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Two-phase Stokes flow by capillarity in the plane: The case of different viscosities

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Abstract. We study the two-phase Stokes flow driven by surface tension for two fluids of different viscosities, separated by an asymptotically flat interface representable as graph of a differentiable function. The flow is assumed to be two-dimensional with the fluids filling the entire space. We prove well-posedness and parabolic smoothing in Sobolev spaces up to critical regularity. The main technical tools are an analysis of nonlinear singular integral operators arising from the hydrodynamic single and double layer potential, spectral results on the corresponding integral operators, and abstract results on nonlinear parabolic evolution equations.

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1. Introduction

In the context of boundary value problems involving elliptic constant-coefficient PDE's like the Laplace equation or the Stokes system, it is often natural to consider two-phase problems in unbounded domains, where the same equation has to be solved on both sides of the boundary, and the boundary conditions typically are of "transmission" type, i.e. they relate limits of the solutions from both sides. The method of layer potentials is a classical technique which is intrinsically suited to such settings. Typically, this method reduces the boundary value problem to a linear, singular integral equation (or system of such equations) on the boundary of the domain, on the basis of well-known jump relations for these potentials across the boundary.

The first applications of layer potentials in the analysis of moving boundary problems of the type described above are from the 1980s, for problems of Hele-Shaw or Muskat type [8] (see also the recent surveys [13,14] on further developments) as well as for Stokes flow problems [5]. In these applications, the interfaces are represented as graphs of a time dependent function $[f \mapsto f(t)]$, with $f(t) \in \mathrm{C}(\mathbb{R})$, for which an evolution equation can be derived. This equation involves singular integral operators originating from the layer potential, depending nonlinearly and nonlocally on $f(t)$. However, in suitable geometries this nonlinearity can be described rather explicitly, and technicalities resulting from transforming the problem to a fixed reference domain can be avoided. More precisely, the operators determining the evolution belong to a class discussed in Sect. 3 below, and results are available concerning mapping properties, smoothness, localization etc. of the operators in this class.

After reducing the moving boundary problem to an evolution equation for f, this equation has to be analyzed. Initially, various approaches have been used that necessitated rather restrictive assumptions on the initial data. Recently, however, more general, in some sense optimal existence, uniqueness, and smoothness results have been obtained. One of the crucial tools for this has been the meanwhile well-developed and versatile abstract theory of nonlinear parabolic evolution equations, cf. [2,17,22].

This paper discusses, along the lines sketched above, the moving boundary problem of two-phase Stokes flow in full 2D space driven by surface tension forces on the interface between the two phases. More precisely, we seek a moving interface $[t \mapsto \Gamma(t)]$ between two liquid phases $\Omega^\pm(t)$, and corresponding functions

$$v^\pm(t) : \Omega^\pm(t) \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad p^\pm(t) : \Omega^\pm(t) \longrightarrow \mathbb{R},$$

representing the velocity and pressure fields in $\Omega^\pm(t)$, respectively, such that the following equations are satisfied:

$$\left.\begin{array}{rll} \mu^\pm \Delta v^\pm - \nabla p^\pm &= 0 & \text{in}\, \Omega^\pm(t), \\ \operatorname{div} v^\pm &= 0 & \text{in}\, \Omega^\pm(t), \\ [v] &= 0 & \text{on}\, \Gamma(t), \\ [T_\mu(v,p)]\tilde\nu &= -\sigma\tilde\kappa\tilde\nu & \text{on}\, \Gamma(t), \\ (v^\pm, p^\pm)(x) &\to 0 & \text{for}\, |x| \to \infty, \\ V_n &= v^\pm \cdot \tilde\nu & \text{on}\, \Gamma(t). \end{array}\right\} \tag{1.1a}$$

Here $\tilde\nu$ is the unit exterior normal to $\partial\Omega^-(t)$ and $\tilde\kappa$ denotes the curvature of the interface. Moreover, $T_\mu(v,p) = (T_{\mu,ij}(v,p))_{1\leq i,\, j\leq 2}$ denotes the stress tensor that is given by

$$T_{\mu,ij}(v,p) := -p\delta_{ij} + \mu(\partial_i v_j + \partial_j v_i), \tag{1.1b}$$

and $[v]$ (respectively $[T_\mu(v,p)]$) is the jump of the velocity (respectively stress tensor) across the moving interface, see (2.3) below. The positive constants $\mu^\pm$ and σ denote the viscosity of the liquids in the two phases and the surface tension coefficient of the interface, respectively. We assume that

$$\Gamma(t) = \partial\Omega^+(t) = \partial\Omega^-(t), \quad \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t) = \mathbb{R}^2, \quad \Gamma(t) = \operatorname{graph} f(t)$$

so that $\Gamma(t)$ is a graph over the real line and $\Omega^+(t)$ (resp. $\Omega^-(t)$) is the unbounded domain above (resp. beneath) the graph $\Gamma(t)$, cf. (2.1). Equation $(1.1a)_6$ determines the motion of the interface by prescribing its normal velocity V_n as coinciding with the normal component of the velocity at $\Gamma(t)$, i.e. the interface is transported by the liquid flow. The interface $\Gamma(t)$ is assumed to be known at time $t = 0$:

$$f(0) = f_0. \tag{1.1c}$$

In the previous paper [20], the authors considered Problem (1.1a) in the case of equal viscosities $\mu^\pm = \mu$. In that case, the solution to the fixed-time problem $(1.1a)_1$–$(1.1a)_5$ can be directly represented as a hydrodynamic single-layer potential [16] with density $-\sigma\tilde{\kappa}\tilde{\nu}$, and the resulting evolution equation represents the time derivative of f as a nonlinear singular integral operator acting on f.

If $\mu^+ \neq \mu^-$ this is not feasible. Instead, we first transform the unknowns such that the same equation holds in both phases, introducing thereby a jump across the interface for the transformed velocity field. In Proposition 5.1, we show that the corresponding fixed-time Stokes problem is uniquely solvable, and we represent the solution by a sum of a hydrodynamic single layer and a double layer potential. While the single layer potential is generated by the same density as in the case of equal viscosities, the density β for the double layer potential is found from solving a linear, singular integral equation of the second kind, cf. (5.8). As $\Gamma(t)$ is unbounded we cannot rely on compactness arguments to show the solvability of this equation. Instead, we modify arguments from [7, 10] to obtain the necessary information on the spectrum of the corresponding integral operator via a Rellich identity. Moreover, we also rely on a further Rellich identity used in [18] in the study of the Muskat problem.

The solution to the fixed-time problem is then used in the formulation of an evolution equation for f

$$\frac{df}{dt}(t) = \Phi(f(t)), \quad t \geq 0, \qquad f(0) = f_0,$$

cf. (5.9), (5.17), (5.18), whose investigation will yield the following main result. Here and further, $H^s(\mathbb{R}) := W^s_2(\mathbb{R})$ denotes the usual Sobolev spaces of integer or noninteger order.

Theorem 1.1. *Let $s \in (3/2, 2)$ be given. Then, the following statements hold true:*

(i) (Well-posedness) *Given $f_0 \in H^s(\mathbb{R})$, there exists a unique maximal solution $(f, v^\pm, p^\pm)$ to (1.1) such that*
- $f = f(\cdot; f_0) \in \mathrm{C}([0, T_+), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_+), H^{s-1}(\mathbb{R}))$,
- $v^\pm(t) \in \mathrm{C}^2(\Omega^\pm(t)) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)})$, $p^\pm(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)})$ *for all* $t \in (0, T_+)$,
- $v(t)^\pm|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$ *for all* $t \in (0, T_+)$,

where $T_+ = T_+(f_0) \in (0, \infty]$ and $\Xi_{f(t)}(\xi) := (\xi, f(t)(\xi))$, $\xi \in \mathbb{R}$. Moreover, the set

$$\mathcal{M} := \{(t, f_0) \mid f_0 \in H^s(\mathbb{R}),\, 0 < t < T_+(f_0)\}$$

is open in $(0, \infty) \times H^s(\mathbb{R})$, and $[(t, f_0) \mapsto f(t; f_0)]$ is a semiflow on $H^s(\mathbb{R})$ which is smooth in $\mathcal{M}$.

(ii) (Parabolic smoothing)
  (iia) *The map* $[(t,\xi)\mapsto f(t)(\xi)] : (0,T_+)\times\mathbb{R}\longrightarrow\mathbb{R}$ *is a* C^∞*-function.*
  (iib) *For any* $k\in\mathbb{N}$*, we have* $f\in \mathrm{C}^\infty((0,T_+),H^k(\mathbb{R}))$.

(iii) (Global existence) *If*

$$\sup_{[0,T]\cap[0,T_+(f_0))}\|f(t)\|_{H^s}<\infty$$

for each $T>0$*, then* $T_+(f_0)=\infty$.

Remark 1.2. Observe that the complete problem (1.1) is encoded in the time evolution of f. Besides, if f is a solution to (1.1), then, given $\lambda>0$, also the function $[t\mapsto f_\lambda(t)]$ given by

$$f_\lambda(t)(\xi):=\lambda^{-1}f(\lambda t)(\lambda\xi),$$

is a solution to (1.1). This identifies $H^{3/2}(\mathbb{R})$ as a critical space for the evolution problem (1.1). Hence, Theorem 1.1 covers all subcritical spaces. To our knowledge, this result is stronger than those found in the literature on the related problems with bounded liquid domain, e.g. [11,12,15,23]. More generally, if the problem is treated using the general strategy described in [22], higher regularity demands on the initial interface are needed than in the approach used here. To be more precise, the authors of [15] establish the local well-posedness of the one phase problem for a bounded fluid domain in $\mathbb{R}^d$ for H^{s+1}-data with $s\geq s_1$, where s_1 is the smallest integer that satisfies $s_1>3+(d-1)/2$. Moreover, it is shown in [15] that balls are exponentially stable under H^{s+1}-perturbations. The exponential stability of balls for the one-phase problem has been also established in $\mathbb{R}^2$ for H^5-initial data, see [12], and in $\mathbb{R}^3$ for H^6-initial data, see [11]. The local well-posedness for $C^{3+\alpha}$-data, with $\alpha>0$, in three space dimensions has been investigated in [23], and the same author has justified in [24] the quasistationary Stokes flow as a limit of the Stokes flow when the Reynolds number vanishes. Finally, the local well-posedness and stability properties for the two-phase Stokes flow (with or without phase transitions) in a bounded geometry in $\mathbb{R}^d$, $d\geq 2$, have been studied in [22] in a $W_p^{2+\mu-2/p}$-setting with $1\geq\mu>(d+1)/p$.

1.1. Outline

The paper is structured as follows: In Sect. 2 we discuss a two-phase Stokes problem with equal viscosities in both phases where the normal stresses are continuous across the interface and the velocity has a prescribed jump there. In fact, the problem is solved by the hydrodynamic double layer potential generated by that jump. Although the boundary behavior of this potential is well-known, we prove the results on this in Appendix A as they do not seem directly available in the literature for our unbounded geometry.

As we rely on the solvability of singular integral equations of the second kind arising from the hydrodynamic double-layer potential, the spectrum of the corresponding operator is investigated in Sects. 3 and 4, first in $L_2(\mathbb{R})^2$ and then in $H^{s-1}(\mathbb{R})^2$, with $s\in(3/2,2)$, and $H^2(\mathbb{R})^2$. The main technical tools in the latter cases are shift invariances and commutator properties for singular integral operators of the type discussed here. In Sect. 5 we reformulate the moving boundary problem (1.1) as a nonlinear and nonlocal evolution equation problem, cf. (5.17). Finally, in Sect. 6 we carry out the linearization of (5.17) and locally approximate the linearization by Fourier multipliers. This enables us to identify the parabolic character of the evolution equation and to prove our main result by invoking abstract results on equations of that type from [17].

1.2. Notation

Slightly deviating from the usual notation, if $E_1, \ldots, E_k$, F, $k \in \mathbb{N}$, are Banach spaces, we write $\mathcal{L}^k(E_1, \ldots, E_k; F)$ for the Banach space of k-linear bounded maps from $\prod_i E_i$ to F. Given two Banach spaces X and Y, we let $\mathcal{L}^k_{\rm sym}(X, Y) \subset \mathcal{L}^k(X, \ldots, X; Y)$ denote the space of bounded, k-linear, and symmetric maps $A: \ X^k \to Y$. Moreover, $\mathrm{C}^{-1}(E, F)$ will denote the space of locally Lipschitz continuous maps from a Banach space E to a Banach space F. Given $k \in \mathbb{N}$, we further let $\mathrm{C}^k(\mathbb{R})$ denote the Banach space of functions with bounded and continuous derivatives up to order k and $\mathrm{C}^{k+\alpha}(\mathbb{R})$, with $\alpha \in (0, 1)$, is its subspace consisting of functions with α-Hölder continuous kth derivative whose α-Hölder modulus is bounded.

2. An auxiliary fixed-time problem

As a preparation for solving the boundary value problem $(1.1a)_1$–$(1.1a)_5$ for fixed time, in this section we consider the related Stokes problem (2.4) with equal viscosities normed to 1. The unique solvability of (2.4) is established in Proposition 2.1 below and in Appendix A. In this section, $f \in H^3(\mathbb{R})$ is fixed. We introduce the following notation:

$$\begin{aligned} \Omega^\pm &:= \Omega^\pm_f := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 \gtrless f(x_1)\}, \\ \Gamma &:= \Gamma_f := \partial\Omega^\pm = \{(\xi, f(\xi)) \,|\, \xi \in \mathbb{R}\}. \end{aligned} \tag{2.1}$$

Note that Γ is the image of $\mathbb{R}$ under the diffeomorphism

$$\Xi := \Xi_f := (\mathrm{id}_{\mathbb{R}}, f).$$

Further, let ν and τ be the componentwise pull-back under Ξ of the unit normal $\tilde\nu$ on Γ exterior to Ω^- and of the unit tangent vector $\tilde\tau$ to Γ, that is

$$\nu := \nu_f := \tfrac{1}{\omega}(-f', 1)^\top, \quad \tau := \tau_f := \tfrac{1}{\omega}(1, f')^\top, \quad \omega := \omega_f := (1 + f'^2)^{1/2}. \tag{2.2}$$

We indicate the dependence of the functions defined in (2.2) on f only where necessary. For any function z defined on $\mathbb{R}^2 \setminus \Gamma$ we set $z^\pm := z|_{\Omega^\pm}$ and if $z^\pm$ have limits at some point $(\xi, f(\xi)) \in \Gamma$ we will write $z^\pm(\xi, f(\xi))$ for the limits, and we set

$$[z](\xi, f(\xi)) := z^+(\xi, f(\xi)) - z^-(\xi, f(\xi)). \tag{2.3}$$

For notational brevity we introduce the function space $X := X_f$ by setting

$$X_f := \left\{ (w, q) : \mathbb{R}^2 \setminus \Gamma \longrightarrow \mathbb{R}^2 \times \mathbb{R} \;\middle|\; \begin{array}{l} w^\pm \in \mathrm{C}^2(\Omega^\pm, \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2) \\ q^\pm \in \mathrm{C}^1(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm}) \end{array} \right\}.$$

For given $\beta = (\beta_1, \beta_2)^\top \in H^2(\mathbb{R})^2$ we seek solutions $(w, q) \in X$ to the Stokes problem

$$\left.\begin{aligned} \Delta w^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} w^\pm &= 0 && \text{in } \Omega^\pm, \\ [w] &= \beta \circ \Xi^{-1} && \text{on } \Gamma, \\ [T_1(w,q)](\nu \circ \Xi^{-1}) &= 0 && \text{on } \Gamma, \\ (w^\pm, q^\pm)(x) &\to 0 && \text{for } |x| \to \infty. \end{aligned}\right\} \tag{2.4}$$

