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Blowup and scattering criteria above the threshold for the focusing inhomogeneous nonlinear Schrödinger equation

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Abstract. We consider the inhomogeneous nonlinear Schrödinger equation (INLS) in $\mathbb{R}^N$

$$i\partial_t u + \Delta u + |x|^{-b}|u|^{p-1}u = 0,$$

with initial data $u_0 \in H^1(\mathbb{R}^N)$ having finite variance. We extend the dichotomy between scattering and blow-up for solutions above the mass-energy threshold (and with arbitrarily large energy). We also show two other blow-up criteria, which are valid in any mass-supercritical setting, given there is local well-posedness.

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1. Introduction

We consider the initial value problem associated to the inhomogeneous nonlinear Schrödinger equation (INLS)

$$\begin{cases} i\partial_t u + \triangle u + |x|^{-b}|u|^{p-1}u = 0, \quad t > 0,\ x \in \mathbb{R}^N, \\ u(\cdot, 0) = u_0 \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

This model arises naturally as a limiting problem in nonlinear optics for the propagation of laser beams. The case $b = 0$ is the classical nonlinear Schrödinger equation (NLS), extensively studied in recent years (see Sulem-Sulem [27], Bourgain [2], Cazenave [6], Linares-Ponce [25], Fibich [15] and the references therein).

The lower Sobolev index where one can expect well-posedness for this model is given by scaling. If $u(x,t)$ is a solution to (1.1), so is $u_\lambda(x,t) =$

$\lambda^{\frac{2-b}{p-1}}u(\lambda x, \lambda^2 t)$, with initial data $u_{0,\lambda}(x)$, for all $\lambda > 0$. Computing the homogeneous Sobolev norm, we get

$$\|u_{0,\lambda}\|_{\dot{H}^s} = \lambda^{s-\frac{N}{2}+\frac{2-b}{p-1}}\|u_0\|_{\dot{H}^s}.$$

Thus, the scale-invariant Sobolev norm is $\dot{H}^{s_c}(\mathbb{R}^N)$, where

$$s_c = \frac{N}{2} - \frac{2-b}{p-1}$$

is called the *critical Sobolev index.*

In this paper, we are interested in the case $s_c > 0$, known as *mass-supercritical.* Rewriting this condition in terms of p, we obtain

$$p > 1 + \frac{2(2-b)}{N}.$$

The local well-posedness for the INLS equation was first studied by Genoud-Stuart in [20] (see also Genoud [17]) by the abstract theory of Cazenave [6], without relying on Strichartz type inequalities. They analyzed the IVP (1.1) in the sense of distributions, that is, $i\partial_t u + \Delta u + |x|^{-b}|u|^{p-1}u = 0$ in $H^{-1}(\mathbb{R}^N)$, $N \geq 1$, with $0 < b < \min\{2, N\}$, and showed it is well-posed

- locally if $1 < p < p_b^*$ ($s_c < 1$);
- globally for any initial data in $H^1(\mathbb{R}^N)$ if $p < 1 + \frac{2(2-b)}{N}$ ($s_c < 0$);
- globally for sufficiently small initial data if $1 + \frac{2(2-b)}{N} \leq p < p_b^*$ ($0 \leq s_c < 1$),

where

$$p_b^* = \begin{cases} \infty, & N \leq 2, \\ 1 + \frac{2(2-b)}{N-2}, & N \geq 3. \end{cases}$$

More recently, Guzmán [23], Cho and Lee [7] and Dinh [8] established local well-posedness of the INLS in $H^s(\mathbb{R}^N)$ based on Strichartz estimates. In particular, they proved that, for $N \geq 2$, $1 < p < p_b^*$ and $0 < b < \min\{N/2, 2\}$, the initial value problem (1.1) is locally well-posed in $H^1(\mathbb{R}^N)$.

Note that the results of Guzmán [23] and Dinh [8] do not treat the case $N = 1$, and the ranges of b are more restrictive than those in the results of Genoud-Stuart [20]. However, Guzmán and Dinh give more detail information on the solutions, showing that there exists $T(\|u_0\|_{H^1}) > 0$ such that $u \in L^q\left([-T, T]; W^{1,r}(\mathbb{R}^N)\right)$ for any L^2-admissible pair (q, r) satisfying

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r},$$

where

$$\begin{cases} 2 \leq r \leq \frac{2N}{N-2} & \text{if} \quad N \geq 3, \\ 2 \leq r < +\infty & \text{if} \quad N = 2, \\ 2 \leq r \leq +\infty & \text{if} \quad N = 1. \end{cases}$$

The solutions to (1.1) have the following conserved quantities

$$M\left[u(t)\right] = \int |u(t)|^2 dx = M[u_0],$$
$$E\left[u(t)\right] = \frac{1}{2}\int |\nabla u(t)|^2 dx - \frac{1}{p+1}\int |x|^{-b}|u(t)|^{p+1} dx = E[u_0].$$

The blow-up theory in the INLS equation is related to the concept of *ground state*, which is the unique positive radial solution of the elliptic problem

$$\Delta Q - Q + |x|^{-b}|Q|^{p-1}Q = 0.$$

The existence of the ground state is proved by Genoud-Stuart [16, 20] for dimension $N \geq 2$, and by Genoud [17] for $N = 1$. Uniqueness was proved in dimension $N \geq 3$ by Yanagida [30] (see also Genoud [16]), in dimension $N = 2$ by Genoud [18] and in dimension $N = 1$ by Toland [28]. The existence and uniqueness hold for $0 < b < \min\{2, N\}$ and $1 < p < p_b^*$.

The ground state satisfies the following Pohozaev's identities (see relations (1.9)-(1.10) in Farah [12])

$$\|\nabla Q\|_{L^2}^2 = \frac{N(p-1)+2b}{2(p+1)}\int |x|^{-b}|Q|^{p+1}\, dx,$$

and

$$E[Q] = \frac{(p-1)s_c}{2(p+1)}\int |x|^{-b}|Q|^{p+1}\, dx. \tag{1.2}$$

Genoud [19] and Farah [12] proved the following sharp Gagliardo-Nirenberg inequality which is valid for $N \geq 1$, $0 < b < \min\{2, N\}$, and $1 < p < p_b^*$

$$\int_{\mathbb{R}^N} |x|^{-b}|f(x)|^{p+1}\, dx \leq C_{p,N}\|\nabla f\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)+2b}{2}}\, \|f\|_{L^2(\mathbb{R}^N)}^{p+1-\frac{(N(p-1)+2b)}{2}}, \tag{1.3}$$

where $C_{p,N} > 0$ is the sharp constant. More precisely,

$$C_{p,N} = \left(\frac{2(p+1)}{N(p-1)+2b}\right)^{\frac{N(p-1)+2b}{4}} \frac{\left(\int |x|^{-b}|Q|^{p+1}\right)^{1-\frac{N(p-1)+2b}{4}}}{\|Q\|_{L^2(\mathbb{R}^N)}^{p+1-\frac{N(p-1)+2b}{2}}}. \tag{1.4}$$

This inequality can be seen as an extension to the case $b > 0$ of the classical Gagliardo-Nirenberg inequality.

If u is a solution to (1.1) and $u_0 \in \Sigma = \left\{f \in H^1(\mathbb{R}^N); |x|f \in L^2(\mathbb{R}^N)\right\}$, we define its variance at time t as

$$V(t) = \int |x|^2|u(x,t)|^2\, dx.$$

The variance satisfies the *virial identities* (see Farah [12, Proposition 4.1])

$$V_t(t) = 4\ \mathrm{Im}\int x\cdot\nabla u(x,t)\overline{u}(x,t)\, dx \tag{1.5}$$

and

$$V_{tt}(t) = 4(N(p-1)+2b)E[u] - 2(N(p-1)+2b-4)\|\nabla u\|_{L^2(\mathbb{R}^N)}^2. \tag{1.6}$$

Together with the variance, a scale-invariant quantity which plays an important role in the global behavior is $M[u_0]^{\frac{1-s_c}{s_c}}E[u_0]$, which we normalize (for $0 < s_c < 1$) as

$$\mathcal{ME}[u] = \mathcal{ME}[u_0] \ = \frac{M[u_0]^{\frac{1-s_c}{s_c}}E[u_0]}{M[Q]^{\frac{1-s_c}{s_c}}E[Q]}$$

and call it the *mass-energy*.

Remark 1.1. From the identity (1.6), if $u_0 \in \Sigma$, $p > 1 + \frac{2(2-b)}{N}$ and $E\left[u_0\right] < 0$, then the graph of $t \mapsto \int |x|^2|u|^2$ lies below an inverted parabola, which becomes negative in finite time. Therefore, the solution cannot exist globally and blows up in finite time. Recently, [9] extended this blow-up result to the radial case, and to the case $N = 1$ without symmetry or decaying assumptions. Since we only consider initial data in Σ, in this work we focus on data with non-negative energy, but avoid rising the energy to a fractional power in the definition of mass-energy so that this quantity makes sense in any setting.

Other useful scale-invariant quantities are the *mass-potential-energy*

$$\mathcal{MP}[u(t)] \ = \frac{M[u_0]^{\frac{1-s_c}{s_c}}\displaystyle\int |x|^{-b}|u(t)|^{p+1}}{M[Q]^{\frac{1-s_c}{s_c}}\displaystyle\int |x|^{-b}|Q|^{p+1}}$$

and the *mass-kinetical-energy*

$$\mathcal{MK}[u(t)] = \frac{M[u_0]^{\frac{1-s_c}{s_c}}\displaystyle\int |\nabla u(t)|^2}{M[Q]^{\frac{1-s_c}{s_c}}\displaystyle\int |\nabla Q|^2}.$$

1.1. Dichotomy above the mass-energy threshold

In previous works, [9,14] and [4] studied the global behavior of solutions to (1.1) below the *mass-energy threshold*, i.e, in the case $\mathcal{ME}[u_0] < 1$. They proved a dichotomy between blow-up and scattering, depending on the quantity $\mathcal{MK}[u_0]$.

We summarize the global behavior of solutions to (1.1) with $\mathcal{ME}[u_0] < 1$ in the following theorem

Theorem 1.2. ([3,4,9,14]). *Let $u(x,t)$ be a solution of* (1.1) *and $0 < s_c < 1$. Assume $\mathcal{ME}[u_0] < 1$. Then*

(i) *If $\mathcal{MP}[u_0] > 1$, and either $u_0 \in \Sigma$, or u_0 is radial, or $N = 1$, then the solution blows up in finite time, in both time directions.*
(ii) *If $\mathcal{MP}[u_0] < 1$, $N \geq 2$, $b < \min\left\{\frac{N}{2}, 2\right\}$, then u scatters in both time directions in $H^1(\mathbb{R}^N)$.*

Remark 1.3. In [14] and [9], Theorem 1.2 was proven using $\mathcal{MK}[u_0]$ instead of $\mathcal{MP}[u_0]$. We show the equivalence, in the case $\mathcal{ME}[u_0] \leq 1$, in Proposition 2.1. Therefore, by Theorem 1.2 and Theorem 1.5 below, as in the case $\mathcal{ME}[u_0] >$

1 the equivalence does not hold, the quantity that governs the dichotomy between blow-up and scattering is, in any case, $\mathcal{MP}[u_0]$.

Remark 1.4. The case $\mathcal{MP}[u_0] = 1$ cannot occur if $\mathcal{ME}[u_0] < 1$ (see [14, Lemma 4.2, item (ii)]).

