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# A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on halfspaces of the Heisenberg group

Francesco UGUZZONI
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato 5, I-40127 Bologna, Italy
*e-mail*: uguzzoni@dm.unibo.it

**Abstract**

Let $\Delta_{\mathbb{H}^n}$ be the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $Q = 2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. In this note, completing a recent result obtained with E. Lanconelli [9], we prove that, if $\Pi$ is a halfspace of $\mathbb{H}^n$, then the critical Dirichlet problem

$$(*) \qquad -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \Pi, \qquad u = 0 \quad \text{in } \partial\Pi,$$

has no nontrivial nonnegative weak solutions. This result enables to improve a representation theorem by Citti [2], for Palais-Smale sequences related to the equation in (*).

## 1 Introduction

In a recent paper with E. Lanconelli [9] we have proved the following uniqueness result. Let $u$ be a nonnegative weak solution of the following boundary value problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} & \text{in } \Pi \\ u \geq 0 & \text{in } \Pi \\ u = 0 & \text{in } \partial\Pi \end{cases} \tag{1.1}$$

where $\Delta_{\mathbb{H}^n}$ denotes the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$ and $\Pi$ is a halfspace with boundary *parallel to the center* of $\mathbb{H}^n$. Then $u \equiv 0$.

In this note, by means of a different technique, we extend this result to the other halfspaces: the ones with boundary *transverse to the center* of $\mathbb{H}^n$. Therefore, we are able to state the following theorem which completely extends to the context of the Heisenberg group a result by Esteban and P.L. Lions related to the classical Laplacian in $\mathbb{R}^N$ [4].

**Theorem 1.1** *Let* $\Pi$ *be an arbitrary halfspace of* $\mathbb{H}^n$*. Then the Dirichlet problem*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} & \text{in } \Pi \\ u \in S_0^1(\Pi) \end{cases} \tag{1.2}$$

*has no nontrivial nonnegative weak solutions.*

While we refer to section 2 for the notation used in Theorem 1.1 and throughout the paper, here we only recall that

$$\frac{Q+2}{Q-2} \quad \text{is the critical exponent for } \Delta_{\mathbb{H}^n} \text{ as well as } \quad \frac{N+2}{N-2}$$

is the critical exponent for semilinear Poisson equations in $\mathbb{R}^N$, $N \geq 3$. As proved by Citti in [2], the non-existence result of Theorem 1.1 plays a crucial role in the characterization of the Palais-Smale sequences related to the equation $-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}}$ on bounded domains of $\mathbb{H}^n$. Additional remarks and further references can be found in [9]. Other non-existence results related to subcritical equations on cones of $\mathbb{H}^n$ have been recently proved by Birindelli, Capuzzo Dolcetta and Cutrì[1].

We would like to stress that the techniques employed in [9] are not applicable to the halfspaces $\Pi$ of the type $\{t > 0\}$ considered here. Moreover, unlike in [9], in our case the boundary $\partial\Pi$ contains characteristic points: as a consequence, a more careful analysis of the behavior of the solution $u$ at the boundary is required.

Our approach is based on a systematic use of cylindrically symmetric barrier functions. The starting point is the following result on the asymptotic behavior of a solution $u$ to (1.2), proved in [9]:

$$u(\xi) = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty,\ \xi \in \Pi_t = \{t > 0\}$$

(see (2.4) and (2.5) for this notation). From this estimate we first deduce a behavior at infinity of the trace of $\partial_t u$ on $\partial\Pi_t$:

$$|\partial_t u(z, 0)| = O(|z|^{2-Q}), \quad \text{as } |z| \to \infty.$$

Then, by comparison with suitable barrier functions and by exploiting the fact that $\Delta_{\mathbb{H}^n}$ and $\partial_t$ commute, we are able to extend the above estimate inside $\Pi_t$ and obtain

$$|\partial_t u(\xi)| = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty,\ \xi \in \Pi_t.$$

In a similar way we study the behavior of $u$ at the origin, where the regularity may fail due to the characteristic nature of 0 for $\partial\Pi_t$. The main steps of our scheme are the proofs of propositions 3.2, 3.8, 3.9 and 3.15. The proof of Theorem 1.1 easily follows from these propositions by using, as in [9], the Rellich-Pohozaev type identity proved by Garofalo and Lanconelli in [7].

We thank Prof. E. Lanconelli for his continuous interest in this work.

## 2 Notation

The Heisenberg group $\mathbb{H}^n$, whose points will be denoted by $\xi = (z,t) = (x,y,t)$, is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ with composition law defined by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle)) \tag{2.1}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^n$. The Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} ({X_j}^2 + {Y_j}^2) \quad \text{where} \quad X_j = \partial_{x_j} + 2y_j\partial_t\ ,\quad Y_j = \partial_{y_j} - 2x_j\partial_t$$

for all $j \in \{1, \dots, n\}$. We set

$$\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

A natural group of dilations on $\mathbb{H}^n$ is given by

$$\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)\ ,\quad \lambda > 0. \tag{2.2}$$

The Jacobian determinant of $\delta_\lambda$ is $\lambda^Q$ where

$$Q = 2n + 2$$

is the *homogeneous dimension* of $\mathbb{H}^n$. The operators $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$ are invariant w.r.t. the left translations $\tau_\xi$ of $\mathbb{H}^n$ and homogeneous w.r.t. the dilations $\delta_\lambda$ of degree one and of degree two, respectively. More precisely, if we set

$$\tau_\xi(\xi') = \xi \circ \xi' \tag{2.3}$$

we have

$$\nabla_{\mathbb{H}^n}(u \circ \tau_\xi) = (\nabla_{\mathbb{H}^n} u) \circ \tau_\xi\ ,\quad \nabla_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda(\nabla_{\mathbb{H}^n} u) \circ \delta_\lambda,$$

$$\Delta_{\mathbb{H}^n}(u \circ \tau_\xi) = (\Delta_{\mathbb{H}^n} u) \circ \tau_\xi\ ,\quad \Delta_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda^2(\Delta_{\mathbb{H}^n} u) \circ \delta_\lambda.$$

