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Sub-harmonics for two-dimensional Hamiltonian systems

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Abstract. Using variational methods, we prove the existence of infinitely many sub-harmonics for two-dimensional asymptotically linear Hamiltonian systems with at least two periodic orbits, under non-resonance conditions. Such a result is strictly related to a recent result by Franks, proved by completely different methods.

1 Introduction

Poincaré's celebrated 'last geometric theorem' asserts that every area-preserving homeomorphism of the annulus which twists the two boundaries into opposite directions must have at least two fixed points [11]. This theorem was first proved by Birkhoff [3, 4], using strictly two-dimensional tools (see [5] for a modern exposition of Birkhoff's proof).

In 1984 Conley and Zehnder proved the following result: an asymptotically linear T-periodic Hamiltonian system in $\mathbf{R}^{2n}$ which has the origin as an equilibrium point and such that the Maslov index of the origin differs from the Maslov index at infinity, must have at least one T-periodic orbit, two under non-resonance conditions [6].

Since the flow determined by a Hamiltonian system is volume-preserving and the Maslov index is a number which measures the twist of the flow near a given equilibrium point, the two results are strictly connected. Notice, however, that Conley and Zehnder's theorem holds in every dimension. Their proof is based on the study of the action functional via Morse theory.

More recently, Franks has proved a somewhat astonishing result: every area-preserving homeomorphism of the two-dimensional open disk with at least two fixed points must have infinitely many periodic points [8]. Having two fixed points is of course essential: a rotation of an angle not commensurable with 2π has one fixed point and no other periodic points. Again, Franks's proof is strictly two-dimensional.

In this paper we prove the following result: every asymptotically linear T-periodic Hamiltonian system in $\mathbf{R}^2$ which has at least two periodic orbits must have infinitely many sub-harmonics. Again, we need non-resonance conditions. As an immediate corollary we get that a two-dimensional system having an equilibrium point with Maslov index different from the Maslov index at infinity has not only one periodic orbit, as proved by Conley and Zehnder, but infinitely many sub-harmonics.

The analytical tool in our existence result is the Morse theory for the action functional developed by Conley and Zehnder. Then our theorem follows from a careful study of the Maslov index in dimension two and, in particular, of a real invariant called mean winding number or twist number. Such a number was widely used by Benci [1] and Benci-Fortunato [2], because it contains a lot of relevant information about the periodic orbit. We show that in dimension two it actually determines uniquely the Maslov index of a periodic orbit and of all its iterations. As a side remark, we also show that it determines the linear stability of the solution.

The first two sections are devoted to the study of the symplectic group and of the Maslov index in dimension two. Our exposition is different from the traditional one, which covers the general case (see for example [12]), and in some sense the analysis is more accurate. Excluding the proof of the Morse relations of Conley and Zehnder, which are only stated, the exposition is self-contained.

2 The structure of $Sp(1)$ and the rotation function

The symplectic group of the plane $Sp(1)$ consists of the real matrices two by two A such that $A^T JA = J$, where A^T is the transpose of A and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Equivalently

$$Sp(1) = \{A \in GL(2, \mathbf{R}) \mid \det A = 1\}.$$

Every invertible matrix A can be decomposed into polar form

$$A = PO, \quad P = (AA^T)^{\frac{1}{2}}, \quad O = P^{-1}A$$

where P is symmetric and positive definite, O is orthogonal. This decomposition is unique. If A is symplectic, both P and O are symplectic. Therefore O belongs to the group of rotations

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbf{R} \right\}.$$

As a topological group, $SO(2)$ is isomorphic to $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. Therefore we can define the map $u : Sp(1) \mapsto S^1$ as

$$u(PO) = e^{i\theta} \quad if \quad O = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Since P and O in the polar decomposition depend continuously on A, $Sp(1)$ is homeomorphic to the product space between S^1 and the set of symmetric, positive definite, symplectic two by two matrices. Call $\mathcal{P}$ the latter set.

Using the fact that every positive definite symmetric matrix is the exponential of one and only one symmetric matrix, it can be shown that $\mathcal{P}$ is homeomorphic to the plane. However we are going to build a different homeomorphism, which will make some later calculations easier.

