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The asymptotic behaviors of normalized ground states for nonlinear Schrödinger equations

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Abstract. In this paper, we study the relation between the least energy levels and between the minimizers of the following minimization problems

$$E_\sigma(\rho) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |w|^{2\sigma+2}\Big| \int_{\mathbb{R}^N} w^2 = \rho\right\}$$

and

$$Z(\rho) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{2}\int_{\mathbb{R}^N} w^2 \log w^2 \Big| \int_{\mathbb{R}^N} w^2 = \rho\right\}.$$

We show that as $\sigma \to 0^+$, the minimizers for $E_\sigma(\rho)$, after rescaling, converge to the minimizers of $Z(\rho)$. Besides, we also give estimates for $E_\sigma(\rho)$ and the corresponding Lagrange multiplier when σ is small.

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1. Introduction

The aim of this work is to explore the asymptotic behaviors of the minimizer and the least energy level to the following minimization problem with L^2-constraint:

$$E_\sigma(\rho) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 \mathrm{d}x - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |w|^{2\sigma+2}\mathrm{d}x \Big| w \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} w^2 \mathrm{d}x = \rho\right\}, \tag{1.1}$$

where $N \geq 1$, $\rho > 0$, and the power σ satisfies the so-called L^2-subcritical condition that $0 < \sigma < \frac{2}{N}$ (see [5]). By the classical results from [1,2,5,8,10], for each $\rho > 0$, we know that $E_\sigma(\rho)$ is achieved at some $w_\sigma(x) = w_\sigma(|x|) > 0$ depending on σ and ρ. Moreover, the minimizer $w_\sigma(x)$ is unique up to translations, decreases in $r = |x|$, and decays exponentially at infinity. The Euler-Lagrange equation corresponding to problem (1.1) is as follows:

$$\begin{cases} -\Delta w + \mu w = |w|^{2\sigma} w, \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} w(x) = 0, \int_{\mathbb{R}^N} w^2 \mathrm{d}x = \rho, \end{cases} \tag{1.2}$$

where $\mu = \mu_\sigma(\rho) \in \mathbb{R}$ appears as a Lagrange multiplier depending on σ and ρ. Solutions with prescribed L^2-norms are known as *normalized* solutions. w_σ is a *normalized ground state* solution of (1.2), since it is a nontrivial solution to (1.2) having the least energy $E_\sigma(\rho)$. Problem (1.1) and (1.2) are motivated in particular by the search for stationary states in nonlinear Schrödinger equation, that is, the following time-dependent nonlinear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t}(t,x) + \Delta\Psi(t,x) + |\Psi(t,x)|^{2\sigma}\Psi(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

which appears in nonlinear optics and the theory of Bose-Einstein condensates (see [7,9,15]). The constraint $\int_{\mathbb{R}^N} w^2 \mathrm{d}x = \rho$ in the stationary problem is introduced due to the mass conservation property of the time-dependent nonlinear Schrödinger equation. In applications, the prescribed mass represents the power supply in nonlinear optics or the number of particles in Bose-Einstein condensates. The problem (1.2) also appears in the Mean Field Games theory in Lasry and Lions [11] as a case of the mean field limit equations in a stationary setting.

The present paper is invoked by [19] that uncovers a relation between power-law nonlinear scalar field equations and logarithmic-law scalar field equations. In [19], Wang and Zhang consider the following power-law nonlinear Schrödinger equation:

$$\begin{cases} -\Delta v + \lambda v = |v|^{p-2} v \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.3}$$

and logarithmic-law nonlinear Schrödinger equation:

$$\begin{cases} -\Delta v = \lambda v \log |v| \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1.4}$$

