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Some remarks on critical point theory for nondifferentiable functionals

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Abstract

In this paper we study the existence of critical points of nondifferentiable functionals J of the kind $J(v) = \int_\Omega A(x,v)|\nabla v|^2 - F(x,v)$ with $A(x,z)$ a Carathéodory function bounded between positive constant and with bounded derivative respect to the variable z, and $F(x,z)$ is the primitive of a (Carathéodory) nonlinearity $f(x,z)$ satisfying suitable hipotheses. Since J is just differentible along bounded directions, a suitable compactness condition is introduced. Its connection with coercivity is discussed. In addition, the case of concave-convex nonlinearities $f(x,z)$, unbounded coefficients $A(x,z)$ and related problems are also studied.

1 Introduction

Usually the critical point theory deals with C^1-functionals defined in Banach spaces. However, simple examples show that this differentiability condition may fail. For example, if one considers

$$I(v) = \int_\Omega A(x,v)|\nabla v|^2\,dx,\ \ v \in W_0^{1,2}(\Omega),$$

with $0 < \alpha \le A(x,z) \le \beta < \infty$, $|A'_z(x,z)| \le \gamma$ and $N > 1$, then I is not Gateaux-differentiable. It is only differentiable along directions of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, even for smooth functions $A(x,z)$ (see [16]).

Part of this paper is devoted to applications of the critical point theory developed in [6] (see also [5, 8, 13, 17]) for functionals which are not differentiable in all directions. The abstract framework is given by the following asumption:

(H) $(X, \|\cdot\|_X)$ is a Banach space and $Y \subset X$ is a subspace which is a normed space endowed with a norm $\|\cdot\|_Y$. Moreover, $J : X \longrightarrow \mathbb{R}$ is a functional on X such that it is continuous in $(Y, \|\cdot\|_X + \|\cdot\|_Y)$ and satisfies the following hypotheses:

a) J has a directional derivative $\langle J'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$.

b) For fixed $u \in X$, the function $\langle J'(u), v\rangle$ is linear in $v \in Y$, and for fixed $v \in Y$, the function $\langle J'(u), v\rangle$ is continuous in $u \in X$.

Thus a function $u \in X$ is called a critical point of J if $\langle J'(u), v\rangle = 0$, $\forall v \in Y$. In this framework a suitable version of the Ambrosetti-Rabinowitz Theorem (with only geometric hypotheses) has been proved in [6]. Specifically, *if we assume* (H) *and that for* $e \in Y$,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) > c_1 = \max\{J(0), J(e)\}$$

with $\Gamma = \{\gamma : [0,1] \longrightarrow (Y, \|\cdot\|_X + \|\cdot\|_Y)/\gamma$ *continuous and* $\gamma(0) = 0$, $\gamma(1) = e\}$, *then there exists a sequence* $\{u_n\}$ *in* Y *satisfying for some* $\{K_n\} \subset \mathbb{R}^+$ *and* $\{\varepsilon_n\} \longrightarrow 0$ *that*

$$\{J(u_n)\} \text{ is bounded,} \tag{1}$$

$$\|u_n\|_Y \le 2K_n \quad \forall n \in \mathbb{N}, \tag{2}$$

$$|\langle J'(u_n), v\rangle| \le \varepsilon_n \left[\frac{\|v\|_Y}{K_n} + \|v\|_X\right] \quad \forall v \in Y. \tag{3}$$

Some *compactness condition* on the functional J is used in order to deduce the existence of a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to some critical point. For instance, in the regular case $X = Y$, it is imposed the well-known Palais-Smale condition: *any sequence* $\{u_n\}$ *in the Banach space* X *for which* $\{J(u_n)\}$ *is bounded and* $\{J'(u_n)\}$ *converges to zero in the dual space* X', *possesses a convergent subsequence.* But, in the general framework the compactness condition have to be different.

The compactness condition we shall consider is:

(C) *Any sequence* $\{u_n\}$ *in the Banach space* Y *satisfying for some* $\{K_n\} \subset \mathbb{R}^+$ *and* $\{\varepsilon_n\} \longrightarrow 0$ *the conditions* (1), (2) *and* (3), *possesses a convergent subsequence in* X.

In Section 2, we give some remarks about the connections of this compactness condition with the coercivity of J extending the previous results for C^1 functionals in [11, 14]. The results in this line are not complete as it is shown in the last example of this second section. Indeed, we prove the existence of minima for a class of integral functionals (motivated by [18]), which are continuous and coercive (without to know if they are weakly lower semicontinuous).

The abstract theorem is applied in the following sections to obtain nontrivial critical points of the functional J defined by

$$J(v) = \int_\Omega A(x,v)|\nabla v|^2\,dx \,-\, \int_\Omega F(x,v)\,dx,\;\; v \in W_0^{1,2}(\Omega), \tag{4}$$

i.e. nontrivial solutions of the boundary value problem:

$$\left.\begin{array}{c} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(\,A(x,u)\,\nabla u) + \frac{1}{2}A'_z(x,u)|\nabla u|^2 = F'_u(x,u) \equiv f(x,u) \end{array}\right\} \qquad (P)$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function with subcritical growth.

For a solution u of (P) we mean

$$\left.\begin{array}{c} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \\ \int_\Omega A(x,u)\nabla u \nabla v dx + \frac{1}{2}\int_\Omega A'_z(x,u)|\nabla u|^2 v dx = \int_\Omega f(x,u) v dx \end{array}\right\}$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We discuss in Section 3 some new examples of nonlinearities $f(x,z)$ which in some sense can be considered complementary of these ones in [6]. These examples constitute models to apply the above Mountain Pass Theorem in different geometrical settings of the functional J associated to (P).

Thus, for bounded coefficients $A(x,z)$, we study nonlinearities which combine a convex term with a concave term in the line developed in [9, 1] for the case of PDE with a Euler C^1 functional.

On the other hand, Section 4 is devoted to study the case of unbounded coefficients $A(x,z)$ with respect to z for superlinear nonlinearities $f(x,z)$. In particular, we show how a suitable sequence of truncatures of $A(x,z)$ enables us to reduce the problem to the case of bounded coefficients.

The general case of functionals

$$\int_\Omega \mathcal{J}(x,v,\nabla v)\,dx \,-\, \int_\Omega F(x,v)\,dx,\;\; v \in W_0^{1,p}(\Omega),\;\; (p > 1)$$

as well as the positiveness of the critical point could be also handle as in [6]. For simplicity reasons, we just present here the case $p = 2$, $\mathcal{J}(x,v,\nabla v) = A(x,v)|\nabla v|^2$.

In the last section we adopt the method of Section 4 to handle a class of nondifferentiable functionals. In this case the nondifferentiability of the functional is due to the fact that it contains a term $\int_\Omega |\nabla v|\,dx$. Using a suitable definition of critical point we prove the existence of it.

2 Coercivity and compactness condition

As it is well known, classical hypotheses (see [16]) imply continuity and weak lower semicontinuity of integral functionals J defined on the (reflexive) Sobolev space $W_0^{1,2}(\Omega)$ and bounded from below. Basically, there are two standard ways to prove existence of (global) minima. One of these consists in using some compactness properties. Indeed, choosing a minimizing sequence $\{u_n\}$ which satisfies $(1)-(3)$, then assuming (C) holds, $\{u_n\}$ is compact, and the continuity of J implies that the cluster points of $\{u_n\}$ are minima. We point out that it is sufficient to show the existence of a compact minimizing sequence.

