized along the section $X \subset Y$. Similarly, we can identify $\mathrm{Bun}_T(Y/X))$ with T-bundles on Y which are trivialized along the section.

Each of the compatibility conditions requires the equality of two given trivializations of some (T- or $\mathbb{G}_m$-) bundle over $D^i_X \underset{X}{\times} Y$. Now our assumption, $\alpha_i \circ v_{|D^i_X} = 1$, together with the assumed trivialization of all objects along the section, guarantees that these equalities hold over the section. The difference between the two trivializations is therefore a global automorphism which equals the identity along the section, so it is the identity everywhere since the fibers of f are integral and proper. $\square$

18.4. By construction, we have a short exact sequence of Picard categories:

$$0 \to \mathrm{Tors}_{T_{\widetilde{X}}} \to f_*(\mathrm{Tors}_{T_{\widetilde{Y}}}) \to \mathrm{Tors}_{Y/X} \to 0.$$

As in Subsection 3.7, an element $v \in \mathrm{Tors}_{Y/X}(X)$ determines a $\mathrm{Tors}_{T_{\widetilde{X}}}$-gerbe which we denote $\mathcal{Q}_v$. In fact, for $(U \to X) \in \mathrm{Sch}_{\mathrm{et}}(X)$, $\mathcal{Q}_v(U)$ is the category of all possible lifts of v to a $T_{\widetilde{Y}}$-torsor on $U \underset{X}{\times} Y$.

The main result of this section is the following analogue of Theorem 17.5:

Theorem 18.5. *A regularized G-bundle on Y is the same as a triple:*

(a) *A cameral cover $\widetilde{X} \to X$.*

(b) *A W-equivariant map $v : \widetilde{X} \to \mathrm{Bun}_T(Y/X)$ (of X-schemes), satisfying: $\alpha_i \circ v_{|D^{\alpha_i}} = 1 \in \mathrm{Pic}(Y/X), \forall$ simple root α_i.*

(c) *An object of* $\mathrm{Higgs}'_{\widetilde{X}}(X) \underset{\mathrm{Tors}_{T_{\widetilde{X}}}}{\otimes} \mathcal{Q}_v$.

Proof. Let us fix a cameral cover $\widetilde{X} \to X$ and consider regularized G-bundles on Y corresponding to this fixed $\widetilde{X}$ as a sheaf of categories over X, denoted by $\mathrm{Reg}_{\widetilde{X}}(Y)$.

By Theorem 4.4, $\mathrm{Reg}_{\widetilde{X}}(Y)$ is a gerbe bound by the sheaf of Picard categories $f_*(\mathrm{Tors}_{T_{\widetilde{Y}}})$. This gerbe is induced from the $\mathrm{Tors}_{T_{\widetilde{X}}}$-gerbe $\mathrm{Higgs}_{\widetilde{X}}$ by the homomorphism $\mathrm{Tors}_{T_{\widetilde{X}}} \to f_*(\mathrm{Tors}_{T_{\widetilde{Y}}})$; cf. Section 3.9.

Thus, according to Lemma 3.10, we have a functor $\mathrm{Reg}_{\widetilde{X}}(Y) \to \mathrm{Tors}_{Y/X}$, and for a given object $v \in \mathrm{Tors}_{Y/X}(X)$, the category-fiber of the above functor is a $\mathrm{Tors}_{T_{\widetilde{X}}}$-gerbe, which can be canonically identified with $\mathrm{Higgs}_{\widetilde{X}}(X) \underset{\mathrm{Tors}_{T_{\widetilde{X}}}}{\otimes} \mathcal{Q}_v$. Finally, according to Lemma 18.3, an object $v \in \mathrm{Tors}_{Y/X}(X)$ is equivalent to data (b) above. □

18.6. Now let us assume that X is projective as well. As our analogue of $\mathbf{Higgs}(X, K)$, we will consider the algebraic stack $\mathbf{Reg}(X, Y)$ which associates to a scheme S the category of regularized G-bundles on $S \underset{X}{\times} Y$ (with respect to the projection $S \underset{X}{\times} Y \to S$).

We can now describe an analogue of the Hitchin map. Indeed, let $\mathbf{B}(X, Y)$ be the stack whose S-points are pairs: $(\widetilde{X}_S, v)$ consisting of a cameral cover of $S \times X$ and a W-equivariant map $v : \widetilde{X}_S \to \mathrm{Bun}_T(Y/X)$ of X-schemes.

We have a natural map of stacks $h : \mathbf{Reg}(X, Y) \to \mathbf{B}(X, Y)$.

Corollary 18.7. *The fiber of the spectral map $\mathbf{Reg}(X, Y) \to \mathbf{B}(X, Y)$ over a cameral point $(\widetilde{X}, v) \in \mathbf{B}(X, Y)$ can be identified with* $\mathrm{Higgs}'_{\widetilde{X}}(X) \underset{\mathrm{Tors}_{T_{\widetilde{X}}}}{\otimes} \mathcal{Q}_v$. *The set of isomorphism classes of objects of this fiber is a torsor over the abelian group $H^1(X, T_{\widetilde{X}})$.*