They show that as $p \downarrow 2$, the ground state solutions of (1.3), after a unique rescaling, converge to the ground state solutions of (1.4). The logarithmic nonlinear Schrödinger equation was introduced as an important model in quantum physics. It admits plenty of applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation, see [3,4,16,17,19] and the references therein. d'Avenia-Montefusco-Squassina[6] and Troy[16] have proved that the ground state solution of (1.4) is unique up to translations and is given by

$$U(x) = e^{\frac{N}{2}} e^{-\frac{\lambda}{4}|x|^2}.$$

Therefore, it is easy to see that the L^2-constrained minimization problem

$$Z(\rho)=\inf\left\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}v^2\log v^2\mathrm{d}x\middle|v\in H^1(\mathbb{R}^N),\int_{\mathbb{R}^N}v^2\mathrm{d}x=\rho\right\} \tag{1.5}$$

is achieved at

$$v^0(x)=\rho^{\frac{1}{2}}\pi^{-\frac{N}{4}}e^{-\frac{|x|^2}{2}}.$$

The logarithmic nonlinear Schrödinger equation corresponding to problem (1.5) is as follows:

$$\begin{cases} -\Delta v+\lambda v=v\log v^2 \quad \text{in } \mathbb{R}^N,\\ \lim\limits_{|x|\to\infty}v(x)=0,\int_{\mathbb{R}^N}v^2\mathrm{d}x=\rho, \end{cases} \tag{1.6}$$

here $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier dependent of ρ. It is obvious that $v^0(x)$ is a normalized ground state solution of (1.6). A direct calculation shows that the Lagrange multiplier is

$$\lambda=\lambda^0(\rho):=\log\rho-N-\frac{N}{2}\log\pi,$$

and the least energy to (1.6) is

$$Z(\rho)=-\frac{\lambda^0\rho}{2}=\frac{\rho}{2}\left(N+\frac{N}{2}\log\pi-\log\rho\right).$$

These explicit formulas give clear information about the Lagrange multiplier and the ground state energy to (1.6). In contrast, very little is known about μ_σ and E_σ of (1.2) except for some obvious knowledge that, as $\sigma\to 0$, $E_\sigma(\rho)\to -\rho/2$ and $\mu_\sigma\to 1$. Further information is rather difficult to obtain.

In order to establish the relation between the two L^2-constrained minimization problems (1.1) and (1.5), and give estimates on $E_\sigma(\rho)$ and $\mu_\sigma(\rho)$, we are dedicated to proving the convergence of $w_\sigma(x)$ to $v^0(x)$ as $\sigma\to 0$.

Theorem 1.1. *Let* $v_\sigma(x)=\sigma^{-\frac{N}{2(2-\sigma N)}}w_\sigma(\sigma^{-\frac{1}{2-\sigma N}}x)$. *Then, as* $\sigma\to 0$, *it holds that* $v_\sigma\to v^0$ *strongly in* $H^1(\mathbb{R}^N)$ *and in* $C^{2,\alpha}(\mathbb{R}^N)$ *for any* $\alpha\in(0,1)$. *Moreover, we have*

$$\rho^{-\frac{2\sigma}{2-\sigma N}}\mu_\sigma(\rho)=1+\frac{N}{2}\sigma\log\sigma-\sigma\left(N+\frac{N}{2}\log\pi\right)+o_\sigma(\sigma),$$

$$\rho^{-1-\frac{2\sigma}{2-\sigma N}}E_\sigma(\rho)=-\frac{1}{2}-\frac{\sigma N}{4}\log\sigma+\left(\frac{N+1}{2}+\frac{N\log\pi}{4}\right)\sigma+o_\sigma(\sigma).$$

In comparison to the recent and interesting work [19] by Wang and Zhang, the major novelty here is given by the mass constraint, that was not considered in [19]. In [19], the authors study the asymptotic behaviors of the least energy level and ground state solutions of probelm (1.3) with a prescribed λ. After a scaling $u_p(x)=\lambda^{-\frac{1}{p-2}}v_p(\frac{x}{\sqrt{p-2}})$(see [19], Theorem 1.1), as $p\downarrow 2$, they derive the convergence of $u_p(x)$ to U. However in this paper, $\mu\in\mathbb{R}$ appears as a Lagrange multiplier, which varies as $\sigma\to 0^+$. So it is difficult to use the result in [19] to obtain the asymptotic behaviors of (μ_σ, w_σ) solving Eq.(1.2). We introduce a new scaling in which μ does not appear to achieve these behaviors.