The trace of P is the sum of its eigenvalues, which are positive and whose product is 1. Therefore tr $P \geq 2$ and we can set tr $P = 2\cosh\tau$, with $\tau \geq 0$. Then P can be written as

$$P = \begin{pmatrix} \cosh\tau + a & b \\ b & \cosh\tau - a \end{pmatrix}.$$

There is a relationship between a and b determined by the determinant condition

$$1 = \det P = \cosh^2\tau - a^2 - b^2.$$

The above equation makes sense if and only if $|a| \leq |\sinh\tau|$. Therefore we can set $a = \cos\sigma \sinh\tau$, with $\sigma \in \mathbf{R}$. Then $b^2 = \sin^2\sigma \sinh^2\tau$ and we get a one-to-one parameterization of $\mathcal{P}$ if we set $b = \sin\sigma \sinh\tau$ and we let σ vary in $[0, 2\pi[$. If we consider $(\tau = |z|, \sigma = \arg z)$ as polar coordinates on the complex plane, a homeomorphism between $\mathbf{C}$ and $\mathcal{P}$ is given by

$$P = \begin{pmatrix} \cosh\tau + \cos\sigma\sinh\tau & \sin\sigma\sinh\tau \\ \sin\sigma\sinh\tau & \cosh\tau - \cos\sigma\sinh\tau \end{pmatrix}.$$

In order to draw better pictures, it is more convenient to parameterize $\mathcal{P}$ with the open unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$. To do this it is enough to consider polar coordinates $(r = |z|, \sigma = \arg z)$ on D and to set $r = \tanh^2\tau$.

Therefore $Sp(1)$ is homeomorphic to the product space between the circle and the open disk, i.e. to the interior of a torus, as it is shown in Fig. 1, (see also [9]). We will always use coordinates (θ, r, σ) as above on $Sp(1)$.

Notice that $SO(2) = \{(\theta, r, \sigma) \in Sp(1) \mid r = 0\}$: thus $SO(2)$ is a deformation retract of $Sp(1)$. Moreover the map u defined before is just

$$u(\theta, r, \sigma) = e^{i\theta}.$$

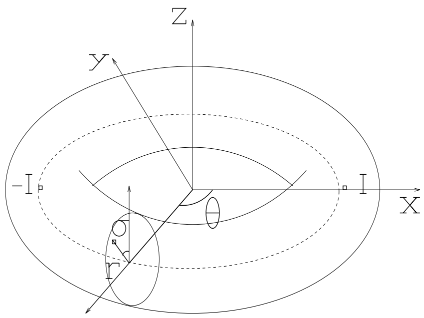

Figure 1 Parameterization of $Sp(1)$

Therefore u restricts to the standard isomorphism from $SO(2)$ to S^1. The problem with the map u is that it is no longer a homomorphism on the whole $Sp(1)$, because in general the orthogonal part and the positive definite part in the polar decomposition do not commute.

In order to avoid this problem we want to define a map

$$\rho : Sp(1) \longmapsto S^1$$

homotopic to u which is still not a homomorphism but has the property that $\rho(A^k) = \rho(A)^k$, for every $A \in Sp(1)$.

The eigenvalues of $A \in Sp(1)$ must be of the form λ, $\frac{1}{\lambda}$, where $\lambda \in \mathbf{R} \cup S^1$. An eigenvalue $\lambda \neq \pm 1$ must be therefore simple. The eigenvalues 1 and -1 are always double.

To define ρ we need to define the Krein sign of the eigenvalues which lie on S^1 (for a complete exposition on this subject see, for example, [9] and [7]). $G = iJ$ is a hermitian matrix, meaning that $G^* = G$, where G^* is the adjoint of G with respect to the standard hermitian product $\langle \cdot, \cdot \rangle$ of $\mathbf{C}^2$: $G^* = \overline{G}^T$. Assume that A has eigenvalues $\lambda \neq \pm 1$ and $\overline{\lambda}$ of modulus one and that ξ and η are the corresponding eigenvectors. Then

$$\langle G\xi, \eta \rangle = \langle A^* G A \xi, \eta \rangle = \langle G A \xi, A \eta \rangle = \lambda^2 \langle G\xi, \eta \rangle.$$

Since $\lambda \neq \pm 1$, we must have $\langle G\xi, \eta \rangle = 0$. So $\{\xi, \eta\}$ is a G-orthonormal basis of $\mathbf{C}^2$. Remembering that G is hermitian and invertible, we conclude that $\langle G\xi, \xi \rangle$ and $\langle G\eta, \eta \rangle$ are real and not zero.