For the construction of the solution to (2.4), let us first point out that for any smooth solution $(U, P) : E \longrightarrow \mathbb{R}^2 \times \mathbb{R}$ to the homogeneous Stokes system

$$\left.\begin{aligned} \Delta U - \nabla P = 0, \\ \operatorname{div} U = 0 \end{aligned}\right\} \qquad \text{in } E, \tag{2.5}$$

where E is a domain in $\mathbb{R}^2$, the functions $(W^i, Q^i) : E \longrightarrow \mathbb{R}^2 \times \mathbb{R}$, $i = 1, 2$, given by

$$W^i_j := T_{1,ij}(U, P) = -P\delta_{ij} + \partial_i U_j + \partial_j U_i, \quad j = 1,\, 2, \quad \text{and} \quad Q^i = 2\partial_i P$$

are solutions to (2.5) as well. In particular, if $E = \mathbb{R}^2 \setminus \{0\}$ and

$$(U, P) = (\mathcal{U}^k, \mathcal{P}^k) : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \times \mathbb{R}, \qquad k = 1, 2,$$

are the fundamental solutions to the Stokes equations (2.5), given by

$$\begin{aligned} \mathcal{U}^k_j(y) &= -\frac{1}{4\pi}\left(\delta_{jk} \ln \frac{1}{|y|} + \frac{y_j y_k}{|y|^2}\right), \quad j = 1,\, 2, \\ \mathcal{P}^k(y) &= -\frac{1}{2\pi}\frac{y_k}{|y|^2}, \qquad y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}, \end{aligned} \tag{2.6}$$

we obtain a system $(\mathcal{W}^{i,k}, \mathcal{Q}^{i,k}) : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \times \mathbb{R}$, $i, k = 1,\, 2$, of solutions to the homogeneous Stokes equations given by

$$\begin{aligned} \mathcal{W}^{i,k}_j(y) &:= (-\mathcal{P}^k \delta_{ij} + \partial_i \mathcal{U}^k_j + \partial_j \mathcal{U}^k_i)(y) = \frac{1}{\pi}\frac{y_i y_j y_k}{|y|^4}, \quad j = 1,\, 2, \\ \mathcal{Q}^{i,k}(y) &:= 2\partial_i \mathcal{P}^k(y) = \frac{1}{\pi}\left(-\frac{\delta_{ik}}{|y|^2} + 2\frac{y_i y_k}{|y|^4}\right), \qquad y \in \mathbb{R}^2 \setminus \{0\}. \end{aligned}$$

We are going to show that $(w, q) := (w, q)[\beta]$ given by

$$\begin{aligned} w_j(x) &:= \int_\Gamma \mathcal{W}^{i,k}_j(x - y)\tilde{\nu}_i(y)\beta_k(y_1)\, d\Gamma_y \\ &= \int_{\mathbb{R}} \mathcal{W}^{i,k}_j(r)\nu_i(s)\beta_k(s)\omega(s)\, ds, \quad j = 1,\, 2, \end{aligned} \tag{2.7}$$

$$\begin{aligned} q(x) &:= \int_\Gamma \mathcal{Q}^{i,k}(x - y)\tilde{\nu}_i(y)\beta_k(y_1)\, d\Gamma_y \\ &= \int_{\mathbb{R}} \mathcal{Q}^{i,k}(r)\nu_i(s)\beta_k(s)\, \omega(s) ds \end{aligned} \tag{2.8}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$ and with $r := r(x, s) := x - (s, f(s))$ solves (2.4). Here and further, we sum over indices appearing twice in a product. We write this more explicitly as

$$
\begin{aligned}
w(x) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{-f'r_1 + r_2}{|r|^4}\begin{pmatrix} r_1^2 & r_1r_2 \\ r_1r_2 & r_2^2 \end{pmatrix}\beta\, ds,\\
q(x) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{|r|^4}(-f' \;\; 1)\begin{pmatrix} r_1^2 - r_2^2 & 2r_1r_2 \\ 2r_1r_2 & r_2^2 - r_1^2 \end{pmatrix}\beta\, ds.
\end{aligned}
\tag{2.9}
$$

The solution (w, q) is the so-called hydrodynamic double-layer potential generated by the density $\beta \circ \Xi^{-1}$ on Γ, see [16].

Proposition 2.1. *The boundary value problem* (2.4) *has precisely one solution* $(w, q) \in X$*. It is given by* (2.7), (2.8)*. Moreover,* $w^{\pm}|_{\Gamma} \circ \Xi \in H^2(\mathbb{R})^2$*.*

Proof. The uniqueness of the solution can be shown as in the proof of [20, Theorem 2.1]. Observe that w and q are defined by integrals of the form

$$
(w, q)(x) = \int_{\mathbb{R}} K(x, s)\beta(s)\, ds
$$

where for every $\alpha \in \mathbb{N}^2$ we have $\partial_x^{\alpha} K(x, s) = O(s^{-1})$ for $|s| \to \infty$ and locally uniformly in $x \in \mathbb{R}^2 \setminus \Gamma$. This shows that w and q are well-defined by (2.7) and (2.8), and that integration and differentiation with respect to x may be interchanged. As $(\mathcal{W}^{i,k}, \mathcal{Q}^{i,k})$ solve the homogeneous Stokes equations, this also holds for (w, q).

To show the decay of q at infinity we obtain from the matrix equality

$$
\frac{1}{\pi|r|^4}(-f' \;\; 1)\begin{pmatrix} r_1^2 - r_2^2 & 2r_1r_2 \\ 2r_1r_2 & r_2^2 - r_1^2 \end{pmatrix} = -2\partial_s(\mathcal{P}^2(r) \;\; -\mathcal{P}^1(r)),
$$

via integration by parts

$$
q(x) = 2\int_{\mathbb{R}} (\mathcal{P}^2 \;\; -\mathcal{P}^1)(r)\beta'\, ds = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{|r|^2}(-r_2 \;\; r_1)\beta'\, ds.
$$

In view of this representation, [18, Lemma 2.1] implies $q(x) \to 0$ as $|x| \to \infty$.

In order to prove the decay of w we rewrite

$$
\begin{aligned}
w(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{-f'r_1 + r_2}{|r|^2}\left(I + \frac{1}{|r|^2}\begin{pmatrix} r_1^2 - r_2^2 & 2r_1r_2 \\ 2r_1r_2 & r_2^2 - r_1^2 \end{pmatrix}\right)\beta\, ds\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{-f'r_1 + r_2}{|r|^2} I + \partial_s\left[\frac{1}{|r|^2}\begin{pmatrix} r_1r_2 & r_2^2 \\ r_2^2 & -r_1r_2 \end{pmatrix}\right]\right)\beta\, ds\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{-f'r_1 + r_2}{|r|^2}\beta - \frac{1}{|r|^2}\begin{pmatrix} r_1r_2 & r_2^2 \\ r_2^2 & -r_1r_2 \end{pmatrix}\beta'\right) ds,
\end{aligned}
$$

where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. In view of [18, Lemma 2.1] and [20, Lemma B.2] we conclude that indeed $w(x) \to 0$ for $|x| \to \infty$.

The boundary conditions $(2.4)_3$ and $(2.4)_4$ together with the properties that $(w, q) \in X$ and $w^{\pm}|_{\Gamma} \circ \Xi \in H^2(\mathbb{R})^2$ are shown in Appendix A. □

3. The L_2-resolvent of the hydrodynamic double-layer potential operator

In this section we study the resolvent set of the hydrodynamic double-layer potential operator $\mathbb{D}(f)$, with $f\in C^1(\mathbb{R})$, introduced in (3.5) below, which we view in this section as an element of $\mathcal{L}(L_2(\mathbb{R})^2)$. The main result of this section is Theorem 3.3 below which provides in particular the invertibility of $\lambda-\mathbb{D}(f)$ for $\lambda\in\mathbb{R}$ with $|\lambda|>1/2$.

To begin, we introduce a general class of singular integral operators suited to our approach via layer potentials, cf. [19,20]. Given n, $m\in\mathbb{N}$ and Lipschitz continuous functions $a_1,\ldots,a_m$, $b_1,\ldots,b_n:\mathbb{R}\longrightarrow\mathbb{R}$, we let $B_{n,m}$ denote the singular integral operator

$$\begin{aligned} &B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,h](\xi)\\ &\quad:=\mathrm{PV}\int_{\mathbb{R}}\frac{h(\xi-\eta)}{\eta}\frac{\prod_{i=1}^n\left(\delta_{[\xi,\eta]}b_i/\eta\right)}{\prod_{i=1}^m\left[1+\left(\delta_{[\xi,\eta]}a_i/\eta\right)^2\right]}\,d\eta, \end{aligned}\tag{3.1}$$

where $\mathrm{PV}\int_{\mathbb{R}}$ denotes the principal value integral and $\delta_{[\xi,\eta]}u:=u(\xi)-u(\xi-\eta)$. For brevity we set

$$B^0_{n,m}(f):=B_{n,m}(f,\ldots f)[f,\ldots,f,\cdot].\tag{3.2}$$

In this section we several times use the following result.

Lemma 3.1. *There exists a constant $C=C(n,m,\max_{i=1,\ldots,m}\|a_i'\|_\infty)$ with*

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\cdot]\|_{\mathcal{L}(L_2(\mathbb{R}))}\leq C\prod_{i=1}^n\|b_i'\|_\infty.$$

Moreover, $B_{n,m}\in C^{1-}(W^1_\infty(\mathbb{R})^m,\mathcal{L}^n_{\mathrm{sym}}(W^1_\infty(\mathbb{R}),\mathcal{L}(L_2(\mathbb{R}))))$.

Proof. See [19, Remark 3.3]. □

As we are concerned exclusively with boundary integral operators in this section, it will be convenient to slightly change notation and write

$$r:=(r^1,r^2):=r(\xi,s):=(\xi,f(\xi))-(s,f(s)),\qquad \xi,s\in\mathbb{R}.\tag{3.3}$$

Given $f\in C^1(\mathbb{R})$, we introduce the linear operators $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$ defined by

$$\begin{aligned} \mathbb{D}(f)[\beta](\xi)&:=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{r_1f'-r_2}{|r|^4}\begin{pmatrix} r_1^2 & r_1r_2\\ r_1r_2 & r_2^2\end{pmatrix}\beta\,ds,\\ \mathbb{D}(f)^*[\beta](\xi)&:=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{-r_1f'(\xi)+r_2}{|r|^4}\begin{pmatrix} r_1^2 & r_1r_2\\ r_1r_2 & r_2^2\end{pmatrix}\beta\,ds, \end{aligned}\tag{3.4}$$

where $\xi\in\mathbb{R}$ and $\beta\in L_2(\mathbb{R})^2$. We note that $\mathbb{D}(f)$ is related to the $B_{n,m}$ via

$$\begin{aligned} \mathbb{D}(f)[\beta]&=\frac{1}{\pi}\begin{pmatrix} B^0_{0,2}(f) & B^0_{1,2}(f)\\ B^0_{1,2}(f) & B^0_{2,2}(f)\end{pmatrix}\begin{pmatrix} f'\beta_1\\ f'\beta_2\end{pmatrix}\\ &\quad-\frac{1}{\pi}\begin{pmatrix} B^0_{1,2}(f) & B^0_{2,2}(f)\\ B^0_{2,2}(f) & B^0_{3,2}(f)\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \end{aligned}\tag{3.5}$$

for $\beta = (\beta_1, \beta_2)^\top$. Therefore, as a consequence of Lemma 3.1, $\mathbb{D}(f)$ is bounded on $L_2(\mathbb{R})^2$. Moreover, up to the sign and the push-forward via Ξ, $\mathbb{D}(f)[\beta](\xi)$ is the "direct value" of the hydrodynamic double-layer potential w generated by β in $(\xi, f(\xi)) \in \Gamma$, cf. $(2.9)_1$. One may also check that $\mathbb{D}(f)^*$ is the L_2-adjoint of $\mathbb{D}(f)$.

Using the same notation, we define the singular integral operators $\mathbb{B}_1(f)$ and $\mathbb{B}_2(f)$ by

$$\mathbb{B}_1(f)[\theta](\xi) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{-r_1 f' + r_2}{|r|^2}\,\theta\, ds$$
$$\mathbb{B}_2(f)[\theta](\xi) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{r_1 + r_2 f'}{|r|^2}\,\theta\, ds,$$

where $\theta \in L_2(\mathbb{R})$. The operators $\mathbb{B}_i(f)$, $i = 1, 2$, play an important role also in the study of the Muskat problem, cf. [18]. Lemma 3.1 implies in particular that also $\mathbb{B}_i(f)$ $i = 1, 2$, is bounded on $L_2(\mathbb{R})$. Moreover, $\mathbb{B}_1(f)[\theta](\xi)$ is the direct value of the double layer potential for the Laplacian corresponding to the density θ in $(\xi, f(\xi)) \in \Gamma$.

We are going to prove in Theorem 3.3 below that the resolvent sets of $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$ contain all real λ with $|\lambda| > 1/2$, with a bound on the resolvent that is uniform in λ away from $\pm 1/2$, and in f as long as $\|f'\|_\infty$ is bounded.

Oriented at [7,10], we obtain this property on the basis of a Rellich identity for the Stokes operator. While eventually the result for $\mathbb{D}(f)$ is needed, it is helpful to consider $\mathbb{D}(f)^*$, as this operator naturally arises from the jump relations for the single-layer hydrodynamic potential generated by β, cf. (3.13) below.

We next derive the Rellich identity (3.14), and based on it we establish an estimate that relates the operator $\mathbb{D}(f)^*$ to the operators $\mathbb{B}_1(f)$ and $\mathbb{B}_2(f)$ introduced above.

