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Non-radial solutions for higher order Hénon-type equation with critical exponent

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Abstract. We consider the following polyharmonic equation with critical exponent

$$\begin{cases}(-\Delta)^m u = K(|y|)u^{m^*-1}, u > 0 \quad \text{in } B_1(0),\\ u \in \mathscr{D}_0^{m,2}(B_1(0)),\end{cases}$$

where $m > 0$ is a integer, $m^* =: \frac{2N}{N-2m}$, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geq 2m+4$, $K : [0,1] \to \mathbb{R}^N$ is a bounded function, $K'(1) > 0$ and $K''(1)$ exists. We prove a non-degeneracy result of the non-radial solutions constructed in Guo and Li (Calc Var PDEs 46(3–4):809–836, 2013) via the local Pohozaev identities for $N \geq 2m+4$. Then we apply the non-degeneracy result to obtain new existence of non-radial solutions for $N \geq 6m$.

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1. Introduction

In this paper, we consider the following polyharmonic equation with critical exponent:

$$\begin{cases}(-\Delta)^m u = K(|y|)u^{m^*-1}, u > 0 \quad \text{in } B_1(0),\\ u \in \mathscr{D}_0^{m,2}(B_1(0)),\end{cases} \tag{1.1}$$

where $m^* = \frac{2N}{N-2m}$ with $m > 0$ being a integer. $B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geq 2m+4$, $K : [0,1] \to \mathbb{R}$ is a bounded function, $K'(1) > 0$ and $K''(1)$ exists. $\mathscr{D}_0^{m,2}(B_1(0))$ denotes the closure of $C_0^\infty(B_1(0))$ with respect to the norm:

$$\|u\| = \begin{cases} |\Delta^{\frac{m}{2}} u|_2, & \text{if m is even,}\\ |\nabla\Delta^{\frac{m-1}{2}} u|_2, & \text{if m is odd,}\end{cases} \tag{1.2}$$

where $|\cdot|_2$ denotes the L^2 norm on $B_1(0)$.

When $m = 1$, $K(|y|) = |y|^\alpha$, $\alpha > 0$, problem (1.1) is reduced to the classical Hénon equation:

$$\begin{cases} -\Delta u = |y|^\alpha u^p, u > 0 & \text{in } B_1(0), \\ u = 0, & \text{on } \partial B_1(0). \end{cases} \tag{1.3}$$

Problem (1.3) was first introduced by Hénon in the study of astrophysics, see [18]. From the view point of mathematics, we are more interesting in the existence of solutions. In the subcritical case, that is $p < \frac{N+2}{N-2}$, the existence of a solution for the problem (1.3) can be proved easily by variational methods. For the critical case, that is, $p = \frac{N+2}{N-2}$, the loss of compactness from $H_0^1(B_1(0))$ to $L^{\frac{2N}{N-2}}(B_1(0))$ makes the problem getting more difficult to study. Ni [25] observed the influence of the non-autonomous term $|y|^\alpha$ and proved that it possesses a positive radial solution when $p \in (1, \frac{N+2+2\alpha}{N-2})$.

It is natural to ask whether (1.3) has a non-radial solution. When $N = 2$, Smets-Su-Willem [25] showed that the ground state solution is non-radial when α is large. When $N \geq 3$, Cao-Peng [7] considered the problem with $p = \frac{N+2}{N-2} - \epsilon$ for ϵ is small, they proved that the mountain pass solution is non-radial and blow up as $\epsilon \to 0$. For the critical case, that is $p = \frac{N+2}{N-2}$, using a variational method, Serra [24] proved that (1.3) has a non-radial solution for $N \geq 4$ and α is large. Later on, using reduction arguments, Wei-Yan [27] proved there exists infinitely many non-radial solutions for $N \geq 4$ and any $\alpha > 0$.

For more any other related results on Hénon equations in the cases of near critical or subcritical, we refer readers to [1,2,9,19,20,22,26] and the references therein.

In this paper, we are concerned with the higher order Hénon type equation. Indeed, the problem with higher order operators have long been of interest due to their application in conformal geometry and elastic mechanics. For example, the conformal covariant operator $P_4 (m = 2)$ was first introduced by Paneitz in 1983 when studying smooth 4-manifolds, and the application of P_4 was generalized to any $N-$manifold by Branson [6] in 1993. We point out that problems relating to polyharmonic operators present new challenges. We refer the reader to [3–5,8,11,12,23] and the references therein for more interesting results related to polyharmonic operators.

In particular, we see that Guo, Li and Li [14] proved that there are infinitely many nonradial solutions for (1.1). The aim of the present paper is two aspects: we first discuss the non-degeneracy of the bubble solution constructed in [14]. Then as an application we prove the existence of new type of non-radial solutions for Eq. (1.1). We would like to mention that the non-degeneracy of the solution is very important for the further study on the construction of new solutions or the existence of positive solution for the problem (1.1) without symmetry assumptions on the curvature function $K(y)$. Moreover, it will be also important in the study of the Morse index of the non-radial solution. We believe our method can be used to construct nonradial solutions for other elliptic problem with higher order operators. Before the statement of the main results, let us first introduce some notations.

It is well known (see [13]) that a family of positive solutions to the following problem

$$(-\Delta)^m u = u^{m^*-1}, \quad u > 0, \text{ in } \mathbb{R}^N, \quad u \text{ in } \mathscr{D}^{1,2}(\mathbb{R}^N), \tag{1.4}$$

are given by

$$\left\{ U_{x,\Lambda}(y) = P_{m,N}^{\frac{N-2m}{4m}} \frac{\Lambda^{\frac{N-2m}{2}}}{(1+\Lambda^2|y-x|^2)^{\frac{N-2m}{2}}} \;\middle|\; x \in \mathbb{R}^N, \Lambda > 0 \right\},$$

where $P_{m,N} = \Pi_{h=-m}^{m-1}(N+2h)$ is a constant, $\Lambda > 0$ is the scaling parameter and $x \in \mathbb{R}^N$. We call $U_{x,\Lambda}$ is single-bubble centered at the point x.

We define the scaling parameter $\mu_k := k^{\frac{N-2\,m+1}{N-2\,m}}, N \geq 2\,m+4$.

Define

$$H_s := \{u : u \text{ is even in } y_h, h = 2, \ldots, N, \text{ and } u(\bar{y}, y'') = u(e^{2\pi \frac{i}{k}} \bar{y}, y''),$$
$$\bar{y} \in \mathbb{R}^2, y'' \in \mathbb{R}^{N-2}\}.$$

Choose $\{x_j\}_{j=1}^k$ as the k vertices of the regular $k-$polygon inside $B_1(0)$, where

$$x_j = \left(r_k \cos\left(\frac{2(j-1)\pi}{k}\right), r_k \sin\left(\frac{2(j-1)\pi}{k}\right), \mathbf{0} \right), \tag{1.5}$$

$\mathbf{0} \in \mathbb{R}^{N-2}$, $r_k \in (1 - \frac{r_0}{k}, 1 - \frac{r_1}{k})$, $r_0 > r_1$ are positive constants.

For a function $u \in H_s \cap \mathscr{D}_0^{m,2}(B_1(0))$, we define the norm $\|u\|_*$ as follows:

$$\|u\|_* := \sup_{y \in B_1(0)} \left(\sum_{j=1}^k \frac{\mu_k^{\frac{N-2m}{2}}}{(1+\mu_k|y-x_j|)^{\frac{N-2m}{2}+\tau}} \right)^{-1} |u(y)|.$$

The nonradial solutions constructed in [14] are stated as following: let $PU_{x_j,\Lambda_k\mu_k}$ denote the solution of the following problem on $B_1(0)$,

$$\begin{cases} (-\Delta)^m (PU_{x_j,\Lambda_k\mu_k}) = U_{x_j,\Lambda_k\mu_k}^{m^*-1} \text{ in } B_1(0), \\ PU_{x_j,\Lambda_k\mu_k} \in \mathscr{D}_0^{m,2}(B_1(0)), \end{cases} \tag{1.6}$$

where $L_0 \leq \Lambda_k \leq L_1$, then we have

Theorem A. *Suppose $N \geq 2m+2$. If $K(1) > 0$, $K'(1) > 0$ and $K''(1)$ exists, then there exists an integer $k_0 > 0$ such that for any integer $k \geq k_0$ the boundary-value problem (1.1) has a solution*

$$u_k = \sum_{j=1}^k PU_{x_j,\Lambda_k\mu_k} + \omega_k,$$

where $\omega_k \in H_s$, $\|\omega_k\|_{L^\infty(B_1(0))} \to 0$ as $k \to \infty$, and $L_0 \leq \Lambda_k \leq L_1$ for some large constants $L_0, L_1 > 0$. In fact there exists an integer $k_0 > 0$ such that for each $k \geq k_0$,

$$\|\omega_k\|_* \leq C\left(\frac{1}{\mu_k}\right)^{\frac{1}{2}+\sigma},$$

where $\sigma > 0$.

Note that the solutions u_k has k bubbles located on the circle in $(y_1, y_2)-$plane with radius near 1. In fact we can also construct solution, for example, namely u_n which has n bubbles located on the circle in $(y_3, y_4)-$plane with radius near 1. One of the aims of the present paper is to get a new solution to (1.1) with main term $u_k + u_n$, where k and n are large integers. However, by careful analysis, we see that it is almost impossible to get the desired solutions with main term as $u_k + u_n$ by using variational method. In this paper, follow the idea in [16], we using a reduction arguments by gluing the n bubble solutions to the k bubble solutions. For this purpose, we need first to prove the non-degeneracy of the k bubble solutions u_k.

We denote the linear operator around u_k by

$$L_k\xi := (-\Delta)^m \xi - (m^* - 1)K(|y|)u_k^{m^*-2}\xi.$$

Our first result is the following.

Theorem 1.1. *Suppose that $K(1) > 0$, $K'(1) > 0$ and $K''(1)$ exists. If $N \geq 2m+4$, then there exists a large constant $K_0 > 0$, such that for any integer $k > K_0$, the positive bubble solution u_k obtained in Theorem A is non-degenerate in the sense that if $\xi \in H_s \cap \mathscr{D}_0^{m,2}(B_1(0))$ is a solution of the following equation:*

$$L_k\xi = 0, \text{ in } B_1(0),$$

then $\xi = 0$.

A direct consequence of Theorem 1.1 is the following.

Theorem 1.2. *Under the assumptions in Theorem 1.1 and $N \geq 6m$. Let u_k be the solution in Theorem A with a large fixed even number $k > 0$. Then there exists an integer $n_0 > 0$, depending on k, such that for any even number $n \geq n_0$, (1.1) has a solution with the form*

$$u_n = u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} + \omega_n,$$

where

$$\omega_n \in X_s \cap \mathscr{D}_0^{m,2}(B_1(0)), \quad \|\omega_n\|_{L^\infty(B_1(0))} \to 0,$$

$$p_{n,j} = (0, 0, t_n \cos\frac{2(j-1)\pi}{n}, t_n \sin\frac{2(j-1)\pi}{n}, 0) \in \mathbb{R}^N, \ j = 1, \dots, n,$$

and $\lambda_n \sim n^{\frac{N-2m+1}{N-2m}}$, $t_n \in (1 - \frac{L_0}{n}, 1 - \frac{L_1}{n})$, $L_0 > L_1 > 0$ are some constants, the definition of X_s is in (3.1).

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 by contraction arguments. We will construct a new type of bubbling solutions by reduction method in Sect. 3. And some important identities for polyharmonic operators and the estimates for the modified Green function are attached in Appendices. We believe the results obtained in this part are also independent of interesting and will be useful to other related problems involving polyharmonic operators.

2. The non-degeneracy of the solutions

In this section, we first establish a fine estimate on the k-bubbling solution u_k obtained in Theorem A. Then with the help of the local Pohozaev identities, we prove a non-degeneracy result by using a contradiction argument.

We introduce the following norms by:

$$||u||_* = \sup_{y\in B_1(0)} |u(y)|\bigg(\sum_{j=1}^{k} \frac{\mu_k^{\frac{N-2m}{2}}}{(1+\mu_k|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau}}\bigg)^{-1},$$

and

$$||f||_{**} = \sup_{y\in B_1(0)} |f(y)|\bigg(\sum_{j=1}^{k} \frac{\mu_k^{\frac{N+2m}{2}}}{(1+\mu_k|y-x_{k,j}|)^{\frac{N+2m}{2}+\tau}}\bigg)^{-1},$$

where $x_{k,j} = (r_k\cos\frac{2(j-1)\pi}{k}, r_k\sin\frac{2(j-1)\pi}{k}, 0)$, and τ is any fixed number in $(\frac{N-2m}{N-2m+1}, 1+\theta)$, $\theta>0$ is a small constant. Noting that $\mu_k = k^{\frac{N-2\,m+1}{N-2\,m}}$ and the choice of $x_{k,j}$ and τ, by definition we find

$$\sum_{j=2}^{k} \frac{1}{(\mu_k|x_{k,1}-x_{k,j}|)^{\tau}} \le \frac{Ck^{\tau}}{\mu_k^{\tau}}\sum_{j=1}^{k}\frac{1}{j^{\tau}} \le \frac{C_1 k}{\mu_k^{\tau}} \le C'.$$

Let

$$\Omega_j := \bigg\{y\in B_1(0) : y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2},\ \Big\langle \frac{(y',0)}{|y'|}, \frac{x_{k,j}}{|x_{k,j}|}\Big\rangle \ge \cos\frac{\pi}{k}\bigg\},$$
$$j=1,\dots,k.$$

First, we will need the following two results.

Lemma 2.1. *Assume $N\ge 2m+2$. Then, for any constant $\sigma\in(0,N-2m)$, there is a constant $C>0$ such that*

$$\int_{\mathbb{R}^N} \frac{dz}{|y-z|^{N-2m}(1+|z|)^{2m+\sigma}} \le \frac{C}{(1+|y|)^{\sigma}}.$$

Let $\tilde{P}U_{x_j,\Lambda_k}$ denote the solution of the following Dirichlet problem on $B_{\mu_k}(0)$:

$$\begin{cases} (-\Delta)^m(\tilde{P}U_{x_j,\Lambda_k}) = U_{x_j,\Lambda_k}^{m^*-1} \text{ in } B_{\mu_k}(0), \\ \tilde{P}U_{x_j,\Lambda_k}\in \mathscr{D}_0^{m,2}(B_{\mu_k}(0)). \end{cases} \tag{2.1}$$

Lemma 2.2. *Assume $N\ge 2m+2, \tau\in(0,2)$. Then there exists a small $\theta>0$ such that*

$$\int_{\mathbb{R}^N}\sum_{j=1}^{k}\frac{W_{r_k,\Lambda_k}^{\frac{4m}{N-2m}}(z)}{|y-z|^{N-2m}(1+|z-x_j|)^{\frac{N-2m}{2}+\tau}}dz\le C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2m}{2}+\tau+\theta}},$$

where $W_{r_k,\Lambda_k}(y)=\sum_{j=1}^{k}\tilde{P}U_{x_j,\Lambda_k}(y)$.

The proofs of the above two lemmas can be found in [13].