Definition 2.1 If $\lambda \in S^1 \setminus \{-1, 1\}$ is an eigenvalue of A and ζ is the corresponding eigenvector, the Krein sign of λ is the sign of $\langle G\zeta, \zeta \rangle$.

Since G has signature $(1, 1)$, if the eigenvalue $\lambda \in S^1 \setminus \{-1, 1\}$ is Krein-positive, the eigenvalue $\overline{\lambda}$ is Krein-negative. Sometimes it is useful to consider the double eigenvalue $\lambda = \pm 1$ as a pair of eigenvalues, one of which is Krein-positive, the other Krein-negative. The rotation of a symplectic matrix A is defined as

$$\rho(A) = \begin{cases} \lambda & \text{if } \lambda \in S^1 \backslash \{-1, 1\} \text{ is the Krein-positive eigenvalue of } A \\ 1 & \text{if the eigenvalues of } A \text{ are real and positive} \\ -1 & \text{if the eigenvalues of } A \text{ are real and negative} \end{cases} .$$

To see that $\rho : Sp(1) \mapsto S^1$ is continuous, notice that

$$\rho(A) = \frac{\lambda}{|\lambda|}$$

where λ is any eigenvalue of A, in the case $\lambda \in \mathbf{R}$, and it is the Krein-positive eigenvalue, in the case $\lambda \in S^1 \setminus \{-1, 1\}$.

The rotation $R(\theta)$ has eigenvalues $e^{i\theta}$, $e^{-i\theta}$.

$$\zeta = \begin{pmatrix} \cos\theta + i \sin\theta \\ \sin\theta - i \cos\theta \end{pmatrix}$$

is an eigenvector corresponding to $e^{i\theta}$. An explicit calculation shows that:

$$\langle G\zeta, \zeta \rangle = 2$$

so $e^{i\theta}$ is Krein-positive and $\rho(R(\theta)) = u(R(\theta)) = e^{i\theta}$.

If λ is an eigenvalue of A, λ^k is an eigenvalue of A^k, the eigenvectors being the same. Therefore

$$\rho(A^k) = \rho(A)^k.$$

In order to study the function ρ we must find the eigenvalues of A, so we must solve

$$\det(\lambda I - A) = \lambda^2 - (\mathrm{tr}\ A)\lambda + 1 = 0$$

for λ. The discriminant of this polynomial is

$$\Delta = (\mathrm{tr}\ A)^2 - 4 = 4 \cosh^2 \tau \cos^2 \theta - 4.$$

A has a double eigenvalue ± 1 if and only if $\Delta = 0$, which is equivalent to

$$r = \sin^2 \theta.$$

So the set of symplectic matrices with double eigenvalue is the checked surface depicted in Fig. 2. It consists of two connected components (remember that the full torus has no boundary). The right component contains I and thus it consists of matrices with eigenvalue 1. The left component contains $-I$ and thus it consists of matrices with eigenvalue -1.

The inequality $\Delta > 0$ is equivalent to

$$r > \sin^2 \theta.$$

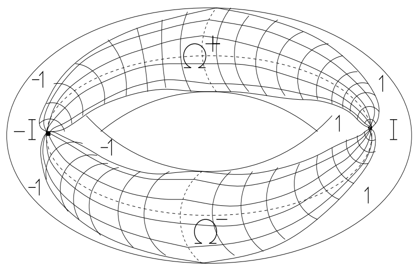

Figure 2 The rotation function

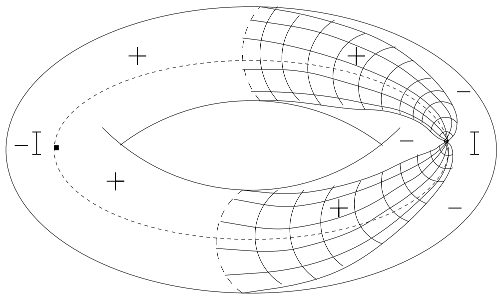

Figure 3 The sets Γ^+, Γ^0 and Γ^-

Therefore Δ is greater than zero outside the checked surface in Fig. 2: so $\rho = \pm 1$ in that region. By continuity, $\rho = 1$ on the right and $\rho = -1$ on the left.

