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Regularity for minimizers of functionals with $p - q$ growth *

Luca ESPOSITO
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata
Università di Salerno, I-84100 Salerno, Italy
posito@matna3.dma.unina.it

Francesco LEONETTI
Dipartimento di Matematica Pura ed Applicata
Università di L'Aquila, I-67100 L'Aquila, Italy
leonetti@univaq.it

Giuseppe MINGIONE
Dipartimento di Matematica dell'Università di Parma
Via D'Azeglio 85/A, I-43100 Parma, Italy
mingione@prmat.math.unipr.it

Abstract

We prove higher integrability for the gradient of vector-valued minimizers of some integral functionals with $p - q$ growth.

1 Introduction

Let us consider the functional

$$F(u) = \int_\Omega f(Du(x))dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$ and $f : \mathbb{R}^{nN} \to \mathbb{R}$ verifies

$$a|z|^p - b \leq f(z) \leq c|z|^q + d, \tag{1.2}$$

for some positive constants a, b, c, d, p, q, with $1 < p \leq q$. Regularity for minimizers u of F has been extensively studied when $p = q$, see [Gia1], [Giu]. When the

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exponents p and q in (1.2) are different, we say that f has nonstandard growth or $p-q$ growth, following Marcellini [Ma5], [Ma1]. Let us remark that minimizers u of F may be not regular, if p and q are too far apart, [Gia2], [Ma6]. On the contrary, when p and q are close enough, some regularity results can be proven: see [Ma5], [FS] where they deal with scalar minimizers $u : \Omega \to \mathbb{R}$; see [AF], [BL], [Le2], [PS] for vector-valued $u : \Omega \to \mathbb{R}^N$. In this paper we prove higher integrability of the gradient for vector-valued minimizers $u \in u^* + W_0^{1,1}(\Omega)$ of (1.1) when f has $p-q$ growth, with $2 \leq p < q < p+2$, the boundary datum $u^* \in L^\infty \cap W^{1,q}$, under suitable assumptions on f. A model functional for our setting is

$$\int_\Omega \Big(|D_1 u|^{q_1} + \cdots + |D_n u|^{q_n} + (e + |Du|^2)^{\alpha + \beta \sin \log \log (e + |Du|^2)} \Big)\, dx, \tag{1.3}$$

where $u : \mathbb{R}^n \to \mathbb{R}^N$, $Du = (D_1 u, \dots, D_n u)$, $D_i u = \partial u / \partial x_i$, $2 \leq q_i \leq 2\alpha + 2\beta$, $0 < \beta < 1/2$, $2 + \beta\sqrt{2} \leq \alpha$. We make a few remarks on the technique we are going to use. We first regularize our functional by adding a q-Dirichlet integral,

$$F_\epsilon(v) = \int_\Omega (f(Dv) + \epsilon |Dv|^q) dx,$$

in order to get the same q growth from above and from below, see Theorem C in [Ma5]. Then we obtain integral estimates for the minimizers u_ϵ of F_ϵ: this can be achieved by a careful use of difference quotient technique, fractional Sobolev spaces and a suitable maximum principle, see [DLM]. Finally, we let ϵ go to zero and we prove that u_ϵ converges to our original minimizer u. A basic tool in this last step is the absence of the so-called Lavrentiev phenomenon, proved in Lemma 2.1. This could be a result of some interest in itself. We remark that, usually, regularity results for minimizers are used to prove the absence of Lavrentiev phenomenon, while in this paper the opposite procedure is followed.

2 Notation and results

Let us consider $\Omega = B(0,1)$ the unit ball in $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \to \mathbb{R}^N, N \geq 1$ and $f : \mathbb{R}^{nN} \to \mathbb{R}$. We shall deal with minimizers of the integral functional

$$F(u) = \int_\Omega f(Du(x))\, dx, \tag{2.1}$$

where $f \in C^2(\mathbb{R}^{nN})$ and, for some positive constants m, L, p, q, ν, σ,

$$2 \leq p < q < p + 2, \tag{2.2}$$

$$m|z|^p \leq f(z) \leq L\,(1 + |z|^q), \tag{2.3}$$

$$|Df(z)| \leq L\,(1 + |z|^{q-1}), \tag{2.4}$$

$$|DDf(z)| \leq L\,(1 + |z|^{q-2}), \tag{2.5}$$

$$\nu\,|z|^{p-2}|\lambda|^2 \le DDf(z)\lambda\lambda, \tag{2.6}$$

$$f(2z) \le \sigma f(z) \tag{2.7}$$

for every $z, \lambda \in \mathbb{R}^{nN}$. The energy density f has the following structure:

$$f(z) = g(|z_1|, \dots, |z_n|, |M_1 z|, \dots, |M_k z|) \tag{2.8}$$

where k is a fixed integer, $1 \le k \le \min\{n, N\}$, $g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is continuous, $M_j Du$ is the vector whose components are all the $j \times j$ minors taken from the $N \times n$ matrix z, thus $|M_1 z| = |z|$, since $M_1 z$ are all the entries z_i^α of the matrix z. We assume that

$$p_i \to g(p_1, \dots, p_i, \dots, p_n, \xi_1, \dots, \xi_k) \qquad \text{is strictly increasing in } [0, +\infty), \tag{2.9}$$

for every $i = 1, \dots, n$,

$$\xi_i \to g(p_1, \dots, p_n, \xi_1, \dots, \xi_i, \dots, \xi_k) \qquad \text{is increasing in } [0, +\infty), \tag{2.10}$$

for every $i = 1, \dots, k$. Let us assume that $u^* : \mathbb{R}^n \to \mathbb{R}^N$ verifies

$$u^* \in W^{1,q}_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n) \tag{2.11}$$