Remark 1.1. We remark that $\rho^{-\frac{2\sigma}{2-\sigma N}}\mu_\sigma(\rho)$, $\rho^{-1-\frac{2\sigma}{2-\sigma N}}E_\sigma(\rho)$ are quantities independent of ρ, see (2.4) and (2.7). Moreover, when ρ belongs to a compact subset of $(0,+\infty)$, there hold

$$\mu_\sigma(\rho) = 1 + \frac{N}{2}\sigma\log\sigma - \sigma\left(N + \frac{N}{2}\log\pi - \log\rho\right) + o_\sigma(\sigma),$$

$$E_\sigma(\rho) = \left(-\frac{1}{2} - \frac{\sigma N}{4}\log\sigma\right)\rho + \left(\frac{N+1}{2} + \frac{N\log\pi}{4} - \rho\log\rho\right)\sigma + o_\sigma(\sigma).$$

The paper is organized as follows: In Sect. 2, we establish the work space and give some preliminaries which will be used in the proof of the main theorems; Sect. 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Throughout this paper, we use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space with the following inner product and norm
$$(u,v) := \int_{\mathbb{R}^N} \nabla u\nabla v + uv\mathrm{d}x, \quad \|u\| := \sqrt{(u,u)}, \quad \forall u,v\in H^1(\mathbb{R}^N).$$
$H^1_{rad}(\mathbb{R}^N)$ denotes the space $\left\{u\in H^1(\mathbb{R}^N) | u(x) = u(|x|)\right\}$.
- $L^p(\mathbb{R}^N)$ $(1\le p<\infty)$ is the Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right)^{\frac{1}{p}}$.
- $o_\sigma(1)$ denotes an infinitesimal with $o_\sigma(1)\to 0$ as $\sigma\to 0$.
- $C(a_1,a_2,\ldots,a_n)$ denotes any positive constant that depends on $a_1, a_2, \ldots, a_n$.
- For $R>0$, $B_R(0)$ denotes the ball of radius R centered at 0. $B_R^c(0)$ denotes the set $\mathbb{R}^N\setminus B_R(0)$.

Consider $I_\sigma : H^1(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$I_\sigma(w) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2\mathrm{d}x - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N}|w|^{2\sigma+2}\mathrm{d}x, \quad w\in H^1(\mathbb{R}^N).$$

We will perform some scaling on w so that we can approach the limit functional $J : H^1(\mathbb{R}^N)\to\mathbb{R}\cup\{+\infty\}$ given by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}v^2\log v^2\mathrm{d}x, \quad v\in H^1(\mathbb{R}^N).$$

First, following [19], for $w\in H^1(\mathbb{R}^N)$ and $|w|_2^2=\rho$, set $u(x) = w(\frac{1}{\sqrt{\sigma}}x)$. Then we get $|u|_2^2 = \rho\sigma^{\frac{N}{2}}$ and

$$I_\sigma(w) = \sigma^{1-\frac{N}{2}}\left(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N}\sigma^{-1}|u|^{2\sigma+2}\mathrm{d}x\right). \tag{2.1}$$

Next, we use a re-scaling that shifts the L^2 norm of functions and preserves some homogeneity on the functional (see also [18,20]). For each $s > 0$, letting $u(x) = s^{\frac{1}{2-\sigma N}} v(s^{\frac{\sigma}{2-\sigma N}} x)$, we have

$$|u|_2^2 = s|v|_2^2, \quad |\nabla u|_2^2 = s^{1+\frac{2\sigma}{2-\sigma N}} |\nabla v|_2^2, \quad |u|_{2\sigma+2}^{2\sigma+2} = s^{1+\frac{2\sigma}{2-\sigma N}} |u|_{2\sigma+2}^{2\sigma+2}.$$

Setting $s = \sigma^{\frac{N}{2}}$, we have $|v|_2^2 = \rho$. By (2.1), we arrive at

$$\begin{aligned} I_\sigma(w) &= \sigma^{1-\frac{N}{2}} \cdot \sigma^{\frac{N}{2}(1+\frac{2\sigma}{2-\sigma N})} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2\sigma+2} \int_{\mathbb{R}^N} \sigma^{-1} |v|^{2\sigma+2} \mathrm{d}x \right) \\ &= \sigma^{1+\frac{N\sigma}{2-\sigma N}} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2\sigma+2} \int_{\mathbb{R}^N} \sigma^{-1} |v|^{2\sigma+2} \mathrm{d}x \right) \\ &= \sigma^{\frac{\sigma N}{2-\sigma N}} \left(\sigma J_\sigma(v) - \frac{\rho}{2\sigma+2} \right), \end{aligned} \tag{2.2}$$

where

$$J_\sigma(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2\sigma+2} \int_{\mathbb{R}^N} \frac{|v|^{2\sigma+2} - v^2}{\sigma} \mathrm{d}x.$$