Lemma 3.2. *Given $K > 0$, there exists a positive constant C, that depends only on K, such that for all $\beta \in L_2(\mathbb{R})^2$, $\lambda \in [-K, K]$, and $f \in \mathrm{C}^1(\mathbb{R})$ which satisfy $\|f'\|_\infty < K$ we have*

$$C\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2\|\beta\|_2 \geq \|(\lambda - \tfrac{1}{2}\mathbb{B}_1(f))[\omega^{-1}\beta\cdot\nu] - \tfrac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau]\|_2^2 + m(\lambda)\|\omega^{-1}\beta\cdot\tau\|_2^2, \tag{3.6}$$

where ω, ν, and τ are defined in (2.2), and with

$$m(\lambda) := \max\left\{\left(\lambda + \tfrac{1}{2}\right)\left(\lambda - \tfrac{3}{2}\right), \left(\lambda - \tfrac{1}{2}\right)\left(\lambda + \tfrac{3}{2}\right)\right\}. \tag{3.7}$$

Proof. Let first $f \in \mathrm{C}^\infty(\mathbb{R})$ and $\beta = (\beta_1, \beta_2)^\top$ with $\beta_k \in \mathrm{C}_0^\infty(\mathbb{R})$, $k = 1, 2$. We define the hydrodynamic single-layer potential u with corresponding pressure Π by

$$u(x) := -\int_{\mathbb{R}} \mathcal{U}^k(x - (s, f(s)))\beta_k(s)\, ds$$
$$\Pi(x) := -\int_{\mathbb{R}} \mathcal{P}^k(x - (s, f(s)))\beta_k(s)\, ds$$

for $x \in \mathbb{R}^2 \setminus \Gamma$, where and $\mathcal{U}^k$, $\mathcal{P}^k$ defined by (2.6). Using the fact that β is compactly supported, is is not difficult to see that the functions (u, Π) are well-defined and smooth in $\Omega^\pm$ and satisfy

$$\left.\begin{array}{r}\Delta u - \nabla\Pi = 0,\\ \operatorname{div} u = 0\end{array}\right\}\qquad \text{in}\,\Omega^\pm, \tag{3.8}$$

as well as

$$\Pi,\, \nabla u = O(|x|^{-1}) \qquad \text{for}\, |x| \to \infty. \tag{3.9}$$

Moreover, [6, Lemma A.1] and the arguments in the proof of [20, Lemma A.1] show that $\Pi|_{\Omega^\pm}$ and $u|_{\Omega^\pm}$ have extensions $\Pi^\pm \in \mathrm{C}(\overline{\Omega^\pm})$ and $u^\pm \in \mathrm{C}^1(\overline{\Omega^\pm})$, and, given $\xi \in \mathbb{R}$, we have

$$\begin{aligned}\partial_i u_j^\pm \circ \Xi(\xi) &= -\mathrm{PV}\int_{\mathbb{R}} \partial_i \mathcal{U}_j^k(r)\beta_k\, ds \pm \frac{-\beta_j\nu^i + \nu^i\nu^j\beta\cdot\nu}{2\omega}(\xi),\\ \Pi^\pm \circ \Xi(\xi) &= -\mathrm{PV}\int_{\mathbb{R}} \mathcal{P}^k(r)\beta_k\, ds \pm \frac{\beta\cdot\nu}{2\omega}(\xi)\\ &= \frac{1}{2}\mathbb{B}_1(f)[\omega^{-1}\beta\cdot\nu](\xi) + \frac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau](\xi) \pm \frac{\beta\cdot\nu}{2\omega}(\xi),\end{aligned} \tag{3.10}$$

where $\nu = (\nu^1, \nu^2)$ and $r = r(\xi, s)$ are defined in (2.2) and (3.3). In particular,

$$\partial_2 u^\pm \circ \Xi(\xi) = \mathbb{T}(f)[\beta](\xi) \mp \frac{(\beta\cdot\tau)\tau}{2\omega^2}(\xi), \tag{3.11}$$

where $\mathbb{T}(f)$ is the singular integral operator given by

$$\mathbb{T}(f)[\beta](\xi) := \frac{1}{4\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{1}{|r|^4}\begin{pmatrix} -r_2^3 - 3r_1^2 r_2 & r_1^3 - r_1 r_2^2\\ r_1^3 - r_1 r_2^2 & r_1^2 r_2 - r_2^3\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} ds.$$

Observe that $\mathbb{T}(f)$ is skew-adjoint on $L_2(\mathbb{R})^2$, i.e. $\mathbb{T}(f)^* = -\mathbb{T}(f)$, and therefore

$$\langle \mathbb{T}(f)[\beta] \,|\, \beta\rangle_2 = 0. \tag{3.12}$$

Here $\langle\cdot\,|\,\cdot\rangle_2$ denotes the inner product of $L_2(\mathbb{R})^2$.

Moreover, for the normal stress at the boundary we find

$$\omega(T_1(u,\Pi)^\pm \circ \Xi)\nu = \Big(\mp\frac{1}{2} - \mathbb{D}(f)^*\Big)[\beta]. \tag{3.13}$$

For convenience we introduce the notation

$$\tau_{ij} := (T_1(u,\Pi))_{ij} = -\Pi\delta_{ij} + \partial_i u_j + \partial_j u_i, \qquad i, j = 1, 2,$$

and observe that due to (3.8)

$$\partial_i \tau_{ij} = 0 \quad \text{in}\,\Omega^\pm, j = 1,\, 2, \qquad \text{and} \qquad \delta_{ij}\partial_i u_j = 0 \qquad \text{in}\,\Omega^\pm.$$

The latter identities lead us to the following identities in $\Omega^\pm$:

$$\partial_i(\tau_{ij}\partial_2 u_j) = \tau_{ij}\partial_i\partial_2 u_j = (\partial_i u_j + \partial_j u_i)\partial_2\partial_i u_j = \frac{1}{4}\sum_{i,\,j=1}^{2} \partial_2(\partial_i u_j + \partial_j u_i)^2.$$

In view of (3.9) we may integrate the latter relation over $\Omega^\pm$ and using Gauss' theorem and (3.13) we get

$$\begin{aligned}\int_\Gamma \frac{1}{\tilde\omega}\sum_{i,\,j=1}^{2}(\partial_i u_j^\pm+\partial_j u_i^\pm)^2\,d\Gamma &= 4\int_\Gamma \tau_{ij}^\pm\tilde\nu_i\partial_2 u_j^\pm\,d\Gamma\\ &= 4\Big\langle\Big(\mp\frac{1}{2}-\mathbb{D}(f)^*\Big)[\beta]\,\Big|\,\partial_2 u^\pm\circ\Xi\Big\rangle_2,\end{aligned}\tag{3.14}$$

where $\tilde\omega:=\omega\circ\Xi^{-1}$.

To estimate the term on the left we observe that the Cauchy-Schwarz inequality and $|\tilde\nu|=1$ yield

$$\sum_{i,\,j=1}^{2}(\partial_i u_j^\pm+\partial_j u_i^\pm)^2\geq\sum_{i=1}^{2}((\partial_i u_j^\pm+\partial_j u_i^\pm)\tilde\nu_j)^2=\sum_{i=1}^{2}(\tau_{ij}^\pm\tilde\nu_j+\Pi^\pm\tilde\nu_i)^2\qquad\text{on }\Gamma.$$

This inequality, the estimate $\|\mathbb{B}_i(f)\|_{\mathcal{L}(L_2(\mathbb{R}))}\leq C(K)$, $i=1,2$, cf. Lemma 3.1, and the representations (3.10) and (3.13), now yield

$$\begin{aligned}&\int_\Gamma\frac{1}{\tilde\omega}\sum_{i,j=1}^{2}(\partial_i u_j^\pm+\partial_j u_i^\pm)^2\,d\Gamma\\
&\quad\geq\Big\|\frac{1}{\omega}\Big(\mp\frac{1}{2}-\mathbb{D}(f)^*\Big)[\beta]+(\Pi^\pm\circ\Xi)\nu\Big\|_2^2\\
&\quad=\Big\|\frac{1}{\omega}\Big(\lambda-\mathbb{D}(f)^*\Big)[\beta]-\frac{1}{\omega}\Big(\lambda\pm\frac{1}{2}\Big)\beta+(\Pi^\pm\circ\Xi)\nu\Big\|_2^2\\
&\quad\geq\Big\|-\frac{1}{\omega}\Big(\lambda\pm\frac{1}{2}\Big)\beta+\Big(\frac{1}{2}\mathbb{B}_1(f)[\omega^{-1}\beta\cdot\nu]+\frac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau]\pm\frac{\beta\cdot\nu}{2\omega}\Big)\nu\Big\|_2^2\\
&\qquad+\Big\|\frac{1}{\omega}(\lambda-\mathbb{D}(f)^*)[\beta]\Big\|_2^2-C\|(\lambda-\mathbb{D}(f)^*)[\beta]\|_2\|\beta\|_2\\
&\quad\geq\Big(\lambda\pm\frac{1}{2}\Big)^2\|\omega^{-1}\beta\cdot\tau\|_2^2+\Big\|\Big(\lambda-\frac{1}{2}\mathbb{B}_1(f)\Big)[\omega^{-1}\beta\cdot\nu]-\frac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau]\Big\|_2^2\\
&\qquad-C\|(\lambda-\mathbb{D}(f)^*)[\beta]\|_2\|\|\beta\|_2\end{aligned}$$

for any $\lambda\in[-K,K]$.

We next consider the term on the right of (3.14). As a direct consequence of Lemma 3.1 we note that $\|\mathbb{T}(f)\|_{\mathcal{L}(L_2(\mathbb{R})^2)}\leq C=C(K)$. This bound together with (3.11) and (3.12) implies

$$\begin{aligned}&4\Big\langle\Big(\mp\frac{1}{2}-\mathbb{D}(f)^*\Big)[\beta]\,\Big|\,\partial_2 u\circ\Xi\Big\rangle_2\\
&\quad=4\Big\langle\Big(\lambda-\mathbb{D}(f)^*\Big)[\beta]-\Big(\lambda\pm\frac{1}{2}\Big)\beta\,\Big|\,\mathbb{T}[\beta]\mp\frac{(\beta\cdot\tau)\tau}{2\omega^2}\Big\rangle_2\\
&\quad\leq C\|(\lambda-\mathbb{D}(f)^*)[\beta]\|_2\|\|\beta\|_2\pm2\Big(\lambda\pm\frac{1}{2}\Big)\|\omega^{-1}\beta\cdot\tau\|_2^2.\end{aligned}$$

For $f \in \mathrm{C}^\infty(\mathbb{R})$, the estimate (3.6) follows from (3.14) and the latter estimates upon rearranging terms and a standard density argument. For general functions $f \in \mathrm{C}^1(\mathbb{R})$ we additionally need to use the continuity of the mappings

$$[f \mapsto \mathbb{D}(f)^*] : \mathrm{C}^1(\mathbb{R}) \to \mathcal{L}(L_2(\mathbb{R})^2),$$
$$[f \mapsto \mathbb{B}_i(f)] : \mathrm{C}^1(\mathbb{R}) \to \mathcal{L}(L_2(\mathbb{R})),\ i = 1,\, 2,$$

which is a straightforward consequence of Lemma 3.1, together with the density of $\mathrm{C}^\infty(\mathbb{R})$ in $\mathrm{C}^1(\mathbb{R})$. □

Based on Lemma 3.2 we now establish the following result.

Theorem 3.3. (Spectral properties of $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$) *Given $\delta \in (0,1)$, there exists a constant $C = C(\delta) > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1/2 + \delta$ and $f \in \mathrm{C}^1(\mathbb{R})$ with $\|f'\|_\infty \leq 1/\delta$ we have*

$$\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2 \geq C\|\beta\|_2 \qquad \text{for all}\, \beta \in L_2(\mathbb{R})^2. \tag{3.15}$$

Moreover, $\lambda - \mathbb{D}(f)^*$, $\lambda - \mathbb{D}(f) \in \mathcal{L}(L_2(\mathbb{R})^2)$ are isomorphisms for all $\lambda \in \mathbb{R}$ with $|\lambda| > 1/2$ and $f \in \mathrm{C}^1(\mathbb{R})$.

Proof. In order to prove (3.15) we assume the opposite. Then we may find sequences (λ_k) in $\mathbb{R}$, (f_k) in $\mathrm{C}^1(\mathbb{R})$, and (β_k) in $L_2(\mathbb{R})^2$ with the property that $|\lambda_k| \geq 1/2 + \delta$, $\|f_k'\|_\infty \leq 1/\delta$, and $\|\beta_k\|_2 = 1$ for all $k \in \mathbb{N}$, and

$$(\lambda_k - \mathbb{D}(f_k)^*)[\beta_k] \to 0 \qquad \text{in}\, L_2(\mathbb{R})^2.$$

Given $k \in \mathbb{N}$, we set $\nu_k := \nu_{f_k}$, $\tau_k := \tau_{f_k}$, and $\omega_k := \omega_{f_k}$, cf. (2.2). As the operators $\mathbb{D}(f_k)^*$ are bounded, uniformly in $k \in \mathbb{N}$, in $\mathcal{L}(L_2(\mathbb{R})^2)$, cf. Lemma 3.1, the sequence (λ_k) is bounded. Observing that for the constant $m = m(\lambda)$ from (3.7) we have $m(\lambda_k) \geq \delta(2+\delta) > 0$ for all $k \in \mathbb{N}$, we get from Lemma 3.2 that

$$\omega_k^{-1}\beta_k \cdot \tau_k \to 0, \quad \big(\lambda_k - \tfrac{1}{2}\mathbb{B}_1(f_k)\big)[\omega_k^{-1}\beta_k \cdot \nu_k] - \tfrac{1}{2}\mathbb{B}_2(f_k)[\omega_k^{-1}\beta_k \cdot \tau_k] \to 0$$

in $L_2(\mathbb{R})$. As the operators $\mathbb{B}_2(f_k)$ are bounded, uniformly with respect to $k \in \mathbb{N}$, in $\mathcal{L}(L_2(\mathbb{R})^2)$, cf. Lemma 3.1, this implies

$$\big(\lambda_k - \tfrac{1}{2}\mathbb{B}_1(f_k)\big)[\omega_k^{-1}\beta_k \cdot \nu_k] \to 0 \qquad \text{in}\, L_2(\mathbb{R}).$$

Let $\mathbb{A}(f) := \mathbb{B}_1(f)^*$. Since $|2\lambda_k| \geq 1$, it follows from the proof of [18, Theorem 3.5] that the operator $2\lambda_k - \mathbb{A}(f_k) \in \mathcal{L}(L_2(\mathbb{R}))$, $k \in \mathbb{N}$, is an isomorphism with

$$\|\,(2\lambda_k - \mathbb{A}(f_k))^{-1}\,\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq C(\delta).$$

This implies that also $2\lambda_k - \mathbb{B}_1(f_k) \in \mathcal{L}(L_2(\mathbb{R}))$, $k \in \mathbb{N}$, is an isomorphism and

$$\big\|(\lambda_k - \tfrac{1}{2}\mathbb{B}_1(f_k))^{-1}\big\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq C(\delta).$$

Thus $\omega_k^{-1}\beta_k \cdot \nu_k \to 0$ in $L_2(\mathbb{R})$, so that

$$\beta_k = \omega_k\big(\omega_k^{-1}(\beta_k \cdot \nu_k)\nu_k + \omega_k^{-1}(\beta_k \cdot \tau_k)\tau_k\big) \to 0 \quad \text{in}\, L_2(\mathbb{R})^2.$$

This contradicts the property that $\|\beta_k\|_2 = 1$ for all $k \in \mathbb{N}$ and (3.15) follows.

To complete the proof we fix $f \in \mathrm{C}^1(\mathbb{R})$ and $\lambda_0 \in \mathbb{R}$ with $|\lambda_0| > 1/2$ and we choose $\delta \in (0,1)$ such that $|\lambda_0| \geq 1/2 + \delta$ and $\|f'\|_\infty \leq 1/\delta$. As $\mathbb{D}(f)^*$ is bounded, $\lambda - \mathbb{D}(f)^* \in \mathcal{L}(L_2(\mathbb{R})^2)$ is an isomorphism if $|\lambda|$ is sufficiently large. The estimate (3.15) together with a standard continuity argument, cf. e.g. [3, Proposition I.1.1.1], now implies that $\lambda_0 - \mathbb{D}(f)^*$ is an isomorphism as well. The result for $\mathbb{D}(f)$ is an immediate consequence of this property. □

4. The resolvent of the hydrodynamic double-layer potential operator in higher order Sobolev spaces

The main goal of this section is to establish spectral properties for $\mathbb{D}(f)$, parallel to those in Theorem 3.3, in the spaces $H^{s-1}(\mathbb{R})^2$, $s \in (3/2, 2)$, and in $H^2(\mathbb{R})^2$. The latter are needed when solving the fixed-time problem (5.1), see Proposition 5.1, and the former are used to derive and study the contour integral formulation (5.17) of the evolution problem (1.1).

For this purpose, we first recall some further results on the singular integral operators $B_{n,m}$ introduced in (3.1).