In what follows, u minimizes the functional (2.1), that is

$$u \in u^* + W^{1,1}_0(\Omega) \tag{2.12}$$

and

$$\int_\Omega f(Du(x))dx \le \int_\Omega f(Dv(x))dx \tag{2.13}$$

for every $v \in u^* + W^{1,1}_0(\Omega)$.

REMARK 1. Minimality of u, growth condition (2.3) and integrability (2.11) of the boundary datum u^* give

$$m \int_\Omega |Du|^p \le \int_\Omega f(Du) \le \int_\Omega f(Du^*) \le \int_\Omega L(1 + |Du^*|^q) < +\infty,$$

thus $Du \in L^p(\Omega)$. We will prove the following

Theorem 2.1 *Under the assumptions (2.2), …, (2.11), if $u \in u^* + W^{1,1}_0(\Omega)$ minimizes the functional (2.1), then*

$$Du \in L^r_{loc}(\Omega), \qquad \forall r < \frac{pn}{n-p+q-2}$$

REMARK 2. Please, note that $p < \frac{pn}{n-p+q-2}$: we improved on the integrability of Du.

REMARK 3. The previous Theorem 2.1 is proven by an approximation argument: we consider $u_\epsilon \in u^* + W_0^{1,q}(\Omega)$ minimizing $\int (f(Du_\epsilon) + \epsilon |Du_\epsilon|^q)dx$: this functional has q growth from above and from below. We are able to deal with its Euler equation and we get estimates independent of ϵ, thus Du_ϵ converges to some Dw, for which the estimates still hold true. Eventually, we prove that $u = w$. The last step $u = w$ can be achieved after proving that no Lavrentiev phenomenon occurs in the present setting. More precisely, we have

Lemma 2.1 *Under the assumptions (2.2), ..., (2.11) we get*

$$\inf_{v \in u^* + W_0^{1,1}(\Omega)} \int_\Omega f(Dv)dx = \inf_{v \in u^* + W_0^{1,q}(\Omega)} \int_\Omega f(Dv)dx. \tag{2.14}$$

In section 3 we collect some known results that we will use later; section 4 contains the proof of the Theorem while section 5 is devoted to Lemma 2.1.

3 Known results

For a vector-valued function $G(x)$, define the difference

$$\tau_{s,h}G(x) = G(x + he_s) - G(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \ldots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R = B_R(x_0)$ be the ball centered at x_0 with radius R. We now state several lemmas that we need later. In the following $G : \mathbb{R}^n \to \mathbb{R}^k$, $k \geq 1$; B_ρ, B_R, $B_{2\rho}$ and B_{2R} are concentric balls.

Lemma 3.1 *If $0 < \rho < R$, $|h| < R - \rho$, $1 \leq t < \infty$, $s \in \{1, \ldots, n\}$, G, $D_sG \in L^t(B_R)$, then*

$$\int_{B_\rho} |\tau_{s,h}G(x)|^t dx \leq |h|^t \int_{B_R} |D_sG(x)|^t dx.$$

(See [Gia1, page 45], [C, page 28])

Lemma 3.2 *If $G \in L^2(B_{3\rho})$ and for some $d \in (0, 1)$ and $C > 0$*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h}G(x)|^2 dx \leq C|h|^{2d},$$

for every h with $|h| < \rho$, then $G \in L^r(B_{\rho/4})$ for every $r < 2n/(n - 2d)$.

Proof. The previous inequality tells us that $G \in W^{b,2}(B_{\rho/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces. [A, chapter VII].

Lemma 3.3 *For every t with $1 \le t < \infty$, for every $G \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$ we have*

$$\int_{B_R} |G(x + he_s)|^t dx \le \int_{B_{2R}} |G(x)|^t dx.$$

Lemma 3.4 *For every $p \ge 2$, $G : B_{2R} \to \mathbb{R}^k$ we have*

$$\left|\tau_{s,h}\left(|G(x)|^{(p-2)/2}\, G(x)\right)\right|^2 \le k^3 \left(\frac{p}{2}\right)^2 \int_0^1 |G(x) + t\,\tau_{s,h}G(x)|^{p-2} |\tau_{s,h}G(x)|^2 dt,$$

for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$, for every $x \in B_R$.

