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Stability estimate for a semilinear elliptic inverse problem

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Abstract. We establish a logarithmic stability estimate for the inverse problem of determining the nonlinear term, appearing in a semilinear boundary value problem, from the corresponding Dirichlet-to-Neumann map. Our result can be seen as a stability inequality for an earlier uniqueness result by Isakov and Sylvester (Commun Pure Appl Math 47:1403–1410, 1994).

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Contents

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1. Introduction

Let Ω be a $C^{1,1}$ bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with boundary Γ. Fix $\mathfrak{c} = (c_0, c_1, c)$ with $c_0 > 0$, $c_1 > 0$ and $0 \leq c < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Laplace operator on Ω with Dirichlet boundary condition.

We denote by $\mathscr{A}(\mathfrak{c}, \alpha)$, with $\alpha \geq 0$, the set of continuously differentiable functions $a : \mathbb{R} \to \mathbb{R}$ satisfying the following two assumptions

$$|a(t)| \leq c_0 + c_1|t|^\alpha \quad \text{for all } t \in \mathbb{R}, \tag{1.1}$$

and

$$a'(t) \geq -c \quad \text{for all } t \in \mathbb{R}. \tag{1.2}$$

In the present article, the ball of a normed space X at center 0 with radius $M > 0$ is denoted by $B_X(M)$. Also, c_Ω denotes a generic constant only depending on Ω.

Unless otherwise stated all the functions we use are assumed to real-valued.

Consider the following non-homogenous semilinear boundary value problem

$$\begin{cases} -\Delta u + a \circ u = 0 \text{ in } \Omega, \\ u = f \qquad\qquad\quad \text{on } \Gamma. \end{cases} \tag{1.3}$$

Henceforth we use the abbreviation BVP for boundary value problem. For the formulation of our inverse problem, we need the well-posedness of the BVP (1.3), which is stated as follows:

Theorem 1.1. *Assume that α is arbitrary if $n = 2$ and $\alpha \leq n/(n-2)$ if $n \geq 3$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$. Then, for any $f \in H^{3/2}(\Gamma)$, the BVP (1.3) has a unique solution $u_a(f) \in H^2(\Omega)$. Furthermore,*

$$\|u_a(f)\|_{H^2(\Omega)} \leq C, \quad \text{for any } f \in B_{H^{3/2}(\Gamma)}(M), \tag{1.4}$$

where $C = C(\Omega, M, \mathfrak{c}, \alpha) > 0$ is a constant. That is $f \to u_a(f)$ maps bounded set of $H^{3/2}(\Gamma)$ into bounded set of $H^2(\Omega)$.

An example of a function a fulfilling the assumptions in the above theorem is the linear case $a(t) = -kt$ with $k < c$, which models the time-harmonic acoustic wave propagation at the wavenumber $k > 0$. The semilinear equation also covers the Schrödinger equation.

Hereafter, the derivative in the direction of the unit exterior normal vector field ν on Γ of a function u is denoted by $\partial_\nu u$.

Theorem 1.2. (i) *Assume that α is arbitrary if $n = 2$ and $\alpha \leq 3$ if $n = 3$. If $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ then we can define the mapping*

$$\Lambda_a : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma) : f \mapsto \partial_\nu u_a(f).$$

Moreover, for arbitrarily given $M > 0$, we have

$$\|\Lambda_a(f)\|_{H^{1/2}(\Gamma)} \leq C, \quad \text{for any } f \in B_{H^{3/2}(\Gamma)}(M), \tag{1.5}$$

where $C = C(\Omega, M, \mathfrak{c}, \alpha)$ is a constant.

(ii) *Assume that* $n > 4$. *Let* $n/2 < p < n$ *and* $\alpha \le q/p$ *with* $q = 2n/(n-4)$. *If* $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ *then we can define*

$$\Lambda_a : W^{2-1/p,p}(\Gamma) \to \partial_\nu u_a(f) \in W^{1-1/p,p}(\Gamma) : f \mapsto \partial_\nu u_a(f).$$

Furthermore, for arbitrarily given $M > 0$, *we have*

$$\|\Lambda_a(f)\|_{W^{1-1/p,p}(\Gamma)} \le C, \quad \text{for any } f \in B_{W^{2-1/p,p}(\Gamma)}(M). \tag{1.6}$$

Here $C = C(\Omega, M, \mathfrak{c}, p, \alpha) > 0$ *is a constant.*

(iii) *Assume that* $n = 4$. *Let* $2 < p < 4$, $1 \le r < 2$, $q = 2r/(2-r)$ *and* $\alpha \le q/p$. *If* $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ *then we can define*

$$\Lambda_a : W^{2-1/p,p}(\Gamma) \to W^{1-1/p,p}(\Gamma) : f \mapsto \partial_\nu u_a(f).$$

Moreover, for any $M > 0$, *we have*

$$\|\Lambda_a(f)\|_{W^{1-1/p,p}(\Gamma)} \le C, \quad \text{for any } f \in B_{W^{2-1/p,p}(\Gamma)}(M), \tag{1.7}$$

where $C = C(\Omega, M, \mathfrak{c}, p, r, \alpha) > 0$ *is a constant.*

We call the (nonlinear) operator Λ_a in Theorem 1.2 the Dirichlet-to-Neumann map associated to a.

We are concerned with the inverse problem of determining the nonlinear term a from the corresponding Dirichlet-to-Neumann map Λ_a. The main purpose is the stability issue.

For most of inverse problems, the solutions of the inverse problem do not necessarily depend on data continuously by conventional choices of topologies even if the uniqueness holds. It is often that if we suitably reduce an admissible set of unknowns, then we can recover the stability for the inverse problem.

Thus we define $\tilde{\mathscr{A}}(\mathfrak{c}, \alpha)$ as an admissible set of functions $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ satisfying the additional condition: for any $R > 0$, there exists a constant $\varkappa_R$ so that

$$|a'(u) - a'(v)| \le \varkappa_R |u - v|, \quad |u|,\ |v| \le R. \tag{1.8}$$

Note that condition (1.8) means that the first derivative of a is Lipschitz continuous on bounded sets of $\mathbb{R}$. Also, we observe that the constant $\varkappa_R$ in (1.8) may depend on a.

Within this class, we can linearize the inverse problem under consideration. Precisely, we have the following proposition in which, for $j = 0, 1$,

$$\mathscr{X}_j = H^{3/2-j}(\Gamma) \text{ if } n = 2, 3 \quad \text{and} \quad \mathscr{X}_j = W^{2-j-1/p,p}(\Gamma) \text{ if } n \ge 4,$$

and the space

$$\mathscr{Y} = \mathscr{B}(\mathscr{X}_0, \mathscr{X}_1)$$

denotes the set of bounded linear operators mapping $\mathscr{X}_0$ into $\mathscr{X}_1$.

The proposition below states that the linearization of the Dirichlet-to-Neumann map Λ_a is the Dirichlet-to-Neumann map of the linearized problem.

Proposition 1.1. *Under the assumptions and the notations of Theorem 1.2, if $a \in \tilde{\mathscr{A}}(\mathfrak{c}, \alpha)$, then Λ_a is Fréchet differentiable at any $f \in \mathscr{X}_0$ with $\Lambda_a'(f)(h) = \partial_\nu v_{a,f}(h)$, where $h \in \mathscr{X}_0$ and $v_{a,f}(h)$ is the unique solution of the BVP*

$$\begin{cases} -\Delta v + a' \circ u_a(f) v = 0 \text{ in } \Omega, \\ v = h \qquad\qquad\qquad\quad \text{on } \Gamma. \end{cases}$$

Moreover, for any $M > 0$, we have

$$\|\Lambda_a'(f)\|_{\mathscr{Y}} \le C, \quad \textit{for any } f \in B_{\mathscr{X}_0}(M).$$

Here the constant $C > 0$ is as Theorem 1.2.

Henceforward $|\Gamma|$ denotes the Lebesgue measure of Γ.

The main result of this paper is the following theorem.

Theorem 1.3. *Assume that $n \ge 3$ and the assumptions of Theorem 1.2 hold for $a, \tilde{a} \in \tilde{\mathscr{A}}(\mathfrak{c}, \alpha)$ satisfying $a(0) = \tilde{a}(0)$ and let $\beta = 1/2$ if $n = 3$ and $\beta = 2 - n/p$ if $n \ge 4$. Let $0 < s < \min(1/2, \beta)$. Then*

$$\max_{|\lambda| \le M} |a(\lambda) - \tilde{a}(\lambda)| \le C_M \Psi \left(\sup_{\|f\|_{\mathscr{X}_0} \le \sqrt{|\Gamma|} M} \|\Lambda_a'(f) - \Lambda_{\tilde{a}}'(f)\|_{\mathscr{Y}} \right),$$

where the constant $C_M = C$ is as in Theorem 1.2, and

$$\Psi(t) = \begin{cases} |\ln t|^{-[2\min(1/2, s/n)\beta]/(n+2\beta)} + t & \textit{if } t > 0, \\ 0 & \textit{if } t = 0. \end{cases}$$

Theorem 1.3 immediately yields

Corollary 1.1. *If $a, \tilde{a} \in \tilde{\mathscr{A}}(\mathfrak{c}, \alpha)$ satisfy $a(0) = \tilde{a}(0)$ and $\Lambda_a = \Lambda_{\tilde{a}}$ then $a = \tilde{a}$.*

This corollary corresponds to the uniqueness result in [12] which considers more general equations $-\Delta u + a(x, u(x)) = 0$.