Lemma 4.1. (i) *Let $n \geq 1$, $s \in (3/2, 2)$, and $a_1, \ldots, a_m \in H^s(\mathbb{R})$ be given. Then, there exists a constant C, depending only on n, m, s, and $\max_{1\leq i\leq m} \|a_i\|_{H^s}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h]\|_2 \leq C\|b_1\|_{H^1}\|h\|_{H^{s-1}} \prod_{i=2}^{n} \|b_i\|_{H^s} \tag{4.1}$$

for all $b_1, \ldots, b_n \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$.
Moreover, $[(a_1, \ldots, a_m) \mapsto B_{n,m}(a_1, \ldots, a_m)]$ is locally Lipschitz continuous as a mapping from $H^s(\mathbb{R})^m$ to

$$\mathcal{L}^{n+1}(H^1(\mathbb{R}), H^s(\mathbb{R}), \ldots, H^s(\mathbb{R}), H^{s-1}(\mathbb{R}); L_2(\mathbb{R})).$$

(ii) *Given $s \in (3/2, 2)$ and $a_1, \ldots, a_m \in H^s(\mathbb{R})$, there exists a constant C, depending only on n, m, s, and $\max_{1\leq i\leq m} \|a_i\|_{H^s}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h]\|_{H^{s-1}} \leq C\|h\|_{H^{s-1}} \prod_{i=1}^{n} \|b_i\|_{H^s}$$

for all $b_1, \ldots, b_n \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$.
Moreover, $B_{n,m} \in \mathrm{C}^{1-}(H^s(\mathbb{R})^m, \mathcal{L}^n_{\mathrm{sym}}(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R}))))$.

(iii) *Let $n \geq 1$, $3/2 < s' < s < 2$, and $a_1, \ldots, a_m \in H^s(\mathbb{R})$ be given. Then, there exists a constant C, which depends only on n, m, s, s', and $\max_{1\leq i\leq m} \|a_i\|_{H^s}$, such that*

$$\begin{aligned}&\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h] - hB_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1']\|_{H^{s-1}}\\ &\quad\leq C\|b_1\|_{H^{s'}}\|h\|_{H^{s-1}} \prod_{i=2}^{n} \|b_i\|_{H^s}\end{aligned}$$

for all $b_1, \ldots, b_n \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$.

Proof. The claims (i) is established in [18, Lemmas 3.2], while (ii) and (iii) are proven in [1, Lemma 5 and Lemma 6]. □

For $\xi \in \mathbb{R}$ we define the left shift operator τ_ξ on $L_2(\mathbb{R})$ by the relation $\tau_\xi u(x) := u(x+\xi)$ and observe the invariance property

$$\tau_\xi B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,h] = B_{n,m}(\tau_\xi a_1,\ldots,\tau_\xi a_m)[\tau_\xi b_1,\ldots,\tau_\xi b_n,\tau_\xi h]. \tag{4.2}$$

Differences of $B_{n,m}$ with respect to the nonlinear arguments a_i can be represented by the identity

$$\begin{aligned} &B_{n,m}(a_1,a_2\ldots,a_m)[b_1,\ldots,b_n,\cdot] - B_{n,m}(\tilde a_1,a_2\ldots,a_m)[b_1,\ldots,b_n,\cdot] \\ &\quad = B_{n+2,m+1}(\tilde a_1,a_1,a_2\ldots,a_m)[b_1,\ldots,b_n,\tilde a_1+a_1,\tilde a_1-a_1,\cdot]. \end{aligned} \tag{4.3}$$

We will also use the interpolation property

$$[H^{s_0}(\mathbb{R}),H^{s_1}(\mathbb{R})]_\theta = H^{(1-\theta)s_0+\theta s_1}(\mathbb{R}), \quad \theta\in(0,1),\ -\infty<s_0\le s_1<\infty, \tag{4.4}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor of exponent θ.

Theorem 4.2. *Given $\delta\in(0,1)$ and $s\in(3/2,2)$, there exists a positive constant $C=C(\delta,s)$ such that*

$$\|(\lambda-\mathbb{D}(f))[\beta]\|_{H^{s-1}} \ge C\|\beta\|_{H^{s-1}} \tag{4.5}$$

for all $\lambda\in\mathbb{R}$ which satisfy $|\lambda|\ge 1/2+\delta$, $f\in H^s(\mathbb{R})$ with $\|f\|_{H^s}\le 1/\delta$, and all $\beta\in H^{s-1}(\mathbb{R})^2$.

Moreover, $\lambda-\mathbb{D}(f)\in\mathcal{L}(H^{s-1}(\mathbb{R})^2)$ is an isomorphism for all $\lambda\in\mathbb{R}$ with $|\lambda|>1/2$ and $f\in H^s(\mathbb{R})$.

Proof. Given $f\in H^s(\mathbb{R})$, the relation (3.5) and Lemma 4.1 (ii) combined imply that $\mathbb{D}(f)\in\mathcal{L}(H^{s-1}(\mathbb{R})^2)$. In order to prove the estimate (4.5), let $\lambda\in\mathbb{R}$ with $|\lambda|\ge 1/2+\delta$ and $f\in H^s(\mathbb{R})$ with $\|f\|_{H^s}\le 1/\delta$ be fixed. Theorem 3.3 together with the embedding $H^s(\mathbb{R})\hookrightarrow L_\infty(\mathbb{R})$ implies there exists $C=C(\delta)>0$ such that $\|(\lambda-\mathbb{D}(\tau_\xi f))^{-1}\|_{\mathcal{L}(L_2(\mathbb{R})^2)}\le C$ for all $\xi\in\mathbb{R}$. It is well-known there exists a constant $C>0$ such that

$$[\beta]_{H^{s-1}} := \|[\xi\mapsto|\xi|^{s-1}\mathcal{F}[\beta](\xi)]\|_2 = C\Big(\int_{\mathbb{R}}\frac{\|\beta-\tau_\xi\beta\|_2^2}{|\xi|^{1+2(s-1)}}\,d\xi\Big)^{1/2} =: [\beta]_{W_2^{s-1}},$$

where $\mathcal{F}[\beta]$ is the Fourier transform of β. Together with (4.2) we then get

$$\begin{aligned} [\beta]_{H^{s-1}}^2 &\le C\int_{\mathbb{R}}\frac{\|(\lambda-\mathbb{D}(\tau_\xi f))[\beta-\tau_\xi\beta]\|_2^2}{|\xi|^{1+2(s-1)}}\,d\xi \\ &\le C\Big(\int_{\mathbb{R}}\frac{\|(\lambda-\mathbb{D}(f))[\beta]-\tau_\xi((\lambda-\mathbb{D}(f))[\beta])\|_2^2}{|\xi|^{1+2(s-1)}}\,d\xi \\ &\quad+\int_{\mathbb{R}}\frac{\|(\mathbb{D}(f)-\mathbb{D}(\tau_\xi f))[\beta]\|_2^2}{|\xi|^{1+2(s-1)}}\,d\xi\Big) \\ &= C[(\lambda-\mathbb{D}(f))[\beta]]_{H^{s-1}}^2 + C\int_{\mathbb{R}}\frac{\|(\mathbb{D}(f)-\mathbb{D}(\tau_\xi f))[\beta]\|_2^2}{|\xi|^{1+2(s-1)}}\,d\xi. \end{aligned} \tag{4.6}$$

The term $\|(\mathbb{D}(f)-\mathbb{D}(\tau_\xi f))[\beta]\|_2$ can be estimated by a finite sum of terms of the form

$$\|(B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f))[\beta_i]\|_2 \quad \text{and} \quad \|B^0_{n,2}(f)[f'\beta_i] - B^0_{n,2}(\tau_\xi f)[(\tau_\xi f')\beta_i]\|_2,$$

where $0 \le n \le 3$ and $i \in \{1,2\}$. Let $s' \in (3/2, s)$ be fixed. We first consider terms of the second type and estimate in view of Lemma 3.1

$$\begin{aligned} &\|B^0_{n,2}(f)[f'\beta_i] - B^0_{n,2}(\tau_\xi f)[(\tau_\xi f')\beta_i]\|_2 \\ &\quad \le \|(B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f))[f'\beta_i]\|_2 + \|B^0_{n,2}(\tau_\xi f)[(\tau_\xi f' - f')\beta_i]\|_2 \\ &\quad \le \|(B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f))[f'\beta_i]\|_2 + C\|\tau_\xi f' - f'\|_2 \|\beta\|_{H^{s'-1}}. \end{aligned} \tag{4.7}$$

Furthermore, using (4.3), we have

$$\begin{aligned} B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f) = &\sum_{\ell=1}^{n} B_{n,2}(f,f)[\underbrace{\tau_\xi f, \ldots, \tau_\xi f}_{\ell-1\,\text{times}}, f - \tau_\xi f, f, \ldots, f, \cdot] \\ &+ B_{n+2,3}(\tau_\xi f, f, f)[\tau_\xi f, \ldots, \tau_\xi f, \tau_\xi - f, \tau_\xi f + f, \cdot] \\ &+ B_{n+2,3}(\tau_\xi f, \tau_\xi f, f)[\tau_\xi f, \ldots, \tau_\xi f, \tau_\xi f - f, \tau_\xi f + f, \cdot], \end{aligned}$$

and together with Lemma 4.1 (i) (with s' instead of s), we conclude that the operator $B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f)$ belongs to $\mathcal{L}(H^{s'-1}(\mathbb{R}), L_2(\mathbb{R}))$ and satisfies

$$\|B^0_{n,2}(f) - B^0_{n,2}(\tau_\xi f)\|_{\mathcal{L}(H^{s'-1}(\mathbb{R}), L_2(\mathbb{R}))} \le C\|f - \tau_\xi f\|_{H^1(\mathbb{R})}.$$

Combining this estimate with (4.7) we get

$$\int_{\mathbb{R}} \frac{\|(\mathbb{D}(f) - \mathbb{D}(\tau_\xi f))[\beta]\|_2^2}{|\xi|^{1+2(s-1)}}\, d\xi \le C\|f\|^2_{H^s} \|\beta\|^2_{H^{s'-1}},$$

and by (4.6) and the interpolation property (4.4) we arrive at

$$\|\beta\|^2_{H^{s-1}} \le C\left([\lambda - \mathbb{D}(f)[\beta]]^2_{H^{s-1}} + \|\beta\|_2^2\right) + \frac{1}{2}\|\beta\|^2_{H^{s-1}}.$$

Finally, using Theorem 3.3 again, we obtain the estimate (4.5). The isomorphism property of $\lambda - \mathbb{D}(f)$, with $\lambda \in \mathbb{R}$ with $|\lambda| > 1/2$ and $f \in H^s(\mathbb{R})$, follows by the same continuity argument as in the L_2 result. □

For the H^2 result we need an additional estimate for the operators $B_{n,m}$ with higher regularity of the arguments.

Lemma 4.3. *Let n, $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in H^2(\mathbb{R})$ be given. Then, there exists a constant C, depending only on n, m, and $\max_{1\le i\le m} \|a_i\|_{H^2}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h]\|_{H^1} \le C\|h\|_{H^1} \prod_{i=1}^{n} \|b_i\|_{H^2} \tag{4.8}$$

for all $b_1, \ldots, b_n \in H^2(\mathbb{R})$ and $h \in H^1(\mathbb{R})$.

Moreover, $B_{n,m} \in \mathrm{C}^{1-}(H^2(\mathbb{R})^m, \mathcal{L}^n_{\mathrm{sym}}(H^2(\mathbb{R}), \mathcal{L}(H^1(\mathbb{R}))))$.

Proof. We first show that the function $\varphi := B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,h]$ belongs to $H^1(\mathbb{R})$. Recalling that the group $\{\tau_\xi\}_{\xi\in\mathbb{R}} \subset \mathcal{L}(H^r(\mathbb{R}))$, $r\geq 0$, has generator $[f\mapsto f'] \in \mathcal{L}(H^{r+1}(\mathbb{R}), H^r(\mathbb{R}))$, it suffices to prove that the quotient $D_\xi\varphi := (\tau_\xi\varphi-\varphi)/\xi$ converges in $L_2(\mathbb{R})$ when letting $\xi\to 0$. In view of (4.3) we write

$$\begin{aligned}D_\xi\varphi &= \sum_{i=1}^n B_{n,m}(\tau_\xi a_1,\ldots,\tau_\xi a_m)\big[b_1,\ldots,b_{i-1},D_\xi b_i,\tau_\xi b_{i+1},\ldots,\tau_\xi b_n,\tau_\xi h\big]\\ &\quad + B_{n,m}(\tau_\xi a_1,\ldots,\tau_\xi a_m)\big[b_1,\ldots,,b_n,D_\xi h\big]\\ &\quad - \sum_{i=1}^m B^i_{n+2,m+1}(\xi)\big[b_1,\ldots,b_n,D_\xi a_i,\tau_\xi a_i+a_i,h\big],\end{aligned}$$

where $B^i_{n+2,m+1}(\xi) := B_{n+2,m+1}(\tau_\xi a_1,\ldots,\tau_\xi a_i,a_i,\ldots,a_m)$ for $1\leq i\leq m$. Lemma 3.1 and Lemma 4.1 (i) enable us to pass to the limit $\xi\to 0$ in $L_2(\mathbb{R})$ in this equality. Hence, $\varphi\in H^1(\mathbb{R})$ and

$$\begin{aligned}\varphi' &= B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,h']\\ &\quad+\sum_{i=1}^n B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_{i-1},b_i',b_{i+1},\ldots b_n,h]\\ &\quad-2\sum_{i=1}^m B_{n+2,m+1}(a_1,\ldots,a_i,a_i,\ldots,a_m)[b_1,\ldots,b_n,a_i',a_i,h].\end{aligned}\tag{4.9}$$

The estimate (4.8) is a consequence of Lemma 3.1 and Lemma 4.1 (i). The local Lipschitz continuity property follows from an repeated application of (4.3) and (4.8). □

As a consequence of Lemma 4.3 and (4.9) we obtain the following result.

Corollary 4.4. $B_{n,m}\in \mathrm{C}^{1-}(H^3(\mathbb{R})^m,\mathcal{L}^n_{\mathrm{sym}}(H^3(\mathbb{R}),\mathcal{L}(H^2(\mathbb{R}))))$ *for* $n,\ m\in\mathbb{N}$.

Theorem 4.5. *The operator* $\lambda-\mathbb{D}(f)\in\mathcal{L}(H^2(\mathbb{R})^2)$ *is an isomorphism for all* $f\in H^3(\mathbb{R})$ *and* $\lambda\in\mathbb{R}$ *with* $|\lambda|>1/2$.