Lemma 3.5 *(Maximum principle) Assume that f and g verify (2.8), …, (2.10), with $g \ge 0$. Consider $v = (v^1, \ldots, v^N)$, $v : A \to \mathbb{R}^N$, A bounded open subset of $\mathbb{R}^n$, such that $v \in W^{1,1}(A)$, $\int_A f(Dv) < +\infty$, $\int_A f(Dv) \le \int_A f(Dv + D\phi)$ for every $\phi \in W_0^{1,1}(A)$. If there exist $\beta \in \{1, \ldots, N\}$ and $t \in \mathbb{R}$ such that*

$$|v^\beta| \le t \qquad \text{on } \partial A \qquad \text{then} \qquad |v^\beta| \le t \qquad \text{in } A.$$

(See [DLM])

4 Proof of Theorem 2.1

This Theorem will be proven using an approximation argument: for $\epsilon \in (0,1)$ we consider the function

$$f_\epsilon(z) = f(z) + \epsilon |z|^q. \tag{4.1}$$

It turns out that $f_\epsilon \in C^2(\mathbb{R}^{nN})$ and, for some positive constant c_1, depending only on n, N, q, we have

$$\epsilon|z|^q + m|z|^p \le f_\epsilon(z) \le (L+1)\,(1 + |z|^q), \tag{4.2}$$

$$|Df_\epsilon(z)| \le (L+q)\,(1 + |z|^{q-1}), \tag{4.3}$$

$$|DDf_\epsilon(z)| \le (L+c_1)\,(1 + |z|^{q-2}), \tag{4.4}$$

$$\epsilon q |z|^{q-2}|\lambda|^2 + \nu\,|z|^{p-2}|\lambda|^2 \le DDf_\epsilon(z)\lambda\lambda, \tag{4.5}$$

for every $z, \lambda \in \mathbb{R}^{nN}$, where m, L, p, q, ν are the same as in (2.2), …,(2.6). Thus f_ϵ has q growth from above and from below. Let $u_\epsilon \in u^* + W_0^{1,q}(\Omega)$ minimize the integral $\int f_\epsilon(Dv)dx$, that is

$$\int_\Omega f_\epsilon(Du_\epsilon)dx \le \int_\Omega f_\epsilon(Dv)dx, \tag{4.6}$$

for every $v \in u^* + W_0^{1,q}(\Omega)$. Direct methods in the calculus of variations guarantee the existence of such u_ϵ and show that inequality (4.6) holds true for $v \in u^* + W_0^{1,1}(\Omega)$ too. From now on, ϵ will denote a sequence $\epsilon_j \in (0,1)$ with $\epsilon_j \to 0$ as $j \to \infty$. Sometimes we will pass to a subsequence that will be again denoted by ϵ. Minimality (4.6) and growth conditions (4.2), (4.3) show that u_ϵ solves the Euler equation,

$$\int_\Omega Df_\epsilon(Du_\epsilon(x))D\phi(x)\,dx = 0, \tag{4.7}$$

for all functions $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,q}(\Omega)$. Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ρ and B_R be concentric balls, $0 < \rho < R$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^\infty(B_R)$ with $\eta \equiv 1$ on B_ρ, $0 \le \eta \le 1$. Fix $s \in \{1, \dots, n\}$, take $0 < |h| < R$. Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u_\epsilon)$ in (4.7) we get

$$(I) = \int_{B_R} \eta^2 \tau_{s,h}\left(Df_\epsilon(Du_\epsilon)\right)\tau_{s,h}Du_\epsilon\,dx =$$

$$-\int_{B_R} \tau_{s,h}\left(Df_\epsilon(Du_\epsilon)\right) 2\eta D\eta\,\tau_{s,h}u_\epsilon\,dx = (II).$$

Moreover

$$\int_{B_R}\int_0^1 DDf_\epsilon(Du_\epsilon + t\tau_{s,h}Du_\epsilon)\,\eta\,\tau_{s,h}Du_\epsilon\,\eta\,\tau_{s,h}Du_\epsilon\,dt\,dx = (I) =$$

$$= (II) = \int_{B_R}\int_0^1 -2DDf_\epsilon(Du_\epsilon + t\tau_{s,h}Du_\epsilon)\,\eta\,\tau_{s,h}Du_\epsilon\,D\eta\,\tau_{s,h}u_\epsilon\,dt\,dx. \tag{4.8}$$

Since f_ϵ is C^2, the bilinear form $(\lambda, \xi) \to DDf_\epsilon(Du_\epsilon + t\tau_{s,h}Du_\epsilon)\,\lambda\,\xi$ is symmetric; moreover, it is positive because of (4.5); therefore we can use Cauchy-Schwartz inequality in order to get

$$\begin{aligned}(II) \le \frac{1}{2}\int_{B_R}\int_0^1 DDf_\epsilon(Du_\epsilon + t\tau_{s,h}Du_\epsilon)\,\eta\,\tau_{s,h}Du_\epsilon\,\eta\,\tau_{s,h}Du_\epsilon\,dt\,dx\,+ \\ +2\int_{B_R}\int_0^1 DDf_\epsilon(Du_\epsilon + t\tau_{s,h}Du_\epsilon)\,D\eta\,\tau_{s,h}u_\epsilon\,D\eta\,\tau_{s,h}u_\epsilon\,dt\,dx \\ = \frac{1}{2}(I) + 2(III).\end{aligned} \tag{4.9}$$