Lemma 2.3. *There exists a constant* $C>0$ *such that for all* $y\in B_1(0)$,

$$|u_k(y)|\le C\sum_{j=1}^{k}\frac{\mu_k^{\frac{N-2m}{2}}}{(1+\mu_k|y-x_{k,j}|)^{N-2m}}.$$

Proof. Let $\widehat{u}_k=\mu_k^{-\frac{N-2\,m}{2}}u_k(\mu_k^{-1}y)$, then

$$\begin{cases}(-\Delta)^m\widehat{u}_k=K(\mu_k^{-1}y)\widehat{u}_k^{m^*-1}, \text{ in } B_{\mu_k}(0),\\ \widehat{u}_k\in H_0^m(B_{\mu_k}(0)).\end{cases}\tag{2.2}$$

Denote $G(y,x)$ is the Green function of $(-\Delta)^m$ in $B_1(0)$ with Dirichlet boundary condition. By [17] we have

$$\widehat{u}_k(y)=\int_{B_{\mu_k}(0)}G(y,z)K(\mu_k^{-1}z)\widehat{u}_k^{m^*-1}(z)dz,\quad |G(y,z)|\le\frac{1}{|y-z|^{N-2m}}.$$

Recall that $u_k=\sum_{j=1}^{k}PU_{x_j,\mu_k}+\omega_k$ with $||\omega_k||_*=O\left(\frac{1}{\mu_k^{\frac{1}{2}+\sigma}}\right)$ for some $\sigma>0$, then

$$\begin{aligned}|\widehat{u}_k(y)|&\le C\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2m}}|\widehat{u}_k^{m^*-1}(z)|dz\\&\le C\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2m}}(\sum_{j=1}^{k}\frac{1}{(1+|z-\widehat{x}_{k,j}|)^{\frac{N-2m}{2}+\tau}})^{m^*-1}dz\\&\le C\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2m}}\sum_{j=1}^{k}\frac{1}{(1+|z-\widehat{x}_{k,j}|)^{\frac{N+2m}{2}+\tau+\frac{4(\tau-\tau_1)}{N-2m}}}\\&\qquad\left(\sum_{j=1}^{k}\frac{1}{(1+|z-\widehat{x}_{k,j}|)^{\tau_1}}\right)^{\frac{4m}{N-2m}}dz\\&\le C\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2m}}\sum_{j=1}^{k}\frac{1}{(1+|z-\widehat{x}_{k,j}|)^{\frac{N+2m}{2}+\tau+\frac{4(\tau-\tau_1)}{N-2m}}}dz\end{aligned}$$

where $\widehat{x}_{k,j}=\mu_k x_{k,j}$, and $\tau_1\in(\frac{N-2\,m}{N-2\,m+1},\tau)$. Noting that

$$\frac{N-2m}{2}+\tau+\frac{4(\tau-\tau_1)}{N-2m}>\frac{N-2m}{2}+\tau,$$

we can choose τ_1 such that $\frac{\frac{N-2m}{2}-\tau}{\frac{4(\tau-\tau_1)}{N-2m}}$ is not an integer. Let $\eta=\frac{4(\tau-\tau_1)}{N-2m}$ and $l=[\frac{\frac{N-2m}{2}-\tau}{\frac{4(\tau-\tau_1)}{N-2m}}]$, then we have

$$|\widehat{u}_k(y)|\le C\sum_{j=1}^{k}\frac{1}{(1+|y-\widehat{x}_{k,j}|)^{\frac{N-2m}{2}+\tau+\eta}}.$$

Continuing this process, we have

$$|\widehat{u}_k(y)|\le C\sum_{j=1}^{k}\frac{1}{(1+|y-\widehat{x}_{k,j}|)^{\frac{N-2m}{2}+\tau+\eta+\frac{4(\tau+\eta-\tau_2)}{N-2m}}},$$

where $\tau_2\in(\frac{N-2m}{N-2m+1},\tau+\eta)$. We can choose τ_2 such that $\frac{4(\tau+\eta-\tau_2)}{N-2m}=\eta$, then we have

$$|\widehat{u}_k(y)|\le C\sum_{j=1}^{k}\frac{1}{(1+|y-\widehat{x}_{k,j}|)^{\frac{N-2m}{2}+\tau+2\eta}}.$$

Repeating this process, we have

$$|\widehat{u}_k(y)|\le C\sum_{j=1}^{k}\frac{1}{(1+|y-\widehat{x}_{k,j}|)^{\frac{N-2m}{2}+\tau+l\eta}}.$$

So

$$\begin{aligned}|\widehat{u}_k(y)|&\le C\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2m}}\sum_{j=1}^{k}\frac{1}{(1+|z-\widehat{x}_{k,j}|)^{\frac{N+2m}{2}+\tau+l\eta+\eta}}dz\\&\le C\sum_{j=1}^{k}\frac{1}{(1+|y-\widehat{x}_{k,j}|)^{N-2m}}.\end{aligned}$$

□

We prove Theorem 1.1 by using contradiction arguments. Suppose that there are $k_n\to+\infty$, satisfying

$$\begin{cases}L_{k_n}\xi_n=0\\ \xi_n\in\mathscr{D}_0^{m,2}(B_1(0)),\end{cases}\tag{2.3}$$

but $\xi_n\ne0$. Without loss of generality, we may assume $||\xi_n||_*=1$ and obtain the contradictions by the following steps. Define

$$\widehat{\xi}_n(y)=(\Lambda_{k_n}\mu_{k_n})^{-\frac{N-2m}{2}}\xi_n((\Lambda_{k_n}\mu_{k_n})^{-1}y+x_{k_n,1}),\quad \Lambda_k\in[L_0,L_1]\tag{2.4}$$

Lemma 2.4. *It holds*

$$\widehat{\xi}_n\to b_0\Phi_0+b_1\Phi_1,\tag{2.5}$$

uniformly in $C^1(B_R(0))$ for any $R>0$, where b_0 and b_1 are some constants,

$$\Phi_0=\frac{\partial U_{0,\mu}}{\partial\mu}|_{\mu=1},\quad \Phi_i=\frac{\partial U_{0,1}}{\partial y_i},\ i=1,\dots,N.$$

Proof. In view of $|\widehat{\xi}_n| \le C$, we may assume that $\widehat{\xi}_n \to \xi$ in $C^m_{loc}(\mathbb{R}^N)$. Then ξ satisfies

$$(-\Delta)^m \xi = (m^*-1)U_{0,1}^{m^*-1}\xi \text{ in } \mathbb{R}^N, \tag{2.6}$$

which gives

$$\xi = \sum_{i=0}^{N} b_i \Phi_i. \tag{2.7}$$

Since ξ_n is even in y_i, $i = 2,\dots,N$, it holds $b_i = 0$, $i = 2,\dots,N$. The results follows. □

We decompose

$$\xi_n(y) = b_{0,n}\mu_{k_n}\sum_{j=1}^{k_n}\frac{\partial PU_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} - b_{1,n}\mu_{k_n}^{-1}\sum_{j=1}^{k_n}\frac{\partial PU_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r} + \xi_n^*, \tag{2.8}$$

where ξ_n^* satisfies

$$\int_{B_1(0)} U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\frac{\partial PU_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}}\xi_n^* = \int_{B_1(0)} U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\frac{\partial PU_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r}\xi_n^* = 0.$$

By the lemma 2.4 know that $b_{0,n}$ and $b_{1,n}$ are bounded.

Lemma 2.5. *It holds*

$$||\xi_n^*||_{**} \le C\mu_{k_n}^{-\frac{1}{2}-\sigma}, \tag{2.9}$$

where $\sigma > 0$ is a small constant.

Proof. Since $L_{k_n}\xi_n = 0$, we first calculate

$$\begin{aligned}
&L_{k_n}\xi_n^* \\
&\quad := (-\Delta)^m\xi_n^* - (m^*-1)K(|y|)u_{k_n}^{m^*-2}\xi_n^* \\
&\quad = (m^*-1)(K(|y|)-1)u_{k_n}^{m^*-2} \\
&\qquad \sum_{j=1}^{k_n}\left(b_{0,n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} - b_{1,n}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r}\right) \\
&\qquad + (m^*-1)\sum_{j=1}^{k_n}\left(u_{k_n}^{m^*-2} - U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\right) \\
&\qquad \left(b_{0,n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} - b_{1,n}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r}\right) \\
&\qquad - (m^*-1)K(|y|)u_{k_n}^{m^*-2} \\
&\qquad \sum_{j=1}^{k_n}\left(b_{0,n}\mu_{k_n}\frac{\partial \psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} - b_{1,n}\mu_{k_n}^{-1}\frac{\partial \psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r}\right)
\end{aligned}$$

$$= J_1 + J_2 + J_3,$$

where $\psi_{x,\lambda} := U_{x,\lambda} - PU_{x,\lambda}$.

In the following, we will estimate J_i, $i = 1, 2, 3$. Without loss of generality, we may assume $y \in \Omega_1$.

First, we have

$$\|J_1\|_{**} \leq \left\| (K(|y|) - 1) \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \Lambda_{k_n} \mu_{k_n}} \right)^{m^*-1} \right\|_{**}, \tag{2.10}$$

and

$$\left| (K(|y|) - 1) \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \Lambda_{k_n} \mu_{k_n}} \right)^{m^*-1} \right|$$
$$\leq C \frac{|(K(|y|) - 1)| \mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{N+2m}} + C \Big(\sum_{j=2}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{N-2m}} \Big)^{\frac{N+2m}{N-2m}}$$
$$:= I_1 + I_2.$$

Noting that $\frac{N-2m}{2} - \tau \frac{N-2m}{N+2m} > 1$, we have

$$\begin{aligned} |I_2| &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}} \left(\frac{k_n}{\mu_{k_n}} \right)^{\frac{N+2m}{2}-\tau} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}} \frac{1}{\mu_{k_n}^{\frac{1}{N-2m+1}(\frac{N+2m}{2}-\tau)}} \\ &\leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}. \end{aligned}$$

Now, we turn to consider I_1. We split the slice Ω_1 into two parts, namely,

$$\Omega_{11} := \{y \in \Omega_1 \mid ||y| - r_{k_n}| > \frac{\delta}{k_n}\}, \text{ and } \Omega_{12} := \{y \in \Omega_1 \mid ||y| - r_{k_n}| \leq \frac{\delta}{k_n}\},$$

where $\delta > 0$ is a fixed small constant.

In the region Ω_{11}, we have

$$|y - x_{k_n,1}| \geq ||y| - |x_{k_n,1}|| \geq \frac{\delta}{k_n},$$

which leads to

$$\begin{aligned} |I_1| &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}} \left(\frac{k_n}{\mu_{k_n}} \right)^{\frac{N+2m}{2}-\tau} \\ &\leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}. \end{aligned}$$

In the region Ω_{12}, noting that

$$||y|-1| \le ||y|-r_{k_n}|+|r_{k_n}-1| \le \frac{C}{k_n},$$

then

$$|K(|y|)-1| = O\left(\frac{1}{k_n}\right).$$

As a result,

$$|I_1| \le \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.$$

For the second term J_2, we denote

$$h_{n,j} := b_{0,n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} - b_{1,n}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r}.$$

Case 1, $N \ge 6m$, then $m^* \le 3$. We have

$$\begin{aligned}
&\left|\left(\left(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\right)^{m^*-2} - U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\right)h_{n,1}\right| \\
&\le CU^{m^*-2}_{x_{k_n,1},\mu_{k_n}} \sum_{j=2}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} \\
&\le C\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}} \sum_{j=2}^{k_n} \frac{1}{(\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N+2m}{2}-\tau}} \\
&\le \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned}$$

Case 2, $2m+4 \le N < 6m$, then $m^* > 3$. We have

$$\begin{aligned}
&\left|\left(\left(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\right)^{m^*-2} - U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\right)h_{n,1}\right| \\
&\le CU^{m^*-2}_{x_{k_n,1},\mu_{k_n}} \sum_{j=2}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} + CU_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\left(\sum_{j=2}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\right)^{m^*-2} \\
&\le \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned}$$

So for all $N \geq 2m+4$, we have

$$\Big|\Big(\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2} - U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\Big)h_{n,1}\Big| \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.$$

On the other hand, we have

$$\begin{aligned}
&\Big|\sum_{j=2}^{k_n}\Big(\Big(\sum_{i=1}^{k_n} U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2} - U^{m^*-2}_{x_{k_nk_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)h_{n,j}\Big| \\
&\leq C\sum_{j=2}^{k_n}\Big(U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}} + \Big(\sum_{i=2}^{k_n} U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\Big)U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} \\
&\leq CU^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\sum_{j=2}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} + C\left(\sum_{j=2}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\right)^{m^*-1} \\
&\leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned}$$

Combining the above two reuslts, we obtained

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k_n}\Big(\Big(\sum_{i=1}^{k_n} U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2} - U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)h_{n,j}\Big\|_{**} \\
&\quad \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}.
\end{aligned} \tag{2.11}$$

For $y \in \Omega_1$, we have

$$\Big|\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\sum_{j=2}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big| \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.$$

For $y \in B_{\frac{\delta}{k_n}}(x_{k_n,1})$, from Lemma A.1 we have

$$\psi_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}(y) = O\left(\frac{k_n^{N-2m}}{\mu_{k_n}^{\frac{N-2m}{2}}}\right),$$

which leads to

$$\begin{aligned}
&\Big|\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\psi_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\Big|\\
&\quad\le C\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\frac{k_n^{N-2m}}{\mu_{k_n}^{\frac{N-2m}{2}}}\\
&\quad\le C\left(U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}+\frac{\mu_{k_n}^2}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{4-\frac{4\eta}{N-2m}}}\Big(\frac{k_n}{\mu_{k_n}}\Big)^{\frac{4\eta}{N-2m}}\right)\frac{k_n^{N-2m}}{\mu_{k_n}^{\frac{N-2m}{2}}}\\
&\quad\le \frac{C\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N-2m}{2}-2-\tau}\Big(\frac{k_n}{\mu_{k_n}}\Big)^{N-2m}\\
&\qquad+\frac{C\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}\\
&\qquad(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N-2m}{2}-2-\tau+\frac{4\eta}{N-2m}}\Big(\frac{k_n}{\mu_{k_n}}\Big)^{N-2m+\frac{4\eta}{N-2m}}\\
&\quad\le \frac{C}{\mu_{k_n}^{\frac12+\sigma}}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}},
\end{aligned}$$

where $\eta>1$ is a constant.

For $y\in\Omega_1\cap B^c_{\frac{\delta}{k_n}}(x_{k_n,1})$,

$$\begin{aligned}
&\Big|\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\psi_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\Big|\\
&\quad\le C\left(U^{m^*-2}_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}+\frac{\mu_{k_n}^2}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{4-\frac{4\eta}{N-2m}}}\Big(\frac{k_n}{\mu_{k_n}}\Big)^{\frac{4\eta}{N-2m}}\right)U_{x_{k_n,1},\Lambda_{k_n}\mu_{k_n}}\\
&\quad\le \frac{C}{\mu_{k_n}^{\frac12+\sigma}}\frac{C\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned}$$

So we have proved

$$\Big\|\Big(\sum_{j=1}^{k_n} U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big\|_{**}\le\frac{C}{\mu_{k_n}^{\frac12+\sigma}}. \tag{2.12}$$

Moreover, for $N \geq 6m$, we have

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k_n}\Big(\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}-u_{k_n}^{m^*-2}\Big)h_{n,j}\Big\|_{**}\\
&\leq C\Big\|\sum_{j=1}^{k_n}\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-3}\\
&\quad\Big(\omega_{k_n}-\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big\|_{**}\\
&\leq C\Big\|\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\omega_{k_n}\Big\|_{**}\\
&\quad+\Big\|\Big(\sum_{j=1}^{k_n}U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big\|_{**}\\
&\leq C||\omega_{k_n}||_*+\frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}\\
&\leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}.
\end{aligned}$$

The penultimate term follows the fact that

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{N-2m}}\Bigg)^{\frac{4m}{N-2m}}\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}}\Bigg)\\
&\leq C\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{(N-2m-\eta)\frac{4m}{N-2m}+\frac{N-2m}{2}+\tau-\eta}}\\
&= C\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{\frac{N+2m}{2}+\tau+2m-\eta\frac{N+2m}{N-2m}}}\\
&\leq C\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N+2m}{2}+\tau}},
\end{aligned}$$

where $\eta=\frac{N-2m}{N-2m+1}$ and the last inequality follows from the fact that $2m-\eta\frac{N+2m}{N-2m}>0$.