Remark 1.1. (a) Consider the Fréchet space $C(\mathbb{R})$ equipped with the family of semi-norms $(\mathfrak{p}_j)_{j \ge 1}$:

$$\mathfrak{p}_j(h) = \max_{|t| \le j} |h(t)|, \quad h \in C(\mathbb{R}).$$

Let $C^1_{\text{loc}}(\mathscr{X}_0, \mathscr{X}_1)$ be the vector space of Fréchet differentiable functions

$$\Lambda : \mathscr{X}_0 \to \mathscr{X}_1$$

so that Λ and Λ' are locally bounded. A natural topology on $C^1_{\text{loc}}(\mathscr{X}_0, \mathscr{X}_1)$ is induced by the following family of semi-norms

$$\mathfrak{q}_j(\Lambda) = \sup_{f \in B_{\mathscr{X}_0}(j|\Gamma|)} \left(\|\Lambda(f)\|_{\mathscr{X}_1} + \|\Lambda'(f)\|_{\mathscr{Y}} \right), \quad \Lambda \in C^1_{\text{loc}}(\mathscr{X}_0, \mathscr{X}_1).$$

We observe that the estimate in Theorem 1.3 can be rewritten in the form

$$\mathfrak{p}_j(a - \tilde{a}) \le C_j \Psi(\mathfrak{q}_j(\Lambda_a - \Lambda_{\tilde{a}})), \quad j \ge 1.$$

(b) A natural distance on $\tilde{\mathscr{A}}(\mathfrak{c}, \alpha)$ is given by

$$\mathbf{d}(a, \tilde{a}) = \sup_{|t| \le 1} |a(t) - \tilde{a}(t)| + \sup_{|t| \ge 1} |t^{-\alpha}(a(t) - \tilde{a}(t))|, \quad a, \tilde{a} \in \tilde{\mathscr{A}}(\mathfrak{c}, \alpha).$$

One can then ask whether it is possible to prove a stability estimate when $\tilde{\mathscr{A}}(\mathfrak{c},\alpha)$ is endowed with distance $\mathbf{d}$. They are two obstructions to get such kind of estimate. The first obstruction is due to the fact the natural space of Dirichlet-to-Neumann maps (defined in (a)) is a locally convex metrizable topological vector space which is not normable. The second obstruction comes from the fact the local modulus of continuity in Theorem 1.3 is logarithmic.

It is worth mentioning that the proof of Theorem 1.3 can be adapted to a partial Dirichlet-to-Neumann map of Λ_a. Here, with fixed compact subsets Γ', Γ'' of Γ, a partial Dirichlet-to-Neumann map means a mapping

$$f \in \{h \in H^{3/2}(\Gamma);\ \mathrm{supp}(h) \subset \Gamma'\} \rightarrow \partial_\nu u_a(f)|_{\Gamma''} \in H^{1/2}(\Gamma'').$$

A double logarithmic stability inequality for the linearized problem, with a partial Dirichlet-to-Neumann map, was recently established by Caro, Dos Santos Ferreira and Ruiz [1]. One can expect by [1] that Theorem 1.3 can be extended with suitable partial Dirichlet-to-Neumann maps. We refer to [13] for the first uniqueness result in determining semilinear terms by partial Cauchy data on arbitrary subboundary.

Uniqueness results for recovering semilinear terms from full Cauchy data were obtained by Isakov and Sylvester [12] in three dimensions and by Isakov and Nachman [11] in two dimensions. These results apply to nonlinearities of the form $a = a(x,u)$. For the sake of simplicity we only consider here the case $a = a(u)$. However we can expect that Theorem 1.3 can be extended to cover completely the uniqueness result in [12], possibly under some additional conditions.

We point out that the uniqueness results for smooth semilinear terms using partial data in $\mathbb{R}^n$ $(n \geq 2)$ were contained in the recent papers by Krupchyk and Uhlmann [15], and Lassas, Liimatainen, Lin and Salo [18]. These two references make use of higher order linearization procedure and contain a detailed overview of semilinear elliptic inverse problems together with a rich list of references. Without being exhaustive, we refer to [13,14,17,20,22,23] for other results concerning the unique determination of the nonlinear term in semilinear and quasilinear elliptic BVP's from boundary measurements. Similar inverse problem was studied in [10] for a semilinear parabolic equation and in [2] for a quasilinear parabolic equation. Inverse problems for hyperbolic equations with various type of nonlinearities were considered in [3,9,16,24].

To our knowledge there are few stability results for the problem of determining the nonlinear term, appearing in partial differential equations, from boundary measurements. The determination of the nonlinear term in a semilinear parabolic equation, from the corresponding Dirichlet-to-Neumann map, was studied by the first author and Kian [5]. In [5] the authors establish a logarithmic stability estimate. A stability inequality of the determination of a nonlinear term in a parabolic equation from a single measurement was proved by the first and third authors and Ouhabaz in [6].

The rest of this article is organized as follows. In Sect. 2 we give the proof of Theorem 1.1 and in Sect. 3 we prove Theorem 1.2. Section 4 is devoted to establish a stability estimate for the linearized inverse problem. In Sect. 5, we give the proof of Proposition 1.1 and Theorem 1.3 on the basis of Sect. 4.

2. Analysis of the semilinear BVP

Prior to introducing the definition of variational solution of the BVP (1.3), we prove the following lemma.

Lemma 2.1. *Assume that α is arbitrary if $n=2$ and $\alpha \le (n+2)/(n-2)$ if $n \ge 3$. Let $a \in \mathscr{A}(\mathfrak{c},\alpha)$ and $\varphi \in L^{\alpha q^*}(\Omega)$, where $q^* = 2n/(n+2)$ denotes the conjugate component of $q = 2n/(n-2)$. Then the linear form on $H_0^1(\Omega)$ given by*

$$\ell(\phi) = \int_\Omega a(\varphi(x))\phi(x)dx, \quad \phi \in H_0^1(\Omega),$$

is bounded with

$$\|\ell\|_{H^{-1}(\Omega)} \le C(1+M^\alpha), \quad \text{for any } \varphi \in B_{L^{\alpha q^*}}(M), \tag{2.1}$$

where $C = C(\Omega, c_0, c_1, \alpha) > 0$ is a constant.

Proof. Consider first the case $n \ge 3$. In that case $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ with $q = 2n/(n-2)$. We have in light of (1.1)

$$\left|\int_\Omega a(\varphi(x))\phi(x)dx\right| \le c_0 \int_\Omega |\phi|dx + c_1 \int_\Omega |\varphi|^\alpha |\phi|dx.$$

Applying Hölder's inequality, we have

$$\int_\Omega |\varphi|^\alpha |\phi|dx \le \left(\int_\Omega |\varphi|^{\alpha q^*} dx\right)^{1/q^*} \left(\int |\phi|^q dx\right)^{1/q}.$$

Hence

$$\begin{aligned}\left|\int_\Omega a(\varphi(x))\phi(x)dx\right| &\le c_0\|\phi\|_{L^1(\Omega)} + c_1\|\varphi\|^\alpha_{L^{\alpha q^*}(\Omega)}\|\phi\|_{L^q(\Omega)} \\ &\le c_\Omega\left(c_0 + c_1\|\varphi\|^\alpha_{L^{\alpha q^*}(\Omega)}\right)\|\phi\|_{H_0^1(\Omega)} \\ &\le c_\Omega(c_0 + c_1 M^\alpha)\|\phi\|_{H_0^1(\Omega)},\end{aligned} \tag{2.2}$$

where we used that $H_0^1(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1, q]$. Taking the supremum over $\phi \in B_{H_0^1(\Omega)}(1)$ in both sides of (2.2) in order to obtain (2.1).

The case $n = 2$ can be carried out similarly by using that $H_0^1(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \ge 1$. □

Let $f \in H^{1/2}(\Gamma)$. We say that $u \in H^1(\Omega)$ is a variational solution of the BVP (1.3) if $u_{|\Gamma} = f$ (in the trace sense) and

$$\int_\Omega \nabla u(x) \cdot \nabla \phi(x)dx + \int_\Omega a(u(x))\phi(x)dx = 0, \quad \phi \in H_0^1(\Omega).$$

For $f \in H^{1/2}(\Omega)$, let $\mathscr{E}f \in H^1(\Omega)$ be its harmonic extension. That is, $v = \mathscr{E}f$ is the unique solution of the BVP

$$\begin{cases} -\Delta v = 0 \text{ in } \Omega, \\ v = f \quad\quad \text{on } \Gamma. \end{cases}$$

Assume that we can find $w \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla w(x) \cdot \nabla \phi(x) = -\int_\Omega a(w(x) + v(x))\phi(x)dx, \quad \text{for any } \phi \in H_0^1(\Omega), \tag{2.3}$$

An integration by parts yields

$$0 = \int_\Omega \Delta v(x)\phi(x)dx = -\int_\Omega \nabla v(x) \cdot \nabla \phi(x)dx, \quad \text{for any } \phi \in C_0^\infty(\Omega).$$

Since $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, we deduce that

$$\int_\Omega \nabla v(x) \cdot \nabla \phi(x)dx = 0, \quad \text{for any } \phi \in H_0^1(\Omega).$$

We then obtain in light of (2.3)

$$\int_\Omega \nabla (w(x) + v(x)) \cdot \nabla \phi(x) = -\int_\Omega a(w(x) + v(x))\phi(x)dx, \quad \text{for any } \phi \in H_0^1(\Omega).$$

In other words, $u = w + v$ is a variational solution of (1.3).

Theorem 2.1. *Assume that α is arbitrary if $n = 2$ and $\alpha < (n+2)/(n-2)$ if $n \geq 3$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ and $f \in H^{1/2}(\Gamma)$. Then the BVP (1.3) has a unique variational solution $u_a(f) \in H^1(\Omega)$. Moreover, for any $M > 0$, we have*

$$\|u_a(f)\|_{H^1(\Omega)} \leq C(1 + M^\alpha), \quad \text{for any } f \in B_{H^{1/2}(\Gamma)}(M), \tag{2.4}$$

where $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a constant.

Proof. In light of the previous discussion, it is enough to prove that (2.3) has a solution $w \in H_0^1(\Omega)$ and (2.4) holds with $u_a(f)$ substituted by w.