Proof. Fix $f\in H^3(\mathbb{R})$. We then infer from (3.5) and Corollary 4.4 that we have $\mathbb{D}(f)\in\mathcal{L}(H^2(\mathbb{R})^2)$. Recalling (4.9), we further compute

$$(\mathbb{D}(f)[\beta])''-\mathbb{D}(f)[\beta''] = T_{\mathrm{lot}}[\beta],\qquad \beta\in H^2(\mathbb{R})^2,\tag{4.10}$$

where each component of $T_{\mathrm{lot}}[\beta]$ is a linear combination of terms

$$\begin{aligned}&B_{n,m}(f,\ldots,f)[f',f',f,\ldots,f,(f')^k\beta_i],\qquad B_{n,m}(f,\ldots,f)[f,\ldots,f,f'''\beta_i],\\ &\quad B_{n,m}(f,\ldots,f)[f',f,\ldots,f,((f')^k\beta_i)'],\quad B_{n,m}(f,\ldots,f)[f'',f,\ldots,f,(f')^k\beta_i],\end{aligned}$$

where $n,\ m\in\mathbb{N}$ satisfy $0\leq n,\ m\leq 7$ and $k\in\{0,\ 1\}$. From Lemma 3.1 and Lemma 4.1 (i) (with $s=7/4$) we conclude that

$$\|T_{\mathrm{lot}}[\beta]\|_2\leq C\|\beta\|_{H^1},\qquad \beta\in H^2(\mathbb{R})^2.\tag{4.11}$$

Given $\lambda\in\mathbb{R}$ with $|\lambda|>1/2$, we pick $\delta\in(0,1)$ such that $|\lambda|\geq 1/2+\delta$ and additionally $\|f'\|_\infty\leq 1/\delta$. Since $\|(\mu-\mathbb{D}(f))^{-1}\|_{\mathcal{L}(L_2(\mathbb{R})^2)}\leq C$ for all $\mu\in\mathbb{R}$

with $|\mu| \geq 1/2 + \delta$, cf. Theorem 3.3, we deduce from (4.10), (4.11), and (4.4) that

$$\begin{aligned}\|\beta\|_{H^2} &\leq C(\|\beta''\|_2 + \|\beta\|_2) \leq C(\|(\mu - \mathbb{D}(f))[\beta'']\|_2 + \|\beta\|_2)\\ &\leq C\big(\|(\mu - \mathbb{D}(f))[\beta]''\|_2 + \|T_{\rm lot}[\beta]\|_2 + \|\beta\|_2\big)\\ &\leq C\big(\|(\mu - \mathbb{D}(f))[\beta]''\|_2 + \|\beta\|_{H^1}\big)\\ &\leq \tfrac{1}{2}\|\beta\|_{H^2} + C\big(\|(\mu - \mathbb{D}(f))[\beta]''\|_2 + \|\beta\|_2\big)\\ &\leq \tfrac{1}{2}\|\beta\|_{H^2} + C\big(\|(\mu - \mathbb{D}(f))[\beta]''\|_2 + \|(\mu - \mathbb{D}(f))[\beta]\|_2\big),\end{aligned}$$

hence

$$\|\beta\|_{H^2} \leq C\|(\mu - \mathbb{D}(f))[\beta]\|_{H^2}$$

for all $\beta \in H^2(\mathbb{R})^2$ and $\mu \in \mathbb{R}$ with $|\mu| \geq 1/2 + \delta$. The result follows now by the same continuity argument as in the proof of Theorem 4.2. □

5. The contour integral formulation

In this section we formulate the Stokes evolution problem (1.1) as an nonlinear evolution problem having only f as unknown, cf. (5.17).

Based on the results established in Sect. 2, Sect. 4, and Appendix A we start by proving that for each $f \in H^3(\mathbb{R})$, the boundary value problem

$$\left.\begin{aligned}\mu^\pm\Delta v^\pm - \nabla p^\pm &= 0 && \text{in } \Omega^\pm,\\ \operatorname{div} v^\pm &= 0 && \text{in } \Omega^\pm,\\ [v] &= 0 && \text{on } \Gamma,\\ [T_\mu(v,p)]\tilde\nu &= -\sigma\tilde\kappa\tilde\nu && \text{on } \Gamma,\\ (v^\pm, p^\pm)(x) &\to 0 && \text{for } |x| \to \infty\end{aligned}\right\} \tag{5.1}$$

has a unique solution $(v,p) \in X_f$ with the property that $v^\pm|_\Gamma \circ \Xi_f \in H^2(\mathbb{R})^2$. This is established in Proposition 5.1 below, where we also provide an implicit formula for $v^\pm|_\Gamma$ in terms of contour integrals on Γ. This representation allows to recast the kinematic boundary condition $(1.1\text{a})_6$ in the form (5.17).

With the substitution $\tilde v^\pm := \mu_\pm v^\pm$, Problem (5.1) is equivalent to

$$\left.\begin{aligned}\Delta\tilde v^\pm - \nabla p^\pm &= 0 && \text{in } \Omega^\pm,\\ \operatorname{div}\tilde v^\pm &= 0 && \text{in } \Omega^\pm,\\ \mu_-\tilde v^+ - \mu_+\tilde v^- &= 0 && \text{on } \Gamma,\\ [T_1(\tilde v,p)]\tilde\nu &= -\sigma\tilde\kappa\tilde\nu && \text{on } \Gamma,\\ (\tilde v^\pm, p^\pm) &\to 0 && \text{for } |x| \to \infty.\end{aligned}\right\} \tag{5.2}$$

We construct the solution to (5.2) by splitting

$$(\tilde v, p) = (w_s, q_s) + (w_d, q_d)$$

where (w_s, q_s), $(w_d, q_d) \in X_f$ satisfy

$$\left.\begin{array}{rll} \Delta w_s^\pm - \nabla q_s^\pm &= 0 & \text{in}\,\Omega^\pm, \\ \operatorname{div} w_s^\pm &= 0 & \text{in}\,\Omega^\pm, \\ w_s^+ - w_s^- &= 0 & \text{on}\,\Gamma, \\ [T_1(w_s, q_s)]\tilde\nu &= -\sigma\tilde\kappa\tilde\nu & \text{on}\,\Gamma, \\ (w_s^\pm, q_s^\pm) &\to 0 & \text{for}\,|x|\to\infty \end{array}\right\} \tag{5.3}$$

and

$$\left.\begin{array}{rll} \Delta w_d^\pm - \nabla q_d^\pm &= 0 & \text{in}\,\Omega^\pm, \\ \operatorname{div} w_d^\pm &= 0 & \text{in}\,\Omega^\pm, \\ \mu_- w_d^+ - \mu_+ w_d^- &= (\mu_+ - \mu_-) w_s & \text{on}\,\Gamma, \\ [T_1(w_d, q_d)]\tilde\nu &= 0 & \text{on}\,\Gamma, \\ (w_d^\pm, q_d^\pm) &\to 0 & \text{for}\,|x|\to\infty. \end{array}\right\} \tag{5.4}$$

The system (5.3) has been studied in [20]. According to [20, Theorem 2.1 and Remark A.2], there exists exactly one solution $(w_s, q_s) := (w_s(f), q_s(f)) \in X_f$ to (5.3). It satisfies

$$w_s \in \mathrm{C}^\infty(\mathbb{R}^2 \setminus \Gamma) \cap \mathrm{C}^1(\mathbb{R}^2) \quad \text{and} \quad q_s^\pm \in \mathrm{C}^\infty(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm}).$$

Moreover, recalling (3.2) and [20, Eqns. (2.2), (2.3), (A.2)], the trace $w_s(f)|_\Gamma$ can be expressed via

$$w_s(f)|_\Gamma \circ \Xi =: G(f) := (G_1(f), G_2(f)), \tag{5.5}$$

with

$$\begin{aligned} 4\pi\sigma^{-1} G_1(f) &:= (B^0_{0,2}(f) - B^0_{2,2}(f))[\phi_1(f) + f'\phi_2(f)] \\ &\quad + B^0_{1,2}(f)[3f'\phi_1(f) - \phi_2(f)] + B^0_{3,2}(f)[f'\phi_1(f) + \phi_2(f)], \\ 4\pi\sigma^{-1} G_2(f) &:= (B^0_{1,2}(f) - B^0_{3,2}(f))[\phi_1(f) + f'\phi_2(f)] \\ &\quad - B^0_{0,2}(f)[f'\phi_1(f) + \phi_2(f)] + B^0_{2,2}(f)[f'\phi_1(f) - 3\phi_2(f)], \end{aligned} \tag{5.6}$$

where $\phi_i(f) \in H^2(\mathbb{R})$, $i \in \{1, 2\}$, are given by

$$\phi_1(f) := \frac{f'^2}{\omega + \omega^2} \qquad \text{and} \qquad \phi_2(f) := \frac{f'}{\omega}. \tag{5.7}$$

We point out that Corollary 4.4 yields $G_i(f) \in H^2(\mathbb{R})$, $i \in \{1, 2\}$.

It remains to show that the boundary value problem (5.4) has a unique solution $(w_d, q_d) \in X_f$ with $w_d^\pm|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$. To construct a solution, we use the ansatz $(w_d, q_d) = (w, q)[\beta]$, where $\beta \in H^2(\mathbb{R})^2$ and $(w, q)[\beta]$ is defined by (2.7), (2.8). We recall from Proposition 2.1 that $(w, q)[\beta]$ is the unique solution to (2.4) in X_f. In view of Lemma A we have

$$(\mu_- w[\beta]^+ - \mu_+[\beta]^-)|_\Gamma \circ \Xi = (\mu_+ + \mu_-)\Big(\frac{1}{2} + a_\mu \mathbb{D}(f)\Big)[\beta].$$

Therefore (w_d, q_d) solves (5.4) if and only if

$$\Big(\frac{1}{2} + a_\mu \mathbb{D}(f)\Big)[\beta] = a_\mu G(f), \tag{5.8}$$

where

$$a_\mu := \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} \in (-1, 1).$$

Theorem 4.5 implies that (5.8) has a unique solution $\beta =: \beta(f) \in H^2(\mathbb{R})^2$. This establishes not only the existence but also the uniqueness of the solution to (5.4).

Summarizing, we have shown the following result:

Proposition 5.1. *Given $f \in H^3(\mathbb{R})$, the boundary value problem (5.1) has a unique solution $(v,p) \in X_f$ such that $v^\pm|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$. Moreover,*

$$v^\pm|_\Gamma \circ \Xi = \frac{G(f)}{\mu_\pm} + \frac{1}{\mu_\pm}\left(-\mathbb{D}(f) \pm \frac{1}{2}\right)[\beta(f)],$$

where $G(f) \in H^2(\mathbb{R})^2$ is defined in (5.5)-(5.6) and $\beta(f) \in H^2(\mathbb{R})^2$ is the unique solution to (5.8).

From this result and (1.1) we infer, under the assumption that $\Gamma(t)$ is at each time instant $t \geq 0$ the graph of a function $f(t) \in H^3(\mathbb{R})$ and that the pair $(v(t), p(t))$ belongs to $X_{f(t)}$ and satisfies $v(t)^\pm|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$, that (1.1a) can be recast as

$$\begin{aligned} \partial_t f &= \frac{1}{\mu_+}\left\langle G(f) - \mathbb{D}(f)[\beta(f)] + \frac{1}{2}\beta(f) \,\middle|\, (-f', 1)^\top \right\rangle \\ &= \frac{1}{\mu_+ - \mu_-}\langle \beta(f) \,|\, (-f', 1)^\top \rangle. \end{aligned} \tag{5.9}$$

Here $\langle \cdot \,|\, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^2$.

Using the results in Sect. 4 and [20] we can formulate the latter equation as an evolution equation in $H^{s-1}(\mathbb{R})^2$, where $s \in (3/2, 2)$ is fixed in the remaining. To this end we first infer from [20, Corollary C.5] that, given n, $m \in \mathbb{N}$, we have

$$[f \mapsto B^0_{n,m}(f)] \in \mathrm{C}^\infty(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R}))). \tag{5.10}$$

Further, [20, Lemma 3.5] ensures for the mappings defined in (5.7) that

$$[f \mapsto \phi_i(f)] \in \mathrm{C}^\infty(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})), \qquad i = 1,\, 2. \tag{5.11}$$

Additionally, for any $f_0 \in H^s(\mathbb{R})$, the Fréchet derivative $\partial\phi_i(f_0)$ is given by

$$\partial\phi_i(f_0) = a_i(f_0)\frac{d}{dx}, \qquad i = 1,\, 2,$$

with

$$a_1(f_0) := \frac{f_0'(2 + f_0'^2 + 2\sqrt{1 + f_0'^2})}{\sqrt{1 + f_0'^2}(\sqrt{1 + f_0'^2} + 1 + f_0'^2)^2} \quad \text{and} \quad a_2(f_0) := \frac{1}{(1 + f_0'^2)^{3/2}}. \tag{5.12}$$

It is easy to check, by arguing as in [20, Lemma C.1], that ϕ_i, $i = 1, 2$, maps bounded sets in $H^s(\mathbb{R})$ to bounded sets in $H^{s-1}(\mathbb{R})$. This observation, the relations (5.6), (5.10), (5.11), and Lemma 4.1 combined enable us to conclude that the map defined in (5.5)–(5.6) satisfies

$$[f \mapsto G(f)] \in \mathrm{C}^\infty(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})^2), \tag{5.13}$$

and also that G maps bounded sets in $H^s(\mathbb{R})$ to bounded sets in $H^{s-1}(\mathbb{R})^2$.

Moreover, recalling (3.5), we infer from (5.10) that

$$\mathbb{D} \in \mathrm{C}^\infty(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R})^2)). \tag{5.14}$$

In view of (5.13) and of Theorem 4.2 we can solve, for given $f \in H^s(\mathbb{R})$, the equation (5.8) in $H^{s-1}(\mathbb{R})^2$. Its unique solution is given by

$$\beta(f) := 2a_\mu(1 + 2a_\mu \mathbb{D}(f))^{-1}[G(f)] \in H^{s-1}(\mathbb{R})^2, \tag{5.15}$$

and, since the mapping which associates to an isomorphism its inverse is smooth, we obtain from Theorem 4.2, (5.13), and (5.14) that

$$\big[f \mapsto \beta(f)]\big] \in \mathrm{C}^\infty(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})^2). \tag{5.16}$$

Furthermore, (5.15) and the estimate (4.5) imply that β inherits from G the property to map bounded sets in $H^s(\mathbb{R})$ to bounded sets in $H^{s-1}(\mathbb{R})^2$. Summarizing, in a compact form, the Stokes flow problem (1.1) can be recast as the evolution problem

$$\frac{df}{dt}(t) = \Phi(f(t)), \quad t \geq 0, \qquad f(0) = f_0, \tag{5.17}$$

where $\Phi : H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ is defined, cf. (5.9), by

$$\Phi(f) := \frac{1}{\mu_+ - \mu_-} \langle \beta(f) | (-f', 1)^\top \rangle. \tag{5.18}$$

Observe that, due to (5.16),

$$\Phi \in \mathrm{C}^\infty(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})), \tag{5.19}$$

and that Φ maps bounded sets in $H^s(\mathbb{R})$ to bounded sets in $H^{s-1}(\mathbb{R})$.

6. Linearization, localization, and proof of the main result

We are going to prove that the nonlinear and nonlocal problem (5.17) is parabolic in $H^s(\mathbb{R})$ in the sense that the Fréchet derivative $\partial\Phi(f_0)$, generates an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$ for each $f_0 \in H^s(\mathbb{R})$. This property then enables us to use the abstract existence results from [17] in the proof of our main result Theorem 1.1.

Theorem 6.1. *For any $f_0 \in H^s(\mathbb{R})$, the Fréchet derivative $\partial\Phi(f_0)$, considered as an unbounded operator in $H^{s-1}(\mathbb{R})$ with dense domain $H^s(\mathbb{R})$, generates an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$.*

The proof of Theorem 6.1 requires some preparation. To start, we fix a function $f_0 \in H^s(\mathbb{R})$, $s' \in (3/2, s)$, and we set $\beta_0 := \beta(f_0) := (\beta_0^1, \beta_0^2)^\top$. We have $\beta_0 \in H^{s-1}(\mathbb{R})^2$.