If A belongs to the interior of the checked surface, Δ is less than zero and A must have two eigenvalues λ, $\overline{\lambda} \in S^1 \setminus \{-1, 1\}$. The interior of the checked surface consists of two components, one which contains the rotations of angle θ with $\sin\theta > 0$, the other which contains the rotations of angle θ with $\sin\theta < 0$. Call these regions Ω^+ and Ω^-, respectively. The function ρ is continuous, it never takes the values ± 1 inside the checked surface and $\rho(O) = e^{i\theta}$ whenever O is a θ-rotation: therefore ρ must take values on the upper half circle in Ω^+, and on the lower half circle in Ω^-. These facts easily imply that the map ρ is homotopic to the map u defined before.

We summarize the above discussion into the following proposition:

Proposition 2.1 *There exists a continuous map*

$$\rho : Sp(1) \longmapsto S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$$

such that:

1. *ρ is homotopic to u;*
2. *$\rho(R(\theta)) = u(R(\theta)) = e^{i\theta}$;*
3. *$\rho(A^k) = \rho(A)^k$ for every $A \in Sp(1)$ and for every integer k.*

3 The Maslov index

The set $Sp(1)$ can be divided into three subsets:

$$\begin{aligned} \Gamma^+ &= \{A \in Sp(1) \mid \det(I - A) > 0\} \\ \Gamma^- &= \{A \in Sp(1) \mid \det(I - A) < 0\} \\ \Gamma^0 &= \{A \in Sp(1) \mid \det(I - A) = 0\}. \end{aligned}$$

Recall that the checked surface of Fig. 2 consists of the matrices with double eigenvalue 1 or -1. Therefore Γ^0 is the component of such surface containing I, that is the right part.

Fig. 3 represents Γ^0, Γ^+ and Γ^-: Γ^0 is a surface with a 2-codimensional singularity and it divides $Sp(1)$ into two connected components, Γ^+ and Γ^-. Notice that both these components are contractible in $Sp(1)$ (also if Γ^- is not contractible in itself).

We want to associate to every continuous path

$$\gamma : [0, T] \longmapsto Sp(1) \quad \gamma(0) = I,\ \gamma(T) \notin \Gamma^0$$

an integer, which will be called the Maslov index of the path γ. Loosely speaking, the Maslov index of γ is the number of half windings made by γ in $Sp(1)$, $Sp(1)$ being divided into two components by Γ^0.

The matrix $-I$ lies in Γ^+; let us fix a matrix W in Γ^-: for instance

$$W = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The path $\gamma : [0, T] \mapsto Sp(1)$ can be extended to a path $\tilde{\gamma} : [0, T+1] \mapsto Sp(1)$ in such a way that

$$\begin{cases} \tilde{\gamma}(t) = \gamma(t) & \text{if } t \in [0, T] \\ \tilde{\gamma}(t) \notin \Gamma^0 & \text{if } t \in [T, T+1] \\ \tilde{\gamma}(T+1) \in \{-I, W\}. \end{cases}$$

There exists a unique function $\tilde{\delta} : [0, T+1] \mapsto \mathbf{R}$ such that:

$$\rho(\tilde{\gamma}(t)) = e^{i\tilde{\delta}(t)}, \quad \tilde{\delta}(0) = 0.$$

Notice that, since $\tilde{\gamma}(t) \notin \Gamma^0$ for every $t \in [T, T+1]$,

$$|\tilde{\delta}(T+1) - \tilde{\delta}(T)| < \pi. \tag{1}$$

Since $\rho(W) = 1$ and $\rho(-I) = -1$, $\tilde{\delta}(T+1)$ is an integer multiple of π.