Fix $w \in L^{\alpha q^*}(\Omega)$ and consider the variational problem: find $\psi \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla \psi(x) \cdot \nabla \phi(x) = -\int_\Omega a(w(x) + v(x))\phi(x)dx, \quad \text{for any } \phi \in H_0^1(\Omega). \tag{2.5}$$

From Lemma 2.1 it follows that

$$\ell : \phi \mapsto \ell(\phi) = -\int_\Omega a(w(x) + v(x))\phi(x)dx$$

defines a bounded linear form on $H_0^1(\Omega)$. Then Lax-Milgram's lemma, which we apply to the functional on the left-hand side, guarantees that (2.5) has a unique solution $\psi \in H_0^1(\Omega)$.

Let $q = 2n/(n-2)$ and $q^* = 2n/(n+2)$ be its conjugate exponent to q and define

$$T : L^{\alpha q^*}(\Omega) \to L^{\alpha q^*}(\Omega) : w \mapsto Tw = \psi,$$

where $\psi \in H_0^1(\Omega)$ is the unique solution of the variational problem (2.5).

Assume that $H_0^1(\Omega)$ is endowed with the norm $\|\nabla h\|_{L^2(\Omega)}$. We obtain by taking $\phi = \psi$ in (2.5)

$$\|\psi\|_{H_0^1(\Omega)} \le \|\ell\|_{H^{-1}(\Omega)}.$$

This and inequality (2.1) in Lemma 2.1 yield

$$\|Tw\|_{H_0^1(\Omega)} = \|\psi\|_{H_0^1(\Omega)} \le C(1+M^\alpha), \quad \text{for any } w \in B_{L^{\alpha q^*}(\Omega)}(M),$$

where $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a constant. That is, T maps each bounded set of $L^{\alpha q^*}(\Omega)$ into a bounded set in $H_0^1(\Omega)$. Hence, according to Rellich-Kondrachov's theorem, $H_0^1(\Omega)$ is compactly embedded in $L^{\alpha q^*}(\Omega)$. Therefore, T is a compact operator.

We are now going to show, with the help of Leray-Schauder's fixed point theorem, that T has a fixed point. The crucial step consists in proving that the set

$$K = \{w \in L^{\alpha q^*}(\Omega);\ \text{there exists } \mu \in [0,1] \text{ so that } w = \mu T w\}$$

is bounded in $L^{\alpha q^*}(\Omega)$.

Pick $w \in K$ and let $\mu \in [0,1]$ so that $w = \mu T w$. According to the definition of T, w $(\in H_0^1(\Omega))$ satisfies

$$\int_\Omega |\nabla w(x)|^2 dx = -\mu \int_\Omega a(w(x)+v(x))w(x)dx. \tag{2.6}$$

On the other hand, we have, for almost everywhere $x \in \Omega$,

$$a(w(x)+v(x)) = a(v(x)) + \int_0^1 a'(sw(x)+v(x))w(x)ds.$$

This in (2.6) yields

$$\int_\Omega |\nabla w(x)|^2 dx = -\mu \int_\Omega a(v(x))w(x)dx - \mu \int_\Omega \left(\int_0^1 a'(sw(x)+v(x))ds\right) w(x)^2 dx.$$

In light of assumption (1.2) we obtain

$$\int_\Omega |\nabla w(x)|^2 dx \le -\mu \int_\Omega a(v(x))w(x)dx + c\int_\Omega w(x)^2 dx$$

which combined with Poincaré's inequality gives

$$\int_\Omega |\nabla w(x)|^2 dx \le -\mu \int_\Omega a(v(x))w(x)dx + c\lambda_1(\Omega)^{-1} \int_\Omega |\nabla w(x)|^2 dx.$$

Or equivalently

$$(1 - c\lambda_1(\Omega)^{-1}) \int_\Omega |\nabla w(x)|^2 dx \le -\mu \int_\Omega a(v(x))w(x)dx.$$

We then apply again Lemma 2.1 in order to obtain

$$\|w\|_{L^{\alpha q^*}(\Omega)} \le C_0 \|w\|_{H_0^1(\Omega)} \le C(1+M^\alpha), \quad \text{for any } w \in K \text{ and } f \in B_{H^{1/2}(\Gamma)}(M), \tag{2.7}$$

where $C_0 = C_0(\Omega, \alpha) > 0$ and $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ are constants.

In light of this inequality we can apply [7, Theorem 11.3, p. 280] to deduce that there exists $w^* \in H_0^1(\Omega)$ so that $w^* = Tw^*$. That is w^* is the solution of the variational problem (1.3). Furthermore, for any $f \in B_{H^{1/2}(\Gamma)}(M)$, we have from (2.7)

$$\|w^*\|_{H_0^1(\Omega)} \le C(1 + M^\alpha),$$

where $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a constant.

We complete the proof by showing that (1.3) has at most one solution. To this end, let $u, \tilde{u} \in H_0^1(\Omega)$ be two solutions of (1.3) and set $v = u - \tilde{u}$. Taking into account that, for almost everywhere $x \in \Omega$, we have

$$a(u(x)) - a(\tilde{u}(x)) = b(x)v(x),$$

with

$$b(x) = \int_0^1 a'(x, \tilde{u}(x) + s(u(x) - \tilde{u}(x)))ds,$$

we find that v is the solution of the BVP

$$\begin{cases} -\Delta v + bv = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

Green's formula then yields

$$\int_\Omega |\nabla v(x)|^2 dx + \int_\Omega b(x)v(x)^2 dx = 0.$$

Hence

$$\int_\Omega |\nabla v(x)|^2 dx = -\int_\Omega b(x)v(x)^2 dx \le c\int_\Omega v(x)^2 dx \le c\lambda_1(\Omega)^{-1}\int_\Omega |\nabla v(x)|^2 dx.$$

By assumption $c\lambda_1(\Omega)^{-1} < 1$, we reach $v = 0$. □

Theorem 1.1 will then follow from the following lemma.

Lemma 2.2. *Assume that α is arbitrary if $n = 2$ and $\alpha \le n/(n-2)$ if $n \ge 3$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ and $f \in H^{3/2}(\Gamma)$. Then $u_a(f) \in H^2(\Omega)$ and*

$$\|u_a(f)\|_{H^2(\Omega)} \le C(1 + M + M^\alpha), \quad \text{for any } f \in B_{H^{3/2}(\Gamma)}(M), \tag{2.8}$$

where $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a constant.

Proof. In this proof $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a generic constant.

Consider the case $n \ge 3$. By (1.1) we have, for almost everywhere $x \in \Omega$,

$$[a \circ u_a(f)(x)]^2 \le 2c_0^2 + 2c_1^2|u_a(f)(x)|^{2\alpha}.$$

Using that $2\alpha \le 2n/(n-2)$ and $H^1(\Omega)$ is continuously embedded in $L^{2\alpha}(\Omega)$, we deduce that $a \circ u_a(f) \in L^2(\Omega)$ and from (2.4), we obtain

$$\|a \circ u_a(f)\|_{L^2(\Omega)} \le C(1 + M^\alpha). \tag{2.9}$$

From the elliptic regularity (e.g., [19, Theorem 5.4, p. 165]), we deduce that $u_a(f) \in H^2(\Omega)$ and

$$\|u_a(f)\|_{H^2(\Omega)} \le c_\Omega \left(\|f\|_{H^{3/2}(\Gamma)} + \|a \circ u_a(f)\|_{L^2(\Omega)}\right). \tag{2.10}$$

Thus, inequalities (2.9) and (2.10) yield (2.8) in a straightforward manner.

The case $n = 2$ can be treated similarly using that $H^1(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \ge 1$. □

3. Dirichlet-to-Neumann map

We first observe that by the help of Theorem 2.1 and Lemma 2.2 we can define the Dirichlet-to-Neumann map associated to $a \in \mathscr{A}(\mathfrak{c}, \alpha)$. Precisely we have the following corollary.

Corollary 3.1. *Assume that α is arbitrary if $n = 2$ and $\alpha \leq n/(n-2)$ if $n \geq 3$. For any $a \in \mathscr{A}(\mathfrak{c}, \alpha)$ and $j = 0, 1$, we can define the mapping*

$$\Lambda_a : H^{j+1/2}(\Gamma) \to H^{j-1/2}(\Gamma) : f \mapsto \partial_\nu u_a(f).$$

Moreover, for any $M > 0$,

$$\|\Lambda_a(f)\|_{H^{j-1/2}(\Gamma)} \leq C(1 + M + M^\alpha), \quad \text{for any } f \in B_{H^{j+1/2}(\Gamma)}(M) \tag{3.1}$$

where $C = C(\Omega, \mathfrak{c}, \alpha)$ is a constant.

We recall that $C^{0,\theta}(\overline{\Omega})$, $0 < \theta \leq 1$, is the usual vector space of functions that are Hölder continuous on $\overline{\Omega}$ with exponent θ. This space is usually endowed with its natural norm

$$\|w\|_{C^{0,\theta}(\overline{\Omega})} = \|w\|_{C(\overline{\Omega})} + \sup_{x,y \in \overline{\Omega},\ x \neq y} \frac{|w(x) - w(y)|}{|x - y|^\theta}.$$

Taking into account that $H^2(\Omega)$ is continuously embedded in $C^{0,1/2}(\overline{\Omega})$, for $n = 2, 3$, in view of Lemma 2.2 we obtain:

Corollary 3.2. *Assume that α is arbitrary if $n = 2$ and $\alpha \leq 3$ if $n = 3$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$, $M > 0$ and $f \in B_{H^{3/2}(\Gamma)}(M)$. Then $u_a(f) \in C^{0,1/2}(\overline{\Omega})$ and*

$$\|u_a(f)\|_{C^{0,1/2}(\overline{\Omega})} \leq C(1 + M + M^\alpha), \tag{3.2}$$

where $C = C(\Omega, \mathfrak{c}, \alpha) > 0$ is a constant.