As for $2m+4\leq N<6m$ similar to the proof of (2.12), we have

$$\Big\|\sum_{j=1}^{k_n}U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big(\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\Big\|_{**}\leq\frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}.$$

So

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k_n}\Big(\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}-u_{k_n}^{m^*-2}\Big)h_{n,j}\Big\|_{**}\\
&\le C\Big\|\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\omega_{k_n}\Big\|_{**}\\
&\quad+C\Big\|\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}(\omega_{k_n})^{m^*-2}\Big\|_{**}\\
&\quad+C\Big\|\sum_{j=1}^{k_n}U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big(\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\Big\|_{**} \qquad (2.13)\\
&\quad+C\Big\|\Big(\sum_{j=1}^{k_n}U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}\sum_{j=1}^{k_n}\psi_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}\Big\|_{**}\\
&\le C\Big(||\omega_{k_n}||_*+||\omega_{k_n}||_*^{m^*-2}+\mu_{k_n}^{-\frac12-\sigma}\Big)\\
&\le \frac{C}{\mu_{k_n}^{\frac12+\sigma}}.
\end{aligned}$$

The inequality (2.13) follows from

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{N-2m}}\Bigg)^{\frac{4m}{N-2m}}\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}}\Bigg)\\
&\le C\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N+2m}{2}+\tau}},
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{N-2m}}\Bigg)\Bigg(\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}}\Bigg)^{\frac{4m}{N-2m}}\\
&\le C\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,1}|)^{N-\frac{N+2m}{N-2m+1}}}\\
&\le C\sum_{j=1}^{k_n}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N+2m}{2}+\tau}},
\end{aligned}$$

since $N\ge 2m+4$. As a consequence, for $N\ge 2m+4$, we have proved

$$\Big\|\sum_{j=1}^{k_n}\Big(\Big(\sum_{i=1}^{k_n}U_{x_{k_n,i},\Lambda_{k_n}\mu_{k_n}}\Big)^{m^*-2}-u_{k_n}^{m^*-2}\Big)h_{n,j}||_{**}\le\frac{C}{\mu_{k_n}^{\frac12+\sigma}}. \qquad (2.14)$$

So combining (2.11) and (2.14), we have

$$||J_2||_{**} \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}. \tag{2.15}$$

Similar to the proof of (2.12), we can obtain

$$\|J_3\|_{**} \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}. \tag{2.16}$$

Combining (2.10), (2.15) and (2.16), we have

$$||L_{k_n}\xi_n^*||_{**} \leq \frac{C}{\mu_{k_n}^{\frac{1}{2}+\sigma}}.$$

On the other hand, from

$$\begin{aligned}&\int_{B_1(0)} U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} \frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial \mu_{k_n}} \xi_n^* \\ &\quad = \int_{B_1(0)} U^{m^*-2}_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}} \frac{\partial U_{x_{k_n,j},\Lambda_{k_n}\mu_{k_n}}}{\partial r} \xi_n^* = 0\end{aligned}$$

and Lemma 2.3, we can see that there exist $\rho > 0$, such that

$$||L_{k_n}\xi_n^*||_{**} \geq \rho||\xi_n^*||_*.$$

Thus, the result follows. □

As before $G(y, x)$ is the Green function of $(-\Delta)^m$ in $B_1(0)$ with Dirichlet boundary condiction.

Denote

$$\partial_j G(y,x) = \frac{\partial G}{\partial y_j}(y,x), \quad \nabla_i G(y,x) = \frac{\partial G}{\partial x_i}(y,x).$$

Then we have the following lemma.

Lemma 2.6. *For a small constant $\delta > 0$ fixed, we have $\forall\ y \in \partial B_{\frac{\delta}{k_n}}(x_{k_n,1})$,*

$$u_{k_n}(y) = \frac{A_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}} \sum_{j=1}^{k_n} G(y, x_{k_n,j}) + O\Big(\frac{1}{\mu_{k_n}^{1+\sigma-\frac{N-2m}{2}}}\Big), \tag{2.17}$$

$$\begin{aligned}\frac{(\partial)^r u_{k_n}(y)}{(\partial)^{i_1}_{y_1}\dots(\partial)^{i_N}_{y_N}} &= \frac{A_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}} \sum_{j=1}^{k_n} \frac{(\partial)^r G(y, x_{k_n,j})}{(\partial)^{i_1}_{y_1}\dots(\partial)^{i_N}_{y_N}} \\ &\quad + O\Big(\frac{k_n^r}{\mu_{k_n}^{1+\sigma-\frac{N-2m}{2}}}\Big),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\xi_n(y) &= b_{0,n}\frac{B_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}} \sum_{j=1}^{k_n} G(y, x_{k_n,j}) \\ &\quad + b_{1,n}\frac{C_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m+2}{2}}} \sum_{j=1}^{k_n}\Big(\cos\theta_j\nabla_1 G(y, x_{k_n,j}) + \sin\theta_j \nabla_2 G(y, x_{k_n,j})\Big) \\ &\quad + O\Big(\frac{1}{\mu_{k_n}^{1+\sigma-\frac{N-2m}{2}}}\Big),\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\frac{(\partial)^r\xi_n}{(\partial)^{i_1}_{y_1}\cdots(\partial)^{i_N}_{y_N}} &= b_{0,n}\frac{B_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}\sum_{j=1}^{k_n}\frac{(\partial)^r G(y,x_{x_{k_n,j}})}{(\partial)^{i_1}_{y_1}\ldots(\partial)^{i_N}_{y_N}} \\ &\quad + b_{1,n}\frac{C_N}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m+2}{2}}}\sum_{j=1}^{k_n}\partial_i\Bigg(\cos\theta_j\nabla_1\frac{(\partial)^r G(y,x_{k_n,j})}{(\partial)^{i_1}_{y_1}\ldots(\partial)^{i_N}_{y_N}}+ \\ &\quad \sin\theta_j\nabla_2\frac{(\partial)^r G(y,x_{k_n,j})}{(\partial)^{i_1}_{y_1}\ldots(\partial)^{i_N}_{y_N}}\Bigg) \\ &\quad + O\Bigg(\frac{k_n^r}{\mu_{k_n}^{1+\sigma-\frac{N-2m}{2}}}\Bigg),\end{aligned} \tag{2.20}$$

where

$$A_N=\int_{\mathbb{R}^N}U_{0,1}^{m^*-1}(x)dx,\ B_N=(m^*-1)\int_{\mathbb{R}^N}U_{0,1}^{m^*-2}(x)\Phi_0(x)dx,$$

and

$$C_N=(m^*-1)\int_{\mathbb{R}^N}U_{0,1}^{m^*-2}(x)\Phi_1(x)x_1dx,$$

$$i_1+\cdots+i_N=r.$$

Proof. First we give the proof of (2.17). We have

$$u_{k_n}(y)=\int_{B_1(0)}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx.$$

Without loss of generality, we assume $y\in\Omega_1$, divide the integral by areas

$$\begin{aligned}u_{k_n}(y)=&\int_{(B_d(y)\cup B_d(x_{k_n,1}))}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx \\ &+\sum_{j=2}^{k_n}\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx \\ &+\sum_{j=2}^{k_n}\int_{\Omega_j\backslash B_{\frac{\delta}{k_n}}(x_{k_n,j})}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx \\ &+\int_{\Omega_1\backslash(B_d(y)\cup B_d(x_{k_n,1}))}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx \\ :=&I_1+I_2+I_3+I_4,\end{aligned}$$

where $d=\frac{|y-x_{k_n,1}|}{2}$. We estimate one by one:

$$\begin{aligned}|I_4|\le& C\int_{\Omega_1\backslash(B_d(y)\cup B_d(x_{k_n,1}))}\frac{1}{|x-y|^{N-2m}}\Big(\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,1}|)^{N+2m}}\Big) \\ &+C\int_{\Omega_1\backslash(B_d(y)\cup B_d(x_{k_n,1}))}\frac{1}{|x-y|^{N-2m}}\sum_{j=2}^{k_n}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,j}|)^{N+2m}}\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{\Omega_1\setminus(B_d(y))\cup B_d(x_{k_n,1})}\frac{1}{|x-y|^{N-2m}}\frac{1}{\mu_{k_n}^{\frac{N+2m}{2}}|x-x_{k_n,1}|^{N+2m}}\\
&\quad+C\left(\frac{k_n}{\mu_{k_n}}\right)^{\eta}\int_{\Omega_1\setminus(B_d(y))\cup B_d(x_{k_n,1})}\frac{1}{|x-y|^{N-2m}}\frac{1}{\mu_{k_n}^{\frac{N+2m}{2}-\eta}|x-x_{k_n,1}|^{N+2m-\eta}}\\
&\leq\frac{C}{\mu_{k_n}^{\frac{N}{N-2m+1}-\frac{N-2m}{2}}},
\end{aligned}
$$

where we choose $\frac{N+2m}{N-2m}<\eta<N+2m$.

$$
\begin{aligned}
|I_3|&\leq C\sum_{j=2}^{k_n}\frac{1}{|x_{k_n,j}-x_{k_n,1}|^{N-2m}}\int_{\Omega_j\setminus B_{\frac{\delta}{k_n}}(x_{k_n,j})}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,j}|)^{N+2m}}\\
&\quad+C\sum_{j=2}^{k_n}\frac{1}{|x_{k_n,j}-x_{k_n,1}|^{N-2m}}\\
&\quad\int_{\Omega_j\setminus B_{\frac{\delta}{k_n}}(x_{k_n,j})}\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,j}|)^{N+2m-\eta}}(\frac{k_n}{\mu_{k_n}})^{\eta}\\
&\leq\frac{C}{\mu_{k_n}^{\frac{N}{N-2m+1}-\frac{N-2m}{2}}}.
\end{aligned}
$$

By Taylor expansion, for $x\in B_{\frac{\delta}{k_n}}(x_{k_n,j})$, we have

$$
\begin{aligned}
G(y,x)&=G(y,x_{k_n,j})+\sum_{i=1}^{N}\nabla_iG(y,x_{k_n,j})(x-x_{k_n,j})_i\\
&\quad+O\left(\frac{|x-x_{k_n,j}|^2}{|x_{k_n,1}-x_{k_n,j}|^{N-2m+2}}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})}G(y,x)u_{k_n}(x)^{m^*-1}dx\\
&=\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})}\left(G(y,x_{k_n,j})+\sum_{i=1}^{N}\nabla_iG(y,x_{k_n,j})(x-x_{k_n,j})_i\right)u_{k_n}(x)^{m^*-1}\\
&\quad+O\left(\frac{1}{|x_{k_n,1}-x_{k_n,j}|^{N-2m+2}}\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})}|x-x_{k_n,j}|^2u_{k_n}(x)^{m^*-1}\right)\\
&=G(y,x_{k_n,j})\left(\frac{1}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}\int_{\mathbb{R}^N}U_{0,1}^{m^*-1}dx+O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+\frac{2m}{N-2m+1}}}\right)\right)\\
&\quad+O\left(\sum_{i=1}^{N}|\nabla_iG(y,x_{k_n,j})|\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+\frac{2m}{N-2m+1}}}\frac{1}{k_n}\right)
\end{aligned}
$$

$$+O\left(\frac{1}{|x_{k_n,1}-x_{k_n,j}|^{N-2m+2}}\left(\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+2}}\right)\right).$$

On the other hand, we have

$$\left|\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})}G(y,x)(K(x)-1)u_{k_n}(x)^{m^*-1}dx\right|$$
$$=O\left(\frac{1}{|x_{k_n,1}-x_{k_n,j}|^{N-2m}}\frac{1}{k_n}\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}}}\right).$$

So

$$I_2=\sum_{j=2}^{k_n}\frac{1}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}G(y,x_{k_n,j})\int_{\mathbb{R}^N}U_{0,1}^{m^*-1}dx+O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m+2}{N-2m+1}-\frac{N-2m}{2}}}\right).$$

Now we compute I_1,

$$\int_{B_d(y)}G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx$$
$$\leq C\int_{B_d(y)}\frac{1}{|y-x|^{N-2m}}\left(\frac{\mu_{k_N}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,1}|)^{N+2m}}\right.$$
$$\left.+\frac{\mu_{k_N}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|x-x_{k_n,1}|)^{N+2m-\eta}}\frac{1}{\mu_{k_n}^{\frac{\eta}{N-2m+1}}}\right)$$
$$=O\left(\frac{1}{\mu_{k_n}^{\frac{N}{N-2m+1}-\frac{N-2m}{2}}}\right),$$

and

$$\int_{B_d(x_{k_n,1})}G(y,x)u_{k_n}(x)^{m^*-1}dx$$
$$=\int_{B_d(x_{k_n,1})}\left(G(y,x_{k_n,1})+\sum_{i=1}^{N}\nabla_iG(y,x_{k_n,1})(x-x_{k_n,1})_i\right)u_{k_n}(x)^{m^*-1}$$
$$+O\left(\int_{B_d(x_{k_n,1})}\frac{|x-x_{k_n,1}|^2}{|d|^N}u_{k_n}(x)^{m^*-1}\right)$$
$$=G(y,x_{k_n,1})\left(\frac{1}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}\int_{\mathbb{R}^N}U_{0,1}^{m^*-1}dx+O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+\frac{2m}{N-2m+1}}}\right)\right)$$
$$+O\left(\sum_{i=1}^{N}|\nabla_iG(y,x_{k_n,1})|\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+\frac{2m}{N-2m+1}}}\frac{1}{k_n}\right)$$
$$+O\left(\frac{1}{d^N}\left(\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}+2}}\right)\right).$$

By simple calculation,

$$\left|\int_{B_d(x_{k_n,1})} G(y,x)(K(x)-1)u_{k_n}(x)^{m^*-1}dx\right| = O\left(\frac{1}{d^{N-2m}}\frac{1}{k_n}\frac{1}{\mu_{k_n}^{\frac{N-2m}{2}}}\right).$$

So

$$I_1 = \frac{1}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}G(y,x_{k_n,1})\int_{\mathbb{R}^N} U_{0,1}^{m^*-1}dx + O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m+2}{N-2m+1}-\frac{N-2m}{2}}}\right),$$

then (2.17) is proved.

Similarly, by

$$\partial_j u_{k_n}(y) = \int_{B_1(0)} \partial_j G(y,x)K(x)u_{k_n}(x)^{m^*-1}dx,$$

we can have (2.18).

Now we compute ξ_n.

$$\begin{aligned}
\xi_n =& \int_{B_d(y)\cup B_d(x_{k_n,1})} (m^*-1)G(y,x)K(x)u_{k_n}(x)^{m^*-2}\xi_n dx \\
&+ \sum_{j=2}^{k_n}\int_{B_{\frac{\delta}{k_n}}(x_{k_n,j})} (m^*-1)G(y,x)K(x)u_{k_n}(x)^{m^*-2}\xi_n dx \\
&+ \sum_{j=2}^{k_n}\int_{\Omega_j\setminus B_{\frac{\delta}{k_n}}(x_{k_n,j})} (m^*-1)G(y,x)K(x)u_{k_n}(x)^{m^*-2}\xi_n dx \\
&+ \int_{\Omega_1\setminus(B_d(y)\cup B_d(x_{k_n,1}))} (m^*-1)G(y,x)K(x)u_{k_n}(x)^{m^*-2}\xi_n dx \\
=:& J_1+J_2+J_3+J_4,
\end{aligned}$$

Similar to the calculation for u_{k_n}, we have

$$\begin{aligned}
J_3 =& O\left(\frac{1}{\mu_{k_n}^{\frac{N}{N-2m+1}-\frac{N-2m}{2}}}\right), \quad J_4 = O\left(\frac{1}{\mu_{k_n}^{\frac{N}{N-2m+1}-\frac{N-2m}{2}}}\right), \\
J_2 =& \frac{b_{0,n}(m^*-1)}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}}\sum_{j=2}^{k_n} G(y,x_{k_n,j})\int_{\mathbb{R}^N} U_{0,1}^{m^*-1}dx \\
&+ \frac{b_{1,n}(m^*-1)}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m+2}{2}}}\sum_{j=2}^{k_n}\Big(\cos\theta_j\nabla_1 G(y,x_{k_n,j}) + \sin\theta_j \nabla_2 G(y,x_{k_n,j})\Big) \\
&\int_{\mathbb{R}^N} U_{0,1}^{m^*-2}\Phi_1 x_1 dx \\
&+ O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m+2}{N-2m+1}-\frac{N-2m}{2}}}\right).
\end{aligned}$$

$$
\begin{aligned}
J_1 =& \frac{b_{0,n}(m^*-1)}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m}{2}}} G(y, x_{k_n,1}) \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} dx \\
&+ \frac{b_{1,n}(m^*-1)}{(\Lambda_{k_n}\mu_{k_n})^{\frac{N-2m+2}{2}}} \nabla_1 G(y, x_{k_n,1}) \\
&\int_{\mathbb{R}^N} U_{0,1}^{m^*-2} \Phi_1 x_1 dx + O\left(\frac{1}{\mu_{k_n}^{\frac{N-2m+2}{N-2m+1}-\frac{N-2m}{2}}}\right),
\end{aligned}
$$

where $\theta_j = \frac{2(j-1)\pi}{k_n}$, $j = 1, \ldots, k_n$.

Combining the estimates of J_1, J_2, J_3, J_4, for the same reason, (2.19) follows.

Noting that

$$
\frac{(\partial)^r \xi_n}{(\partial)_{y_1}^{i_1} \ldots (\partial)_{y_N}^{i_N}} = \int_{B_1(0)} \frac{(\partial)^r G(y,x)}{(\partial)_{y_1}^{i_1} \ldots (\partial)_{y_N}^{i_N}} (m^*-1) K(x) u_{k_n}^{m^*-2}(x) \xi_n(x) dx,
$$

(2.20) can be proved similarly. □

Lemma 2.7. $\widehat{\xi}_n \to 0$ *uniformly in* $C^1(B_R(0))$ *for any* $R > 0$.

Proof. Step1: First we prove $b_{0,n} \to 0$.