Differentiating the relations (5.18) and (5.15), we get

$$\partial\Phi(f_0)[f] = \frac{1}{\mu_+ - \mu_-}\langle \partial\beta(f_0)[f] | (-f_0', 1)^\top\rangle - \frac{\beta_0^1 f'}{\mu_+ - \mu_-} \tag{6.1}$$

and

$$(1 + 2a_\mu \mathbb{D}(f_0))[\partial\beta(f_0)[f]] = 2a_\mu \partial G(f_0)[f] - 2a_\mu \partial\mathbb{D}(f_0)[f][\beta_0]. \tag{6.2}$$

For the computation of $\partial\mathbb{D}(f_0)[f][\beta_0]$ and $\partial G(f_0)[f]$ we use the relation

$$\begin{aligned}\partial B_{n,2}^0(f_0)[f][h] &= nB_{n,2}(f_0, f_0)[f, f_0, \dots f_0, h]\\ &\quad - 4B_{n+2,3}(f_0, f_0, f_0)[f, f_0, \dots, f_0, h], \quad n \in \mathbb{N},\end{aligned}$$

see [20, Lemma C.4]. Additionally we use Lemma 4.1 (iii) to rewrite this expression as

$$\begin{aligned}\partial B_{n,2}^0(f_0)[f][h] &= h\big(nB_{n-1,2}^0(f_0)[f'] - 4B_{n+1,3}^0(f_0)[f']\big) + R_{1,n}[f,h]\\ &= h\big(nB_{n-1,3}^0(f_0)[f'] + (n-4)B_{n+1,3}^0(f_0)[f']\big) + R_{1,n}[f,h],\end{aligned}$$

where $nB_{n-1,3}^0(f_0) := 0$ for $n = 0$ and

$$\|R_{1,n}[f,h]\|_{H^{s-1}} \le C\|h\|_{H^{s-1}}\|f\|_{H^{s'}},$$

with a constant C independent of $f \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$. Using these relations, we infer from (3.5) that

$$\begin{aligned}(\partial\mathbb{D}(f_0)[f][\beta_0])_i = \frac{1}{\pi}\big\{&B_{i+k-2,2}^0[f'\beta_0^k] + \beta_0^k\big((i+k-2)f_0'B_{i+k-3,3}^0\\ &+ (i+k-6)f_0'B_{i+k-1,3}^0 - (i+k-1)B_{i+k-2,3}^0\\ &- (i+k-5)B_{i+k,3}^0\big)[f']\big\} + R_{2,i}[f]\end{aligned} \tag{6.3}$$

for $i = 1, 2$, where we used the shorthand notation $B_{n,m}^0 := B_{n,m}^0(f_0)$ and

$$\|R_{2,i}[f]\|_{H^{s-1}} \le C\|f\|_{H^{s'}}, \quad f \in H^s(\mathbb{R}). \tag{6.4}$$

Taking the derivative of (5.6), the same arguments yield

$$4\pi\sigma^{-1}\partial G_i(f_0)[f] = T_{i,1}(f_0)[f] + T_{i,2}(f_0)[f] + R_{3,i}[f], \quad i = 1, 2, \tag{6.5}$$

where

$$\begin{aligned}T_{1,1}(f_0)[f] :=& (B_{0,2}^0 - B_{2,2}^0)[(a_1 + \phi_2 + f_0'a_2)f'] + B_{1,2}^0[(3(\phi_1 + f_0'a_1) - a_2)f']\\ &+ B_{3,2}^0[(\phi_1 + f_0'a_1 + a_2)f'],\\ T_{1,2}(f_0)[f] :=& \phi_1(3f_0'B_{0,3}^0 - 6B_{1,3}^0 - 6f_0'B_{2,3}^0 + 2B_{3,3}^0 - f_0'B_{4,3}^0)[f']\\ &+ \phi_2(-B_{0,3}^0 - 6f_0'B_{1,3}^0 + 6B_{2,3}^0 + 2f_0'B_{3,3}^0 - B_{4,3}^0)[f'],\\ T_{2,1}(f_0)[f] :=& -B_{0,2}^0[(\phi_1 + f_0'a_1 + a_2)f'] + (B_{1,2}^0 - B_{3,2}^0)[(a_1 + \phi_2 + f_0'a_2)f']\\ &+ B_{2,2}^0[(\phi_1 + f_0'a_1 - 3a_2)f'],\\ T_{2,2}(f_0)[f] :=& \phi_1(B_{0,3}^0 + 6f_0'B_{1,3}^0 - 6B_{2,3}^0 - 2f_0'B_{3,3}^0 + B_{4,3}^0)[f']\\ &+ \phi_2(f_0'B_{0,3}^0 - 2B_{1,3}^0 - 6f_0'B_{2,3}^0 + 6B_{3,3}^0 + f_0'B_{4,3}^0)[f'],\end{aligned} \tag{6.6}$$

cf. [20, Eq. (3.7)-(3.9)]. Here we have used the shortened notation $a_i := a_i(f_0)$ and $\phi_i := \phi_i(f_0)$ for $i = 1, 2$ and

$$\|R_{3,i}[f]\|_{H^{s-1}} \leq C\|f\|_{H^{s'}}, \quad f \in H^s(\mathbb{R}). \tag{6.7}$$

In order to prove Theorem 6.1 we consider the path

$$\Psi : [0,1] \longrightarrow \mathcal{L}(H^s(\mathbb{R}), H^{s-1}(\mathbb{R}))$$

defined by

$$\Psi(\tau)[f] := \frac{1}{\mu_+ - \mu_-}\langle \mathcal{B}(\tau)[f]|(-\tau f_0', 1)^\top\rangle - \frac{\tau\beta_0^1 f'}{\mu_+ - \mu_-} \tag{6.8}$$

for $\tau \in [0,1]$ and $f \in H^s(\mathbb{R})$, where $\mathcal{B}(\tau)[f]$ is defined by

$$(1 + 2\tau a_\mu \mathbb{D}(f_0))[\mathcal{B}(\tau)[f]] = 2a_\mu(\partial G(\tau f_0)[f] - \tau\partial\mathbb{D}(f_0)[f][\beta_0]). \tag{6.9}$$

Theorem 4.2, (6.3)–(6.7), and Lemma 4.1 (ii) combined ensure that the mapping $\mathcal{B} : [0,1] \longrightarrow \mathcal{L}\big(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})^2\big)$ is well-defined, and

$$\|\mathcal{B}(\tau)[f]\|_{H^{s-1}} \leq C\|f\|_{H^s}, \qquad \tau \in [0,1],\ f \in H^s(\mathbb{R}), \tag{6.10}$$

with C independent of f and τ. We also note that both paths $\mathcal{B}$ and Ψ are continuous and $\Psi(1) = \partial\Phi(f_0)$. Besides, since

$$\mathcal{B}(0) = 2a_\mu \partial G(0) = \Big(0, -\frac{2a_\mu\sigma}{4}H \circ \frac{d}{d\xi}\Big)^\top,$$

where $H = \pi^{-1}B_{0,0}$ is the Hilbert transform, we observe that $\Psi(0)$ is the Fourier multiplier

$$\Psi(0) = -\frac{\sigma}{2(\mu_+ + \mu_-)}H \circ \frac{d}{d\xi} = -\frac{\sigma}{2(\mu_+ + \mu_-)}\Big(-\frac{d^2}{d\xi^2}\Big)^{1/2}. \tag{6.11}$$

We next locally approximate the operator $\Psi(\tau)$, $\tau \in [0,1]$, by certain Fourier multipliers $\mathbb{A}_{j,\tau}$, cf. Theorem 6.2. For this purpose, given $\varepsilon \in (0,1)$, we choose $N = N(\varepsilon) \in \mathbb{N}$ and a so-called finite ε-localization family, that is a set

$$\{(\pi_j^\varepsilon, \xi_j^\varepsilon) \,|\, -N+1 \leq j \leq N\}$$

such that

- $\pi_j^\varepsilon \in \mathrm{C}^\infty(\mathbb{R}, [0,1]), -N+1 \leq j \leq N$, and $\sum_{j=-N+1}^{N} (\pi_j^\varepsilon)^2 = 1$;
- $\operatorname{supp} \pi_j^\varepsilon$ is an interval of length ε for all $|j| \leq N-1$;
- $\operatorname{supp} \pi_N^\varepsilon \subset \{|\xi| \geq 1/\varepsilon\}$;
- $\pi_j^\varepsilon \cdot \pi_l^\varepsilon = 0$ if $[|j-l| \geq 2, \max\{|j|,|l|\} \leq N-1]$ or $[|l| \leq N-2, j = N]$;
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \leq C\varepsilon^{-k}$ for all $k \in \mathbb{N}, -N+1 \leq j \leq N$;
- $\xi_j^\varepsilon \in \operatorname{supp} \pi_j^\varepsilon,\ |j| \leq N-1$.

The real number ξ_N^ε plays no role in the analysis below. To each ε-localization family we associate a norm on $H^r(\mathbb{R})$, $r \geq 0$, which is equivalent to the standard norm on $H^r(\mathbb{R})$. Indeed, given $r \geq 0$ and $\varepsilon \in (0,1)$, there exists a constant $c = c(\varepsilon, r) \in (0,1)$ such that

$$c\|f\|_{H^r} \leq \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon f\|_{H^r} \leq c^{-1}\|f\|_{H^r}, \qquad f \in H^r(\mathbb{R}). \tag{6.12}$$

To introduce the aforementioned Fourier multipliers $\mathbb{A}_{j,\tau}$, we first define the coefficient functions α_τ, $\beta_\tau : \mathbb{R} \longrightarrow \mathbb{R}$, $\tau \in [0,1]$, by the relations

$$\alpha_\tau := \frac{\sigma}{2(\mu_+ + \mu_-)}\big(a_2(\tau f_0) + \tau f_0' a_1(\tau f_0)\big), \qquad \beta_\tau := -\frac{\tau\beta_0^1}{\mu_+ - \mu_-}. \tag{6.13}$$

We now set

$$\begin{aligned} \mathbb{A}_{j,\tau} &:= \mathbb{A}_{j,\tau}^\varepsilon := -\alpha_\tau(\xi_j^\varepsilon)\Big(-\frac{d^2}{d\xi^2}\Big)^{1/2} + \beta_\tau(\xi_j^\varepsilon)\frac{d}{d\xi}, \quad |j| \leq N-1, \\ \mathbb{A}_{N,\tau} &:= \mathbb{A}_{N,\tau}^\varepsilon := -\frac{\sigma}{2(\mu_+ + \mu_-)}\Big(-\frac{d^2}{d\xi^2}\Big)^{1/2}. \end{aligned} \tag{6.14}$$

We obviously have

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})), \qquad -N+1 \leq j \leq N, \tau \in [0,1].$$

The following estimate of the localization error is the main step in the proof of Theorem 6.1.

Theorem 6.2. *Let $\mu > 0$ be given and fix $s' \in (3/2, s)$. Then there exist $\varepsilon \in (0,1)$ and a constant $K = K(\varepsilon)$ such that*

$$\|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^{s-1}} \leq \mu\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.15}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $f \in H^s(\mathbb{R})$.

Before proving Theorem 6.2 we first present some auxiliary lemmas which are used in the proof (which is presented below). We start with an estimate for the commutator $[B_{n,m}^0(f), \varphi]$ (we will apply this estimate in the particular case $\varphi = \pi_j^\varepsilon$, $-N+1 \leq j \leq N$).

Lemma 6.3. *Let $n, m \in \mathbb{N}$, $s \in (3/2, 2)$, $f \in H^s(\mathbb{R})$, and $\varphi \in \mathrm{C}^1(\mathbb{R})$ with uniformly continuous derivative φ' be given. Then, there exists a constant K that depends only on n, m, $\|\varphi'\|_\infty$, and $\|f\|_{H^s}$ such that*

$$\|\varphi B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - B_{n,m}(f,\ldots,f)[f,\ldots,f,\varphi h]\|_{H^1} \leq K\|h\|_2 \tag{6.16}$$

for all $h \in L_2(\mathbb{R})$.

Proof. This result is a particular case of [1, Lemma 12]. □

The results in Lemma 6.4-Lemma 6.8 below describe how to "freeze the coefficients" of the multilinear operators $B_{n,m}^0$. For these operators, this technique has been first developed in [19] in the study of the Muskat problem.

Lemma 6.4. *Let n, $m \in \mathbb{N}$, $3/2 < s' < s < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^s(\mathbb{R})$ and $\overline{\omega} \in \{1\} \cup H^{s-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there exists a constant K depending only on ε, n, m, $\|f\|_{H^s}$, and $\|\overline{\omega}\|_{H^{s-1}}$ (if $\overline{\omega} \neq 1$) such that*

$$\Big\|\pi_j^\varepsilon \overline{\omega} B^0_{n,m}(f)[h] - \frac{\overline{\omega}(\xi_j^\varepsilon)(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m} B_{0,0}[\pi_j^\varepsilon h]\Big\|_{H^{s-1}} \leq \nu \|\pi_j^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}$$

for all $|j| \leq N-1$ and $h \in H^{s-1}(\mathbb{R})$.

Proof. See [1, Lemma 13]. □

We now provide a similar result as in Lemma 6.4, the difference to the latter being that the linear argument of $B_{n,m}$ is now multiplied by a function a that also needs to be frozen at ξ_j^ε.

Lemma 6.5. *Let n, $m \in \mathbb{N}$, $3/2 < s' < s < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^s(\mathbb{R})$, $a \in H^{s-1}(\mathbb{R})$, and $\overline{\omega} \in \{1\} \cup H^{s-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there is a constant K depending on ε, n, m, $\|f\|_{H^s}$, $\|a\|_{H^{s-1}}$, and $\|\overline{\omega}\|_{H^{s-1}}$ (if $\overline{\omega} \neq 1$) such that*

$$\begin{aligned}&\Big\|\pi_j^\varepsilon \overline{\omega} B^0_{n,m}(f)[ah] - \frac{a(\xi_j^\varepsilon)\overline{\omega}(\xi_j^\varepsilon)(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m} B_{0,0}[\pi_j^\varepsilon h]\Big\|_{H^{s-1}}\\ &\quad \leq \nu \|\pi_j^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}\end{aligned}$$

for all $|j| \leq N-1$ and $h \in H^{s-1}(\mathbb{R})$.

Proof. See [20, Lemma D.5]. □

Lemma 6.6 and Lemma 6.7 are the analogues of Lemma 6.4 corresponding to the case $j = N$.

Lemma 6.6. *Let n, $m \in \mathbb{N}$, $3/2 < s' < s < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^s(\mathbb{R})$ and $\overline{\omega} \in H^{s-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there is a constant K depending only on ε, n, m, $\|f\|_{H^s}$, and $\|\overline{\omega}\|_{H^{s-1}}$ such that*

$$\|\pi_N^\varepsilon \overline{\omega} B^0_{n,m}(f)[h]\|_{H^{s-1}} \leq \nu \|\pi_N^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}$$

for all $h \in H^{s-1}(\mathbb{R})$.

Proof. See [1, Lemma 14]. □

Lemma 6.7 is the counterpart of Lemma 6.6 in the case when $\overline{\omega} = 1$.

Lemma 6.7. *Let n, $m \in \mathbb{N}$, $3/2 < s' < s < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^s(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there is a constant K depending only on ε, n, m, and $\|f\|_{H^s}$ such that*

$$\|\pi_N^\varepsilon B^0_{0,m}(f)[h] - B_{0,0}[\pi_N^\varepsilon h]\|_{H^{s-1}} \leq \nu \|\pi_N^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}$$

and

$$\|\pi_N^\varepsilon B^0_{n,m}(f)[h]\|_{H^{s-1}} \leq \nu \|\pi_N^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}, \qquad n \geq 1,$$

for all $h \in H^{s-1}(\mathbb{R})$.