The two integrals in (4.9) are finite, so we can subtract $\frac{1}{2}(I)$ from both sides of (4.8) in order to get

$$\frac{1}{2}(I) \le 2(III). \tag{4.10}$$

In order to estimate (I) from below, with constants independent of ϵ, we look at (4.5): we drop $\epsilon q|z|^{q-2}|\lambda|^2$ and we keep $\nu|z|^{p-2}|\lambda|^2$. If we also use Lemma 3.4 and we recall that $\eta = 1$ in B_ρ, we get

$$c_2 \int_{B_\rho} \left|\tau_{s,h}(|Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon)\right|^2 dx \leq (I), \tag{4.11}$$

for some positive constant c_2 indepedent of ϵ and h. On the other hand, if we use the growth condition (4.4), we have

$$(III) \leq c_3 \int_{B_R} (1 + |Du_\epsilon|^{q-2} + |\tau_{s,h} Du_\epsilon|^{q-2})|\tau_{s,h} u_\epsilon|^2 dx, \tag{4.12}$$

for some positive constant c_3 indepedent of ϵ and h. The inequalities (4.10-12) merge into the following Caccioppoli's estimate

$$\int_{B_\rho} \left|\tau_{s,h}(|Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon)\right|^2 dx \leq c_4 \int_{B_R} (1 + |Du_\epsilon|^{q-2} + |\tau_{s,h} Du_\epsilon|^{q-2})|\tau_{s,h} u_\epsilon|^2 dx, \tag{4.13}$$

for some positive constant c_4 indepedent of ϵ and h. In order to control the right hand side of the previous inequality, we consider the growth condition (4.2): again, we drop $\epsilon|z|^q$ and we keep $m|z|^p$. Using the minimality (4.6) of u_ϵ, we are able to control the L^p norm of u_ϵ by means of the L^q norm of the boundary datum u^*:

$$m \int_\Omega |Du_\epsilon|^p dx \leq \int_\Omega f_\epsilon(Du_\epsilon) dx \leq \int_\Omega f_\epsilon(Du^*) dx \leq \int_\Omega (L+1)(1+|Du^*|^q) dx. \tag{4.14}$$

Since we are able to control only the L^p norm of Du_ϵ, we need $q-2 < p$ in (4.13): assumption (2.2) allows us to go on. By Hölder's inequality with exponents $p/(q-2)$ and $p/(p-q+2)$, we get

$$\begin{aligned} &\int_{B_R} (1 + |Du_\epsilon|^{q-2} + |\tau_{s,h} Du_\epsilon|^{q-2})|\tau_{s,h} u_\epsilon|^2 dx \\ &\leq c_5 \left(\int_{B_R} (1 + |Du_\epsilon|^p + |\tau_{s,h} Du_\epsilon|^p) dx \right)^{\frac{q-2}{p}} \left(\int_{B_R} |\tau_{s,h} u_\epsilon|^{\frac{2p}{p-q+2}} dx \right)^{\frac{p-q+2}{p}}, \end{aligned}$$

for some positive constant c_5 indepedent of ϵ and h. Application of Lemma 3.3 with $t = p$ and $G = Du_\epsilon$ gives

$$\left(\int_{B_R} (1 + |Du_\epsilon|^p + |\tau_{s,h} Du_\epsilon|^p) dx \right)^{\frac{q-2}{p}} \leq c_6 \left(\int_{B_{2R}} (1 + |Du_\epsilon|^p) dx \right)^{\frac{q-2}{p}}$$

for some positive constant c_6 indepedent of ϵ and h. Because of assumptions (2.8-11), the maximum principle is available, [DLM], thus

$$\|u_\epsilon\|_{L^\infty(\Omega)} \le c_7 < \infty, \tag{4.15}$$

for some positive constant c_7 indepedent of ϵ and h, so application of (4.15) and Lemma 3.1 give

$$\int_{B_R} |\tau_{s,h}u_\epsilon|^{\frac{2p}{p-q+2}}\,dx = \int_{B_R} |\tau_{s,h}u_\epsilon|^p |\tau_{s,h}u_\epsilon|^{\frac{2p}{p-q+2}-p}dx \le$$
$$\le c_8 \int_{B_R} |\tau_{s,h}u_\epsilon|^p dx \le c_8|h|^p \int_{B_{2R}} |D_s u_\epsilon|^p dx,$$

for some positive constant c_8 indepedent of ϵ and h. The L^p bound (4.14) and the previous inequalities merge into

$$\sum_{s=1}^{n} \int_{B_\rho} \left|\tau_{s,h}(|Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon)\right|^2 dx \le c_9|h|^{p-q+2}, \tag{4.16}$$

for some positive constant c_9 indepedent of ϵ and h. Now we recall that $u_\epsilon \in u^* + W_0^{1,q}(\Omega)$, thus, the L^p bound (4.14) implies that, passing to some subsequence still labelled by u_ϵ,