On the other hand, we can argue in a different way by using the weakly lower semicontinuity of J if it is coercive. Indeed, this assumption yields the boundedness of the minimizing sequences $\{u_n\}$. Then a bounded sequence is weakly compact. Hence, the weak cluster points of $\{u_n\}$ are minima since J is w.l.s.c.

Thus in our setting, the existence of minima is obtained if we assume either the condition (C) for the minimizing sequences or the coercivity of the functional. These two kinds of arguments are not very different. In fact, in [11, 14], for C^1 functionals bounded from below, it is proved that the usual Palais-Smale condition implies the coercivity of the functional. The following theorem extends this result to our setting. We follow closely the proof of [14].

Theorem 2.1 *In addition to (H), assume that Y is dense in X and that J is continuous in X and bounded from below. If J satisfies condition (C) then J is coercive; i.e., $\lim_{\|u\|_X \longrightarrow \infty} J(u) = \infty$.*

Proof. Denote $J^d = \{u \in X \ / \ J(u) \le d\}$ for all $d \in \mathbb{R}$, and consider the set $\mathcal{D} = \{d \in \mathbb{R} \ / \ J^d \text{ is bounded }\}$. Since J is bounded from below, we have that $(-\infty, \inf_X \ J) \subset \mathcal{D}$, and thus $\mathcal{D}$ is not empty. To prove that J is coercive, is equivalent to show that $\mathcal{D} = \mathbb{R}$. We argue by contradiction assuming that $d_0 \equiv \sup \mathcal{D} < \infty$. We reach a contradiction by obtaining sequences $\{u_n\} \subset Y$ and $\{K_n\} \subset \mathbb{R}^+$ satisfying

$$\{J(u_n)\} \longrightarrow d_0, \tag{5}$$

$$\|u_n\|_Y \le 2K_n \quad \forall n \in \mathbb{N}, \tag{6}$$

$$\langle J'(u_n), v\rangle \le \frac{2}{\sqrt{n}} \left[\frac{\|v\|_Y}{K_n} + \|v\|_X\right] \quad \forall v \in Y, \tag{7}$$

and

$$\|u_n\|_X \longrightarrow \infty; \tag{8}$$

which clearly contradicts that J satisfies (C). Indeed, since $d_0 - \frac{1}{n} \in \mathcal{D}$ by the definition of d_0, there exists a sequence $\{R_n\} \subset \mathbb{R}^+$ such that $R_n \longrightarrow \infty$ and

$$J^{d_0 - \frac{1}{n}} \subset B_{R_n}(0, \|\cdot\|_X).$$

where $B_{R_n}(0, \|\cdot\|_X)$ is the open ball in X centered at zero with radius R_n. Let φ_n be the restriction of J to $M_n \equiv Y - B_{R_n}(0, \|\cdot\|_X)$. Then

$$m_n \equiv \inf_{M_n} \varphi_n \geq d_0 - \frac{1}{n}.$$

On the other hand, using that $J^{d_0+\frac{1}{n}}$ is unbounded, we can consider a sequence $\{\hat{u}_n\}$ in X satisfying $J(\hat{u}_n) \leq d_0 + \frac{1}{n}$ and $\|\hat{u}_n\|_X \geq R_n + 1 + \frac{1}{n}$. By the density of Y in X and the continuity of J in X, there is no loss of generality in assuming that $\hat{u}_n \in Y$ and, hence

$$\hat{u}_n \in M_n, \quad m_n \leq \varphi(\hat{u}_n) \leq d_0 + \frac{1}{n} \leq m_n + \frac{2}{n}.$$

Now, take $K_n = \|\hat{u}_n\|_Y$ and consider the space Y equipped with the norm $\|\cdot\|_n = \|\cdot\|_Y/K_n + \|\cdot\|_X$. The Ekeland Variational Principle [15] allows us to deduce the existence of a sequence $\{u_n\}$ in M_n such that

$$d_0 - \frac{1}{n} \leq m_n \leq J(u_n) \leq J(\hat{u}_n) = \varphi(\hat{u}_n) \leq d_0 + \frac{1}{n} \leq m_n + \frac{2}{n}, \tag{9}$$

$$J(u_n) \leq J(u) + \frac{2}{\sqrt{n}}\|u - u_n\|_n, \quad \forall u \in M_n \tag{10}$$

and

$$\|u_n - \hat{u}_n\|_n \leq \frac{1}{\sqrt{n}} \tag{11}$$

for all $n \in \mathbb{N}$. Observe that (5) follows from (9). Moreover, from the inequality (11),

$$\begin{aligned} \|u_n\|_Y &\leq \|u_n - \hat{u}_n\|_Y + \|\hat{u}_n\|_Y \\ &\leq K_n\|u_n - \hat{u}_n\|_n + \|\hat{u}_n\|_Y \leq 2K_n, \end{aligned}$$

i.e., (6) holds.

In addition,

$$\begin{aligned} \|u_n\|_X &\geq \|\hat{u}_n\|_X - \|u_n - \hat{u}_n\|_X \\ &\geq \|\hat{u}_n\|_X - \|u_n - \hat{u}_n\|_n \geq R_n + 1, \end{aligned} \tag{12}$$

which implies $u_n \in Y - \overline{B}_{R_n}(0, \|\cdot\|_X)$, and thus, for fixed $n \in \mathbb{N}$ and $v \in Y$, there exists $\delta > 0$ such that $u_n + tv \in M_n$ if $|t| < \delta$. Taking into account (10), we obtain

$$\frac{J(u_n) - J(u_n + tv)}{t} \leq \frac{2}{\sqrt{n}}\|v\|_n \ \ \forall t \in (0, \delta)$$

and

$$\frac{J(u_n) - J(u_n + tv)}{t} \geq -\frac{2}{\sqrt{n}}\|v\|_n \ \ \forall t \in (-\delta, 0).$$

Taking limits as n goes to infinity we conclude that

$$|\langle J'(u_n), v\rangle| \leq \frac{2}{\sqrt{n}}\|v\|_n,$$

which is (7). Finally, (8) is deduced from (12) and the fact that R_n tends to infinity. □

It has to be pointed out that the connections between coercivity and compactness is not completely studied by the above theorem. In fact, the following functional I, taken from [18], shows that this theorem does not cover all the cases The functional I is defined by

$$I(v) = \alpha \int_\Omega |\nabla v|^2\,dx - a\int_\Omega \frac{|v|^2}{|x|^2}\,dx - \int_\Omega f(x)v(x)\,dx,\ \forall v \in W_0^{1,2}(\Omega),$$

where Ω is an open set containing the origin, $f \in L^2(\Omega)$, $a < \chi\alpha$ and χ is the best constant in the Hardy inequality:

$$\chi \int_\Omega \frac{|v|^2}{|x|^2}\,dx \leq \int_\Omega |\nabla v|^2\,dx,\ \forall v \in W_0^{1,2}(\Omega).$$

I is continuous and coercive, but it is not known if it is weak lower semicontinuous. However the authors show the existence of a compact minimizing sequence by using the Ekeland Variational Principle and the homogeneity of the principal part of I. We give a different proof which allows us to handle the nonhomogeneous case.