A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by [5]

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}}\ , \tag{2.4}$$

where $c_Q$ is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4}. \tag{2.5}$$

Moreover, if we define $d(\xi, \xi') = d({\xi'}^{-1} \circ \xi)$, then $d$ is a distance on $\mathbb{H}^n$ (see [3] for a complete proof of this statement). We shall denote by $B_d(\xi, r)$ the $d$-ball of center $\xi$ and radius $r$.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$\|\varphi\|_{Q^*}^2 \le B_Q \|\nabla_{\mathbb{H}^n}\varphi\|_2^2 \qquad \forall \varphi \in C_0^\infty(\mathbb{H}^n) \tag{2.6}$$

where

$$Q^* := \frac{2Q}{Q-2} \tag{2.7}$$

and $B_Q$ is a positive constant whose best value has been determined by Jerison and Lee in [8]. If $\Omega$ is an open subset of $\mathbb{H}^n$, we shall denote by $S^1(\Omega)$ the Sobolev space of the functions $u \in L^{Q^*}(\Omega)$ such that $\nabla_{\mathbb{H}^n} u \in L^2(\Omega)$. The norm in $S^1(\Omega)$ is given by

$$\|u\|_{S^1(\Omega)} = \|u\|_{Q^*} + \|\nabla_{\mathbb{H}^n} u\|_2. \tag{2.8}$$

We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to (2.8). By means of (2.6), this norm is equivalent in $S_0^1(\Omega)$ to that generated by the inner product

$$\langle u, v\rangle_{S_0^1} = \int_\Omega \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v\rangle.$$

A nonnegative weak solution of the Dirichlet problem (1.1) is a function $u \in S_0^1(\Omega)$, $u \ge 0$, such that

$$\int_\Omega \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \varphi\rangle = \int_\Omega u^{Q^*-1}\varphi \qquad \forall \varphi \in S_0^1(\Omega). \tag{2.9}$$

We explicitly remark that, for every $u, \varphi \in S_0^1(\Omega)$, $u \ge 0$, we have $u^{\frac{Q+2}{Q-2}}\varphi \in L^1(\Omega)$. Indeed $\varphi \in L^{\frac{2Q}{Q-2}}(\Omega)$, $u^{\frac{Q+2}{Q-2}} \in L^{\frac{2Q}{Q+2}}(\Omega)$ and $\frac{Q-2}{2Q} + \frac{Q+2}{2Q} = 1$. We also remark that every classical solution of (1.1) satisfies the integral identity (2.9) since $X_j^* = -X_j$ and $Y_j^* = -Y_j$, for $j = 1, ..., n$.

We conclude by recalling that a boundary point $\xi$ of a smooth domain $\Omega$ is called characteristic if the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ are all tangent to $\partial\Omega$ at $\xi$.

# 3 Proof of Theorem 1.1

We know from [9] that Theorem 1.1 holds when $\partial\Pi$ is parallel to the $t$-axis. Then, we only have to study the other case. It is not restrictive to assume

$$\Pi = \Pi_t := \{\xi = (z, t) \in \mathbb{H}^n \,|\, t > 0\}. \tag{3.1}$$

Indeed, for every halfspace $\Pi$ with boundary transverse to the $t$-axis, there exists a left translation $\tau_{\xi_0}$ such that either $\Pi = \tau_{\xi_0}(\Pi_t)$ or $\Pi = \tau_{\xi_0}(-\Pi_t)$. Moreover, setting $\sigma(x, y, t) = (y, x, -t)$, we have $-\Pi_t = \sigma(\Pi_t)$ and the operators $\Delta_{\mathbb{H}^n}$ and $|\nabla_{\mathbb{H}^n}|$ are invariant with respect to $\sigma$, i.e. $\Delta_{\mathbb{H}^n}(f\circ\sigma) = (\Delta_{\mathbb{H}^n} f)\circ\sigma$ and $|\nabla_{\mathbb{H}^n}(f\circ\sigma)| = |\nabla_{\mathbb{H}^n} f|\circ\sigma$.

Throughout this section we will then assume (3.1) and denote by $u$ a (fixed) nonnegative weak solution of (1.2). Moreover we define

$$\begin{aligned} w &= \Gamma * (u^{\frac{Q+2}{Q-2}}) : \mathbb{H}^n \to \mathbb{R} \\ w(\xi) &= \int_{\mathbb{H}^n} \Gamma(\xi,\xi')u(\xi')^{\frac{Q+2}{Q-2}} d\xi' \end{aligned} \tag{3.2}$$

and

$$v = w - u \qquad \text{in } \Pi. \tag{3.3}$$

In (3.2) we have set $u = 0$ outside $\Pi$ and denoted $\Gamma(\xi,\xi') = \Gamma({\xi'}^{-1}\circ\xi)$. Many properties of the above introduced functions were established in [9] for arbitrary halfspaces, then, in particular, for $\Pi = \Pi_t$. The following proposition collects the ones we will need here.