$$Du_\epsilon \to Dw \qquad \text{weakly in } L^p(\Omega), \tag{4.17}$$

for some $w \in u^* + W_0^{1,p}(\Omega)$. Moreover

$$\||Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon\|^2_{L^2(\Omega)} = \|Du_\epsilon\|^p_{L^p(\Omega)} \le \frac{L+1}{m}\int_\Omega (1+|Du^*|^q)dx, \tag{4.18}$$

thus, up to a subsequence,

$$|Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon \to v \qquad \text{weakly in } L^2(\Omega), \tag{4.19}$$

for some $v \in L^2(\Omega)$. The estimate (4.16) shows that the convergence (4.19) is actually strong:

$$|Du_\epsilon|^{\frac{p-2}{2}} Du_\epsilon \to v \qquad \text{strongly in } L^2_{loc}(\Omega), \tag{4.20}$$

then, passing again to some subsequence,

$$|Du_\epsilon|^{\frac{p-2}{2}}(x) Du_\epsilon(x) \to v(x) \qquad \text{for almost every } x \in \Omega. \tag{4.21}$$

Now (4.17), (4.20) and (4.21) guarantee that

$$v = |Dw|^{\frac{p-2}{2}} Dw, \tag{4.22}$$

thus, (4.21) and (4.22) give

$$Du_\epsilon(x) \to Dw(x) \qquad \text{for almost every } x \in \Omega. \tag{4.23}$$

We use (4.20) and (4.22) in order to pass to the limit in the estimate (4.16): we have

$$\sum_{s=1}^{n} \int_{B_\rho} \left| \tau_{s,h}(|Dw|^{\frac{p-2}{2}} Dw) \right|^2 dx \leq c_9 |h|^{p-q+2}, \tag{4.24}$$

for every h with $|h| < \rho$, where c_9 is independent of h. Application of Lemma 3.2 with $G = |Dw|^{\frac{p-2}{2}} Dw$ gives $|Dw|^{\frac{p-2}{2}} Dw \in L^t(B_{\rho/4})$ for every $t < 2n/(n-p+q-2)$. Since $||Dw|^{\frac{p-2}{2}} Dw| = |Dw|^{p/2}$, a covering argument shows that

$$Dw \in L^r_{loc}(\Omega), \qquad \forall r < \frac{pn}{n-p+q-2}. \tag{4.25}$$

We claim that

$$u = w. \tag{4.26}$$

In order to prove that, we consider any $v \in u^* + W_0^{1,q}(\Omega)$; we keep in mind the relation between f and f_ϵ, so, using the minimality (4.6), we get

$$\int_\Omega f(Du_\epsilon) \leq \int_\Omega f_\epsilon(Du_\epsilon) \leq \int_\Omega f_\epsilon(Dv) = \int_\Omega f(Dv) + \epsilon \int_\Omega |Dv|^q.$$

We use the pointwise convergence (4.23), the continuity of f and Fatou's lemma, thus

$$\int_\Omega f(Dw) \leq \liminf_{\epsilon \to 0} \int_\Omega f(Du_\epsilon) \leq \liminf_{\epsilon \to 0} \left(\int_\Omega f(Dv) + \epsilon \int_\Omega |Dv|^q \right) = \int_\Omega f(Dv),$$

so that

$$\int_\Omega f(Dw) \leq \int_\Omega f(Dv), \qquad \forall v \in u^* + W_0^{1,q}(\Omega),$$

then

$$\int_\Omega f(Dw) \leq \inf_{v \in u^* + W_0^{1,q}(\Omega)} \int_\Omega f(Dv).$$

Now we use the minimality property (2.13) of u:

$$\int_\Omega f(Du) = \inf_{v \in u^* + W_0^{1,1}(\Omega)} \int_\Omega f(Dv) \leq \int_\Omega f(Dw) \leq \inf_{v \in u^* + W_0^{1,q}(\Omega)} \int_\Omega f(Dv). \tag{4.27}$$

In our setting no Lavrentiev phenomenon occurs, as Lemma 2.1 shows, thus (2.14) and (4.27) give

$$\int_\Omega f(Du) = \int_\Omega f(Dw).$$

Since f is strictly convex, see (2.6), we get $Du = Dw$. Moreover, $u - w \in W_0^{1,1}(\Omega)$, so $u = w$. (4.25) and (4.26) end the proof. $\square$

5 Proof of Lemma 2.1

Direct methods in the calculus of variations give us the existence of $u \in u^* + W_0^{1,1}(\Omega)$ such that

$$\int_\Omega f(Du)dx = \inf_{v \in u^* + W_0^{1,1}(\Omega)} f(Dv)dx. \tag{5.1}$$

