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Convergence to periodic fronts in a class of semilinear parabolic equations

Gawtum NAMAH
Mathématiques Appliquées de Bordeaux, Université Bordeaux I,
33405 Talence Cedex, France

Jean-Michel ROQUEJOFFRE
UFR MIG, Université Paul Sabatier, 118 route de Narbonne,
31062 Toulouse Cedex, France

Abstract

A class of scalar autonomous parabolic equations, nonlinear with respect to the unknown and its gradient, is investigated. The main topic of this paper is the convergence of the solutions of the Cauchy Problem towards solutions which exhibit, *modulo* a linear growth in time, periodic spatio-temporal oscillations. Different generalizations are discussed.

1 Introduction, main results

Let a function $u(x_1, \ldots, x_N) \in C(\mathbb{R}^N)$ be called "1-periodic in x" if it is 1-periodic with respect to x_i, for every $i \in \{1, \ldots, N\}$. The purpose of this paper is to investigate the long-time behaviour of the solutions of parabolic equations of the form

$$u_t - \Delta u = f(x, u, \nabla u), \quad x \in \mathbb{R}^N \tag{1.1}$$

and its following u-independent version

$$u_t - \Delta u = f(x, \nabla u), \quad x \in \mathbb{R}^N \tag{1.2}$$

with $f(x, r, p)$ 1−periodic in x and r. In addition to the fact that the above problem is interesting on its own, our motivation for this study comes from a physical problem, namely the propagation of flame fronts in a solid medium having periodic striations, cf. [2] and [3], where equations similar to (1.1) and (1.2) are encountered. See Appendix for a more detailed justification.

The function f shall be assumed to satisfy the following assumptions:

$(\mathcal{H}_1)$ • there exist $0 < m < M$ such that

$$m \le f(x,u,p) \le M(1+|p|),$$
$$|f_x(x,u,p)| + |f_{xx}(x,u,p)| + |f_u(x,u,p)| + |f_{uu}(x,u,p)| \le M(1+|p|)$$

for every $(x,u,p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.
$(\mathcal{H}_2)$ • The derivatives f_p and f_{pp} are uniformly bounded.

The reader will notice that the nonlinearity $f(x,u,p) = R(x,u)\sqrt{1+|p|^2}$ of the Appendix fits into the above assumptions. Much less smoothness would produce the same results, but being interested in qualitative properties, we may afford to be very lavish with smoothness. What is important is the existence of a global nonlinear semigroup associated to (1.1), denoted by $\mathcal{S}(t)$; a property which is implied by $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$.

Let us denote, for $u(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^N)$:

$$\text{for any } t_0 > 0: \quad \tau_{t_0} u(t,x) = u(t+t_0, x)$$

$$\text{for any vector } E \in \mathbb{R}^N: \quad \tau_e u(t,x) = u(t, x+e)$$

Also, for the same $u(t,x)$, let us set

$$< u > (t) = \int_{[0,1]^N} u(t,x)\, dx.$$

These two notations will be of constant use in the sequel.

Theorem 1.1 *Assume that, for every $p \in \mathbb{R}^N$, the function $f_u(.,.,p)$ is nonzero. Then there exists a unique $T > 0$, and a unique – up to the translations in time – function $\varphi(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N)$, T-periodic in t, and 1-periodic in x, such that the function*

$$\phi(t,x) = \frac{t}{T} + \varphi(t,x) \tag{1.3}$$

is a solution of (1.1).
Furthermore, let $u_0(x)$ be a C^2 function, 1-periodic in x. There exists $t_0 \in \mathbb{R}$ and $\omega > 0$ such that

$$\|\mathcal{S}(t)u_0 - \tau_{t_0}\phi(t)\|_{C^1(\mathbb{R}^N)} \le Ce^{-\omega t}.$$

The function ϕ will be called a *periodic front*.

When f does not depend on u anymore, periodic in time solutions become stationary. This is accounted for by the next result.

Theorem 1.2 *Assume that $f(x,u,p) = f(x,p)$ for all $u \in \mathbb{R}$. Then there exists a unique $\lambda > 0$, and a unique – up to constants – function $\varphi(x) \in C^2(\mathbb{R}^N)$, 1-periodic in x, such that the function*

$$\phi(t,x) = \lambda t + \varphi(x) \tag{1.4}$$

is a solution of (1.2). Furthermore, let $u_0(x)$ be a C^2 function, 1-periodic in x. There exists $t_0 \in \mathbb{R}$ and $\omega > 0$ such that

$$\|\mathcal{S}(t)u_0 - \tau_{t_0}\phi(t)\|_{C^1(\mathbb{R}^N)} \le Ce^{-\omega t}.$$

As we shall see, the function $\varphi(x)$ satisfies a nonlinear elliptic equation, but it will be more convenient to work on the evolution equation. We also see that all the translations in time for $\phi(t,x)$ are solutions of (1.2).

The above-mentioned convergence results deal with solutions that are initially 1-periodic in x. When this is not so, we may still say something about the long-time behaviour of the solutions, although the results are less precise. We only give the result when f is truly dependent on u. In this case, the function $v(t,x) = u(t,x) - \frac{t}{T}$ is a solution of

$$v_t - \Delta v = f(x, \frac{t}{T} + v, \nabla v) - \frac{1}{T} = g(t,x,v,\nabla v) \tag{1.5}$$

The function g is T-time periodic, thus (1.5) defines a discrete dynamical system, defined by $\mathcal{T}u_0 = \mathcal{S}(T)u_0$.