Note that in (2.2)

$$v(x) = \sigma^{-\frac{N}{2(2-\sigma N)}} w(\sigma^{-\frac{1}{2-\sigma N}} x).$$

Remark 2.1. If we set $s = \rho\sigma^{\frac{N}{2}}$, then $|v|_2^2 = 1$ and

$$I_\sigma(w) = \rho^{1+\frac{2\sigma}{2-\sigma N}} \sigma^{\frac{\sigma N}{2-\sigma N}} \left(\sigma J_\sigma(v) - \frac{1}{2\sigma+2} \right).$$

Setting $\mathcal{M}_\rho := \{ v \in H^1(\mathbb{R}^N) \mid |v|_2^2 = \rho \}$, we introduce another L^2-constrained minimization problem

$$Z_\sigma(\rho) := \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2\sigma+2} \int_{\mathbb{R}^N} \frac{|v|^{2\sigma+2} - v^2}{\sigma} \mathrm{d}x \;\middle|\; v \in \mathcal{M}_\rho \right\}. \tag{2.3}$$

Then, by (2.2) and Remark 2.1, it is clear that

$$E_\sigma(\rho) = \sigma^{\frac{\sigma N}{2-\sigma N}} \left(\sigma Z_\sigma(\rho) - \frac{\rho}{2\sigma+2} \right) = \rho^{1+\frac{2\sigma}{2-\sigma N}} \sigma^{\frac{\sigma N}{2-\sigma N}} \left(\sigma Z_\sigma(1) - \frac{1}{2\sigma+2} \right). \tag{2.4}$$

Since for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$J'_\sigma(v)\varphi = \int_{\mathbb{R}^N} \nabla v \nabla \varphi \mathrm{d}x - \int_{\mathbb{R}^N} \frac{|v|^{2\sigma} v - v}{\sigma} \varphi \mathrm{d}x - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} v\varphi \mathrm{d}x, \tag{2.5}$$

the minimizer corresponding to problem (2.3) satisfies:

$$\begin{cases} -\Delta v + \lambda_\sigma v = \dfrac{|v|^{2\sigma} v - v}{\sigma}, & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = 0, \displaystyle\int_{\mathbb{R}^N} v^2 \mathrm{d}x = \rho. \end{cases} \tag{2.6}$$

Here $\lambda_\sigma = \lambda_\sigma(\rho) \in \mathbb{R}$ appears as a Lagrange multiplier. From [20, Lemma 2.1], we know $(\lambda_\sigma, v_\sigma)$ is unique up to translations. So λ_σ depends only on N, σ and ρ. It is easy to verify that

$$\mu_\sigma(\rho) = \sigma^{\frac{\sigma N}{2-\sigma N}}(\sigma\lambda_\sigma(\rho)+1) = \rho^{\frac{2\sigma}{2-\sigma N}}\sigma^{\frac{\sigma N}{2-\sigma N}}(\sigma\lambda_\sigma(1)+1). \tag{2.7}$$

In the sequel, we will compare the above equation (2.6) with the logarithmic Schrödinger equation (1.6).