Theorem 2.2 *Assume that Ω is an open set containing the origin, $f \in L^2(\Omega)$ and $a < \chi\alpha$. Let $j(x,\xi)$ be a Carathéodory function convex with respect to the variable ξ such that for some $\alpha, \beta > 0$,*

$$\alpha|\xi|^2 \leq j(x,\xi) \leq \beta|\xi|^2,\ \forall \xi \in \mathbb{R}^N,\ a.e.\ x \in \Omega.$$

Then the functional J defined in $W_0^{1,2}(\Omega)$ by

$$J(v) = \int_\Omega j(x,\nabla v)\,dx - a\int_\Omega \frac{|v|^2}{|x|^2}\,dx - \int_\Omega f(x)v(x)\,dx,\ \forall v \in W_0^{1,2}(\Omega)$$

has a minimum.

Proof. Consider the sequence of modified functionals

$$J_n(v) = \int_\Omega j(x,\nabla v)\,dx - a\int_\Omega b_n(x)|v|^2\,dx - \int_\Omega fv\,dx,\ \forall v \in W_0^{1,2}(\Omega),$$

where

$$b_n(x) = \begin{cases} \dfrac{1}{|x|^2}, & \text{if } \dfrac{1}{|x|^2} \leq n \\ n, & \text{if } \dfrac{1}{|x|^2} > n. \end{cases}$$

Thus the term $\int_\Omega b_n(x)|v|^2\,dx$ is continuous and the existence of a minimum u_n of I_n is a consequence of the coercivity by the De Giorgi semicontinuity theorem (see [16]).

The assumptions $\alpha\chi > a$ and $j(x,\xi) \geq \alpha|\xi|^2$ imply that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. So, for some $u \in W_0^{1,2}(\Omega)$, and some subsequence $\{u_{n_k}\}$ we have that

$$\{u_{n_k}\} \rightharpoonup u \text{ in } W_0^{1,2}(\Omega) \text{ and } \{u_{n_k}\} \longrightarrow u \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{13}$$

By definition of minimum

$$J_n(u_n) \leq J_n(u_n - T_k(u_n - v)),\ \forall v \in W_0^{1,2}(\Omega), \tag{14}$$

where $T_k(s)$ is the real function

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\dfrac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

We recall that, up to a countable set of k, we have

$$\operatorname{meas}\Omega_k = 0, \tag{15}$$

where

$$\Omega_k = \{x \in \Omega \,/\, |u(x) - v(x)| = k\}.$$

Indeed, the familly $\sum_{k\geq 0} \operatorname{meas}\,\Omega_k$ is summable since for any finite subset $F \subset [0,\infty)$ we have $\sum_{k\in F} \operatorname{meas}\,\Omega_k \leq \operatorname{meas}\,\Omega$. Then $\{k \geq 0 \,/\, \operatorname{meas}\,\Omega_k \neq 0\}$ is countable (see [20, pg. 84]). In the sequel, we choose k such that (15) holds.

The inequality (14) gives us

$$\begin{aligned}\int_{\Omega_{n,k}} j(x,\nabla u_n)\,dx \quad &\leq \quad \int_{\Omega_{n,k}} j(x,\nabla v)\,dx + \int_\Omega f T_k(u_n - v)\,dx \\ &\quad + a\int_\Omega b_n(x)\left[|u_n|^2 - |u_n - T_k(u_n - v)|^2\right] dx, \end{aligned} \tag{16}$$

with $\Omega_{n,k} = \{x \in \Omega \,/\, |u_n(x) - v(x)| \leq k\}$. In order to pass to the limit in (16), for the second integral, by (15), we deduce the convergence a.e. of the characteristic function of $\Omega_{n,k}$ to the characteristic function of Ω_k and the Lebesgue theorem works. With respect to the third integral, we use the Vitali Theorem. To be more precise, we observe that $b_n(x)\left[|u_n|^2 - |u_n - T_k(u_n - v)|^2\right]$ converges a.e. in Ω to $\frac{1}{|x|^2}\left[|u|^2 - |u - T_k(u - v)|^2\right]$ and, in addition,

$$\begin{aligned} b_n(x)\left[|u_n|^2 - |u_n - T_k(u_n - v)|^2\right] \quad &\leq \quad \frac{c_1}{|x|^2}\left[|T_k(u_n - v)|\left\{|u_n| + k\right\}\right] \\ &\leq \quad \frac{kc_1}{|x|^2}\left[|u_n| + k\right]. \end{aligned}$$

Thus for any measurable subset E of Ω we have by the Hölder inequality and the boundedness of $\{u_n\}$ that

$$\begin{aligned}\int_E b_n(x)\left[|u_n|^2-|u_n-T_k(u_n-v)|^2\right]dx &\le \int_E \frac{kc_1}{|x|^2}\left[|u_n|+k\right]dx\\ &\le c(k)\left(\int_E \frac{1}{|x|^{\frac{4N}{N+2}}}\,dx\right)^{\frac{N+2}{2N}}\end{aligned}$$

and

$$\lim_{|E|\to 0}\int_E b_n(x)\left[|u_n|^2-|u_n-T_k(u_n-v)|^2\right]dx=0,$$

uniformly with respect to n, since $\frac{4N}{N+2}<N$. Therefore, passing to the limit in (16) and using the weak lower semicontinuity in the first integral, we deduce that u is a solution of the following integral inequality

$$\begin{aligned}\int_{\Omega_k} j(x,\nabla u)\,dx \le& \int_{\Omega_k} j(x,\nabla v)\,dx + a\int_\Omega \frac{|u|^2-|u-T_k(u-v)|^2}{|x|^2}\,dx\\ &+\int_\Omega fT_k(u-v)\,dx,\end{aligned}$$

for all $v\in W_0^{1,2}(\Omega)$, with $\Omega_k=\{x\in\Omega\,/\,|u(x)-v(x)|\le k\}$. Now, we take the limit as k tends to infinity and we use the same techniques of the previous steps together to the fact $\frac{|u|}{|x|}\in L^2(\Omega)$, thanks to the Hardy's inequality, to deduce that u satisfies the classical inequality of the minima, i.e.

$$\int_\Omega j(x,\nabla u)\,dx - a\int_\Omega \frac{|u|^2}{|x|^2}\,dx-\int_\Omega fu\,dx \le \int_\Omega j(x,\nabla v)\,dx - a\int_\Omega \frac{|v|^2}{|x|^2}\,dx-\int_\Omega fv\,dx,$$

for all $v\in W_0^{1,2}(\Omega)$. □

3 Concave-convex nonlinearities

In this section, we present some results of existence of nontrivial solutions of (P). In the proofs, we use the version of the Mountain Pass Theorem given in [6, Theorem 2.1], with $X=W_0^{1,2}(\Omega)$ and $Y=W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Consider the functional J_λ defined in $W_0^{1,2}(\Omega)$ by

$$J_\lambda(v)=\frac{1}{2}\int_\Omega A(x,v)|\nabla v|^2\,dx-\frac{\lambda}{\theta+1}\int_\Omega |v|^{\theta+1}\,dx-\frac{1}{s+1}\int_\Omega |v|^{s+1}\,dx, \qquad (17)$$

for $v\in W_0^{1,2}(\Omega)$, where $0<\theta<1<s<2^*-1$ and $A:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function satisfying:

(A_1) There exist $\beta > \alpha > 0$ such that

$$\alpha \le A(x,z) \le \beta,$$

for almost every $x \in \Omega$ and $z \in \mathbb{R}$.