Definition 3.1 The Maslov index of the path γ at time T is defined as

$$\mu_T(\gamma) = \frac{1}{\pi}\tilde{\delta}(T+1).$$

The above definition does not depend on the extension $\tilde{\gamma}$ chosen, provided $\tilde{\gamma}(t) \notin \Gamma^0$: this follows from the fact that the closed loops in Γ^+ and in Γ^- are contractible in $Sp(1)$ (see Fig. 3).

A linear T-periodic Hamiltonian system in $\mathbf{R}^2$ has the form

$$\dot{z}(t) = JB(t)z(t) \tag{2}$$

where $B(t)$ is a T-periodic path of symmetric matrices.

Let $\gamma(t)$ be the fundamental solution of (2), i.e. the solution of the matrix differential problem

$$\begin{cases} \dot{\gamma}(t) = JB(t)\gamma(t) \\ \gamma(0) = I. \end{cases}$$

Then $\gamma(t)$ is symplectic for every $t \in \mathbf{R}$.

Definition 3.2 The T-Floquet multipliers of system (2) are the eigenvalues of $\gamma(T)$.

Definition 3.3 The linear Hamiltonian system (2) is said to be T-resonant if 1 is a T-Floquet multiplier.

If system (2) is T-non resonant, the path γ has a well defined Maslov index $\mu_T(\gamma)$, which will be also the T-Maslov index of system (2). Moreover the system will be kT-non resonant at least for every large prime k: this follows from the formula $\gamma(kT) = \gamma(T)^k$. We would like to study the behavior of the kT-Maslov index of system (2) as k grows up. As before, let $\delta : \mathbf{R} \mapsto \mathbf{R}$ be the only continuous function such that

$$\rho(\gamma(t)) = e^{i\delta(t)}, \quad \delta(0) = 0.$$

Proposition 3.1 *$\delta(kT) = k\delta(T)$ for every $k \in \mathbf{Z}$.*

Proof. Since the system (2) is T-periodic, its fundamental solution γ satisfies

$$R(kT + t) = R(t)R(T)^k \quad \forall t \in \mathbf{R},\ \forall k \in \mathbf{Z}.$$

In the previous section we have seen that $SO(2)$ is a deformation retract of $Sp(1)$: let $\gamma_\lambda : [0, T] \mapsto Sp(1)$, $\lambda \in [0, 1]$, be a homotopy such that $\gamma_0 = \gamma$, $\gamma_1(t) \in SO(2)$ for every $t \in [0, T]$ and $\gamma_\lambda(0) = I$. We can extend the paths γ_λ over all $\mathbf{R}$ by setting

$$\gamma_\lambda(t) = \gamma_\lambda(t - kT)\gamma_\lambda(T)^k \quad \text{for } t \in [kT, (k+1)T].$$

Then $\gamma_\cdot(\cdot)$ is continuous on $(\lambda, t) \in [0, 1] \times \mathbf{R}$. Let $\delta_\lambda : \mathbf{R} \mapsto \mathbf{R}$ be the only continuous function such that

$$\rho(\gamma_\lambda(t)) = e^{i\delta_\lambda(t)}, \quad \delta_\lambda(0) = 0.$$

By construction

$$\gamma_\lambda(kT+t) = \gamma_\lambda(t)\gamma_\lambda(T)^k \quad \forall t \in \mathbf{R},\ \forall k \in \mathbf{Z} \tag{3}$$

and $\gamma_1(t) \in SO(2)$ for every $t \in \mathbf{R}$. By (3) and Proposition 2.1 (3)

$$\rho(\gamma_\lambda(kT)) = \rho(\gamma_\lambda(T)^k) = \rho(\gamma_\lambda(T))^k$$

and so

$$\delta_\lambda(kT) - k\delta_\lambda(T) \tag{4}$$

must be an integer multiple of 2π. Moreover the quantity (4) depends continuously on λ and therefore

$$\delta(kT) - k\delta(T) = \delta_0(kT) - k\delta_0(T) = \delta_1(kT) - k\delta_1(T). \tag{5}$$

So it is enough to show that the right-hand member of (5) vanishes for every $k \in \mathbf{Z}$.