Note that I_σ and J_σ are C^2 functionals but J is not continuous(see [6] or [17]). In fact, by using the following standard logarithmic Sobolev inequality(see Lieb and Loss [12])

$$\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x \leq \frac{c^2}{\pi}|\nabla u|_2^2 + \left[\log|u|_2^2 - N(1+\log c)\right]|u|_2^2, \quad u \in H^1(\mathbb{R}^N) \text{ and } c > 0, \tag{2.8}$$

where the equality holds if and only if $u(x) = e^{-\frac{\pi|x|^2}{2c^2}}$, it is obvious that $\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x < +\infty$ for all $u \in H^1(\mathbb{R}^N)$. Indeed there exists $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x = -\infty$ (see [13]). Thus, in general, the functional J fails to be finite and lacks C^1-smoothness on $H^1(\mathbb{R}^N)$. However as in [17], the L^2-constrained minimization problem (1.5) can also be considered in the following space

$$\begin{aligned}\mathcal{D} &= \left\{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} |u^2 \log u^2| \mathrm{d}x < +\infty\right\} \\ &= \left\{u \in H^1(\mathbb{R}^N) | \int_{|u|\leq 1} |u^2 \log u^2| \mathrm{d}x < +\infty\right\}.\end{aligned}$$

In addition, we give some important notations:

Definition 2.1. (1) We say $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}$ in the following sense: for any $v \in \mathcal{D}$, there exists a sequence $\varphi_n \in C_0^\infty(\mathbb{R}^N)$, such that

$$\|\varphi_n - v\| \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \varphi_n^2 \log \varphi_n^2 \mathrm{d}x \to \int_{\mathbb{R}^N} v^2 \log v^2 \mathrm{d}x.$$

(2) For $v, \varphi \in \mathcal{D}$, we define

$$J'(v)\varphi := \int_{\mathbb{R}^N} \nabla v \nabla \varphi \mathrm{d}x - \int_{\mathbb{R}^N} v\varphi(1+\log v^2)\mathrm{d}x$$

(3) We say $v \in \mathcal{D} \cap \mathcal{M}_\rho$ is a critical point of J on $\mathcal{D} \cap \mathcal{M}_\rho$ if and only if $J'(v)\varphi = 0$ for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} v\varphi = 0$.

Remark 2.2. Note that $v \in \mathcal{D} \cap \mathcal{M}_\rho$ is a critical point of J on $\mathcal{D} \cap \mathcal{M}_\rho$ if and only if there is $\lambda \in \mathbb{R}$ such that

$$J'(v)\varphi = \lambda \int_{\mathbb{R}^N} v\varphi \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Moreover, $\lambda = \rho^{-1} J'(v)v$.

The following lemma (see [19, Lemma 2.1]) describes the behavior of the nonlinear term as σ near 0.

Lemma 2.1. (i) *For any* $\sigma' > 0$, *there exists* $C(\sigma') > 0$ *such that*

$$\frac{x^{2\sigma}-1}{\sigma} \le C(\sigma')x^{2\sigma'}$$

holds for all $\sigma \in (0, \sigma')$ *and* $x \ge 0$.

(ii) *Let* $s > 0$, $\sigma > 0$, *then*

$$\frac{x^s(x^\sigma - 1)}{\sigma} \to x^s \log x \quad in \quad C^{m,\alpha}_{loc}[0, +\infty)$$

as $\sigma \to 0$, *where* m *is the largest integer with* $m < s$, *and* $\alpha \in (0, s-m)$.

3. Proofs of the main results

We first consider the case $\rho = 1$ and write $Z_\sigma = Z_\sigma(1)$.

Lemma 3.1. Z_σ *is achieved at*

$$v_\sigma(x) = \sigma^{-\frac{N}{2(2-\sigma N)}} w_\sigma(\sigma^{-\frac{1}{2-\sigma N}} x) \in \mathcal{M}_1.$$

Moreover, as $\sigma \to 0^+$, Z_σ, $\|v_\sigma\|$, *and* $\int_{\mathbb{R}^N} \sigma^{-1} ||v_\sigma|^{2\sigma+2} - v_\sigma^2|$ *are bounded.*

Proof. Recalling (2.2), for $w \in H^1(\mathbb{R}^N)$ and $|w|_2^2 = 1$, by rescaling

$$v(x) = \sigma^{-\frac{N}{2(2-\sigma N)}} w(\sigma^{-\frac{1}{2-\sigma N}} x),$$

we derive that $|v|_2^2 = 1$ and

$$I_\sigma(w) = \sigma^{\frac{2}{2-\sigma N}} J_\sigma(v) - \frac{1}{2\sigma+2} \cdot \sigma^{\frac{N\sigma}{2-\sigma N}}.$$