When m is even: We apply the first Pohozaev identity in lemma B.1 with $B = B_{\frac{\delta}{k_n}}(x_{k_n,1})$ then we have

$$
\begin{aligned}
&\int_{B_{\frac{\delta}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \frac{\partial K(|y|)}{\partial y_1} \\
&= \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} K(|y|) u_{k_n}^{m^*-1} \xi_n \nu_1 - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{\frac{m}{2}} u_{k_n} \Delta^{\frac{m}{2}} \xi \nu_1 \\
&\quad + \sum_{i=1}^{\frac{m}{2}} \Bigg(\int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i} u_{k_n} \frac{\partial^2 \Delta^{i-1} \xi_n}{\partial y_1 \partial \nu} \\
&\quad - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial \Delta^{m-i} u_{k_n}}{\partial \nu} \frac{\partial \Delta^{i-1} \xi_n}{\partial y_1} \\
&\quad + \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i} \xi_n \frac{\partial^2 \Delta^{i-1} u_{k_n}}{\partial y_1 \partial \nu} \\
&\quad - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial \Delta^{m-i} \xi_n}{\partial \nu} \frac{\partial \Delta^{i-1} u_{k_n}}{\partial y_1} \Bigg).
\end{aligned}
\tag{2.21}
$$

Note

$$I_{1,1}(u,v,d) := -\int_{\partial B_d(x_{k_n,1})} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} v \nu_1$$
$$+ \sum_{i=1}^{\frac{m}{2}} \Big(\int_{\partial B_d(x_{k_n,1})} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} v}{\partial y_1 \partial \nu} - \int_{\partial B_d(x_{k_n,1})} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} v}{\partial y_1}$$
$$+ \int_{\partial B_d(x_{k_n,1})} \Delta^{m-i} v \frac{\partial^2 \Delta^{i-1} u}{\partial y_1 \partial \nu} - \int_{\partial B_d(x_{k_n,1})} \frac{\partial \Delta^{m-i} v}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_1} \Big),$$

we have

$$\int_{B_{\frac{\delta}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \frac{\partial K(|y|)}{\partial y_1} = O\left(\frac{1}{\mu_{k_n}^{\sigma}}\right),$$

and

$$\int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} K(|y|) u_{k_n}^{m^*-1} \xi_n \nu_1 = O\left(\frac{1}{\mu_{k_n}^{\frac{2m}{N-2m+1}}}\right).$$

By Lemmas 2.6 and B.3 we have:

$$\begin{aligned}
& I_{1,1}(u_{k_n}, \xi_n, \frac{\delta}{k_n}) \\
&= \frac{A_N B_N b_{0,n}}{(\Lambda_{k_n}\mu_{k_n})^{N-2m}} \Big(I_{1,1}\big(G(y,x_{k_n,1}), G(y,x_{k_n,1}), \frac{\delta}{k_n}\big) + 2 I_{1,1}\big(G(y,x_{k_n,1}), \\
&\quad \sum_{j=2}^{k_n} G(y,x_{k_n,j}), \frac{\delta}{k_n}\big)\Big) + O\left(\frac{1}{\mu_{k_n}^{\sigma}}\right) \\
&= \frac{2A_N B_N b_{0,n}}{(\Lambda_{k_n}\mu_{k_n})^{N-2m}} \Big(-\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1}) + \sum_{j=2}^{k_n} \frac{\partial G}{\partial y_1}(x_{k_n,1}, x_{k_n,j}) \Big) \\
&\quad + O\left(\frac{1}{\mu_{k_n}^{\sigma}}\right) \\
&= \frac{(N-2m) A_N B_N b_{0,n} K'(1)}{N} \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*}}{\int_{\mathbb{R}^N} U_{0,1}^{m^*-1}} + O\left(\frac{1}{\mu_{k_n}^{\sigma}}\right),
\end{aligned}$$

the last equal comes from Proposition 3.1 of [14], which says

$$\begin{aligned}
& \frac{1}{(\Lambda_{k_n}\mu_{k_n})^{N-2m}} \Big(\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1}) - \sum_{j=2}^{k_n} \frac{\partial G}{\partial y_1}(x_{k_n,1}, x_{k_n,1}) \Big) \\
&\quad = -\frac{(N-2m)K'(1)}{2N} \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*}}{(\int_{\mathbb{R}^N} U_{0,1}^{m^*-1})^2} + O(\mu_{k_n}^{-\sigma}).
\end{aligned} \tag{2.22}$$

Thus by (2.21) we have $b_{0,n} \to 0$, as for m is odd, we have same result.

Step2:

Now we prove $b_{1,n} \to 0$. We apply the second Pohozaev identity of Lemma B.1

$$\begin{aligned}
&\int_{B_{\frac{\delta}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1}\xi_n\langle\nabla K(|y|), y-x_{x_{k_n},1}\rangle \\
&= \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} K(|y|)u_{k_n}^{m^*-1}\xi_n\langle\nu, y-x_{x_{k_n},1}\rangle \\
&\quad - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{\frac{m}{2}}u_{k_n}\Delta^{\frac{m}{2}}\xi_n\langle\nu, y-x_{x_{k_n},1}\rangle \\
&\quad - \sum_{i=1}^{\frac{m}{2}}\Bigg(\int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial\Delta^{m-i}u_{k_n}}{\partial\nu}\Delta^{i-1}\langle\nabla\xi_n, y-x_{x_{k_n},1}\rangle \\
&\quad + \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i}u_{k_n}\frac{\partial\Delta^{i-1}\langle\nabla\xi_n, y-x_{x_{k_n},1}\rangle}{\partial\nu} \\
&\quad - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial\Delta^{m-i}\xi_n}{\partial\nu}\Delta^{i-1}\langle\nabla u_{k_n}, y-x_{x_{k_n},1}\rangle \\
&\quad + \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i}\xi_n\frac{\partial\Delta^{i-1}\langle\nabla u_{k_n}, y-x_{x_{k_n},1}\rangle}{\partial\nu}\Bigg) \\
&\quad + \frac{N-2m}{2}\sum_{i=1}^{\frac{m}{2}}\Bigg(\int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i}u_{k_n}\frac{\partial\Delta^{i-1}\xi_n}{\partial\nu} \\
&\quad + \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \Delta^{m-i}\xi_n\frac{\partial\Delta^{i-1}u_{k_n}}{\partial\nu} \\
&\quad - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial\Delta^{m-i}u_{k_n}}{\partial\nu}\Delta^{i-1}\xi_n - \int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})} \frac{\partial\Delta^{m-i}\xi_n}{\partial\nu}\Delta^{i-1}u_{k_n}\Bigg).
\end{aligned} \tag{2.23}$$

We note:

$$\begin{aligned}
&I_{2,1}(u,v,d) \\
&\quad := -\int_{\partial B_d(x_{k_n,1})} \Delta^{\frac{m}{2}}u\Delta^{\frac{m}{2}}v\langle\nu, y-x_{x_{k_n},1}\rangle \\
&\qquad - \sum_{i=1}^{\frac{m}{2}}\Bigg(\int_{\partial B_d(x_{k_n,1})} \frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\langle\nabla v, y-x_{x_{k_n},1}\rangle \\
&\qquad + \int_{\partial B_d(x_{k_n,1})} \Delta^{m-i}u\frac{\partial\Delta^{i-1}\langle\nabla v, y-x_{x_{k_n},1}\rangle}{\partial\nu} \\
&\qquad - \int_{\partial B_d(x_{k_n,1})} \frac{\partial\Delta^{m-i}v}{\partial\nu}\Delta^{i-1}\langle\nabla u, y-x_{x_{k_n},1}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial B_d(x_{k_n,1})}\Delta^{m-i}v\frac{\partial\Delta^{i-1}\langle\nabla u,y-x_{x_{k_n},1}\rangle}{\partial\nu}\Bigg)\\
&+\frac{N-2m}{2}\sum_{i=1}^{\frac{m}{2}}\Bigg(\int_{\partial B_d(x_{k_n,1})}\Delta^{m-i}u\frac{\partial\Delta^{i-1}v}{\partial\nu}\\
&+\int_{\partial B_d(x_{k_n,1})}\Delta^{m-i}v\frac{\partial\Delta^{i-1}u}{\partial\nu}\\
&-\int_{\partial B_d(x_{k_n,1})}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}v-\int_{\partial B_d(x_{k_n,1})}\frac{\partial\Delta^{m-i}v}{\partial\nu}\Delta^{i-1}u\Bigg).
\end{aligned}
$$

We estimate both sides of the equation separately: direct computation shows that

$$
\begin{aligned}
&\int_{B_{\frac{\delta}{k_n}}(x_{k_n,1})}u_{k_n}^{m^*-1}\xi_n\langle\nabla K(|y|),y-x_{x_{k_n},1}\rangle\\
&\quad=\frac{b_{1,n}K'(1)}{\Lambda_{k_n}\mu_{k_n}}\int_{\mathbb{R}^N}U_{0,1}^{m^*-1}\Phi_1 y_1+O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right).
\end{aligned}
\tag{2.24}
$$

Let’s estimate the right-hand side of the equation. A direct calculation leads to

$$
\int_{\partial B_{\frac{\delta}{k_n}}(x_{k_n,1})}K(|y|)u_{k_n}^{m^*-1}\xi_n\langle\nu,y-x_{x_{k_n},1}\rangle=O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right).
$$

According to Lemma B.3 we have

$$
\begin{aligned}
&I_{2,1}\Bigg(\sum_{j=1}^{k_n}G(y,x_{k_n,j}),\sum_{j=1}^{k_n}G(y,x_{k_n,j}),\frac{\delta}{k_n}\Bigg)\\
&\quad=I_{2,1}\Bigg(G(y,x_{k_n,1}),G(y,x_{k_n,1}),\frac{\delta}{k_n}\Bigg)\\
&\qquad+2I_{2,1}\Bigg(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}G(y,x_{k_n,j}),\frac{\delta}{k_n}\Bigg)\\
&\qquad+I_{2,1}\Bigg(\sum_{j=2}^{k_n}G(y,x_{k_n,j}),\sum_{j=2}^{k_n}G(y,x_{k_n,j})\frac{\delta}{k_n}\Bigg)\\
&\quad=(N-2m)\Bigg(H(x_{k_n,1},x_{k_n,1})-\sum_{j=2}^{k_n}G(x_{k_n,1},x_{k_n,j})\Bigg).
\end{aligned}
\tag{2.25}
$$

From Proposition 3.1 in [14] we have

$$
\frac{1}{\Lambda_{k_n}^{N-2m}\mu_{k_n}^{N-2m}}\Bigg(H(x_{k_n,1},x_{k_n,1})-\sum_{j=2}^{k_n}G(x_{k_n,1},x_{k_n,j})\Bigg)=O\bigg(\frac{1}{\mu_{k_n}^{1+\sigma}}\bigg),
$$

which leads to

$$I_{2,1}\Big(\sum_{j=1}^{k_n} G(y, x_{k_n,j}), \sum_{j=1}^{k_n} G(y, x_{k_n,j}), \frac{\delta}{k_n}\Big) = O\left(\frac{1}{\mu_{k_n}^{-N+2m+1+\sigma}}\right), \quad (2.26)$$

and

$$\begin{aligned} & I_{2,1}\Big(\sum_{j=1}^{k_n} G(y, x_{k_n,j}), \sum_{j=1}^{k_n} \cos\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}) + \sin\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}), \frac{\delta}{k_n}\Big) \\ &= I_{2,1}\Big(G(y, x_{k_n,1}), \frac{\partial G}{\partial x_1}(y, x_{k_n,j}), \frac{\delta}{k_n}\Big) \\ &\quad + I_{2,1}\Big(G(y, x_{k_n,1}), \sum_{j=2}^{k_n} \cos\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}) + \sin\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}), \frac{\delta}{k_n}\Big) \quad (2.27) \\ &\quad + I_{2,1}\Big(\sum_{j=2}^{k_n} G(y, x_{k_n,j}), \frac{\partial G}{\partial x_1}(y, x_{k_n,1}), \frac{\delta}{k_n}\Big) \\ &= (N-2m+1)\Big(\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1}) - \sum_{j=2}^{k_n} \frac{\partial G}{\partial y_1}(x_{k_n,1}, x_{k_n,1})\Big). \end{aligned}$$

By (2.22) we get

$$\begin{aligned} & I_{2,1}\Big(\sum_{j=1}^{k_n} G(y, x_{k_n,j}), \sum_{j=1}^{k_n} \cos\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}) + \sin\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}), \frac{\delta}{k_n}\Big) \\ &= \frac{(N-2m+1)(N-2m)K'(1)}{2N} \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*}}{(\int_{\mathbb{R}^N} U_{0,1}^{m^*-1})^2} (\Lambda_{k_n}\mu_{k_n})^{N-2m} + O(\mu_{k_n}^{N-2m-\sigma}). \end{aligned} \quad (2.28)$$

Combining (2.17), (2.19), (2.26) and (2.28) we have

$$\begin{aligned} & I_{2,1}(u_{k_n}, \xi_n, \frac{\delta}{k_n}) \\ &= \frac{A_N B_N b_{0,n}}{(\Lambda_{k_n}\mu_{k_n})^{N-2m}} I_{2,1}\left(\sum_{j=1}^{k_n} G(y, x_{k_n,j}), \sum_{j=1}^{k_n} G(y, x_{k_n,j}), \frac{\delta}{k_n}\right) \\ &\quad + \frac{A_N C_N b_{1,n}}{(\Lambda_{k_n}\mu_{k_n})^{N-2m+1}} I_{2,1}\left(\sum_{j=1}^{k_n} G(y, x_{k_n,j}), \sum_{j=1}^{k_n} \cos\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j})\right. \quad (2.29) \\ &\quad \left. + \sin\theta_j \frac{\partial G}{\partial x_1}(y, x_{k_n,j}), \frac{\delta}{k_n}\right) + O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right) \\ &= \frac{A_N C_N (N-2m+1)(N-2m)K'(1)}{\Lambda\mu_{k_n} N} \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*}}{(\int_{\mathbb{R}^N} U_{0,1}^{m^*-1})^2} b_{1,n} + O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right). \end{aligned}$$

Combining (2.23), (2.24) and (2.29):

$$\begin{aligned} \Big[\int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \Phi_1 y_1\Big] b_{1,n} &= \Big[\frac{C_n(N-2m+1)(N-2m)}{N} \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*}}{\int_{\mathbb{R}^N} U_{0,1}^{m^*-1}}\Big] b_{1,n} \\ &\quad + O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right). \end{aligned}$$

Compare the coefficients of $b_{1,n}$ on both sides of the equation where

$$C_N = (m^* - 1)\int_{\mathbb{R}^N} U_{0,1}^{m^*-2}\Phi_1 y_1,$$

we will need this classical integration

$$\int_0^{+\infty} \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{1}{2}\frac{\Gamma(-\frac{1}{2}+\beta-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\alpha+\frac{1}{2})}{\Gamma(\beta)}, \quad 2\beta-\alpha>1$$

Then the ratio of coefficients will be

$$\begin{aligned}&\frac{(N+2m)(N-2m+1)\int_{\mathbb{R}^N} U_{0,1}^{m^*-2}\Phi_1 y_1 \int_{\mathbb{R}^N} U_{0,1}^{m^*}}{\int_{\mathbb{R}^N} U_{0,1}^{m^*-1}\Phi_1 y_1 \int_{\mathbb{R}^N} U_{0,1}^{m^*-1}}\\ &= 2(N-2m+1)\\ &\neq 1.\end{aligned}$$

since $Q_{M,n}$ isn't a transcendental number, where $\pi^{\frac{N}{2}}$ is. When m is odd we can get almost the same expression, the ratio of coefficients will become

$$P_{M,N}\pi^{\frac{N}{2}-\frac{1}{2}} \neq 1$$

where $P_{M,N}$ isn't a transcendental number. Then $b_{1,n} \to 0$. □

Now, we give the proof of Theorem 1.1.

Proof. With the aid of the above lemmas, it is sufficient to get a contradiction with $||\xi_n||_* = 1$.