We start with the case $k > 0$ and we argue by induction on k. The thesis is true for $k = 1$. Assume that it is true for $k = \overline{k}$. Since ρ is a homomorphism on $SO(2)$

$$\rho(\gamma_1(T+t)) = \rho(\gamma_1(t)) \cdot \rho(\gamma_1(T)).$$

Therefore the quantity

$$\delta_1(T+t) - \delta_1(t) - \delta_1(T)$$

must be an integer multiple of 2π. It depends continuously on t and it vanishes for $t = 0$. So

$$\delta_1(T+t) - \delta_1(t) - \delta_1(T) = 0. \tag{6}$$

For $t = \overline{k}T$, (6) gives:

$$\delta_1((\overline{k}+1)T) - \delta_1(\overline{k}T) - \delta_1(T) = 0.$$

Since the thesis is supposed to be true for $k = \overline{k}$, we have

$$\delta_1((\overline{k}+1)T) - (\overline{k}+1)\delta_1(T) = 0.$$

This proves the induction step and the thesis for $k > 0$. To conclude the proof notice that formula (6) with $t = -T$ gives

$$\delta_1(-T) = -\delta_1(T).$$

Then the same induction argument for negative k's proves the Proposition.

□

The above Proposition implies that the kT-Maslov index grows linearly with k. This fact allows to give the following definition (see also [7]):

Definition 3.4 The mean winding number of the linear Hamiltonian system (2) is

$$\tau = \lim_{k\to+\infty} \frac{\mu_{kT}}{kT} = \lim_{k\to+\infty} \frac{\delta(kT)}{\pi kT} = \frac{1}{\pi T}\delta(T).$$

Here the limit is taken over all $k \in \mathbf{N}$ such that system (2) is kT-non resonant and we have been able to write $\frac{1}{\pi}\delta(kT)$ instead of $\mu_{kT} = \frac{1}{\pi}\tilde{\delta}(kT+1)$ because of estimate (1).

A crucial remark is that the mean winding number of a T-periodic linear Hamiltonian system in dimension 2 completely determines its Maslov index:

Proposition 3.2 *Let τ be the mean winding number of the T-non resonant linear Hamiltonian system* (2). *Let $\nu = \frac{1}{T}$ be the frequency of the system.*

1. *If $\tau = 2k\nu$, with $k \in \mathbf{Z}$, then the Maslov index μ_T of system* (2) *is equal to* $2k$.
2. *If $\tau = (2k+\sigma)\nu$, with $k \in \mathbf{Z}$, $\sigma \in]0,2[$, then μ_T is equal to $2k+1$,*

In fact, if $\pi T\tau = \delta(T) = 2k\pi$, $R(T)$ must belong to Γ^- (see Figs. 2, 3). When we extend R so to reach W, remaining in Γ^-, we find that $\tilde{\delta}(T+1) = 2k\pi$. Therefore $\mu_T = 2k$.

If $\pi T\tau = \delta(T) = 2k\pi + \sigma\pi$, with $\sigma \in]0,2[$, $R(T)$ must belong to Γ^+ (see Figs. 2, 3). By (1), $\mu_T = \frac{1}{\pi}\tilde{\delta}(T+1)$ must be the odd integer such that $|\pi\mu_T - \pi T\tau| < \pi$ and so $\mu_T = 2k+1$.

We end up this section showing, as a side remark, that the mean winding number determines also the stability of the system. Recall that a T-periodic linear system is said to be elliptic if the eigenvalues of $R(T)$ have modulus 1. If the eigenvalues have modulus 1 and they are different from ± 1, the system is strongly elliptic. The system is hyperbolic if the eigenvalues have modulus different from 1.

Proposition 3.3 *Let τ be the mean winding number of the T-non resonant linear Hamiltonian system* (2). *Let $\nu = \frac{1}{T}$ be the frequency of the system.*

1. *If $\tau = 2k\nu$, with $k \in \mathbf{Z}$, then system* (2) *is hyperbolic.*
2. *If $\tau = (2k+1)\nu$, with $k \in \mathbf{Z}$, and system* (2) *is also $2T$-non resonant, then system* (2) *is hyperbolic.*
3. *If $\tau \notin \nu\mathbf{Z}$ then system* (2) *is strongly elliptic.*

Again this Proposition follows from the fact that $\pi T\tau = \delta(T)$ and from the dependence of the function ρ on the eigenvalues (see Fig. 2).