(A_2) There exists the partial derivative $A'_z(x,z)$ of $A(x,z)$ which is also assumed to be a Carathéodory function such that

$$|A'_z(x,z)| \le \gamma, \text{ for almost every } x \in \Omega, \forall z \in \mathbb{R},$$

for some $\gamma > 0$.

(A_3) There exists $R_1 > 0$ such that $zA'_z(x,z) \ge 0$ for almost every $x \in \Omega$, for every $|z| \ge R_1$.

Even under these conditions, the functional J_λ is, in general, not Gateaux differentiable. However, it may be shown (see [16]) that J_λ has a directional derivative $\langle J'_\lambda(v), w\rangle$ at each $v \in W_0^{1,2}(\Omega)$ along any direction $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Indeed,

$$\begin{aligned}\langle J'_\lambda(v), w\rangle &= \int_\Omega A(x,v)\nabla u \cdot \nabla w \, dx + \int_\Omega \frac{1}{2} A'_z(x,v)|\nabla v|^2 w \, dx \\ &\quad -\lambda \int_\Omega |v|^{\theta-1} v \, w \, dx - \int_\Omega |v|^{s-1} v \, w \, dx\end{aligned}$$

for every $v \in W_0^{1,2}(\Omega)$ and $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Hence, if we take $X = W_0^{1,2}(\Omega)$ equiped with the usual norm $\|\cdot\|$ and $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ endowed with the norm $\|\cdot\|_\infty$ then J_λ satisfies (H).

In addition, we see in the following lemma that every critical point of J_λ is bounded provided that the above hypotheses hold. This assertion can be proved (see [6, Lemma 1.4]) by using the inequality (18) (see below) which is (3.4) of [19]. Then we can conclude thanks to the results of Chapter 5 in [19] (which are based in previous results in this book). However, we prefer to give here a more selfcontained proof based on a simpler result due to G. Stampacchia (see [21, Lemma 4.1]).

Lemma 3.1 *Assume (A_{1-3}) and $0 < \theta < 1 < s < 2^* - 1$. If $u \in W_0^{1,2}(\Omega)$ is a critical point of J_λ, then $u \in L^\infty(\Omega)$.*

Proof. Let $u \in W_0^{1,2}(\Omega)$ be a critical point of J_λ. Consider $r \ge 2^*$ such that $u \in L^r(\Omega)$. For $k > \max(R_1, 1)$, put $v = G_k(u)$ as test function to deduce that

$$\int_\Omega A(x,u)\nabla u \cdot \nabla G_k(u) \, dx + \frac{1}{2}\int_\Omega A'_z(x,u)|\nabla u|^2 G_k(u) \, dx \le$$

$$\le (1+\lambda)\int_\Omega |u|^{s-1} u \, G_k(u) \, dx = (1+\lambda)\int_{\Omega(k)} |u|^{s-1} u \, G_k(u) \, dx,$$

where $\Omega(k) = \{x \in \Omega \,/\, |u(x)| \geq k\}$. Hence, by the Sobolev and Hölder inequalities, (A_1) and (A_3) we have

$$\begin{aligned} \|G_k(u)\|_{2^*}^2 &\leq C_1 \|G_k(u)\|^2 \\ &\leq C_2 \int_{\Omega(k)} |u|^s \, G_k(u)\, dx \\ &\leq C_2 \|u\|_r^s \, \|G_k(u)\|_{2^*} \, [\text{ meas } \Omega(k)]^{[(2^*-1)r - 2^* s]/2^* r}, \end{aligned} \tag{18}$$

i.e. $\|G_k(u)\|_{2^*}^2 \leq C_2 \|u\|_r^s \, [\text{ meas } \Omega(k)]^{[(2^*-1)r-2^*s]/2^*r}$. Since for $h > k$, $\Omega(h) \subset \Omega(k)$ and $G_k(u(x)) \geq h - k \;\forall x \in \Omega(h)$, we get

$$\begin{aligned} (h-k)\,[\text{ meas } \Omega(h)]^{1/2^*} &\leq \|G_k(u)\|_{2^*} \\ &\leq C_2 \|u\|_r^s \, [\text{ meas } \Omega(k)]^{[(2^*-1)r-2^*s]/2^*r} \end{aligned}$$

which implies

$$\text{meas } \Omega(h) \leq \frac{C_2 \|u\|_r^{2^* s}}{(h-k)^{2^*}} \, [\text{ meas } \Omega(k)]^{[(2^*-1)r-2^*s]/r}.$$

Therefore, by [21, Lemma 4.1] we obtain one of the next possibilities for r:

i) If $r > N/2$ then $u \in L^\infty(\Omega)$.
ii) If $r = N/2$ then $u \in L^s(\Omega)$ for $s \in [1, \infty)$.
iii) If $r < N/2$ then $u \in L^s(\Omega)$ for $s = 2^* r/[(2-2^*)r + 2^* s] - \delta$ for arbitrary small $\delta > 0$.

Now, we can argue by iteration. First, take $r_0 = 2^*$ and observe that with this choice we are in the case iii). It follows that $u \in L^{s(r_0)}(\Omega)$ with

$$s(r_0) = \frac{2^* r_0}{(2-2^*)r_0 + 2^* s} - \delta_1 > r_0.$$

Choosing $r_1 = s(r_0)$, it can occur again the three cases i), ii) and iii). If we would be either in the case i) or in the case ii), then either we have finished or it is easy to conclude the proof. In contrast, if we are in the case iii) then we have that $u \in L^{s(r_1)}(\Omega)$ with

$$s(r_1) = \frac{2^* r_1}{(2-2^*)r_1 + 2^* s} - \delta_2.$$

We claim that in a finite number of steps we can prove that $u \in L^\infty(\Omega)$. Indeed, on the contrary, we would obtain (by induction) sequences $\{r_n\}$ and $\{\delta_n\}$ verifying

$$\lim_{n \to \infty} \delta_n = 0$$

$$\begin{cases} r_0 = 2^* \\ r_{n+1} = \dfrac{2^* r_n}{(2-2^*)r_n + 2^* s} - \delta_{n+1}, \quad n \geq 0. \end{cases}$$

But then, since the function $y(r) = \frac{2^* r}{(2-2^*)r+2^* s}$ is increasing in the interval $\left(\frac{N}{2}(s-1), \frac{N}{2}s\right)$ and $y(2^*) > 2^*$, i.e. $y(r_0) > r_0$, we deduce that $y(r_n)$ is increasing and thus also r_n. Let $r \in [2^*, Ns/2]$ be the limit of this sequence. Then $r = y(r)$, i.e. $r = N(s-1)/2 < r_0$ which is a contradiction concluding the proof. □

In order to prove the condition (C), we observe that the nonlinearity $f(x,z) = \lambda |z|^{\theta-1} z + |z|^{s-1} z$ satisfies the condition of Ambrosetti-Rabinowitz [3] because fixing $\overline{s} \in (1, s)$ we have

$$(\overline{s}+1) F(x,z) \leq z f(x,z) \text{ a.e. } x \in \Omega,$$

if $|z|$ is large enough. We need the following hypothesis on A:

(A_4) There exists $\alpha_1 > 0$ such that

$$\left(\frac{\overline{s}-1}{2}\right) A(x,z) - \frac{1}{2} z A'_z(x,z) \geq \alpha_1,$$

for almost every $x \in \Omega$, $z \in \mathbb{R}$.