We are interested here in criteria that include initial data *above* the threshold $\mathcal{ME}[u_0] = 1$. The first theorem we prove is a dichotomy

Theorem 1.5. *Let u be a solution of (1.1), where $1 + \frac{2(2-b)}{N} < p < p_b^*$. Assume $N \geq 2$, $u_0 \in \Sigma$ and*

$$\mathcal{ME}[u_0]\left(1 - \frac{(V_t(0))^2}{32E[u_0]V(0)}\right) \leq 1. \tag{1.7}$$

(i) *(Blow-up) If*

$$\mathcal{MP}[u_0] > 1 \tag{1.8}$$

and

$$V_t(0) \leq 0, \tag{1.9}$$

then $u(t)$ blows-up in finite positive, $T_+ < \infty$.

(ii) *(Boundedness and scattering) If*

$$\mathcal{MP}[u_0] < 1 \tag{1.10}$$

and

$$V_t(0) \geq 0, \tag{1.11}$$

then

$$\limsup_{t \to T_+(u)} M[u_0]^{1-s_c}\left(\int |x|^{-b}|u(t)|^{p+1}\right)^{s_c} < M[Q]^{1-s_c}\left(\int |x|^{-b}|Q|^{p+1}\right)^{s_c}. \tag{1.12}$$

In particular, $T_+ = +\infty$. Moreover, u scatters forward in time in $H^1(\mathbb{R}^N)$.

Remark 1.6. If $\mathcal{ME}[u_0] < 1$, the conclusion of Theorem 1.5 follows from Theorem 1.2. Theorem is new only in the case $\mathcal{ME}[u_0] \geq 1$.

Remark 1.7. The proof of Theorem 1.5 shows that there are two disjoint subsets (defined by (1.7), (1.8) and (1.9); and by (1.7), (1.10) and (1.11)) that are stable under the INLS flow and contain solutions with arbitrary mass and energy (see, for example, Remark 1.11 below).

Remark 1.8. We prove in Sect. 3 that any solution of (1.1) that satisfies (1.12) scatters for positive time. Replacing $\mathcal{MP}[u_0]$ by $\mathcal{MK}[u_0]$, this result is already known (see [14]). Due to the one-sided implication (2.1), our assumption is weaker. Therefore, Theorem 1.5 improves known results.

Remark 1.9. The scattering statement of Theorem 1.5 is optimal in the following sense: If $u_0 \in H^1(\mathbb{R}^N)$ has finite variance and scatters forward in time, then there exists $t_0 \geq 0$ such that (1.7), (1.10) and (1.11) are satisfied by $u(t)$ for all $t \geq t_0$. In fact, if $u(t)$ scatters forward in time, then $\displaystyle\int |x|^{-b}|u(t)|^{p+1} \to 0$. This implies $E[u_0] > 0$ and, by (1.6),

$$V_t(t) \approx 16E[u_0]t \qquad \text{and } V(t) \approx 8E[u_0]t^2$$

which implies

$$\mathcal{ME}[u_0]\left(1 - \frac{(V_t(t))^2}{32E[u_0]V(t)}\right) \to 0, \text{ as } t \to +\infty.$$

As a consequence of Theorem 1.5, we obtain

Corollary 1.10. *Let $\gamma \in \mathbb{R}\backslash\{0\}$, $v_0 \in \Sigma$ such that $\mathcal{ME}[v_0] < 1$, and u^γ be the solution of (1.1) with initial data*

$$u_0^\gamma = e^{i\gamma|x|^2}v_0.$$

(i) *If $\mathcal{MP}[v_0] > 1$, then for any $\gamma < 0$, u^γ blows up in finite positive time;*
(ii) *If $\mathcal{MP}[v_0] < 1$, then for any $\gamma > 0$, u^γ satisfies (1.12). Moreover, u^γ scatters forward in time in $H^1(\mathbb{R}^N)$.*

Remark 1.11. With the above corollary, we can predict the behavior of a class of solutions with arbitrarily large energy. If $\mathcal{ME}[v_0] < 1$, then

$$E[u_0^\gamma] = 4\gamma^2\|xv_0\|_{L^2}^2 + 4\gamma \operatorname{Im}\int x \cdot \nabla v_0 \bar{v}_0 + E[v_0]$$

and $E[u_0^\gamma] \to +\infty$ as $\gamma \to \pm\infty$.

Remark 1.12. Note that the statement of Theorem 1.5 is not symmetric in time as the statement of Theorem 1.2. Indeed, Corollary 1.13 below shows solutions with different behaviors in positive and negative times.

Corollary 1.13. *Let $\gamma \in \mathbb{R}$ and Q^γ be the solution to (1.1) with initial data*

$$Q_0^\gamma = e^{i\gamma|x|^2}Q.$$

(i) *If $\gamma > 0$, then Q^γ is globally defined on $[0, +\infty)$, scatters forward in time and blows up backwards in time.*
(ii) *If $\gamma < 0$, then Q^γ is globally defined on $(-\infty, 0]$, scatters backward in time and blows up forward in time.*

1.2. Blow-up criteria

The blow up criterion of [29], Zakharov [31] and Glassey [21] for the NLS use the second derivative of the variance $V(t)$ to show that finite variance, negative energy solutions blow up in finite time. The second derivative of the variance is also used in [26], but with an approach based on classical mechanics, resulting in a finer blow-up criterion. This and another criteria were proven in [24] for the 3D cubic NLS. The argument was extended in [10] to the focusing mass-supercritical NLS in any dimension, and examples were given to show that these new criteria are not equivalent to the previous ones. We extend these criteria for the focusing, mass-supercritical INLS equation in any dimension:

Theorem 1.14. *Suppose that $u_0 \in \Sigma$ and $N \geq 1$. The following inequality is a sufficient condition for blow-up in positive finite time for solutions to (1.1) with $0 < s_c < 1$ and $E[u_0] > 0$*

$$\frac{V_t(0)}{M[u_0]} < \sqrt{8Ns_c}\, g\left(\frac{4}{Ns_c}\frac{E[u_0]V(0)}{M[u_0]^2}\right),$$

where

$$g(x) = \begin{cases} \sqrt{\frac{1}{kx^k} + x - (1+\frac{1}{k})} & \text{if } 0 < x \leq 1 \\ -\sqrt{\frac{1}{kx^k} + x - (1+\frac{1}{k})} & \text{if } x \geq 1 \end{cases} \quad \text{with } k = \frac{(p-1)s_c}{2}. \tag{1.13}$$

Theorem 1.15. *Suppose that $u_0 \in \Sigma$ and $N \geq 1$. The following inequality is a sufficient condition for blow-up in positive finite time for solutions to (1.1) with $0 < s_c < 1$ and $E[u_0] > 0$*

$$\frac{V_t(0)}{M[u_0]} < \frac{4\sqrt{2}M[u_0]^{\frac{1}{2}-\frac{p+1}{N(p-1)+2b}}E[u_0]^{\frac{s_c}{N}}}{C} g\left(C^2 \frac{E[u_0]^{\frac{4}{N(p-1)+2b}}V(0)}{M[u_0]^{1+\frac{2(p+1)}{N(p-1)+2b}}}\right),$$

where g is defined in (1.13),

$$C = \left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{2}{N(p-1)+2b}}$$

and $C_{p,N}$ the a sharp constant in the interpolation inequality (1.3).

Remark 1.16. For real-valued initial data, Theorem 1.15 is an improvement of Theorem 1.14 if

$$\mathcal{ME}[u_0] > \left(\frac{Ns_cC^2}{4}\right)^{\frac{N(p-1)+2b}{N(p-1)+2b-4}},$$

since in this case the right-hand side of Theorem 1.15 is bigger.

Remark 1.17. In both theorems, the restriction $s_c < 1$ is only needed to ensure the local well-posedness.

This paper is structured as follows: In Sect. 2, we prove the boundedness and blow-up part of Theorem 1.5. The scattering part is proven in Sect. 3. In Sect. 4, we show two non-equivalent blow-up criteria for the INLS (Theorems 1.14 and 1.15).

2. Boundedness and blow-up

We start this section with the proof of the equivalence between using $\mathcal{MK}[u_0]$ and $\mathcal{MP}[u_0]$ in the dichotomy when $\mathcal{ME}[u_0] \leq 1$.

Proposition 2.1. *If $f \in H^1(\mathbb{R}^N)$, then*

$$\mathcal{MK}[f] < 1 \Longrightarrow \mathcal{MP}[f] < 1. \tag{2.1}$$

Furthermore, assume $\mathcal{ME}[f] \leq 1$. Then

$$\mathcal{MK}[f] < 1 \Longleftrightarrow \mathcal{MP}[f] < 1. \tag{2.2}$$

Proof. We write the sharp Gagliardo-Nirenberg inequality (1.3) as

$$(\mathcal{MP}[f])^{\frac{4}{N(p-1)+2b}} \leq \mathcal{MK}[f],$$

and (2.1) follows. Now, if $\mathcal{MP}[f] < 1$ and $\mathcal{ME}[f] \leq 1$, then

$$M[Q]^{\frac{1-s_c}{s_c}} E[Q] \geq M[f]^{\frac{1-s_c}{s_c}} E[f] > \frac{1}{2} M[f]^{\frac{1-s_c}{s_c}} \int |\nabla f|^2 \, dx - \frac{1}{p+1} M[Q]^{\frac{1-s_c}{s_c}} \int |x|^{-b} |Q|^{p+1} \, dx$$

taking the first and last member, we conclude $\mathcal{MK}[f] < 1$. $\square$

We also point out that the inequalities in (2.2) can be replaced by equalities: we can scale f so that $M[f] = M[Q]$. By similar arguments as the ones used in proving (2.1) and (2.2), $\mathcal{MP}[f] = 1$ or $\mathcal{MK}[f] = 1$ in the case $\mathcal{ME}[f] \leq 1$, implies $\mathcal{MP}[f] = \mathcal{MK}[f] = \mathcal{ME}[f] = 1$. In this case, f is equal to Q up to scaling and phase.

We now turn to the proof of Theorem 1.5. Start rewriting the Gagliardo-Nirenberg inequality (1.3) as

$$\left(\int |x|^{-b} |f|^{p+1} \, dx \right)^{\frac{4}{N(p-1)+2b}} \leq C_Q M[f]^{\kappa} \int |\nabla u|^2 \, dx, \tag{2.3}$$

where

$$\kappa = \frac{2(p+1)}{N(p-1)+2b} - 1$$

and

$$C_Q := (C_{p,N})^{\frac{4}{N(p-1)+2b}} = \frac{2(p+1)}{N(p-1)+2b} \frac{\left(\int |x|^{-b} |Q|^{p+1} \, dx \right)^{\frac{4}{N(p-1)+2b} - 1}}{M[Q]^{\kappa}}$$

$$= \left(\frac{8(p+1)}{A} \right)^{\frac{4}{N(p-1)+2b}} \frac{s_c(p-1)}{N(p-1)+2b} \cdot \frac{E[Q]}{M[Q]^{\kappa}}^{\frac{4}{N(p-1)+2b} - 1}$$

and

$$A := 2(N(p-1) + 2b - 4) = 4(p-1)s_c.$$

We use the following Cauchy-Schwarz inequality, proved by Banica [1]. We include the proof here for the sake of completeness.