**Proposition 3.1** *1) $u$ is a classical solution of*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} & \text{in } \Pi \\ u \geq 0 & \text{in } \Pi \\ u = 0 & \text{in } \partial\Pi \end{cases}, \tag{3.4}$$

$$u \in C^\infty(\overline{\Pi} \setminus \{0\}) \cap C(\overline{\Pi}) \tag{3.5}$$

*and*

$$u(\xi) = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty,\ \xi \in \Pi. \tag{3.6}$$

*Moreover, if we continue $u$ on $\mathbb{H}^n$ by setting $u = 0$ outside $\Pi$, there exist $\beta_0 \in ]0,1[$ and $M > 0$ such that*

$$|u(\xi) - u(\xi')| \leq M d(\xi,\xi')^{\beta_0} \qquad \forall \xi,\xi' \in \mathbb{H}^n \tag{3.7}$$

*(i.e., following Folland-Stein [6], $u$ belongs to the Hölder space $\Gamma^{\beta_0}(\mathbb{H}^n)$).*
*2) $w$ has the following properties:*

$$w \geq u, \tag{3.8}$$

$$w \in C^1(\mathbb{H}^n), \tag{3.9}$$

$$w(\xi) = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty, \tag{3.10}$$

$$|\partial_t w(\xi)| = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty. \tag{3.11}$$

*3) $v$ is a classical solution of*

$$\begin{cases} -\Delta_{\mathbb{H}^n} v = 0 & \text{in } \Pi \\ v \geq 0 & \text{in } \Pi \\ v = w & \text{in } \partial\Pi \end{cases} \tag{3.12}$$

*and*

$$v(\xi) = O(\Gamma(\xi)), \quad \text{as } d(\xi) \to \infty, \ \xi \in \Pi. \tag{3.13}$$

The proof of these statements is contained in [9]. See: Remark 2.10, Proposition 2.9, Theorem 1.1, Corollary 2.8, (3.9), (3.25), Proposition 3.2, Proposition 3.4, (3.8), respectively.

The first step of our approach consists in finding an estimate of the normal derivative of $u$ at the boundary of $\Pi$.

**Proposition 3.2** *We have $|\partial_t u(\xi)| = O(\Gamma(\xi))$, as $d(\xi) \to \infty$, $\xi \in \partial\Pi$.*

We will make use of the following cylindrically symmetric barrier functions. For every $R \geq 1$ we set

$$A_R = \{\xi = (z,t) \in \mathbb{H}^n \,|\, 0 < t < 1, |z| > R\}$$

and for every $\alpha > 0$ and $\beta \in ]0,1]$ we define

$$F_{\alpha,\beta} : A_1 \to \mathbb{R}^+, \quad F_{\alpha,\beta}(z,t) = \frac{(\sin t)^\beta}{|z|^\alpha}.$$

We also set

$$F_\alpha = F_{\alpha,1}.$$

**Remark 3.3** *If $\Phi(z,t) = \varphi(|z|,t) = \varphi(r,t)$ is a cylindrically symmetric regular function, then, as it has been noticed by Garofalo-Lanconelli in [7], we have*

$$\Delta_{\mathbb{H}^n} \Phi = \partial_r^2 \varphi + \frac{Q-3}{r} \partial_r \varphi + 4r^2 \partial_t^2 \varphi. \tag{3.14}$$

**Lemma 3.4** *For every $\beta \in ]0,1[$ there exist $\delta_\beta > 0$ and $R_\beta \geq 1$ such that*

$$-\Delta_{\mathbb{H}^n} F_{2,\beta} \geq \delta_\beta \qquad \text{in } A_{R_\beta}. \tag{3.15}$$

*Moreover, for every $\alpha > 0$ there exists $R_\alpha \geq 1$ such that*

$$-\Delta_{\mathbb{H}^n} F_\alpha \geq \frac{\sin t}{|z|^{\alpha-2}}. \qquad \text{in } A_{R_\alpha}. \tag{3.16}$$

**Proof** We set $\delta_\beta = 4\beta(1-\beta)$. Using formula (3.14) a computation yields

$$\begin{aligned} -\Delta_{\mathbb{H}^n} F_{\alpha,\beta}(z,t) &= \frac{\delta_\beta}{|z|^{\alpha-2}(\sin t)^{2-\beta}} + \left(4\beta^2 - \frac{\alpha(\alpha+4-Q)}{|z|^4}\right)\frac{(\sin t)^\beta}{|z|^{\alpha-2}} \\ &\geq \frac{\delta_\beta}{|z|^{\alpha-2}} + \left(4\beta^2 - \frac{\alpha^2}{R^4}\right)\frac{(\sin t)^\beta}{|z|^{\alpha-2}} \end{aligned} \tag{3.17}$$

for every $(z,t) \in A_R$. If $\alpha = 2$ and $\beta \in ]0,1[$ then $\delta_\beta > 0$ and, choosing $R_\beta \geq \frac{1}{\sqrt{\beta}}$, from (3.17) we get (3.15). On the other hand, if $\beta = 1$ then $\delta_\beta = 0$ and, choosing $R_\alpha \geq \sqrt{\alpha}$, (3.17) gives (3.16).

**Lemma 3.5** *For every $\alpha \in ]0, Q-2]$ and $R \geq 1$ there exists $M > 0$ such that*

$$u \leq MF_{\alpha,\beta} \quad \text{in } \partial A_R \cup \{\infty\} \qquad \forall \beta \in ]0,1]. \tag{3.18}$$

**Proof** Since $u = 0$ in $\partial\Pi$ and $u \to 0$ at infinity (see (3.4) and (3.6)), we only need to prove (3.18) in $\partial_1 = \{t = 1, |z| \geq R\}$ and $\partial_2 = \{0 \leq t \leq 1, |z| = R\}$. From (3.6) it follows that for every $\xi = (z,1) \in \partial_1$ we have

$$\begin{aligned} u(\xi) &\leq \frac{c}{d(\xi)^{Q-2}} \leq \frac{c}{|z|^{Q-2}} \leq \frac{c}{|z|^\alpha} \\ &\leq M\frac{\sin 1}{|z|^\alpha} = MF_\alpha(\xi) \leq MF_{\alpha,\beta}(\xi). \end{aligned}$$

On the other hand, (3.5) implies $\partial_t u \in C(\partial_2)$. Hence for every $\xi \in \partial_2$

$$\begin{aligned} u(\xi) &= u(z,t) - u(z,0) \leq (\max_{\partial_2} |\partial_t u|)t \\ &\leq c_R(\sin t)^\beta = M_{R,\alpha}\frac{(\sin t)^\beta}{R^\alpha} = MF_{\alpha,\beta}(\xi). \end{aligned}$$

We now exploit the equation $-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}}$ and compare $u$ with the functions $F_{\alpha,\beta}$. Since $\frac{Q+2}{Q-2} > 1$ we will be able to perform a kind of boot-strap process.

