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Boundary controllability for a coupled system of parabolic equations with singular potentials

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Abstract. This paper deals with the boundary controllability for a coupled system of singular parabolic equations by means of one control force. In particular, we consider well posedness of the problem and then we prove both approximate and null controllability results. Moreover, an estimate on the null-control cost is provided. Our proofs rely on the use of the moment method together with some properties of Bessel functions and their zeros.

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1. Introduction

The goal of this paper is to analyze controllability properties for a singular system of two equations when we apply just one control on a part of the boundary. More precisely, we consider the following linear control system:

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2} y = Ay, & (t,x) \in Q := (0,T) \times (0,1), \\ y(t,1) = Bv, & t \in (0,T), \\ y(t,0) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $y_0 \in H^{-1,\mu}(0,1)^2$ (that will be defined later in sect. 2), μ is a real parameter such that $\mu \leq \frac{1}{4}$, $v = v(t)$ represents the control force which is exerted at point $x = 1$ by means of the boundary Dirichlet condition, and $y = (y_1, y_2)^*$ is the state variable.

Moreover, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are, respectively, a suitable coupling matrix and a control operator, chosen so that the Kalman rank condition is satisfied. Indeed, as highlighted in [25], it is well-known that a necessary condition for the controllability of this kind of systems is given by the so-called Kalman's rank condition:

$$rank[B|AB] = 2. \tag{1.2}$$

On the other hand, as explained in [25], by taking $P = [B|AB]$, the change of variables

$$\tilde{y} = P^{-1}y,$$

leads to the following reformulation of (1.1):

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} - \frac{\mu}{x^2}\tilde{y} = \tilde{A}\tilde{y}, & (t,x) \in (0,T) \times (0,1), \\ \tilde{y}(t,1) = \tilde{B}v, & t \in (0,T), \\ \tilde{y}(t,0) = 0, & t \in (0,T), \\ \tilde{y}(0,x) = P^{-1}y_0(x), & x \in (0,1), \end{cases} \tag{1.3}$$

with

$$\tilde{A} = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \text{ and } \tilde{B} = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, passing through this mentioned change of variables, the situation reduces to the case where

$$A = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \quad \text{and} \quad B = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.4}$$

For simplicity, it will be assumed in the rest of the paper that A and B are given by (1.4). Let us observe that this choice amounts to suppose that we are exerting only one control force on the system but we want to control the corresponding state $y = (y_1, y_2)$ which has two components. In fact, y_1 is directly controlled by the boundary control and the second equation in (1.1) is indirectly controlled by means of the coupling term y_1.

We are particularly interested in the study of system (1.1) under the assumption that the coupling matrix A admits two distinct eigenvalues, that is:

$$a_2^2 + 4a_1 \neq 0. \tag{1.5}$$

Prior to controllability issues is the well-posedness of problem (1.1), a question we address in sect. 2. We will see that, for every $v \in L^2(0,T)$ and $y_0 \in H^{-1,\mu}(0,1)^2$, system (1.1) admits a unique weak solution defined by transposition that satisfies

$$y \in L^2(Q)^2 \cap C^0\big([0,T], H^{-1,\mu}(0,1)^2\big).$$

Observe that the previous regularity permits to pose the boundary controllability of the singular system (1.1) in the space $H^{-1,\mu}(0,1)^2$.

Then, we pass to see whether one can force the solution of system (1.1) to have certain desired properties by choosing appropriate control inputs. In particular, we analyze both approximate and null controllability issues. So, we use the following notions:

Definition 1.1. 1. It will be said that system (1.1) is approximately controllable in $H^{-1,\mu}(0,1)^2$ at time $T>0$ if for every $y_0, y_d \in H^{-1,\mu}(0,1)^2$ and any $\varepsilon > 0$, there exists a control function $v \in L^2(0,T)$ such that the solution y to system (1.1) satisfies

$$\|y(T,\cdot) - y_d\|_{H^{-1,\mu}(0,1)^2} \leq \varepsilon.$$

2. It will be said that system (1.1) is null controllable at time $T>0$ if for every $y_0 \in H^{-1,\mu}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that the solution y to system (1.1) satisfies

$$y(T,\cdot) = 0, \quad \text{in} \quad H^{-1,\mu}(0,1)^2.$$

Our first result for the boundary controllability of system (1.1) concerns the approximate controllability and is proved in sect. 4 under the rank condition (1.2) together with a condition (see (3.13)) that is equivalent to the simplicity of the spectrum associated with system (1.1). We refer to Theorem 4.3 for a precise statement of this result. In order to prove this result, we argue by duality reducing the problem to the obtention of an unique continuation property for the corresponding adjoint system.

In this work, we will also prove the boundary null controllability of (1.1) assuming the previous conditions which characterize the approximate controllability property (see Theorem 5.1 for a rigorous statement). The strategy for proving our null controllability result for system (1.1) is based on the well-known moment method initially developed in [23,24]. In the literature, this moment method has been successfully applied for treating the controllability of (nonsingular) parabolic systems of PDEs, see [4–6,9,10,20,28].

Before dealing with problem (1.1), let us first review some previous results concerning the theory of singular PDEs. In this framework, the Hardy inequality (see (2.1)) has a crucial role in the analysis of these equations. The intimate relation between Hardy's inequality and the nonexistence results of positive solutions of parabolic equations with a singular potential was discovered by Baras and Goldstein in [7]. For better contextualise this fact, let us consider the following heat operator with a singular potential

$$Pu = u_t - u_{xx} - \frac{\mu}{x^L}u, \quad x \in (0,1), \tag{1.6}$$

with Dirichlet boundary conditions. The case $L=2$, we have the so-called inverse-square potential (that appear in particular in the context of quantum mechanics or in linearized combustion problems). Baras and Goldstein [7,8] proved that the Cauchy-Dirichlet problem for Eq. (1.6) has a global positive solution (for any value of $\mu \in \mathbb{R}$) if $L<2$ whereas instantaneous and complete blow-up occurs (for any value of μ) if $L>2$. Next, when the exponent is critical i.e. when $L=2$, it is the value of the parameter μ that determines the behavior of the equation: if $\mu \leq 1/4$ (which is the optimal constant of the Hardy inequality, see (2.1)) global positive solutions exist, while, if $\mu > 1/4$, instantaneous and complete blow-up occurs (for other comments on this argument we refer to [40]). Later on, this result has been improved by J. L. Vazquez and E. Zuazua [43] where the authors gave a complete description of the functional framework in which heat equations perturbed by such inverse-square potentials are well-posed.

Recently, the null controllability properties of the linear heat equation with an inverse-square potential began to be studied. In [42], using Carleman estimates it has been proved that such equations can be controlled (in any time $T > 0$) by a locally distributed control under the condition $\mu \leq \frac{1}{4}$. On the contrary, if $\mu > \frac{1}{4}$, the null controllability fails as shown in [22]. After these first results, several other works followed extending them in various situations. See for instance [26,27,29,38,39] for parabolic problems in divergence and non divergence form with interior singularity and [15,16,40] for boundary singularity. It is also worth to mention the work [13] for the case of a potential with singularity distributed all over the boundary.

It is interesting to point out that all the previous papers are dealing with locally distributed controls and contributions are mainly based on a Carleman approach, suitably adapted for taking into account the singularity in the equation. To our best knowledge, the first results on controllability in the case of a boundary control, have been established in [11,37]. In more detail, the boundary controllability from $x = 1$ has been studied in [37] whereas the case of a control acting at $x = 0$ is treated in [11]. Different from the distributed case, the approach of these mentioned papers is based on decomposition in series and the moment method.

In the present work, we are interested in studying the more complex situation case in which we are exerting only one control force on the system (a boundary control) but we want to control the corresponding state $y = (y_1, y_2)^*$ that has two components. As far as we know, the analysis of problem (1.1) that we are presenting has never been treated in precedence, although it is a natural extension of the results achieved in the articles presented above.

Throughout this paper, we shall use the following notations. Given $D \in \mathcal{L}\left(\mathbb{R}^N; \mathbb{R}^M\right)$, $N, M \geq 1$, $D^* \in \mathcal{L}\left(\mathbb{R}^M; \mathbb{R}^N\right)$ stands for the transpose of D. For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z.

Let us now precise the main technical tools for obtaining our main controllability results. For the proof of the approximate controllability result, we are going to apply the following known result which relates the existence and bounds of biorthogonal families to complex exponentials to some gap conditions (see [6] or [25]).

Theorem 1.2. *Let $T > 0$. Suppose that $\{\Lambda_n\}_{n\geq 1}$ is a sequence of complex numbers such that, for some $\delta, \rho > 0$, one has*

$$\begin{cases} \Re(\Lambda_n) \geq \delta|\Lambda_n|, \qquad |\Lambda_n - \Lambda_m| \geq \rho|n-m|, \quad \forall n, m \geq 1, \\ \sum\limits_{n\geq 1} \dfrac{1}{|\Lambda_n|} < +\infty. \end{cases} \tag{1.7}$$

Then, there exists a family $\{q_n\}_{n\geq 1} \subset L^2(0,T)$ biorthogonal to $\{e^{-\Lambda_n t}\}_{n\geq 1}$ i.e., a family $\{q_n\}_{n\geq 1}$ in $L^2(0,T)$ such that

$$\int_0^T q_n(t) e^{-\Lambda_m t}\, dt = \delta_{nm}, \quad \forall n, m \geq 1.$$

Moreover, for every $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *for which*

$$\|q_n\|_{L^2(0,T)} \le C_\varepsilon e^{\varepsilon \Re(\Lambda_n)}, \qquad \forall n \ge 1.$$

It is worth mentioning that the above Theorem can also be applied to get the null controllability result for the system (1.1). However, it does not permit to deduce the required exponential estimate on the null-control cost.

For this reason, to obtain the null controllability result together with an estimate of the control cost, we will need the next result provided in [10].

Theorem 1.3. *Let* $\{\Lambda_n\}_{n\ge1}$ *be a sequence of complex numbers fulfilling the following assumptions:*

1. $\Lambda_n \ne \Lambda_m$ *for all* $n, m \ge 1$ *with* $n \ne m$*;*
2. $\Re(\Lambda_n) > 0$ *for every* $n \ge 1$*;*
3. *for some* $\delta > 0$
$$|\Im(\Lambda_n)| \le \delta\sqrt{\Re(\Lambda_n)} \quad \forall n \ge 1;$$
4. $\{\Lambda_n\}_{n\ge1}$ *is nondecreasing in modulus,*
$$|\Lambda_n| \le |\Lambda_{n+1}| \quad \forall n \ge 1;$$
5. $\{\Lambda_n\}_{n\ge1}$ *satisfies the following gap condition: for some* $\varrho, q > 0$,
$$\begin{cases} |\Lambda_n - \Lambda_m| \ge \varrho|n^2 - m^2| \quad \forall n, m : |n-m| \ge q, \\ \inf\limits_{n\ne m,\ |n-m|<q} |\Lambda_n - \Lambda_m| > 0; \end{cases} \tag{1.8}$$
6. *for some* $p, s > 0$,
$$|p\sqrt{r} - \mathcal{N}(r)| \le s, \quad \forall r > 0, \tag{1.9}$$
where $\mathcal{N}$ *is the counting function associated with the sequence* $\{\Lambda_n\}_{n\ge1}$*, that is the function defined by*
$$\mathcal{N}(r) = \#\{n : |\Lambda_n| \le r\}, \qquad \forall r > 0.$$

Then, there exists $T_0 > 0$*, such that for any* $T \in (0, T_0)$*, we can find a family* $\{q_n\}_{n\ge1} \subset L^2(-T/2, T/2)$ *biorthogonal to* $\{e^{-\Lambda_n t}\}_{n\ge1}$ *i.e., a family* $\{q_n\}_{n\ge1}$ *in* $L^2(-T/2, T/2)$ *such that*

$$\int_{-T/2}^{T/2} q_n(t) e^{-\Lambda_m t}\, dt = \delta_{nm}.$$

Moreover, there exists a positive constant $C > 0$ *independent of* T *for which*

$$\|q_n\|_{L^2(-T/2,T/2)} \le C e^{C\sqrt{\Re(\Lambda_n)} + \frac{C}{T}}, \qquad \forall n \ge 1. \tag{1.10}$$

The rest of the paper is organized as follows. In Sect. 2, we prove the well-posedness of the problem (1.1) in appropriate weighted spaces using the transposition method and recall some characterizations of the controllability. In sect. 3, we discuss the spectral analysis related to scalar singular operators and present a description of the spectrum associated with system (1.1) which will be useful for developing the moment method. Section 4 is devoted to studying the boundary approximate controllability problem for the system (1.1). Finally, in sect. 5, we prove the boundary null controllability result.