Lemma 2.2. *Let $f \in H^1(\mathbb{R}^N)$ such that $|x| f \in L^2(\mathbb{R}^N)$. Then,*

$$\left(Im \int x \cdot \nabla f \, \overline{f} \, dx \right)^2 \leq \int |x|^2 |f|^2 \, dx \left[\int |\nabla f|^2 \, dx - \frac{1}{C_Q M^{\kappa}} \left(\int |x|^{-b} |f|^{p+1} \, dx \right)^{\frac{4}{N(p-1)+2b}} \right].$$

Proof. Given $f \in H^1(\mathbb{R}^N)$ and $\lambda > 0$, we have

$$\nabla\left(e^{i\lambda|x|^2} f\right) = 2i\lambda e^{i\lambda|x|^2} xf + e^{i\lambda|x|^2}\nabla f = e^{i\lambda|x|^2}(2i\lambda\, xf + \nabla f).$$

Thus,

$$\begin{aligned}\int \left|\nabla\left(e^{i\lambda|x|^2} f\right)\right|^2 dx &= \int e^{i\lambda|x|^2}(2i\lambda\, xf + \nabla f)e^{-i\lambda|x|^2}(-2i\lambda\, x\overline{f} + \nabla\overline{f})\, dx\\ &= 4\lambda^2\int |x|^2|f|^2\, dx + 4\lambda\,\mathrm{Im}\int x\cdot\nabla f\,\overline{f}\, dx\\ &\quad + \int |\nabla f|^2\, dx\end{aligned}$$

and from the Gagliardo-Niremberg inequality (2.3), for all $\lambda \in \mathbb{R}$ we get

$$\begin{aligned}C_Q M[f]^{\kappa}&\left[4\lambda^2\int |x|^2|f|^2\, dx + 4\lambda\,\mathrm{Im}\int x\cdot\nabla f\,\overline{f}\, dx + \int |\nabla f|^2\, dx\right]\\ &\qquad - \left(\int |x|^{-b}|f|^{p+1}\, dx\right)^{\frac{4}{N(p-1)+2b}} \geq 0.\end{aligned}$$

Note that the left-hand side of inequality above is a quadratic polynomial in λ . The discriminant of this polynomial is non-positive, wich yields the conclusion of the lemma. □

Proof of Theorem 1.5. We will assume

$$\mathcal{ME}[u_0] \geq 1, \tag{2.4}$$

as the case $\mathcal{ME}[u_0] < 1$ has been proven by [14]. By (1.6), we have

$$\int |\nabla u|^2\, dx = \frac{4(N(p-1)+2b)E[u_0] - V_{tt}}{A}. \tag{2.5}$$

Furthermore,

$$\begin{aligned}\int |x|^{-b}|u|^{p+1}\, dx &= (p+1)\frac{8\|\nabla u\|_2^2 - V_{tt}}{4(N(p-1)+2b)}\\ &= (p+1)\frac{16E[u_0] - V_{tt}}{4(N(p-1)+2b)}\\ &\quad + \frac{16}{4(N(p-1)+2b)}\int |x|^{-b}|u|^{p+1}\, dx.\end{aligned}$$

Solving the equality above for $\int |x|^{-b}|u|^{p+1}\, dx$, we have

$$\int |x|^{-b}|u|^{p+1}\, dx = (p+1)\frac{16E[u_0] - V_{tt}}{2A}. \tag{2.6}$$

Note that the expression (2.6) implies that $V_{tt} \leq 16E[u_0]$ for all t. In view of the equation (1.5), the derivative of variance $V(t)$, and Lemma 2.2 we get,

$$(V_t(t))^2 = 16\left(\mathrm{Im}\int x\cdot\nabla u(t)\,\overline{u}(t)\,dx\right)^2$$
$$\leq 16\int V(t)\Bigg[\int|\nabla u(t)|^2\,dx$$
$$-\frac{1}{C_Q M[u_0]^{\kappa}}\left(\int |x|^{-b}|u(t)|^{p+1}\,dx\right)^{\frac{4}{N(p-1)+2b}}\Bigg]. \qquad (2.7)$$

If $z(t)=\sqrt{V(t)}$, then

$$z_t(t)=\frac{1}{2}\frac{V_t(t)}{\sqrt{V(t)}}.$$

Dividing (2.7) by $V(t)$, using (2.5), (2.6) and (2.7), we have

$$z_t^2(t)=\frac{1}{4}\frac{(V_t(t))^2}{V(t)}$$
$$\leq 4\Bigg[\frac{4(N(p-1)+2b)E[u_0]-V_{tt}}{A}$$
$$-\frac{1}{C_Q M[u_0]^{\kappa}}\left(\frac{(p+1)(16E[u_0]-V_{tt})}{2A}\right)^{\frac{4}{N(p-1)+2b}}\Bigg],$$

that is,

$$z_t^2(t)\leq 4\varphi(V_{tt}), \qquad (2.8)$$

where

$$\varphi(\alpha)=\frac{4(N(p-1)+2b)E[u_0]-\alpha}{A}$$
$$-\frac{1}{C_Q M[u_0]^{\kappa}}\left(\frac{(p+1)(16E[u_0]-\alpha)}{2A}\right)^{\frac{4}{N(p-1)+2b}}$$

is defined for $\alpha\in(-\infty,16E[u_0]]$. We have

$$\varphi'(\alpha)=-\frac{1}{A}+\frac{4(16E[u_0]-\alpha)^{\frac{4}{N(p-1)+2b}-1}}{C_Q M[u_0]^{\kappa}(N(p-1)+2b)}\left(\frac{p+1}{2A}\right)^{\frac{4}{N(p-1)+2b}}.$$

Consider $\alpha_m\in(-\infty,16E[u_0])$ such that $\varphi'(\alpha_m)=0$, that is,

$$\frac{1}{A}=\frac{4(16E[u_0]-\alpha_m)^{\frac{4}{N(p-1)+2b}-1}}{C_Q M[u_0]^{\kappa}(N(p-1)+2b)}\left(\frac{p+1}{2A}\right)^{\frac{4}{N(p-1)+2b}}. \qquad (2.9)$$

Since $s_c>0$,

$$\frac{4}{N(p-1)+2b}-1=\frac{4-N(p-1)-2b}{N(p-1)+2b}=-\frac{2s_c}{(p-1)(N(p-1)+2b)}<0,$$

therefore φ is decreasing on $(-\infty, \alpha_m)$ and increasing on $(\alpha_m, 16E[u_0]]$. Note that (2.9) implies

$$\frac{\alpha_m}{8} = \frac{(\alpha_m - 16E)(N(p-1)+2b)}{4A} + \frac{4(N(p-1)+2b)E}{A} - \frac{\alpha_m}{A} = \varphi(\alpha_m).$$

Using (2.9) and (1.4), we have

$$\frac{E[Q]}{M[Q]^{\kappa}}^{\frac{4}{N(p-1)+2b}-1} = \frac{\left(E[u_0] - \frac{\alpha_m}{16}\right)}{M[u_0]^{\kappa}}^{\frac{4}{N(p-1)+2b}-1},$$

hence raising both sides to $\frac{2(p-1)}{N(p-1)+2b}$, we get

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0] - \frac{\alpha_m}{16}}{E[Q]} = 1. \tag{2.10}$$

As a consequence of (2.4)

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0] - \frac{\alpha_m}{16}}{E[Q]} = 1 \leq \mathcal{ME}[u_0] = \left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0]}{E[Q]},$$

i.e,

$$\alpha_m \geq 0,$$

and by (1.7) and (2.10),

$$\begin{aligned}
(z_t(0))^2 &= -\left(1 - \frac{(V_t(0))^2}{32E[u_0]V(0)}\right) \frac{8E[u_0]\mathcal{ME}[u_0]}{\mathcal{ME}[u_0]} + 8E[u_0] \\
&\geq -\frac{8E[u_0]}{\mathcal{ME}[u_0]} \left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0] - \frac{\alpha_m}{16}}{E[Q]} + 8E[u_0] \\
&= \frac{\alpha_m}{2} = 4\varphi(\alpha_m).
\end{aligned} \tag{2.11}$$

We first prove case (i) of Theorem 1.5. Suppose that $u \in \Sigma$ satisfies (1.8) and (1.9). Note that (1.9) is equivalent to

$$z_t(0) = \frac{V_t(0)}{2\sqrt{V(0)}} \leq 0. \tag{2.12}$$

In view of (1.2), the assumption (1.8) means

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{A\displaystyle\int |x|^{-b}|u_0|^{p+1}\,dx}{(p+1)E[Q]} = \left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{\displaystyle\int |x|^{-b}|u_0|^{p+1}\,dx}{\displaystyle\int |x|^{-b}|Q|^{p+1}\,dx} > 1$$

and consequently, from (2.6)

$$V_{tt}(0) = -\frac{2A}{p+1}\int |x|^{-b}|u_0|^{p+1} + 16E[u_0] < \alpha_m. \tag{2.13}$$

Note that, for all $t > 0$

$$z_{tt}(t) = \frac{d}{dt}\left[\frac{V_t(t)}{2\sqrt{V(t)}}\right] = \frac{V_{tt}(t)}{2\sqrt{V(t)}} - \frac{(V_t(t))^2}{4\sqrt{V(t)^3}}$$
$$= \frac{1}{z(t)}\left(\frac{V_{tt}(t)}{2} - z_t^2(t)\right). \tag{2.14}$$

Hence from (2.11) and (2.13), we have

$$z_{tt}(0) = \frac{1}{z(0)}\left(\frac{V_{tt}(0)}{2} - z_t^2(0)\right) < \frac{1}{z(0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right) = 0.$$

Suppose that $z_{tt}(\tilde{t}) \geq 0$ for some $\tilde{t}$ belonging to $[0, T_+(u))$. Then, as z_{tt} is continuous on $[0, T_+(u))$, by the intermediate value theorem there exists $t_0 \in (0, T_+(u))$ such that

$$\forall t \in [0, t_0),\ z_{tt}(t) < 0 \text{ and } z_{tt}(t_0) = 0.$$

Thus for (2.11) and (2.12)

$$\forall t \in (0, t_0],\ z_t(t) < z_t(0) \leq -\sqrt{4\varphi(\alpha_m)}.$$

We have, thus,

$$\forall t \in (0, t_0],\ z_t^2(t) > 4\varphi(\alpha_m).$$

Using the inequality above and (2.8),

$$\forall t \in (0, t_0],\ 4\varphi(V_{tt}(t)) \geq z_t^2(t) > 4\varphi(\alpha_m).$$

Therefore, $V_{tt}(t) \neq \alpha_m$ for $t \in (0, t_0]$. Since $V_{tt}(0) < \alpha_m$ and by the continuity of V_{tt},

$$\forall t \in [0, t_0],\ V_{tt}(t) < \alpha_m. \tag{2.15}$$

Since $V_{tt}(t) \neq \alpha_m$ and by (2.15), we get

$$z_{tt}(t_0) = \frac{1}{z(t_0)}\left(\frac{V_{tt}(t_0)}{2} - z_t^2(t_0)\right) < \frac{1}{z(t_0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right),$$

contradicting the definition of t_0. Therefore,

$$z_{tt} < 0 \text{ for all } t \in [0, T_+(u)). \tag{2.16}$$

By contradiction, suppose that $T_+(u) = +\infty$. From (2.12) and (2.16),

$$\forall t > 0,\ z_t(t) < z_t(0) \leq 0,$$

a contradiction with nonnegativity of $z(t)$.