Lemma 3.1. (i) *Assume that $n > 4$, $n/2 < p < n$ and $\alpha \leq q/p$ with $q = 2n/(n-4)$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$, $M > 0$ and $f \in B_{W^{2-1/p,p}(\Gamma)}(M)$. Then $u_a(f) \in W^{2,p}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$, with $\beta = 2 - n/p$, and*

$$\|u_a(f)\|_{W^{2,p}(\Omega)} + \|u_a(f)\|_{C^{0,\beta}(\overline{\Omega})} \leq C(1 + M + M^\alpha), \tag{3.3}$$

where $C = C(\Omega, \mathfrak{c}, \alpha, p)$ is a constant.

(ii) *Assume that $n = 4$, $2 < p < 4$, $1 \leq r < 2$, $q = 2r/(2-r)$ and $\alpha \leq q/p$. Let $a \in \mathscr{A}(\mathfrak{c}, \alpha)$, $M > 0$ and $f \in B_{W^{2-1/p,p}(\Gamma)}(M)$. Then $u_a(f) \in W^{2,p}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$, with $\beta = 2 - 4/p$, and*

$$\|u_a(f)\|_{W^{2,p}(\Omega)} + \|u_a(f)\|_{C^{0,\beta}(\overline{\Omega})} \leq C(1 + M + M^\alpha), \tag{3.4}$$

where $C = C(\Omega, \mathfrak{c}, \alpha, p, r) > 0$ is a constant.

Proof. (i) In this part $C = C(\Omega, \mathfrak{c}, \alpha, p) > 0$ is a generic constant.

Noting that $q/p < n/(n-2)$, we obtain from Lemma 2.2 that $u_a(f) \in H^2(\Omega)$ and, since $H^2(\Omega)$ is continuously embedded in $L^q(\Omega)$ with $q = 2n/(n-4)$, $u_a(f) \in L^q(\Omega)$. Consequently, using (1.1), (2.8) and the assumption on α, we obtain $a \circ u_a(f) \in L^p(\Omega)$ and

$$\|a \circ u_a(f)\|_{L^p(\Omega)} \leq C(1 + M + M^\alpha). \tag{3.5}$$

We obtain by applying [7, Theorem 9.15, p. 241] that $u_a(f) \in W^{2,p}(\Omega)$ and, since $W^{2,p}(\Omega)$ is continuously embedded in $C^{0,\beta}(\overline{\Omega})$, we conclude that $u_a(f) \in C^{0,\beta}(\overline{\Omega})$.

A combination of [7, (9.46), p. 242] and (3.5) yields in straightforward manner

$$\|u_a(f)\|_{W^{2,p}(\Omega)} \le C(1 + M + M^\alpha).$$

Hence (3.3) follows.

(ii) Let $n = 4$ and $1 \le r < 2$. As $q/p < 2$, we obtain from Lemma 2.2 that $u_a(f) \in H^2(\Omega)$. Since $H^2(\Omega)$ is continuously embedded in $W^{2,r}(\Omega)$ and $W^{2,r}(\Omega)$ is continuously embedded in $L^q(\Omega)$ with $q = 2r/(2-r)$, we conclude that $H^2(\Omega)$ is continuously embedded in $L^q(\Omega)$. Hence, if $\alpha p \le q$, for some $2 < p < 4$, then $u \circ u_a(f)$ is in $L^p(\Omega)$. The rest of the proof is quite similar to that of (i). □

We end this section by noting that Theorem 1.2 follows readily from Corollary 3.2 and Lemma 3.1.

4. Linearized inverse problem

Some parts of this section are borrowed from [4]. The main novelty of the results in this section consists in constructing complex geometric optic solutions in $W^{2,r}(\Omega)$ for any $r \in [2, \infty)$.

All functions we consider in this section are assumed to be complex-valued.

Fix $\xi \in \mathbb{S}^n$, $\mathfrak{q} \in L^\infty(\Omega)$ and, for $h > 0$, consider the operator

$$P_h = P_h(\mathfrak{q}, \xi) = e^{x\cdot\xi/h} h^2(-\Delta + \mathfrak{q}) e^{-x\cdot\xi/h}.$$

Clearly we can write P_h in the form

$$P_h = -h^2\Delta + 2h\xi \cdot \nabla - 1 + h^2\mathfrak{q}.$$

Lemma 4.1. (Carleman inequality) *Let $M > 0$. Then there exists a constant $c_\Omega > 0$ so that, for any $q \in B_{L^\infty(\Omega)}(M)$, $0 < h < h_0 = c_\Omega/(2M)$ and $u \in C_0^\infty(\Omega)$, we have*

$$h\|u\|_{L^2(\Omega)} \le 2c_\Omega^{-1}\|P_h u\|_{L^2(\Omega)}. \tag{4.1}$$

Proof. Let $P_h^0 = P_h(0, \xi)$. For $u \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\|P_h^0 u\|_{L^2(\Omega)}^2 &= \|(h^2\Delta + 1)u\|_{L^2(\Omega)}^2 \\ &\quad - 4h\Re((h^2\Delta + 1)u, \xi \cdot \nabla u)_{L^2(\Omega)} + h^2\|\xi \cdot \nabla u\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.2}$$

Simple integrations by parts yields

$$\Re((h^2\Delta + 1)u, \xi \cdot \nabla u)_{L^2(\Omega)} = 0.$$

This in (4.2) gives

$$\|P_h^0 u\|_{L^2(\Omega)}^2 \ge h^2\|\xi \cdot \nabla u\|_{L^2(\Omega)}^2. \tag{4.3}$$

From Poincaré's inequality and its proof, we have

$$\|\xi\cdot\nabla u\|^2_{L^2(\Omega)} \geq c_\Omega \|u\|_{L^2(\Omega)}.$$

This and (4.3) imply

$$\|P_h^0 u\|_{L^2(\Omega)} \geq c_\Omega h\|u\|_{L^2(\Omega)}. \tag{4.4}$$

Pick $\mathfrak{q}\in B_{L^\infty(\Omega)}(M)$. Since

$$\|P_h^0\|_{L^2(\Omega)} \leq \|P_h u\|_{L^2(\Omega)} + h^2 M\|u\|_{L^2(\Omega)},$$

we obtain from (4.6)

$$c_\Omega h\|u\|_{L^2(\Omega)} \leq \|P_h u\|_{L^2(\Omega)} + h^2 M\|u\|_{L^2(\Omega)}.$$

This inequality yields (4.1) in a straightforward manner. □

Proposition 4.1. *Let $M>0$. There exists a constant $c_\Omega>0$ so that, for any $\mathfrak{q}\in B_{L^\infty(\Omega)}(M)$ and $0<h<h_0=c_\Omega/(2M)$, we find $w\in L^2(\Omega)$ satisfying*

$$\left[e^{x\cdot\xi/h}(-\Delta+\mathfrak{q})e^{-x\cdot\xi/h}\right]w = f$$

and

$$\|w\|_{L^2(\Omega)} \leq 2c_\Omega^{-1}h\|f\|_{L^2(\Omega)}. \tag{4.5}$$

Proof. Pick $\mathfrak{q}\in B_{L^\infty(\Omega)}(M)$ and $\xi\in\mathbb{S}^{n-1}$. Let $H=P_h^*(C_0^\infty(\Omega))$ that we consider as a subspace of $L^2(\Omega)$. We observe that if $P_h=P_h(\mathfrak{q},\xi)$ then $P_h^*=P_h(\overline{\mathfrak{q}},-\xi)$. Therefore inequality (4.1) holds when P_h is substituted by P_h^*.

Let $f\in L^2(\Omega)$ and define on H the linear form

$$\ell(P_h^*v) = (v, h^2 f)_{L^2(\Omega)}, \quad v\in C_0^\infty(\Omega).$$

From Lemma 4.1, ℓ is bounded with

$$|\ell(P_h^*v)| \leq h^2\|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq 2c_\Omega^{-1}h\|f\|_{L^2(\Omega)}\|P_h^*v\|_{L^2(\Omega)}.$$

Hence, according to the Hahn-Banach extension theorem, there exists a linear form L extending ℓ to $L^2(\Omega)$ so that $\|L\|_{[L^2(\Omega)]'}=\|\ell\|_H$. In consequence

$$\|L\|_{[L^2(\Omega)]'} \leq 2c_\Omega^{-1}h\|f\|_{L^2(\Omega)}. \tag{4.6}$$

Applying Riesz's representation theorem, we find $w\in L^2(\Omega)$ such that

$$\|w\|_{L^2(\Omega)} = \|L\|_{[L^2(\Omega)]'} \tag{4.7}$$

and

$$(P_h^*v, w)_{L^2(\Omega)} = L(P_h^*v) = \ell(P_h^*v) = (v, h^2 f)_{L^2(\Omega)}, \quad v\in C_0^\infty(\Omega).$$

Hence

$$\left[e^{x\cdot\xi/h}(-\Delta+\mathfrak{q})e^{-x\cdot\xi/h}\right]w = f.$$

We complete the proof by noting that (4.5) is obtained by combining (4.6) and (4.7). □

Proposition 4.2. *Let $\mathcal{O} \Supset \Omega$, $M > 0$, $\mathfrak{q} \in B_{L^\infty(\Omega)}(M)$ and $u \in L^2(\mathcal{O})$ satisfying*

$$(-\Delta + \mathfrak{q}\chi_\Omega)u = 0 \text{ in } \mathscr{D}'(\mathcal{O}).$$

(i) We have $u \in H^1_{\rm loc}(\mathcal{O})$ and, for any $\Omega \Subset \Omega_1 \Subset \Omega_2 \Subset \mathcal{O}$, we have the following interior Caccioppoli type inequality

$$\|u\|_{H^1(\Omega_1)} \le C(1+M)\|u\|_{L^2(\Omega_2)},$$

where $C = C(\Omega, \mathcal{O}, d) > 0$ is a constant with $d = dist(\overline{\Omega_1}, \partial\Omega_2)$.
(ii) We have $u \in W^{2,r}_{\rm loc}(\mathcal{O})$ for any $1 < r < \infty$,

$$\|u\|_{W^{2,r}(\Omega)} \le C(1+M)^2\|u\|_{L^2(\mathcal{O})},$$

where $C = C(\Omega, \mathcal{O}, r) > 0$ is a constant.