**Lemma 3.6** *There exist $R \geq 1$ and $M > 0$ such that*

$$u \leq MF_{Q-2} \qquad \text{in } A_R. \tag{3.19}$$

**Proof** We set

$$\beta = \frac{Q-2}{Q+2}.$$

Since $u \in L^\infty(\Pi)$ (see Proposition 3.1), from (3.15) and (3.18) we deduce the existence of $R' \geq 1$ and $M' > 0$ such that

$$\begin{cases} u \leq M'F_{2,\beta} & \text{in } \partial A_{R'} \cup \{\infty\} \\ -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} \leq M'\delta_\beta \leq -\Delta_{\mathbb{H}^n}(M'F_{2,\beta}) & \text{in } A_{R'} \end{cases}$$

Hence, the maximum principle for $\Delta_{\mathbb{H}^n}$ yields

$$u \leq M'F_{2,\beta} \qquad \text{in } A_{R'}. \tag{3.20}$$

We now define for every $\alpha > 0$

$$(P)_\alpha = (\exists R_\alpha \geq 1\ \exists M_\alpha > 0 \text{ such that } u \leq M_\alpha F_\alpha \text{ in } A_{R_\alpha}).$$

From (3.16), (3.18) and (3.20) we can deduce $(P)_2$. In deed (3.20) and (3.16) yield

$$-\Delta_{\mathbb{H}^n} u = u^{\frac{1}{\beta}} \le M'^{\frac{1}{\beta}} F_{2,\beta}^{\frac{1}{\beta}} = c\frac{\sin t}{|z|^{\frac{2}{\beta}}} \le c \sin t \le -\Delta_{\mathbb{H}^n}(M_2 F_2)$$

in $A_{R_2}$, for $R_2$ large enough, and (3.18) gives

$$u \le M_2 F_2 \qquad \text{in } \partial A_{R_2} \cup \{\infty\}$$

since $Q \ge 4$. Hence, by the maximum principle, $(P)_2$ holds. We now set, for every $\alpha > 0$, $\alpha' = \min\{Q-2, \alpha+6\}$. Since $(P)_2$ holds, it is sufficient to prove that

$$(P)_\alpha \Rightarrow (P)_{\alpha'} \qquad \forall \alpha > 0$$

and we will get (3.19)=$(P)_{Q-2}$. Let us then fix $\alpha > 0$ and assume $(P)_\alpha$. We have

$$\begin{aligned} -\Delta_{\mathbb{H}^n} u = u u^{\frac{4}{Q-2}} &\le M_\alpha F_\alpha u^{\frac{4}{Q-2}} \qquad (\text{by } (P)_\alpha) \\ &\le c\frac{\sin t}{|z|^\alpha d^4} \qquad (\text{by } (3.6)) \\ &\le c\frac{\sin t}{|z|^{\alpha'-2}} \le -\Delta_{\mathbb{H}^n}(M_{\alpha'} F_{\alpha'}) \qquad (\text{by } (3.16)) \end{aligned}$$

in $A_{R_{\alpha'}}$, for $R_{\alpha'}$ large enough. Moreover, by (3.18), we can choose $M_{\alpha'}$ such that

$$u \le M_{\alpha'} F_{\alpha'} \qquad \text{in } \partial A_{R_{\alpha'}} \cup \{\infty\}.$$

Therefore, from the maximum principle $(P)_{\alpha'}$ follows.

**Proof of Proposition 3.2** Since $u \ge 0$ in $\Pi$ and $u = 0$ in $\partial\Pi$, from (3.19) we get

$$0 \le \frac{u(z,t) - u(z,0)}{t} = \frac{u(z,t)}{t} \le M\frac{F_{Q-2}(z,t)}{t} \le \frac{M}{|z|^{Q-2}}$$

for $|z| > R$ and $0 < t < 1$. Letting $t \to 0$ and recalling (3.5) we finally obtain

$$0 \le \partial_t u(z,0) \le \frac{M}{|z|^{Q-2}} = \frac{M}{d(z,0)^{Q-2}}, \qquad \text{for } |z| > R.$$

Now, we want to estimate $\partial_t u$ in a neighborhood of the origin, the only characteristic point of $\partial\Pi$. We recall that $\partial_t u$ is smooth up to the boundary at any non-characteristic point (see (3.5)). We define

$$A = \{\xi = (z,t) \in \mathbb{H}^n \,|\, 0 < t < 1, |z| < \frac{1}{2}\}.$$

**Lemma 3.7** *There exists $M > 0$ such that $u(z,t) \le Mt$ for every $(z,t) \in A$.*

**Proof** For every $\beta \in ]0,1]$ we define

$$(P)_\beta = (\exists M_\beta > 0 \text{ such that } u \leq M_\beta t^\beta \text{ in } A).$$