Theorem 1.3 *For any C^2 function u_0 such that u_0 and its gradient are bounded and uniformly continuous on $\mathbb{R}^N$, the ω-limit set of u_0 – with respect to the local uniform convergence, and for the discrete dynamical system $\mathcal{T}$ – contains only periodic fronts. Furthermore, there exists a finite time $t_*(u_0)$ and $\delta > 0$ such that*

$$\forall t \ge t_*, \quad \partial_t(\mathcal{S}(t)u_0) \ge \delta. \tag{1.6}$$

This time-monotonicity in finite time implies the L^∞-stability for every trajectory.

We wish to point out that Theorem 1.3 is optimal. To see this, let us write down the solution of the heat equation on $\mathbb{R}$; for any bounded uniformly continuous function u_0 there holds

$$\mathcal{S}(t)u_0(x) = \int_{-\infty}^{+\infty} e^{-z^2} u_0(x + 2\sqrt{t}z)\, dz;$$

an expression for which, in the case of a general u_0, no compactness – with respect to the uniform topology – holds.

Let us also point out that the optimal space for the initial data u_0 in the above theorems will be defined in the course of the proofs.

The paper is organized as follows. The second section is devoted to the discussion of existence of periodic, or steady solutions to (1.1) and (1.2). The long-time behaviour of the solutions is examined in Section 3. Finally, a survey of the physical background that motivated our study is given in Appendix.

2 Existence and uniqueness

Let us first deal with the functions ϕ, defined by (1.3), solutions of (1.1), and let us prove their uniqueness.

Lemma 2.1 *There is at most one* $T > 0$ *and one – up to additive constants – function* $\varphi(t,x)$, *T-periodic in time and 1-periodic in* x, *such that the function* $\phi(t,x)$ *defined by (1.3) is a solution of (1.1).*

Proof. Assume that there are two such functions: $\phi_i(t,x)$ associated to the periods T_i and functions φ_i. Let t_1 and t_2 be defined as

$$t_1 = \sup\{t \in \mathbb{R} : \tau_t \phi_1(0) < \phi_2(0)\}$$

$$t_2 = \inf\{t \in \mathbb{R} : \tau_t \phi_1(0) > \phi_2(0)\}$$

There exist x_1 and x_2 such that $\phi_1(t_1,x_1) = \phi_2(0,x_1)$, and $\phi_1(t_2,x_2) = \phi_2(0,x_2)$. From the maximum principle we have $\mathcal{S}(t)\tau_{t_1}\phi_1(0) \le \mathcal{S}(t)\phi_2(0) \le \mathcal{S}(t)\tau_{t_2}\phi_1(0)$, in other words:

$$\frac{t+t_1}{T_1} + \varphi_1(t+t_1,x) \le \frac{t}{T_2} + \varphi_2(t,x) \le \frac{t+t_2}{T_1} + \varphi_1(t+t_2,x). \tag{2.1}$$

Dividing (2.1) by t and letting $t \to +\infty$ yields: $T_1 = T_2$. But this implies that, at $t = T_1$, we have, by periodicity and the expression of ϕ_i: $\mathcal{S}(T_1)\phi_1(t_1,x_1) = \mathcal{S}(T_1)\phi_2(0,x_1)$. This situation is forbidden by the strong maximum principle applied to the linear parabolic equation satisfied by

$$\psi(t,x) = \phi_1(t+t_1,x) - \phi_2(t,x).$$ □

Let us now deal with the existence proof of Theorem (1.1). The basic ingredient is a topological degree argument applied to a suitable mapping. We shall look for our solution $\phi(t,x)$ under a slightly different form to what was announced, namely we decompose our unknown as

$$\phi(t,x) = p(t) + \xi(t,x); \qquad \text{with } p(t) := <\phi>(t).$$

It will be our task to show *a posteriori* that $p(t)$ can be written as the sum of a linear function and a periodic one, and that $\xi(t,x)$ is time-periodic. System (1.1) is rewritten as

$$\begin{cases} \dot{p} & = <f(.,p+\xi,\nabla\xi)> \\ \xi_t - \Delta\xi & = f(x,p+\xi,\nabla\xi) - <f(.,p+\xi,\nabla\xi)> \\ <\xi>(t) & = 0 \end{cases} \tag{2.2}$$

This helps us see that, in order to have ξ time periodic of period T, we must not merely demand the equality $\xi(T) = \xi(0)$; we have to add a requirement on the first equation of (2.2). Namely, we wish to have $\dot{p}(T) = \dot{p}(0)$; this would ensure the time periodicity of the whole function ϕ, in view of the uniqueness for the Cauchy problem associated to (2.2). A sufficient condition to achieve this is to have, due to the 1-periodicity in u of the function f, $p(T) \in \mathbb{N}$; the simplest way to achieve this is therefore to require

$$\int_0^T < f(.,p+\xi,\nabla\xi) > \ dt = 1. \tag{2.3}$$

A solution to (2.2) satisfying this condition will be referred to as "time-periodic", without any other precision. In view of Assumption $(\mathcal{H}_1)$, we get an upper bound for T:

$$T \leq \frac{1}{m}. \tag{2.4}$$

Remark 2.2 At this stage, nothing excludes solutions of (1.1) having zero time-derivatives, as the solutions of (1.2). Let us only notice that we may, using the assumptions on f, plug (1.4) into (1.1), and realize that such a function cannot be a solution. Hence we have true time-periodicity. □