Then,

$$\inf_{w \in \mathcal{M}_1} I_\sigma(w) = \sigma^{\frac{2}{2-\sigma N}} \inf_{v \in \mathcal{M}_1} J_\sigma(v) - \frac{1}{2\sigma+2} \cdot \sigma^{\frac{N\sigma}{2-\sigma N}},$$

implying that

$$E_\sigma(1) = \sigma^{\frac{2}{2-\sigma N}} Z_\sigma(1) - \frac{1}{2\sigma+2} \cdot \sigma^{\frac{N\sigma}{2-\sigma N}},$$

and $E_\sigma(1)$ is attained at w_σ if and only if $Z_\sigma(1)$ is attained at v_σ.

We next prove that Z_σ is bounded as $\sigma \to 0^+$. Fixing any $v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ satisfying $\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = 1$, from Lemma 2.1(ii), one gets $J_\sigma(v) \to J(v)$ as $\sigma \to 0$. Therefore, $Z_\sigma(1)$ is bounded from above.

Applying Lemma 2.1(i) and Gagliardo-Nirenberg inequality

$$|v|_{2\sigma'+2}^{2\sigma'+2} \le C(\sigma', N)|\nabla v|_2^{\sigma' N} |v|_2^{2\sigma'+2-\sigma' N},$$

where σ' is a fixed constant such that $0 < \sigma' N < 2$, we obtain

$$\begin{aligned} Z_\sigma &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x - C(\sigma')\int_{\mathbb{R}^N} |v_\sigma|^{2\sigma'+2} \mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x - C(\sigma', N)\left(\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x\right)^{\frac{\sigma' N}{2}} \left(\int_{\mathbb{R}^N} |v_\sigma|^2 \mathrm{d}x\right)^{\frac{2\sigma'+2-\sigma' N}{2}} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x - C(\sigma', N)\left(\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x\right)^{\frac{\sigma' N}{2}}. \end{aligned} \tag{3.1}$$

Then (3.1) implies that Z_σ is bounded from below. Thus, Z_σ is bounded. In addition, the boundedness of Z_σ and (3.1) assert that $|\nabla v_\sigma|_2$ is bounded. Together with $|v_\sigma|_2^2 = 1$, we derive that $\|v_\sigma\|$ is bounded. By the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$, we have

$$\int_{\mathbb{R}^N} \frac{(|v_\sigma|^{2\sigma+2} - v_\sigma^2)_+}{\sigma} \mathrm{d}x \leq C(\sigma')|v_\sigma|_{2\sigma'+2}^{2\sigma'+2} \leq C(\sigma', N)\|v_\sigma\|^{2\sigma'+2}.$$

By

$$Z_\sigma = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x - \frac{1}{2\sigma+2}\left[\int_{\mathbb{R}^N} \frac{(|v_\sigma|^{2\sigma+2} - v_\sigma^2)_+}{\sigma} \mathrm{d}x - \int_{\mathbb{R}^N} \frac{(|v_\sigma|^{2\sigma+2} - v_\sigma^2)_-}{\sigma} \mathrm{d}x\right],$$

we know $\int_{\mathbb{R}^N} \sigma^{-1}(|v_\sigma|^{2\sigma+2} - v_\sigma^2)_-$ is bounded. Thus, $\int_{\mathbb{R}^N} \sigma^{-1}||v_\sigma|^{2\sigma+2} - v_\sigma^2|$ is bounded. Then we complete the proof. □

Theorem 3.1. *As $\sigma \to 0^+$, we have $\lambda_\sigma \to \lambda^0$, $Z_\sigma(1) \to Z(1)$, and $v_\sigma \to v^0$ strongly in $H^1(\mathbb{R}^N)$.*

Proof. Since v_σ satisfies (2.6) with $\rho = 1$, i.e.

$$\begin{cases} -\Delta v_\sigma + \lambda_\sigma v_\sigma = \dfrac{|v_\sigma|^{2\sigma} v_\sigma - v_\sigma}{\sigma} \\ \lim\limits_{|x|\to\infty} v_\sigma(x) = 0, \displaystyle\int_{\mathbb{R}^N} v_\sigma^2 \mathrm{d}x = 1, \end{cases} \tag{3.2}$$

by Lemma 3.1, we know that

$$\begin{aligned} |\lambda_\sigma| &= \left|\int_{\mathbb{R}^N} \frac{|v_\sigma|^{2\sigma+2} - v_\sigma^2}{\sigma} \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla v_\sigma|^2 \mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}^N} \sigma^{-1}||v_\sigma|^{2\sigma+2} - v_\sigma^2|\mathrm{d}x + \|v_\sigma\|^2 \end{aligned} \tag{3.3}$$

is bounded.