The following lemma verifiying the condition (C) and some remarks about the meaning of (A_4) may be found in [6, Lemma 3.2 and Remarks 3.1].

Lemma 3.2 (*Compactness condition*) *Assume* (A_{1-4}). *If* $0 < \theta < 1 < s < 2^* - 1$ *then the functional* J_λ *defined by* (17) *satisfies* (C). □

The following result may be considered complementary of those obtained in [4, 6]. It is concerned with a combination of concave and convex nonlinearities and constitutes a partial generalization of the results in [1, 2, 9].

Theorem 3.3 *Assume* (A_{1-4}), $\lambda > 0$ *and* $0 < \theta < 1 < s < 2^* - 1$. *Then there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$ *the boundary value problem*

$$\left.\begin{aligned} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -div(\, A(x,u)\, \nabla u) + \tfrac{1}{2} A'_z(x,u) |\nabla u|^2 = \lambda |u|^{\theta-1} u + |u|^{s-1} u \end{aligned}\right\} \tag{19}$$

has at least two nonzero solutions $u_\lambda, v_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Consider the functional J_λ defined in $W_0^{1,2}(\Omega)$ by (17). For $\lambda = 0$, $u = 0$ is a strict local minimum of J_0. In fact, since $0 < \theta < 1 < s < 2^* - 1$ we deduce by the arguments in [3, Lemma 3.3] that $\int_\Omega \left(|u|^{\theta+1} + |u|^{s+1}\right) dx = o\left(\|u\|^2\right)$ at $u = 0$ and, by (a_1), we find that there exist $\rho, R > 0$ such that

$$J_0(v) \geq \rho \quad \forall \|v\| = R$$

Take

$$\lambda_0 = \left(\sup \left\{ \frac{2\int_\Omega |v|^{\theta+1}\, dx}{\rho(\theta+1)} \ / \ \|v\| = R \right\} \right)^{-1}$$

and observe that for $\lambda \in (0, \lambda_0)$

$$\begin{aligned} J_\lambda(v) &= J_0(v) - \frac{\lambda}{\theta+1}\int_\Omega |v|^{\theta+1}\, dx \\ &\geq \rho - \frac{\rho}{2} > 0 = J_\lambda(0) \\ &\geq \min_{\|v\|\leq R} J_\lambda(v) \end{aligned}$$

for $\|v\| = R$. Hence, if $v_\lambda \in W_0^{1,2}(\Omega)$ satisfies $\|v_\lambda\| \leq R$ and

$$J_\lambda(v_\lambda) = \min_{\|v\|\leq R} J_\lambda(v)$$

then $\|v_\lambda\| < R$ and v_λ is a local minimum of J_λ. We point out that $\theta < 1$ implies that $J(t\varphi_1) < 0$ for $t > 0$ small enough. Thus $v_\lambda \not\equiv 0$ and it is a nontrivial solution of (19). In order to find a second solution, observe that

$$J_\lambda(v_\lambda) < \inf_{\|v\|=R} J_\lambda(v), \tag{20}$$

and

$$\lim_{t\to\infty} J_\lambda(t\varphi_1) = -\infty$$

imply that J_λ has the geometry of the Mountain Pass Theorem. Now, consider $e = t_0\varphi_1$ such that $J_\lambda(t_0\varphi_1) < J_\lambda(v_\lambda)$ and

$$\|t_0\varphi_1\| \geq 2R. \tag{21}$$

Let Γ be the set of the (cont.) paths $\gamma : [0,1] \to \Big(W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \|\cdot\|_\infty + \|\cdot\| \Big)$ which joint 0 and e; i.e. $\gamma(0) = 0$ and $\gamma(1) = t_0\varphi_1$. Note that every $\gamma \in \Gamma$ is continuous from $[0,1]$ to $W_0^{1,2}(\Omega)$, so that, by (20) and (21), for every $\gamma \in \Gamma$ there exists $\bar{t} \in [0,1]$ such that

$$\|\gamma(\bar{t})\| = R.$$

Thus, by ii),

$$c \equiv \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) \geq \inf_{\|v\|=R} J_\lambda(v) > \max\{J_\lambda(0), J_\lambda(t_0\varphi_1)\}. \tag{22}$$

Let $\{\gamma_n\} \subset \Gamma$ be a sequence of paths for which

$$c \leq \max_{t\in[0,1]} J_\lambda(\gamma_n(t)) \leq c + \frac{1}{2n}, \quad \forall n \in \mathbb{N}.$$

For fixed $n \in \mathbb{N}$, consider $K_n = \max_{t\in[0,1]} \|\gamma_n(t)\|_\infty \geq t_0\|\varphi_1\|_\infty$, and observe that $|||\cdot|||_n \equiv \dfrac{\|\cdot\|_\infty}{K_n}$ is a norm in $Y = L^\infty(\Omega)$ which is equivalent to $\|\cdot\|_\infty$. By applying [6, Theorem 2.1], we deduce the existence of a path $\overline{\gamma}_n \in \Gamma$ and a function $u_n = \overline{\gamma}_n(t_n) \in \overline{\gamma}_n([0,1])$ satisfying

$$c \leq \max_{t\in[0,1]} J_\lambda(\overline{\gamma}_n(t)) \leq \max_{t\in[0,1]} J_\lambda(\gamma_n(t)) \leq c + \frac{1}{2n},$$

$$\max_{t\in[0,1]} |||\overline{\gamma}_n(t) - \gamma_n(t)|||_n + \|\overline{\gamma}_n(t) - \gamma_n(t)\| \leq \sqrt{\frac{1}{n}},$$

$$c - \frac{1}{n} \leq J_\lambda(u_n) \leq c + \frac{1}{2n},$$

$$|\langle J'_\lambda(u_n), v\rangle| \leq \sqrt{\frac{1}{n}}\left(|||v|||_n + \|v\|\right), \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

and for $n \in \mathbb{N}$ large enough,

$$\|u_n\|_\infty = \|\overline{\gamma}_n(t_n)\|_\infty \leq K_n|||\overline{\gamma}_n(t_n) - \gamma_n(t_n)|||_n + \|\gamma_n(t_n)\|_\infty \leq 2K_n.$$

We conclude the proof by observing that the Lemma 3.2 imply the existence of a subsequence of $\{u_n\}$ converging in $W_0^{1,2}(\Omega)$ to some $u \in W_0^{1,2}(\Omega)$, which, necessarily is a critical point of J_λ. Thus, by Lemmma 3.1, $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. In addition, since J_λ is continuous in $W_0^{1,2}(\Omega)$, $J_\lambda(u) = c > 0$, and u is different from zero and v_λ. □

We present now results of existence and multiplicity of nontrivial solutions of (P) for a particular kind of nonlinearities $f(x,z)$. We are concerned with nonlinearities satisfying

$$f(x,z) \leq 0 \text{ for almost every } x \in \Omega \text{ and } |z| \geq k, \tag{23}$$

for some positive constant k. Note that this condition implies that the functional J given by (4) is coercive (besides w.l.s.c.). In addition, $F(x,z)$ is nonincreasing in $(k,+\infty)$ and we deduce that

$$J(T_k(u)) \leq J(u), \quad \forall u \in W_0^{1,2}(\Omega). \tag{24}$$

As a consequence of this inequality, the sequence of minimizing paths of the Mountain Pass that we consider in the sequel can be chosen in such a way that they are bounded in $W_0^{1,2}(\Omega)$ and in $L^\infty(\Omega)$. For this reason, the compactness condition that we need is weaker than (C) (see [6] for the details). It does not require condition (A_3) and is given by the following lemma which was also proved in [6].