Claim: for every $\tau > 0$ there exists $w \in u^* + W_0^{1,q}(\Omega)$ such that

$$\int_\Omega f(Dw)dx \leq \int_\Omega f(Du)dx + \tau. \tag{5.2}$$

This and (5.1) will end the proof. We are going to build such a function w. Since the minimizer u belongs to $u^* + W_0^{1,1}(\Omega)$, there exists $u_0 \in W_0^{1,1}(\Omega)$ such that $u = u^* + u_0$. We extend u_0 by zero to all of $\mathbb{R}^n$: $u_0 = 0$ in $\mathbb{R}^n \setminus \Omega$, thus $u_0 \in W^{1,1}(\mathbb{R}^n)$. Keep in mind that $u^* \in W^{1,q}_{loc}(\mathbb{R}^n)$. We recall that Ω is the unit ball $B(0,1)$. We write B_t instead of $B(0,t)$. For $0 < \delta < 1$, let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^1(B_{1-\delta/2})$ with $\eta \equiv 1$ on $B_{1-3\delta/4}$, $0 \leq \eta \leq 1$. For $0 < \epsilon < \frac{\delta}{4}$ we define $r_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$r_\epsilon(x) = \frac{x}{1-\epsilon}. \tag{5.3}$$

Thus

$$u_0 \circ r_\epsilon(x) = u_0(r_\epsilon(x)) = 0 \qquad \text{when } |x| > 1 - \epsilon. \tag{5.4}$$

Let ρ_ϵ be a radial mollifier with support in the ball $\overline{B(0, \epsilon/4)}$. We write $G \star \rho_\epsilon$ for the mollification of G:

$$G \star \rho_\epsilon(x) = \int_{B(x,\epsilon/4)} G(y)\rho_\epsilon(x-y)dy.$$

Because of (5.4), we get

$$(u_0 \circ r_\epsilon) \star \rho_\epsilon \in C_0^\infty(\Omega). \tag{5.5}$$

We define v_ϵ to be

$$v_\epsilon = \eta(u \star \rho_\epsilon) + (1-\eta)(u^* + (u_0 \circ r_\epsilon) \star \rho_\epsilon) \tag{5.6}$$

and we have

$$v_\epsilon = u^* + \eta[(u \star \rho_\epsilon) - (u_0 \circ r_\epsilon) \star \rho_\epsilon - u^*] + (u_0 \circ r_\epsilon) \star \rho_\epsilon \in u^* + W_0^{1,q}(\Omega). \tag{5.7}$$

It will turn out that, for a suitable choice of δ and ϵ, v_ϵ will be the desired function w in (5.2). In order to check it, we need to compute the energy of v_ϵ:

$$\int_\Omega f(Dv_\epsilon) = \int_{B_{1-3\delta/4}} f(Dv_\epsilon) + \int_{B_{1-\delta/2} \setminus B_{1-3\delta/4}} f(Dv_\epsilon) + \int_{B_1 \setminus B_{1-\delta/2}} f(Dv_\epsilon) = (I) + (II) + (III). \tag{5.8}$$

Let us recall that f is convex and positive, thus Jensen's inequality gives

$$\int_A f(D\tilde{u}\star\rho_\epsilon(x))dx = \int_A f(\int_{B(x,\epsilon/4)} D\tilde{u}(y)\rho_\epsilon(x-y)dy)dx \leq$$

$$\leq \int_A \int_{B(x,\epsilon/4)} f(D\tilde{u}(y))\rho_\epsilon(x-y)dydx = \int_A \int_{A^{\epsilon/4}} f(D\tilde{u}(y))\rho_\epsilon(x-y)dydx$$

$$= \int_{A^{\epsilon/4}} \int_A f(D\tilde{u}(y))\rho_\epsilon(x-y)dxdy \leq \int_{A^{\epsilon/4}} f(D\tilde{u}(y))dy, \tag{5.9}$$

for $A^{\epsilon/4} = \bigcup_{\tilde{x}\in A} B(\tilde{x},\epsilon/4)$ and suitable $\tilde{u}$. Now we are ready to deal with (I) in (5.8): in $B_{1-3\delta/4}$ we have $Dv_\epsilon = Du\star\rho_\epsilon$, thus, applying (5.9) with $\tilde{u}=u$ and $A = B_{1-3\delta/4}$, we get

$$(I) \leq \int_\Omega f(Du). \tag{5.10}$$

Let us note that convexity (2.6) and Δ_2 property (2.7) imply

$$f(v+w) \leq \frac{\sigma}{2}(f(v)+f(w)), \tag{5.11}$$

$$f(tz) \leq \frac{\sigma}{2}(tf(z)+(2-t)f(0)), \tag{5.12}$$

for every $v,w,z\in\mathbb{R}^{nN}$, for every t with $1<t<2$. We now use (5.11), the property of the "cut off" function $0\leq\eta\leq 1$ and the convexity of f:

$$(II) \leq \frac{\sigma}{2}\int_{B_{1-\delta/2}\setminus B_{1-3\delta/4}} \eta f(Du\star\rho_\epsilon) +$$

$$+\frac{\sigma}{2}\int_{B_{1-\delta/2}\setminus B_{1-3\delta/4}} (1-\eta)f(D(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon))$$

$$+\frac{\sigma}{2}\int_{B_{1-\delta/2}\setminus B_{1-3\delta/4}} f(D\eta[u\star\rho_\epsilon-(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon)])$$