2. Preliminary results

2.1. Functional framework

In the study of evolution PDEs containing singular inverse-square potentials, it is by now classical that of great importance is the Hardy inequality, guaranteeing that for any function $z \in H_0^1(0,1)$ we have $\frac{z}{x} \in L^2(0,1)$ and the following estimate holds (see, for example, [30, Theorem 327] or [19, Lemma 5.3.1]):

$$\frac{1}{4}\int_0^1 \frac{z^2}{x^2}\,dx \leq \int_0^1 z_x^2\,dx. \tag{2.1}$$

Let us fix $\mu \leq \frac{1}{4}$. We introduce the associated functional space:

$$H_0^{1,\mu}(0,1) := \Big\{ z \in L^2(0,1) \cap H^1_{loc}((0,1]) \mid z(0) = z(1) = 0, \\ \text{and} \quad \int_0^1 (z_x^2 - \mu \frac{z^2}{x^2})\,dx < +\infty \Big\}.$$

Note that $H_0^{1,\mu}(0,1)$ is a Hilbert space obtained as the closure of $C_c^\infty(0,1)$, or $H_0^1(0,1)$, with respect to the norm

$$\forall z \in H_0^1(0,1), \qquad \|z\|_\mu := \Big(\int_0^1 (z_x^2 - \mu \frac{z^2}{x^2})\,dx \Big)^{\frac{1}{2}}.$$

In the case of a sub-critical parameter $\mu < \frac{1}{4}$, thanks to Hardy inequality (2.1), one can see that $\|\cdot\|_\mu$ is equivalent to the standard norm of $H_0^1(0,1)$, and thus $H_0^{1,\mu}(0,1) = H_0^1(0,1)$. In the critical case $\mu = \frac{1}{4}$, it has been proved (see [43]) that this identification does not hold anymore and the space $H_0^{1,\mu}(0,1)$ is slightly (but strictly) larger than $H_0^1(0,1)$. Further, in both cases, one can define $H^{-1,\mu}(0,1)$ the dual space of $H_0^{1,\mu}(0,1)$ with respect to the pivot space $L^2(0,1)$, endowed with the natural norm

$$\|f\|_{H^{-1,\mu}(0,1)} := \sup_{\|g\|_{H_0^{1,\mu}(0,1)}=1} \langle f, g\rangle_{H^{-1,\mu}(0,1), H_0^{1,\mu}(0,1)}.$$

Observe that, if we denote $H^{1,\mu}(0,1)$ the Hilbert space obtained as the completion of $H^1(0,1)$ with respect to the norm $\|\cdot\|_{L^2(0,1)} + \|\cdot\|_\mu$, we have

$$H_0^{1,\mu}(0,1) = \big\{ z \in H^{1,\mu}(0,1) \quad | \quad z(0) = z(1) = 0 \big\}.$$

We also define

$$H^{2,\mu}(0,1) = \big\{ z \in H^{1,\mu}(0,1) \cap H^2_{loc}((0,1]) \quad | \quad z_{xx} + \frac{\mu}{x^2} z \in L^2(0,1) \big\}.$$

We recall (see [41,43]) the following fundamental embedding result.

Theorem 2.1. *Let $\mu \leq \frac{1}{4}$ be given. Then $H_0^{1,\mu}(0,1) \hookrightarrow L^2(0,1)$ with compact embedding.*

Notice besides that, as $C_c^\infty(0,1)$ is dense both in $L^2(0,1)$ and in $H_0^{1,\mu}(0,1)$, $H_0^{1,\mu}(0,1)$ is dense in $L^2(0,1)$.

In what follows, for simplicity, we will always denote by $\langle \cdot, \cdot \rangle$ the standard scalar product of either $L^2(0,1)$ or $L^2(0,1)^2$, by $\langle \cdot, \cdot \rangle_{X', X}$ the duality pairing between the Hilbert space X and its dual X'. On the other hand, we will use $\|\cdot\|_\mu$ (resp. $\|\cdot\|_{H^{-1,\mu}}$) for denoting the norm of $H_0^{1,\mu}(0,1)^2$ (resp. $H^{-1,\mu}(0,1)^2$).

2.2. Well-posedness

Now, we are ready to give some results related to the existence, uniqueness and continuous dependence with respect to the data of the singular problem (1.1). To this aim, let us consider the nonhomogeneous adjoint problem:

$$\begin{cases} -\varphi_t - \varphi_{xx} - \frac{\mu}{x^2}\varphi = A^*\varphi + g, & \text{in } Q, \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_0, & \text{in } (0,1), \end{cases} \tag{2.2}$$

where A is given in (1.4) and φ_0 and g are functions in appropriate spaces.

Let us start with a first result on existence and uniqueness of strict solutions to system (2.2). One has (see [37, Definition 2.2] or [2, Theorem 2.1]):

Proposition 2.2. *Assume that $\varphi_0 \in H_0^{1,\mu}(0,1)^2$ and $g \in L^2(Q)^2$. Then, system (2.2) admits a unique strict solution*

$$\varphi \in \mathcal{W} := C^0([0,T]; H_0^{1,\mu}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \\ \cap L^2(0,T; H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)$$

such that

$$\|\varphi\|_{C^0([0,T];H_0^{1,\mu}(0,1)^2)} + \|\varphi\|_{H^1(0,T;L^2(0,1)^2)} + \|\varphi\|_{L^2(0,T;H^{2,\mu}(0,1)^2\cap H_0^{1,\mu}(0,1)^2)} \\ \leq C\Big(\|\varphi_0\|_\mu + \|g\|_{L^2(Q)^2}\Big), \tag{2.3}$$

for some positive constant C.

In view of Proposition 2.2, the following definition makes sense:

Definition 2.3. Let $y_0 \in H^{-1,\mu}(0,1)^2$ and $v \in L^2(0,T)$ be given. It will be said that $y \in L^2(Q)^2$ is a solution by transposition to (1.1) if, for each $g \in L^2(Q)^2$, the following identity holds

$$\iint_Q y\cdot g\,dx\,dt = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1,\mu},H_0^{1,\mu}} - \int_0^T B^*\varphi_x(t,1)\,v(t)\,dt, \tag{2.4}$$

where $\varphi \in \mathcal{W}$ is the solution of (2.2) associated to g and $\varphi_0 = 0$.

With this definition, we can state the result of existence and uniqueness of solution to system (1.1) by the transposition method in the spirit of [34].

Proposition 2.4. *Assume that $y_0 \in H^{-1,\mu}(0,1)^2$ and $v \in L^2(0,T)$. Then, system (1.1) admits a unique solution by transposition y that satisfies*

$$\begin{cases} y \in L^2(Q)^2 \cap C^0\big([0,T], H^{-1,\mu}(0,1)^2\big), \\ y_t \in L^2\big(0,T; (H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big), \\ y_t - y_{xx} - \frac{\mu}{x^2}y = Ay \quad \text{in} \quad L^2\big(0,T; (H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big), \\ y(0,\cdot) = y_0 \quad \text{in} \quad H^{-1,\mu}(0,1)^2 \end{cases} \tag{2.5}$$

and

$$\begin{aligned} &\|y\|_{L^2(Q)^2} + \|y\|_{C^0(H^{-1,\mu})} + \|y_t\|_{L^2((H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)')} \\ &\quad \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big), \end{aligned} \tag{2.6}$$

for a constant $C = C(T) > 0$.

Proof. Let $y_0 \in H^{-1,\mu}(0,1)^2$, $v \in L^2(0,T)$ and consider the following functional $\mathcal{T} : L^2(Q)^2 \to \mathbb{R}$ given by

$$\mathcal{T}(g) = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1,\mu},H_0^{1,\mu}} - \int_0^T B^*\varphi_x(t,1)v(t)\,dt,$$

where $\varphi \in \mathcal{W}$ is the solution of the adjoint system (2.2) associated to $g \in L^2(Q)^2$ and $\varphi_0 = 0$. In view of the results of Proposition 2.2, the mapping $\mathcal{T}$ is well defined and it is linear. Next, observe that

$$\big|\mathcal{T}(g)\big| \le \|y_0\|_{H^{-1,\mu}}\|\varphi(0,\cdot)\|_\mu + \|B^*\varphi_x(t,1)\|_{L^2(0,T)}\|v\|_{L^2(0,T)}.$$

Since $\varphi \in L^2(0,T; H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2) \subset L^2(0,T; H^2_{loc}((0,1])^2)$, by invoking the theory of traces (see, for example [1, Theorem 4.3.28]), one has:

$$\begin{aligned} \|B^*\varphi_x(t,1)\|_{L^2(0,T)} \le C\|\varphi_x(t,1)\|_{L^2(0,T)} &\le C\|\varphi\|_{L^2(0,T;H^2_{loc}((0,1])^2)} \\ &\le C\|\varphi\|_{L^2(0,T;H^{2,\mu}(0,1)^2\cap H_0^{1,\mu}(0,1)^2)} \\ &\text{(by (2.3))} \\ &\le C\|g\|_{L^2(Q)^2}. \end{aligned}$$

Using again (2.3), we also have:

$$\|\varphi(0,\cdot)\|_\mu \le C\|g\|_{L^2(Q)^2}.$$

Therefore,

$$\big|\mathcal{T}(g)\big| \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big)\|g\|_{L^2(Q)^2},$$

for all $g \in L^2(Q)^2$. We infer that $\mathcal{T}$ is bounded. Hence, by Riesz-Fréchet representation theorem, there exists a unique $y \in L^2(Q)^2$ satisfying (2.4), i.e., a solution by transposition of (1.1) in the sense of Definition 2.3. Moreover,

$$\|y\|_{L^2(Q)^2} = \|\mathcal{T}\| \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big).$$

Moreover, this solution satisfies the equality $y_t - y_{xx} - \frac{\mu}{x^2}y = Ay$ in $\mathcal{D}'(Q)^2$. Indeed, it suffices to multiply (2.2) associated to g and $\varphi_0 = 0$ by any regular solution y of (1.1) (corresponding to regular data (y_0, v)) and integrate on Q, obtaining

$$\begin{aligned} &\iint_Q y\cdot g\,dx\,dt + \iint_Q Ay\cdot\varphi\,dx\,dt \\ &\quad = \iint_Q (y_t - y_{xx} - \frac{\mu}{x^2}y)\varphi\,dx\,dt \\ &\quad + \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1,\mu},H_0^{1,\mu}} - \int_0^T B^*\varphi_x(t,1)\,v(t)\,dt. \end{aligned} \tag{2.7}$$

By comparing (2.4) and (2.7), we deduce that

$$\iint_Q (y_t - y_{xx} - \frac{\mu}{x^2} y)\varphi \, dx \, dt = \iint_Q Ay \cdot \varphi \, dx \, dt, \quad \forall \varphi \in \mathcal{W}.$$

Since $\mathcal{D}(Q)^2 \subset \mathcal{W}$, the claim follows.

Next, we are going to prove that the solution y of system (1.1) is more regular. To be precise, let us show that $y_{xx} + \frac{\mu}{x^2} y \in L^2\big(0,T; (H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big)$ and

$$\|y_{xx} + \frac{\mu}{x^2} y\|_{L^2((H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)')} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big). \quad (2.8)$$

To this end, let us consider two sequences $\{y_0^m\}_{m\geq 1} \subset H_0^{1,\mu}(0,1)^2$ and $\{v^m\}_{m\geq 1} \subset H_0^1(0,T)$ such that

$$y_0^m \to y_0 \quad \text{in} \quad H^{-1,\mu}(0,1)^2 \quad \text{and} \quad v^m \to v \quad \text{in} \quad L^2(0,T).$$

Now, the strategy consists in transforming our original system (1.1) (as done for instance in [37] in the context of a scalar singular parabolic equation) into a problem with homogeneous boundary conditions and a source term. To this end, let us introduce the following function:

$$\forall x \in [0,1], \qquad p(x) := x^{q_\mu} \qquad \text{where} \qquad q_\mu := \frac{1+\sqrt{1-4\mu}}{2}.$$

Formally, if y^m is the solution of (1.1) associated to y_0^m and v^m, then the function defined by

$$\tilde{y}^m(t,x) = y^m(t,x) - Bp(x)v^m(t),$$

is solution of

$$\begin{cases} \tilde{y}_t^m - \tilde{y}_{xx}^m - \frac{\mu}{x^2}\tilde{y}^m = A\tilde{y}^m + \tilde{f}^m(t,x), & \text{on } (0,T)\times(0,1), \\ \tilde{y}^m(t,0) = \tilde{y}^m(t,1) = 0, & \text{in} \quad (0,T), \\ \tilde{y}^m(0,x) = y_0^m(x), & \text{in} \quad (0,1), \end{cases} \quad (2.9)$$

where $\tilde{f}^m(t,x) = p(x)v^m(t)AB - p(x)v_t^m(t)B \in L^2(Q)^2$. With the previous regularity assumptions on the data, we can apply Proposition 2.2, to deduce that system (2.9) has a unique strict solution

$$\tilde{y}^m \in C^0([0,T]; H_0^{1,\mu}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2).$$

By setting

$$\tilde{v}^m(t,x) := Bp(x)v^m(t),$$

we observe that $\tilde{v}^m$ satisfies

$$\tilde{v}^m \in C^0([0,T]; H^{1,\mu}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}(0,1)^2).$$

Therefore, the problem (1.1) for v^m and y_0^m has a unique solution

$$y^m \in C^0([0,T]; H^{1,\mu}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}(0,1)^2)$$

which satisfies

$$\iint_Q y^m \cdot g\, dtdx = \langle y_0^m, \varphi(0,x)\rangle_{H^{-1,\mu},H_0^{1,\mu}} - \int_0^T B^*\varphi_x(t,1)v^m(t)\,dt, \quad \forall m \geq 1,$$

for all $g \in L^2(Q)^2$, where φ is the solution of the system (2.2) associated to g and $\varphi_0 = 0$. Using this last identity and (2.4), we get

$$\begin{cases} \|y^m\|_{L^2(Q)^2} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big) \quad \text{and} \\ y^m \to y \quad \text{in } L^2(Q)^2. \end{cases} \tag{2.10}$$

From this, we also deduce that $y_{xx}^m \to y_{xx}$ in $\mathcal{D}'(Q)^2$. Indeed, for every $\phi \in \mathcal{D}(Q)^2$, we have

$$\begin{aligned} &\Big|\iint_Q y_{xx}^m \cdot \phi\, dt\, dx - \iint_Q y_{xx}\cdot\phi\, dt\, dx\Big| \\ &\qquad = \Big|\iint_Q y^m \cdot \phi_{xx}\, dt\, dx - \iint_Q y\cdot \phi_{xx}\, dt\, dx\Big| \to 0, \end{aligned}$$

as claimed.