Lemma 3.4 *Assume* (A_{1-2}) *and suppose that* $f(x,z)$ *has a subcritical growth. Let* $\{u_n\} \subset W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *be a sequence such that*

$$\|u_n\| \le C_1, \quad \|u_n\|_\infty \le C_2,$$

$$|\langle J'(u_n), v\rangle| \le \varepsilon_n \left[\|v\| + \|v\|_\infty\right], \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \forall n \in \mathbb{N},$$

where $C_1, C_2 > 0$ *and* $\{\varepsilon_n\}$ *is a sequence in* $\mathbb{R}$ *which converges to zero. Then there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *converging strongly in* $W_0^{1,2}(\Omega)$ *to some* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. □

Consider first the case in which $f(x,z)$ is given by

$$f_\lambda(x,z) = \lambda |z|^{r-1} z - |z|^{s-1} z, \quad (\lambda > 0,\ 1 < r < s < 2^* - 1)$$

for $x \in \Omega$ and $z \in \mathbb{R}$. Thus we are concerned with the boundary value problem

$$\left.\begin{array}{r} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \\ -\mathrm{div}(\, A(x,u)\, \nabla u) + \frac{1}{2} A'_z(x,u) |\nabla u|^2 = \lambda |u|^{r-1} u - |u|^{s-1} u \end{array}\right\} \tag{25}$$

Theorem 3.5 *Let* (A_{1-2}) *hold and suppose that* $1 < r < s < 2^* - 1$*. There exists* $\overline{\lambda} > 0$ *such that for every* $\lambda > \overline{\lambda}$ *problem* (25) *has, at least, two nonzero solutions* $u_\lambda, \overline{u}_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Consider the functional J_λ defined in $W_0^{1,2}(\Omega)$ by

$$J_\lambda(u) = \int_\Omega A(x,u) |Du|^2 \, dx - \frac{\lambda}{r+1} \int_\Omega |u|^{r+1} \, dx \ + \ \frac{1}{s+1} \int_\Omega |u|^{s+1} \, dx,$$

Since f_λ satisfies (23) with $k = k_\lambda = \lambda^{1/(s-r)}$, there exists $u_\lambda \in W_0^{1,2}(\Omega)$ such that

$$J_\lambda(u_\lambda) = \min_{u \in W_0^{1,2}(\Omega)} J_\lambda(u).$$

By (24) with $k = k_\lambda$, we have $J_\lambda(u_\lambda) = J_\lambda(T_{k_\lambda}(u_\lambda))$. Hence by (A_1) we obtain the inequality

$$\begin{aligned} \alpha \int_{\{|u| > k_\lambda\}} |\nabla u_\lambda|^2 \, dx \quad &\le \quad \int_{\{|u| > k_\lambda\}} A(x, u_\lambda) |\nabla u_\lambda|^2 \, dx \\ &\le \quad \int_{\{u > k_\lambda\}} [F_\lambda(u_\lambda) - F_\lambda(k_\lambda)] \, dx \le 0. \end{aligned}$$

This means that u_λ belongs to $L^\infty(\Omega)$, with $\|u_\lambda\|_\infty \le k_\lambda$. Therefore u_λ is a bounded critical point of J_λ, that is, a solution of (25).

On the other hand, we show that u_λ is nonzero for $\lambda > 0$ large enough. Indeed we deduce from (A_1)

$$J_\lambda(\varphi_1) \le \beta - \frac{\lambda}{r+1} \|\varphi_1\|_{r+1}^{r+1} + \frac{1}{s+1} \|\varphi_1\|_{s+1}^{s+1} < 0$$

for $\lambda > \overline{\lambda}$ with sufficiently large $\overline{\lambda}$.

Using similar arguments to those of the proof of Theorem 3.3 we show that $u = 0$ is a local minimum of J_λ. The existence of a second critical point of J_λ is then proved by applying again [6, Theorem 2.1]. In fact, consider in this theorem, $J = J_\lambda$, $X = W_0^{1,2}(\Omega)$ with the usual norm, $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ endowed with the norm $\|\cdot\|_Y = \|\cdot\|_\infty$, $e = u_\lambda$ and

$$0 < c \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

Take a sequence $\{\gamma_n\}$ of paths in Γ such that

$$c \leq \max_{t \in [0,1]} J_\lambda(\gamma_n(t)) \leq c + \frac{1}{2n}, \quad \forall n \in \mathbb{N}.$$

Since $\|u_\lambda\|_\infty \leq k_\lambda$, it follows from (24) that $T_{k_\lambda} \circ \gamma_n \in \Gamma$ and

$$c \leq \max_{t \in [0,1]} J_\lambda(T_{k_\lambda}(\gamma_n(t))) \leq c + \frac{1}{2n}, \quad \forall n \in \mathbb{N}.$$

Applying [6, Theorem 2.1], there exists a sequence $\{\overline{\gamma}_n\}$ of paths in Γ and a sequence $u_n = \overline{\gamma}_n(t_n) \in \overline{\gamma}_n([0,1])$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\max_{t \in [0,1]} \|\overline{\gamma}_n(t) - T_{k_\lambda}(\gamma_n(t))\| + \|\overline{\gamma}_n(t) - T_{k_\lambda}(\gamma_n(t))\|_\infty \leq \sqrt{\frac{1}{n}}, \tag{26}$$

$$c - \frac{1}{n} \leq J_\lambda(u_n) \leq c + \frac{1}{2n},$$

$$|\langle J_\lambda'(u_n), v \rangle| \leq \sqrt{\frac{1}{n}}\,(\|v\| + \|v\|_\infty), \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{27}$$

Note that by (26),

$$\begin{aligned} \|u_n\|_\infty &\leq \|T_{k_\lambda}(\gamma_n(t_n))\|_\infty + \|\overline{\gamma}_n(t_n) - T_{k_\lambda}(\gamma_n(t_n))\|_\infty \\ &\leq k_\lambda + \sqrt{\frac{1}{n}}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Now, by Lemma 3.4 extract from $\{u_n\}$ a subsequence converging strongly in $W_0^{1,2}(\Omega)$ to some $\hat{u}_\lambda$. Of course, by (27), $\hat{u}_\lambda$ is a solution of (25) with $J_\lambda(\hat{u}_\lambda) = c > 0$, that is, $\hat{u}_\lambda$ is a nontrivial solution of (P_λ) and $\hat{u}_\lambda \neq u_\lambda$. □

Remarks 3.6 i) Let Σ be the set of these $\lambda > 0$ for which the problem (25) has at least one nonzero solution. Using the method of lower and upper solutions it is possible to show [6] that Σ is an interval of $\mathbb{R}$. Then $(0, \alpha) \subset \Sigma \subset (0, \alpha]$, for some $\alpha > 0$. One open question is to study if $\alpha = \overline{\lambda}$.

ii) Observe that $\|u\|_\infty \le k_\lambda$ for every solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (25). Indeed, choosing $\eta > [\gamma/\alpha]^2$ and $\varphi(z) = ze^{\eta z^2}$ and taking $v = \varphi(G_{k_\lambda}(u))$ as test function in (25), we conclude from (a_1) and (a_2),

$$\frac{\alpha}{2}\int_\Omega |\nabla G_{k_\lambda}(u)|^2\,dx \le \int_\Omega [\alpha\,\varphi'(G_{k_\lambda}(u)) - \gamma|\varphi(G_{k_\lambda}(u))|]\,|\nabla G_{k_\lambda}(u)|^2\,dx$$

$$\le \int_\Omega \left[A(x,u)|\nabla\varphi(G_{k_\lambda}(u))|^2 + A'_z(x,u)|\nabla u|^2\varphi(G_{k_\lambda}(u))\right]\,dx$$

$$= \int_\Omega f_\lambda(u)\varphi(G_{k_\lambda}(u))\,dx \le 0,$$

which clearly implies the assertion.