We next re-scale v_σ in order to avoid the possible difficulties that $\lambda_\sigma < 0$ poses to the subsequent proofs. Setting $\tilde{v}_\sigma = a v_\sigma$, from (3.2), we can get

$$-\Delta \tilde{v}_\sigma + \mu_\sigma \tilde{v}_\sigma = \frac{a^{-2\sigma}(|\tilde{v}_\sigma|^{2\sigma}\tilde{v}_\sigma - \tilde{v}_\sigma)}{\sigma} \tag{3.4}$$

where

$$\mu_\sigma = \lambda_\sigma - \frac{a^{-2\sigma} - 1}{\sigma}. \tag{3.5}$$

Since

$$\frac{a^{-2\sigma} - 1}{\sigma} \to -2\log a,$$

we can fix $a > 0$ sufficiently large such that $\mu_\sigma > 1$ for every small σ. By Lemma 3.1 and (3.3) and (3.5), $\|\tilde{v}_\sigma\|$ and μ_σ are both bounded. Then up to a subsequence, we assume

$$\begin{aligned} &\tilde{v}_\sigma \rightharpoonup \tilde{v} \quad \text{weakly in } H^1(\mathbb{R}^N),\\ &\tilde{v}_\sigma \to \tilde{v} \quad \text{strongly in } L^p_{loc}(\mathbb{R}^N), p \in (2, 2^*),\\ &\tilde{v}_\sigma \to \tilde{v} \quad \text{a.e. in } \mathbb{R}^N, \end{aligned}$$

and $\mu_\sigma \to \mu^0 \in [1, +\infty)$. If $N \geq 2$, then by the radial lemma of Strauss [14], $\tilde{v}_\sigma(x) < 1$, $|x| \geq R$ for some R independent of σ. If $N = 1$, we have $\tilde{v}_\sigma \to \tilde{v}$ in $C_{loc}(\mathbb{R})$. Then we also get $\tilde{v}_\sigma(x) < 1$ when $|x| \geq R$ for some R independent of σ. Hence for $N \geq 1$, $\tilde{v}_\sigma$ satisfies $-\Delta\tilde{v}_\sigma + \tilde{v}_\sigma \leq 0$ in $\mathbb{R}^N \setminus B_R(0)$. By comparison theorem, we obtain

$$\tilde{v}_\sigma(x) \leq Ce^{-c|x|}, \quad |x| \geq R,$$

for $C, c > 0$ independent of σ. Necessarily,

$$\tilde{v}_\sigma \to \tilde{v} \quad \text{strongly in } L^p(\mathbb{R}^N), p \in [1, 2^*).$$

Especially,

$$\frac{|\tilde{v}_\sigma|^{2\sigma}\tilde{v}_\sigma - \tilde{v}_\sigma}{\sigma}(\tilde{v_\sigma} - \tilde{v}) \to 0 \quad \text{strongly in } L^1(\mathbb{R}^N).$$

Multiplying (3.4) by $\tilde{v_\sigma} - \tilde{v}$ and integrating, we get

$$\int_{\mathbb{R}^N} \nabla\tilde{v_\sigma}\nabla(\tilde{v_\sigma} - \tilde{v}) + \mu_\sigma\tilde{v_\sigma}(\tilde{v_\sigma} - \tilde{v})\mathrm{d}x = \int_{\mathbb{R}^N} \frac{a^{-2\sigma}(|\tilde{v_\sigma}|^{2\sigma}\tilde{v_\sigma} - \tilde{v_\sigma})}{\sigma}(\tilde{v_\sigma} - \tilde{v})\mathrm{d}x. \tag{3.6}$$