On the other hand, integrations by parts that may be justified as described in a detailed manner in [17], lead to

$$\iint_Q (y_{xx}^m + \frac{\mu}{x^2}y^m)\cdot\psi\, dt\, dx = \iint_Q y^m\cdot(\psi_{xx} + \frac{\mu}{x^2}\psi)\, dt\, dx - \int_0^T B^*\psi_x(t,1)\, v^m(t)\, dt,$$

for every $\psi \in L^2\big(0,T; H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2\big)$. From this equality we deduce that the sequence $\{y_{xx}^m + \frac{\mu}{x^2}y^m\}_{m\geq 1}$ is bounded in $L^2\big(0,T;(H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big)$. This property together with (2.10) implies that $y_{xx} + \frac{\mu}{x^2}y \in L^2\big(0,T;(H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big)$ and satisfies the estimate (2.8).

Combining the identity $y_t = y_{xx} + \frac{\mu}{x^2}y + Ay$ and the regularity property for $y_{xx} + \frac{\mu}{x^2}y$, we also see that $y_t \in L^2\big(0,T;(H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)'\big)$ and

$$\|y_t\|_{L^2((H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2)')} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big),$$

for some positive constant C. Therefore $y \in C([0,T];X^2)$, where X is the interpolation space $X = [L^2(0,1), (H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1))']_{1/2} = H^{-1,\mu}(0,1)$ (see [35, Proposition 2.1, p. 22]). In conclusion, we get

$$\|y\|_{C(H^{-1,\mu})} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1,\mu}}\big).$$

Finally, one can easily check that $y(0,\cdot) = y_0$ in $H^{-1,\mu}(0,1)^2$. This ends the proof. □

2.3. Duality

Let us consider the adjoint of system (1.1)

$$\begin{cases} -\varphi_t - \varphi_{xx} - \frac{\mu}{x^2}\varphi = A^*\varphi, & \text{in } Q, \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_0, & \text{in } (0,1), \end{cases} \tag{2.11}$$

where $\varphi_0 \in H_0^{1,\mu}(0,1)^2$. In the sequel, the solution to (2.11) will be called the adjoint state associated to φ_0. The controllability of system (1.1) can be characterized in terms of appropriate properties of the solutions to (2.11). In order to provide these characterizations, we use the following result which relates the solutions of systems (1.1) and (2.11). One has:

Proposition 2.5. *Let $y_0 \in H^{-1,\mu}(0,1)^2$, $v \in L^2(0,T)$ and $\varphi_0 \in H_0^{1,\mu}(0,1)^2$ be given. Let y be the state associated to y_0 and v and let φ be the adjoint state associated to φ_0. Then:*

$$\int_0^T B^*\varphi_x(t,1)v(t)\,dt = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1,\mu},H_0^{1,\mu}} - \langle y(T), \varphi_0\rangle_{H^{-1,\mu},H_0^{1,\mu}}. \tag{2.12}$$

This result is a straightforward consequence of the properties of y stated in Proposition 2.4.

One important result that will be useful for treating the approximate controllability of the system (1.1) is the following characterization in terms of the unique continuation property for the corresponding adjoint system (2.11). More precisely, we have:

Theorem 2.6. *Let us consider $T > 0$. Then, system (1.1) is approximately controllable at time T if and only if for all initial condition $\varphi_0 \in H_0^{1,\mu}(0,1)^2$ the solution to system (2.11) satisfies the unique continuation property*

$$B^*\varphi_x(\cdot,1) = 0 \quad \text{on} \quad (0,T) \Rightarrow \varphi_0 = 0 \quad \text{in} \quad (0,1) \quad (i.e., \quad \varphi = 0 \quad \text{in} \quad Q).$$

This result is well known. For a proof see, for instance [18,25] and [45].

3. Spectral analysis

3.1. Spectral properties of scalar singular operators

In this section, we discuss some preliminary results related to a spectral analysis of the operator $y \mapsto -y_{xx} - \frac{\mu}{x^2}y$, i.e., the nontrivial solutions (λ, Φ) of

$$\begin{cases} -\Phi''(x) - \frac{\mu}{x^2}\Phi(x) = \lambda\Phi(x), & x \in (0,1), \\ \Phi(0) = \Phi(1) = 0, \end{cases} \tag{3.1}$$

that will be essential for our purposes. For this reason, we will start by giving a brief account of some results concerning the Bessel functions that will be useful in the rest of the paper. For a complete treatise on Bessel functions, see [44].

For a real number $\nu \in \mathbb{R}_+$, we denote by J_ν the Bessel function of the first kind of order ν defined by the following Taylor series expansion around $x = 0$:

$$J_\nu(x) = \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m+\nu},$$

where $\Gamma(.)$ is the Gamma function.

We recall that the Bessel function J_ν satisfies the following differential equation

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \qquad x \in (0, +\infty).$$

Moreover, the function J_ν has an infinite number of real zeros which are simple with the possible exception of $x = 0$ (see [21,33]). We denote by $(j_{\nu,n})_{n\geq 1}$ the strictly increasing sequence of the positive zeros of J_ν:

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots$$

and we recall that

$$j_{\nu,n} \to +\infty \quad \text{as} \quad n \to +\infty$$

and the following bounds on the zeros $j_{\nu,n}$, which are provided in [36]:

- $\forall \nu \in \left[0, \frac{1}{2}\right]$, $\forall n \geq 1$,
$$\left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi \leq j_{\nu,n} \leq \left(n + \frac{\nu}{4} - \frac{1}{8}\right)\pi. \tag{3.2}$$
- $\forall \nu \geq \frac{1}{2}$, $\forall n \geq 1$,
$$\left(n + \frac{\nu}{4} - \frac{1}{8}\right)\pi \leq j_{\nu,n} \leq \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi. \tag{3.3}$$

In our investigation we need the following classical result (see [32, Proposition 7.8]):

Lemma 3.1. *Let $j_{\nu,n}, n \geq 1$ be the positive zeros of the Bessel function J_ν. Then, the following hold:*

- *If $\nu \in \left[0, \frac{1}{2}\right]$, the difference sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is nondecreasing and converges to π as $n \longrightarrow +\infty$.*
- *If $\nu \geq \frac{1}{2}$, the sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is nonincreasing and converges to π as $n \longrightarrow +\infty$.*

We also have that the Bessel functions enjoy the following integral formula (see [44]):

$$\int_0^1 xJ_\nu(j_{\nu,n}x)J_\nu(j_{\nu,m}x)\,dx = \frac{\delta_{nm}}{2}[J_\nu'(j_{\nu,n})]^2, \quad n, m \in \mathbb{N}^*,$$

where, δ_{nm} is the Kronecker symbol.

Next we recall the expression of the eigenvalues and eigenfunctions related to problem (3.1) that have been computed in [37]. To this end, given $\mu \leq \frac{1}{4}$, let us introduce the quantity

$$\nu_\mu := \sqrt{\frac{1}{4} - \mu}.$$

With the previous notation, we have the following result:

Proposition 3.2. *Assume $\mu \leq \frac{1}{4}$. Then the admissible eigenvalues λ for problem (3.1) are given by*

$$\lambda_{\mu,n} = j^2_{\nu_\mu,n}, \qquad \forall n \geq 1. \tag{3.4}$$

and the associated normalized (in $L^2(0,1)$) eigenfunctions are

$$\Phi_{\mu,n}(x) = \frac{\sqrt{2}}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|}\sqrt{x}J_{\nu_\mu}(j_{\nu_\mu,n}x), \qquad x \in (0,1), \quad n \geq 1. \tag{3.5}$$

Moreover, the family $(\Phi_{\mu,n})_{n\geq 1}$ forms an orthonormal basis of $L^2(0,1)$.

We end this subsection with the following lemma which will be used later. One has:

Lemma 3.3. *The sequence of eigenvalues $(\lambda_{\mu,n})_{n\geq 1}$ satisfies the following gap condition: there is a constant $\rho > 0$ such that*

$$|\lambda_{\mu,n} - \lambda_{\mu,m}| \geq \rho|n^2 - m^2|, \qquad \forall n, m \geq 1. \tag{3.6}$$

Proof. Let $n, m \in \mathbb{N}^\star$ with $n \geq m$. We have

$$\begin{aligned}\lambda_{\mu,n} - \lambda_{\mu,m} &= (j^2_{\nu_\mu,n} - j^2_{\nu_\mu,m})\\ &= (j_{\nu_\mu,n} - j_{\nu_\mu,m})(j_{\nu_\mu,n} + j_{\nu_\mu,m})\\ &= \Big((j_{\nu_\mu,n} - j_{\nu_\mu,n-1}) + \ldots + (j_{\nu_\mu,m+1} - j_{\nu_\mu,m})\Big)(j_{\nu_\mu,n} + j_{\nu_\mu,m}).\end{aligned}$$

We can now distinguish the two different cases $\nu_\mu \in \left[0, \dfrac{1}{2}\right]$ and $\nu_\mu \geq \dfrac{1}{2}$, depending on the parameter μ.

- if $\nu_\mu \in \left[0, \dfrac{1}{2}\right]$ (i.e. $\mu \in \left[0, \frac{1}{4}\right]$), by virtue of Lemma 3.1 we immediately have that

  $$j_{\nu_\mu,n} - j_{\nu_\mu,n-1} \geq j_{\nu_\mu,2} - j_{\nu_\mu,1}, \quad \forall n \geq 2.$$

  Therefore,

  $$\lambda_{\mu,n} - \lambda_{\mu,m} \geq (n-m)(j_{\nu_\mu,2} - j_{\nu_\mu,1})(j_{\nu_\mu,n} + j_{\nu_\mu,m}).$$

  Using (3.2), the last inequality becomes:

  $$\lambda_{\mu,n} - \lambda_{\mu,m} \geq \frac{7}{8}\pi^2(n-m)\big(n+m+\nu_\mu - \frac{1}{2}\big).$$

  Moreover, we have

  $$\big(n+m+\nu_\mu - \frac{1}{2}\big) > \frac{n+m}{2},$$

  and thus, that there exists $\rho = \frac{7}{16}\pi^2$ such that

  $$\lambda_{\mu,n} - \lambda_{\mu,m} \geq \rho(n^2 - m^2).$$

- Let us now see the case $\nu_\mu \geq \frac{1}{2}$ (i.e. $\mu \leq 0$). Here we use the fact that the sequence $(j_{\nu_\mu,n+1} - j_{\nu_\mu,n})_n$ is nonincreasing and converges to π. This ensures that

$$j_{\nu_\mu,n+1} - j_{\nu_\mu,n} \geq \pi, \qquad \forall n \geq 1.$$

Therefore:

$$\lambda_{\mu,n} - \lambda_{\mu,m} \geq \pi(n-m)(j_{\nu_\mu,n} + j_{\nu_\mu,m}).$$

Owing to (3.3), we also have

$$j_{\nu_\mu,n} + j_{\nu_\mu,m} \geq \big(n + m + \frac{\nu_\mu}{2} - \frac{1}{4}\big)\pi \geq \pi(n+m).$$

Combining the above last two estimates, the thesis follows with $\rho = \pi^2$.