From the Hölder continuity of $u$ (see (3.7)) we obtain

$$u(z,t) = |u(z,t) - u(z,0)| \leq Md((z,t),(z,0))^{\beta_0} = Mt^{\frac{\beta_0}{2}},$$

i.e. $(P)_{\frac{\beta_0}{2}}$ holds. Therefore it is sufficient to show that, setting $\beta' = \frac{Q-2}{Q+2}\beta$,

$$(P)_{\beta'} \Rightarrow (P)_\beta \qquad \forall \beta \in ]0,1]$$

and we will get $(P)_1$ and prove the lemma. Let us then fix $\beta \in ]0,1]$ and assume $(P)_{\beta'}$. We set

$$F : A \to \mathbb{R}^+, \quad F(z,t) = t^\beta \exp\{-|z|^2\}.$$

Using formula (3.14) a computation yields

$$-\Delta_{\mathbb{H}^n} F = (2Q - 4 - 4|z|^2)F + 4\beta(1-\beta)\frac{|z|^2}{t^2}F.$$

Hence

$$-\Delta_{\mathbb{H}^n} F \geq F \qquad \text{in } A$$

so that, from $(P)_{\beta'}$ we obtain

$$-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} \leq (M_{\beta'} t^{\beta'})^{\frac{Q+2}{Q-2}} = ct^\beta \leq cF \leq -\Delta_{\mathbb{H}^n}(M_\beta F) \qquad \text{in } A.$$

On the other hand, using (3.5) it is easily seen that $M_\beta$ can be chosen such that

$$u \leq M_\beta F \qquad \text{in } \partial A.$$

Therefore, from the maximum principle $(P)_\beta$ follows.

**Proposition 3.8** *There exists* $M > 0$ *such that* $|\partial_t u(z,0)| \leq M$ *for every* $z \in \mathbb{R}^{2n}$ *such that* $0 < |z| < \frac{1}{2}$.

**Proof** It is an immediate consequence of (3.4), (3.5) and Lemma 3.7.

The next step is to extend inside $\Pi$ the estimates obtained in propositions 3.2 and 3.8. We first evaluate the behavior of $\partial_t u$ at infinity.

**Proposition 3.9** *We have* $|\partial_t u(\xi)| = O(\Gamma(\xi))$, *as* $d(\xi) \to \infty$, $\xi \in \Pi$.

To prove Proposition 3.9 we need some lemmas. Let us define

$$\varrho : \Pi \to \mathbb{R}^+, \quad \varrho(z,t) = \frac{t}{|z| + \sqrt{t + |z|^2}}. \tag{3.21}$$

**Lemma 3.10** *For every* $\xi \in \Pi$ *we have*

$$B_d(\xi, \varrho(\xi)) \subseteq \{(z',t') \in \mathbb{H}^n \mid 0 < t' < 2t\} \subseteq \Pi. \tag{3.22}$$

**Proof** We fix $\xi \in \Pi$ and for sake of brevity we set $\varrho = \varrho(\xi)$. Let $\xi' \in B_d(\xi, \varrho)$ and let us denote $\zeta = (h,k) = (x-x', y-y') = z - z'$, $s = t - t'$. We have (see (2.1) and (2.5))

$$\varrho^4 > d({\xi'}^{-1} \circ \xi)^4 = |\zeta|^4 + (s + 2(\langle x-h, y\rangle - \langle x, y-k\rangle))^2 = |\zeta|^4 + (s + 2\langle z, (k,-h)\rangle)^2.$$

Hence $|\zeta| < \varrho$ and

$$|s| \le |s + 2\langle z, (k,-h)\rangle| + 2|z||\zeta| \le \varrho^2 + 2|z|\varrho.$$

On the other hand (3.21) yields $\varrho^2 + 2|z|\varrho = t$. Therefore $|t - t'| = |s| \le t$.
We now consider the function $v$ defined in (3.3). Since $\Delta_{\mathbb{H}^n} v = 0$ in $\Pi$ (see (3.12)) and (3.22) holds, we can give an estimate of the derivatives of $v$ in terms of $v$ and $\varrho$.

**Lemma 3.11** *There exists $c > 0$ such that for every $\xi \in \Pi$ we have*

$$|\nabla_{\mathbb{H}^n} v(\xi)| \le \frac{c}{\varrho(\xi)} \sup_{B_d(\xi, \frac{\varrho(\xi)}{2})} |v| \tag{3.23}$$

$$|\partial_t v(\xi)| \le \frac{c}{\varrho(\xi)^2} \sup_{B_d(\xi, \frac{\varrho(\xi)}{2})} |v| \tag{3.24}$$

**Proof** It follows from the $\Delta_{\mathbb{H}^n}$-harmonicity of $v$. We refer to [10], Proposition 2.1, for a complete proof.
We define

$$D = \{\xi = (z,t) \in \mathbb{H}^n \,|\, 0 < t < |z|,\ |z| > 1\},$$
$$E = \{\xi = (z,t) \in \mathbb{H}^n \,|\, t \ge |z|,\ t \ge 1\}.$$

From (3.13), (3.21) and (3.24) the next lemma easily follows.

**Lemma 3.12** *There exists $M > 0$ such that*

$$|\partial_t v| \le \frac{M}{t^2} \qquad \text{in } D, \tag{3.25}$$

$$|\partial_t v| \le M\Gamma \qquad \text{in } E. \tag{3.26}$$

We now set, for every $R > 1$,

$$D_R = \{\xi = (z,t) \in \mathbb{H}^n \,|\, 0 < t < |z|,\ 1 < |z| < 2R - t\}.$$

We also set $\beta = \frac{1}{2}$, $\alpha = 2 + \beta$ and define

$$G_R : D_R \to \mathbb{R}^+, \quad G_R(z,t) = \frac{t^\beta}{(2R - |z|)^\alpha}.$$