The main step is an *a priori* estimate for the time-periodic solutions ξ of (2.2). In this scope, let X be the space of all continuous functions on $\mathbb{R}^N$, 1-periodic. For $\alpha \in [0,1]$ we denote by X^α the usual domain of $(-\Delta+1)^\alpha$ in X; see [5]. We select once and for all a real number $\alpha \in]0,1[$ such that

$$X^\alpha \subset C^1(\mathbb{R}^N).$$

Lemma 2.3 *Let $(f_\tau(x,u,p))_{\tau\in[0,1]}$ be a family of functions satisfying Assumptions $\mathcal{H}_i$, $i \in 1,2$, with common upper and lower bounds. There exists a constant $C > 0$, independent of τ, such that, for every solution (p^τ,ξ^τ) of System (2.2) with nonlinearity f_τ, with period T^τ, we have*

$$\|\xi^\tau\|_{L^\infty([0,\frac{2}{m}],X^\alpha)} \leq C. \tag{2.5}$$

Let us notice that we have allowed, in our definitions of "periodic functions", functions $\xi(t,x)$ that may identically vanish. This will be of help in the course of the existence proof.

Proof of Lemma 2.3. Assume by contradiction the existence of a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that the corresponding solutions (p^n,ξ^n), with periods T^n, satisfy, in addition to the assumptions of the lemma, the property

$$\lim_{n\to+\infty} \|\xi^n\|_{L^\infty([0,\frac{2}{m}],X^\alpha)} = +\infty.$$

Let us therefore set

$$\zeta_n(t,x) = \frac{\xi_n(t,x)}{\|\xi^n\|_{L^\infty([0,\frac{2}{m}],X^\alpha)}}.$$

We may always assume that the periods T^n are bounded away from 0; and therefore – up to extraction of a subsequence – converge to some $T > 0$. If this were not so, we would consider the sequence $\tilde{T}^n := \gamma[\frac{1}{T^n}]T^n$, where γ is chosen so as to render that quantity less than $\frac{2}{m}$ and where $[a]$ stands for the integer part of a. The sequence $\tilde{T}^n$ is this time bounded from above and below, independently of n.

On the interval $[0, \frac{2}{m}]$, the following convergence results hold, up to a subsequence.

- $\zeta_n \to \zeta_\infty$ in $C^{1+\beta, \frac{1+\beta}{2}}([0, \frac{2}{m}] \times \mathbb{R}^N)$, for all $\beta \in]0, 1[$
- $\zeta_n \to \zeta_\infty$ in $L^\infty([0, \frac{2}{m}], X^\beta)$, for all $\beta \geq \alpha$
- $\frac{f(.,p^n+\xi^n,\nabla\xi^n)-f(.,p^n+\xi^n,0)}{\|\xi^n\|_{L^\infty([0,\frac{2}{m}],X^\alpha)}} \rightharpoonup \tilde{f}(t,x)$ in L^p weak $*$, for $1 < p < +\infty$.

The first convergence result holds true because of smoothness estimates for parabolic equations with nonsmooth coefficients – see [6]; Chapter 3 – and the time-periodicity of the solutions. Moreover there holds $|\tilde{f}(t,x)| \leq \|f\|_{\mathrm{Lip}}$.

To sum up, we have found a solution to the following inequality:

$$\begin{aligned} \partial_t\zeta_\infty - \Delta\zeta_\infty - \tilde{f}\frac{\nabla\zeta_\infty}{|\nabla\zeta_\infty|}.\nabla\zeta_\infty &\leq 0 \\ \zeta_\infty(0) &= \zeta_\infty(T) \\ \zeta_\infty \text{ is 1-periodic in } & x \end{aligned}$$

In the above parabolic inequality, the coefficient $\frac{\nabla\xi_\infty}{|\nabla\xi_\infty|}$ trivially belongs to $L^\infty(\mathbb{R}_+ \times \mathbb{R})$; therefore the Harnack inequalities are applicable and imply that ζ_∞ is a constant. Therefore ζ_∞ has to be zero because it has zero mean. However, the second convergence result implies that $\|\zeta_\infty\|_{L^\infty([0,\frac{2}{m}],X^\alpha)} = 1$, a contradiction. □

Corollary 2.4 *Let $(f_\tau)_{\tau\in[0,1]}$ be as above. There exists $M > 0$, independent of τ such that, for every solution (p^τ, ξ^τ) of System (2.2) with nonlinearity f_τ, with period T^τ, and which satisfies $p^\tau(0) = 0$, the following estimates hold:*

$$\frac{1}{M} \leq T^\tau \leq \frac{1}{m}, \qquad \|\xi^\tau\|_{C^2([0,\frac{2}{m}]\times\mathbb{R}^N)} \leq M.$$

Proof. The lower bound follows from the gradient estimate for ξ^τ and the condition (2.3). The additional smoothness follows from the smoothness of f – Assumption $\mathcal{H}_2$ – and the standard parabolic estimates. □