In fact, we have

$$|\xi_n(y)| \le (m^*-1)\int_{B_1(0)} \frac{1}{|z-y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z)\xi_n(z)dz, \tag{2.30}$$

and

$$\begin{aligned}&|\int_{B_1(0)} \frac{1}{|z-y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z)\xi_n(z)dz|\\ &\le C||\xi_n||_* \int_{B_1(0)} \frac{1}{|z-y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z)\\ &\quad \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|z-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}} dz\\ &\le C||\xi_n||_* \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+\theta}},\end{aligned}$$

for some $\theta > 0$. So we obtain

$$\frac{|\xi_n(y)|}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau}}} \le C||\xi_n||_* \frac{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+\theta}}}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}}}.$$

Since $\xi_n \to 0$ in $B_{R(\Lambda_{k_n}\mu_{k_n})^{-1}}(x_{k_m,j})$ and $||\xi_n||_* = 1$, we know that

$$\frac{|\xi_n(y)|}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k_n,j}|)^{\frac{N-2m}{2}+\tau}}}$$

attains its maximum in $\mathbb{R}^N \setminus \cup_{j=1}^{k_n} B_{R(\Lambda_{k_n}\mu_{k_n})^{-1}}(x_{k_n,j})$. Thus

$$||\xi_n||_* \le o(1)||\xi_n||_*.$$

So $||\xi_n||_* \to 0$ as $n \to +\infty$. This is a contradiction to $||\xi_n||_* = 1$. $\square$

3. Construction of new bubbling solution

Let u_k be the k-bubbling solution in Theorem A with a large even integer $k > 0$. Then u_k is even in each component y_i, $i = 1, \dots, N$ and u_k is radial in $y'' = (y_3, \dots, y_N)$.

Let $n \ge k$ be a large even integer. Let

$$p_{n,j} = (0, 0, t_n \cos \frac{2(j-1)\pi}{n}, t_n \sin \frac{2(j-1)\pi}{n}, 0) \in \mathbb{R}^N,\ j = 1, \dots, n,$$

where $t_n \in (1 - \frac{C_1}{n}, 1 - \frac{C_2}{n})$, for some constants $C_1 > C_2 > 0$.

Define

$$\begin{aligned} X_s = \{u : u \in H_s, u \text{ is even in } y_h,\ h = 1, \dots, N,\\ u(y_1, y_2, t\cos(\theta + \frac{2\pi j}{n}), t\sin(\theta + \frac{2\pi j}{n}), y^*) = u(y_1, y_2, t\cos\theta, t\sin\theta, y^*)\}, \end{aligned} \tag{3.1}$$

where $y^* = (y_5, \dots, y_N)$.

In this section, we are devoted to construct a solution of (1.1) with the form

$$u_n = u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} + \omega_n,$$

where $\omega_n \in X_s \cap \mathscr{D}_0^{m,2}(B_1(0))$ is a small perturbed term.

We first introduce the weighted norms:

$$\|u\|_{*,n} = \sup_{y \in B_1(0)} \left(\sum_{j=1}^{n} \frac{\lambda_n^{\frac{N-2m}{2}}}{(1 + \lambda_n |y - p_{n,j}|)^{\frac{N-2m}{2}+\tau}} \right)^{-1} |u(y)|,$$

and

$$\|f\|_{**,n} = \sup_{y \in B_1(0)} \left(\sum_{j=1}^{n} \frac{\lambda_n^{\frac{N+2m}{2}}}{(1 + \lambda_n |y - p_{n,j}|)^{\frac{N+2m}{2}+\tau}} \right)^{-1} |f(y)|,$$

where τ is any fixed number satisfying $\frac{N-2\,m}{N-2\,m+1} < \tau < 1 + \eta$, $\eta > 0$ is a small constant.

Let

$$Z_{j,1} = \frac{\partial PU_{p_{n,j},\lambda_n}}{\partial t_n},\ Z_{j,2} = \frac{\partial PU_{p_{n,j},\lambda_n}}{\partial \lambda_n},\ j = 1, \dots, n.$$

Consider the following linearized problem of (1.1) around $u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}$:

$$\begin{cases} (-\Delta)^m \omega_n - (m^*-1)K(|y|)(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n})^{m^*-2}\omega_n \\ = h_n + \sum_{i=1}^{2} a_{n,i} \sum_{j=1}^{n} U^{m^*-2}_{p_{n,j},\lambda_n} Z_{j,i}, \text{ in } B_1(0), \\ \omega_n \in X_s \cap \mathscr{D}^{m,2}_0(B_1(0)), \quad \int_{B_1(0)} U^{m^*-2}_{p_{n,j},\lambda_n} Z_{j,i}\omega_n = 0, \quad i=1,2, \quad j=1,\dots,n, \end{cases} \tag{3.2}$$

for some numbers $a_{n,i}$, depending on ω_n.

Lemma 3.1. *Assume ω_n solves the problem (3.2) for $h = h_n$. If $\|h_n\|_{**,n} \to 0$ as $n \to +\infty$, so does $\|\omega_n\|_{*,n}$.*

Proof. We argue by contradiction. Suppose that there exist $p_{n,j}$, λ_n, h_n, ω_n satisfying (3.2), $\lambda_n \to +\infty$, $\|h_n\|_{**,n} \to 0$ and $\|\omega_n\|_{*,n} \geq c_0 > 0$. Without loss of generality, we may assume $\|\omega_n\|_{*,n} = 1$.

We write

$$\begin{aligned} L_k\omega_n &= (m^*-1)K(|y|)\big[(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n})^{m^*-2} - u_k^{m^*-2}\big]\omega_n + h_n \\ &\quad + \sum_{i=1}^{2} a_{n,i} \sum_{j=1}^{n} U^{m^*-2}_{p_{n,j},\lambda_n} Z_{j,i}. \end{aligned}$$

Then by Proposition C.1, we have

$$\begin{aligned} &|\omega_n(x)| \\ &\leq C\int_{B_1(0)} |G_k(y,x)| \Bigg(\Big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} \Big)^{m^*-2} |\omega_n| + |h_n| \\ &\quad + \Big| \sum_{i=1}^{2} a_{n,i} \sum_{j=1}^{n} U^{m^*-2}_{p_{n,j},\lambda_n} Z_{j,i} \Big| \Bigg) dy \\ &\leq C\int_{B_1(0)} \frac{1}{|y-x|^{N-2m}} \Bigg(\Big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} \Big)^{m^*-2} |\omega_n| + |h_n| \\ &\quad + \Big| \sum_{i=1}^{2} a_{n,i} \sum_{j=1}^{n} U^{m^*-2}_{p_{n,j},\lambda_n} Z_{j,i} \Big| \Bigg) dy. \end{aligned}$$

Then, similar to the computation in Proposition 2.3 in [14], we can obtain

$$\begin{aligned} &\left(\sum_{j=1}^{n} \frac{\lambda_n^{\frac{N-2m}{2}}}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N-2m}{2}+\tau}} \right)^{-1} |\omega_n(y)| \\ &\leq C\left(\|h_n\|_{**,n} + \frac{\sum_{j=1}^{n} \frac{1}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N-2m}{2}+\tau+\theta}}}{\sum_{j=1}^{n} \frac{1}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N-2m}{2}+\tau}}} \|\omega_n\|_{*,n} \right), \end{aligned} \tag{3.3}$$

for some $\theta > 0$ small enough.

From (3.3) and $\|\omega_n\|_{*,n} = 1$, we obtain that there exists $L > 0$ large enough such that

$$||\lambda_n^{-\frac{N-2m}{2}} \omega_n||_{L^\infty(B_{\frac{L}{\lambda_n}}(p_{n,i}))} \geq c_0 > 0, \tag{3.4}$$

for some i. Furthermore, the dilation $\tilde{\omega}_n(y) = \lambda_n^{-\frac{N-2m}{2}} \omega_n(\lambda_n^{-1} y + p_{n,i})$ converges uniformly on any compact set to a solution u of the following equation:

$$(-\Delta)^m u - (m^* - 1) U_{0,1}^{m^*-2} u = 0, \text{ in } \mathbb{R}^N.$$

On the other hand, noting the orthogonality in (3.2), we have that u is perpendicular to the kernel of this equation. As a result, $u = 0$, which is a contradiction to (3.4). □

With the help of Lemma 3.1, similar to Proposition 4.1 in [10], we have the following proposition.

Proposition 3.2. *There exist $n_0 > 0$ and a constant $C > 0$, independent of n, such that for all $n \geq n_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (3.2) has a unique solution $\omega = \mathcal{L}_n(h)$ with $\|\mathcal{L}_n(h)\|_{*,n} \leq C\|h\|_{**,n}$, $|a_{n,i}| \leq \frac{C}{\lambda_n^{\bar{n}_i}} \|h\|_{**,n}$, $i = 1, 2$, where $\bar{n}_1 = 1, \bar{n}_2 = -1$.*

Now we consider the following perturbed problem of (1.1):

$$\begin{cases} (-\Delta)^m \Big(u_k + \sum_{j=1}^n PU_{p_{n,j},\lambda_n} + \omega_n \Big) \\ = K(|y|) \Big(u_k + \sum_{j=1}^n PU_{p_{n,j},\lambda_n} + \omega_n \Big)^{m^*-1} + \sum_{i=1}^2 a_{n,i} \sum_{j=1}^n U_{p_{n,j},\lambda_n}^{m^*-2} Z_{j,i}, \text{ in } B_1(0), \\ \omega_n \in X_s \cap \mathscr{D}_0^{m,2}(B_1(0)), \quad \int_{B_1(0)} U_{p_{n,j},\lambda_n}^{m^*-2} Z_{j,i} \omega_n = 0, \quad i = 1, 2, \quad j = 1, \ldots, n, \end{cases} \tag{3.5}$$

for some numbers $a_{n,i}$.

Noting that the problem (3.5) can be rewritten as

$$\begin{cases} (-\Delta)^m \omega_n - (m^* - 1) K(|y|) \Big(u_k + \sum_{j=1}^n PU_{p_{n,j},\lambda_n} \Big)^{m^*-2} \omega_n \\ = \mathcal{F}_n(\omega_n) + l_n + \sum_{i=1}^2 a_{n,i} \sum_{j=1}^n U_{p_{n,j},\lambda_n}^{m^*-2} Z_{j,i}, \text{ in } B_1(0), \\ \omega_n \in X_s \cap \mathscr{D}_0^{m,2}(B_1(0)), \quad \int_{B_1(0)} U_{p_{n,j},\lambda_n}^{m^*-2} Z_{j,i} \omega_n = 0, \quad i = 1, 2, \quad j = 1, \ldots, n, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} \mathcal{F}_n(\omega_n) = & K(|y|) \Big(u_k + \sum_{j=1}^n PU_{p_{n,j},\lambda_n} + \omega_n \Big)^{m^*-1} \\ & - K(|y|) \Big(u_k + \sum_{j=1}^n PU_{p_{n,j},\lambda_n} \Big)^{m^*-1} \end{aligned}$$

$$- (m^* - 1)K(|y|)\Big(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-2}\omega_n,$$

and

$$l_n = K(|y|)\Big(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-1} - K(|y|)u_k^{m^*-1} - \sum_{j=1}^{n} U_{p_{n,j},\lambda_n}^{m^*-1}.$$

A standard argument leads to

Lemma 3.3. $\|\mathcal{F}_n(\omega_n)\|_{**,n} \le C\|\omega_n\|_{*,n}^{\min\{m^*-1,2\}}$.

Next, we estimate l_n.

Lemma 3.4. $\|l_n\|_{**,n} \le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}$.

Proof. Define

$$D_{n,j} := \Big\{y \in B_1(0) : y = (y', y_3, y_4, y^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4}, \\ \Big\langle \frac{(0,0,y_3,y_4,,0,\dots,0)}{|(y_3,y_4)|}, \frac{p_{n,j}}{|p_{n,j}|}\Big\rangle \ge \cos\frac{\pi}{n}\Big\}.$$

We have

$$\begin{aligned} l_n &= K(|y|)\Big(\Big(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-1} - u_k^{m^*-1} - \Big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-1}\Big) \\ &\quad + \Big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-1} - \sum_{j=1}^{n} U_{p_{n,j},\lambda_n}^{m^*-1} \\ &\quad + \Big(K(|y|) - 1\Big)\Big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\Big)^{m^*-1} \\ &:= \bar{J}_1 + \bar{J}_2 + \bar{J}_3. \end{aligned}$$

We will estimate these terms one by one.

For $\bar{J}_1$, in the case $y \in \cup_{j=1}^{n}(D_{n,j} \cap B_{\frac{\delta}{n}}(p_{n,j}))$, where $\delta > 0$ is a small constant. Without loss of generality, we may assume $y \in D_{n,1} \cap B_{\frac{\delta}{n}}(p_{n,1})$, then

$$u_k \le CPU_{p_{n,1},\lambda_n}.$$

We have

$$\begin{aligned} |\bar{J}_1| &\le C\left(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*-2} u_k \\ &\le \lambda_n^{2m}\left(\frac{1}{(1+\lambda_n|y-p_{n,1}|)^{N-2m-\eta}}\right)^{m^*-2}\left(\frac{n}{\lambda_n}\right)^{\eta(m^*-2)} \\ &\le C\frac{\lambda_n^{2m}}{(1+\lambda_n|y-p_{n,1}|)^{4m-\frac{4m}{N-2m+1}}} \\ &\le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}} \\ &\le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}\sum_{j=1}^{n}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N+2m}{2}+\tau}}. \end{aligned} \tag{3.7}$$

In the case $y \notin \cup_{j=1}^{n}(D_{n,j}\cap B_{\frac{\delta}{n}}(p_{n,j}))$, without loss of generality, we may assume $y \in D_{n,1}\backslash B_{\frac{\delta}{n}}(p_{n,1})$, then

$$\begin{aligned} |\bar{J}_1| \le& C u_k^{m^*-2}\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} + C\left(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*-1} \\ \le& C\sum_{j=1}^{n}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N+2m}{2}+\tau}}\frac{1}{\lambda_n^{2m}(1+\lambda_n|y-p_{n,j}|)^{\frac{N-6m}{2}-\tau}} \\ &+ C\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}}\left(\frac{n}{\lambda_n}\right)^{\frac{N+2m}{2}-\tau} \\ \le& \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}\sum_{j=1}^{n}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,j}|)^{\frac{N+2m}{2}+\tau}}, \end{aligned} \tag{3.8}$$

where $\eta > 1$ is a constant. Combining (3.7) and (3.8), we have

$$\|\bar{J}_1\|_{**,n} \le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}.$$

For $\bar{J}_2$, we may assume $y \in D_1$. For $y \in B_{\frac{\delta}{n}}(p_{n,1})$, from Lemma A.1 we have

$$\psi_{p_{n,1},\lambda_n} = O\left(\frac{n^{N-2m}}{\lambda_n^{\frac{N-2m}{2}}}\right),$$

which leads to

$$\begin{aligned}
|\bar{J}_2| \le\ & CPU_{p_{n,1},\lambda_n}^{m^*-2}\left(\sum_{j=2}^{n} PU_{p_{n,j},\lambda_n} + \psi_{p_{n,1},\lambda_n}\right) + C\left(\sum_{j=2}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*-1} \\
& + C\psi_{p_{n,1},\lambda_n}^{m^*-1} + C\sum_{j=2}^{n} U_{p_{n,j},\lambda_n}^{m^*-1} \\
\le\ & C\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}} \sum_{j=2}^{n} \frac{1}{(\lambda_n|p_{n,j}-p_{n,1}|)^{\frac{N+2m}{2}-\tau}} \\
& + C\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}} \\
& \left(\sum_{j=2}^{n} \frac{1}{(\lambda_n|p_{n,j}-p_{n,1}|)^{N-2m-\frac{N-2m}{N+2m}(\frac{N+2m}{2}+\tau)}}\right)^{m^*-1} \\
& + O\left(\frac{n^{N+2m}}{\lambda_n^{\frac{N+2m}{2}}}\right) + \frac{\lambda_n^{2m}}{(1+\lambda_n|y-p_{n,1}|)^{4m}} O\left(\frac{n^{N-2m}}{\lambda_n^{\frac{N-2m}{2}}}\right) \\
\le\ & \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}} \frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned} \tag{3.9}$$

For $y \notin B_{\frac{\delta}{n}}(p_{n,1})$, we have

$$\begin{aligned}
|\bar{J}_2| \le\ & C\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m}} + C\left(\sum_{j=2}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*-1} + C\sum_{j=2}^{n} U_{p_{n,j},\lambda_n}^{m^*-1} \\
\le\ & \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}} \frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\|\bar{J}_2\|_{**,n} \le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}.$$

For $\bar{J}_3$, we assume $y \in D_1$. For $y \in B_{\frac{\delta}{n}}(p_{n,1})$, noting that

$$K(y) - 1 = O\left(\frac{1}{n}\right),$$

we have

$$\begin{aligned}
|\bar{J}_3| \le\ & \frac{C}{n}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m}} + \frac{C}{n}\left(\sum_{j=2}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*-1} \\
\le\ & \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}} \frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}}.
\end{aligned} \tag{3.11}$$

For $y \notin B_{\frac{\delta}{n}}(p_{n,1})$, similar to (3.10), we have

$$|\bar{J}_3| \le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}} \frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{\frac{N+2m}{2}+\tau}}. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\|\bar{J}_3\|_{**,n} \le \frac{C}{\lambda_n^{\frac{1}{2}+\sigma}}.$$

Combining the above estimates, the result follows. □

With the help of Proposition 3.2, Lemmas 3.3 and 3.4, a standard argument with the Fixed Point Theorem leads to the following proposition:

Proposition 3.5. *There exist $n_0 > 0$ and a constant $C > 0$, independent of n, such that for all $n \ge n_0$, $\lambda_n \in [\Lambda_0 n^{\frac{N-2\,m+1}{N-2\,m}}, \Lambda_1 n^{\frac{N-2\,m+1}{N-2\,m}}]$, $t_n \in (1-\frac{C_1}{n}, 1-\frac{C_2}{n})$, where $\Lambda_1 > \Lambda_0 > 0$ and $C_1 > C_2 > 0$ are some constants, problem (3.5) has a unique solution ω_n for some constant $a_{n,i}$, satisfying*

$$\|\omega_n\|_{*,n} \le \frac{C}{\lambda^{\frac{1}{2}+\sigma}}, \ |a_{n,i}| \le \frac{C}{\lambda^{\frac{1}{2}+\bar{n}_i+\sigma}},$$

where $\sigma > 0$ small enough.