$$= (IV)+(V)+(VI). \tag{5.13}$$

Application of (5.9) with $\tilde{u}=u$ and $A = B_{1-\delta/2}\setminus B_{1-3\delta/4}$ gives us

$$(IV) \leq \frac{\sigma}{2}\int_{B_1\setminus B_{1-\delta}} f(Du). \tag{5.14}$$

In order to deal with (V), we recall that $u = u^*+u_0$, thus

$$u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon = (u\circ r_\epsilon)\star\rho_\epsilon + u^* - (u^*\circ r_\epsilon)\star\rho_\epsilon,$$

then, with the aid of (5.11) we get

$$(V) \leq \frac{\sigma^2}{4} \int_{B_{1-\delta/2} \setminus B_{1-3\delta/4}} f(D((u \circ r_\epsilon) \star \rho_\epsilon)) +$$

$$+ \frac{\sigma^2}{4} \int_{B_{1-\delta/2} \setminus B_{1-3\delta/4}} f(Du^* - D((u^* \circ r_\epsilon) \star \rho_\epsilon)) = (VII) + (VIII). \tag{5.15}$$

We use (5.9) with $\tilde{u} = u \circ r_\epsilon$ and $A = B_{1-\delta/2} \setminus B_{1-3\delta/4}$:

$$(VII) \leq \frac{\sigma^2}{4} \int_{B_{1-\delta/4} \setminus B_{1-\delta}} f(D(u \circ r_\epsilon)) = \frac{\sigma^2}{4} \int_{B_{1-\delta/4} \setminus B_{1-\delta}} f(\frac{1}{1-\epsilon} Du \circ r_\epsilon),$$

where we computed $Dr_\epsilon = \frac{1}{1-\epsilon} Id$. Now we use (5.12) and we change variables; eventually, we get

$$(VII) \leq \frac{\sigma^3}{4} \int_{B_1 \setminus B_{1-\delta}} [f(Du) + f(0)]. \tag{5.16}$$

Nearly in the same way, we obtain

$$(VIII) \leq \int_{B_1 \setminus B_{1-\delta}} \{ \frac{\sigma^3}{8} f(Du^*) + \frac{\sigma^4}{8} [f(-Du^*) + f(0)] \}. \tag{5.17}$$

The inequalities (5.15-17) yield

$$(V) \leq \int_{B_1 \setminus B_{1-\delta}} [\frac{\sigma^3}{4} f(Du) + \frac{\sigma^3}{8} f(Du^*) + \frac{\sigma^4}{8} f(-Du^*) + (\frac{\sigma^3}{4} + \frac{\sigma^4}{8}) f(0)]. \tag{5.18}$$

In order to deal with (VI), we recall the growth condition (2.3):

$$(VI) \leq \int_{B_1 \setminus B_{1-\delta}} \frac{\sigma}{2} L + \int_{B_{1-\delta/2}} \frac{\sigma}{2} L |D\eta|^q |u \star \rho_\epsilon - (u^* + (u_0 \circ r_\epsilon) \star \rho_\epsilon)|^q. \tag{5.19}$$

Now we use the minimality property (2.13) of u, boundedness (2.11) of the boundary datum u^* and assumptions (2.8-10): maximum principle is available, [DLM], thus

$$u \in L^\infty(\Omega).$$

Since $u_0 = u - u^*$, we get

$$u_0 \in L^\infty(\Omega), \tag{5.20}$$

thus

$$\begin{aligned}
&\|u\star\rho_\epsilon-(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon)\|_{L^q(B_{1-\delta/2})}=\\
&=\|u^*\star\rho_\epsilon-u^*+(u_0-u_0\circ r_\epsilon)\star\rho_\epsilon\|_{L^q(B_{1-\delta/2})}\\
&\le\|u^*\star\rho_\epsilon-u^*\|_{L^q(B_{1-\delta/2})}+\|(u_0-u_0\circ r_\epsilon)\star\rho_\epsilon\|_{L^q(B_{1-\delta/2})}\\
&\le\|u^*\star\rho_\epsilon-u^*\|_{L^q(B_{1-\delta/2})}+\|u_0-u_0\circ r_\epsilon\|_{L^q(B_{1-7\delta/16})}. \qquad (5.21)
\end{aligned}$$

Because of the integrability properties (2.11) and (5.20), inequality (5.21) shows that

$$(D\eta)(u\star\rho_\epsilon-(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon))\to 0 \qquad \text{in } L^q(B_{1-\delta/2}), \qquad \text{as } \epsilon\to 0. \quad (5.22)$$

Inequalities (5.13-14), (5.18-19) merge into

$$\begin{aligned}
&(II)\le(\sigma(1+L)+\frac{\sigma^3}{2}+\frac{\sigma^4}{4})\times\\
&\times\int_{B_1\setminus B_{1-\delta}}[f(Du)+f(Du^*)+f(-Du^*)+1+f(0)]\\
&+\frac{\sigma}{2}L\int_{B_{1-\delta/2}}|(D\eta)(u\star\rho_\epsilon-(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon))|^q. \qquad (5.23)
\end{aligned}$$