Proof of Theorem 1.1, *existence part.* We choose the simplest deformation available for f_τ, namely:

$$f_\tau = \tau f + 1 - \tau.$$

Let us set

$$Y = \{(T, \xi_0) \in]\frac{1}{2M}, +\infty[\times X^\alpha\}$$

and let (p^τ, ξ^τ) be the solution of the Cauchy problem for (2.2) with $f = f_\tau$ and $(p^\tau(0), \xi^\tau(0)) = (0, \xi_0)$. Let $\mathcal{F}_\tau$ be the mapping from Y to Y such that

$$\mathcal{F}_\tau(T, \xi_0) = (T + 1 - p^\tau(T), \xi^\tau(T)).$$

Our problem is reduced to finding a zero of the mapping $\mathrm{Id}_Y - \mathcal{F}_\tau$. From Corollary 2.4, the operator $\mathcal{F}_\tau$ is a compact operator of Y; moreover, from the same corollary, there exists a bounded open subset V of Y such that the zeroes of $\mathrm{Id}_Y - \mathcal{F}_\tau$ lie in V. Therefore, by homotopy invariance,

$$\deg(\mathrm{Id}_Y - \mathcal{F}_1, 0, V) = \deg(\mathrm{Id}_Y - \mathcal{F}_0, 0, V).$$

However, finding the zeroes of $\mathrm{Id}_Y - \mathcal{F}_0$ amounts to studying the totally uncoupled problem

$$\left\{ \begin{array}{rl} \dot{p} & = 1 \\ \xi_t - \Delta \xi & = 0 \end{array} \right. \tag{2.6}$$

From the strong maximum principle, the only zero of $\mathrm{Id}_Y - \mathcal{F}_0$ is the couple $(1, 0)$; moreover, by Fredholm alternative, the operator $\mathrm{Id}_Y - \mathcal{F}_0$ is an isomorphism of Y. Therefore

$$\deg(\mathrm{Id}_Y - \mathcal{F}_0, 0, V) = 1,$$

which implies the existence of T and φ. □

Let us notice that the above proof is valid for Theorem 1.2. Let us however give a more direct proof.

Proof of Theorem 1.2, *existence part.* Plugging (1.4) into (1.2) yields the following elliptic system:

$$\begin{array}{rl} -\Delta\varphi & = f(x, \nabla\varphi) - < f(x, \nabla\varphi) > \\ \varphi & \text{is 1-periodic in } x \\ < \varphi > & = 0 \end{array} \tag{2.7}$$

At a point x_0 where φ reaches its maximum, we have

$$< f(x, \nabla\varphi) > \leq f(x_0, 0) \leq \|f(., 0)\|_\infty.$$

Then assume that there exists a sequence of solutions $(\varphi_n)_n$ such that $\|\varphi_n\|_\infty \to +\infty$; reasoning as above we see that the sequence $(\tilde{\varphi}_n)_n$, with

$$\tilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|_\infty} \tag{2.8}$$

converges strongly to a solution φ_∞ of an equation of the form

$$\begin{array}{rl} -\Delta\varphi_\infty - \tilde{f}|\nabla\varphi_\infty| & = 0 \\ \varphi_\infty & \text{is 1-periodic in } x \\ < \varphi_\infty > & = 0 \end{array} \tag{2.9}$$

This implies $\varphi_\infty = 0$, contradicting (2.8). This allows us to apply a topological degree argument to equation (2.7). □

We shall come back to this problem in a future paper, in the framework of Hamilton-Jacobi equations.

3 Convergence to periodic or steady solutions

We start with the most complicated case (1.1), then explain what modifications are necessary to handle (1.2). Therefore let us denote by $\phi(t,x)$ the solution constructed in Section 3, defined by (1.3), such that $<\phi>(0)=0$.

Lemma 3.1 *There holds $\phi_t > 0$ in $\mathbb{R}\times\mathbb{R}^N$.*

Proof. Let us recall that $\frac{\phi(t,.)}{t} \to \frac{1}{T}$ as $t\to+\infty$; therefore there exists $t_0>0$ such that $\phi(t_0,.)\geq\phi(0,.)$. As a consequence there holds $\phi(t+t_0,.)\geq\phi(t,.)$ for every $t>0$. Let us set

$$\tau_0 = \inf\{t_0>0: \ \forall t\geq 0, \ \phi(t+t_0,.)\geq\phi(t,.)\}.$$

Let us assume $\tau_0>0$. From the spatial periodicity of ϕ, there exists $x_0\in[0,1]^N$ such that $\phi(\tau_0,x_0)\geq\phi(0,x_0)$. From the time periodicity of ϕ, we have $\phi(T+\tau_0,x_0)\geq\phi(T,x_0)$. However, the function $\psi(t,x):=\phi(t+\tau_0,x)-\phi(t,x)$ satisfies a linear parabolic equation, to which the strong maximum principle may be applied, implying that ψ is a constant in space and time; a contradiction. □

Let us then consider the linearized operator about ϕ, that we will denote by $L(t)$:

$$L(t) = -\Delta - f_p(x,\phi,\nabla\phi).\nabla - f_u(x,\phi,\nabla\phi),$$

Its domain is the space X^1, as defined in Section 2. We introduce the evolution system associated to $L(t)$, denoted by $U(t,s)$, namely: for every $\psi\in X^\alpha$, the function $u(t)=U(t,s)\psi$ is a solution of

$$\dot{u}+L(t)u=0, \quad u(s)=\psi. \tag{3.1}$$

The Poincaré map is then $U(t,t+T)$; its spectrum shall be denoted by $\sigma(U(t))$, and its range by $R(U(t))$. The main feature to be noticed is that $L(t)$ has *time-bounded* coefficients, although ϕ is unbounded in time. Therefore, the standard stability theorems may be used, without one single change in their proofs. Let us denote by $B_\rho(0)$ the ball of the complex plane of radius ρ and centre 0.