Then we check energy expansion. The idea of the energy expansion comes from the observation that the nonlinear energy can be approximated by a linear combination of simple terms with the parameters t_n and λ_n.

Define

$$F : \mathbb{R}^2 \to \mathbb{R}, \quad F(t_n, \lambda_n) := I(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} + \omega_n),$$

where ω_n is the function obtained in Proposition 3.2, and I is the functional of problem (1.1), that is

$$I(u) = \begin{cases} \frac{1}{2}\int_{B_1(0)} |\Delta^{\frac{m}{2}} u|^2 - \frac{1}{(m)^*}\int_{B_1(0)} K(|y|)|u|^{m^*} & m \text{ even}, \\ \frac{1}{2}\int_{B_1(0)} |\nabla\Delta^{\frac{m-1}{2}} u|^2 - \frac{1}{m^*}\int_{B_1(0)} K(|y|)|u|^{m^*} & m \text{ odd}. \end{cases} \tag{3.13}$$

To obtain a solution with the form $u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n} + \omega_n$, we just need to find a critical point of $F(t_n, \lambda_n)$ in the domain $[1-\frac{C_1}{n}, 1-\frac{C_2}{n}] \times [\Lambda_0 n^{\frac{N-2\,m+1}{N-2\,m}}, \Lambda_0 n^{\frac{N-2\,m+1}{N-2\,m}}]$, where $0 < \Lambda_0 < \Lambda_1 < \infty$ and $C_1 > C_2 > 0$ are some constants.

Proof of Theorem 1.2. **When m is odd,**

$$F(t_n, \lambda_n) = I(u_k + \sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}) + nO\left(\frac{1}{\lambda_n^{1+\sigma}}\right), \tag{3.14}$$

and by symmetry,

$$\begin{aligned} & I\Big(u_k+\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\Big) \\ &= I(u_k)+I\Big(\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\Big)+\sum_{j=1}^{n}\int_{B_1(0)}U_{p_{n,j},\lambda_n}^{m^*-1}u_k \\ &\quad -\frac{1}{m^*}\int_{B_1(0)}K(|y|)\Big(\big(u_k+\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\big)^{m^*} \\ &\quad -\left(\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\right)^{m^*}-u_k^{m^*}\Big) \\ &= I(u_k)+I\Big(\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\Big)+n\int_{B_1(0)}U_{p_{n,1},\lambda_n}^{m^*-1}u_k \\ &\quad -\frac{1}{m^*}\int_{B_1(0)}K(|y|)\Big(\big(u_k+\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\big)^{m^*} \\ &\quad -\big(\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\big)^{m^*}-u_k^{m^*}\Big). \end{aligned} \tag{3.15}$$

We can check

$$\begin{aligned} \Big|\int_{B_1(0)}U_{p_{n,j},\lambda_n}^{m^*-1}u_k\Big| \le& C\Big[\int_{B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m}} \\ &+\int_{\mathbb{R}^N\setminus B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m}}\Big] \\ \le& \frac{C}{\lambda_n^{\frac{N-2m}{2}}}+\frac{C\lambda_n}{\lambda_n^{\frac{N+2m}{2}}}=O\left(\frac{1}{\lambda_n^{\frac{N-2m}{2}}}\right). \end{aligned}$$

For $y\in\mathbb{R}^N\setminus\cup_{j=1}^{n}(D_{n,j}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,j}))$ we have

$$\begin{aligned} &\left|\Big(u_k+\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\Big)^{m^*}-\Big(\sum_{j=1}^{n}PU_{p_{n,j},\lambda_n}\Big)^{m^*}-u_k^{m^*}\right| \\ &\quad\le Cu_k^{m^*-1}\sum_{j=1}^{n}U_{p_{n,j},\lambda_n}+C\Big(\sum_{j=1}^{n}U_{p_{n,j},\lambda_n}\Big)^{m^*-1}u_k, \end{aligned} \tag{3.16}$$

which leads to

$$\begin{aligned}
&\Big|\int_{B_1(0)\setminus\cup_{j=1}^n(D_{n,j}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1}))} K(|y|)\bigg(\Big(u_k+\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\Big)^{m^*}\\
&\quad-\bigg(\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\bigg)^{m^*}-u_k^{m^*}\bigg)\Big|\\
&\quad\le C\int_{\mathbb{R}^N\setminus\cup_{j=1}^n(D_{n,j}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})} u_k^{m^*-1}\sum_{j=1}^n U_{p_{n,j},\lambda_n}\\
&\quad\quad+C\int_{\mathbb{R}^N\setminus\cup_{j=1}^n(D_{n,j}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})} \bigg(\sum_{j=1}^n U_{p_{n,j},\lambda_n}\bigg)^{m^*-1}u_k\\
&\quad\le \frac{Cn}{\lambda_n^{\frac{N-2m}{2}}}+n\int_{D_{n,1}\setminus B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}\frac{1}{(1+|y|)^{N-2}}\frac{\lambda_n^{\frac{N+2}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m-\frac{N+2m}{N-2m+1}}}\\
&\quad\le \frac{Cn}{\lambda_n^{\frac{N-2m}{2}}}+\frac{Cn}{\lambda_n^{\frac{N+2m}{2}-\frac{N+2m}{N-2m+1}}}\int_{\mathbb{R}^N\setminus B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}\frac{1}{|y-p_{n,1}|^{N+2m-\frac{N+2m}{N-2m+1}}}\\
&\quad=O\bigg(\frac{n}{\lambda_n^{\frac{n-2m}{2}}}\bigg).
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{\cup_{j=1}^n(D_{n,j}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1}))} K(|y|)\bigg(\Big(u_k+\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\Big)^{m^*}\\
&\quad-\bigg(\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\bigg)^{m^*}-u_k^{m^*}\bigg)\\
&\quad=n\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}K(|y|)\left(\bigg(u_k+\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\bigg)^{m^*}\right.\\
&\quad\quad\left.-\bigg(\sum_{j=1}^n PU_{p_{n,j},\lambda_n}\bigg)^{m^*}-u_k^{m^*}\right).
\end{aligned} \tag{3.17}$$

Noting that

$$\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac12}}}(p_{n,1})}K(|y|)u_k^{m^*}=O\bigg(\frac{1}{\lambda_n^{\frac{N}{2}}}\bigg),$$

and

$$\begin{aligned}
&\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac{1}{2}}}}(p_{n,1})} K(|y|)\Big(\big(u_k+\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\big)^{m^*}-\big(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\big)^{m^*}\Big)\\
&\le C\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac{1}{2}}}}(p_{n,1})}\Big(\sum_{j=1}^{n} U_{p_{n,j},\lambda_n}\Big)^{m^*-1} u_k + C\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac{1}{2}}}}(p_{n,1})} u_k^{m^*}\\
&\le C\int_{D_{n,1}\cap B_{\frac{1}{\lambda_n^{\frac{1}{2}}}}(p_{n,1})}\frac{\lambda_n^{\frac{N+2m}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N+2m-\frac{N+2m}{N-2m+1}}}+\frac{C}{\lambda^{\frac{N}{2}}}\\
&\le \frac{C}{\lambda_n^{\frac{N-2m}{2}}}\int_{B_{\lambda_n^{\frac{1}{2}}}(0)}\frac{1}{(1+|y|)^{N+2m-\frac{N+2m}{N-2m+1}}}+\frac{C}{\lambda^{\frac{N}{2}}}\\
&\le \frac{C}{\lambda_n^{\frac{N-2m}{2}}},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{B_1(0)} K(|y|)\left(\left(u_k+\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*}-\left(\sum_{j=1}^{n} PU_{p_{n,j},\lambda_n}\right)^{m^*}-u_k^{m^*}\right)\\
&=O\left(\frac{n}{\lambda_n^{\frac{N-2m}{2}}}\right).
\end{aligned}$$

Then we have

$$I\Big(u_k+\sum_{j=1}^{n} U_{p_{n,j},\lambda_n}\Big)=I(u_k)+I\Big(\sum_{j=1}^{n} U_{p_{n,j},\lambda_n}\Big)+O\Big(\frac{n}{\lambda_n^{\frac{N-2m}{2}}}\Big). \tag{3.18}$$

Combining (3.14), (3.18) and a standard procedure as in [14], we obtain

$$\begin{aligned}
&F(t_n,\lambda_n)\\
&\quad=I(u_k)+I\Big(\sum_{j=1}^{n} U_{p_{n,j},\lambda_n}\Big)+O\Big(\frac{n}{\lambda_n^{1+\sigma}}\Big)\\
&\quad=I(u_k)+n\Big(A+\frac{B_1H(p_{n,1},p_{n,1})}{\lambda_n^{N-2m}}+B_2K'(1)(1-t_n)\\
&\qquad-\sum_{j=2}^{n}\frac{B_3G(p_{n,j},p_{n,1})}{\lambda_n^{N-2m}}\Big)+nO\Big(\frac{1}{\lambda_n^{1+\sigma}}\Big),
\end{aligned} \tag{3.19}$$

where $A, B_i, i\in\{1,2,3\}$ are some positive constants, $\sigma>0$ is a small constant.

Then similar to the argument as in [14], we can find a critical point of $F(t_n,\lambda_n)$, the result follows. The process is similar when m is even, therefore we have proven Theorem 1.2. □

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Declarations

Conflict of interest The authors declare no competing interests.

Appendix A. The estimate of U-PU

In the following, we always assume that

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, \mathbf{0}\right), \quad j = 1, \dots k,$$

$r \in [\mu(1-\frac{r_0}{k}), \mu(1-\frac{r_1}{k})]$. Let $\bar{x}_j = \frac{1}{\mu}x_j$, $G(x,y)$ be the Green function of $(-\Delta)^m$ in $B_1(0)$ with homogenous Dirichlet boundary condition, and $H(x,y)$ be the regular part of Green function. We use $PU_{x_j,\Lambda_k\mu_k}$ to denote the solution of (1.6) and $r_3 = min(r_0, 1)$.

For $l = 1, \dots, N$, denote

$$\partial_l PU_{x,\Lambda\mu} = \frac{\partial PU_{x,\Lambda\mu}(y)}{\partial x_l}, \quad \partial_l U_{x,\Lambda\mu} = \frac{\partial U_{x,\Lambda\mu}(y)}{\partial x_l};$$

for $l = N+1$, we set

$$\partial_l PU_{x,\Lambda\mu} = \frac{\partial PU_{x,\Lambda\mu}(y)}{\partial \mu}, \quad \partial_l U_{x,\Lambda\mu} = \frac{\partial U_{x,\Lambda\mu}(y)}{\partial \mu}.$$

Lemma A.1. *Assume $N \geq 2m+4$, for any $i = 1, \dots, k$, if $y \in B_{\frac{\mu r_3}{8k}(x_j)}$ and $j \neq i$, then*

$$\begin{aligned} & U_{x_i,\Lambda\mu}(y) - PU_{x_i,\Lambda\mu}(y) \\ & = \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{N-2m}} + O\left(\frac{1}{\mu^{\frac{N-2m}{2}+2}|x_i-y|^{N-2m+2}} + \frac{k^{2m}}{\mu^{\frac{N+2m}{2}}|x_i-y|^{N-2m}}\right), \end{aligned} \tag{A.1}$$

$$\partial_l U_{x_i,\Lambda\mu}(y) - \partial_l PU_{x_i,\Lambda\mu}(y)$$

$$= \frac{A_{N,m}\partial_l H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{1}{\mu^{\frac{N-2m}{2}+2}|x_i-y|^{N-2m+3}} + \frac{k^{2m+1}}{\mu^{\frac{N+2m}{2}}|x_i-y|^{N-2m}}\right),$$
$$\textit{for } l=1,\dots,N, \tag{A.2}$$

and

$$\partial_{N+1}U_{x_i,\Lambda\mu}(y) - \partial_{N+1}PU_{x_i,\Lambda\mu}(y)$$
$$= \frac{-(N-2m)A_{N,m}H(x_i,y)}{2\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}+1}} + O\left(\frac{1}{\mu^{\frac{N-2m}{2}+3}|x_i-y|^{N-2m+2}}\right.$$
$$\left.+\frac{k^{2m}}{\mu^{\frac{N+2m}{2}+1}|x_i-y|^{N-2m}}\right), \tag{A.3}$$

where $A_{N,m}$ is a constant only depend on N and m.

If $y\in B_{\frac{\mu r_3}{8k}}(x_i)$, then

$$U_{x_i,\Lambda\mu}(y) - PU_{x_i,\Lambda\mu}(y) = \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{k^{N-2m+2}}{\mu^{\frac{N-2m}{2}+2}}\right), \tag{A.4}$$

$$\partial_l U_{x_i,\Lambda\mu}(y) - \partial_l PU_{x_i,\Lambda\mu}(y) = \frac{A_{N,m}\partial_l H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}}$$
$$+O\left(\frac{k^{N-2m+3}}{\mu^{\frac{N-2m}{2}+2}}\right),\ \textit{for } l=1,\dots,N, \tag{A.5}$$

and

$$\partial_{N+1}U_{x_i,\Lambda\mu}(y) - \partial_{N+1}PU_{x_i,\Lambda\mu}(y) = \frac{-(N-2m)A_{N,m}H(x_i,y)}{2\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}+1}}$$
$$+O\left(\frac{k^{N-2m+2}}{\mu^{\frac{N-2m}{2}+3}}\right). \tag{A.6}$$

Proof. By the potential theory,

$$\begin{aligned} &U_{x_i,\Lambda\mu} - PU_{x_i,\Lambda\mu}(y)\\ &= \int_{\mathbb{R}^N} \frac{C_{1,N,m}}{|x-y|^{N-2m}} \frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\\ &\quad - \int_{B_1(0)} G(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\\ &= \int_{\mathbb{R}^N\setminus B_1(0)} \frac{C_{1,N,m}}{|x-y|^{N-2m}} \frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\\ &\quad + \int_{B_1(0)} H(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\\ &= M_1 + M_2. \end{aligned} \tag{A.7}$$

Case 1: $y\in B_{\frac{r_3}{8k}}(x_i)$, it is easy to check

$$|M_1| = O\left(\frac{k^N}{\mu^{\frac{N+2m}{2}}}\right), \tag{A.8}$$

and

$$\begin{aligned} M_2 &= \int_{B_1(0)\setminus B_{\frac{r_0}{2k}}(x_i)} H(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &\quad + \int_{B_{\frac{r_0}{2k}}(x_i)} H(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &= \int_{B_{\frac{r_0}{2k}}(x_i)} H(x_i,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &\quad + O\left(\int_{B_{\frac{r_0}{2k}}(x_i)} |x_i-x|^2|\nabla^2 H(x_i+t(x_i-x),y)|\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\right) \\ &\quad + O\left(\frac{k^N}{\mu^{\frac{N+2m}{2}}}\right) = \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{k^{N-2m+2}}{\mu^{\frac{N-2m}{2}+2}}\right). \end{aligned} \tag{A.9}$$

So,

$$U_{x_i,\Lambda\mu}(y) - PU_{x_i,\Lambda\mu}(y) = \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{k^{N-2m+2}}{\mu^{\frac{N-2m}{2}+2}}\right). \tag{A.10}$$