Proof. See [1, Lemma 15]. □

Finally, Lemma 6.8 below is the analogue of Lemma 6.5 corresponding to the case $j = N$.

Lemma 6.8. *Let $n, m \in \mathbb{N}$, $3/2 < s' < s < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^s(\mathbb{R})$, $a \in H^{s-1}(\mathbb{R})$, and $\overline{\omega} \in \{1\} \cup H^{s-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there is a constant K depending on ε, n, m, $\|f\|_{H^s}$, $\|a\|_{H^{s-1}}$, and $\|\overline{\omega}\|_{H^{s-1}}$ (if $\overline{\omega} \neq 1$) such that*

$$\|\pi_N^\varepsilon \overline{\omega} B^0_{n,m}(f)[ah]\|_{H^{s-1}} \leq \nu \|\pi_N^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}$$

for all $h \in H^{s-1}(\mathbb{R})$.

Proof. See [20, Lemma D.6]. □

We are now in a position to prove Theorem 6.2.

Proof of Theorem 6.2. Fix $\mu > 0$ and let $\varepsilon \in (0,1)$. We next choose a finite ε-localization family $\{(\pi_j^\varepsilon, \xi_j^\varepsilon) \,|\, -N+1 \leq j \leq N\}$ and, associated to it, a second family $\{\chi_j^\varepsilon \,|\, -N+1 \leq j \leq N\}$ with the following properties:

- $\chi_j^\varepsilon \in \mathrm{C}^\infty(\mathbb{R},[0,1])$ and $\chi_j^\varepsilon = 1$ on $\operatorname{supp} \pi_j^\varepsilon$, $-N+1 \leq j \leq N$;
- $\operatorname{supp} \chi_j^\varepsilon$ is an interval of length 3ε, $|j| \leq N-1$;
- $\operatorname{supp} \chi_N^\varepsilon \subset \{|\xi| \geq 1/\varepsilon - \varepsilon\}$.

In the arguments that follow we repeatedly use the estimate

$$\|gh\|_{H^{s-1}} \leq C(\|g\|_\infty \|h\|_{H^{s-1}} + \|h\|_\infty \|g\|_{H^{s-1}}) \tag{6.17}$$

which holds for g, $h \in H^{s-1}(\mathbb{R})$ and $s \in (3/2, 2)$, with a constant C independent of g and h.

Below we denote by C constants that do not depend on ε and by K constants that may depend on ε. We need to approximate the linear operators $\big[f \mapsto \mathcal{B}_2(\tau)[f] - \tau f_0' \mathcal{B}_1(\tau)[f]\big]$ and $[f \mapsto \beta_0^1 f']$, see (6.8)-(6.9), where we set $\mathcal{B}(\tau) =: (\mathcal{B}_1(\tau), \mathcal{B}_2(\tau))^\top$. The proof is divided in several steps.
Step 1. We consider the operator $[f \mapsto \beta_0^1 f']$. Since $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, (6.17) yields

$$\begin{aligned}\|\pi_j^\varepsilon(\beta_0^1 f') - \beta_0^1(\xi_j^\varepsilon)(\pi_j^\varepsilon f)'\|_{H^{s-1}} \leq{}& C\|\chi_j^\varepsilon(\beta_0^1 - \beta_0^1(\xi_j^\varepsilon))\|_\infty \|(\pi_j^\varepsilon f)'\|_{H^{s-1}}\\ &+ K\|f\|_{H^{s'}}\end{aligned}$$

for $|j| \leq N-1$ and

$$\|\pi_N^\varepsilon(\beta_0^1 f')\|_{H^{s-1}} \leq C\|\chi_N^\varepsilon \beta_0^1\|_\infty \|(\pi_N^\varepsilon f)'\|_{H^{s-1}} + K\|f\|_{H^{s'}}.$$

From (5.16) we have $\beta_0^1 \in \mathrm{C}^{s-3/2}(\mathbb{R})$ and $\beta_0^1(\xi) \to 0$ for $|\xi| \to \infty$. Hence, if ε is sufficiently small, then

$$\begin{aligned}\|\pi_j^\varepsilon(\beta_0^1 f') - \beta_0^1(\xi_j^\varepsilon)(\pi_j^\varepsilon f)'\|_{H^{s-1}} &\leq \frac{\mu|\mu_+ - \mu_-|}{3}\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}},\\ \|\pi_N^\varepsilon(\beta_0^1 f')\|_{H^{s-1}} &\leq \frac{\mu|\mu_+ - \mu_-|}{3}\|\pi_N^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}.\end{aligned} \tag{6.18}$$

for $|j| \leq N-1$.

The approximation procedure for $\big[f \mapsto \mathcal{B}_2(\tau)[f] - \tau f_0'\mathcal{B}_1(\tau)[f]\big]$ is more involved.

Step 2. We prove there exists a constant $C_{\mathcal{B}}$ such that

$$\|\pi_j^\varepsilon \mathcal{B}(\tau)[f]\|_{H^{s-1}} \leq C_{\mathcal{B}}\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.19}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $f \in H^s(\mathbb{R})$. To start, we infer from (6.9) that

$$\begin{aligned}(1+2\tau a_\mu \mathbb{D}(f_0))[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] &= 2a_\mu \pi_j^\varepsilon \partial G(\tau f_0)[f] - 2\tau a_\mu \pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0] \\ &\quad + 2\tau a_\mu\big(\mathbb{D}(f_0)[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] - \pi_j^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]\big).\end{aligned} \tag{6.20}$$

In order to estimate the terms on the right, we use the representations and estimates (6.3)–(6.7) together with the commutator estimate from Lemma 6.3 and the H^{s-1}-estimate for the operators $B_{m,n}$ provided in Lemma 4.1 (ii). So we get

$$\|\pi_j^\varepsilon \partial G(\tau f_0)[f]\|_{H^{s-1}} + \|\pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]\|_{H^{s-1}} \leq C\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}, \tag{6.21}$$

and similarly, using (3.5) and (6.10) with s replaced by s',

$$\|\mathbb{D}(f_0)[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] - \pi_j^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]\|_{H^{s-1}} \leq K\|\mathcal{B}(\tau)[f]\|_2 \leq K\|f\|_{H^{s'}}. \tag{6.22}$$

The estimate (6.19) follows now from (6.20)–(6.22) and Theorem 4.2.

Step 3. Given $\tau \in [0,1]$ and $-N+1 \leq j \leq N$, let $\mathbb{B}_{j,\tau} \in \mathcal{L}(H^s(\mathbb{R})^2, H^{s-1}(\mathbb{R})^2)$ denote the Fourier multipliers

$$\begin{aligned}\mathbb{B}_{j,\tau} &:= \frac{a_\mu\sigma}{2\pi}\begin{pmatrix} a_1(\tau f_0)(\xi_j^\varepsilon)B_{0,0}\circ(d/d\xi) \\ -a_2(\tau f_0)(\xi_j^\varepsilon)B_{0,0}\circ(d/d\xi)\end{pmatrix}, \qquad |j| \leq N-1, \\ \mathbb{B}_{N,\tau} &:= \frac{a_\mu\sigma}{2\pi}\begin{pmatrix} 0 \\ -B_{0,0}\circ(d/d\xi)\end{pmatrix}.\end{aligned}$$

We next prove that given $\nu > 0$, we have

$$\|\pi_j^\varepsilon \mathcal{B}(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^{s-1}} \leq \nu\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.23}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, $f \in H^s(\mathbb{R})$ and all sufficiently small ε. To start, we multiply (6.9) by π_j^ε and get

$$\pi_j^\varepsilon \mathcal{B}(\tau)[f] = 2a_\mu \pi_j^\varepsilon\big[\partial G(\tau f_0)[f] - \tau\big(\mathbb{D}(f_0)[\mathcal{B}(\tau)[f]] + \partial\mathbb{D}(f_0)[f][\beta_0]\big)\big] \tag{6.24}$$

We consider the terms on the right hand side of (6.24) one by one. To deal with the first term we recall (6.5)–(6.7). Repeated use of Lemma 6.4 and Lemma 6.5 then shows that

$$\|2a_\mu \pi_j^\varepsilon \partial G(\tau f_0)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^{s-1}} \leq \frac{\nu}{3}\|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.25}$$

for $|j| \leq N-1$, while Lemma 6.6, Lemma 6.7, and Lemma 6.8 yield

$$\|2a_\mu \pi_N^\varepsilon \partial G(\tau f_0)[f] - \mathbb{B}_{N,\tau}[\pi_N^\varepsilon f]\|_{H^{s-1}} \leq \frac{\nu}{3}\|\pi_N^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.26}$$

provided that ε is sufficiently small.

We estimate the second term on the right of (6.24) and let $|j| \leq N-1$ first. Combining (3.5), Lemma 6.4, Lemma 6.5, (6.10) with s replaced by s', and (6.19) we obtain

$$\begin{aligned}
&\|\pi_j^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]\|_{H^{s-1}} \\
&\leq \Bigg\| \pi_j^\varepsilon \begin{pmatrix} B_{0,2}^0 & B_{1,2}^0 \\ B_{1,2}^0 & B_{2,2}^0 \end{pmatrix} \begin{pmatrix} f_0'\mathcal{B}_1(\tau)[f] \\ f_0'\mathcal{B}_2(\tau)[f] \end{pmatrix} \\
&\quad - \frac{f_0'(\xi_j^\varepsilon)}{(1+f_0'^2(\xi_j^\varepsilon))^2} \begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'^2(\xi_j^\varepsilon) \end{pmatrix} \begin{pmatrix} B_{0,0}[\pi_j^\varepsilon \mathcal{B}_1(\tau)[f]] \\ B_{0,0}[\pi_j^\varepsilon \mathcal{B}_2(\tau)[f]] \end{pmatrix} \Bigg\|_{H^{s-1}} \\
&\quad + \Bigg\| \pi_j^\varepsilon \begin{pmatrix} B_{1,2}^0 & B_{2,2}^0 \\ B_{2,2}^0 & B_{3,2}^0 \end{pmatrix} \begin{pmatrix} \mathcal{B}_1(\tau)[f] \\ \mathcal{B}_2(\tau)[f] \end{pmatrix} \\
&\quad - \frac{f_0'(\xi_j^\varepsilon)}{(1+f_0'^2(\xi_j^\varepsilon))^2} \begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'^2(\xi_j^\varepsilon) \end{pmatrix} \begin{pmatrix} B_{0,0}[\pi_j^\varepsilon \mathcal{B}_1(\tau)[f]] \\ B_{0,0}[\pi_j^\varepsilon \mathcal{B}_2(\tau)[f]] \end{pmatrix} \Bigg\|_{H^{s-1}} \\
&\leq \frac{\nu}{6|a_\mu|} \|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}
\end{aligned} \tag{6.27}$$

provided that ε is sufficiently small. Similarly, if $j = N$, then Lemma 6.7, Lemma 6.8, (6.10) with s replaced by s', and (6.19) imply that

$$\|\pi_N^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]\|_{H^{s-1}} \leq \frac{\nu}{6|a_\mu|} \|\pi_N^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.28}$$

provided that ε is sufficiently small.

It remains to consider the term $\pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]$ on the right of (6.24). To this end we argue similarly as in the proof of (6.27) by adding and subtracting suitable localization operators. Recalling (6.3)-(6.4), we get from Lemma 6.4 and Lemma 6.5 if $|j| \leq N-1$, respectively from Lemma 6.6 and Lemma 6.8 if $j = N$, that

$$\|\pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]\|_{H^{s-1}} \leq \frac{\nu}{6|a_\mu|} \|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}} \tag{6.29}$$

provided that ε is sufficiently small. The estimate (6.23) follows now from (6.24)–(6.29).

Step 4. We now localize the operators $\big[f \mapsto \mathcal{B}_2(\tau)[f] - \tau f_0'\mathcal{B}_1(\tau)[f]\big]$. The estimate (6.23) shows that, choosing ε sufficiently small, we have

$$\begin{aligned}
&\Big\| \pi_j^\varepsilon \mathcal{B}_2(\tau)[f] + \frac{a_\mu \sigma}{2\pi} a_2(\tau f_0)(\xi_j^\varepsilon) B_{0,0}[(\pi_j^\varepsilon f)'] \Big\|_{H^{s-1}} \\
&\quad \leq \frac{\mu|\mu_+ - \mu_-|}{3} \|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}
\end{aligned} \tag{6.30}$$

for $|j| \leq N-1$ and

$$\Big\| \pi_N^\varepsilon \big(\mathcal{B}_2(\tau)[f] + \frac{a_\mu \sigma}{2\pi} B_{0,0}[(\pi_N^\varepsilon f)'] \Big\|_{H^{s-1}} \leq \frac{\mu|\mu_+ - \mu_-|}{3} \|\pi_N^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}. \tag{6.31}$$

Moreover, for $|j|\le N-1$, we write in view of $\chi_j^\varepsilon\pi_j^\varepsilon=\pi_j^\varepsilon$

$$\begin{aligned}&\Big\|\pi_j^\varepsilon f_0'\mathcal{B}_1(\tau)[f]-\frac{a_\mu\sigma}{2\pi}f_0'(\xi_j^\varepsilon)a_1(\tau f_0)(\xi_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)']\Big\|_{H^{s-1}}\\&\quad\le\|\chi_j^\varepsilon(f_0'-f_0'(\xi_j^\varepsilon))\pi_j^\varepsilon\mathcal{B}_1(\tau)[f]\|_{H^{s-1}}\\&\qquad+C\Big\|\pi_j^\varepsilon\mathcal{B}_1(\tau)[f]-\frac{a_\mu\sigma}{2\pi}a_1(\tau f_0)(\xi_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)']\Big\|_{H^{s-1}}.\end{aligned}$$

The first term on the right hand side may be estimated by using (6.10) (with s replaced by s'), (6.17), (6.19), and the fact that $f_0'\in \mathrm{C}^{s-3/2}(\mathbb{R})$. For the second term we rely on (6.23). Hence, if ε is sufficiently small then

$$\begin{aligned}&\Big\|\pi_j^\varepsilon f_0'\mathcal{B}_1(\tau)[f]-\frac{a_\mu\sigma}{2\pi}f_0'(\xi_j^\varepsilon)a_1(\tau f_0)(\xi_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)']\Big\|_{H^{s-1}}\\&\quad\le\frac{\mu|\mu_+-\mu_-|}{3}\|\pi_j^\varepsilon f\|_{H^s}+K\|f\|_{H^{s'}}.\end{aligned}\tag{6.32}$$

For $j=N$, it follows from (6.10) (with s replaced by s'), (6.17), (6.19), and the fact that f_0' vanishes at infinity that

$$\|\pi_N^\varepsilon f_0'\mathcal{B}_1(\tau)[f]\|_{H^{s-1}}\le\frac{\mu|\mu_+-\mu_-|}{3}\|\pi_N^\varepsilon f\|_{H^s}+K\|f\|_{H^{s'}}.\tag{6.33}$$

The desired claim (6.15) follows now from (6.8), (6.18), (6.30), and (6.32) if $|j|\le N-1$, respectively from (6.8), (6.18), (6.31), and (6.33) if $j=N$. ☐

We now investigate the Fourier multipliers $\mathbb{A}_{j,\tau}$ found in Theorem 6.2. We recall the definitions (5.12), (6.13), and (6.14) and observe that as the functions f_0', β_0^1, and $a_i(\tau f_0)$ belong to $H^{s-1}(\mathbb{R})$, $i=1$, 2 and $\tau\in[0,1]$, there is a constant $\eta\in(0,1)$ such that

$$\eta\le\alpha_\tau\le\frac{1}{\eta}\quad\text{and}\quad|\beta_\tau|\le\frac{1}{\eta},\qquad\tau\in[0,1].$$

Based on this, it can be shown as in [19, Proposition 4.3], that there is a constant $\kappa_0\ge1$ such that for all $\varepsilon\in(0,1)$, $-N+1\le j\le N$, and $\tau\in[0,1]$ we have

- $\lambda-\mathbb{A}_{j,\tau}\in\mathcal{L}(H^s(\mathbb{R}),H^{s-1}(\mathbb{R}))$ is an isomorphism for all $\operatorname{Re}\lambda\ge1$, (6.34)
- $\kappa_0\|(\lambda-\mathbb{A}_{j,\tau})[f]\|_{H^{s-1}}\ge|\lambda|\cdot\|f\|_{H^{s-1}}+\|f\|_{H^s},\quad f\in H^s(\mathbb{R}),\ \operatorname{Re}\lambda\ge1.$ (6.35)

The properties (6.34)-(6.35) together with Theorem 6.2 enable us to prove Theorem 6.1.