To conclude this section we discuss the case

$$f(x,z) = \lambda\left(|z|^{\theta-1}z - |z|^{r-1}z\right) - |z|^{s-1}z, \quad (\lambda > 0,\ \theta < 1 < s < r < 2^* - 1) \tag{28}$$

for $x \in \Omega$ and $z \in \mathbb{R}$. We have

Theorem 3.7 *Assume* (A_{1-2}) *and let* $f(x,z)$ *be given by* (28). *Then there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0,\lambda_0)$ *the problem* (P) *has at least three nonzero solutions.*

Proof. Let J_λ be defined in $W_0^{1,2}(\Omega)$ by

$$J_\lambda(v) = \frac{1}{2}\int_\Omega A(x,v)|\nabla v|^2\,dx - \lambda\int_\Omega\left[\frac{|v|^{\theta+1}}{\theta+1} - \frac{|v|^{r+1}}{r+1}\right]\,dx$$
$$+\frac{1}{s+1}\int_\Omega |v|^{s+1}\ dx,\ \ v \in W_0^{1,2}(\Omega).$$

We prove that in this case, if λ is small enough, then the functional J_λ has two distinct local minima different from 0. These are two solutions of (P). The third may be found by applying [6, Theorem 2.1] and the compactness condition given in Lemma 3.4.

For $\lambda = 0$, it was seen in the proof of Theorem 3.3 that there exist $\rho, R > 0$ such that

$$J_0(v) \ge \rho \quad \forall \|v\| = R.$$

In addition, since $\lim_{t\to\infty} J_0(t\varphi_1) = -\infty$, we may choose $t_0 > 0$ such that

$$J_0(t_0\varphi_1) < -\frac{\rho}{2}.$$

Consider

$$\lambda_0 = \left[\sup\left\{\frac{2}{\rho}\int_\Omega\left(\frac{|v|^{\theta+1}}{\theta+1} - \frac{|v|^{r+1}}{r+1}\right)\,dx\ /\ \|v\| \le R\right\}\right]^{-1}.$$

Then for every $\lambda \in (0, \lambda_0)$

$$J_\lambda(v) \geq J_0(v) - \frac{\rho}{2} \geq \frac{\rho}{2} > 0 = J_\lambda(0) > J_\lambda(t_\lambda \varphi_1) \geq \min_{\|v\| \leq R} J_\lambda(v)$$

for $\|v\| = R$ and $t_\lambda > 0$ small enough. Hence, if $v_\lambda \in W_0^{1,2}(\Omega)$ satisfies $0 < \|v_\lambda\| \leq R$ and $J_\lambda(v_\lambda) = \min_{\|v\| \leq R} J_\lambda(v)$ then $\|v_\lambda\| < R$ and v_λ is a local minimum of J_λ with

$$J_\lambda(v_\lambda) < \inf_{\|v\|=R} J_\lambda(v).$$

In addition, taking into account that $r > s > \theta$, the functional J_λ is bounded from below and attains its infimum at some $\overline{v_\lambda}$ with

$$\begin{aligned} J_\lambda(v_\lambda) &\geq J_0(v_\lambda) - \frac{\rho}{2} > -\frac{\rho}{2} \\ &> J_\lambda(t_0 \varphi_1) \geq \min_{v \in W_0^{1,2}(\Omega)} J_\lambda(v) = J_\lambda(\overline{v_\lambda}) \quad \forall \lambda \in (0, \lambda_0), \end{aligned}$$

which yields $\overline{v_\lambda} \neq v_\lambda$ and the existence of two local minima different from zero has been shown. The third critical point is obtained by the Mountain Pass Theorem as it has been mentioned. □

4 Unbounded coefficients

This last section is devoted to study the case in which the Carathéodory coefficient $A(x, z)$ is unbounded from above with respect to z. More precisely, in addition to (A_{3-4}), we just suppose the following weakness of (A_1):

(A_1') There exists $\alpha > 0$ such that

$$\alpha \leq A(x, z),$$

for almost every $x \in \Omega$ and $z \in \mathbb{R}$.

In this case, if, in addition, the nonlinear term $f(x, z)$ satisfies (23), then the constant k is an *a priori* estimate of $\|u\|_\infty$ for all the solutions obtained in Theorems 3.5 and 3.7, (see Remark 3.6–ii)). Hence all these theorems hold also if we substitute (A_1) by (A_1').

The case of a sublinear term (e.g. a pure power $f(x, z) = |z|^{s-1} z$ with $1 < s < 2$ has been studied in [7], where the authors consider unbounded and degenerated coefficients $A(x, z)$ and find nontrivial solutions by (global) minimization of a suitable truncated functional. We remark explicitely that, in contrast with [7], our thecniques work also to find solutions through the Mountain Pass Theorem for unbounded coefficients.

The following theorem is an extension of the main result in [6, 12] to the case of coefficient $A(x, z)$ unbounded from above with respect to z. We show the existence of nonzero solution of the problem (P) when the function f is given by $f(x,z) = |z|^{s-1}z$, $z \in \mathbb{R}$, with $1 < s < 2^* - 1$. In order to do this, we consider the associated Euler functional J defined in $W_0^{1,2}(\Omega)$ by setting

$$J(v) = \int_\Omega A(x,v)|\nabla v|^2\,dx \;-\; \frac{1}{s+1}\int_\Omega |v|^{s+1}\,dx,\;\; v \in W_0^{1,2}(\Omega).$$

Now the difficulty is that J is not defined in all $W_0^{1,2}(\Omega)$ and that there is a lack of differentiability of J even for directions $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. However, we overcome this problem by considering a suitable sequence of truncated functionals J_n. Indeed, for these the techniques developed previously work and so every J_n has a nonzero critical point u_n obtained by the Mountain Pass Theorem. We prove that this sequence $\{u_n\}$ is convergent to some $u \neq 0$ which is a critical point of J.

Theorem 4.1 *Assume* (A_1') *and* (A_{3-4}). *If* $1 < s < 2^* - 1$, *then the problem*

$$\left.\begin{array}{c} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -div(\,A(x,u)\,\nabla u) + \frac{1}{2}A_z'(x,u)|\nabla u|^2 = |u|^{s-1}u \end{array}\right\}$$

has, at least, one nontrivial solution.

Remark 4.2 The existence result of the above theorem is also true for more general superlinear nonlinear terms $f(x,z)$ satisfying the condition introduced in [3]:

$$\exists z_0 > 0, s > 1 \text{ such that } 0 < (s+1)F(x,z) \le z f(x,z)\;\; \forall |z| \ge z_0,\;\text{a.e. } x \in \Omega.$$

We have preferred to present here a less general version and to leave to the reader the details of the general case.