Case 2: $y \in B_{\frac{r_3}{8k}}(x_j)$, where $j \neq i$. In this case, it is easy to check

$$M_1 = O\left(\frac{1}{\mu^{\frac{N+2m}{2}}|x_i-y|^N} + \frac{k^{2m}}{\mu^{\frac{N+2m}{2}}|x_i-y|^{N-2m}}\right),$$

and

$$\begin{aligned} M_2 &= \int_{B_1(0)\setminus B_{\frac{|y-x_i|}{2}}(x_i)} H(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &\quad + \int_{B_{\frac{|y-x_i|}{2}}(x_i)\cap B_1(0)} H(x,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &= \int_{B_{\frac{r_3}{8k}}(x_i)} H(x_i,y)\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}} \\ &\quad + O\left(\int_{B_{\frac{|y-x_i|}{2}}(x_i)} |x_i-x|^2|\nabla^2 H(x_i+t(x_i-x),y)|\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\right) \\ &\quad + O\left(\int_{B_{\frac{|y-x_i|}{2}}(x_i)\setminus B_{\frac{r_3}{8k}}(x_i)} |H(x,y)|\frac{C_{2,N,m}(\Lambda\mu)^{\frac{N+2m}{2}}}{(1+(\Lambda\mu|x-x_i|)^2)^{\frac{N+2m}{2}}}\right) \\ &\quad + O\left(\frac{1}{\mu^{\frac{N+2m}{2}}|x_i-y|^N}\right) \end{aligned}$$

$$= \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{1}{\mu^{\frac{N-2m}{2}+2}|x_i-y|^{N-2m+2}} + \frac{k^{2m}}{\mu^{\frac{N+2m}{2}}|x_i-y|^{N-2m}}\right). \tag{A.11}$$

So

$$\begin{aligned} &U_{x_i,\Lambda\mu}(y) - PU_{x_i,\Lambda\mu}(y) \\ &= \frac{A_{N,m}H(x_i,y)}{\Lambda^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}}} + O\left(\frac{1}{\mu^{\frac{N-2m}{2}+2}|x_i-y|^{N-2m+2}} + \frac{k^{2m}}{\mu^{\frac{N+2m}{2}}|x_i-y|^{N-2m}}\right). \end{aligned} \tag{A.12}$$

Since for $l = 1, ..., N+1$,

$$\begin{cases} (-\Delta)^m \partial_l PU_{x_i,\Lambda\mu}(x) = \partial_l (U_{x_i,\Lambda\mu}(x))^{m^*-1}, & \text{in } B_1(0), \\ PU_{x_i,\Lambda\mu}(x) = 0, & x \in \partial B_1(0). \end{cases} \tag{A.13}$$

Similar to (A.1), (A.2), we can prove (A.4)–(A.6). □

For the completeness of the proof, we supplement the estimation of the derivative of H. In area $B_1(0)$ the Green function for the Dirichlet problem is positive and given by

$$G(x,y) = k_{m,N}\frac{1}{|x-y|^{N-2m}}\int_1^{\frac{[xy]}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{N-1}}dv,$$

where

$$[xy] = \big||x|y - \frac{x}{|x|}\big|, k_{m,N} = \frac{1}{w_n 4^{m-1}}((m-1)!)^2,$$

w_n is the n-dimensional unit sphere surface area. Using the following identity

$$\int_1^\infty \frac{(v^2-1)^{m-1}}{v^{N-1}}dv = \frac{2^{m-1}(m-1)!}{(N-2)\dots(N-2m)},$$

and (B.13) we can get the expression of H

$$H(x,y) = \Gamma(x,y) - G(x,y) = k_{m,N}\frac{1}{|x-y|^{N-2m}}\int_{\frac{[xy]}{|x-y|}}^\infty \frac{(v^2-1)^{m-1}}{v^{N-1}}dv, \tag{A.14}$$

we have

Lemma A.2. *The above function H is satisfied*

$$\left|\frac{\partial^\alpha H(x,y)}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_N}x_N}\right| \le \frac{C_{m,N}}{[xy]^{N-2m+\alpha}}, \quad \alpha \ge 0 \quad \alpha_1+\dots+\alpha_s = \alpha. \tag{A.15}$$

Proof. By definition after a variable substitution, we have

$$\begin{aligned} H(x,y) =& \frac{k_{m,N}}{[xy]^{N-2m}}\int_1^\infty \frac{(v^2 - \frac{|x-y|^2}{[xy]^2})^{m-1}}{v^{N-1}}dv \\ \le& \frac{k_{m,N}}{[xy]^{N-2m}}\int_1^\infty \frac{1}{v^{N-2m+1}}dv \\ \le& C\frac{1}{[xy]^{N-2m}}, \end{aligned}$$

repeated derive the function and similarity calculation lead us to the result. □

Appendix B. Pohozaev indentities and the estimates of Green's function

In this part, we will establish the Pohozaev indentities for polyharmonic operator in the small domain $B_{\frac{\delta}{k_n}}(x_{k_n,1})$, where $\delta > 0$ is a small fixed constant. And then we give some expression related to Green's function. We will use the same notations as before.

Recall that the equation and its linearized equation read as:

$$(-\Delta)^m u = K(|y|)u^{m^*-1}, \text{ in } B_1(0); \tag{B.1}$$

and

$$(-\Delta)^m \xi = (m^*-1)K(|y|)u^{m^*-2}\xi, \text{ in } B_1(0). \tag{B.2}$$

Lemma B.1. *If m is even and $B \subset B_1(0)$ is a smooth area, then*

$$\begin{aligned}\int_B u^{m^*-1}\xi \frac{\partial K(|y|)}{\partial y_s} &= \int_{\partial B} K(|y|)u^{m^*-1}\xi\nu_s - \int_{\partial B} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}}\xi\nu_s \\ &+ \sum_{i=1}^{\frac{m}{2}}\Big(\int_{\partial B}\Delta^{m-i}u\frac{\partial^2\Delta^{i-1}\xi}{\partial y_s\partial\nu} - \int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\frac{\partial\Delta^{i-1}\xi}{\partial y_s} \\ &+ \int_{\partial B}\Delta^{m-i}\xi\frac{\partial^2\Delta^{i-1}u}{\partial y_s\partial\nu} - \int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\frac{\partial\Delta^{i-1}u}{\partial y_s}\Big),\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}&\int_B u^{m^*-1}\xi\langle\nabla K(|y|), y-x_0\rangle \\ &= \int_{\partial B} K(|y|)u^{m^*-1}\xi\langle\nu, y-x_0\rangle - \int_{\partial B}\Delta^{\frac{m}{2}}u\Delta^{\frac{m}{2}}\xi\langle\nu, y-x_0\rangle \\ &- \sum_{i=1}^{\frac{m}{2}}\Big(\int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\langle\nabla\xi, y-x_0\rangle + \int_{\partial B}\Delta^{m-i}u\frac{\partial\Delta^{i-1}\langle\nabla\xi, y-x_0\rangle}{\partial\nu} \\ &- \int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\Delta^{i-1}\langle\nabla u, y-x_0\rangle + \int_{\partial B}\Delta^{m-i}\xi\frac{\partial\Delta^{i-1}\langle\nabla u, y-x_0\rangle}{\partial\nu}\Big) \\ &+ \frac{N-2m}{2}\sum_{i=1}^{\frac{m}{2}}\Big(\int_{\partial B}\Delta^{m-i}u\frac{\partial\Delta^{i-1}\xi}{\partial\nu} + \int_{\partial B}\Delta^{m-i}\xi\frac{\partial\Delta^{i-1}u}{\partial\nu} \\ &- \int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\xi - \int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\Delta^{i-1}u\Big),\end{aligned} \tag{B.4}$$

where $x_0 \in B_1(0)$, ν is the out-of-unit normal derivative of B, $s = 1,\ldots,N$ and ν_s is the s-th component of ν.

Lemma B.2. *If m is odd and $B \subset B_1(0)$ is a smooth area, then*

$$\begin{aligned} &\int_B u^{m^*-1}\xi\frac{\partial K(|y|)}{\partial y_s} \\ &\quad = \int_{\partial B} K(|y|)u^{m^*-1}\xi\nu_s - \int_{\partial B}\langle\nabla\Delta^{\frac{m-1}{2}}u, \nabla\Delta^{\frac{m-1}{2}}\xi\rangle\nu_s \\ &\qquad - \sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B}\Delta^{m-i}u\frac{\partial^2\Delta^{i-1}\xi}{\partial y_s\partial\nu} + \sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\frac{\partial\Delta^{i-1}\xi}{\partial y_s} \\ &\qquad - \sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B}\Delta^{m-i}\xi\frac{\partial^2\Delta^{i-1}u}{\partial y_s\partial\nu} + \sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\frac{\partial\Delta^{i-1}u}{\partial y_s}, \end{aligned} \tag{B.5}$$

and

$$\begin{aligned} &\int_B u^{m^*-1}\xi\langle\nabla K(|y|), y-x_0\rangle \\ &= \int_{\partial B} K(|y|)u^{m^*-1}\xi\langle\nu, y-x_0\rangle - \int_{\partial B}\langle\nabla\Delta^{\frac{m-1}{2}}u, \nabla\Delta^{\frac{m-1}{2}}\xi\rangle\langle\nu, y-x_0\rangle \\ &\quad + \sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\langle\nabla\xi, y-x_0\rangle - \sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B}\Delta^{m-i}u\frac{\partial\Delta^{i-1}\langle\nabla\xi, y-x_0\rangle}{\partial\nu} \\ &\quad + \int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\Delta^{i-1}\langle\nabla u, y-x_0\rangle - \sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B}\Delta^{m-i}\xi\frac{\partial\Delta^{i-1}\langle\nabla u, y-x_0\rangle}{\partial\nu} \\ &\quad - \frac{N-2m}{2}\sum_{i=1}^{\frac{m-1}{2}}\left(\int_{\partial B}\Delta^{m-i}u\frac{\partial\Delta^{i-1}\xi}{\partial\nu} + \int_{\partial B}\Delta^{m-i}\xi\frac{\partial\Delta^{i-1}u}{\partial\nu}\right) \\ &\quad + \frac{N-2m}{2}\sum_{i=1}^{\frac{m+1}{2}}\left(\int_{\partial B}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\xi - \int_{\partial B}\frac{\partial\Delta^{m-i}\xi}{\partial\nu}\Delta^{i-1}u\right), \end{aligned}$$

where $x_0 \in B_1(0)$, ν is the out-of-unit normal derivative of B, $s = 1, \ldots, N$ and ν_s is the s-th component of ν.

The proof of the above two lemmas are similar. We only give the outline of proofs. For the first lemma, we multiply the two Eqs. (B.1) and (B.2) by $\frac{\partial\xi}{\partial y_s}$ and $\frac{\partial u}{\partial y_s}$ respectively; for the second part, we multiply the two Eqs. (B.1) and (B.2) by $\langle\nabla\xi, y-x_0\rangle$ and $\langle\nabla u, y-x_0\rangle$ respectively, finally use integral by parts to get the result. For specific details, please refer to [15] Lemma 2.1 and 2.2.

Next, let us discuss the following identities involving Green function, which is very important when proving non-degenerate properties. We add definitions for N is odd:

$$I_{1,2}(u,v,d) = \int_{\partial B_d} K(|y|)u^{m^*-1}v\nu_1 - \int_{\partial B_d}\langle\nabla\Delta^{\frac{m-1}{2}}u, \nabla\Delta^{\frac{m-1}{2}}v\rangle\nu_1$$

$$
\begin{aligned}
&-\sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B_d}\Delta^{m-i}u\frac{\partial^2\Delta^{i-1}v}{\partial y_1\partial\nu}+\sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B_d}\frac{\partial\Delta^{m-i}u}{\partial\nu}\frac{\partial\Delta^{i-1}v}{\partial y_1}\\
&-\sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B_d}\Delta^{m-i}v\frac{\partial^2\Delta^{i-1}u}{\partial y_1\partial\nu}+\sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B_d}\frac{\partial\Delta^{m-i}v}{\partial\nu}\frac{\partial\Delta^{i-1}u}{\partial y_1},
\end{aligned}
$$

$$
\begin{aligned}
I_{2,2}(u,v,d)=&\int_{\partial B_d}K(|y|)u^{m^*-1}v\langle\nu,y-x_{x_{k_n},1}\rangle\\
&-\int_{\partial B_d}\langle\nabla\Delta^{\frac{m-1}{2}}u,\nabla\Delta^{\frac{m-1}{2}}v\rangle\langle\nu,y-x_{x_{k_n},1}\rangle\\
&+\sum_{i=1}^{\frac{m+1}{2}}\int_{\partial B_d}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}\langle\nabla v,y-x_{x_{k_n},1}\rangle\\
&-\sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B_d}\Delta^{m-i}u\frac{\partial\Delta^{i-1}\langle\nabla v,y-x_{x_{k_n},1}\rangle}{\partial\nu}\\
&+\int_{\partial B_d}\frac{\partial\Delta^{m-i}v}{\partial\nu}\Delta^{i-1}\langle\nabla u,y-x_{x_{k_n},1}\rangle\\
&-\sum_{i=1}^{\frac{m-1}{2}}\int_{\partial B_d}\Delta^{m-i}v\frac{\partial\Delta^{i-1}\langle\nabla u,y-x_{x_{k_n},1}\rangle}{\partial\nu}\\
&-\frac{N-2m}{2}\sum_{i=1}^{\frac{m-1}{2}}\left(\int_{\partial B_d}\Delta^{m-i}u\frac{\partial\Delta^{i-1}v}{\partial\nu}+\int_{\partial B_d}\Delta^{m-i}v\frac{\partial\Delta^{i-1}u}{\partial\nu}\right)\\
&+\frac{N-2m}{2}\sum_{i=1}^{\frac{m+1}{2}}\left(\int_{\partial B_d}\frac{\partial\Delta^{m-i}u}{\partial\nu}\Delta^{i-1}v-\int_{\partial B_d}\frac{\partial\Delta^{m-i}v}{\partial\nu}\Delta^{i-1}u\right).
\end{aligned}
$$

Lemma B.3. *For any* $d\in(0,\frac{\delta}{k_n})$*, where* $\delta>0$ *is a fixed small constant, we have*

$$
\begin{aligned}
&I_{1,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)=I_{1,2}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)\\
&\quad=-2\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1}),
\end{aligned}\tag{B.6}
$$

$$
\begin{aligned}
&I_{1,1}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}G(y,x_{k_n,j}),d)=I_{1,2}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}G(y,x_{k_n,j}),d)\\
&\quad=\sum_{j=2}^{k_n}\frac{\partial G}{\partial y_1}(x_{k_n,1},x_{k_n,j}),
\end{aligned}\tag{B.7}
$$

$$
\begin{aligned}
&I_{2,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)=I_{2,2}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)\\
&\quad=(N-2m)H(x_{k_n,1},x_{k_n,1}),
\end{aligned}\tag{B.8}
$$

$$I_{2,1}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}G(y,x_{k_n,j}),d)=I_{2,2}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}G(y,x_{k_n,j}),d)$$
$$=-\frac{N-2m}{2}\sum_{j=2}^{k_n}G(x_{k_n,1},x_{k_n,j}), \tag{B.9}$$
$$I_{2,1}(G(y,x_{k_n,1}),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),d)=I_{2,2}(G(y,x_{k_n,1}),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),d)$$
$$=(N-2m+1)\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1}), \tag{B.10}$$
$$I_{2,1}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}(cos\theta_j\frac{\partial G}{\partial x_1}(x_{k_n,1},x_{k_n,j})+sin\theta_j\frac{\partial G}{\partial x_2}(x_{k_n,1},x_{k_n,j})),d)$$
$$=I_{2,2}(G(y,x_{k_n,1}),\sum_{j=2}^{k_n}(cos\theta_j\frac{\partial G}{\partial x_1}(x_{k_n,1},x_{k_n,j})+sin\theta_j\frac{\partial G}{\partial x_2}(x_{k_n,1},x_{k_n,j})),d)$$
$$=-\frac{N-2m}{2}\sum_{j=2}^{k_n}\frac{\partial G}{\partial y_1}(x_{k_n,1},x_{k_n,j}), \tag{B.11}$$

and

$$I_{2,1}\left(\sum_{j=2}^{k_n}G(y,x_{k_n,j}\right),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),d)=I_{2,2}\left(\sum_{j=2}^{k_n}G(y,x_{k_n,j}\right),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),d)$$
$$=-\frac{N-2m+2}{2}\sum_{j=2}^{k_n}\frac{\partial G}{\partial y_1}(x_{k_n,1},x_{k_n,j}). \tag{B.12}$$

Proof. We give the proving process of (B.6), (B.8) and (B.10). The rest can be deduced similarly. In the first place, in the domain $B_d(x_{k_n,1})\backslash B_\epsilon(x_{k_n,1}), 0<\epsilon<d$, we have

$$\begin{aligned}&I_{1,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)-I_{1,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),\epsilon)\\&\quad=\int_{B_d(x_{k_n,1})\backslash B_\epsilon(x_{k_n,1})}(-\Delta)^mG(y,x_{k_n,1})\frac{\partial G}{\partial y_1}(y,x_{k_n,1})\\&\qquad+(-\Delta)^mG(y,x_{k_n,j})\frac{\partial G}{\partial y_1}(y,x_{k_n,1})\\&\quad=0.\end{aligned}$$

Thus

$$I_{1,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),d)=\lim_{\epsilon\to0}I_{1,1}(G(y,x_{k_n,1}),G(y,x_{k_n,1}),\epsilon),$$

while noting that $G(y,x)=\Gamma(y,x)-H(y,x)$, where $\Gamma(y,x)$ is the foundamental solution:

$$\begin{aligned}\Gamma(y,x)&=\Gamma_{m,N}(y,x)\\&=\frac{1}{\omega_N2^{m-1}(N-2)\dots(N-2m)(m-1)!|y-x|^{N-2m}},\end{aligned} \tag{B.13}$$

and H is a regular function. Then by the linear of $I_{i,j}, i, j \in \{1,2\}$ and the symmetry of the integration region we have

$$\begin{aligned}
&I_{1,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), \epsilon) \\
&\quad = I_{1,1}(\Gamma(y, x_{k_n,1}) - H(y, x_{k_n,1}), \Gamma(y, x_{k_n,1}) - H(y, x_{k_n,1}), \epsilon) \\
&\quad = I_{1,1}(\Gamma(y, x_{k_n,1}), \Gamma(y, x_{k_n,1}), \epsilon) - 2I_{1,1}(\Gamma(y, x_{k_n,1}), H(y, x_{k_n,1}), \epsilon) \\
&\qquad + I_{1,1}(H(y, x_{k_n,1}), H(y, x_{k_n,1}), \epsilon) \\
&\quad = -2\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1}) + o_\epsilon(1).
\end{aligned}$$

Let $\epsilon \to 0$, $I_{1,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) = -2\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1})$,by the exactly same way we can have $I_{1,2}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) = -2\frac{\partial H}{\partial y_1}(x_{k_n,1}, x_{k_n,1})$ so (B.6) is proved.