**Lemma 3.13** *There exist $M > 0$ and $R_0 > 1$ such that for every $R > R_0$ we have*

$$|\partial_t v| \le M(\Gamma + G_R) \qquad \text{in } D_R. \tag{3.27}$$

**Proof** Using (3.14) it is not difficult to verify that, for large $R$,

$$-\Delta_{\mathbb{H}^n} G_R \geq 0 \qquad \text{in } D_R.$$

Since $\Delta_{\mathbb{H}^n}(\partial_t v) = \partial_t(\Delta_{\mathbb{H}^n} v)$, by (3.12) $\partial_t v$ is $\Delta_{\mathbb{H}^n}$-harmonic in $\Pi$. Then, if we prove that

$$|\partial_t v| \leq M(\Gamma + G_R) \qquad \text{in } \partial D_R, \tag{3.28}$$

(3.27) will follow from the maximum principle. From (3.11) and Proposition 3.2 we get

$$|\partial_t v| \leq |\partial_t w| + |\partial_t u| \leq M\Gamma \qquad \text{in } \partial D_R \cap \partial\Pi.$$

On the other hand (3.25) yields

$$|\partial_t v| \leq \frac{M}{t^2} = MG_R \qquad \text{in } \partial D_R \cap \{|z| = 2R - t\}$$

and (3.26) gives

$$|\partial_t v| \leq M\Gamma \qquad \text{in } \partial D_R \cap \{|z| = t\}$$

where $M$ is a constant not depending on $R$. Moreover $|\partial_t v|$ is a continuous function on the set $\{0 \leq t \leq 1,\ |z| = 1\}$. Therefore (3.28) holds.

**Corollary 3.14** *We have* $|\partial_t v(\xi)| \to 0$, *as* $d(\xi) \to \infty$, $\xi \in \Pi$.

**Proof** From (3.27) it follows that, for $R$ large enough,

$$|\partial_t v| \leq M(\Gamma + G_R) \qquad \text{in } D \cap \{|z| = R\}.$$

Then, if $\xi = (z, t) \in D$ and $|z|$ is sufficiently big, we have

$$|\partial_t v(\xi)| \leq M\Big(\Gamma(\xi) + \frac{t^\beta}{(2|z| - |z|)^{2+\beta}}\Big) \leq M\Big(\Gamma(\xi) + \frac{1}{|z|^2}\Big).$$

Hence $|\partial_t v(\xi)| \to 0$, as $d(\xi) \to \infty$, $\xi \in D$. Keeping in mind (3.26), the corollary is proved.

**Proof of Proposition 3.9** We set $\widetilde{\Pi} = \Pi \setminus B_d(0,1)$. Due to (3.11), (3.12) and Proposition 3.2 we have

$$|\partial_t v| \leq M\Gamma \quad \text{in } \partial\widetilde{\Pi}$$

and

$$\Delta_{\mathbb{H}^n}(\partial_t v) = \partial_t(\Delta_{\mathbb{H}^n} v) = 0 = \Delta_{\mathbb{H}^n}(M\Gamma) \quad \text{in } \widetilde{\Pi}.$$

Then, by using Corollary 3.14 and the maximum principle, we get

$$|\partial_t v| \leq M\Gamma \quad \text{in } \widetilde{\Pi}.$$

This estimate and (3.11) finally give $|\partial_t u| \leq M\Gamma$ in $\widetilde{\Pi}$.

We now examine the behavior of $\nabla_{\mathbb{H}^n} u$ and $\partial_t u$ at the origin. We shall prove the following statement.

**Proposition 3.15** *We have* $|\nabla_{\mathbb{H}^n} u|,\ |\partial_t u| \in L^\infty(\Pi \cap B_d(0,1))$.

Let us define

$$K = \{\xi = (z,t) \in \mathbb{H}^n \mid 0 < t < 1,\ |z| < 1\}.$$

**Lemma 3.16** *There exists* $M > 0$ *such that*

$$|\nabla_{\mathbb{H}^n} u| \le M \qquad \text{in } K, \tag{3.29}$$

$$|\partial_t u| \le \frac{M}{\varrho} \qquad \text{in } K. \tag{3.30}$$

*The function* $\varrho$ *has been defined in (3.21).*

**Proof** We fix $\xi_0 \in K$ and define

$$v_0 = v - v(z_0, 0).$$

For sake of brevity we also set $\varrho = \varrho(\xi_0)$ and $B = B_d(\xi_0, \varrho)$. For every $\xi \in B$ we have

$$\begin{aligned} |v_0(\xi)| &= |w(\xi) - w(z_0,0) - u(\xi)| \\ &\le \Big(\max_{Bd(0,3)} |\nabla w|\Big)|\xi - (z_0,0)| + u(\xi) \qquad (\text{see } (3.9)) \\ &\le c(|z - z_0| + t) + ct \qquad (\text{by Lemma } 3.7) \\ &\le c(\varrho + t_0) \qquad (\text{by } (3.22)). \end{aligned}$$

Hence

$$\sup_B |v_0| \le c(\varrho + t_0).$$

Since $v_0$ is $\Delta_{\mathbb{H}^n}$-harmonic as well as $v$, (3.23) and (3.24) hold also replacing $v$ with $v_0$. Therefore we obtain

$$|\nabla_{\mathbb{H}^n} v(\xi_0)| = |\nabla_{\mathbb{H}^n} v_0(\xi_0)| \le \frac{c}{\varrho} \sup_B |v_0| \le c(1 + \frac{t_0}{\varrho}) \le c$$

and

$$|\partial_t v(\xi_0)| = |\partial_t v_0(\xi_0)| \le \frac{c}{\varrho^2} \sup_B |v_0| \le \frac{c}{\varrho},$$

where $c$ is a positive constant not depending on $\xi_0$. Recalling (3.9) we finally get (3.29) and (3.30).