We now prove case (ii) of Theorem 1.5. We assume, besides the conditions (1.7) and (2.4), that (1.10) and (1.11) hold. That implies, in the same way as we did in case (i),

$$z_t(0) \geq 0 \tag{2.17}$$

and

$$V_{tt}(0) > \alpha_m. \tag{2.18}$$

We affirm that there is $t_0 \geq 0$ such that

$$z_t(t_0) > 2\sqrt{\varphi(\alpha_m)}. \tag{2.19}$$

Indeed, by (2.11) and (2.17),

$$z_t(0) \geq 2\sqrt{\varphi(\alpha_m)}. \tag{2.20}$$

If $z_t(0) > 2\sqrt{\varphi(\alpha_m)}$, then choose $t_0 = 0$ and we have the result. If not,

$$z_{tt}(0) = \frac{1}{z(0)}\left(\frac{V_{tt}(0)}{2} - z_t^2(0)\right) > \frac{1}{z(0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right) = 0,$$

by (2.18) and (2.20). Hence, there is a small $t_0 > 0$ satisfying (2.19).

Let ε_0 be a positive small number and assume

$$z_t(t_0) \geq 2\sqrt{\varphi(\alpha_m)} + 2\varepsilon_0. \tag{2.21}$$

We will show that, for all $t \leq t_0$

$$z_t(t) > 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0. \tag{2.22}$$

Suppose (2.22) is false, and define

$$t_1 = \inf\{t \geq t_0;\ z_t(t) \leq 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0\}.$$

By (2.21) $t_1 > t_0$. By continuity of z_t,

$$z_t(t_1) = 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0 \tag{2.23}$$

and

$$\forall \in [t_0, t_1], \quad z_t(t) \geq 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0. \tag{2.24}$$

In view of (2.8),

$$\forall t \in [t_0, t_1],\ (2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2 \leq z_t^2(t) \leq 4\varphi(V_{tt}(t)). \tag{2.25}$$

Hence, $\varphi(V_{tt}(t)) > \varphi(\alpha_m)$ for all $t \in [t_0, t_1]$, so, $V_{tt}(t) \neq \alpha_m$ and by continuity $V_{tt}(t) > \alpha_m$ for $t \in [t_0, t_1]$. Using the Taylor expansion of φ around $\alpha = \alpha_m$, there exists $a > 0$ such that, if $|\alpha - \alpha_m| \leq 1$, then

$$\varphi(\alpha) \leq \varphi(\alpha_m) + a(\alpha - \alpha_m)^2. \tag{2.26}$$

We show that there exists a universal constant $D > 0$ such that

$$\forall\, t \in [t_0, t_1]\ V_{tt}(t) \geq \alpha_m + \frac{\sqrt{\varepsilon_0}}{D}. \tag{2.27}$$

Consider two cases:

a) If $V_{tt}(t) \geq \alpha_m + 1$, then for $D > 0$ large, we get (2.27)

b) If $\alpha_m < V_{tt}(t) \leq \alpha_m + 1$, then by (2.25) and (2.26), we obtain

$$(2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2 \leq (z_t(t))^2 \leq 4\varphi(V_{tt}(t)) \leq 4\varphi(\alpha_m) + 4a(V_{tt}(t) - \alpha_m)^2.$$

Thus,

$$4\sqrt{\varphi(\alpha_m)}\varepsilon_0 < 4\sqrt{\varphi(\alpha_m)}\varepsilon_0 + \varepsilon_0^2 \leq 4a(V_{tt} - \alpha_m)^2,$$

and choosing $D = \sqrt{a}(\varphi(\alpha_m))^{-\frac{1}{4}}$, (2.27) holds.

Furthermore, by (2.14) and (2.24)

$$\begin{aligned} z_{tt}(t_1) &= \frac{1}{z(t_1)}\left(\frac{V_{tt}(t_1)}{2} - z_t^2(t_1)\right) \\ &\geq \frac{1}{z(t_1)}\left(\frac{\alpha_m}{2} + \frac{\sqrt{\varepsilon_0}}{2D} - (2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2\right) \\ &\geq \frac{1}{z(t_1)}\left(\frac{\sqrt{\varepsilon_0}}{2D} - 4\varepsilon\sqrt{\varphi(\alpha_m)} - \varepsilon_0^2\right) > 0, \end{aligned}$$

if ε_0 is small enough. That is, z_t is increasing close to t_1, contradicting (2.23) and(2.24). This shows (2.22). Note that we have also shown that the inequality (2.27) holds for all $t \in [t_0, T_+(u))$. Hence, by (2.6), (1.2) and (2.10)

$$\begin{aligned} M[u_0]^{1-s_c}\left(\int |x|^{-b}|u(t)|^{p+1}\,dx\right)^{s_c} &= M[u_0]^{1-s_c}\left[\frac{p+1}{2A}(16E[u_0] - V_{tt}(t))\right]^{s_c} \\ &< M[u_0]^{1-s_c}\left[\frac{p+1}{2A}(16E[u_0] - \alpha_m)\right]^{s_c} \\ &= M[u_0]^{1-s_c}\left[\frac{8(p+1)}{A}E[Q]\right]^{s_c} \\ &= M[Q]^{1-s_c}\left[\int |x|^{-b}|Q|^{p+1}\,dx\right]^{s_c}. \end{aligned}$$

□

2.1. Dichotomy for quadratic phase initial data

We now prove Corollary 1.10, except for the scattering statement, which will follow from the results in Sect. 3.

Proof of Corollary 1.10. Let v_0 satisfy $\mathcal{ME}[v_0] < 1$, $\gamma \in \mathbb{R}\backslash\{0\}$ and u be the solution with initial data $u_0 = e^{i\gamma|x|^2}v_0$. We assume

$$\mathcal{ME}[u_0] \geq 1$$

(otherwise the result follows from Theorem 1.2).

We will now show that u_0 satisfies the assumption of Theorem 1.5. We need to calculate

$$E[u_0] = E[v_0] + 2\gamma \ \mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx + 2\gamma^2\int |x|^2|v_0|^2\,dx \tag{2.28}$$

and

$$\mathrm{Im}\int \bar{u}_0\,x\cdot\nabla u_0\,dx = \ \mathrm{Im}\int \bar{v}_0\,x\cdot\nabla v_0\,dx + 2\gamma\int |x|^2|v_0|^2\,dx.$$

Rewriting the above equations,

$$\begin{aligned} E[u_0] - \frac{\left(\mathrm{Im}\int \bar{u}_0\,x\cdot\nabla u_0\,dx\right)^2}{2\int |x|^2|u_0|^2\,dx} &= E[v_0] - \frac{\left(\mathrm{Im}\int \bar{v}_0\,x\cdot\nabla v_0\,dx\right)^2}{2\int |x|^2|v_0|^2\,dx} \\ &\leq E[v_0], \end{aligned} \tag{2.29}$$

or,

$$\mathcal{ME}[u_0]\left[1-\frac{\left(\mathrm{Im}\int \bar{u}_0 x\cdot\nabla u_0\right)^2}{2E[u_0]\int |x|^2|u_0|^2}\right]=\mathcal{ME}[v_0]\le 1. \tag{2.30}$$

Therefore, the assumption (1.7) follows from (1.5) and (2.30).

We will assume here $\gamma > 0$ and $\mathcal{MP}[v_0] < 1$, as the proof of the other case is very similar. First note that, since $\mathcal{ME}[v_0] < 1$ and $\int |x|^2|v_0|^2 > 0$, there is only one positive solution of

$$M[v_0]^{\frac{1-s_c}{s_c}}\left(E[v_0]+2\gamma\ \mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx + 2\gamma^2\int |x|^2|v_0|^2\,dx\right)=M[Q]^{\frac{1-s_c}{s_c}}E[Q]. \tag{2.31}$$

Now, since $\mathcal{ME}[u_0]\ge 1$ and $\gamma>0$, (2.28), we have $\gamma\ge\gamma_c^+$, where γ_c^+ is the positive solution of (2.31). Rewriting (2.31), we have

$$\frac{\gamma_c^+\ \mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx+(\gamma_c^+)^2\int |x|^2|v_0|^2\,dx}{M[Q]^{\frac{1-s_c}{s_c}}E[Q]}=\frac{1-\mathcal{ME}[v_0]}{2M[v_0]^{\frac{1-s_c}{s_c}}}>0,$$

which implies

$$\mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx+\gamma_c^+\int |x|^2|v_0|^2\,dx>0.$$

Using that $\gamma\ge\gamma_c^+$, we see that

$$\mathrm{Im}\int x\cdot\nabla u_0\bar{u}_0\,dx=\ \mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx+\gamma\int |x|^2|v_0|^2\,dx>0,$$

which yields (1.11). Since Theorem 1.5 applies, we conclude the proof. □

We next prove prove Corollary 1.13, except for the scattering statement.

Proof of Corollary 1.13. Given that $\bar{u}(x,-t)$ is a solution of (1.1) if $u(x,t)$ is a solution, we can assume $\gamma>0$. We only need to prove that

$$\mathrm{Im}\int x\cdot\nabla Q^\gamma(t_0)\overline{Q^\gamma}(t_0)\,dx\ge 0,$$
$$\mathcal{MP}[Q^\gamma(t_0)]<1$$

and

$$ME[Q^\gamma(t_0)]\left(1-\frac{(V_t(t_0))^2}{32E[Q^\gamma(t_0)]V(t_0)}\right)\le 1,$$

for some $t_0>0$, where $V(t)=\int |x|^2|Q^\gamma(x,t)|^2\,dx$. First note that, for $Q_0^\gamma=e^{i\gamma|x|^2}Q$, we have

$$\nabla Q_0^\gamma=(2i\gamma xQ+\nabla Q)e^{i\gamma|x|^2}$$

and

$$\triangle Q_0^\gamma = e^{i\gamma|x|^2}(2iN\gamma Q + 4i\gamma x\cdot\nabla Q - 4\gamma^2|x|^2Q + \triangle Q). \tag{2.32}$$

Thus,

$$\begin{aligned}\operatorname{Im}\int x\cdot\nabla Q_0^\gamma Q_0^\gamma\,dx &= \operatorname{Im}\int x\cdot(2i\gamma xQ+\nabla Q)e^{i\gamma|x|^2}e^{-i\gamma|x|^2}Q\,dx\\ &= \operatorname{Im}\int x\cdot(2i\gamma xQ+\nabla Q)Q\,dx\\ &= 2\gamma\int|x|^2Q^2\,dx>0. \end{aligned}\tag{2.33}$$

which shows $\operatorname{Im}\displaystyle\int x\cdot\nabla Q^\gamma(t_0)\overline{Q^\gamma}(t_0)\,dx>0$ for sufficiently small t_0. Moreover, using the fact that Q^γ is a solution to (1.1), we have

$$\begin{aligned}\frac{d}{dt}\int|x|^{-b}|Q^\gamma|^{p+1}\,dx &= (p+1)\operatorname{Re}\int|x|^{-b}(\partial_tQ^\gamma\overline{Q^\gamma})|Q^\gamma|^{p-1}\,dx\\ &= (p+1)\operatorname{Re}\int|x|^{-b}(i\triangle Q^\gamma\overline{Q^\gamma})|Q^\gamma|^{p-1}\,dx\\ &= -(p+1)\operatorname{Im}\int|x|^{-b}|Q^\gamma|^{p-1}\triangle Q^\gamma\overline{Q^\gamma}\,dx.\end{aligned}$$

Consequently, from (2.32),

$$\begin{aligned}\left[\frac{d}{dt}\int|x|^{-b}|Q^\gamma|^{p+1}\,dx\right]\Bigg|_{t=0} &= \left[-(p+1)\operatorname{Im}\int|x|^{-b}|Q^\gamma|^{p-1}\triangle Q^\gamma\overline{Q^\gamma}\,dx\right]\Bigg|_{t=0}\\ &= -2N\gamma(p-1)\int|x|^{-b}Q^{p+1}\,dx<0.\end{aligned}$$