Proof. Fix $\phi \in C_0^\infty(\mathcal{O})$. Then $v = \phi u$ is the solution of the BVP

$$\begin{cases} -\Delta v = -\mathfrak{q}\chi_\Omega \phi u - 2\nabla u \cdot \nabla\phi - \Delta\phi u & \text{in } \mathcal{O}, \\ v = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

Since

$$-\mathfrak{q}\chi_\Omega \phi u - 2\nabla u \cdot \nabla\phi - \Delta\phi u \in H^{-1}(\mathcal{O}),$$

we obtain $\phi u \in H^1_0(\mathcal{O})$.

Next, pick $\psi \in C_0^\infty(\Omega_2)$ satisfying $0 \le \psi \le 1$, $\psi = 1$ in a neighborhood of $\overline{\Omega_1}$ and $|\nabla\psi| \le \kappa$, where $\kappa > 0$ is a constant only depending on $\text{dist}(\overline{\Omega_1}, \partial\Omega_2)$. Let (v_k) be a sequence in $C_0^\infty(\Omega_2)$ converging to $\psi^2 u$ in $H^1(\Omega_2)$. We pass to the limit in the identity

$$\int_{\Omega_2} \nabla u \cdot \nabla\overline{v_k} dx + \int_{\Omega_2} qu\overline{v_k} dx = 0$$

in order to obtain

$$\int_{\Omega_2} \nabla u \cdot \nabla(\psi^2\overline{u}) dx + \int_{\Omega_2} \mathfrak{q}\psi^2|u|^2 dx = 0.$$

Hence

$$\int_{\Omega_2} |\psi\nabla u|^2 dx = -2\int_{\Omega_2} \psi\nabla u \cdot \overline{u}\nabla\psi - \int_{\Omega_2} q\psi^2|u|^2 dx. \tag{4.8}$$

For any $\epsilon > 0$, we have

$$|\psi\nabla u \cdot \overline{u}\nabla\psi| \le (\epsilon/2)|\psi\nabla u|^2 + (1/(2\epsilon))|u|^2|\nabla\psi|^2.$$

The particular choice $\epsilon = 1/2$ yields

$$|\psi\nabla u \cdot \overline{u}\nabla\psi| \le (1/4)|\psi\nabla u|^2 + |u|^2|\nabla\psi|^2.$$

This inequality together with (4.8) give

$$\int_{\Omega_1} |\nabla u|^2 dx \le \int_{\Omega_2} |\psi\nabla u|^2 dx \le 2(M+\kappa^2)\int_{\Omega_2} |u|^2 dx.$$

(ii) Let $\Omega \Subset \Omega_1 \Subset \Omega' \Subset \Omega_2 \Subset \mathcal{O}$ be subdomains. Let $\psi \in C_0^\infty(\Omega')$ satisfying $0 \le \psi \le 1$, $\psi = 1$ in a neighborhood of $\overline{\Omega_1}$. Then ψu is the solution of the BVP

$$\begin{cases} -\Delta(\psi u) = -\mathfrak{q}\chi_\Omega u - 2\nabla u \cdot \nabla\psi - \Delta\psi u & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega'. \end{cases}$$

From H^2 interior estimates (see for instance [21, Section 8.5]) $u = \psi u \in H^2(\Omega_1)$ and there exists a constant $c_\Omega > 0$ so that

$$\|u\|_{H^2(\Omega_1)} \le c_\Omega \|q\chi_\Omega u + 2\nabla u \cdot \nabla\psi + \Delta\psi u\|_{L^2(\Omega')}.$$

Hence

$$\|u\|_{H^2(\Omega_1)} \le C(1+M)\|u\|_{H^1(\Omega')},$$

where $C = C(\Omega, \mathcal{O}, \Omega_1, \Omega') > 0$ is a constant.

This inequality combined with (i) yields

$$\|u\|_{H^2(\Omega_1)} \le C(1+M)\|u\|_{L^2(\Omega_2)},$$

where $C = C(\Omega, \mathcal{O}, \Omega_1, \Omega', \Omega_2) > 0$ is a constant.

Assume $n > 2$ and set $r_0 = (2n)/(n-2)$. As $H^1(\Omega')$ is continuously embedded in $L^r(\Omega')$ for $r \in [1, r_0]$, we have

$$-\mathfrak{q}\chi_\Omega u - 2\nabla u \cdot \nabla\psi - \Delta\psi u \in L^r(\Omega'),$$

We then obtain by applying [7, Theorem 9.15, p. 241] that $u \in W^{2,r}(\Omega)$. Furthermore, [7, Lemma 9.17, p. 242] gives

$$\begin{aligned} \|u\|_{W^{2,r}(\Omega_1)} &\le C\|\mathfrak{q}\chi_\Omega u + 2\nabla u \cdot \nabla\psi + \Delta\psi u\|_{L^r(\Omega')} \\ &\le C(1+M)\|u\|_{H^2(\Omega')} \\ &\le C(1+M)^2\|u\|_{L^2(\Omega_2)}, \end{aligned}$$

where $C = C(\Omega, \mathcal{O}, \Omega_1, \Omega', \Omega_2) > 0$ is a constant.

If $r_0 < n$, we set $r_1 = (nr_0)/(n-r_0)$ and we repeat the preceding step where r_0 is substituted by r_1. We obtain that $u \in W^{2,r}(\Omega_1)$ for $r \in [1, r_1]$ and

$$\|u\|_{W^{2,r}(\Omega_1)} \le C(1+M)^2\|u\|_{L^2(\Omega_2)}.$$

If $r_0 < n$ and $r_1 < n$, r_2 given by $r_2 = (nr_1)/(n-r_1)$ satisfies

$$r_2 = r_1 + \frac{r_1^2}{n-r_1} \ge r_0 + 2\frac{r_0^2}{n-r_0},$$

where we used that the mapping $t \in [0, n[\mapsto t^2/(n-t)$ is increasing. By induction in $k \ge 1$, if $r_j < n$ for $0 \le j \le k$ we set $r_{k+1} = (nr_k)/(n-r_k)$. In that case we have

$$r_{k+1} \ge r_0 + (k+1)\frac{r_0^2}{n-r_0}.$$

Since the right hand side of this inequality tends to ∞ when k goes to ∞, we find a non negative integer k_n so that $r_j < n$ if $0 \le j \le k_n - 1$ and $r_{k_n} \ge n$.

We repeat the preceding arguments from r_0 until r_{k_n-1}. We obtain $u \in W^{2,r}(\Omega_1)$ with $r \in [1, r_{k_n-1}]$. If $r_{k_n} > n$, we complete the proof since $W^{1,r_{k_n}}(\Omega)$

is continuously embedded in $L^\infty(\Omega)$. Otherwise $r_{k_n+1} > n$ and we end up getting the expected result by a last step. □

Theorem 4.1. *Let $M > 0$ and $1 < r < \infty$. Then there exist $C = C(\Omega, r)$, $c_\Omega > 0$, $\kappa = \kappa(\Omega)$ so that, for any $\mathfrak{q} \in B_{L^\infty(\Omega)}(M)$, $\xi, \zeta \in \mathbb{S}^{n-1}$ satisfying $\xi \perp \zeta$ and $0 < h \le h_0 = c_\Omega/(2M)$, the equation*

$$(-\Delta + \mathfrak{q})u = 0 \quad \text{in } \Omega$$

admits a solution $u \in W^{2,r}(\Omega)$ of the form

$$u = e^{-x\cdot(\xi + i\zeta)/h}(1 + v),$$

where $v \in W^{2,r}(\Omega)$ satisfies

$$\|v\|_{L^2(\Omega)} \le 2c_\Omega^{-1} h.$$

Moreover, we have

$$\|u\|_{W^{2,r}(\Omega)} \le C(1 + M)^2 e^{\kappa/h}.$$

Proof. Fix $\mathcal{O} \Supset \Omega$ arbitrary. We first consider the equation

$$(-\Delta + \mathfrak{q}\chi_\Omega)u = 0 \quad \text{in } \mathcal{O}. \tag{4.9}$$

If $u = e^{-x\cdot(\xi+i\zeta)/h}(1 + v)$ then v should verify

$$\begin{aligned} &\left[e^{x\cdot\xi/h}(-\Delta + \mathfrak{q}\chi_\Omega)e^{-x\cdot\xi/h}\right]\left(e^{-ix\cdot\zeta/h} v\right) \\ &\quad = -\left[e^{x\cdot\xi/h}(-\Delta + \mathfrak{q})e^{-x\cdot\xi/h}\right]\left(e^{-ix\cdot\zeta/h}\right) = -\mathfrak{q}\chi_\Omega e^{-ix\cdot\zeta/h}. \end{aligned}$$

By Proposition 4.1, with Ω and $\mathfrak{q}$ substituted respectively by $\mathcal{O}$ and $\mathfrak{q}\chi_\Omega$, we find $w \in L^2(\mathcal{O})$ so that

$$\left[e^{x\cdot\xi/h}(-\Delta + \mathfrak{q}\chi_\Omega)e^{-x\cdot\xi/h}\right] w = -\mathfrak{q}\chi_\Omega e^{-ix\cdot\zeta/h}$$

and

$$\|w\|_{L^2\mathcal{O})} \le 2c_\Omega^{-1} h.$$

Let $v = e^{ix\cdot\zeta/h} w$. Then

$$\|v\|_{L^2(\mathcal{O})} \le 2c_\Omega^{-1} h$$

and $u = e^{-x\cdot(\xi+i\zeta)/h}(1 + v)$ is a solution of (4.9). Furthermore, we apply Proposition 4.2 in order to obtain

$$\begin{aligned} \|u\|_{W^{2,r}(\Omega)} &\le C(1 + M)\|e^{-x\cdot(\xi+i\zeta)/h}(1 + v)\|_{L^2(\mathcal{O})} \\ &\le C(1 + M)e^{\kappa/h}. \end{aligned}$$

This completes the proof. □

When $\mathfrak{q} \in L^\infty(\Omega)$ satisfies $\mathfrak{q} \ge -c$ almost everywhere, we can easily verify, with the help of Poincaré's inequality, that 0 does not belong to the spectrum of $-\Delta + \mathfrak{q}$ under Dirichlet boundary condition. For notational convenience we set

$$\mathcal{Q}_c = \{\mathfrak{q} \in L^\infty(\Omega);\ \mathfrak{q} \ge -c \text{ almost everywhere}\}.$$

Theorem 4.2. *Let $M > 0$ and $2 \le r < \infty$. For any $\mathfrak{q} \in \mathcal{Q}_c \cap B_{L^\infty(\Omega)}(M)$ and $f \in W^{2-1/r,r}(\Gamma)$, the BVP*

$$\begin{cases} (-\Delta + \mathfrak{q})u = 0 \ \textit{in}\ \Omega, \\ u = f \qquad\qquad \textit{on}\ \Gamma. \end{cases} \tag{4.10}$$

admits a unique solution $u_{\mathfrak{q}}(f) \in W^{2,r}(\Omega)$. Furthermore

$$\|u_{\mathfrak{q}}(f)\|_{W^{2,r}(\Omega)} \le C(1+M)\|f\|_{W^{2-1/r,r}(\Gamma)}, \tag{4.11}$$

where $C = C(\Omega, c, r) > 0$ is a constant.