Lemma 3.2 *The following properties are true.*
(i). ϕ_t is a simple eigenvector for $U(t)$, associated to the eigenvalue 1.
(ii). The rest of the spectrum of $U(t)$ is made up of eigenvalues; moreover there exists $\rho\in]0,1[$ such that $\sigma((U(t))\backslash\{0\}$ lies in $B_\rho(0)$.

Proof. From [5], Lemma 7.2.2, the set $\sigma((U(t))\backslash\{0\}$ does not depend on time. On the other hand, $U(t)$ is a compact positive operator; it even sends the positive cone of X into its interior. Finally we see, just by differentiating (1.1), that $U(t)\phi_t=\phi_t$. However, from Lemma 3.1, the function ϕ_t is positive; a single application of Krein-Rutman's Theorem yields both (i) and (ii). □

We effortlessly end up with an orbital stability result, namely

Theorem 3.3 *Let $u(t,x)$ be a solution of (1.1), such that $u_0 \in X^\alpha$. Set*

$$\varepsilon = \|u_0 - \phi(0,.)\|_\alpha.$$

If ε is small enough, then there exists $t_0 \in \mathbb{R}$ such that

$$\|\mathcal{S}(t)u_0 - \tau_{t_0}\phi(t,x)\|_\alpha = O(e^{-\omega t}).$$

Proof. Once Lemma 3.2 is proved, a direct application of Theorem 8.2.3 f [5] yields the result. □

Here, we notice that we are in quite an unusual situation. Indeed, the periodic solution of an autonomous parabolic equation is systematically unstable, due to the same Krein-Rutman's Theorem that we have used a few lines above. The reason why we have a stable solution comes of course from the addition of a nonzero mean velocity.

Let us end the local stability problem by saying that the above considerations apply word by word to the u-independent problem. Indeed we may consider a Poincaré map as defined in (3.1) by choosing any $T > 0$ to our convenience. However, if we really wish to give an independent proof to the corresponding version of Theorem 3.3, we may proceed as follows. We simply rewrite equation (1.2) in πX, where π represents the projection onto the space of the functions with zero mean. We obtain, still denoting the unknown by u:

$$\begin{aligned} u_t - \Delta u &= f(x,\nabla u) - < f(x,\nabla u) > \\ u(t) &\in \pi X \end{aligned}$$

The linearized operator is this time

$$Lh = -\Delta h - f_p(x,\nabla\varphi(x)).\nabla h - \int_{[0,1]^N} f_p(x,\nabla\varphi(x)).\nabla h \; dx$$

Lemma 3.4 *There exist $\omega > 0$, $\theta \in]0,\frac{\pi}{2}[$ such that*

$$\sigma(L) \subset \{\lambda \in \mathcal{C} : \; |\arg\,(\lambda - \omega)| \le \theta\}.$$

Proof. The operator L has compact resolvent; therefore it is enough to study its eigenvalues. Let $\lambda \in \mathcal{C}\backslash\{0\}$ be an eigenvalue of L, with nonpositive real part; let h be an eigenfunction. The spectrum of the operator

$$h \mapsto L_1 h := -\Delta h - f_p(x,\nabla\varphi(x)).\nabla h$$

with domain in the functions with zero mean, lies in the strict right half-plane. A possible solution of $Lh = \lambda h$ being

$$h = \int_{[0,1]^N} f_p(x,\nabla\varphi(x)).\nabla h \; dx,$$

it is therefore the only solution. But this implies that $h = 0$.

If $\lambda = 0$, we just multiply the equation $Lh = \lambda h$ by a positive function e^* satisfying $L_1^* e^* = 0$; this implies

$$\int_{[0,1]^N} f_p(x, \nabla\varphi(x)).\nabla h \; dx = 0,$$

this implies in turn that $L_1 h = 0$ and therefore $h = 0$, since h has zero mean. □

Proof of Theorem 1.1. The semigroup $\mathcal{T}$ is strongly order-preserving in X^α; see [4] for a definition. Moreover, every periodic front is stable by Theorem 3.3. Theorem 2 of [4] states that any trajectory staying in an interval of X^α converges to a fixed point of $\mathcal{T}$, which concludes the proof. The proof of Theorem 1.2 is contained in the above arguments. □

Let us turn to the proof of Theorem 1.3. For this let Y be the set of all bounded, uniformly continuous real-valued functions of $\mathbb{R}^N$, endowed with its natural topology. For $\alpha \in [0,1]$ let Y^α be the domain of $(-\Delta + 1)^\alpha$ in Y. Then $\mathcal{T}$ defines a discrete semigroup on Y^α, for $\alpha \in]\frac{1}{2}, 1[$. Let us recall that, for any initial datum u_0, the ω-limit set of u_0 is defined by

$$\omega(u_0) = \{\psi \in Y^\alpha : \quad \exists n_k \to +\infty, \; \mathcal{T}^{n_k} u_0 \to \psi\}.$$

In the above innocent looking convergence arrow for $\mathcal{T}^{n_k} u_0$ lie all our difficulties: if we take the convergence in the Y^α norm, we may well find that $\omega(u_0)$ is empty, due to the lack of control of the solution as $x \to \pm\infty$. Moreover the semigroup $\mathcal{T}$ is not strongly order-preserving; it is only order preserving. Therefore the convergences will always be assumed to hold, unless otherwise specified, in the C^1_{loc} topology.

Proposition 3.5 *For any $u_0 \in Y^\alpha$, every element of $\omega(u_0)$ is a 1-periodic function.*

Proof. The idea is the same as in [7]; it is the parabolic version of the sliding method of Berestycki and Nirenberg [1]. This time we make translations both in time and space. Let $\mathcal{B}$ denote the canonical basis of $\mathbb{R}^N$.