Thus, in every case there holds

$$\lambda_{\mu,n} - \lambda_{\mu,m} \geq \rho(n^2 - m^2).$$

In both cases, after reversing the roles of n and m, one has

$$\lambda_{\mu,m} - \lambda_{\mu,n} \geq \rho(m^2 - n^2).$$

Hence,

$$|\lambda_{\mu,n} - \lambda_{\mu,m}| \geq \rho|n^2 - m^2|, \qquad \forall n, m \geq 1,$$

for a constant $\rho > 0$. □

3.2. Spectral properties of vectorial singular operators

Let A be given by (1.4) and consider the singular vectorial operator

$$\begin{aligned} L : D(L) \subset L^2(0,1)^2 &\to L^2(0,1)^2 \\ y &\mapsto -y_{xx} - \frac{\mu}{x^2} y - Ay, \end{aligned} \tag{3.7}$$

with domain $D(L) = H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2$ and also its adjoint L^*.

This section will be devoted to giving some properties of the eigenvalues and eigenfunctions of the operators L and L^* which will be useful for developing the moment method. Let us first analyze the spectrum of the operators L and L^*:

Proposition 3.4. *Let us consider the operator L given by (3.7) and its adjoint L^*. Then,*

1. *The spectra of L and L^* are given by $\sigma(L) = \sigma(L^*) = \{\lambda_{\mu,n}^{(1)}, \lambda_{\mu,n}^{(2)}\}_{n\geq 1}$ with*

$$\lambda_{\mu,n}^{(1)} = \lambda_{\mu,n} - \alpha_1, \quad \lambda_{\mu,n}^{(2)} = \lambda_{\mu,n} - \alpha_2, \quad \forall n \geq 1, \tag{3.8}$$

where α_1 and α_2 are the eigenvalues of the matrix A defined by :

- *Case 1: $a_2^2 + 4a_1 > 0$,*

$$\alpha_1 = \frac{1}{2}\Big(a_2 - \sqrt{a_2^2 + 4a_1}\Big) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\Big(a_2 + \sqrt{a_2^2 + 4a_1}\Big). \tag{3.9}$$

- *Case 2:* $a_2^2 + 4a_1 < 0$,

$$\alpha_1 = \frac{1}{2}\Big(a_2 + i\sqrt{-(a_2^2+4a_1)}\Big) \text{ and } \alpha_2 = \frac{1}{2}\Big(a_2 - i\sqrt{-(a_2^2+4a_1)}\Big). \tag{3.10}$$

2. *For each* $n \geq 1$, *the corresponding eigenfunctions of* L *(resp.,* L^**) associated to* $\lambda_{\mu,n}^{(1)}$ *and* $\lambda_{\mu,n}^{(2)}$ *are respectively given by*

$$\psi_n^{(1)} = U_1\Phi_{\mu,n}, \qquad \psi_n^{(2)} = U_2\Phi_{\mu,n}, \tag{3.11}$$

with

$$U_1 = \frac{1}{\alpha_1-\alpha_2}\begin{pmatrix}-\alpha_2\\ 1\end{pmatrix} \quad \text{and} \quad U_2 = \frac{1}{\alpha_2-\alpha_1}\begin{pmatrix}-\alpha_1\\ 1\end{pmatrix}$$

(resp.,

$$\Psi_n^{(1)} = V_1\Phi_{\mu,n}, \qquad \Psi_n^{(2)} = V_2\Phi_{\mu,n}, \tag{3.12}$$

with

$$V_1 = \begin{pmatrix}1\\ \alpha_1\end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix}1\\ \alpha_2\end{pmatrix}.$$

Proof. We will prove the result for the operator L. The same reasoning provides the proof for its adjoint L^*.

Using the fact that the function $\Phi_{\mu,n}$ is the eigenfunction of the Dirichlet-singular elliptic operator $(-\partial_{xx} - \frac{\mu}{x^2})$ associated to the eigenvalue $\lambda_{\mu,n}$, one can see that the eigenvalues of the operator L correspond to the eigenvalues of the matrices

$$\lambda_{\mu,n}Id - A, \qquad \forall n \geq 1,$$

($Id \in \mathcal{L}(\mathbb{C}^2)$ is the identity matrix) and the associated eigenfunctions of L are given under the form $\psi_n(\cdot) = z_n\Phi_{\mu,n}(\cdot)$, where $z_n \in \mathbb{C}^2$ is the associated eigenvector of the matrix $\lambda_{\mu,n}Id - A$.

Taking into account the expression of the characteristic polynomial of $\lambda_{\mu,n}Id - A$:

$$P(z) = z^2 - z(2\lambda_{\mu,n} - a_2) + \lambda_{\mu,n}(\lambda_{\mu,n} - a_2) - a_1, \qquad n \geq 1,$$

a direct computation provides the formulas (3.8) and (3.11) as eigenvalues and associated eigenfunctions of the operator L. This ends the proof. □

Let us now check that the sequence of eigenvalues of L and L^* fulfills the conditions in Theorem 1.3. One has

Proposition 3.5. *Assume that the following condition holds*

$$\lambda_{\mu,n} - \lambda_{\mu,l} \neq \alpha_1 - \alpha_2, \quad \forall n, l \in \mathbb{N}^*, \quad \text{with} \quad n \neq l. \tag{3.13}$$

Then, one can construct a family from the spectrum $\{\lambda_{\mu,n}^{(1)}, \lambda_{\mu,n}^{(2)}\}_{n\geq 1}$, *defined by*

$$\begin{aligned}\{\Lambda_{\mu,n}\}_{n\geq 1} &= \{\lambda_{\mu,n}^{(1)} + \alpha_2, \lambda_{\mu,n}^{(2)} + \alpha_2\}_{n\geq 1}\\ &= \{\lambda_{\mu,n} + \alpha_2 - \alpha_1 : n \geq 1\} \cup \{\lambda_{\mu,n} : n \geq 1\},\end{aligned} \tag{3.14}$$

which satisfies the hypotheses in Theorem 1.3.

Here, we note that the sequence $\{\lambda_{\mu,n}^{(1)}, \lambda_{\mu,n}^{(2)}\}_{n\geq 1}$ *refers to the union of two families of sequences.*

Proof. We distinguish between two cases depending on the spectrum of matrix A.

Case 1: A **has two real eigenvalues** α_1 **and** α_2, **chosen such that** $\alpha_1 < \alpha_2$.

Let us introduce the sequence $\{\Lambda_{\mu,n}\}_{n\geq 1}$, where

$$\{\Lambda_{\mu,n} : n \geq 1\} := \{\lambda_{\mu,n}^{(1)} + \alpha_2, \lambda_{\mu,n}^{(2)} + \alpha_2\}_{n\geq 1}.$$

The hypothesis 1) holds true if and only if the condition (3.13) is satisfied. In addition, the hypotheses 2) and 3) are obviously satisfied by definition.

Let us now show the hypothesis 4). Since $\alpha_2 - \alpha_1 > 0$, observe that $\{\lambda_{\mu,n}^{(1)} + \alpha_2\}_{n\geq 1}$ and $\{\lambda_{\mu,n}^{(2)} + \alpha_2\}_{n\geq 1}$ are increasing sequences satisfying

$$0 < \lambda_{\mu,n}^{(2)} + \alpha_2 < \lambda_{\mu,n}^{(1)} + \alpha_2, \quad \forall n \geq 1.$$

Thus, we deduce that the sequence $\{\Lambda_{\mu,n}\}_{n\geq 1}$ can be rearranged into a positive increasing sequence.

Let us move to prove hypothesis 5). For this purpose, we are going to give an explicit rearrangement of the sequence $\{\lambda_{\mu,n}^{(1)} + \alpha_2, \lambda_{\mu,n}^{(2)} + \alpha_2\}_{n\geq 1}$. Firstly, observe that there exists an integer $n_0 \geq 1$ and a constant $C > 0$ such that

$$\begin{aligned} &\lambda_{\mu,n-1}^{(1)} < \lambda_{\mu,n}^{(2)} < \lambda_{\mu,n}^{(1)} < \lambda_{\mu,n+1}^{(2)} < \cdots, \quad \forall n \geq n_0, \text{ and} \\ &\min_{n\geq n_0} \{\lambda_{\mu,n}^{(2)} - \lambda_{\mu,n-1}^{(1)}, \lambda_{\mu,n}^{(1)} - \lambda_{\mu,n}^{(2)}\} > C. \end{aligned} \tag{3.15}$$

Indeed, using (3.6), one has

$$\begin{aligned} \lambda_{\mu,n}^{(2)} - \lambda_{\mu,n-1}^{(1)} &= \lambda_{\mu,n} - \lambda_{\mu,n-1} + \alpha_1 - \alpha_2 \\ &\geq \rho(2n-1) + \alpha_1 - \alpha_2 \underset{n\to+\infty}{\longrightarrow} +\infty. \end{aligned} \tag{3.16}$$

From (3.16) and the fact that $\lambda_{\mu,n}^{(1)} - \lambda_{\mu,n}^{(2)} = \alpha_2 - \alpha_1 > 0$, we can conclude (3.15).

Therefore, if $1 \leq n \leq 2n_0 - 2$, we define $\Lambda_{\mu,n}$ such that

$$\begin{gathered} \{\Lambda_{\mu,n}\}_{1\leq n\leq 2n_0-2} = \{\lambda_{\mu,n}^{(1)} + \alpha_2\}_{1\leq n\leq n_0-1} \cup \{\lambda_{\mu,n}^{(2)} + \alpha_2\}_{1\leq n\leq n_0-1} \quad \text{and} \\ \Lambda_{\mu,n} < \Lambda_{\mu,n+1} \quad \forall n : 1 \leq n \leq 2n_0 - 3. \end{gathered}$$

Moreover, from $(2n_0-1)$-th term, we choose to arrange the sequence as follows:

$$\Lambda_{\mu,2n-1} = \lambda_{\mu,n}^{(2)} + \alpha_2 \quad \text{and} \quad \Lambda_{\mu,2n} = \lambda_{\mu,n}^{(1)} + \alpha_2, \ \forall n \geq n_0. \tag{3.17}$$

Since the elements of the sequence $\{\Lambda_{\mu,n}\}_{n\geq 1}$ are pairwise different and from (3.15), one has:

$$\inf_{n,m\geq 1: n\neq m} |\lambda_{\mu,n}^{(1)} - \lambda_{\mu,m}^{(2)}| > 0. \tag{3.18}$$

Hence, thanks to (3.18), the sequence $\{\Lambda_{\mu,n}\}_{n\geq 1}$ satisfies the second inequality in (1.8) for every $q \geq 1$.

Our next task will be to prove the first inequality of (1.8) for appropriate $q > 0$ and $\varrho > 0$. To this aim, as it has been remarked in [28], it is enough to prove the existence of $q > 0$ and $\tilde{\varrho} > 0$ such that

$$|\Lambda_n - \Lambda_m| \geq \tilde{\varrho}|n^2 - m^2| \quad \forall n, m \geq q,\ |n - m| \geq q. \tag{3.19}$$

We divide the proof of (3.19) into two steps.

1. Observe that, if $n, m \in \mathbb{N}^\star$ are such that $n, m \geq n_0$ and $|n - m| \geq n_0$, then by (3.17) and using (3.6) we have

$$|\Lambda_{\mu,2n} - \Lambda_{\mu,2m}| = |\lambda_{\mu,n} - \lambda_{\mu,m}| \geq \rho|n^2 - m^2| = \frac{\rho}{4}|(2n)^2 - (2m)^2|$$

and

$$\begin{aligned}|\Lambda_{\mu,2n-1} - \Lambda_{\mu,2m-1}| = |\lambda_{\mu,n} - \lambda_{\mu,m}| &\geq \rho|n^2 - m^2| \\ &\geq \frac{\rho}{4}|(2n-1)^2 - (2m-1)^2|.\end{aligned}$$

We obtain thus the proof of (3.19) for $q = n_0$ and $\tilde{\varrho} = \frac{\rho}{4}$.