Since

$$M[Q_0^\gamma]^{\frac{1-s_c}{s_c}}\int|x|^{-b}|Q_0^\gamma|^{p+1}\,dx = M[Q]^{\frac{1-s_c}{s_c}}\int|x|^{-b}|Q|^{p+1}\,dx,$$

we get, for sufficiently small t_0

$$\mathcal{MP}[Q^\gamma(t_0)]<1.$$

Now, define the function F as

$$\begin{aligned}F(t) &= M[Q^\gamma]^{\frac{1-s_c}{s_c}}\left[E[Q^\gamma]-\frac{\left(\operatorname{Im}\displaystyle\int x\cdot\nabla Q^\gamma(t)\overline{Q^\gamma}(t)\,dx\right)^2}{2\displaystyle\int|x|^2|Q^\gamma(t)|^2\,dx}\right]\\ &\quad - M[Q]^{\frac{1-s_c}{s_c}}E[Q].\end{aligned}\tag{2.34}$$

In view of (2.29), with $v_0=Q$, we conclude $F(0)=0$. We just need to check that $F(t)\le 0$ for small positive t. Let

$$V(t)=\int|x|^2|Q^\gamma(x,t)|^2\,dx,\qquad z(t)=\sqrt{V(t)}.$$

We can rewrite (2.34) as

$$F(t) = M[Q^\gamma]^{\frac{1-s_c}{s_c}} \left(E[Q^\gamma] - \frac{1}{8}(z_t(t)^2) \right) - M[Q]^{\frac{1-s_c}{s_c}} E[Q],$$

and thus,

$$F_t(t) = -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} z_t(t) z_{tt}(t).$$

Using (1.5), (1.6) and the fact that Gagliardo-Nirenberg inequality (1.3) is an equality for $f = Q = e^{-i\gamma|x|^2} Q_0^\gamma$, we conclude that $z_{tt}(0) = 0$. Therefore,

$$\begin{aligned} F_{tt}(0) &= -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} \left(z_t(0) z_{ttt}(0) + (z_{tt}(0))^2 \right) \\ &= -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} z_t(0) z_{ttt}(0). \end{aligned}$$

On the other hand,

$$V_{tt} = 2(z_t)^2 + 2z z_{tt} \quad \text{and} \quad V_{ttt} = 6 z_t z_{tt} + 2z z_{ttt}.$$

Thus, $V_{ttt}(0) = 2z(0) z_{ttt}(0)$. Hence, $F_{tt}(0)$ and $-V_{ttt}(0)$ have the same sign, but from (2.33) $z_t(0) > 0$. By (2.6), we get that this sign is the same as the one of

$$\left[\frac{d}{dt} \int |x|^{-b} |Q^\gamma|^{p+1} \, dx \right] \Bigg|_{t=0} = -\frac{(p+1)}{2A} V_{ttt}(0).$$

Therefore, $F_{tt}(0) < 0$, which shows that $F(t)$ is negative for small $t > 0$. This completes the proof. $\square$

3. Scattering

We now prove the scattering part of theorem 1.5. We start with a lemma:

Lemma 3.1. *Let* $0 < a < A < \left(\int |x|^{-b} |Q|^{p+1} \right)^{s_c} M[Q]^{1-s_c}$. *Then, there exists* $\epsilon_0 = \epsilon_0(a, A)$ *such that for all* $f \in H^1(\mathbb{R}^N)$ *with*

$$a \leq \left(\int |x|^{-b} |f|^{p+1} \, dx \right)^{s_c} M[f]^{1-s_c} \leq A,$$

one has

$$\int |\nabla f|^2 \, dx - \frac{N(p-1)+2b}{2(p+1)} \int |x|^{-b} |f|^{p+1} \, dx \geq \epsilon_0 M[f]^{1-\frac{1}{s_c}} \tag{3.1}$$

and

$$E[f] \geq \frac{\epsilon_0}{2} M[f]^{1-\frac{1}{s_c}}. \tag{3.2}$$

Proof. Recalling the sharp Gagliardo-Nirenberg inequality, we have:

$$\begin{aligned} M[f]^{\frac{1}{s_c}-1}&\left[\int |\nabla f|^2\,dx - \frac{N(p-1)+2b}{2(p+1)}\int |x|^{-b}|f|^{p+1}\,dx\right]\\ &\geq \frac{1}{c_Q}M[f]^{\frac{1}{s_c}-1-\kappa}\left(\int |x|^{-b}|f|^{p+1}\,dx\right)^{\frac{4}{N(p-1)+2b}}\\ &\qquad - M[f]^{\frac{1}{s_c}-1}\frac{N(p-1)+2b}{2(p+1)}\int |x|^{-b}|f|^{p+1}\,dx\\ &= \frac{y^{\frac{4}{N(p-1)+2b}}}{c_Q} - \frac{N(p-1)+2b}{2(p+1)}y. \end{aligned} \tag{3.3}$$

where $y = M[f]^{\frac{1}{s_c}-1}\int |x|^{-b}|f|^{p+1}\,dx$. One can check, by direct calculations, that the function $y \mapsto \frac{y^{\frac{4}{N(p-1)+2b}}}{c_Q} - \frac{N(p-1)+2b}{2(p+1)}y$ has only one zero y^* on $(0,+\infty)$ and is positive in $(0,y^*)$. Since the inequality (3.3) is an equality when $f=Q$, y^* is exactly $M[Q]^{\frac{1}{s_c}-1}\int |x|^{-b}|Q|^{p+1}\,dx$, and (3.1) follows. Noting that

$$E[f] \geq \frac{1}{2}\left(\int |\nabla f|^2\,dx - \frac{N(p-1)+2b}{2(p+1)}\int |x|^{-b}|f|^{p+1}\,dx\right),$$

we get (3.2), because $\frac{N(p-1)+2b}{4} \geq 1$. □

Definition 3.2. If $N \geq 1$ and $s \in (0,1)$, the pair (q,r) is called *$\dot{H}^s$-admissible* if it satisfies the condition

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s,$$

where

$$2 \leq q, r \leq \infty, \text{ and } (q,r,N) \neq (2,\infty,2).$$

Also, considering the following closed subset of H^s-admissible pairs

$$\mathcal{A}_s = \left\{ (q,r) \text{ is } \dot{H}^s\text{-admissible} \;\middle|\; \begin{cases} \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^-, & N\geq 3\\ \left(\frac{2}{1-s}\right)^+ \leq r \leq \left(\left(\frac{2}{1-s}\right)^+\right)', & N=2\\ \frac{2}{1-2s} \leq r \leq \infty, & N=1 \end{cases} \right\}$$

where $a^+ = a+\epsilon$, for a fixed, small $\epsilon>0$ and $(a^+)'$ is defined as the number such that

$$\frac{1}{a} = \frac{1}{a^+} + \frac{1}{(a^+)'},$$

we define the scattering norm

$$\|u\|_{S(\dot{H}^{s_c})} = \sup_{(q,r)\in\mathcal{A}_{s_c}} \|u\|_{L_t^q L_x^r}.$$

It is already known that scattering follows from the uniform boundedness of the H^1 norm and the finiteness of the $S(\dot{H}^{s_c})$ norm (see [14, Proposition

1.4]). The following proposition was proved in [5] and covers for the broken translation symmetry and lack of momentum conservation.

Proposition 3.3. *Suppose* $N \geq 2$, $0 < b < \min\{\frac{N}{2}, 2\}$, *and* $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. *Let* $\psi \in H^1(\mathbb{R}^N)$. *Suppose that* $t_n \equiv 0$ *or* $t_n \to \pm\infty$ *and that* $|x_n| \to \infty$. *Then for all* n *sufficiently large, there exists a global solution* v_n *to* (1.1) *with*

$$v_n(0) = \psi_n := e^{it_n\Delta}\psi(x - x_n)$$

that scatters in H^1 *and obeys*

$$\|v_n\|_{S(\dot{H}^{s_c})} + \|v_n\|_{S(L^2)} + \|\nabla v_n\|_{S(L^2)} \leq C$$

for some $C = C(\|\psi\|_{H^1})$. *Moreover, for any* $\varepsilon > 0$, *there exists* $K > 0$ *and* $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^N)$ *such that*

$$\|v_n - \phi(\cdot - x_n, \cdot + t_n)\|_{S(\dot{H}^{s_c})} < \varepsilon \text{ for } n \geq K.$$

We now have all tools to upgrade the global existence to scattering.

Proposition 3.4. *Define* $S(L, A)$ *as the supremum of* $\|u\|_{S(\dot{H}^{s_c})}$ *such that* u *is a solution to (1.1) on* $[0, +\infty)$ *with*

$$\mathcal{ME}[u_0] \leq L \tag{3.4}$$

and

$$\sup_{t\in[0,+\infty)} \left(\int |x|^{-b}|u(t)|^{p+1}\, dx\right)^{s_c} M[u]^{1-s_c} \leq A. \tag{3.5}$$

If $A < \left(\int |x|^{-b}Q^{p+1}\, dx\right)^{s_c} M[Q]^{1-s_c}$, *then* $S(L, A) < +\infty$.

Proof. The proof goes along the spirit of [10,14] and (see also [22]). We give an outline of the proof, highlighting the main differences.

First we note that, if $0 < L < 1$, by Theorem 1.2, then $S(L, A) < +\infty$. Assume, by contradiction, that $S(L, A) = +\infty$ for some $L \in \mathbb{R}$. Note that, if a nonzero u satisfies the equation (3.5), with $A < \left(\int |x|^{-b}Q^{p+1}\, dx\right)^{s_c} M[Q]^{1-s_c}$, then by Lemma 3.1, $E[u] > 0$. Thus, the quantity L_c given by

$$L_c = L_c(A) := \inf\{L \in \mathbb{R} \text{ s.t. } S(L, A) = +\infty\}$$

is well-defined and positive.