We now fix $\beta \in ]0,1[$ and set $\alpha = 2 + \beta$ and $\gamma = \sqrt{\frac{4\beta(1-\beta)}{\alpha(\alpha+1)}}$. Moreover for every $\varepsilon \in ]0, \frac{1}{2}[$ we define

$$K_\varepsilon = \{\xi = (z,t) \in \mathbb{H}^n \mid \varepsilon < |z| < 1,\ 0 < t < \gamma |z|(|z| - \varepsilon)\}$$

and

$$\Psi_\varepsilon : K_\varepsilon \to \mathbb{R}^+, \quad \Psi_\varepsilon(z,t) = \frac{1}{\varepsilon} + \frac{\varepsilon^{1-\beta} t^\beta}{(|z| - \varepsilon)^\alpha}.$$

**Lemma 3.17** *There exists $M > 0$ such that*

$$|\partial_t v| \le M\Psi_\varepsilon \quad in\ K_\varepsilon \qquad \forall \varepsilon \in ]0, \frac{1}{2}[. \tag{3.31}$$

**Proof** Using (3.14) it is easy to check that

$$-\Delta_{\mathbb{H}^n}\Psi_\varepsilon \ge 0 \qquad \text{in } K_\varepsilon.$$

Then, since $\Delta_{\mathbb{H}^n}(\partial_t v) = \partial_t(\Delta_{\mathbb{H}^n} v) = 0$, if we prove that

$$|\partial_t v| \le M\Psi_\varepsilon \qquad \text{in } \partial K_\varepsilon, \tag{3.32}$$

the maximum principle will give (3.31). From (3.9) and Proposition 3.8 we get

$$|\partial_t v| \le |\partial_t w| + |\partial_t u| \le M \le M\Psi_\varepsilon \qquad \text{in } \partial K_\varepsilon \cap \partial\Pi.$$

Moreover

$$|\partial_t v| \le \max_{\{0\le t\le 1,\ |z|=1\}} |\partial_t v| \le M\Psi_\varepsilon \qquad \text{in } \partial K_\varepsilon \cap \{|z| = 1\}.$$

On the other hand from (3.9) and (3.30) it follows that

$$|\partial_t v| \le M\Psi_\varepsilon \qquad \text{in } \partial K_\varepsilon \cap \{t = \gamma|z|(|z| - \varepsilon)\},$$

with $M$ not depending on $\varepsilon$. Indeed, keeping in mind the very definition (3.21) of the function $\varrho$, if $t = \gamma|z|(|z| - \varepsilon)$ and $\varepsilon \le |z| \le 2\varepsilon$, we have

$$\frac{1}{\varrho(\xi)} \le c\frac{|z|}{t} \le c\frac{\varepsilon^{1-\beta}t^\beta}{(|z| - \varepsilon)^{2+\beta}} \le c\Psi_\varepsilon(\xi).$$

Moreover, if $t = \gamma|z|(|z| - \varepsilon)$ and $2\varepsilon \le |z| \le 1$, then

$$\frac{1}{\varrho(\xi)} \le c\frac{|z|}{t} \le \frac{c}{\varepsilon} \le c\Psi_\varepsilon(\xi).$$

Therefore (3.32) holds.

**Corollary 3.18** *We have $|\partial_t v(\xi)| = O(\frac{1}{d(\xi)})$, as $d(\xi) \to 0$, $\xi \in \Pi$.*

**Proof** From (3.31) it follows that for every $\varepsilon \in ]0, \frac{1}{2}[$

$$|\partial_t v| \le M\Psi_\varepsilon \qquad \text{in } \{|z| = 2\varepsilon,\ 0 < t < \frac{\gamma}{2}|z|^2\}.$$

This means that for every $\xi \in K \cap \{t < \frac{\gamma}{2}|z|^2\}$ we have

$$|\partial_t v(\xi)| \le M\Psi_{\frac{|z|}{2}}(\xi) = \frac{2M}{|z|} + c\frac{|z|^{1-\beta}t^\beta}{|z|^\alpha} \le \frac{c'}{|z|} \le \frac{c}{d(\xi)}.$$

On the other hand (3.21) and (3.30) give

$$|\partial_t v(\xi)| \le \frac{M}{\varrho(\xi)} \le \frac{c'}{\sqrt{t}} \le \frac{c}{d(\xi)} \qquad \text{in } K \cap \{t \ge \frac{\gamma}{2}|z|^2\}.$$

**Proof of Proposition 3.15** Thanks to (3.29) and (3.9) we only need to prove that $|\partial_t v| \in L^\infty(B)$, $B = \Pi \cap B_d(0,1)$. From Proposition 3.8 it follows that there exists $M > 0$ such that

$$|\partial_t v| \le M \quad \text{in } \partial B \setminus \{0\}.$$

Moreover, by Corollary 3.18, for every $\varepsilon > 0$ we have

$$\lim_{\xi \to 0} (M + \varepsilon\Gamma - |\partial_t v|)(\xi) = +\infty.$$

Since $\Delta_{\mathbb{H}^n}(\partial_t v) = 0 = \Delta_{\mathbb{H}^n}(M + \varepsilon\Gamma)$ in $B$, the maximum principle gives $|\partial_t v| \le M + \varepsilon\Gamma$ in $B$. Letting $\varepsilon \to 0$ we finally obtain $|\partial_t v| \le M$ in $B$.

**Proof of Theorem 1.1** We only sketch the proof which is similar to that of Theorem 1.2 in [9] for the halfspace $\{x_1 > 0\}$. It follows from Proposition 3.9 and Proposition 3.15, by using the Rellich-Pohozaev type integral identity proved in [7].