Proof. For every $n \in \mathbb{N}$, let h_n a nondecreasing C^1 function in $[0,\infty)$ satisfying

$$h_n(s) = s,\;\; \forall s \in [0, n-1],$$

$$h_n(s) \le s,\;\; \forall s \in (n-1, n),$$

$$h_n(s) = n,\;\; \forall s \ge n.$$

Consider the coefficients $A_n(x,z) \equiv h_n(A(x,z))$, $x \in \Omega$, $z \in \mathbb{R}$. Clearly, A_n satisfies (A_{1-4}) and thus, by Theorem 3.3 there exists a nontrivial solution u_n of the problem

$$\left.\begin{array}{c} u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(\,A_n(x,u_n)\,\nabla u_n) + \frac{1}{2}A_n'(x,u_n)|\nabla u_n|^2 = |u_n|^{s-1}u_n. \end{array}\right\} \tag{29}$$

(By simplicity in the notation we denote here by A'_n the partial derivative of $A_n(x,z)$ with respect to the variable z). Remind that this u_n is obtained by applying the version of Mountain Pass Theorem given in [6, Theorem 2.1]. This means that u_n is a critical point of the functional J_n defined by

$$J_n(v) = \int_\Omega A_n(x,v)|\nabla v|^2\,dx \;-\; \frac{1}{s+1}\int_\Omega |v|^{s+1}\,dx,\;\; v \in W_0^{1,2}(\Omega),$$

with critical level

$$J_n(u_n) = c_n \equiv \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_n(\gamma(t)),$$

where $\Gamma = \{\gamma : [0,1] \longrightarrow W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,/\,\gamma(0)=0, \gamma(1)=e_n\}$, $e_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that $J_n(e_n) < 0$. Taking into account that $A_n(x,z) \le A(x,z)$, we observe that $J_n(t\varphi_1) \le J(t\varphi_1) < 0$ for all $t \in [t_0,\infty)$ if $t_0 > 0$ is large enough. This allows us to choose as $e_n = t_0\varphi_1$ (independent of $n \in \mathbb{N}$). On the other hand, by the Mountain Pass geometry of J_1 there exist $\delta, r > 0$ such that

$$J_n(v) \ge J_1(v) \ge \delta,\;\; \forall \|v\| \le r,$$

(i.e., roughly speaking, $v = 0$ is a strict local minimum of J_n uniformly in $n \in \mathbb{N}$). This implies that

$$J_n(u_n) = c_n \ge \delta. \tag{30}$$

We claim that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Indeed, using again that $A_n(x,z) \le A(x,z)$, we deduce

$$\begin{aligned} J_n(u_n) &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_n(\gamma(t)) \\ &\le \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \\ &\le \max_{t\in[0,1]} J(tt_0\varphi_1) \equiv C_1. \end{aligned}$$

Subtracting $\dfrac{1}{s+1}\langle J'_n(u_n), u_n\rangle = 0$ we derive

$$\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega A_n(x,u_n)|\nabla u_n|^2\,dx - \frac{1}{2(s+1)}\int_\Omega A'_n(x,u_n)u_n|\nabla u_n|^2\,dx \le C_2$$

which, by (A_4) implies that $\|u_n\|$ is bounded proving the claim.

Then, passing to a subsequence, if necessary, we can assume that $\{u_n\}$ is weakly convergent to some $u \in W_0^{1,2}(\Omega)$. Now, we prove that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Indeed, we can use $v = G_k(u_n)$, $k > R_1$, as test function in (29) to deduce that $\{u_n\}$ satisfies the inequality (18). As a consequence, there is a constant $C_3 > 0$ such that $\|u_n\|_\infty \le C_3$. The boundedness in $W_0^{1,2}(\Omega)$ and $L^\infty(\Omega)$ of the sequence $\{u_n\}$ and the subcritical growth of the lower order term imply that $\{u_n\}$ is compact in $W_0^{1,2}(\Omega)$, thanks to the results of [10]. Therefore, $\{u_n\} \longrightarrow u$ and $u \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a critical point of J. In addition, $\{J(u_n)\} \longrightarrow J(u)$ and we get from (30) that $J(u) \ge \delta$ and $u \ne 0$. Thus u is the weak nontrivial solution we are searching. □

5 Critical points and variational inequalities

In this section, we follow the outline of Theorem 4.1 in order to study existence of critical points of the nondifferentiable functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v| - \frac{1}{p}\int_\Omega |v|^p, \qquad 2 < p < 2^*, \tag{31}$$

where the nondifferentiability is due to the term $\int_\Omega |\nabla v|$. This is the reason why we can not hope to deduce an Euler equation for the *critical points* of J. Instead of it, we obtain a variational inequality.

Theorem 5.1 *The functional defined in* (31) *has a nontrivial critical point in the sense that it is a solution of the following variational inequality*

$$\left.\begin{array}{c} u \in W_0^{1,2}(\Omega), \quad u \neq 0, \\[2ex] \int_\Omega \nabla u \nabla (v-u)\,dx + \int_\Omega |\nabla v|\,dx - \int_\Omega |\nabla u|\,dx \geq \int_\Omega |u|^{p-2}u(v-u)\,dx \\[2ex] \forall v \in W_0^{1,2}(\Omega). \end{array}\right\} \tag{32}$$

Proof. We consider the sequence of approximate functionals

$$J_n(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \int_\Omega \sqrt{\frac{1}{n} + |\nabla v|^2}\,dx - \frac{1}{p}\int_\Omega |v|^p\,dx.$$

The classical Ambrosetti-Rabinowitz Theorem yields the existence of critical points of J_n. As in Theorem 4.1, we prove

$$\delta \leq J_n(u_n) \leq C_1, \tag{33}$$

for some $\delta, C_1 > 0$. The sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, since

$$\begin{aligned} C_1 &\geq J_n(u_n) = J_n(u_n) - \frac{1}{p}\langle J_n'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega \sqrt{\frac{1}{n} + |\nabla u_n|^2}\,dx - \frac{1}{p}\int_\Omega \frac{|\nabla u_n|^2}{\sqrt{\frac{1}{n} + |\nabla u_n|^2}}\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |\nabla u_n|^2\,dx. \end{aligned}$$

Thus, up to subsequence, u_n converges weakly in $W_0^{1,2}(\Omega)$ to some u. Using $u_n - u$ as test function in the equation $J_n'(u_n) = 0$, thanks to the monotonicity of the operator

$$-\text{div}\left(\frac{\nabla v}{\sqrt{\dfrac{1}{n} + |\nabla v|^2}}\right)$$

it is easy to prove that u_n converges strongly in $W_0^{1,2}(\Omega)$ to u.

Moreover, the inequality (33) implies that $J(u) \geq \delta$ and, thus, u is not zero.

Now, let v any function in $W_0^{1,2}(\Omega)$, take $v-u_n$ as test function in $J_n'(u_n)=0$ and use the convexity of $\int_\Omega \sqrt{\frac{1}{n}+|\nabla u_n|^2}\,dx$ to obtain

$$\int_\Omega |u_n|^{p-2}u_n(v-u_n)\,dx \leq \int_\Omega \nabla u_n \nabla(v-u_n)\,dx + \int_\Omega \sqrt{\frac{1}{n}+|\nabla v|^2}\,dx - \int_\Omega \sqrt{\frac{1}{n}+|\nabla u_n|^2}\,dx.$$

Then we pass to the limit and we deduce that u is a nontrivial critical point of (31) in the sense of (32). □

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