Proof of Theorem 6.1. Let $s'\in(3/2,s)$ and let $\kappa_0\ge1$ be the constant in (6.35). Theorem 6.2 with $\mu:=1/2\kappa_0$ implies that there are $\varepsilon\in(0,1)$, a constant $K=K(\varepsilon)>0$ and bounded operators $\mathbb{A}_{j,\tau}\in\mathcal{L}(H^s(\mathbb{R}),H^{s-1}(\mathbb{R}))$, for $-N+1\le j\le N$ and $\tau\in[0,1]$, satisfying

$$2\kappa_0\|\pi_j^\varepsilon\Psi(\tau)[f]-\mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^{s-1}}\le\|\pi_j^\varepsilon f\|_{H^s}+2\kappa_0K\|f\|_{H^{s'}},\quad f\in H^s(\mathbb{R}).$$

Moreover, (6.35) yields

$$2\kappa_0\|(\lambda-\mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\|_{H^{s-1}}\ge2|\lambda|\cdot\|\pi_j^\varepsilon f\|_{H^{s-1}}+2\|\pi_j^\varepsilon f\|_{H^s}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq 1$, and $f \in H^s(\mathbb{R})$. The latter estimates combined lead us to

$$\begin{aligned}2\kappa_0\|\pi_j^\varepsilon(\lambda-\Psi(\tau))[f]\|_{H^{s-1}} &\geq 2\kappa_0\|(\lambda-\mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\|_{H^{s-1}}\\ &\quad -2\kappa_0\|\pi_j^\varepsilon\Psi(\tau)[f]-\mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^{s-1}}\\ &\geq 2|\lambda|\cdot\|\pi_j^\varepsilon f\|_{H^{s-1}}+\|\pi_j^\varepsilon f\|_{H^s}-2\kappa_0 K\|f\|_{H^{s'}}.\end{aligned}$$

Summing over j, we deduce from (6.12), Young's inequality, and the interpolation property (4.4) that there exist constants $\kappa \geq 1$ and $\omega > 1$ such that

$$\kappa\|(\lambda-\Psi(\tau))[f]\|_{H^{s-1}} \geq |\lambda|\cdot\|f\|_{H^{s-1}}+\|f\|_{H^s} \tag{6.36}$$

for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq \omega$, and $f \in H^s(\mathbb{R})$.

From (6.11) we also deduce that $\omega-\Psi(0) \in \mathcal{L}(H^s(\mathbb{R}), H^{s-1}(\mathbb{R}))$ is an isomorphism. This together with method of continuity [3, Proposition I.1.1.1] and (6.36) implies that also

$$\omega-\Psi(1)=\omega-\partial\Phi(f_0) \in \mathcal{L}(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})) \tag{6.37}$$

is an isomorphism. The estimate (6.36) (with $\tau = 1$) and (6.37) finally imply that $\partial\Phi(f_0)$ generates an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$, cf. [3, Chapter I], and the proof is complete. □

We are now in a position to prove the main result, for which we can exploit abstract theory for fully nonlinear parabolic problems from [17].

Proof of Theorem 1.1. Well-posedness: Given $\alpha \in (0,1)$, $T > 0$, and a Banach space X we set

$$\mathrm{C}^\alpha_\alpha((0,T],X) := \{f : (0,T] \longrightarrow X \mid \|f\|_{C^\alpha_\alpha} < \infty\},$$

where

$$\|f\|_{C^\alpha_\alpha} := \sup_t \|f(t)\| + \sup_{s\neq t}\frac{\|t^\alpha f(t)-s^\alpha f(s)\|}{|t-s|^\alpha}.$$

The property (5.19) together with Theorem 6.1 shows that the assumptions of [17, Theorem 8.1.1] are satisfied for the evolution problem (5.17). According to this theorem, (5.17) has, for each $f_0 \in H^s(\mathbb{R})$, a local solution $f(\cdot; f_0)$ such that

$$f \in \mathrm{C}([0,T], H^s(\mathbb{R})) \cap \mathrm{C}^1([0,T], H^{s-1}(\mathbb{R})) \cap \mathrm{C}^\alpha_\alpha((0,T], H^s(\mathbb{R})),$$

where $T = T(f_0) > 0$ and $\alpha \in (0,1)$ is fixed (but arbitrary). This solution is unique within the set

$$\bigcup_{\alpha\in(0,1)} \mathrm{C}^\alpha_\alpha((0,T], H^s(\mathbb{R})) \cap \mathrm{C}([0,T], H^s(\mathbb{R})) \cap \mathrm{C}^1([0,T], H^{s-1}(\mathbb{R})).$$

We improve the uniqueness property by showing that the solution is unique within

$$\mathrm{C}([0,T], H^s(\mathbb{R})) \cap \mathrm{C}^1([0,T], H^{s-1}(\mathbb{R})).$$

Indeed, let f now be any solution to (5.17) in that space, let $s' \in (3/2, s)$ be fixed and set $\alpha := s - s' \in (0,1)$. Using (4.4), we find a constant $C > 0$ such that

$$\|f(t_1) - f(t_2)\|_{H^{s'}} \leq C|t_1 - t_2|^\alpha, \qquad t_1, t_2 \in [0,T], \tag{6.38}$$

which shows in particular that $f \in \mathrm{C}^\alpha_\alpha((0,T], H^{s'}(\mathbb{R}))$. The uniqueness statement of[17, Theorem 8.1.1] applied in the context of the evolution problem (5.17) with $\Phi \in \mathrm{C}^\infty(H^{s'}(\mathbb{R}), H^{s'-1}(\mathbb{R}))$ establishes the uniqueness claim. This unique solution can be extended up to a maximal existence time $T_+(f_0)$, see [17, Section 8.2]. Finally, [17, Proposition 8.2.3] shows that the solution map defines a semiflow on $H^s(\mathbb{R})$ which, according to [17, Corollary 8.3.8], is smooth in the open set $\{(t, f_0) \,|\, 0 < t < T_+(f_0)\}$. This proves (i).
Parabolic smoothing: The uniqueness result established in (i) enables us to use a parameter trick applied also to other problems, cf., e.g., [4,9,19,21], in order to establish (iia) and (iib). The proof details are similar to those in [18, Theorem 1.2 (v)] or [1, Theorem 2 (ii)] and therefore we omit them.

Global existence: We prove the statement by contradiction. Assume there exists a maximal solution $f \in \mathrm{C}([0,T_+), H^s(\mathbb{R})) \cap \mathrm{C}^1([0,T_+), H^{s-1}(\mathbb{R}))$ to (5.17) with $T_+ < \infty$ and such that

$$\sup_{[0,T_+)} \|f(t)\|_{H^s} < \infty. \tag{6.39}$$

Recalling that Φ maps bounded sets in $H^s(\mathbb{R})$ to bounded sets in $H^{s-1}(\mathbb{R})$, we get

$$\sup_{t\in[0,T_+)} \Big\|\frac{df}{dt}(t)\Big\|_{H^{s-1}} = \sup_{t\in[0,T_+)} \|\Phi(f(t))\|_{H^{s-1}} < \infty. \tag{6.40}$$

Let $s' \in (3/2, s)$ be fixed. Arguing as above, see (6.38), from the bounds (6.39) and (6.40) we get that $f : [0,T_+) \longrightarrow H^{s'}(\mathbb{R})$ is uniformly continuous. Applying [17, Theorem 8.1.1] to (5.17) with $\Phi \in \mathrm{C}^\infty(H^{s'}(\mathbb{R}), H^{s'-1}(\mathbb{R}))$, we may extend the solution f to a time interval $[0, T'_+)$ with $T_+ < T'_+$ and such that

$$f \in \mathrm{C}([0,T'_+), H^{s'}(\mathbb{R})) \cap \mathrm{C}^1([0,T'_+), H^{s'-1}(\mathbb{R})).$$

Since by (iib) (with s replaced by s') we have $f \in \mathrm{C}^1((0,T'_+), H^s(\mathbb{R}))$, this contradicts the maximality property of f and the proof is complete. □

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Appendix A. The hydrodynamic double-layer potential near Γ

Given $f \in H^3(\mathbb{R})$ and $\beta \in H^2(\mathbb{R})$, we let (w, q) be given by (2.7) and (2.8). We recall the definitions (3.4) of $\mathbb{D}(f)$ and (3.2) of the operators $B^0_{n,m}$.

Lemma A.1. *We have* $w^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$, $q^\pm \in \mathrm{C}(\overline{\Omega^\pm})$, $w^\pm|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$, *and*

$$\left.\begin{aligned} w^\pm &= \Big(-\mathbb{D}(f)[\beta] \pm \frac{1}{2}\beta\Big) \circ \Xi^{-1} && \text{on } \Gamma, \\ [T_1(w,q)](\nu\circ\Xi^{-1}) &= 0 && \text{on } \Gamma. \end{aligned}\right\} \tag{A.1}$$

Proof. For $j = 0, \ldots, 3$, let $\mathcal{Z}_j \in \mathrm{C}^1(\mathbb{R}^2 \setminus \{0\})$ be given by

$$\mathcal{Z}_j(y) := \frac{y_1^{3-j} y_2^j}{|y|^4}, \qquad y \in \mathbb{R}^2 \setminus \{0\}.$$

Given $\phi \in H^1(\mathbb{R})$, we define the function $Z_j[\phi] : \mathbb{R}^2 \setminus \Gamma \longrightarrow \mathbb{R}$, $j = 0, \ldots, 3$, by

$$Z_j[\phi](x) := \int_{\mathbb{R}} \mathcal{Z}_j(r)\phi\, ds, \quad x \in \mathbb{R}^2 \setminus \Gamma,\ r := x - (s, f(s)).$$

Recalling $(2.9)_1$, we have

$$w = \frac{1}{\pi}\left(-\begin{pmatrix} Z_0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix}[f'\beta] + \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & Z_3 \end{pmatrix}[\beta]\right).$$

It is shown in [20, Lemma A.1] that $Z_j[\phi]^\pm \in \mathrm{C}(\overline{\Omega^\pm})$, with

$$\begin{pmatrix} Z_0[\phi]^\pm \\ Z_1[\phi]^\pm \\ Z_2[\phi]^\pm \\ Z_3[\phi]^\pm \end{pmatrix} \circ \Xi = \begin{pmatrix} B^0_{0,2}(f)[\phi] \\ B^0_{1,2}(f)[\phi] \\ B^0_{2,2}(f)[\phi] \\ B^0_{3,2}(f)[\phi] \end{pmatrix} \mp \frac{\pi}{2\omega^4} \begin{pmatrix} f'^3 + 3f' \\ f'^2 - 1 \\ f'^3 - f' \\ -3f'^2 - 1 \end{pmatrix} \phi. \tag{A.2}$$

Consequently, $w^\pm \in \mathrm{C}(\overline{\Omega^\pm}, \mathbb{R}^2)$, and the jump relations (A.2) imply $(\mathrm{A.1})_1$. Moreover, recalling Corollary 4.4, we get $w^\pm|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$. Further, the property $q^\pm \in \mathrm{C}(\overline{\Omega^\pm})$ follows from [18, Lemma 2.1].

Exchanging integration with respect to s and differentiation with respect to x by dominated convergence we find from (1.1b), (2.7), and (2.8) that

$$\partial_l w_j(x) = \int_{\mathbb{R}} \partial_l \mathcal{W}_j^{i,k}(r)\nu_i\beta_k\omega\, ds, \tag{A.3}$$

$$(T_1(w,q))_{jl}(x) = \int_{\mathbb{R}} (-\delta_{jl}\mathcal{Q}^{i,k} + \partial_l \mathcal{W}_j^{i,k} + \partial_j \mathcal{W}_j^{i,k})(r)\nu_i\beta_k\omega\, ds \tag{A.4}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$ and $l,\ j = 1,\ 2$.

For $E \subset \mathbb{R}^2$ open, $\mathcal{Z} \in \mathrm{C}^1(E)$, $i = 1, 2$, we let $\operatorname{rot}\mathcal{Z} := (\operatorname{rot}^1\mathcal{Z}, \operatorname{rot}^2\mathcal{Z})$, with $\operatorname{rot}\mathcal{Z} \in \mathrm{C}(E, \mathbb{R}^2)$, be defined by

$$\operatorname{rot}^i\mathcal{Z} := \begin{cases} -\partial_2\mathcal{Z} & \text{if}\, i = 1, \\ \quad\partial_1\mathcal{Z} & \text{if}\, i = 2. \end{cases}$$

With this notation, we find from integration by parts

$$\begin{aligned}\int_{\mathbb{R}} (\operatorname{rot}^i\mathcal{Z}_j)(r)\nu_i\phi\omega\, ds &= \int_{\mathbb{R}} (f'\partial_2\mathcal{Z}_j(r) + \partial_1\mathcal{Z}_j(r))\phi\, ds = -\int_{\mathbb{R}} \partial_s(\mathcal{Z}_j(r))\phi\, ds \\ &= Z_j[\phi'].\end{aligned}$$

Together with (A.3), (A.4), and the identities

$$\begin{aligned}
\partial_1\mathcal{W}_1^{i,1} &= -\partial_2\mathcal{W}_2^{i,1} = -\partial_2\mathcal{W}_1^{i,2} = \frac{1}{\pi}\operatorname{rot}^i\mathcal{Z}_1, \\
\partial_1\mathcal{W}_1^{i,2} &= -\partial_2\mathcal{W}_2^{i,2} = \partial_1\mathcal{W}_2^{i,1} = \frac{1}{\pi}\operatorname{rot}^i\mathcal{Z}_2, \\
\partial_2\mathcal{W}_1^{i,1} &= -\frac{1}{\pi}\operatorname{rot}^i\mathcal{Z}_0, \\
\partial_1\mathcal{W}_2^{i,2} &= \frac{1}{\pi}\operatorname{rot}^i\mathcal{Z}_3
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}^{i,1} &= \frac{1}{\pi}\operatorname{rot}^i(-\mathcal{Z}_1 - \mathcal{Z}_3), \\
\mathcal{Q}^{i,2} &= \frac{1}{\pi}\operatorname{rot}^i(\mathcal{Z}_0 + \mathcal{Z}_2),
\end{aligned}$$

this yields $w^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$ and $(\text{A.1})_2$.

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