2. Let $n, m \in \mathbb{N}^\star$ such that $n, m \geq n_0$. From (3.17), by denoting $\tilde{n} = 2n$ and $\tilde{m} = 2m - 1$ and using again (3.6), we readily see that

$$\begin{aligned}|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}| &= \left|\lambda^{(1)}_{\mu,n} - \lambda^{(2)}_{\mu,m}\right| \\ &= \left|\lambda_{\mu,n} - \lambda_{\mu,m} + (\alpha_2 - \alpha_1)\right| \\ &\geq \rho|n^2 - m^2| - (\alpha_2 - \alpha_1) \\ &= \frac{\rho}{4}|\tilde{n}^2 - (\tilde{m}+1)^2| - (\alpha_2 - \alpha_1) \\ &= \frac{\rho}{4}|\tilde{n}^2 - \tilde{m}^2 - 2\tilde{m} - 1| - (\alpha_2 - \alpha_1).\end{aligned}$$

Now, observe that if $\tilde{n} < \tilde{m}$, we have

$$|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}| \geq \frac{\rho}{4}(\tilde{m}^2 - \tilde{n}^2)\Big(1 - \frac{4(\alpha_2 - \alpha_1)}{\rho(\tilde{m}^2 - \tilde{n}^2)}\Big).$$

Let us take an integer $q_0 \geq \max\{2n_0 - 1, \frac{4(\alpha_2 - \alpha_1)}{\rho}\}$. Then, $\forall \tilde{m}, \tilde{n} \geq q_0$ with $|\tilde{m} - \tilde{n}| \geq q_0$, one has

$$\begin{aligned}|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}| &\geq \frac{\rho}{4}(\tilde{m}^2 - \tilde{n}^2)\Big(1 - \frac{4(\alpha_2 - \alpha_1)}{\rho(\tilde{m}+\tilde{n})q_0}\Big) \\ &\geq \frac{\rho}{4}(\tilde{m}^2 - \tilde{n}^2)\Big(1 - \frac{2(\alpha_2 - \alpha_1)}{\rho q_0}\Big) \\ &\geq \frac{\rho}{8}(\tilde{m}^2 - \tilde{n}^2).\end{aligned}$$

On the other hand, if $\tilde{n} > \tilde{m}$, we have

$$|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}| \geq \frac{\rho}{4}(\tilde{n}^2 - \tilde{m}^2)\Big(1 - \Big(\frac{4(\alpha_2 - \alpha_1)}{\rho} + 2\tilde{m} + 1\Big)\frac{1}{(\tilde{n}^2 - \tilde{m}^2)}\Big).$$

Let us work with an integer q_1 given by

$$q_1 \geq \max\{2n_0 - 1, \frac{4(\alpha_2 - \alpha_1)}{\rho} + 4\}.$$

Thus, if $\tilde{n}, \tilde{m} \in \mathbb{N}^*$ are such that $\tilde{n}, \tilde{m} \geq q_1$ and $|\tilde{n} - \tilde{m}| \geq q_1$, then one has

$$\begin{aligned}|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}| &\geq \frac{\rho}{4}(\tilde{n}^2 - \tilde{m}^2)\Big(1 - \Big(\frac{4(\alpha_2-\alpha_1)}{\rho} + 2\tilde{m} + 1\Big)\frac{1}{2\tilde{m}q_1}\Big)\\ &\geq \frac{\rho}{4}(\tilde{n}^2 - \tilde{m}^2)\Big(1 - \frac{1}{q_1}\Big(\frac{2(\alpha_2-\alpha_1)}{\rho} + 2\Big)\Big)\\ &\geq \frac{\rho}{8}(\tilde{n}^2 - \tilde{m}^2).\end{aligned}$$

Hence, choosing $q = max\{q_0, q_1\}$, (3.19) follows immediately for $\tilde{\varrho} = \frac{\rho}{8}$. In conclusion, we have proved the existence of a number $q \geq 1$ such that (3.19) holds.

Let us now show the hypothesis 6). From the definition of $\{\Lambda_{\mu,n}\}_{n\geq 1}$, for any $r > 0$, we can write:

$$\begin{aligned}\mathcal{N}(r) &= \#\{k: \ \lambda_{\mu,k} + \alpha_2 - \alpha_1 \leq r\} + \#\{k: \ \lambda_{\mu,k} \leq r\}\\ &= \#\mathcal{A}_1(r) + \#\mathcal{A}_2(r) = n_1 + n_2,\end{aligned}$$

where $\mathcal{A}_i(r) = \{k: \ \lambda^{(i)}_{\mu,k} + \alpha_2 \leq r\}$ and $n_i = \#\mathcal{A}_i(r)$, i=1,2. Our purpose is to prove suitable estimates for n_1 and n_2.

From the definition of $\mathcal{A}_2(r)$ and n_2, we deduce that n_2 is a natural number which is characterized by $\lambda_{\mu,n_2} \leq r$ and $\lambda_{\mu,n_2+1} > r$. We distinguish two cases depending on the value of ν_μ. Let us start by the case $\nu_\mu \leq \frac{1}{2}$. From the inequality $\lambda_{\mu,n_2} \leq r$ and by (3.2), we have $\big(n_2 + \frac{\nu_\mu}{2} - \frac{1}{4}\big)^2\pi^2 \leq r$ so that

$$n_2 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu_\mu}{2} + \frac{1}{4}. \tag{3.20}$$

On the other hand, from the inequality $\lambda_{\mu,n_2+1} > r$, we get

$$n_2 > \frac{\sqrt{r}}{\pi} - \frac{\nu_\mu}{4} - \frac{7}{8}.$$

Summarizing, n_2 is a nonnegative integer such that

$$\frac{\sqrt{r}}{\pi} - \frac{\nu_\mu}{4} - \frac{7}{8} < n_2 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu_\mu}{2} + \frac{1}{4}, \quad \forall r > 0. \tag{3.21}$$

Next we are going to estimate n_1. Using arguments similar to the ones used above, we can see that

$$\lambda_{\mu,n_1} + \alpha_2 - \alpha_1 \leq r$$

and

$$\lambda_{\mu,n_1+1} + \alpha_2 - \alpha_1 > r$$

imply that

$$\frac{\sqrt{r+\alpha_1-\alpha_2}}{\pi} - \frac{\nu_\mu}{4} - \frac{7}{8} < n_1 \leq \frac{\sqrt{r+\alpha_1-\alpha_2}}{\pi} - \frac{\nu_\mu}{2} + \frac{1}{4}.$$

Then, using the fact that $\sqrt{a}-\sqrt{b}\le\sqrt{a-b}$ and $\sqrt{a-b}\le\sqrt{a}$ provided $a\ge b>0$, one gets

$$\frac{\sqrt{r}}{\pi}-\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}-\frac{\nu_\mu}{4}-\frac{7}{8}<n_1\le\frac{\sqrt{r}}{\pi}-\frac{\nu_\mu}{2}+\frac{1}{4},\quad \forall r>0. \tag{3.22}$$

Recall that $\mathcal{N}(r)=n_1+n_2$. Thus, combining (3.21) and (3.22), it follows that for $\nu_\mu\le\frac{1}{2}$:

$$\frac{2\sqrt{r}}{\pi}-\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}-\frac{\nu_\mu}{2}-\frac{7}{4}<\mathcal{N}(r)\le\frac{2\sqrt{r}}{\pi}-\nu_\mu+\frac{1}{2},\quad \forall r>0,$$

and deduce (1.9) with

$$p=\frac{2}{\pi}\quad\text{and}\quad s=\max\{\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}+\frac{\nu_\mu}{2}+\frac{7}{4},\ -\nu_\mu+\frac{1}{2}\}=\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}+\frac{\nu_\mu}{2}+\frac{7}{4}.$$

The case $\nu_\mu\ge\frac{1}{2}$ can be treated in a similar way, but, instead of working with the bounds (3.2), we will use (3.3) to obtain

$$\frac{\sqrt{r}}{\pi}-\frac{\nu_\mu}{2}-\frac{3}{4}<n_2\le\frac{\sqrt{r}}{\pi}-\frac{\nu_\mu}{4}+\frac{1}{8},\quad \forall r>0, \tag{3.23}$$

and

$$\frac{\sqrt{r}}{\pi}-\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}-\frac{\nu_\mu}{2}-\frac{3}{4}<n_1\le\frac{\sqrt{r}}{\pi}-\frac{\nu_\mu}{4}+\frac{1}{8},\quad \forall r>0. \tag{3.24}$$

From the inequalities (3.23) and (3.24), we obtain that:

$$\frac{2\sqrt{r}}{\pi}-\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}-\nu_\mu-\frac{3}{2}<\mathcal{N}(r)\le\frac{2\sqrt{r}}{\pi}-\frac{\nu_\mu}{2}+\frac{1}{4},\quad \forall r>0,$$

and again deduce (1.9) with

$$p=\frac{2}{\pi}\quad\text{and}\quad s=\max\left\{\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}+\nu_\mu+\frac{3}{2},\ -\frac{\nu_\mu}{2}+\frac{1}{4}\right\}=\frac{\sqrt{\alpha_2-\alpha_1}}{\pi}+\nu_\mu+\frac{3}{2}.$$

We thus obtain the last hypothesis 6) of Theorem 1.3. This ends the proof in this case.

Case 2: A has two complex eigenvalues α_1 and α_2.

In this case $a_2^2+4a_1<0$,

$$\alpha_1=\frac{a_2}{2}+i\beta,\quad\text{and}\quad \alpha_2=\frac{a_2}{2}-i\beta,$$

where $\beta:=\frac{1}{2}\sqrt{-(a_2^2+4a_1)}$.

Now, we consider the complex sequence $\{\Lambda_{\mu,n}\}_{n\ge1}$, with

$$\begin{aligned}\Lambda_{\mu,2n-1}&=\lambda^{(2)}_{\mu,n}+\alpha_2=\lambda_{\mu,n},\quad \forall n\ge1,\\ \Lambda_{\mu,2n}&=\lambda^{(1)}_{\mu,n}+\alpha_2=\lambda_{\mu,n}-2i\beta,\quad \forall n\ge1.\end{aligned} \tag{3.25}$$

Let us check if the hypotheses in Theorem 1.3 hold true for $\left\{\Lambda_{\mu,n}\right\}_{n\ge1}$. First, it is clearly that the sequence $\left\{\Lambda_{\mu,n}\right\}_{n\ge1}$ always satisfies the hypothesis 1). Furthermore, the hypothesis 2) follows directly from the fact that

$$\Re(\Lambda_{\mu,2n})=\Re(\Lambda_{\mu,2n-1})=\lambda_{\mu,n}>0.$$

The hypothesis 3) is clearly fulfilled. Indeed, one can find $\delta > 0$ (which depends on β) such that

$$|\Im(\Lambda_{\mu,2n})| = 2\beta \leq \delta\sqrt{\Re(\Lambda_{\mu,2n})}$$

and

$$|\Im(\Lambda_{\mu,2n-1})| = 0 \leq \delta\sqrt{\Re(\Lambda_{\mu,2n-1})}.$$

Let us now prove hypothesis 4). To this end, it suffices to prove that there exists an integer $\tilde{n}_0 \geq 1$ such that for all $n \geq \tilde{n}_0$ $|\Lambda_{\mu,2n}| \leq |\Lambda_{\mu,2n+1}|$. Using (3.6), we have

$$\begin{aligned} |\Lambda_{\mu,2n+1}|^2 - |\Lambda_{\mu,2n}|^2 &= \lambda_{\mu,n+1}^2 - \lambda_{\mu,n}^2 - 4\beta^2 \\ &\geq (\lambda_{\mu,n+1} - \lambda_{\mu,n})^2 - 4\beta^2 \\ &\geq \rho^2|(n+1)^2 - n^2|^2 - 4\beta^2 \\ &= \rho^2(2n+1)^2 - 4\beta^2, \end{aligned}$$

which implies that

$$\lim_{n\to+\infty} \left(|\Lambda_{\mu,2n+1}|^2 - |\Lambda_{\mu,2n}|^2\right) = +\infty.$$

Therefore, there exists $\tilde{n}_0 \geq 1$ such that $\{\Lambda_{\mu,n}\}_{n\geq 2\tilde{n}_0}$ is nondecreasing in modulus. This shows hypothesis 4).