In order to deal with (III), we keep in mind that $\eta=0$ in $B_1\setminus B_{1-\delta/2}$, thus $Dv_\epsilon=D(u^*+(u_0\circ r_\epsilon)\star\rho_\epsilon)=D(u\circ r_\epsilon)\star\rho_\epsilon+Du^*-D(u^*\circ r_\epsilon)\star\rho_\epsilon$. We now use (5.11) and we obtain

$$\begin{aligned}
(III)\le{}&\frac{\sigma}{2}\int_{B_1\setminus B_{1-\delta/2}}f(D(u\circ r_\epsilon)\star\rho_\epsilon)+\\
&+\frac{\sigma}{2}\int_{B_1\setminus B_{1-\delta/2}}f(Du^*-D(u^*\circ r_\epsilon)\star\rho_\epsilon)\\
&=(X)+(XI). \qquad (5.24)
\end{aligned}$$

We use (5.9) with $\tilde u=u\circ r_\epsilon$ and $A=B_1\setminus B_{1-\delta/2}$, then we apply (5.12), eventually we change variables and we get

$$\begin{aligned}
(X)&\le\frac{\sigma}{2}\int_{B_{1+\delta/4}\setminus B_{1-3\delta/4}}f(D(u\circ r_\epsilon))=\frac{\sigma}{2}\int_{B_{1+\delta/4}\setminus B_{1-3\delta/4}}f(\frac{1}{1-\epsilon}Du\circ r_\epsilon)\\
&\le\frac{\sigma^2}{2}\int_{B_{1+\delta/4}\setminus B_{1-3\delta/4}}[f(Du\circ r_\epsilon)+f(0)]\\
&\le\frac{\sigma^2}{2}\int_{B_{1+\delta}\setminus B_{1-\delta}}[f(Du)+f(0)]. \qquad (5.25)
\end{aligned}$$

Using growth condition (2.3), L^q estimates for mollification and changing variables, we obtain

$$(XI) \leq \frac{\sigma}{2} \int\limits_{B_1 \setminus B_{1-\delta/2}} (1 + L2^q|Du^*|^q + L2^q|D(u^* \circ r_\epsilon) \star \rho_\epsilon|^q) \leq$$
$$\leq \sigma \int\limits_{B_{1+\delta} \setminus B_{1-\delta}} (1 + L3^q|Du^*|^q). \qquad (5.26)$$

Estimates (5.24–26) merge into

$$(III) \leq (\sigma + \sigma^2) \int\limits_{B_{1+\delta} \setminus B_{1-\delta}} [f(Du) + f(0) + 1 + L3^q|Du^*|^q]. \qquad (5.27)$$

We put together the inequalities (5.8), (5.10), (5.23), (5.27):

$$\int\limits_{\Omega} f(Dv_\epsilon) \leq \int\limits_{\Omega} f(Du) +$$
$$+c_{10} \int\limits_{B_{1+\delta} \setminus B_{1-\delta}} [f(Du) + f(Du^*) + f(-Du^*) + f(0) + 1 + L3^q|Du^*|^q]$$
$$+\sigma L \int\limits_{B_{1-\delta/2}} |(D\eta)(u \star \rho_\epsilon - (u^* + (u_0 \circ r_\epsilon) \star \rho_\epsilon))|^q, \qquad (5.28)$$

where $c_{10} = \sigma(2+L) + \sigma^2 + \sigma^3 + \sigma^4$. For every $\tau > 0$, there exists $\delta = \delta_\tau > 0$ such that

$$c_{10} \int\limits_{B_{1+\delta} \setminus B_{1-\delta}} [f(Du) + f(Du^*) + f(-Du^*) + f(0) + 1 + L3^q|Du^*|^q] \leq \frac{\tau}{2}. \qquad (5.29)$$

For such $\delta = \delta_\tau$, because of (5.22), there exists $\epsilon = \epsilon_\tau > 0$ such that

$$\sigma L \int\limits_{B_{1-\delta/2}} |(D\eta)(u \star \rho_\epsilon - (u^* + (u_0 \circ r_\epsilon) \star \rho_\epsilon))|^q \leq \frac{\tau}{2}. \qquad (5.30)$$

With these parameters $\delta = \delta_\tau$ and $\epsilon = \epsilon_\tau$, the resulting function v_ϵ verifies (5.7), thus it belongs to $u^* + W_0^{1,q}(\Omega)$. Moreover, collecting (5.28-30) yields

$$\int\limits_{\Omega} f(Dv_\epsilon) \leq \int\limits_{\Omega} f(Du) + \tau,$$

thus, (5.2) is proven with $w = v_\epsilon$. This ends the proof. $\square$

REMARK. We quote [B] and its references for more information on the Lavrentiev phenomenon. Let us mention the approximation technique of Remark 3.4 in [B]: unfortunately, such a method does not seem to preserve the boundary value u^*. In our Lemma 2.1, the approximating functions v_ϵ are required to agree with u^* on $\partial\Omega$: this requirement makes the proof a little bit difficult.

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