We start to prove (B.8), a direct calulation leads to

$$\begin{aligned}
&I_{2,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) - G(y, x_{k_n,1}), G(y, x_{k_n,1}), \epsilon) \\
&\quad = \int_{B_d(x_{k_n,1}) \setminus B_\epsilon(x_{k_n,1})} (\Delta)^m G(y, x_{k_n,1}) \langle \nabla G(y, x_{k_n,1}), y - x_{k_n,1} \rangle \\
&\qquad + \int_{B_d(x_{k_n,1}) \setminus B_\epsilon(x_{k_n,1})} (\Delta)^m G(y, x_{k_n,j}) \langle \nabla G(y, x_{k_n,1}), y - x_{k_n,1} \rangle \\
&\qquad + (N - 2m) \int_{B_d(x_{k_n,1}) \setminus B_\epsilon(x_{k_n,1})} (\Delta)^m G(y, x_{k_n,1}) G(y, x_{k_n,1}) \\
&\quad = 0.
\end{aligned}$$

We still have

$$I_{2,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) = \lim_{\epsilon \to 0} I_{2,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), \epsilon),$$

using the same calculation method,

$$I_{2,1}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) = (N - 2m)H(x_{k_n,1}, x_{k_n,1}).$$

As for the $I_{2,2}(G(y, x_{k_n,1}), G(y, x_{k_n,1}), d) = (N-2m)H(x_{k_n,1}, x_{k_n,1})$ is directly availble. In the following, we give a brief calculation process of (B.10). For $I_{2,1}$: by (B.13) and the symmetry of the area $B_d(x)$ and the regularity of H we have:

$$\begin{aligned}
&I_{2,1}(G(y, x_{k_n,1}), \frac{\partial G}{\partial x_1}(y, x_{k_n,1}), \epsilon) \\
&\quad = I_{2,1}(\Gamma(y, x_{k_n,1}) - H(y, x_{k_n,1}), \frac{\partial(\Gamma - H)}{\partial x_1}(y, x_{k_n,1}), \epsilon) \\
&\quad = I_{2,1}(\Gamma(y, x_{k_n,1}), \frac{\partial \Gamma}{\partial x_1}(y, x_{k_n,1}), \epsilon) - I_{2,1}(\Gamma(y, x_{k_n,1}), \frac{\partial H}{\partial x_1}(y, x_{k_n,1}), \epsilon) \\
&\qquad - I_{2,1}(H(y, x_{k_n,1}), \frac{\partial \Gamma}{\partial x_1}(y, x_{k_n,1}), \epsilon) + I_{2,1}(H(y, x_{k_n,1}), \frac{\partial H}{\partial x_1}(y, x_{k_n,1}), \epsilon) \\
&\quad = -I_{2,1}(\Gamma(y, x_{k_n,1}), \frac{\partial H}{\partial x_1}(y, x_{k_n,1}), \epsilon) - I_{2,1}(H(y, x_{k_n,1}), \frac{\partial \Gamma}{\partial x_1}(y, x_{k_n,1}), \epsilon) + o_\epsilon(1),
\end{aligned}$$

By direct computations,

$$
\begin{aligned}
& I_{2,1}(\Gamma(y,x_{k_n,1}),\frac{\partial H}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&\quad = -\frac{N-2m}{2}\int_{\partial B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}\Gamma}{\partial\nu}\frac{\partial H}{\partial x_1}(y,x_{k_n,1})+o_\epsilon(1)\\
&\quad = -\frac{N-2m}{2}\frac{\partial H}{\partial x_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1),
\end{aligned}
$$

since for m is even we have

$$
\int_{\partial B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}\Gamma}{\partial\nu}=1,
$$

For the second one:

$$
\begin{aligned}
& I_{2,1}(H(y,x_{k_n,1}),\frac{\partial \Gamma}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&\quad = -\int_{B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}\langle\nabla H,y-x_{k_n,1}\rangle\\
&\qquad + \int_{B_\epsilon(x_{k_n,1})}\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})\frac{\partial\langle\nabla H,y-x_{k_n,1}\rangle}{\partial\nu}\\
&\qquad + \frac{N-2m}{2}\int_{B_\epsilon(x_{k_n,1})}\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})\frac{\partial H}{\partial\nu}-\frac{N-2m}{2}\\
&\qquad \int_{B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}H+o_\epsilon(1),
\end{aligned}
$$

we will need the following basic results:

$$
\begin{aligned}
\Delta^{m-1}\Gamma = & \frac{(-1)^{m-1}}{(N-2)w_N|y-x|^{N-2}},\quad \int_{\partial B_\epsilon(x)}\frac{\partial\Delta^{m-1}\Gamma}{\partial\nu}=(-1)^m,\\
& \frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}=(-1)^m\frac{(N-1)(y-x_{k_n,1})_1}{w_N|y-x_{k_n,1}|^{N+1}}.
\end{aligned}
$$

A direct calculation shows

$$
-\int_{B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}\langle\nabla H,y-x_{k_n,1}\rangle=-\frac{N-1}{N}\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1),
$$

$$
\begin{aligned}
&\int_{B_\epsilon(x_{k_n,1})}\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})\frac{\partial\langle\nabla H,y-x_{k_n,1}\rangle}{\partial\nu}=\\
&\quad -\int_{B_\epsilon(x_{k_n,1})}\frac{(y-x_{k_n,1})_1^2}{|y-x_{k_n,1}|^2}\frac{\partial H}{\partial y_1}(y,x_{k_n,1})+o_\epsilon(1)\\
&\quad = -\frac{1}{N}\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1).
\end{aligned}
$$

therefore we have

$$
\begin{aligned}
&I_{2,1}(G(y,x_{k_n,1}),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&=\left(\frac{N-2m}{2}+\frac{N-1}{N}+\frac{1}{N}+\frac{N-2m}{2}\left(\frac{1}{N}+\frac{N-1}{N}\right)\right)\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1)\\
&=(N-2m+1)\frac{\partial H}{\partial y_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1).
\end{aligned}
$$

Let $\epsilon\to 0$ we get (B.10) for m is even. As for m is odd, similar calculations show that:

$$
\begin{aligned}
&I_{2,2}(\Gamma(y,x_{k_n,1}),\frac{\partial H}{\partial x_1}(y,x_{k_n,1}),\epsilon)=-\frac{N-2m}{2}\frac{\partial H}{\partial x_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1),\\
&I_{2,2}(H(y,x_{k_n,1}),\frac{\partial \Gamma}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&\quad=\int_{B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}\langle\nabla H,y-x_{k_n,1}\rangle\\
&\qquad-\int_{B_\epsilon(x_{k_n,1})}\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})\frac{\partial\langle\nabla H,y-x_{k_n,1}\rangle}{\partial\nu}\\
&\qquad-\frac{N-2m}{2}\int_{B_\epsilon(x_{k_n,1})}\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})\frac{\partial H}{\partial\nu}+\frac{N-2m}{2}\\
&\qquad\int_{B_\epsilon(x_{k_n,1})}\frac{\partial\Delta^{m-1}(\frac{\partial\Gamma}{\partial x_1})}{\partial\nu}H+o_\epsilon(1)\\
&\quad=\left(\frac{1-N}{N}-\frac{N-2m}{2}\frac{1}{N}-\frac{1}{N}+\frac{N-2m}{2}\frac{1-N}{N}\right)\frac{\partial H}{\partial x_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1),
\end{aligned}
$$

so

$$
\begin{aligned}
&I_{2,2}(G(y,x_{k_n,1}),\frac{\partial G}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&=I_{2,2}(\Gamma(y,x_{k_n,1}),\frac{\partial \Gamma}{\partial x_1}(y,x_{k_n,1}),\epsilon)-I_{2,2}(\Gamma(y,x_{k_n,1}),\frac{\partial H}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&\quad-I_{2,2}(H(y,x_{k_n,1}),\frac{\partial \Gamma}{\partial x_1}(y,x_{k_n,1}),\epsilon)+I_{2,2}(H(y,x_{k_n,1}),\frac{\partial H}{\partial x_1}(y,x_{k_n,1}),\epsilon)\\
&=-I_{2,2}(\Gamma(y,x_{k_n,1}),\frac{\partial H}{\partial x_1}(y,x_{k_n,1}),\epsilon)-I_{2,2}(H(y,x_{k_n,1}),\frac{\partial \Gamma}{\partial x_1}(y,x_{k_n,1}),\epsilon)+o_\epsilon(1)\\
&=(N-2m+1)\frac{\partial H}{\partial x_1}(x_{k_n,1},x_{k_n,1})+o_\epsilon(1).
\end{aligned}
$$

(B.10) is proved. □

Appendix C. Green function

In this part, we give the estimate of modified Green function G_k, which is used in Lemma 3.1. It's necessary for the construction of new bubble solutions.

In general, for any function f defined in $\mathbb{R}^N$, we define its corresponding function $f^* \in H_s$ as follows. We first define A_j as

$$A_j z = \left(r\cos\left(\theta + \frac{2j\pi}{k}\right), r\sin\left(\theta + \frac{2j\pi}{k}\right), z''\right), \quad j = 1, \ldots, k,$$

where $z = (z', z'') \in \mathbb{R}^N$, $z' = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$, $z'' \in \mathbb{R}^{N-2}$, while

$$B_i z = \big(z_1, \ldots, z_{i-1}, -z_i, z_{i+1}, \ldots, z_N\big), \quad i = 1, \ldots, N.$$

Let

$$\bar{f}(y) = \frac{1}{k}\sum_{j=1}^{k} f(A_j y),$$

and

$$f^*(y) = \frac{1}{N-1}\sum_{i=2}^{N} \frac{1}{2}\big(\bar{f}(y) + \bar{f}(B_i y)\big).$$

Then one can easily check that $f^* \in H_s$.

In the following, we discuss the Green's function of L_k. Since δ_x is not in the space H_s, we consider

$$L_k u = \delta_x^*, \text{ in } B_1(0), \quad u \in H_s \cap \mathscr{D}_0^{m,2}(B_1(0)). \tag{C.1}$$

The solution of (C.1) is denoted as $G_k(y,x)$, which we call it the Green function of L_k. Let

$$\delta_x^* = \frac{1}{N-1}\sum_{i=2}^{N}\frac{1}{2}\Big(\frac{1}{k}\sum_{j=1}^{k}\delta_{A_j x} + \frac{1}{k}\sum_{j=1}^{k}\delta_{B_i A_j x}\Big).$$

We have

Proposition C.1. *The solution $G_k(y,x)$ satisfies*

$$|G_k(y,x)| \le \frac{C}{N-2m+1}\sum_{i=2}^{N}\Big(\frac{1}{k}\sum_{j=1}^{k}\frac{C}{|y - A_j x|^{N-2m}} + \frac{1}{k}\sum_{j=1}^{k}\frac{C}{|y - B_i A_j x|^{N-2m}}\Big).$$

Proof. Let $v_1 = G(x,y)$ be the Green's function of $(-\Delta)^m$ in $B_1(0)$ with Dirichlet boundary condition, which can constructed the solution of following polyharmonic equation

$$\begin{cases} (-\Delta)^m u = f, \text{ in } B_1(0) \\ D^\alpha u|_{\partial\Omega} = 0, \text{ for } |\alpha| \le m-1. \end{cases} \tag{C.2}$$

f is a datum in a suitable functional space and u is the unknown solution, then

$$u(x) = \int_{B_1(0)} G(x,y) f(y)\,dy, \quad x \in \Omega,$$

holds ture. As before we denote $[xy] = \left| |x|y - \frac{x}{|x|} \right|$ for domain $B_1(0)$, we have

$$G(x, y) = k_{m,N}|x - y|^{2m-N} \int_1^{\frac{[xy]}{|x-y|}} (v^2 - 1)^{m-1} v^{1-N} dv,$$
$$G(x, y) \simeq |x - y|^{2m-N} \min\left\{1, \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right\}, \text{ where } d(x) = dist(x, \partial B_1(0)).$$

Let v_2 be the solution of

$$\begin{cases}(-\Delta)^m v = (m^* - 1)K(y)u_k^{m^*-2} v_1, & \text{in } B_1(0),\\ D^\alpha v|_{\partial\Omega} = 0 \text{ for } |\alpha| \leq m - 1.\end{cases}$$

Then $v_2 \geq 0$ and

$$\begin{aligned} v_2(y) &= \int_{B_1(0)} G(z, y)(m^* - 1)u_k^{m^*-2} K(y)v_1 \\ &\leq C \int_{B_1(0)} \frac{1}{|y - z|^{N-2m}} \frac{1}{|z - x|^{N-2m}} \, dz. \end{aligned}$$

We can continue this process to find v_i, which is the solution of

$$\begin{cases}(-\Delta)^m v = (m^* - 1)u_k^{m^*-2} K(y)v_{i-1}, & \text{in } B_1(0),\\ D^\alpha v|_{\partial\Omega} = 0 \text{ for } |\alpha| \leq m - 1.\end{cases}$$

And satisfies

$$\begin{aligned} 0 \leq & v_i(y) \\ = & \int_{B_1(0)} G(z, y)(m^* - 1)u_k^{m^*-2} K(y)v_{i-1} \\ \leq & C \int_{B_1(0)} \frac{1}{|y - z|^{N-2m}} \frac{1}{|z - x|^{N-2m(i-1)}} \, dz \\ \leq & \frac{C}{|y - x|^{N-2mi}}. \end{aligned}$$

Let i be large so that $v_i \in L^\infty(B_1(0))$. Define

$$v = \sum_{l=1}^{i} v_l \quad \text{ and } w = G_k(y, x) - v^*,$$

We then have

$$\begin{cases} L_k w = f, & \text{in } B_1(0),\\ w = 0, & \text{on } \partial B_1(0),\end{cases} \tag{C.3}$$

where $f \in L^\infty \cap H_s$. By Theorem 1.1, (C.3) has a solution $w \in H_s \cap \mathscr{D}_0^{m,2}(B_1(0))$.

By the regularity results of polyharmonic Dirichlet boundary conditions (Theorem 2.20 of [17]), we have

$$||w||_{L^\infty(B_1(0))} \leq C||f||_{L^\infty(B_1(0))}.$$

Thus the conclusion is proved. □

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