Taking limits as $\sigma \to 0^+$, we get

$$|\nabla\tilde{v_\sigma}|_2^2 \to |\nabla\tilde{v}|_2^2.$$

It follows that $\lim_{\sigma\to 0^+} \|\tilde{v_\sigma}\| = \|\tilde{v}\|$. Combining with $\tilde{v_\sigma} \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^N)$, we deduce that $\tilde{v_\sigma} \to \tilde{v}$ strongly in $H^1(\mathbb{R}^N)$. Up to a translation, we can assume $\tilde{v}(0) = \max_{\mathbb{R}^N} \tilde{v}$. Since $(\mu^0, \tilde{v})$ solves

$$-\Delta\tilde{v} + \mu^0\tilde{v} = \tilde{v}\log\tilde{v}^2, \quad |\tilde{v}|_2^2 = a^2,$$

we conclude that $(\mu^0, \tilde{v})$ is unique. Therefore, the convergence $(\mu_\sigma, \tilde{v_\sigma}) \to (\mu^0, \tilde{v})$ is independent of subsequences.

By uniqueness of the solution to (2.6), we derive that $\mu^0 = \lambda^0 + 2\log a$ and $\tilde{v} = av^0$. Thus, $\lambda_\sigma \to \lambda^0$, and $v_\sigma \to v^0$ strongly in $H^1(\mathbb{R}^N)$. □

Proof of Theorem 1.1. Combining (2.2) with Lemma 3.1, one can show that

$$E_\sigma(\rho) = \sigma^{\frac{2}{2-\sigma N}} Z_\sigma(\rho) - \frac{\rho}{2\sigma + 2} \cdot \sigma^{\frac{N\sigma}{2-\sigma N}},$$

and $Z_\sigma(\rho)$ is attained at $v_\sigma(x) = \sigma^{-\frac{N}{2(2-\sigma N)}} w_\sigma(\sigma^{-\frac{1}{2-\sigma N}}x) \in \mathcal{M}_\rho$. By arguments similar to those in the proof of Theorem 3.1, we also have $\lambda_\sigma \to \lambda^0$ and $v_\sigma \to v^0$ strongly in $H^1(\mathbb{R}^N)$ as $\sigma \to 0$. By regularity theory, it follows that $v_\sigma \to v^0$ in $C^{2,\alpha}(\mathbb{R}^N)$, $\alpha \in (0, 1)$. Applying Lemma 2.1(ii) again, we deduce that $J_\sigma(v_\sigma) \to J(v^0)$ as $\sigma \to 0$. That is, $Z_\sigma(\rho) \to Z(\rho)$ as $\sigma \to 0$. Hence, by (2.7), one gets

$$\rho^{-\frac{2\sigma}{2-\sigma N}}\mu_\sigma(\rho) = \sigma^{\frac{\sigma N}{2-\sigma N}}(\sigma\lambda_\sigma(1)+1) = \sigma\lambda^0(1) + \frac{N}{2}\sigma\log\sigma + 1 + o_\sigma(\sigma)$$
$$= 1 + \frac{N}{2}\sigma\log\sigma - (N + \frac{N}{2}\log\pi)\sigma + o_\sigma(\sigma),$$

and by (2.4),

$$\begin{aligned}\rho^{-1-\frac{2\sigma}{2-\sigma N}}E_\sigma(\rho) &= \sigma^{\frac{\sigma N}{2-\sigma N}}\left(\sigma Z_\sigma(1) - \frac{1}{2\sigma+2}\right)\\ &= \sigma^{\frac{\sigma N}{2-\sigma N}}\left(\sigma Z_\sigma(1) - \frac{1}{2} + \frac{\sigma}{2} + o_\sigma(\sigma)\right)\\ &= \sigma Z(1) - \frac{1}{2} - \frac{\sigma N}{4}\log\sigma + \frac{\sigma}{2} + o_\sigma(\sigma)\\ &= -\frac{1}{2} - \frac{\sigma N}{4}\log\sigma + (\frac{N+1}{2} + \frac{N\log\pi}{4})\sigma + o_\sigma(\sigma).\end{aligned}$$

□

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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