We assume $\Pi = \Pi_t$ (see (3.1)) and, as usual, denote by $\xi = (z,t)$ the point $\xi \in \mathbb{H}^n$. The outer unit normal to $\partial\Pi$ is

$$N = (0, -1). \tag{3.33}$$

Let $P$ be the vector field

$$P = -\partial_t \equiv (0, -1). \tag{3.34}$$

Then $\langle P, N\rangle = 1$ on $\partial\Pi$ and $\Pi$ is $\tau$-starshaped with respect to $(0,-1)$ (see Definition 2.2 in [7]). We also remark that, in the notation of [7], $P = P^{(0,-1)}$. We set $B_r = B_d(0,r)$ for every $r > 0$ and $B_{R,\varepsilon} = B_R \setminus B_\varepsilon$ for every $R > \varepsilon > 0$. Recalling (3.5), using the integral identity (2.7) of [7] and proceeding as on page 83 of the same paper, we obtain

$$\begin{aligned}\int\limits_{B_{R,\varepsilon}\cap\partial\Pi} |\nabla_{\mathbb{H}^n} u|^2 d\sigma &= \int\limits_{B_{R,\varepsilon}\cap\partial\Pi} |\nabla_{\mathbb{H}^n} u|^2 \langle P, N\rangle d\sigma \\ &= \int\limits_{\Pi\cap\partial B_{R,\varepsilon}} \Big( \big(|\nabla_{\mathbb{H}^n} u|^2 - \frac{2}{Q^*} u^{Q*}\big)\langle P, \nu\rangle - 2\langle A\nabla u, \nu\rangle Pu \Big) d\sigma.\end{aligned} \tag{3.35}$$

Here $\nu = \pm \frac{\nabla d}{|\nabla d|}$ is the outer unit normal to $\partial B_{R,\varepsilon}$, $\sigma$ denotes the surface measure and $A$ is the matrix which allows us to represent $\Delta_{\mathbb{H}^n}$ in the divergence form $\Delta_{\mathbb{H}^n} = \operatorname{div}(A\nabla)$. Since

$$|\langle P, \nu\rangle(\xi)| = \left| \left\langle P, \frac{\nabla d}{|\nabla d|} \right\rangle(\xi) \right| = \frac{|\langle (0,-1), d(\xi)^{-3}(|z|^2 z, \frac{t}{2})\rangle|}{|\nabla d(\xi)|} \le \frac{1}{d(\xi)|\nabla d(\xi)|}$$

and

$$|\langle A\nabla u, \nu\rangle| = \frac{|\langle A\nabla u, \nabla d\rangle|}{|\nabla d|} = \frac{|\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} d\rangle|}{|\nabla d|} \le \frac{|\nabla_{\mathbb{H}^n} u|}{|\nabla d|},$$

(3.34) and (3.35) yield

$$\int_{B_{R,\varepsilon}\cap\partial\Pi} |\nabla_{\mathbb{H}^n} u|^2 d\sigma \le c \int_{\Pi\cap\partial B_{R,\varepsilon}} \frac{|\nabla_{\mathbb{H}^n} u|^2 + u^{Q*}}{d|\nabla d|} d\sigma + c \int_{\Pi\cap\partial B_{R,\varepsilon}} \frac{|\nabla_{\mathbb{H}^n} u||\partial_t u|}{|\nabla d|} d\sigma. \tag{3.36}$$

By Federer's coarea formula, for every $g \in L^1(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} g = \int_0^{+\infty} \Big( \int_{\partial B_r} \frac{g}{|\nabla d|} d\sigma \Big) dr. \tag{3.37}$$

Letting $g = |\nabla_{\mathbb{H}^n} u|^2 + u^{Q^*}$, (3.37) implies that there exists a divergent sequence $(R_k)_{k\in\mathbb{N}}$ such that

$$\int_{\Pi\cap\partial B_{R_k}} \frac{|\nabla_{\mathbb{H}^n} u|^2 + u^{Q*}}{|\nabla d|} d\sigma = o\Big(\frac{1}{R_k}\Big), \quad \text{as } k \to +\infty. \tag{3.38}$$

Moreover, letting $g$ be the characteristic function of the set $B_r$, (3.37) yields

$$\int_0^r \Big( \int_{\partial B_\varrho} \frac{d\sigma}{|\nabla d|} \Big) d\varrho = \int_{B_r} d\xi = cr^Q$$

and, by differentiation,

$$\int_{\partial B_r} \frac{d\sigma}{|\nabla d|} = cQr^{Q-1}. \tag{3.39}$$

By means of Proposition 3.9 and Proposition 3.15, choosing a sequence $\varepsilon_k \to 0$, from (3.36), (3.38) and (3.39) we finally obtain

$$\begin{aligned}
\int_{B_{R_k,\varepsilon_k}\cap\partial\Pi} |\nabla_{\mathbb{H}^n} u|^2 d\sigma &\le o\Big(\frac{1}{R_k^2}\Big) + c\varepsilon_k^{Q-2} \\
&\quad + \frac{c}{R_k^{Q-2}} \Big( \int_{\Pi\cap\partial B_{R_k}} \frac{|\nabla_{\mathbb{H}^n} u|^2}{|\nabla d|} d\sigma \Big)^{\frac{1}{2}} \Big( \int_{\Pi\cap\partial B_{R_k}} \frac{1}{|\nabla d|} d\sigma \Big)^{\frac{1}{2}} + c\varepsilon_k^{Q-1} \\
&\le o(1) + \frac{1}{R_k^{Q-2}} \Big( o\Big(\frac{1}{R_k}\Big) \Big)^{\frac{1}{2}} R_k^{\frac{Q-1}{2}} = o(1) + \frac{1}{R_k^{\frac{Q-2}{2}}} o(1).
\end{aligned}$$

Since $Q = 2n + 2 \ge 2$, as $k$ goes to infinity we obtain $\nabla_{\mathbb{H}^n} u \equiv 0$ in $\partial\Pi \setminus \{0\}$ and, as in [7], we can conclude that $u \equiv 0$ in $\Pi$.

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