Let us now check if the hypothesis 5) holds true. To this aim, we choose to arrange the sequence $\{\Lambda_{\mu,n}\}_{n\geq 1}$ defined in (3.25) as follows:

$$\begin{aligned} \left\{\Lambda_{\mu,n}\right\}_{1\leq n\leq 2\tilde{n}_0-2} = \{\lambda_{\mu,n}^{(1)} + \alpha_2\}_{1\leq n\leq \tilde{n}_0-1} \cup \{\lambda_{\mu,n}^{(2)} + \alpha_2\}_{1\leq n\leq \tilde{n}_0-1} \quad \text{and} \\ |\Lambda_{\mu,n}| < |\Lambda_{\mu,n+1}| \quad \forall n : 1 \leq n \leq 2\tilde{n}_0 - 3. \end{aligned}$$

Moreover, from $(2\tilde{n}_0 - 1)$-th term, we set:

$$\Lambda_{\mu,2n-1} = \lambda_{\mu,n}^{(2)} + \alpha_2 \quad \text{and} \quad \Lambda_{\mu,2n} = \lambda_{\mu,n}^{(1)} + \alpha_2, \ \forall n \geq \tilde{n}_0. \tag{3.26}$$

First, observe that the second property is actually satisfied for any q. Our next objective will be to prove the first inequality in (1.8). Arguing as done in the real case, by Lemma 3.3, there exists $\rho > 0$ such that

$$|\Lambda_{\mu,2n} - \Lambda_{\mu,2m}| \geq \frac{\rho}{4}|(2n)^2 - (2m)^2|, \quad \forall n, m \geq \tilde{n}_0$$

and

$$|\Lambda_{\mu,2n-1} - \Lambda_{\mu,2m-1}| \geq \frac{\rho}{4}|(2n-1)^2 - (2m-1)^2|, \quad \forall n, m \geq \tilde{n}_0.$$

Moreover, denoting $\tilde{n} = 2n$ and $\tilde{m} = 2m - 1$, one can prove that there exists $q \geq \max\{4, 2\tilde{n}_0 - 1\}$ such that $\forall \tilde{n}, \tilde{m} \geq q$ with $|\tilde{n} - \tilde{m}| \geq q$, we have

$$|\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}|^2 = |\Lambda_{\mu,2n} - \Lambda_{\mu,2m-1}|^2 \geq \left(\frac{\rho}{8}|\tilde{n}^2 - \tilde{m}^2|\right)^2.$$

Indeed, by (3.6), for $\tilde{n}, \tilde{m} \geq 2\tilde{n}_0 - 1$ we have

$$\begin{aligned} |\Lambda_{\mu,\tilde{n}} - \Lambda_{\mu,\tilde{m}}|^2 &= |\Lambda_{\mu,2n} - \Lambda_{\mu,2m-1}|^2 \\ &= \left|\lambda_{\mu,n} - \lambda_{\mu,m}\right|^2 + 4\beta^2 \\ &\geq \left|\lambda_{\mu,n} - \lambda_{\mu,m}\right|^2 \\ &\geq \left(\rho|n^2 - m^2|\right)^2 = \left(\frac{\rho}{4}|\tilde{n}^2 - \tilde{m}^2 - 2\tilde{m} - 1|\right)^2. \end{aligned}$$

Next, if $|\tilde{n} - \tilde{m}| \geq 4$, simple computation gives

$$\begin{aligned} |\tilde{n}^2 - \tilde{m}^2 - 2\tilde{m} - 1| &= |(\tilde{n}^2 - \tilde{m}^2)(1 - \frac{2\tilde{m} + 1}{\tilde{n}^2 - \tilde{m}^2})| \\ &\geq \frac{1}{2}|\tilde{n}^2 - \tilde{m}^2|. \end{aligned}$$

Hence, the conclusion follows by working with q given by

$$q \geq \max\{4, 2\tilde{n}_0 - 1\}.$$

Finally, proceeding as in the real case, it is not difficult to obtain some suitable parameters p and s for which the inequality (1.9) holds. This completes the proof of Proposition 3.5. □

We will finish this section giving a result on the set of eigenfunctions of the operators L and L^*. It reads as follows:

Proposition 3.6. *Let us consider the sequences*

$$\mathcal{B} = \left\{\psi_n^{(1)}, \psi_n^{(2)}, \quad n \geq 1\right\} \quad \text{and} \quad \mathcal{B}^* = \left\{\Psi_n^{(1)}, \Psi_n^{(2)}, \quad n \geq 1\right\}. \tag{3.27}$$

Then,

1. *$\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal families in $L^2(0,1)^2$.*
2. *$\mathcal{B}$ and $\mathcal{B}^*$ are complete sequences in $L^2(0,1)^2$.*
3. *The sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz bases of $L^2(0,1)^2$.*
4. *The sequence $\mathcal{B}^*$ is a basis of $H_0^{1,\mu}(0,1)^2$ and $\mathcal{B}$ is its biorthogonal basis in $H^{-1,\mu}(0,1)^2$.*

Proof. From the expressions of $\psi_n^{(i)}$ and $\Psi_n^{(i)}$ (see (3.11) and (3.12)), we can write

$$\psi_n^{(i)} = U_i \Phi_{\mu,n} \quad \text{and} \quad \Psi_n^{(i)} = V_i \Phi_{\mu,n}, \quad i = 1, 2, \quad n \geq 1,$$

where $U_i, V_i \in \mathbb{R}^2$ and $\Phi_{\mu,n}$ is given (3.5).

1. It is not difficult to check that $\{U_i\}_{i=1,2}$ and $\{V_i\}_{i=1,2}$ are biorthogonal families of $\mathbb{R}^2$. Moreover, since $(\Phi_{\mu,n})_{n\geq 1}$ is an orthonormal basis for $L^2(0,1)$, we readily deduce

$$\langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle = U_i \cdot V_j \langle \Phi_{\mu,n}, \Phi_{\mu,k} \rangle = \delta_{ij}\delta_{nk}, \quad \forall n, k \geq 1, \quad i, j = 1, 2.$$

This proves the claim.

2. We will use [31, Lemma 1.44]. For this purpose, let us consider $f = (f_1, f_2) \in L^2(0,1)^2$ such that
$$\langle f, \psi_n^{(i)} \rangle = 0, \quad \forall n \geq 1, \quad i = 1, 2.$$
If we denote $f_{i,n}$ $(i = 1, 2)$ the corresponding Fourier coefficients of the function $f_i \in L^2(0,1)$ with respect to the basis $(\Phi_{\mu,n})_{n\geq 1}$, then the previous equality can be written as
$$(f_{1,n}, f_{2,n})[U_1|U_2] = 0_{\mathbb{R}^2}, \quad \forall n \geq 1.$$
Using the fact that $det[U_1|U_2] \neq 0$, we deduce $f_{1,n} = f_{2,n} = 0$, for all $n \geq 1$. This implies that $f_1 = f_2 = 0$ (since $(\Phi_{\mu,n})_{n\geq 1}$ is an orthonormal basis in $L^2(0,1)$) and, therefore, $f = 0$ which proves the completeness of $\mathcal{B}$. A similar argument can be used for $\mathcal{B}^*$ and the conclusion follows immediately.
3. By [31, Theorem 7.13], we know that $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ is a Riesz basis for $L^2(0,1)^2$ if and only if $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ is a complete Bessel sequence and possesses a biorthogonal system that is also a complete Bessel sequence. Using the previous properties 1) and 2), we only have to prove that the sequence $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ and $\{\Psi_n^{(1)}, \Psi_n^{(2)}\}_{n\geq 1}$ are Bessel sequences. This amounts to prove that the series
$$S_1(f) = \sum_{n\geq 1} \left[\langle f, \psi_n^{(1)} \rangle^2 + \langle f, \psi_n^{(2)} \rangle^2\right] \text{ and}$$
$$S_2(f) = \sum_{n\geq 1} \left[\langle f, \Psi_n^{(1)} \rangle^2 + \langle f, \Psi_n^{(2)} \rangle^2\right]$$
converge for any $f = (f_1, f_2) \in L^2(0,1)^2$.

From the definition of the functions $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, it is easy to see that there exists some constant $C > 0$ such that
$$S_1(f) \leq C \sum_{n\geq 1} \left(|f_{1,n}|^2 + |f_{2,n}|^2\right) \quad \text{and}$$
$$S_2(f) \leq C \sum_{n\geq 1} \left(|f_{1,n}|^2 + |f_{2,n}|^2\right).$$
Recall that $f_{i,n}$ is the Fourier coefficient of the function $f_i \in L^2(0,1)$ $(i = 1, 2)$ with respect to $\Phi_{\mu,n}$. Accordingly, the series $S_1(f)$ and $S_2(f)$ converge since $(\Phi_{\mu,n})_{n\geq 1}$ is an orthonormal basis for $L^2(0,1)$. We obtain thus the proof of desired result.
4. For showing item 4) we make use of [31, Theorem 5.12]. First, using Theorem 2.1, one has
$$H_0^{1,\mu}(0,1) \subset L^2(0,1) \subset \left(H_0^{1,\mu}(0,1)\right)' = H^{-1,\mu}(0,1).$$
Furthermore, observe that $\mathcal{B}^* \subset H_0^{1,\mu}(0,1)^2$ and is complete in this space since it is in $L^2(0,1)^2$. On the other hand, by the definition of the duality pairing, we have
$$\langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle_{H^{-1,\mu}, H_0^{1,\mu}} = \langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle = \delta_{ij}\delta_{nk}, \quad \forall n, k \geq 1, \quad i, j = 1, 2.$$

Thus, $\mathcal{B} \subset H^{-1,\mu}(0,1)^2$ and is biorthogonal to $\mathcal{B}^*$, which also yields that $\mathcal{B}^*$ is minimal in $H_0^{1,\mu}(0,1)^2$ thanks to [31, Lemma 5.4]. To conclude the proof, it remains to prove that for any $f = (f_1, f_2) \in H_0^{1,\mu}(0,1)^2$, the series

$$S(f) = \sum_{n\geq 1} \left[\langle \psi_n^{(1)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(1)} + \langle \psi_n^{(2)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(2)}\right]$$

converges in $H_0^{1,\mu}(0,1)^2$.

Using again the definitions of $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, one can prove that

$$\langle \psi_n^{(1)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(1)} = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -\alpha_2 f_{1,n} + f_{2,n} \\ \\ -\alpha_1\alpha_2 f_{1,n} + \alpha_1 f_{2,n} \end{pmatrix} \Phi_{\mu,n}$$

and

$$\langle \psi_n^{(2)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(2)} = \frac{1}{\alpha_2 - \alpha_1} \begin{pmatrix} -\alpha_1 f_{1,n} + f_{2,n} \\ \\ -\alpha_1\alpha_2 f_{1,n} + \alpha_2 f_{2,n} \end{pmatrix} \Phi_{\mu,n}$$

where $f_{i,n}$ is the Fourier coefficient of the function $f_i \in H_0^{1,\mu}(0,1)$, $i = 1, 2$.

But, we know that the series $\sum\limits_{n\geq 1} f_{i,n}\Phi_{\mu,n}$, $i = 1, 2$ converges in $H_0^{1,\mu}(0,1)$ since $(\Phi_{\mu,n})_{n\geq 1}$ is an orthogonal basis for $H_0^{1,\mu}(0,1)$ and $f_1, f_2 \in H_0^{1,\mu}(0,1)$. This implies that, the series

$$\sum_{n\geq 1} \langle \psi_n^{(1)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(1)} \text{ and } \sum_{n\geq 1} \langle \psi_n^{(2)}, f\rangle_{H^{-1,\mu},H_0^{1,\mu}} \Psi_n^{(2)}$$

converge in $H_0^{1,\mu}(0,1)^2$ and assure the convergence of $S(f)$ in $H_0^{1,\mu}(0,1)^2$. This concludes the proof of the result.

□

4. Boundary approximate controllability

We will devote this section to proving the approximate controllability at time $T > 0$ of system (1.1). To this aim, we are going to use Theorem 1.2. Firstly, we give the following result:

Lemma 4.1. *Let $(\lambda_{\mu,k})_{k\geq 1}$ be the sequence of eigenvalues of the spectral problem (3.1). Then, the following properties hold:*

1. *For all $n, m \in \mathbb{N}^\star$, there is a constant $\rho > 0$ such that the sequence of eigenvalues $(\lambda_{\mu,n})_{n\geq 1}$ satisfy the separation condition:*

$$|\lambda_{\mu,n} - \lambda_{\mu,m}| \geq \rho|n - m|, \qquad \forall n, m \geq 1.$$

2. *The series $\sum\limits_{n\geq 1} \dfrac{1}{\lambda_{\mu,n}}$ is convergent.*

The proof of the above Lemma is similar to the one given, for instance, in [3, Lemma 2], so we omit it.

Using the previous result and similar techniques as in Proposition 3.5, we also have the following result.

Proposition 4.2. *Assume that condition* (3.13) *holds. Then, the family defined in* (3.14) *satisfies* (1.7).

Now, we are ready to state our first main result on approximate controllability. One has:

Theorem 4.3. *Assume that condition* (1.5) *holds. Let* $\mu \leq \frac{1}{4}$ *and let us denote by* α_1 *and* α_2 *the eigenvalues of the matrix* A*. Then, system* (1.1) *is approximately controllable in* $H^{-1,\mu}(0,1)^2$ *at time* $T > 0$ *if and only if condition* (3.13) *is satisfied.*

Remark 1. We highlight that the above approximate controllability result can be derived as a byproduct of the null controllability one (see, Theorem 5.1). Here, we will provide a direct proof, which is interesting in itself and potentially useful in other situations such as pointwise control problem where the approximate controllability holds at any time $T > 0$, whereas a minimal time of control appears for the null controllability result.

Proof. As said in sect. 2, in order to prove this theorem we will follow a duality approach leading us to study the unique continuation property for the adjoint system.