In the case of K-valued Higgs bundles, we saw in Corollary 17.8 that the fiber of the Hitchin map $\mathbf{Higgs}(X, K) \to \mathbf{B}(X, K)$ is independent of the line bundle K. Note in contrast that the fiber $\mathrm{Higgs}'_{\widetilde{X}}(X) \underset{\mathrm{Tors}_{T_{\widetilde{X}}}}{\otimes} \mathcal{Q}_v$ of the spectral map could depend on the original map $f : Y \to X$. This dependence is mild though: it affects only the second factor, $\mathcal{Q}_v$. A simplification occurs when $f : Y \to X$ has a global section: in this case $\mathcal{Q}_v$ is always trivial because its defining short exact sequence of Picard categories 18.4 is split. It follows that the category $\mathrm{Reg}_{\widetilde{X}}(Y)$ of regularized bundles with a specified cameral cover $\widetilde{X}$ factors:

$$\mathrm{Reg}_{\widetilde{X}}(Y) = \mathrm{Tors}_{Y/X} \times \mathrm{Higgs}'_{\widetilde{X}}(X).$$

18.8. In addition to the stack $\mathbf{B}(X, Y)$, one can also define an analogue of $\mathbf{Hitch}(X, K)$: we let the space $\mathbf{Hitch}(X, Y)$ denote the scheme of all sections of the fibration

$$(\mathrm{Bun}_T(Y/X))/W \to X.$$

As before, we have an obvious retraction $\mathbf{B}(X, Y) \to \mathbf{Hitch}(X, Y)$. The analogue of the embedding $\mathbf{Hitch}(X, K) \to \mathbf{B}(X, K)$ can be described as follows:

Consider the W-cover $\mathrm{Bun}_T(Y/X) \to (\mathrm{Bun}_T(Y/X))/W$ and let $(\mathrm{Bun}_T(Y/X))^0/W$ be the maximal open subscheme over which this cover is cameral; let $\mathrm{Bun}_T(Y/X)^0$ denote its preimage in $\mathrm{Bun}_T(Y/X)$.

We will have to shrink $(\mathrm{Bun}_T(Y/X))^0/W$ to a still smaller open subscheme: For a simple root α_i consider the corresponding ramification divisor

$$D^{\alpha_i}_{(\mathrm{Bun}_T(Y/X))^0/W} \subset \mathrm{Bun}_T(Y/X)^0.$$

Under the map $\mathrm{Bun}_T(Y/X) \to \mathrm{Pic}(Y/X)$ given by α_i, the image of $D^{\alpha_i}_{(\mathrm{Bun}_T(Y/X))^0/W}$ is contained in the set of $\mathbb{Z}_2$-torsion points of $\mathrm{Pic}(Y/X)$.

We define the open subscheme $(\mathrm{Bun}_T(Y/X))^{00}/W$ of $(\mathrm{Bun}_T(Y/X))^0/W$ by removing those points, whose preimage in $\mathrm{Bun}_T(Y/X)^0$ maps to a *non-unit* point in $\mathrm{Pic}(Y/X)$ by means of the above map. Let $\mathrm{Bun}_T(Y/X))^{00} \to (\mathrm{Bun}_T(Y/X))^{00}/W$ denote the corresponding cameral cover.

Finally, let $\mathbf{Hitch}(X,Y)^0$ be the open subscheme of $\mathbf{Hitch}(X,Y)$, which corresponds to sections whose values belong to $(\mathrm{Bun}_T(Y/X))^{00}/W$. The fiber product construction gives the desired map $i : \mathbf{Hitch}(X,Y)^0 \subset \mathbf{B}(X,Y)$. Its image is the open substack corresponding to the locus where the map $v : \widetilde{X} \to \mathrm{Bun}_T(Y/X)$ is an embedding.

18.9. The case of an elliptic fiber. The main relevance of the above results is to the case that $f : Y \to X$ is an elliptic fibration. This is due to the existence in this case of a good notion of a regular bundle, analogous to the notion of a regular K-valued Higgs bundle. Take the group G to be semisimple, and consider the case of a single elliptic curve Z.

For any semistable G-bundle E_G on Z, the dimension of the group $H := \mathrm{Aut}_G(E_G)$ of (global) automorphisms of E_G is $\geq r$. We say that E_G is regular if $\dim(H) = r$. In this case, H is commutative and there exists an embedding $H \to G$ and a principal H-bundle E_H on Z such that $E_G \simeq G \overset{H}{\times} E_H$. A regular bundle has a unique regularization.

These results can be found in [10, 18, 16, 23] and elsewhere. In fact, the moduli space $M_G(Z)$ of (S-equivalence classes of) semistable, topologically trivial G-bundles on the elliptic curve Z is well understood. As a complex variety, it is isomorphic to $M_T(Z)/W$. (This is proved analytically (e.g., [16]) using Borel's result that in a simply connected compact group, any two commuting elements are contained in a maximal torus. An algebraic proof was given in [23].) Each S-equivalence class contains a unique regular representative as well as a unique semisimple representative (i.e., one whose structure group can be reduced to T). For a generic point of the moduli space, the S-equivalence class consists of a unique isomorphism class, which is both regular and semisimple. A similar but somewhat more complicated description exists for all reductive G; cf. [18]. Returning to an elliptic family $f : Y \to X$, we find ourselves in a situation analogous to that which we had for K-valued Higgs bundles: a "generic" G-bundle on Y which is semistable along the elliptic fibers should be regular on the generic fiber, and therefore its restriction to a dense open $X_0 \subset X$ should admit a unique regularization to which we can apply our results.

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