Moreover, there exists a sequence $\{u_n\}$ of (global) solutions such that

$$M[u_n] = 1,$$
$$\|u_n\|_{S(\dot{H}^{s_c})} \to +\infty,$$
$$\mathcal{ME}[u_n] \searrow L_c,$$

and

$$\sup_{t\in[0,+\infty)} \int |x|^{-b}|u_n|^{p+1}\, dx \leq A.$$

Therefore, using the linear profile decomposition ([11, Theorem 5.1]) for the initial conditions $u_{n,0}$ (note that $\{u_{n,0}\}$ is bounded in $H^1(\mathbb{R}^N)$), the existence of wave operators for large times (see [14,22]) and Proposition 3.3, to deal with the unbounded translation parameters, we obtain for each $M \in \mathbb{N}$ (passing, if necessary, to a subsequence) a nonlinear profile decomposition of the form:

$$u_{n,0} = \sum_{j=1}^{M} \tilde{u}_n^j(0) + \tilde{W}_n^M \tag{3.6}$$

and an approximate solution

$$\tilde{u}_n(t) = \sum_{j=1}^{M} \tilde{u}_n^j(t),$$

where, for each pair (j, n), $\tilde{u}_n^j$ is a solution to (1.1) and

1. for each (j, n), there exists $T_n^j > 0$ such that $[0, T_n^j)$ is the maximal (positive) interval of existence of $\tilde{u}_n^j$;
2. $\lim\limits_{M\to+\infty} \left[\lim\limits_{n\to+\infty} \left\| e^{it\Delta} \tilde{W}_n^M \right\|_{S\left(\dot{H}^{s_c}\right)} \right] = 0;$
3. The profiles can be ordered in such a way that
   (a) The first nonlinear profile $\tilde{u}_n^1$ (corresponding to bounded space and time translation parameters in the linear profile decomposition), may or may not be global in time, but we can assume that it is independent of n and often write it as $\tilde{u}^1$ and its maximal time of existence as $[0, T^1)$;
   (b) $\tilde{u}_n^j$, for $2 \leq j \leq M_1$ corresponding to bounded space translation, but unbounded time translation, are obtained from the wave operators, therefore scattering forward in time in H^1 to their correspondent linear profile and satisfying, $\|\tilde{u}_n^j\|_{S(\dot{H}^{s_c},[0,+\infty))} \to 0$ as $n \to \infty$;
   (c) $\tilde{u}_n^j$, for $M_1 + 1 \leq j \leq M$, corresponding to unbounded space translation, are obtained from Lemma 3.3, scattering in both time directions in H^1 and satisfying the global space-time bounds $\|\nabla \tilde{u}_n^j\|_{L_t^\infty L_x^2} \leq \|\nabla \tilde{u}_n^j\|_{S(L^2)} \leq C\|\nabla \tilde{u}_n^j(0)\|_{L^2} + o_n(1)$;
4. for fixed $M \in \mathbb{N}$ and any $0 \leq s \leq 1$, the asymptotic Pythagorean expansion holds for the $\dot{H}^s$ norm

$$\|u_{n,0}\|_{\dot{H}^s}^2 = \sum_{j=1}^{M} \left\|\tilde{u}_n^j(0)\right\|_{\dot{H}^s}^2 + \left\|\tilde{W}_n^M\right\|_{\dot{H}^s}^2 + o_n(1)$$

   and for the energy

$$E[u_{n,0}] = \sum E[\tilde{u}_n^j] + E[\tilde{W}_n^M] + o_n(1).$$

These items follow from the usual approach, as in [5,13] and [14]. Items 1-4 follow from the construction of the nonlinear profiles together with the so-called asymptotic orthogonality of the space and time translation parameters from the linear profile decomposition. The major difference is that we do not have information for $t > T^1$ because it is not clear whether all the nonlinear profiles evolve into global solutions, since the quantity $\mathcal{ME}[\tilde{u}_n^1(0)]$ may not be small. $\square$

Thus, in order to prove that all $\tilde{u}_n^j(t)$ exist on $[0,+\infty)$, we need to track $\|\nabla\tilde{u}_n^j(t)\|_{L^2}$. Denoting by INLS$(t)v_0$ the evolution of the datum v_0 under the flow of (1.1), we prove the following:

Lemma 3.5. *(Pythagorean expansion along the bounded INLS flow). Suppose $u_{n,0}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Let $T\in(0,+\infty)$ be a fixed time. Assume that $u_n(t) = INLS(t)u_{n,0}$ exists up to time T for all n and $\lim_n \|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x} < +\infty$. Consider the nonlinear profile decomposition (3.6) and write $\tilde{W}_n^M(t) = INLS(t)\tilde{W}_n^M$. Then given any $T>0$, for all j, the nonlinear profiles $\tilde{u}_n^j(t)$ exist up to time T and for all $t\in[0,T]$,*

$$\|\nabla u_n(t)\|_{L^2_x}^2 = \sum_{j=1}^{M}\|\nabla\tilde{u}_n^j(t)\|_{L^2_x}^2 + \|\nabla\tilde{W}_n^M(t)\|_{L^2_x}^2 + o_{M,n}(1),$$

where $o_{M,n}(1)\to 0$ uniformly on $0\le t\le T$.

Proof. For fixed $T>0$, define $B=\max\{1,\lim_n\|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x}\}$ and let $\tilde{T}^1$ be the maximal time of existence of $\tilde{u}_n^1$ such that $\tilde{T}^1\le T$ and $\|\nabla\tilde{u}_n^1\|_{L^\infty_{[0,\tilde{T}^1]}L^2_x}\le 2B$. This is the only possibly "ill-behaved" profile, and we aim to show the converse inequality $\tilde{T}^1\ge T$. From the items 1-4 above, we estimate

$$\begin{aligned}\|\nabla\tilde{u}_n\|_{L^\infty_{[0,\tilde{T}^1]}L^2_x} &\le \|\nabla\tilde{u}_n^1\|_{L^\infty_{[0,\tilde{T}^1]}L^2_x} + \sum_{j=2}^{M_1}\|\nabla\tilde{u}_n^j\|_{L^\infty_{[0,\tilde{T}^1]}L^2_x}\\ &\quad + \sum_{j=M_1+1}^{M}\|\nabla\tilde{u}_n^j\|_{L^\infty_{[0,\tilde{T}^1]}L^2_x}\\ &\le 2B + 2\sum_{j=2}^{M_1}\|\nabla\tilde{u}_n^j(0)\|_{L^2_x} + C\sum_{M_1+1}^{M}\|\nabla\tilde{u}_n^j(0)\|_{L^2_x} + o_n(1)\\ &\le 2B + C\|\nabla u_{n,0}\|_{L^2} + o_n(1),\end{aligned}$$

by interpolation and Sobolev embedding,

$$\begin{aligned}\|\tilde{u}_n\|_{S(\dot{H}^{s_c},[0,\tilde{T}^1])} &\le c\|\tilde{u}_n\|_{L^\infty_{[0,\tilde{T}^1]}L^{\frac{2N}{N-2s_c}}} + c\|\tilde{u}_n\|_{L^{\frac{2}{1-s_c}}[0,\tilde{T}^1]L^{\frac{2N}{N-2}}}\\ &\le c(1+(\tilde{T}^1)^{\frac{1-s_c}{2}})\|\tilde{u}_n\|_{L^\infty_{[0,\tilde{T}^1]}H^1_x},\end{aligned}$$

and, by construction,

$$\|e^{it\Delta}[u_n(0)-\tilde{u}_n(0)]\|_{S(\dot{H}^{s_c})} = \|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{s_c})} = o_{M,n}(1).$$

Defining the error of the approximation

$$e_n^M = (i\partial_t+\Delta)\tilde{u}_n + |x|^{-b}f(\tilde{u}_n) = |x|^{-b}\left[f\left(\sum_{j=1}^{M}\tilde{u}_n^j\right) - \sum_{j=1}^{M}f(\tilde{u}_n^j)\right],$$

where $f(z) = |z|^\alpha z$, we have

$$\|e_n^M\|_{S'(\dot H^{-s_c},[0,T^1])} + \|e_n^M\|_{S'(L^2,[0,T^1])} + \|\nabla e_n^M\|_{S'(L^2,[0,T^1])} = o_{M,n}(1).$$

These estimates are obtained from the pointwise linear estimates of the difference in the right-hand side, also making use of the asymptotic orthogonality and the individual space-times bounds of each $\tilde u_n^j$ on $[0, T^1)$.

Note all the profiles are defined at least for $t \in [0, T^1)$, since the only profile with possibly finite time of existence is $\tilde u_n^1$. By using long-time perturbation and interpolation,

$$\sup_{t\in[0,\tilde T^1]} \int |x|^{-b}|u_n(t) - \tilde u_n(t)|^{p+1}\,dx \le c\|u_n - \tilde u_n\|^{p-1}_{L^\infty_{[0,\tilde T^1]} L^{\frac{2N}{N-2s_c}}} \|\nabla u_n\|^2_{L^\infty_{[0,\tilde T^1]} L^2}$$
$$= o_{M,n}.$$

(Recall that $T > 0$ is fixed and $\tilde T^1 \le T$). Thus, by the asymptotic orthogonality, we get

$$\int |x|^{-b}|u_n(t)|^{p+1}\,dx = \sum_{j=1}^M \int |x|^{-b}|\tilde u_n^j(t)|^{p+1}\,dx + o_{M,n}(1).$$

Now, energy conservation and the Pythagorean expansion for the energy at $t = 0$ gives

$$0 = E[u_{n,0}] - E[u_n(t)] = \sum_{j=1}^M E[\tilde u_n^j(t)] + E[\tilde W_n^M(t)] - E[u_n(t)] + o_n(1),$$

which in turn proves

$$\|\nabla u_n\|^2_{L^\infty_{[0,\tilde T^1]} L^2_x} = \sum_{j=1}^M \|\nabla \tilde u_n^j\|^2_{L^\infty_{[0,\tilde T^1]} L^2_x} + \|\nabla \tilde W_n^M\|^2_{L^\infty_{[0,\tilde T^1]} L^2_x} + o_{M,n}(1). \quad (3.7)$$

The last bound shows that $\|\nabla \tilde u_n^1(t)\|_{L^2_x} \le B + o_{M,n}(1)$ for all $t \in [0, \tilde T^1]$ which in turn, by maximality of $\tilde T^1$, shows that $\tilde T^1 \ge T$. In particular, $u_n^1(t)$ exists up to time T, and we can replace $\tilde T^1$ by T in (3.7), finishing the proof of the lemma. □

Invoking (3.4) and (3.5) and using the orthogonality along the INLS flow, one is able to prove that $\tilde u_n^j(t)$ is defined on $[0, +\infty)$ as well, and satisfies, for every j and every large n,

$$M[\tilde u_n^j] \le 1,$$
$$\mathcal{ME}[\tilde u_n^j] \le L_c$$

and

$$\sup_{t\in[0,+\infty)} \left(\int |x|^{-b}|\tilde u_n^j(t)|^{p+1}\,dx\right)^{s_c} M[\tilde u_n^j]^{1-s_c} \le A.$$

The rest of the proof follows the same lines as [10] and [14], using the criticality of L_c to show that only the first profile, $\tilde u^1$, can be non-zero, and letting $u_c(t) = \tilde u^1(t)$. This criticality also shows that $M[u_c] = 1$ and $\mathcal{ME}[u_c] =$

L_c. Long-time perturbation theory yields $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$. At this point, the classical compactness lemma follows.

Lemma 3.6. (Compactness). *Assume that there exists $L_0 \in \mathbb{R}$ and a positive number*

$$A < \left(\int |x|^{-b} |Q|^{p+1} \, dx \right)^{s_c} M[Q]^{1-s_c}$$

such that $S(L_0, A) = +\infty$. Then there exists a global solution u_c of (1.1) such that the set

$$K = \{u_c(t), t \in [0, +\infty)\}$$

has a compact closure in $H^1(\mathbb{R}^N)$.

Using this compactness lemma and the virial identity (1.6), we also have the classic rigidity lemma.

Lemma 3.7. (Rigidity). *There exists no solution u_c of* (1.1) *satisfying the conclusion of Lemma* 3.6.

The proof goes on the same lines as in [10] and [14]. □

4. Proof of the blowup criteria

In this section we prove two criteria for blow up in finite time. The first one is a generalization of Lushnikov's criterion in [26] and of Holmer-Platte-Roudenko criteria in [24] for the INLS, and the second one is the modification of the first approach, where the generalized uncertainty principle is replaced by the interpolation inequality (4.10). The two criteria are the INLS versions of the ones proved by Duyckaerts and Roudenko in [10].