Sketch of the proof. Let $2 \le r \le \infty$, $f \in W^{2-1/r,r}(\Gamma)$ and pick $F \in W^{2,r}(\Omega)$ so that $F = f$ on Γ and $\|F\|_{W^{2,r}(\Omega)} \le 2\|f\|_{W^{2-1/r,r}(\Gamma)}$. If u is a solution of (4.10) then $v = u - F$ must be a solution of the BVP

$$\begin{cases} (-\Delta + \mathfrak{q})v = g := \Delta F - \mathfrak{q}F \ \text{in}\ \Omega, \\ v = 0 \qquad\qquad\qquad\qquad\quad \text{on}\ \Gamma. \end{cases} \tag{4.12}$$

According to [21, Sections 8.5 and 8.6], the BVP (4.12) has a unique solution $v \in H^2(\Omega)$ so that

$$\begin{aligned} \|v\|_{H^2(\Omega)} &\le c_\Omega \left(\|\mathfrak{q}u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right) \\ &\le C\left(\|\mathfrak{q}u\|_{L^2(\Omega)} + \|g\|_{L^r(\Omega)}\right) \\ &\le C(1+M)\left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{2-1/r,r}(\Gamma)}\right), \end{aligned} \tag{4.13}$$

where $C = C(\Omega, c, r) > 0$ is a constant.

On the other hand from (4.12) we obtain

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &= -\int_\Omega \mathfrak{q}|u|^2 dx + \int_\Omega g\bar{v} dx \\ &\le c\int_\Omega |u|^2 dx + \|g\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}. \end{aligned}$$

From Poincaré's inequality

$$\lambda_1(\Omega)\int_\Omega |u|^2 dx \le \int_\Omega |\nabla u|^2 dx.$$

Hence

$$\|u\|_{L^2(\Omega)} \le (\lambda_1(\Omega) - c)^{-1}\|g\|_{L^2(\Omega)}.$$

This in (4.13) gives

$$\|v\|_{H^2(\Omega)} \le C(1+M)\|f\|_{W^{2-1/r,r}(\Gamma)}.$$

Here and henceforward $C = C(\Omega, c, r) > 0$ is a generic constant.

As in Proposition 4.2 we discuss separately cases $n = 2, 3$, $n = 4$ and $n > 4$. If $n > 4$, we know that $H^2(\Omega)$ is continuously embedded in $L^s(\Omega)$ for

$s \in [2, (2n)/(n-4)]$. We then apply [21, Theorem 9.15 p. 241 and Theorem 9.17 p. 242]. We conclude that $v \in W^{2,s}(\Omega)$ with

$$\begin{aligned}\|v\|_{W^{2,s}(\Omega)} &\le C\|-\mathfrak{q}v + g\|_{L^s(\Omega)}\\ &\le C(\|v\|_{L^s(\Omega)} + \|g\|_{L^r(\Omega)})\\ &\le C(\|v\|_{H^2(\Omega)} + \|g\|_{L^r(\Omega)})\\ &\le C(1+M)\|f\|_{W^{2-1/r,r}(\Gamma)}.\end{aligned}$$

The rest of the proof is quite similar to that Proposition 4.2. That is based on the iterated $W^{2,s}$ regularity and the corresponding a priori estimate. Finally, once we proved

$$\|v\|_{W^{2,r}(\Omega)} \le C(1+M)\|f\|_{W^{2-1/r,r}(\Gamma)},$$

we end up getting the expected inequality by noting that

$$\|u\|_{W^{2,r}(\Omega)} \le \|v\|_{W^{2,r}(\Omega)} + \|F\|_{W^{2,r}(\Omega)}.$$

The proof in then complete. □

In light of Theorem 4.2, we can define the Dirichlet-to-Neumann map associated to $r \in [2,\infty)$ and $\mathfrak{q} \in \mathcal{Q}_c$ as follows

$$\Lambda_{\mathfrak{q}}^r : f \in W^{2-1/r,r}(\Gamma) \mapsto \partial_\nu u_{\mathfrak{q}}(f) \in W^{1-1/r,r}(\Gamma).$$

Additionally, estimate (4.11) yields

$$\|\Lambda_{\mathfrak{q}}^r\| \le C(1+M), \quad \text{for any } \mathfrak{q} \in \mathcal{Q}_c \cap B_{L^\infty(\Omega)}(M),$$

where $C = C(\Omega, c, r) > 0$ is a constant and $\|\Lambda_q^r\|$ denotes the norm of $\Lambda_{\mathfrak{q}}^r$ in $\mathscr{B}(W^{2-1/r,r}(\Gamma), W^{1-1/r,r}(\Gamma))$.

We also define, for $\mathfrak{q} \in \mathcal{Q}_c$ and $r \in [2,\infty)$,

$$\mathscr{S}_{\mathfrak{q}}^r = \{u \in W^{2,r}(\Omega);\ (-\Delta + \mathfrak{q})u = 0 \text{ in } \Omega\}.$$

Lemma 4.2. *(Integral identity) For $r \in [2,\infty)$, $\mathfrak{q}, \tilde{\mathfrak{q}} \in \mathcal{Q}_c$, $u \in \mathscr{S}_{\mathfrak{q}}^r$ and $\tilde{u} \in \mathscr{S}_{\tilde{\mathfrak{q}}}^r$, we have*

$$\int_\Omega (\tilde{\mathfrak{q}} - \mathfrak{q})u\tilde{u}dx = \int_\Gamma (\Lambda_{\tilde{\mathfrak{q}}}^r - \Lambda_{\mathfrak{q}}^r)(u_{|\Gamma})\tilde{u}d\sigma(x). \tag{4.14}$$

Proof. Let $v = u_{\tilde{\mathfrak{q}}}(u_{|\Gamma})$. We obtain by applying Green's formula

$$\int_\Gamma \partial_\nu(u-v)\tilde{u}d\sigma(x) = \int_\Omega (\mathfrak{q}u - \tilde{\mathfrak{q}}v)\tilde{u}dx + \int_\Omega \nabla(u-v)\cdot\nabla\tilde{u}dx \tag{4.15}$$

and

$$0 = \int_\Gamma \partial_\nu\tilde{u}(u-v)d\sigma(x) = \int_\Omega \tilde{\mathfrak{q}}\tilde{u}(u-v)dx + \int_\Omega \nabla\tilde{u}\cdot\nabla(u-v)dx. \tag{4.16}$$

Identity (4.16) yields

$$\int_\Omega \nabla\tilde{u}\cdot\nabla(u-v)dx = -\int_\Omega \tilde{\mathfrak{q}}\tilde{u}(u-v)dx.$$

This inequality in (4.15) gives

$$\int_\Gamma \partial_\nu(u-v)\tilde{u}d\sigma(x) = \int_\Omega (\mathfrak{q} - \tilde{\mathfrak{q}})u\tilde{u}dx.$$

We end up getting the expected identity because

$$\int_\Gamma \partial_\nu (u-v)\tilde{u}d\sigma(x) = \int_\Gamma (\Lambda_{\tilde{\mathfrak{q}}}^r - \Lambda_{\mathfrak{q}}^r)(u_{|\Gamma})\tilde{u}d\sigma(x).$$

□

The following observation will be useful in the sequel: if $w \in H^t(\Omega)$, $0 < t < 1/2$, then $w\chi_\Omega \in H^t(\mathbb{R}^n)$ (see [8, p. 31]).

Theorem 4.3. *Let $M > 0$, $r \in [2,\infty)$ and $0 < s < 1/2$ and assume that $n \geq 3$. Then there exist two constants $C = C(\Omega, r, s) > 0$ and $\rho_0 = \rho_0(\Omega, M)$ so that, for any $\mathfrak{q}, \tilde{\mathfrak{q}} \in B_{H^s(\Omega)\cap L^\infty(\Omega)}(M) \cap \mathscr{Q}_c$, we have*

$$C\|\mathfrak{q} - \tilde{\mathfrak{q}}\|_{L^2(\Omega)} \leq 1/\rho^\gamma + \mathfrak{D}(1+M)^4 e^{\kappa\rho}, \quad \rho \geq \rho_0.$$

with $\gamma = \min(1/2, s/n)$ and

$$\mathfrak{D} = \|\Lambda_{\mathfrak{q}}^r - \Lambda_{\tilde{\mathfrak{q}}}^r\|_{\mathscr{B}\left(W^{2-1/r,r}(\Gamma), W^{1-1/r,r}(\Gamma)\right)}.$$

Proof. Pick $\mathfrak{q}, \tilde{\mathfrak{q}} \in B_{H^s(\Omega)\cap L^\infty(\Omega)}(M) \cap \mathscr{Q}_c$. Let $k, \tilde{k} \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{S}^{n-1}$ so that $k \perp \tilde{k}$, $k \perp \xi$ and $\tilde{k} \perp \xi$ (this is possible because $n \geq 3$). We assume that $|\tilde{k}| = \rho$ with $\rho \geq \rho_0 = h_0^{-1}$ where h_0 is as Theorem 4.1. Let then

$$h = h(\rho) = \frac{1}{(|k|^2/4 + \rho^2)^{1/2}} \ (\leq h_0).$$

Set

$$\zeta = h(k/2 + \tilde{k}), \quad \tilde{\zeta} = h(k/2 - \tilde{k})$$

As we have seen in the proof of Theorem 4.1, $\zeta, \tilde{\zeta} \in \mathbb{S}^{n-1}$, $\zeta \perp \xi$, $\tilde{\zeta} \perp \xi$ and $\zeta + \tilde{\zeta} = hk$.