We emphasize the fact that Propositions 3.2 and 3.3 are a special feature of dimension two and that they definitely do not hold for larger dimension.

4 Asymptotically linear 2-dimensional systems

A T-periodic Hamiltonian system in $\mathbf{R}^2$ takes the form:

$$\dot{z}(t) = J\nabla H(z(t), t) \tag{7}$$

where $H : \mathbf{R}^2 \times \mathbf{R} \mapsto \mathbf{R}$ is T-periodic in the last variable and ∇ denotes the Jacobian with respect to the first two variables.

A kT-periodic solution of (7), for $k \in \mathbf{N}$, is called a sub-harmonic. The minimum of the numbers hT such that z is hT-periodic is the minimal period of z.

Let z be a kT-periodic solution of (7). We can consider the linearization of (7) along z:

$$\dot{w}(t) = JH''(z(t), t)w(t) \tag{8}$$

where H'' is the Hessian of H with respect to the first two variables. (8) is a kT-periodic linear Hamiltonian system.

Definition 4.1 The kT-Floquet multipliers of the kT-periodic solution z are the kT-Floquet multipliers of system (8).

Definition 4.2 The kT-periodic solution z of (7) is said to be kT-resonant if system (8) is kT-resonant.

Definition 4.3 The Maslov index $\mu_{kT}(z)$ and the mean winding number $\tau(z)$ of z are defined as the kT-Maslov index and the mean winding number of (8), respectively.

A kT-periodic solution of (7) is said to be elliptic, strongly elliptic or hyperbolic if the kT-periodic system (8) is elliptic, strongly elliptic or hyperbolic.

The ellipticity, strong ellipticity or hyperbolicity of a kT-periodic solution is determined by its mean winding number, exactly as in Proposition 3.3.

Definition 4.4 System (7) is said to be asymptotically linear if:

$$\lim_{|z|\to+\infty} |\nabla H(z, t) - A_\infty(t)z| = 0 \quad \text{uniformly in } t \in \mathbf{R}$$

where A_∞ is a T-periodic path of symmetric matrices.

Moreover we assume that the hessian of H with respect to the space variables is bounded:

$$\sup_{(z,t)\in\mathbf{R}^2\times\mathbf{R}} |H''(z, t)| < +\infty \tag{9}$$

Definition 4.5 The asymptotically linear T-periodic Hamiltonian system (7) is said to be T-resonant at infinity if its linearization at infinity:

$$\dot{w}(t) = JA_\infty(t)w(t) \tag{10}$$

is T-resonant.

The basic tool to determine the existence of periodic solutions for asymptotically linear Hamiltonian systems is provided by the Morse relations proved by Conley and Zehnder [6] (see also [10] where the two-dimensional case, omitted by Conley and Zehnder, is considered):

Theorem 4.1 *Assume that the T-periodic asymptotically linear Hamiltonian system* (7) *satisfies* (9) *and that it is T-non resonant at infinity. Assume moreover that all its T-periodic solutions are T-non resonant. Let $\mu_T(\infty)$ be the Maslov index of the linearization at infinity* (10). *Then there is only a finite number of T-periodic solutions and the following equality holds:*

$$\sum_z \lambda^{\mu_T(z)} = \lambda^{\mu_T(\infty)} + (1+\lambda)Q(\lambda)$$

where the sum is taken over all the T-periodic solutions and Q is a Laurent polynomial with non-negative integer coefficients.