Let us introduce $\sigma \geq 0$ defined as follows:

$$\begin{aligned} \Sigma(t) &= \{s \geq 0 : \quad \forall s' \geq s, \; \forall e \in \mathcal{B}, \; \tau_{s'} \tau_e \mathcal{S}(t) u_0 \geq \mathcal{S}(t) u_0\} \\ \sigma(t) &= \inf \Sigma(t) \\ \sigma &= \lim_{t \to +\infty} \sigma(t) \end{aligned} \tag{3.2}$$

It is obvious that $\Sigma(t)$ is closed, nonempty, that its infimum is time-decreasing, and thus that σ is finite. Let us assume that $\sigma > 0$. There exists a sequence $(t_n)_n$, going to $+\infty$, a sequence $(x_n)_n$ and a vector e of $\mathcal{B}$, such that

$$\begin{aligned} &\lim_{n \to +\infty} \left(\tau_{\sigma(t_n)} \tau_e u_0(x_n) - \mathcal{S}(t) u_0(x_n) \right) \\ &\quad = \lim_{t \to +\infty} \inf_{e \in \mathcal{B}, \; x \in \mathbb{R}^N} \left(\tau_{\sigma(t)} \tau_e u_0(x) - \mathcal{S}(t) u_0(x) \right) = 0 \end{aligned} \tag{3.3}$$

Let us write $x_n = [x_n] + y_n$, denoting by $[x_n]$ the integer part of x_n; let us also set $t_n = p_n T + q_n$, $0 \leq q_n < T$. The point y_n is thus confined in the unit cube; hence the sequence $(y_n)_n$ may be assumed to converge towards some y_∞ belonging to the unit cube, as may be assumed to converge – towards q_∞ – the sequence $(q_n)_n$. The sequence $\big(\tau_{[x_n]}\mathcal{S}(t+p_nT-1)u_0 - p_n\big)_n$ may also be assumed to converge, in $C^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^N)$, towards a function $u_\infty(t,x)$, solution of (1.1), and which is uniformly bounded in time.

$$u_\infty(t,x) = \frac{t}{T} + v_\infty(t,x),$$

the function v_∞ being bounded on $\mathbb{R}_+ \times \mathbb{R}^N$. However, the sequence of equalities (3.3) imply that

$$\begin{aligned} \tfrac{t+\sigma}{T} + v_\infty(t+\sigma, x+e) &\geq \tfrac{t}{T} + v_\infty(t,x) \\ \tfrac{1+q_\infty+\sigma}{T} + v_\infty(1+q_\infty+\sigma, y_\infty+e) &= \tfrac{1}{T} + v_\infty(1,y_\infty) \end{aligned}$$

The strong maximum principle implies that there holds everywhere:

$$\frac{t+q_\infty+\sigma}{T} + v_\infty(t+q_\infty+\sigma, x+e) = \frac{t}{T} + v_\infty(t,x).$$

This is impossible because the function v_∞ is bounded.

As a consequence, $\sigma = 0$. This means exactly that, for every $\psi \in \omega(u_0)$ and every $e \in \mathcal{B}$: $\tau_e\psi \geq \psi$. This implies the 1-periodicity of ψ. □

Corollary 3.6 *Every element of $\omega(u_0)$ is 1-periodic in x.*

Proof. From Theorem 1.1, for every $\psi \in \omega(u_0)$, the ω-limit set of ψ contains a unique periodic front. We conclude by using the invariance of $\omega(u_0)$ under the discrete semigroup. □

Let us end this section by proving time-monotonicity in finite time of the solutions of (1.1). We point out that any of the properties that are displayed here would imply the convergence of all trajectories if compactness in Y^α held. In particular, their conjunction may be viewed as an alternative method to the result of Dancer and Hess in the periodic case.

Lemma 3.7 *For any $u_0 \in Y^\alpha$ there holds*

$$\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}^N} \partial_t \mathcal{S}(t)u_0 \geq 0.$$

Proof. This is basically the same idea as in the preceding proposition. Define σ as in (3.2), but this time we do not take into account the translations in space. We end up with $\sigma = 0$, which is the desired property. □

Proof of Theorem 1.3 (*end*). It is sufficient to prove that

$$\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}^N} \partial_t \mathcal{S}(t)u_0 > 0.$$

Assume this is false. Then there exists a sequence $(x_n)_n$ and a sequence $(t_n)_n$, $\lim_{n\to+\infty} t_n = +\infty$, such that

$$\lim_{n\to+\infty} \partial_t\big(\mathcal{S}(t_n)u_0(x_n)\big) = 0.$$

Writing once again $x_n = [x_n] + y_n$, $t_n = p_nT + q_n$ – with $0 \leq q_n < T$ – we may assume that the sequence $(u_n)_n$, defined by

$$u_n(t,x) = \tau_{[x_n]}\mathcal{S}(t-1+p_nT)u_0(x) - p_n$$

converges to a function $u_\infty(t,x)$ such that there exists a point $y_\infty \in \mathbb{R}^N$ for which $u_t(1+q_\infty, y_\infty) = 0$. The strong maximum principle implies that u_∞ does not depend on time. It is therefore a steady solution of (1.1), which is clearly impossible due to Assumption $\mathcal{H}_1$. □

As said in the introduction, Theorem 1.3 admits a corollary which will be our conclusion.