By Theorem 4.1, the equation

$$(-\Delta + \mathfrak{q})u = 0 \quad \text{in } \Omega$$

admits a solution $u \in W^{2,r}(\Omega)$ of the form

$$u = e^{-x\cdot(\xi + i\zeta)/h}(1+v)$$

so that, for some constants $C = C(\Omega, r) > 0$ and $\kappa = \kappa(\Omega)$,

$$\|v\|_{L^2(\Omega)} \leq Ch \tag{4.17}$$

and

$$\|u\|_{W^{2,r}(\Omega)} \leq C(1+M)^2 e^{\kappa/h}. \tag{4.18}$$

Similarly, the equation

$$(-\Delta + \tilde{\mathfrak{q}})u = 0 \quad \text{in } \Omega$$

admits a solution $\tilde{u} \in W^{2,r}(\Omega)$ of the form

$$\tilde{u} = e^{-x\cdot(-\xi + i\tilde{\zeta})/h}(1+\tilde{v}),$$

with

$$\|\tilde{v}\|_{L^2(\Omega)} \leq Ch \tag{4.19}$$

and

$$\|\tilde{u}\|_{W^{2,r}(\Omega)} \leq C(1+M)^2 e^{\kappa/h}. \tag{4.20}$$

We use the following temporary notations

$$w = (v + \tilde{v} + v\tilde{v})e^{-ix\cdot k}, \quad g = u_{|\Gamma}, \quad \tilde{g} = \tilde{u}_{|\Gamma}.$$

We find by applying the integral identity (4.14)

$$\int_\Omega (\mathfrak{q} - \tilde{\mathfrak{q}})e^{-ix\cdot k}dx = -\int_\Omega (\mathfrak{q} - \tilde{\mathfrak{q}})w dx + \int_\Gamma (\Lambda^r_{\mathfrak{q}} - \Lambda^r_{\tilde{\mathfrak{q}}})(g)\tilde{g} d\sigma(x).$$

Hence, in light of (4.17) and (4.19), we deduce that

$$|\hat{\mathfrak{p}}(k)| \leq Ch(\rho) + \mathfrak{D}\|g\|_{W^{2-1/r,r}(\Gamma)}\|\tilde{g}\|_{W^{2-1/r,r}(\Gamma)}, \quad k \in \mathbb{R}^n \setminus \{0\},\ \rho \geq \rho_0, \tag{4.21}$$

with $\mathfrak{p} = (\mathfrak{q} - \tilde{\mathfrak{q}})\chi_\Omega$ (in $H^s(\mathbb{R}^n)$).

On the other hand, inequalities (4.18) and (4.20) yield

$$\|g\|_{W^{2-1/r,r}(\Gamma)} \leq C_0\|u\|_{W^{2,r}(\Omega)} \leq C(1+M)^2 e^{\kappa/h},$$
$$\|\tilde{g}\|_{W^{2-1/r,r}(\Gamma)} \leq C_0\|\tilde{u}\|_{W^{2,r}(\Omega)} \leq C(1+M)^2 e^{\kappa/h},$$

where $C_0 = C_0(\Omega, r) > 0$ is a constant

These estimates in (4.21) yield

$$C|\hat{\mathfrak{p}}(k)| \leq h(\rho) + \mathfrak{D}(1+M)^4 e^{\kappa/h(\rho)}, \quad k \in \mathbb{R}^n \setminus \{0\},\ \rho \geq \rho_0.$$

That is we have

$$C|\hat{\mathfrak{p}}(k)| \leq 1/\rho + \mathfrak{D}(1+M)^4 e^{\kappa(|k|/2+\rho)}, \qquad k \in \mathbb{R}^n \setminus \{0\},\ \rho \geq \rho_0.$$

Hence

$$C\int_{|k|\leq \rho^{1/n}} |\hat{p}(k)|^2 dk \leq 1/\rho + \mathfrak{D}(1+M)^4 e^{\kappa\rho}, \quad \rho \geq \rho_0. \tag{4.22}$$

Moreover,

$$\begin{aligned}\int_{|k|\geq \rho^{1/n}} |\hat{\mathfrak{p}}(k)|^2 dk &\leq \rho^{-2s/n}\int_{|k|\geq h^{-\alpha}} |k^{2s}|\hat{\mathfrak{p}}(k)|^2 dk \\ &\leq \rho^{-2s/n}\|p\|^2_{H^s(\mathbb{R}^n)}.\end{aligned} \tag{4.23}$$

Now inequalities (4.22) and (4.23) together with Planchel-Parseval's identity give

$$C\|\mathfrak{q} - \tilde{\mathfrak{q}}\|_{L^2(\Omega)} \leq 1/\rho^\gamma + \mathfrak{D}(1+M)^4 e^{\kappa\rho}, \quad \rho \geq \rho_0. \tag{4.24}$$

with $\gamma = \min(1/2, s/n)$ and $C = C(\Omega, r, s) > 0$ is a constant. □

5. Proof of the main result

Before we proceed to the proof of Proposition 1.1, we establish a lemma. To this end, let $\mathfrak{X} = H^2(\Omega)$ if $n \le 3$ and $\mathfrak{X} = W^{2,p}(\Omega)$ if $n \ge 4$, where p is as in Theorem 1.2.

Lemma 5.1. *For any $a \in \mathscr{A}(\mathfrak{c}, \alpha)$, the mapping*

$$f \in \mathscr{X}_0 \mapsto u_a(f) \in \mathfrak{X}$$

is continuous.

Proof. Pick $f, h \in \mathscr{X}_0$. If $u = u_a(f+h) - u_a(f)$ then simple computations give that u is the solution of the BVP

$$\begin{cases} -\Delta u + ru = 0 \text{ in } \Omega, \\ u = h \qquad\qquad \text{on } \Gamma, \end{cases}$$

where

$$r(x) = \int_0^1 a'(u_a(f)(x) + s[u_a(f+h) - u_a(f)](x))ds.$$

We can then mimic the proof of Theorem 4.2 in order to find a constant $C > 0$ independent on h so that

$$\|u\|_{\mathfrak{X}} \le \|h\|_{\mathscr{X}_0}.$$

Thus the continuity of $f \in \mathscr{X}_0 \mapsto u_a(f) \in \mathfrak{X}$ follows. □

Proof of Proposition 1.1. We give the proof in case (i). The proof for cases (ii) and (iii) is quite similar.

Since the trace operator

$$w \in H^2(\Omega) \mapsto \partial_\nu w_\Gamma \in H^{1/2}(\Gamma)$$

is bounded, it is sufficient to prove that

$$f \in H^{3/2}(\Gamma) \mapsto u_a(f) \in H^2(\Omega)$$

is Fréchet differentiable.

Fix $N > 0$ and let $f \in B_{H^{3/2}(\Gamma)}(N)$. Then, for any $h \in B_{H^{3/2}(\Gamma)}(1)$, we have $f + h \in B_{H^{3/2}(\Gamma)}(M)$, with $M = N + 1$.

Let $v = v_{a,f}(h)$ and

$$w = u_a(f+h) - u_a(f) - v.$$

It is then straightforward to verify that w is the solution of the BVP

$$\begin{cases} -\Delta w = F \text{ in } \Omega, \\ w = 0 \qquad \text{on } \Gamma, \end{cases}$$

with

$$\begin{aligned} F(x) &= -a(u_a(f+h))(x) + a(u_a(f))(x) + a'(u_a(f)(x))v(x) \\ &= -\int_0^1 \{a'(u_a(f)(x) + s[u_a(f+h)(x) - u_a(f)(x)]) \\ &\qquad\qquad \times [u_a(f+h)(x) - u_a(f)(x)] - a'(u_a(f)(x))v(x)\}ds \\ &= -\int_0^1 a'(u_a(f)(x) + s[u_a(f+h)(x) - u_a(f)(x)])w(x)ds \\ &\quad + \int_0^1 \{a'(u_a(f)(x) + s[u_a(f+h)(x) - u_a(f)(x)]) \\ &\qquad\qquad - a'(u_a(f)(x))\}v(x)ds, \end{aligned}$$

where $v = v_{a,f}(h)$.

We decompose F as $F = -\mathfrak{q}w + G$, where

$$\begin{aligned} \mathfrak{q}(x) &= \int_0^1 a'(u_a(f)(x) + s[u_a(f+h)(x) - u_a(f)(x)])ds, \\ G(x) &= \int_0^1 \{a'(u_a(f)(x) + s[u_a(f+h)(x) - u_a(f)(x)]) \\ &\qquad\qquad - a'(u_a(f)(x))\}v(x)ds. \end{aligned}$$

Under these new notations, we see that w is the solution of the BVP

$$\begin{cases} -\Delta w + \mathfrak{q}w = G \text{ in } \Omega, \\ w = 0 \qquad\qquad \text{on } \Gamma. \end{cases}$$

According to Corollary 3.2, we have

$$\|u_a(f+h)\|_{L^\infty(\Omega)} \le C,$$

where $C = C(\Omega, \mathfrak{c}, \alpha, M) > 0$ is a constant.