Proof of Theorem 1.14. Integrating by parts,

$$\begin{aligned}
\|u\|_{L^2}^2 &= \int |u|^2 \, dx = \frac{1}{N} \sum_{j=1}^{N} \int \partial_j x_j |u|^2 \, dx = -\frac{1}{N} \sum_{j=1}^{N} \int x_j \partial_j (|u|^2) \, dx \\
&= -\frac{1}{N} \sum_{j=1}^{N} \int x_j (\partial_j u \overline{u} + u \partial_j \overline{u}) \, dx = -\frac{2}{N} \sum_{j=1}^{N} \mathrm{Re} \int x_j \partial_j u \overline{u} \, dx \\
&= -\frac{2}{N} \mathrm{Re} \int (x \cdot \nabla u) \overline{u} \, dx.
\end{aligned}$$

Since $|z|^2 = |\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2$, using Hölder's inequality

$$\begin{aligned}
\|xu\|_{L_2}^2 \|\nabla u\|_{L^2}^2 &\geq \left| \int (x \cdot \nabla u) \overline{u} \, dx \right|^2 \\
&= \left| \mathrm{Re} \int (x \cdot \nabla u) \overline{u} \, dx \right|^2 + \left| \mathrm{Im} \int (x \cdot \nabla u) \overline{u} \, dx \right|^2 \\
&= \frac{N^2}{4} \|u\|_{L^2}^4 + \left| \mathrm{Im} \int (x \cdot \nabla u) \overline{u} \, dx \right|^2.
\end{aligned}$$

From the definition of variance and the identity for the first derivative of the variance (1.5), we get the uncertainty principle

$$\frac{N^2}{4}\|u_0\|_{L^2}^2 + \left|\frac{V_t}{4}\right|^2 \leq V(t)\|\nabla u(t)\|_{L^2}^2. \tag{4.1}$$

Using the equation (1.6) for the second derivative of the variance, we obtain

$$V_{tt}(t) = 4(N(p-1)+2b)E[u_0] - 4(p-1)s_c\|\nabla u(t)\|_{L^2}^2. \tag{4.2}$$

Substituting (4.2) in the uncertainty principle (4.1), we have

$$V_{tt}(t) \leq 4(N(p-1)+2b)E[u_0] - N^2(p-1)s_c\frac{(M[u_0])^2}{V(t)} - \frac{(p-1)s_c}{4}\frac{|V_t(t)|^2}{V(t)}. \tag{4.3}$$

Now, we rewrite equation (4.3) in order to cancel the term V_t^2. For this, define

$$V = B^{\frac{1}{\alpha+1}}, \qquad \alpha = \frac{(p-1)s_c}{4} = \frac{N(p-1)-4+2b}{8}. \tag{4.4}$$

Then,

$$V_t = \frac{1}{\alpha+1}B^{-\frac{\alpha}{\alpha+1}} \quad \text{and} \quad V_{tt} = -\frac{\alpha}{(\alpha+1)^2}B^{-\frac{2\alpha+1}{\alpha+1}}B_t^2 + \frac{1}{\alpha+1}B^{-\frac{\alpha}{\alpha+1}}B_{tt},$$

which gives

$$B_{tt} \leq 4(\alpha+1)N(p-1)E[u_0]B^{\frac{\alpha}{\alpha+1}} - (\alpha+1)N^2(p-1)s_c(M[u_0])^2B^{\frac{\alpha-1}{\alpha+1}},$$

that is, for all $t \in [0, T_+(u))$

$$B_{tt} \leq \frac{N(p-1)(N(p-1)+4+2b)}{2}\left(E[u_0]B^{\frac{N(p-1)-4+2b}{N(p-1)+4+2b}} - \frac{Ns_c}{4}(M[u_0])^2B^{\frac{N(p-1)-12+2b}{N(p-1)+4+2b}}\right).$$

In order to further simplify the inequality, let us make a rescaling. Define $B(t) = \mu\Phi(\lambda t)$, with

$$\mu = \left(\frac{Ns_c(M[u_0])^2}{4E[u_0]}\right)^{\frac{N(p-1)+4+2b}{8}}, \qquad \lambda = \frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]}. \tag{4.5}$$

Then letting $s = \lambda t$, we obtain

$$\omega\Phi_{ss} \leq \Phi^\gamma - \Phi^\delta, \quad s \in [0, T_+/a), \tag{4.6}$$

where

$$\gamma = \frac{N(p-1)-4+2b}{N(p-1)+4+2b}, \qquad \delta = \frac{N(p-1)-12+2b}{N(p-1)+4+2b} = 2\gamma - 1,$$

$$\omega = \frac{64}{N(p-1)(N(p-1)+4+2b)}$$

and since $p > 1 + \frac{4}{N}$,

$$0 < \gamma < 1, \quad -1 < \delta < \gamma.$$

We rewrite (4.6) as

$$\omega\Phi_{ss} + \frac{\partial U}{\partial \Phi} \leq 0, \tag{4.7}$$

for $t \in [0, T_+/a)$, where $U(\Phi) = \frac{\Phi^{\delta+1}}{\delta+1} - \frac{\Phi^{\gamma+1}}{\gamma+1}$. Define the energy of the particle

$$\mathcal{E}(s) = \frac{\omega}{2}\Phi_s^2(s) + U(\Phi(s))$$

which is conserved for solutions of

$$\omega\Phi_{ss} + \frac{\partial U}{\partial \Phi} = 0.$$

Based on the ideas of Lushnikov [26], Duyckaerts and Roudenko [10] studied this model and showed the following proposition □

Proposition 4.1. *Let Φ be a nonnegative solution of (4.7) such that one of the following holds:*

(**A**) $\mathcal{E}(0) < U_{max}$ *and* $\Phi(0) < 1$,
(**B**) $\mathcal{E}(0) > U_{max}$ *and* $\Phi_s(0) < 0$,
(**C**) $\mathcal{E}(0) = U_{max}, \Phi_s(0) < 0$ *and* $\Phi(0) < 1$,

where U_{max} is the absolute maximum of U on the interval $[0, +\infty)$. Then we have $T_+ < \infty$.

Proof. For the sake of completeness of this work, we will give the proof of the proposition. Multiplying equation (4.7) by Φ_s, we get

$$\Phi_s(s) > 0 \Rightarrow \mathcal{E}_s(s) < 0, \quad \Phi_s(s) < 0 \Rightarrow \mathcal{E}_s(s) > 0. \tag{4.8}$$

We argue by contradiction, assuming $T_+ = T_+(u) = +\infty$.

We first assume (A). Let us prove by contradiction that

$$\exists\, s > 0, \ \ \Phi_s(s) < 0.$$

If not, $\Phi_s(s) \geq 0$ for all s, and (4.8) implies that the energy decays. By (A), $\mathcal{E}(s) \leq \mathcal{E}(0) < U_{max}$ for all s. Thus, $|\Phi(s) - 1| \geq \varepsilon_0$ (where $\varepsilon_0 > 0$ depends on $\mathcal{E}(0)$) for all s. Since by (A) $\Phi(0) < 1$, we obtain by continuity of Φ that $\Phi(s) \leq 1 - \varepsilon_0$ for all s. By equation (4.6), we deduce $\Phi_{ss} \leq -\varepsilon_1$ for all s, where $\varepsilon_1 > 0$ depends on ε_0. Thus, Φ is strictly concave, a contradiction with the fact that Φ is positive and $T_+ = +\infty$.

We have proved that there exists $s > 0$ such that $\Phi_s(s) < 0$. Letting

$$t_1 = \inf\{s > 0; \Phi_s(s) < 0\},$$

we get by (4.8) that the energy is nonincreasing on $[0, t_1]$. Thus, $\mathcal{E}(s) < \mathcal{E}(0) \leq U_{max}$ on $[0, t_1]$, which proves that $\Phi(s) \neq 1$ on $[0, t_1]$. Since $\Phi(0) < 1$, we deduce by the intermediate value theorem that $\Phi(t_1) < 1$ and by (4.6) that $\Phi_{ss}(t_1) < 0$. Since $\Phi_s(t_1) \leq 0$, an elementary bootstrap argument, together with equation (4.6) shows that $\Phi(s) \leq 1 - \varepsilon_0$, $\Phi_s(s) < 0$ and $\Phi_{ss}(s) \leq -\varepsilon_1$ for $s > t_1$, for some positive constants $\varepsilon_0, \varepsilon_1$. This is again a contradiction with the positivity of Φ.

We next assume (B). Let t_1 be such that $\Phi_s(s) < 0$ on $[0, t_1]$. By (4.8), $\mathcal{E}$ is nondecreasing on $[0, t_1]$, and thus, $\mathcal{E}(s) \geq \mathcal{E}(0) > U_{max}$ for all s on $[0, t_1]$. As a consequence, $\frac{1}{2}\Phi_s(s)^2 \geq \mathcal{E}(0) - U_{max} > 0$ for all s in $[0, t_1]$, which shows that the inequality $\Phi_s(s) \leq -\sqrt{\mathcal{E}(0) - U_{max}}$ holds on $[0, t_1]$. Finally, an elementary bootstrap argument shows that the inequality $\Phi_s(s) \leq -\sqrt{\mathcal{E}(0) - U_{max}}$ is valid for all $s \geq 0$, a contradiction with the positivity of Φ.

Finally, we assume (C). By bootstrap again, $\Phi_s(s) < 0$, $\Phi(s) < 1$ and $\Phi_{ss}(s) < 0$ for all positive s, proving again that Φ is a strictly concave function, a contradiction. □

Since

$$\alpha = \frac{(p-1)s_c}{4} = \frac{N(p-1) - 4 + 2b}{8},$$

we have

$$2\alpha + 1 = \frac{N(p-1) + 2b}{4}, \quad \alpha + 1 = \frac{N(p-1) + 4 + 2b}{8},$$

$$(\alpha + 1)(\delta + 1) = 2\alpha, \quad (\alpha + 1)(\gamma + 1) = 2\alpha + 1 \text{ and } \omega = \frac{2}{(2\alpha + 1)(\alpha + 1)}.$$

By making $\Phi = v^{\alpha+1}$, then

$$\mathcal{E} = \frac{\omega}{2}\Phi_s^2(s) + U(\Phi(s)) = \frac{\alpha + 1}{2\alpha + 1}(v')^2 v^{2\alpha} + \frac{\alpha + 1}{2\alpha} v^{2\alpha} - \frac{\alpha + 1}{2\alpha + 1} v^{2\alpha+1}$$

and

$$U_{max} = \frac{1}{2\alpha}\frac{\alpha + 1}{2\alpha + 1}.$$

Consider the function f given for

$$f(x) = \sqrt{\frac{1}{kx^k} + x - \left(1 + \frac{1}{k}\right)}, \tag{4.9}$$

where $k = \frac{(p-1)s_c}{2} = 2\alpha$. Hence, if $v_s(0)$ satisfies the condition

$$v_s(0) < \begin{cases} +f(v(0)), & \text{if } v(0) < 1, \\ -f(v(0)), & \text{if } v(0) \geq 1, \end{cases}$$

then $\Phi = v^{\alpha+1}$ satisfies the conditions of Proposition 4.1. Indeed, the condition $\mathcal{E} < U_{max}$ is equivalent to

$$2\alpha(v')^2 v^{2\alpha} + (2\alpha + 1)v^{2\alpha} - 2\alpha v^{2\alpha+1} < 1$$

that is,

$$|v_s| < f(v).$$

Hence, the condition (A) means

$$v(0) < 1 \quad \text{and} \quad -f(v(0)) < v_s(0) < f(v(0))$$

and the condition (B) holds if and only if

$$|v_s(0)| > f(v(0)) \quad \text{and} \quad v_s(0) < 0.$$

More precisely,

$$v_s(0) < -f(v(0))$$

and the condition (C) is equivalent to

$$v(0) < 1 \quad \text{and} \quad v_s(0) = -f(v(0)).$$

Therefore, from (4.4), (4.5) and from the definition of v, we have

$$\begin{aligned} V(0) = (\mu\Phi(\lambda t))^{\frac{1}{\alpha+1}}\Big|_{t=0} &= \mu^{\frac{8}{N(p-1)+4+2b}} v\left(\frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]}t\right)\Big|_{t=0} \\ &= \mu^{\frac{8}{N(p-1)+4+2b}} v(0) = \frac{Ns_c M^2}{4E[u_0]} v(0) \end{aligned}$$

and

$$\begin{aligned} V_t(0) &= \mu^{\frac{8}{N(p-1)+4+2b}} \frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]} v_s(0) \\ &= \frac{Ns_c M^2}{4E[u_0]}\frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]} v_s(0) \\ &= M[u_0]\sqrt{8Ns_c}\, v_s(0). \end{aligned}$$

Furthermore,

$$\frac{V_t(0)}{M[u_0]} = \sqrt{8Ns_c}\, v_s(0) < \sqrt{8Ns_c}\, g(v(0)) = \sqrt{8Ns_c}\, g\left(\frac{4}{Ns_c}\frac{V(0)E[u_0]}{M[u_0]^2}\right),$$

which completes the proof of Theorem 1.14. □

We now proceed to the proof of Theorem 1.15. For that, we consider the following proposition.