Necessary condition: By contradiction, let us assume that condition (3.13) does not hold, i.e., that there is $n_0, l_0 \in \mathbb{N}^*$ with $n_0 \neq l_0$ such that

$$\lambda^{(1)}_{\mu,n_0} = \lambda^{(2)}_{\mu,l_0} := \lambda.$$

Let us see that the unique continuation property for the adjoint system (2.11) is no longer valid. Indeed, let us take $\varphi_0 = a\Psi^{(1)}_{n_0} + b\Psi^{(2)}_{l_0} \in H^{1,\mu}_0(0,1)^2$, with $a, b \in \mathbb{R}$ to be determined. In this case, it is not difficult to see that the corresponding solution to the adjoint problem (2.11) is given by

$$\varphi(t,x) = (a\Psi^{(1)}_{n_0}(x) + b\Psi^{(2)}_{l_0}(x))e^{-\lambda(T-t)}, \quad \forall (t,x) \in Q.$$

On the other hand, direct computations show that

$$\Phi_{\mu,n,x}(1) = \frac{\sqrt{2} j_{\nu_\mu,n}}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|} J'_{\nu_\mu}(j_{\nu_\mu,n}).$$

Coming back to the definition of $\Psi^{(i)}_n$, and taking into account the previous property, one obtains

$$\begin{aligned}
B^*\varphi_x(t,1) &= B^*\big(a\Psi^{(1)}_{n_0,x}(1) + b\Psi^{(2)}_{l_0,x}(1)\big)e^{-\lambda(T-t)} \\
&= \sqrt{2}\Big(aB^*V_1 j_{\nu_\mu,n_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,n_0})|} + bB^*V_2 j_{\nu_\mu,l_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,l_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,l_0})|}\Big)e^{-\lambda(T-t)} \\
&= \sqrt{2}\Big(a j_{\nu_\mu,n_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,n_0})|} + b j_{\nu_\mu,l_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,l_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,l_0})|}\Big)e^{-\lambda(T-t)}.
\end{aligned}$$

Choosing

$$a = j_{\nu_\mu,l_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,l_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,l_0})|} \quad \text{and}$$
$$b = -j_{\nu_\mu,n_0}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n_0})}{|J'_{\nu_\mu}(j_{\nu_\mu,n_0})|},$$

we have that $B^*\varphi_x(t,1) = 0$ but $\varphi_0 \neq 0$, which proves that the unique continuation property for the adjoint system (2.11) fails to be true. This ends the proof of the necessary part.

Sufficient condition: Let us now assume that the condition (3.13) holds and prove that the unique continuation property for the solutions of the adjoint system (2.11) holds.

Let us consider $\varphi_0 \in H_0^{1,\mu}(0,1)^2$ and suppose that the corresponding solution φ of the adjoint problem (2.11) satisfies

$$B^*\varphi_x(t,1) = 0, \quad \forall t \in (0,T). \tag{4.1}$$

From Proposition 3.6, we know that $\mathcal{B}^*$ is a basis for $H_0^{1,\mu}(0,1)^2$ and thus $\varphi_0 \in H_0^{1,\mu}(0,1)^2$ can be written as

$$\varphi_0 = \sum_{n\geq 1}(b_n\Psi_n^{(1)} + c_n\Psi_n^{(2)}),$$

where

$$b_n = \langle \psi_n^{(1)}, \varphi_0\rangle_{H^{-1,\mu},H_0^{1,\mu}} \quad \text{and} \quad c_n = \langle \psi_n^{(2)}, \varphi_0\rangle_{H^{-1,\mu},H_0^{1,\mu}}, \quad \text{for any} \quad n \geq 1.$$

Using Proposition 3.4, the corresponding solution φ of system (2.11) associated to φ_0 is given by

$$\varphi(t,\cdot) = \sum_{n\geq 1}\left(b_n\Psi_n^{(1)}e^{-\lambda_{\mu,n}^{(1)}(T-t)} + c_n\Psi_n^{(2)}e^{-\lambda_{\mu,n}^{(2)}(T-t)}\right), \quad \forall t \in (0,T).$$

Therefore,

$$\begin{aligned}
0 &= B^*\varphi_x(T-t,1)\\
&= \sum_{n\geq 1} B^*\big(b_n\Psi_{n,x}^{(1)}(1)e^{-\lambda_{\mu,n}^{(1)}t} + c_n\Psi_{n,x}^{(2)}(1)e^{-\lambda_{\mu,n}^{(2)}t}\big)\\
&= \sqrt{2}\sum_{n\geq 1}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n})}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|}j_{\nu_\mu,n}\big(b_nB^*V_1e^{-\lambda_{\mu,n}^{(1)}t} + c_nB^*V_2e^{-\lambda_{\mu,n}^{(2)}t}\big)\\
&= \sqrt{2}\sum_{n\geq 1}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n})}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|}j_{\nu_\mu,n}\big(b_ne^{-\lambda_{\mu,n}^{(1)}t} + c_ne^{-\lambda_{\mu,n}^{(2)}t}\big)\\
&= \sqrt{2}\sum_{n\geq 1}\frac{J'_{\nu_\mu}(j_{\nu_\mu,n})}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|}j_{\nu_\mu,n}e^{\alpha_2 t}\big(b_ne^{-(\lambda_{\mu,n}^{(1)}+\alpha_2)t} + c_ne^{-(\lambda_{\mu,n}^{(2)}+\alpha_2)t}\big).
\end{aligned}$$

From Proposition 4.2, we can apply Theorem 1.2 in order to deduce the existence of a biorthogonal family $\{q_n^{(1)}, q_n^{(2)}\}_{n\geq 1}$ to $\{e^{-(\lambda_{\mu,n}^{(1)}+\alpha_2)t}, e^{-(\lambda_{\mu,n}^{(2)}+\alpha_2)t}\}_{n\geq 1}$

in $L^2(0,T)$. Thus, the previous identity, in particular, implies

$$\begin{cases} \int_0^T B^*\varphi_x(T-t,1)\, e^{-\alpha_2 t}\, q_n^{(1)}(t)\, dt \\ = \sqrt{2}\dfrac{J'_{\nu_\mu}(j_{\nu_\mu,n})}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|} j_{\nu_\mu,n} b_n = 0, \quad \forall n \geq 1 \\ \int_0^T B^*\varphi_x(T-t,1)\, e^{-\alpha_2 t}\, q_n^{(2)}(t)\, dt \\ = \sqrt{2}\dfrac{J'_{\nu_\mu}(j_{\nu_\mu,n})}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|} j_{\nu_\mu,n} c_n = 0, \quad \forall n \geq 1. \end{cases}$$

Then $b_n = c_n = 0$ for any $n \geq 1$. In conclusion, $\varphi_0 = 0$. This proves the continuation property for the solutions to the adjoint problem (2.11) and, thanks to Theorem 2.6, the approximate controllability of system (1.1) at any positive time T holds. □

5. Boundary null controllability

In this section, we will address the main achievement of this work which is the boundary null controllability result of system (1.1), providing an estimate of the control cost as a function of T. In this sense, one has:

Theorem 5.1. *Assume that condition* (1.5) *holds. Let* $\mu \leq \frac{1}{4}$ *and let us denote by* α_1 *and* α_2 *the eigenvalues of* A. *Then, system* (1.1) *is null controllable in* $H^{-1,\mu}(0,1)^2$ *at time* $T > 0$ *if and only if condition* (3.13) *is satisfied.*

Moreover, for every $T > 0$ *and* $y_0 \in H^{-1,\mu}(0,1)^2$ *there exists a null control* $v \in L^2(0,T)$ *for system* (1.1) *which, in addition, satisfies*

$$\|v\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}}, \tag{5.1}$$

where $C > 0$ *does not depend on* T.

Proof. To prove Theorem 5.1, we transform the controllability problem into a moment problem. Using Proposition 2.5, we deduce that the control $v \in L^2(0,T)$ drives the solution of (1.1) to zero at time T if and only if $v \in L^2(0,T)$ fulfills

$$\int_0^T B^*\varphi_x(t,1)\, v(t)\, dt = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1,\mu},H_0^{1,\mu}}, \quad \forall \varphi_0 \in H_0^{1,\mu}(0,1)^2 \tag{5.2}$$

where $\varphi \in C^0\big([0,T]; H_0^{1,\mu}(0,1)^2\big) \cap L^2\big(0,T; H^{2,\mu}(0,1)^2 \cap H_0^{1,\mu}(0,1)^2\big)$ is the solution of the adjoint system (2.11) associated to φ_0.

Using Proposition 3.4, the corresponding solution φ of system (2.11) associated to φ_0 is given by

$$\varphi(t,x) = \sum_{k\geq 1}\Big(\langle \psi_k^{(1)}, \varphi_0\rangle_{H^{-1,\mu},H_0^{1,\mu}}\Psi_k^{(1)} e^{-\lambda_{\mu,k}^{(1)}(T-t)} + \langle \psi_k^{(2)}, \varphi_0\rangle_{H^{-1,\mu},H_0^{1,\mu}}\Psi_k^{(2)} e^{-\lambda_{\mu,k}^{(2)}(T-t)}\Big).$$

From Proposition 3.6, we have that $\mathcal{B}^*$ is a basis for $H_0^{1,\mu}(0,1)^2$. In particular, we also deduce that $\varphi(t,x) = \Psi_n^{(i)}(x)e^{-\lambda_{\mu,n}^{(i)}(T-t)}$ is the solution of system

(2.11) corresponding to $\varphi_0 = \Psi_n^{(i)} \in H_0^{1,\mu}(0,1)^2$. Therefore, we can deduce that the identity (5.2) is equivalent to

$$\int_0^T B^*\Psi_{n,x}^{(i)}(1)v(t)e^{-\lambda_{\mu,n}^{(i)}(T-t)}dt = e^{-\lambda_{\mu,n}^{(i)}T}\langle y_0, \Psi_n^{(i)}\rangle_{H^{-1,\mu},H_0^{1,\mu}},\ \forall n\geq 1,\ i=1,2.$$

Taking into account the expressions of $\Psi_n^{(i)}$ (see (3.12)), we infer that $v \in L^2(0,T)$ is a null control for system (1.1) associated to y_0 if and only if

$$\frac{\sqrt{2}j_{\nu_\mu,n}}{|J'_{\nu_\mu}(j_{\nu_\mu,n})|}J'_{\nu_\mu}(j_{\nu_\mu,n})B^*V_i\int_0^T v(t)e^{-\lambda_{\mu,n}^{(i)}(T-t)}dt$$
$$= e^{-\lambda_{\mu,n}^{(i)}T}\langle y_0, \Psi_n^{(i)}\rangle_{H^{-1,\mu},H_0^{1,\mu}},\ \forall n\geq 1,\ i=1,2$$

and equivalently,

$$\int_0^T v(t)e^{-\lambda_{\mu,n}^{(i)}(T-t)}dt = C_{\nu_\mu,n}^{(i)},\ \forall n\geq 1,\ i=1,2, \tag{5.3}$$

where $C_{\nu_\mu,n}^{(i)}$ is given by

$$C_{\nu_\mu,n}^{(i)} = \frac{|J'_{\nu_\mu}(j_{\nu_\mu,n})|e^{-\lambda_{\mu,n}^{(i)}T}}{\sqrt{2}j_{\nu_\mu,n}J'_{\nu_\mu}(j_{\nu_\mu,n})B^*V_i}\langle y_0, \Psi_n^{(i)}\rangle_{H^{-1,\mu},H_0^{1,\mu}},\ \forall n\geq 1,\ i=1,2.$$

Performing the change of variable $s = T/2-t$ in (5.3), the controllability problem reduces then to the following moment problem: Given $y_0 \in H^{-1,\mu}(0,1)^2$ find $v \in L^2(0,T)$ such that $u(s) = v(T/2-s)e^{\alpha_2 s} \in L^2(-T/2,T/2)$ satisfies

$$\int_{-T/2}^{T/2} u(s)e^{-(\lambda_{\mu,n}^{(i)}+\alpha_2)s}ds = \widehat{C}_{\nu_\mu,n}^{(i)}, \quad \forall n\geq 1, \quad i=1,2, \tag{5.4}$$

with

$$\widehat{C}_{\nu_\mu,n}^{(i)} = e^{\lambda_{\mu,n}^{(i)}T/2}C_{\nu_\mu,n}^{(i)}. \tag{5.5}$$

At this stage, the strategy to solve the moment problem (5.4) is to use the concept of biorthogonal family. In fact, Proposition 3.5 and Theorem 1.3 guarantee the existence of $T_0>0$, such that for any $T\in(0,T_0)$, there exists a biorthogonal family $\{q_n^{(1)}, q_n^{(2)}\}_{n\geq 1}$ to $\{e^{-(\lambda_{\mu,n}^{(1)}+\alpha_2)t}, e^{-(\lambda_{\mu,n}^{(2)}+\alpha_2)t}\}_{n\geq 1}$ in $L^2(-T/2,T/2)$ which also satisfies

$$\|q_n^{(i)}\|_{L^2(-T/2,T/2)} \leq Ce^{\sqrt{\Re(\lambda_{\mu,n}^{(i)}+\alpha_2)}+\frac{C}{T}}, \qquad \forall n\geq 1, \quad i=1,2. \tag{5.6}$$

for some positive constant C independent of T.