Proposition 3.8 *For every $u_0 \in Y^\alpha$, the trajectory $(\mathcal{S}(t)u_0)_{t\geq 0}$ is L^∞-stable in Y^α.*

Proof. Let us start by deriving the following estimate for $\partial_t(\mathcal{S}(t)u_0)$:

$$\|\partial_t(\mathcal{S}(t)u_0)\|_\alpha \leq M \tag{3.4}$$

for some $M > 0$. Indeed there exist t_1 and t_2 such that

$$\phi(t+t_1,x) \leq S(t)u_0 \leq \phi(t+t_2,x),$$

with ϕ given by Theorem 1.1. Now if we set

$$w(t,x) = S(t)u_0 - \frac{t}{T},$$

one gets the following uniform bound for w

$$\|w(t,.)\|_\infty \leq C_1, \tag{3.5}$$

and it is easy to see that w verifies

$$w_t - \Delta w - b_i(t,x)w_{x_i} = c(t,x), \tag{3.6}$$

with

$$b_i(x,t) = \frac{f(x,S(t)u_0,\nabla w) - f(x,S(t)u_0,0)}{\|\nabla w\|}\frac{\nabla w}{\|\nabla w\|}w_{x_i}.$$

Due to the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, it is clear that the equation (3.6), considered as a linear parabolic one, has uniformly bounded coefficients, so that one can use parabolic estimates with nonsmooth coefficients (chapter 3, [6]) to get uniform bounds of w in Holder estimates which lead to

$$\|w(t,.)\|_\alpha \leq C_2. \tag{3.7}$$

By using (3.7) and parabolic estimates with smooth coefficients, we then get

$$\|\partial_t w(t,.)\|_\alpha \leq C_3,$$

and finally (3.4).

Note that, being interested in large values of t, we do not bother about singularities which may occur near $t = 0$.

Now let v_0 belong to Y^α, and satisfy $\|u_0 - v_0\|_\alpha \leq \varepsilon$. From the continuity with respect to the initial datum, there holds

$$\|\mathcal{S}(t_0)u_0 - \mathcal{S}(t_0)v_0\|_\alpha \leq C\varepsilon,$$

for some sufficiently large value of $t_0 > t_*$. If $\varepsilon > 0$ is small enough – namely $\frac{\varepsilon C}{\delta} < 1$ – there holds

$$\mathcal{S}(t_0 - \frac{\varepsilon C}{\delta})u_0 \leq \mathcal{S}(t_0)v_0 \leq \mathcal{S}(t_0 + \frac{\varepsilon C}{\delta})u_0$$

Therefore, for every $t \geq 0$, there holds

$$\mathcal{S}(t + t_0 - \frac{\varepsilon C}{\delta})u_0 \leq \mathcal{S}(t + t_0)v_0 \leq \mathcal{S}(t + t_0 + \frac{\varepsilon C}{\delta})u_0.$$

The above inequalities together with (3.4) yield

$$\|\mathcal{S}(t)u_0 - \mathcal{S}(t)v_0\|_\infty \leq \frac{MC\varepsilon}{\delta};$$

to get this inequality in the Y^α norm we once again apply the parabolic estimates with smooth coefficients [6]. $\square$

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Appendix

Let $\Gamma(t)$ represent a propagating interface at time t. Then, in its classical formulation, Γ propagates normal to itself with velocity

$$V_n = R + k$$

where k is a curvature term and R, a given driving force. In the case of flame propagation, R can be viewed as the combustion rate and is therefore an intrinsic property of the material. In our case, the latter will be heterogeneous so that R will have a spatial dependence.

Consider now $\Gamma(t)$ as the zero level set of a function $\Phi(x,t)$ defined in the whole domain, i.e.

$$\Gamma(t) = \{x \in \mathbb{R}^N / \Phi(x,t) = 0\}.$$

Then if $x(t)$ is the trajectory of a particle located on this level set, the normal velocity will be given by

$$V_n = \frac{dx}{dt}.\mathbf{n}, \text{ where } \mathbf{n} = \frac{\nabla\Phi}{|\nabla\Phi|}.$$

By the chain rule, we then have

$$\Phi_t + \nabla\Phi\frac{dx}{dt} = 0,$$

and substitution yields the following Hamilton-Jacobi type equation

$$\Phi_t - (R+k)|\nabla\Phi| = 0, \tag{A.1}$$

with the curvature term given by $k = \nabla.\mathbf{n}$. Now if we suppose a parametric representation of Γ, *i.e*

$$\Gamma(t) = \{x = (x', x_N)/\Phi(x,t) = x_N - u(x',t) = 0\},$$

then (A.1) becomes

$$u_t - R(x',u)\sqrt{1+|\nabla u|^2} = G(\nabla u, \nabla^2 u), \tag{A.2}$$

the curvature k being contained in the second order term G. Note that in this parametric formulation, R depends also on u, so that (1.1) appears as an approximated version of (A.2) with

$$f(x,u,p) = R(x,u)\sqrt{1+|p|^2}$$

and the second order term G replaced by the viscosity term Δu. That this modified version (1.1) is reasonable, at least for qualitative behaviour of propagating fronts, is motivated by the following numerical simulations.

Consider the two dimensional case so that $x' = x$ and $x_N = y$ and $G = \dfrac{u_{xx}}{1+u_x^2}$. Then $y = u(x,t)$ gives the position of the front and we are interested in the equations

$$\begin{aligned} u_t - R(x,u)\sqrt{1+u_x^2} &= g \\ u(x,0) &= u_0(x) \end{aligned} \tag{A.3}$$

where R is periodic in both directions and u_0 periodic in x. A typical set up would be a medium with periodically displayed oblique striations, Fig. 1. Note that a medium with horizontal striations would lead to an equation of type (1.2).