Now we want to prove the result stated in the introduction:

Theorem 4.2 *Assume that the T-periodic asymptotically linear Hamiltonian system* (7) *satisfies* (9) *and that it is T-non resonant at infinity. Assume moreover that all its T-periodic solutions are T-non resonant. If there are two or more T-periodic solutions, then for every large prime p there is at least a sub-harmonic with minimal period pT. If such a sub-harmonic is pT-non resonant, then there must be a second sub-harmonic with minimal period pT.*

Proof. By Theorem 4.1 there is a finite number of T-periodic solutions. We can group them into subsets Z_i, $i = 1, \dots, k$, according on the value of their mean winding number:

$$\tau(z) = \tau_i \quad \forall z \in Z_i \quad \text{and} \quad \tau_i \neq \tau_j \quad \text{if } i \neq j.$$

Let $\{\lambda_\infty, \lambda_\infty^{-1}\}$ be the T-Floquet multipliers of system (10) and let $\{\lambda_z, \lambda_z^{-1}\}$ be the T-Floquet multipliers of z, for every T-periodic solution z. By the non-resonance assumptions

$$\lambda_\infty \neq 1 \quad \text{and} \quad \lambda_z \neq 1 \ \forall z.$$

Assume that $\{\alpha_1, \dots, \alpha_r\} \subset \{\lambda_\infty\} \cup \{\lambda_z \mid z \text{ is a } T\text{-periodic solution}\}$ are those Floquet multipliers which are roots of 1:

$$\alpha_i^{m_i} = 1, \quad \alpha_i^n \neq 1 \text{ if } 1 \leq n < m_i, \quad i = 1, \dots, r.$$

Let $M = \max\{m_1, \ldots, m_r\}$. Let p be a prime number greater than M. Since p can not be an integer multiple of m_i, for any $i = 1, \ldots, r$, we have

$$\alpha_i^p \neq 1 \quad \forall i = 1, \ldots, r.$$

Therefore (7) is pT-non resonant at infinity and all the T-periodic solutions are pT-non resonant.

If $z \in Z_i$ is a T-periodic solution, then Proposition 3.2 implies that

$$|\mu_{pT}(z) - pT\tau_i| < 1.$$

If z is in Z_i and w is in Z_j

$$\begin{aligned}|\mu_{pT}(z) - \mu_{pT}(w)| &\geq pT|\tau_i - \tau_j| - |\mu_{pT}(z) - pT\tau_i \\ &+ pT\tau_j - \mu_{pT}(w)| > pT|\tau_i - \tau_j| - 2.\end{aligned}$$

So we can find a large number $N > M$ such that, for every $p \geq N$

$$|\mu_{pT}(z) - \mu_{pT}(w)| \geq 2 \quad \text{if } z \in Z_i,\ w \in Z_j,\ i \neq j. \tag{11}$$

Let $p \geq N$ be a prime number and assume, by contradiction, that there are no sub-harmonics with minimal period pT. Since p is prime, all the pT-periodic solutions must be T-periodic.

So (7) is pT-non resonant at infinity and all its pT-periodic solutions are pT-non resonant; therefore the Morse relations hold

$$\sum_z \lambda^{\mu_{pT}(z)} = \lambda^{\mu_{pT}(\infty)} + (1+\lambda)Q(\lambda) \tag{12}$$

where the sum is taken over all the T-periodic solutions.

Since there are at least two T-periodic solutions, $Q \neq 0$. Let $n \in \mathbf{Z}$ be such that the coefficient of λ^n in Q is not zero. Then the Morse relations imply that there exist T-periodic solutions z and w such that

$$\mu_{pT}(z) = n \quad \text{and} \quad \mu_{pT}(w) = n+1. \tag{13}$$

But this is impossible: by (11) these solution must belong to the same Z_i and so

$$\tau(z) = \tau(w).$$

But then $\mu_{pT}(z) = \mu_{pT}(w)$, because Proposition 3.2 implies that the pT-Maslov index is uniquely determined by the mean winding number.

The existence of the second sub-harmonic in the pT-non resonant case follows again from relation (12). □

From the above theorem we get immediately the following corollary:

Corollary 4.3 *Assume that the T-periodic asymptotically linear Hamiltonian system* (7) *satisfies* (9) *and that it is T-non resonant at infinity. Assume that it has an equilibrium point at the origin and that all the T-periodic solutions are T-non resonant. If the Maslov index $\mu_T(\infty)$ of the linearization at infinity* (10) *is different from the Maslov index $\mu_T(0)$ of the origin, then for every large prime p there is at least a sub-harmonic with minimal period pT. If such a sub-harmonic is pT-non resonant, then there must be a second sub-harmonic with minimal period pT.*

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