For $T<T_0$, a solution to the moment problem (5.4) is then given for every $t\in(0,T)$ by

$$u(t) = \sum_{n\geq 1}(\widehat{C}_{\nu_\mu,n}^{(1)}q_n^{(1)}(t) + \widehat{C}_{\nu_\mu,n}^{(2)}q_n^{(2)}(t)).$$

Thus

$$v(t) = \sum_{n\geq 1}\left(\widehat{C}_{\nu_\mu,n}^{(1)}q_n^{(1)}(T/2-t) + \widehat{C}_{\nu_\mu,n}^{(2)}q_n^{(2)}(T/2-t)\right)e^{-\alpha_2(T/2-t)}. \tag{5.7}$$

The only remaining point is to prove that $v \in L^2(0,T)$ and to estimate its norm with respect to T and y_0. This can be achieved thanks to the estimate (5.6). Indeed, from the expression of $\Psi_n^{(i)}$ and $\lambda_{\mu,n}^{(i)}$, we can easily deduce the existence of constants $C_1, C_2 > 0$ such that for $i = 1, 2$:

$$\|\Psi_n^{(i)}\|_\mu \leq C_1\sqrt{\lambda_{\mu,n}} = C_1 j_{\nu_\mu,n}, \quad |\widehat{C}_{\nu_\mu,n}^{(i)}| \leq C_2 e^{-\lambda_{\mu,n}^{(i)}T/2}\|y_0\|_{H^{-1,\mu}}, \ \forall n \geq 1.$$

From (5.5), it is easy to see that there exists a new constants C not depending on n and T such that

$$|\widehat{C}_{\nu_\mu,n}^{(i)}| \leq Ce^{-\lambda_{\mu,n}^{(i)}T/2}\|y_0\|_{H^{-1,\mu}}, \quad \forall n \geq 1, \quad i = 1, 2. \tag{5.8}$$

Coming back to the expression (5.7) of the null control v, taking into account the definition of $\lambda_{\mu,n}^{(i)}$ and using the estimates (5.6) and (5.8), we get

$$\|v\|_{L^2(0,T)} \leq Ce^{CT}\|y_0\|_{H^{-1,\mu}} \sum_{n\geq 1} e^{-\lambda_{\mu,n}T/2} e^{C\sqrt{\lambda_{\mu,n}}+\frac{C}{T}}. \tag{5.9}$$

Moreover, Young's inequality gives

$$C\sqrt{\lambda_{\mu,n}} \leq \frac{\lambda_{\mu,n}T}{4} + \frac{C^2}{T}$$

for every $n \geq 1$ and $T > 0$, so that

$$\|v\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}} \sum_{n\geq 1} e^{-\lambda_{\mu,n}T/4}.$$

On the other hand, by (3.2) and (3.3), it can be easily checked that there exists a constant $C > 0$ such that

$$Cn^2 \leq \lambda_{\mu,n} = j_{\nu_\mu,n}^2, \quad \forall n \geq 1.$$

Finally, for every $T < T_0$, we then have

$$\begin{aligned}
\|v\|_{L^2(0,T)} &\leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}} \sum_{n\geq 1} e^{-Cn^2T} \\
&\leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}} \int_0^\infty e^{-CTs^2}\, ds \\
&= Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}} \sqrt{\frac{\pi}{T}} \\
&\leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H^{-1,\mu}},
\end{aligned}$$

where C is independent of T. This inequality shows that $v \in L^2(0,T)$ and yields the desired estimate on the null control in the case where $T < T_0$. The case $T \geq T_0$ is actually reduced to the previous one. Indeed, any continuation by zero of a control on $(0, T_0/2)$ is a control on $(0,T)$ and the estimate follows from the decrease of the cost with respect to the time. This completes the proof of Theorem 5.1. □

6. Conclusions and open problems

In this work we have dealt with the boundary controllability for two coupled parabolic equations with singular potentials. In particular, we provided a well-posedness analysis of the corresponding system and using the method of moments of Fattorini-Russell, conditions for its controllability through a boundary action are derived. Moreover, an estimate for the null controllability cost with respect to $T > 0$ was given.

The methods and results in this paper lead to some interesting open problems, which we will briefly mention.

1. As a first thing, we recall that in the present work we are not treating the case of a coupling matrix admitting a double eigenvalue. However, we expect that null controllability results can be obtained using the ideas in [6] (see also, [14, Theorem V.4.17]). Moreover, we point out that combining our proofs with the ideas of [12], null controllability result can be obtained for degenerate/singular parabolic systems in the case of a boundary control acting away from the degenerate and singular point. When the control acts at the degenerate/singular point, instead, this is an open question. Indeed, as explained in [12], in this case the difficulty comes from the fact that it is not possible to impose a standard non homogeneous Dirichlet boundary condition. See [11,12] for some results in the range of subcritical coefficients of the singular potential.
2. In this work, we have addressed the problem of boundary controllability for a coupled system of parabolic equations with the same parameter of singularity. It would be interesting to consider a coupled system with two different parameters of singularity. In this case the difficulty lies in the behavior of the sequence of eigenvalues of the associated operator which may not satisfy the gap condition appearing in (1.7), and, consequently, this is completely an open problem.
3. Inspired by the results in [37], it would also be interesting to give sharp estimates of the cost of controllability in terms of the two parameters $T > 0$ and μ.
4. Finally, it would be of interest to extend these boundary controllability results to the more general case of a system of n singular parabolic equations by means of m controls. Of course, the most interesting case is the case in which the number of controls is less than the number of equations: $m < n$.

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References

[1] Allaire, G., Craig, A.: Numerical Analysis and Optimization: An Introduction to Mathematical Modelling and Numerical Simulation. Oxford University Press, Oxford, Numerical mathematics and scientific computation (2007)

[2] Allal, B., Fragnelli, G., Salhi, J.: Null controllability for a singular heat equation with a memory term. Electron. J. Qual. Theory Differ. Equ. **14**(14), 1–24 (2021)

[3] Allal, B., Salhi, J.: Pointwise controllability for degenerate parabolic equations by the moment method. J. Dyn. Control Syst. **26**, 349–362 (2020)

[4] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M., de Teresa, L.: New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence. J. Math. Anal. Appl. **444**, 1071–1113 (2016)

[5] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M., de Teresa, L.: Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences. J. Funct. Anal. **267**, 2077–2151 (2014)

[6] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M., de Teresa, L.: The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials. J. Math. Pures Appl. **96**, 555–590 (2011)

[7] Baras, P., Goldstein, J.A.: The heat equation with a singular potential. Trans. Amer. Math. Soc. **284**, 121–139 (1984)

[8] Baras, P., Goldstein, J. A.: Remarks on the inverse square potential in quantum mechanics, Differential Equations, North-Holland Math. Stud. 92, North-Holland, Amsterdam 31–35 (1984)

[9] Benabdallah, A., Boyer, F., Morancey, M.: A block moment method to handle spectral condensation phenomenon in parabolic control problems. Annal. Henri Lebesgue **3**, 717–793 (2020)

[10] Benabdallah, A., Boyer, F., Gonzalez-Burgos, M., Olive, G.: Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains. SIAM J. Control. Optim. **52**, 2970–3001 (2014)

[11] Biccari, U.: Boundary controllability for a one-dimensional heat equation with a singular inverse-square potential. Math. Control Relat. Fields **9**, 191–219 (2019)

[12] Biccari, U., Hernández-Santamaría, V., Vancostenoble, J.: Existence and cost of boundary controls for a degenerate/singular parabolic equation, Mathematical Control & Related Fields, https://doi.org/10.3934/mcrf.2021032.

[13] Biccari, U., Zuazua, E.: Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function. J. Differential Equations **261**, 2809–2853 (2016)

[14] Boyer, F.: Controllability of linear parabolic equations and systems, Lecture Notes, https://hal.archives-ouvertes.fr/hal-02470625 (2020)

[15] Cannarsa, P., Guglielmi, R.: Null controllability in large time for the parabolic Grushin operator with singular potential, In Geometric control theory and sub-Riemannian geometry, volume 5 of Springer INdAM Ser., pages 87–102. Springer, Cham, (2014)

[16] Cazacu, C.: Controllability of the heat equation with an inverse-square potential localized on the boundary. SIAM J. Control. Optim. **52**, 2055–2089 (2014)

[17] Cazacu, C.: Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results. J. Funct. Anal. **263**, 3741–3783 (2012)

[18] Coron, J.-M.: Control and Nonlinearity, Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI (2007)

[19] Davies, E.B.: Spectral theory and differential operators, Cambridge Studies in Advanced Mathematics, vol. 42. Cambridge University Press, Cambridge (1995)

[20] Duprez, M.: Controllability of a 2×2 parabolic system by one force with space-dependent coupling term of order one. ESAIM: COCV **23**, 1473–1498 (2017)

[21] Elbert, A.: Some recent results on the zeros of Bessel functions and orthogonal polynomials. J. Comput Appl. **133**, 65–83 (2001)

[22] Ervedoza, S.: Control and stabilization properties for a singular heat equation with an inverse-square potential. Comm. Partial. Differ. Equ. **33**, 1996–2019 (2008)

[23] Fattorini, H.O., Russell, D.L.: Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, Quart. Appl. Math. **32**, 45–69 (1974/75)

[24] Fattorini, H.O., Russell, D.L.: Exact controllability theorems for linear parabolic equations in one space dimension. Arch. Ration. Mech. Anal. **43**, 272–292 (1971)

[25] Fernández-Cara, E., González-Burgos, M., de Teresa, L.: Boundary controllability of parabolic coupled equations. J. Funct. Anal. **259**, 1720–1758 (2010)

[26] Fragnelli, G.: Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates. J. Differ. Equ. **260**, 1314–1371 (2016)

[27] Fragnelli, G., Mugnai, D.: Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients. Adv. Nonlinear Anal. **6**, 61–84 (2017)

[28] González-Burgos, M., Sousa-Neto, G.R.: Boundary controllability of a one-dimensional phase-field system with one control force. J. Differ. Equ. (2020). https://doi.org/10.1016/j.jde.2020.03.036

[29] Hajjaj, A., Maniar, L., Salhi, J.: Carleman estimates and null controllability of degenerate/singular parabolic systems, Electron. J. Differential Equations, 1–25 (2016)

[30] Hardy, G. H., Littlewood, J. E., Pólya, G.: Inequalities, 2nd ed., Cambridge, at the University Press, (1952)

[31] Heil, C.: A basis theory primer, expanded Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York (2011)

[32] Komornik, V., Loreti, P.: Fourier series in control theory. Springer, Berlin (2005)

[33] Lebedev, N.N.: Special Functions and their Applications. Dover Publications, New York (1972)

[34] Lions, J.-L.: Contrôlabilité exacte, perturbation et stabilisation de Systèmes Distribués, 1. Masson, Paris (1988)

[35] Lions, J.L., Magenes, E.: Problèmes aux limites non homogènes et applications, vol. 1, Travaux et Rech. Math., vol. 17, Dunod, Paris, (1968)

[36] Lorch, L., Muldoon, M.E.: Monotonic sequences related to zeros of Bessel functions. Numer. Algor. **49**, 221–233 (2008)

[37] Martinez, P., Vancostenoble, J.: The cost of boundary controllability for a parabolic equation with inverse square potential. Evol. Equ. Control Theory **8**, 397–422 (2019)

[38] Morancey, M.: Approximate controllability for a 2D Grushin equation with potential having an internal singularity. Ann. Inst. Fourier (Grenoble) **65**, 1525–1556 (2015)

[39] Salhi, J.: Null controllability for a singular coupled system of degenerate parabolic equations in nondivergence form. Electron. J. Qual. Theory Differ. Equ. **13**, 1–28 (2018)

[40] Vancostenoble, J.: Improved Hardy-Poincaré inequalities and sharp Carleman estimates for degenerate/singular parabolic problems. Discrete Contin. Dyn. Syst. Ser. S **4**, 761–790 (2011)

[41] Vancostenoble, J.: Global Non-negative Approximate Controllability of Parabolic Equations with Singular Potentials, In Trends in Control Theory and Partial Differential Equations. Springer INdAM Series, vol 32. Springer, Cham (2019)

[42] Vancostenoble, J., Zuazua, E.: Null controllability for the heat equation with singular inverse-square potentials. J. Funct. Anal. **254**, 1864–1902 (2008)

[43] Vazquez, J.L., Zuazua, E.: The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J. Funct. Anal. **173**, 103–153 (2000)

[44] Watson, G.N.: A treatise on the theory of Bessel functions, 2nd edn. Cambridge University Press, Cambridge (1944)

[45] Zabczyk, J.: Mathematical control theory: an introduction. Birkhäuser, Boston (1995)

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