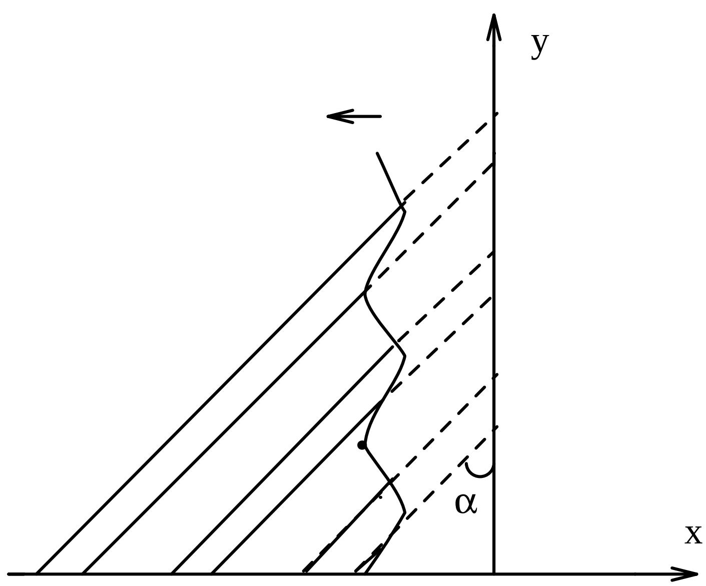

Figure 1: Propagation of a front in a medium with oblique striations.

Figures 2 and 3 show the numerical results obtained by solving equation with $g = G$ and $g = \Delta u$ respectively. We have used schemes of Godunov type for the first order terms while g is approximated by central differences. In both cases, one

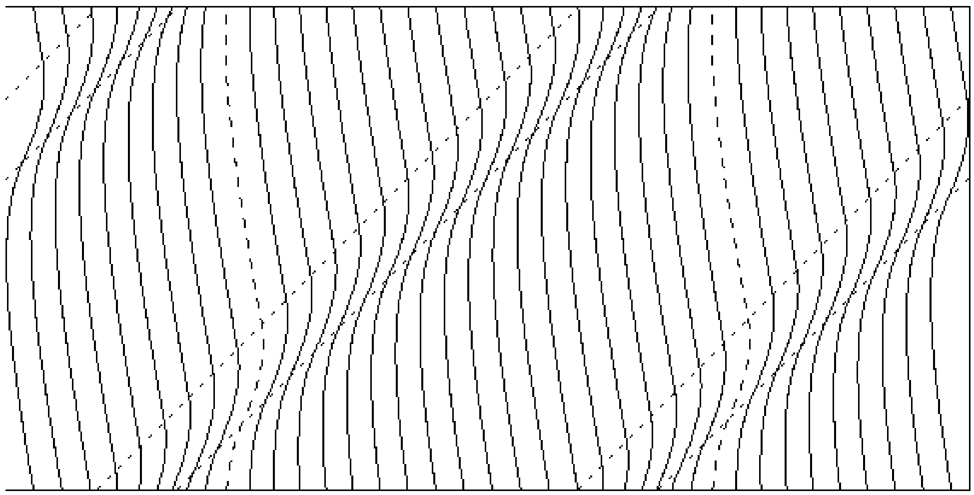

Figure 2: Evolution of the front under curvature effect.
$(u_t + R(x,u)\sqrt{1+u_x^2} = G)$

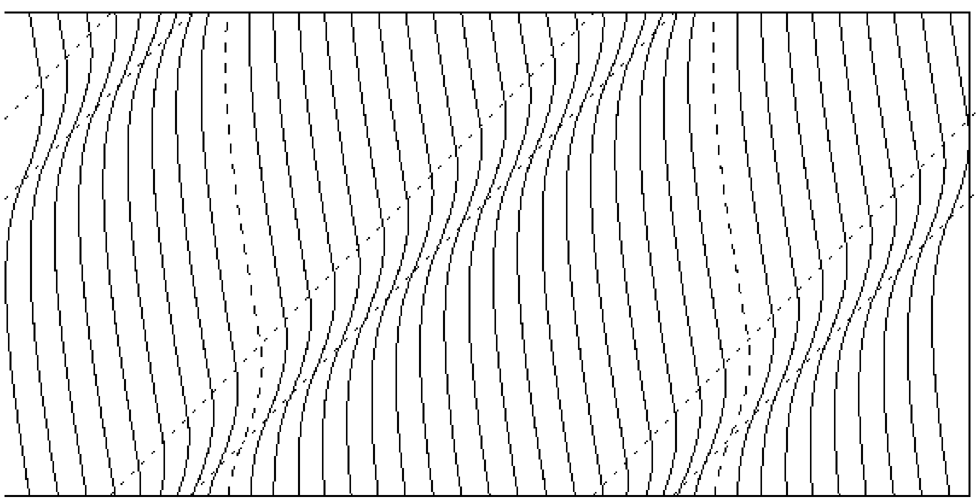

Figure 3: Evolution of the front with viscosity approximation.
$(u_t + R(x,u)\sqrt{1+u_x^2} = u_{xx})$

can observe the setting up of a periodic pattern in time, *i.e.* the existence of a period T such that $u(x,t+T) = u(x,t) + L$ for some constant L – in other words, the speed of the front is T-periodic in time.

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