Using (1.8) for estimating the integrand of the definition of $\mathfrak{q}(x)$ and applying triangle's inequality, we obtain

$$\|\mathfrak{q}\|_{L^\infty(\Omega)} \le \|a'(0)\|_{L^\infty(\Omega)} + \varkappa_C C := \tilde{C}.$$

We obtain from the usual a priori H^2-estimate (e.g., [21, Sections 8.5 and 8.6]) that

$$\|w\|_{H^2(\Omega)} \le \hat{C}\|G\|_{L^2(\Omega)},$$

where $\hat{C} = \hat{C}(\Omega, a, \mathfrak{c}, \alpha, M)$ is a constant. But

$$\begin{aligned} \|G\|_{L^2(\Omega)} &\le \varkappa_C \|u_a(f+h) - u_a(f)\|_{L^2(\Omega)} \|v\|_{L^\infty(\Omega)} \\ &\le \varkappa_C c_\Omega \|u_a(f+h) - u_a(f)\|_{L^2(\Omega)} \|v\|_{H^2(\Omega)}. \end{aligned}$$

Therefore, again from H^2 a priori estimates for v, we have

$$\|w\|_{H^2(\Omega)} \le \hat{C}\varkappa_C c_\Omega \|u_a(f+h) - u_a(f)\|_{L^2(\Omega)} \|h\|_{H^{3/2}(\Gamma)}.$$

Now we complete the proof of the differentiability of $f \mapsto u_a(f)$ by using that, according to Lemma 5.1, the mapping

$$f \in H^{3/2}(\Gamma) \mapsto u_a(f) \in H^2(\Omega)$$

is continuous. □

Define

$$\mathfrak{q}_{a,f} := a' \circ u_a(f).$$

In order to apply the results of the preceding section we need to extend $\Lambda_{\mathfrak{q}_{a,f}}$ to complex-valued functions from $H^{3/2}(\Gamma)$. As $\mathfrak{q}_{a,f}$ is real-valued, this extension is obviously given by

$$\Lambda_{\mathfrak{q}_{a,f}}(f+ig) = \Lambda_{\mathfrak{q}_{a,f}}(f) + i\Lambda_{\mathfrak{q}_{a,f}}(g), \quad f,g \in H^{3/2}(\Gamma) \text{ real-valued.}$$

It is then useful to observe that this extension is entirely determined by it restriction to real-valued functions from $H^{3/2}(\Gamma)$.

Proceeding as in the proof of Proposition 1.1, we prove the following result.

Lemma 5.2. *Let β be as in Theorem 1.3. Under the assumptions and the notations of Proposition 1.1, we have $\mathfrak{q}_{a,f} \in C^{0,\beta}(\overline{\Omega})$ and*

$$\|\mathfrak{q}_{a,f}\|_{C^{0,\beta}(\overline{\Omega})} \le C. \tag{5.1}$$

Here the constant $C > 0$ is so that $C = C(\Omega, \mathfrak{c}, \alpha, M)$ if $n = 2$ or $n = 3$; $C = C(\Omega, \mathfrak{c}, \alpha, p, r)$ if $n = 4$; $C = C(\Omega, \mathfrak{c}, \alpha, M, p)$ if $n > 4$.

Following [8, Definition 1.3.2.1, p. 16], the space $H^t(\Omega)$, $0 < t < 1$, consists of functions $w \in L^2(\Omega)$ satisfying

$$\int_\Omega \int_\Omega \frac{|w(x) - w(y)|^2}{|x-y|^{n+2t}} dxdy < \infty.$$

Let $0 < t < \theta \le 1$ and $w \in C^{0,\theta}(\overline{\Omega})$. Then

$$\frac{|w(x) - w(y)|^2}{|x-y|^{n+2t}} \le \frac{[w]_\theta^2}{|x-y|^{n+2t-2\theta}}, \tag{5.2}$$

where

$$[w]_\theta = \sup\{|w(x) - w(y)|/|x-y|^\theta;\ x, y \in \overline{\Omega},\ x \ne y\}.$$

On the other hand, for any $\epsilon > 0$, we have

$$\int_{B(x,\epsilon)} \frac{1}{|x-y|^{n+2t-2\theta}} dy = \int_{\mathbb{S}^{n-1}} d\omega \int_0^\epsilon \frac{1}{t^{2t-2\theta+1}} dt. \tag{5.3}$$

Consequently, since the integral in (5.3) is convergent by $2t - 2\theta + 1 < 1$, in terms of inequality (5.2) we can directly see that $C^{0,\theta}(\overline{\Omega})$ is continuously embedded in $H^t(\Omega)$. Hence an immediate consequence of the previous lemma is the following corollary.

Corollary 5.1. *Let β be as in Theorem 1.3. Under the assumptions and the notations of Proposition 1.1, we have $\mathfrak{q}_{a,f} \in C^{0,\beta}(\overline{\Omega}) \cap H^s(\Omega)$ for $0 < s < \min(1/2, \beta)$ and*

$$\|\mathfrak{q}_{a,f}\|_{C^{0,\beta}(\overline{\Omega})} + \|\mathfrak{q}_{a,f}\|_{H^s(\Omega)} \le C, \tag{5.4}$$

where the constant $C > 0$ can be described as $C = C(\Omega, \mathfrak{c}, \alpha, M)$ if $n = 2$ or $n = 3$; $C = C(\Omega, \mathfrak{c}, \alpha, p, r)$ if $n = 4$; $C = C(\Omega, \mathfrak{c}, \alpha, M, p)$ if $n > 4$.

Proof of Theorem 1.3. In this proof $C>0$, $\rho_0>0$ and $\kappa>0$ are generic constants only depending : on $(\Omega,\mathfrak{c},\alpha,M,s)$ if $n=2,3$, $(\Omega,\mathfrak{c},\alpha,M,s,p,r)$ if $n=4$, $(\Omega,\mathfrak{c},\alpha,M,s,p)$ if $n>4$. The constants p and r are the same as in Theorem 1.3.

Using (5.4) for both a and $\tilde{a}$, we obtain by applying Theorem 4.3

$$C\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{L^2(\Omega)}\le 1/\rho^\gamma+\mathfrak{D}(f)e^{\kappa\rho},\quad \rho\ge\rho_0, \tag{5.5}$$

where $\gamma=\min(1/2,s/n)$ and

$$\mathfrak{D}(f)=\|\Lambda_a'(f)-\Lambda_{\tilde{a}}'(f)\|_{\mathscr{Y}}.$$

Now the interpolation inequality in [5, Lemma B.1] gives

$$\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{C(\overline{\Omega})}\le C_0\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{C^{0,\beta}(\Omega)}^{n/(n+2\beta)}\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{L^2(\Omega)}^{2\beta/(n+2\beta)}. \tag{5.6}$$

Inequalities (5.6) and (5.4) both for a and $\tilde{a}$ imply

$$\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{C(\overline{\Omega})}\le C\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{L^2(\Omega)}^{2\beta/(n+2\beta)}. \tag{5.7}$$

We find by putting (5.7) in (5.5)

$$C\|\mathfrak{q}_{a,f}-\tilde{\mathfrak{q}}_{\tilde{a},f}\|_{C(\overline{\Omega})}^{(n+2\beta)/(2\beta)}\le 1/\rho^\gamma+\mathfrak{D}(f)e^{\kappa\rho},\quad \rho\ge\rho_0.$$

Using this inequality with $f=\lambda$ such that $|\lambda|\le M$, we have

$$C\left[\max_{|\lambda|\le M}|a'(\lambda)-\tilde{a}'(\lambda)|\right]^{(n+2\beta)/(2\beta)}\le 1/\rho^\gamma+\mathfrak{D}_Me^{\kappa\rho},\quad \rho\ge\rho_0, \tag{5.8}$$

with

$$\mathfrak{D}_M=\sup_{\|f\|_{\mathscr{X}_0}\le\sqrt{|\Gamma|}M}\mathfrak{D}(f).$$

Since $a(0)=\tilde{a}(0)$, we have

$$\max_{|\lambda|\le M}|a(\lambda)-\tilde{a}(\lambda)|\le\max_{|\lambda|\le M}|a'(\lambda)-\tilde{a}'(\lambda)|.$$

This in (5.8) yields

$$C\left[\max_{|\lambda|\le M}|a(\lambda)-\tilde{a}(\lambda)|\right]^{(n+2\beta)/(2\beta)}\le 1/\rho^\gamma+\mathfrak{D}_Me^{\kappa\rho},\quad \rho\ge\rho_0. \tag{5.9}$$

For completing the proof we choose $\rho\ge\rho_0$ which makes the right-hand side nearly minimum. Let $\tau=\rho_0e^{\kappa_0}$. Since the mapping $\rho\in[0,\infty)\mapsto\rho^\gamma e^{\kappa\rho}$ is increasing, we see that if $\mathfrak{D}_M<\mu=\min(1,\tau^{-1})$, then we can find $\rho_1\ge\rho_0$ so that $1/\rho_1^\gamma=\mathfrak{D}_Me^{\kappa\rho_1}$. Therefore, by taking $\rho=\rho_1$ in (5.9), we find

$$C\left[\max_{|\lambda|\le M}|a(\lambda)-\tilde{a}(\lambda)|\right]^{(n+2\beta)/(2\beta)}\le 1/\rho_1^\gamma.$$

Now elementary computations show that $\rho_1^{-1}\le(\kappa+\gamma)|\ln\mathfrak{D}_M|^{-1}$. Hence

$$\max_{|\lambda|\le M}|a(\lambda)-\tilde{a}(\lambda)|\le C|\ln\mathfrak{D}_M|^{-(2\beta\gamma/(n+2\beta)}.$$

When $\mathfrak{D}_M\ge\mu$, we have

$$\max_{|\lambda|\le M}|a(\lambda)-\tilde{a}(\lambda)|\le C\le C\mu^{-1}\mathfrak{D}_M.$$

We complete the proof by putting together the last two inequalities. □

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