Proposition 4.2. *Let $p > 1$ and $N \geq 1$. Then, the following inequality*

$$\|u\|_{L^2}^2 \leq C_{p,N}\left(\|xu\|_{L^2}^{\frac{N(p-1)+2b}{2}} \||\cdot|^{\frac{-b}{p+1}} u\|_{L^{p+1}}^{p+1}\right)^{\frac{2}{N(p-1)+2(p+1)+2b}} \tag{4.10}$$

holds with the sharp constant $C_{p,N}$ (depending on the nonlinearity p and dimension N) given by (4.14). Moreover, the equality occurs if and only if there exists $\beta \geq 0$, $\alpha \leq 0$ such that $|u(x)| = \beta\phi(\alpha x)$, where

$$\phi(x) = \begin{cases} |x|^{\frac{b}{p-1}}(1-|x|^2)^{\frac{1}{p-1}} & \text{if } 0 \leq |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

The proof of Proposition 4.2 follows the ideas of [10].

Proof. Let $R > 0$ to be specified later. Split the mass of u as follows

$$\int |u(x)|^2\,dx = \frac{1}{R^2}\int_{|x|\leq R}(R^2-|x|^2)|u(x)|^2\,dx + \frac{1}{R^2}\int_{|x|\leq R}|x|^2|u(x)|^2\,dx + \int_{|x|\geq R}|u(x)|^2\,dx.$$

By Hölder inequality we have

$$\begin{aligned}
&\frac{1}{R^2}\int (R^2-|x|^2)|u(x)|^2\,dx\\
&\leq \frac{1}{R^2}\left(\int_{|x|\leq R} |x|^{\frac{2b}{p-1}}(R^2-|x|^2)^{\frac{p+1}{p-1}}\,dx\right)^{\frac{p-1}{p+1}}\\
&\quad\times\left(\int |x|^{-b}|u(x)|^{p+1}\,dx\right)^{\frac{2}{p+1}}\\
&\leq \frac{1}{R^2}\left(\int_{|x|\leq 1} R^{\frac{2b}{p-1}}|y|^{\frac{2b}{p-1}}(R^2-R^2|y|^2)^{\frac{p+1}{p-1}}R^N\,dy\right)^{\frac{p-1}{p+1}}\\
&\quad\times\left(\int |x|^{-b}|u(x)|^{p+1}\,dx\right)^{\frac{2}{p+1}}\\
&= R^{\frac{N(p-1)+2b}{p+1}}D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{p+1}^2,
\end{aligned}\tag{4.11}$$

where

$$D_{p,N}=\left(\int_{|y|\leq 1}|y|^{\frac{2b}{p-1}}(1-|y|^2)^{\frac{p+1}{p-1}}\,dy\right)^{\frac{p-1}{p+1}}.$$

Furthermore,

$$\frac{1}{R^2}\int_{|x|\leq R}|x|^2|u(x)|^2\,dx+\int_{|x|\geq R}|u(x)|^2\,dx\leq\frac{1}{R^2}\int|x|^2|u(x)|^2\,dx. \tag{4.12}$$

Combining (4.11) and (4.12), we get

$$\forall R>0,\quad \|u\|_{L^2}^2\leq D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{L^{p+1}}^2 R^{\frac{N(p-1)+2b}{p+1}}+\frac{1}{R^2}\|xu\|_{L^2}^2. \tag{4.13}$$

Let $F:(0,+\infty)\to\mathbb{R}$ given by $F(R)=AR^\alpha+BR^{-2}$, where $A,B>0$ and $\alpha>0$. The minimum value of F is reached at $R=\left(\frac{2B}{\alpha A}\right)^{\frac{1}{\alpha+2}}$ and

$$F\left(\left(\frac{2B}{\alpha A}\right)^{\frac{1}{\alpha+2}}\right)=A\left(\frac{2B}{\alpha A}\right)^{\frac{\alpha}{\alpha+2}}+B\left(\frac{\alpha A}{2B}\right)^{\frac{2}{\alpha+2}}=\frac{2+\alpha}{\alpha}(\alpha A)^{\frac{2}{\alpha+2}}(2B)^{\frac{\alpha}{\alpha+2}}.$$

Thus, by taking

$$R=\left(\frac{p+1}{N(p-1)+2b}\frac{2\|xu\|_{L^2}^2}{D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{L^{p+1}}^2}\right)^{\frac{p+1}{N(p-1)+2(p+1)+2b}}$$

in (4.13), we have

$$\|u\|_{L^2}^2\leq C_{p,N}^2\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{L^{p+1}}^{\frac{4(p+1)}{N(p-1)+2(p+1)+2b}}\|xu\|_{L^2}^{\frac{2N(p-1)+4b}{N(p-1)+2(p+1)+2b}},$$

where

$$C_{p,N} = \left(\frac{N(p-1)+2(p+1)+2b}{2N(p-1)+4b}\right)^{\frac{1}{2}}$$
$$\times \left(\frac{N(p-1)+2b}{p+1} D_{p,N}\right)^{\frac{(p+1)}{N(p-1)+2(p+1)+2b}}$$
$$\times 2^{\frac{N(p-1)+2b}{2N(p-1)+4(p+1)+4b}}. \tag{4.14}$$

Note that equality in (4.10) holds if and only if there exists $R > 0$ such that (4.13) is an equality. This is equivalent to the fact that for some $R > 0$, both (4.11) and (4.12) are equalities. The inequality (4.11) is an equality if and only if, for $|x| < R$, $|x|^{-b}|u(x)|^{p+1} = c|x|^{\frac{2b}{p-1}}(R^2 - |x|^2)^{\frac{p+1}{p-1}}$ for some constant $c \geq 0$, and inequality (4.12) is an equality if and only if $u(x) = 0$ for $|x| \geq R$. This completes the proof of Proposition 4.2. □

Proof of Theorem 1.15. Since the energy is

$$E[u_0] = \frac{1}{2}\|\nabla u(t)\|^2_{L^2} - \frac{1}{p+1}\left\| |\cdot|^{-\frac{b}{p+1}} u(t)\right\|^{p+1}_{L^{p+1}},$$

from (1.6), we obtain

$$V_{tt}(t) = 4(N(p-1)+2b)E[u_0] - 2(N(p-1)+2b-4)\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)}$$
$$= 16E[u_0] - \frac{8(p-1)s_c}{p+1}\left\| |\cdot|^{-\frac{b}{p+1}} u(t)\right\|^{p+1}_{L^{p+1}}.$$

Using the sharp interpolation inequality (4.10)

$$V_{tt}(t) \leq 16E[u_0] - \frac{8(p-1)s_c}{(p+1)(C_{p,N})^{\frac{N(p-1)}{2}+(p+1)+b}} \frac{M[u_0]^{\frac{N(p-1)}{4}+\frac{(p+1)}{2}+\frac{b}{2}}}{V(t)^{\frac{N(p-1)+2b}{4}}}, \tag{4.15}$$

with $C_{p,N}$ from (4.10). As done in the proof of Proposition 1.14, take $v(s)$ with $s = at$ such that

$$V(t) = \mu v(\lambda t), \quad \lambda = \sqrt{\frac{32E[u_0]}{\mu}},$$

where

$$\mu = \left(\frac{s_c(p-1)}{2(p+1)}\right)^{\frac{4}{N(p-1)+2b}} \frac{M[u_0]^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}}{(C_{p,N})^{2+(p+1)\left(\frac{4}{N(p-1)+2b}\right)} E[u_0]^{\frac{4}{N(p-1)+2b}}}.$$

Hence, applying in the inequality (4.15), we have

$$v_{ss}(s) \leq \frac{1}{2}\left(1 - v^{-\frac{N(p-1)+2b}{4}}(s)\right).$$

If the inequality in the above expression is replaced by an equality, then we have that the following energy is conserved

$$\mathcal{E}(s) = \frac{k}{1+k}\left((v(s))^2 - v(s) - \frac{1}{kv(s)^k}\right),$$

where as before $k = \frac{(p-1)s_c}{2} = \frac{N(p-1)+2b}{4} - 1$. The maximum of the function

$$f(x) = \frac{k}{1+k}\left(x + \frac{1}{kx^k}\right),$$

attained at $x = 1$, is -1. As we did to (A), (B) and (C), we identify the three sufficient conditions for blow-up in finite time.

(A^*) $\mathcal{E}(0) < -1$ and $v(0) < 1$,
(B^*) $\mathcal{E}(0) > -1$ and $v_s(0) < 0$,
(C^*) $\mathcal{E}(0) = -1$, $v_s(0) < 0$ and $v(0) < 1$.

If $v_s(0)$ satisfies the condition

$$v_s(0) < \begin{cases} +f(v(0)), & \text{if } v(0) < 1 \\ -f(v(0)), & \text{if } v(0) \geq 1, \end{cases}$$

then v satisfies one of the conditions (A*), (B*) and (C*). Indeed, recalling the function f from (4.9) and using the definition of $\mathcal{E}$, we obtain

a) $\mathcal{E} < -1$ if and only if $|v_s| < f(v)$.
b) $\mathcal{E} \geq -1$ if and only if $|v_s| \geq f(v)$.

Then the previous conditions can be written in the following form:

$(A^*) \Leftrightarrow v(0) < 1$ and $-f(v(0)) < v_s(0) < f(v(0))$,
$(B^*) \Leftrightarrow v_s(0) < -f(v(0))$,
$(C^*) \Leftrightarrow v_s(0) = -f(v(0)), \ \ v(0) < 1$.

Substituting back $V(t)$, we obtain

$$\frac{V_t(0)}{\lambda\mu} < g\left(\frac{V(0)}{\mu}\right),$$

where g is defined in (1.13). Hence,

$$\frac{V_t(0)}{4\sqrt{2}}\left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{2}{N(p-1)+2b}} \times \frac{(C_{p,N})^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}}{E[u_0]^{\frac{s_c}{N}} M[u_0]^{\frac{1}{2}+(p+1)\left(\frac{1}{N(p-1)+2b}\right)}} < g(\theta),$$

with

$$\theta = \left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{4}{N(p-1)+2b}} \times \frac{E[u_0]^{\frac{4}{N(p-1)+2b}}}{M[u_0]^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}} V(0).$$

This completes the proof of Theorem 1.15. □

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